

# Ungleichungen und ähnlich verwirrende Konzepte

## Coffee

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# Outline

## 1 Introduction

- ksjladüys

## 2 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices

## 3 Grothendieck Inequality

# Let's get on the same page

- We should know what a *state* is
- We should know what the *tensor product* does
- We should be familiar with the *Dirac notation*

# Quantum systems

- A quantum system is a portion of the whole universe. For example a set of electrons.
- A quantum system  $X$  is associated with a copy of  $\mathbb{C}^k$
- It may consist of subsystems  $X_1, \dots, X_N$  each of which is associated with a copy of  $\mathbb{C}^{n_i}$ . In this case  $k = n_1 \dots n_N$

# Measurements

- A measurement can be performed on a system  $X$  that is in state  $\rho$
- Let  $\mathcal{A}$  be a finite set of outcomes of the measurement
- The measurement itself is defined by a set of psd matrices  $\{F^a\}_{a \in \mathcal{A}} \subseteq \mathbb{C}^{n \times n}$  that sum up to the identity matrix, i.e.  $\sum_{a \in \mathcal{A}} F^a = I$

# Measurements

- A projective measurement is defined by psd matrices that satisfy
$$F^a F^b = \delta_{ab} F^a \quad \forall a, b \in \mathcal{A}$$
- The outcome of a measurement is a random variable  $\chi$  with probability distribution:  $\mathbb{P}[\chi = a] = \text{Tr}(\rho F^a)$
- To define an expected value we define outcomes in  $\mathcal{A}$  as real numbers

# Measurements

- $\mathbb{E}[\chi] = \sum_{a \in \mathcal{A}} a \text{Tr}(\rho F^a) = \text{Tr}(\rho(\sum_{a \in \mathcal{A}} a F^a))$
- $\sum_{a \in \mathcal{A}} a F^a$  is called observable
- A simple case we will use later are  $\{-1, 1\}$ -valued observables
- if we consider projective measurements we have

$$(F^+ - F^-)^2 = \underbrace{F^{+2}}_{=F^+} - \underbrace{F^+ F^-}_{\delta_{+-}=0} + \underbrace{F^{-2}}_{F^-} = F^+ + F^- = I$$

- i.e. a  $\{-1, 1\}$ -valued observable is both unitary and Hermitian

# Doling out subsystems

- Consider a system  $X$  consisting of subsystems  $X_1, \dots, X_N$  which we distribute among  $N$  parties, which may be located anywhere in the universe
- The parties *share* the state  $X$  is in
- Every party may perform a measurement on their subsystem  $X_i$ , i.e. there are  $N$  sets of psd matrices  $\{F^{a_1}\}_{a_1 \in \mathcal{A}_1} \in \mathbb{C}^{n_1 \times n_1}, \dots, \{F^{a_N}\}_{a_N \in \mathcal{A}_N} \in \mathbb{C}^{n_N \times n_N}$



The joint probability distribution of the  $N$  measurement outcomes  $\chi_1, \dots, \chi_N$  is

$$\mathbb{P}[\chi_1 = a_1, \chi_2 = a_2, \dots, \chi_N = a_N] = \text{Tr}(\rho F_1^{a_1} \otimes \dots \otimes F_N^{a_N})$$

# Entanglement

- We will only consider pure states meaning states that they have rank 1 and therefore can be written as  $\rho = |\psi\rangle\langle\psi|$
- A state is called product state if it can be written as  $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle \dots |\psi_N\rangle$
- When a vector  $|\psi\rangle$  is referred to as a state we mean the matrix  $|\psi\rangle\langle\psi|$
- A state that is not a product state is called entangled

# Example

- Let  $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$  be a system and give  $|\psi_A\rangle$  to Alice and  $|\psi_B\rangle$  to Bob
- Let them perform measurements  $\{G^b\}_{b \in \mathcal{B}}$  and  $\{F^a\}_{a \in \mathcal{A}}$  on their respective quantum systems
- What is the probability of Alice getting measurement outcome  $\chi_A = a$  and Bob getting  $\chi_B = b$ ?

## Example

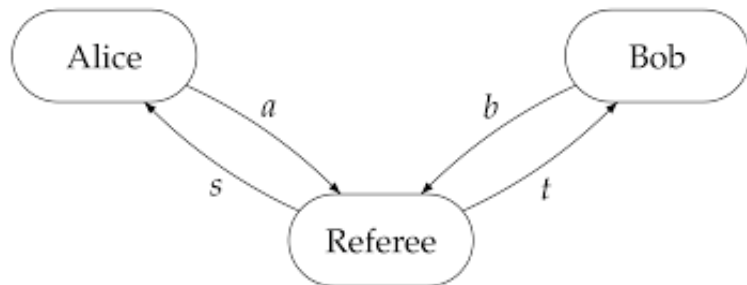
$$\begin{aligned}\mathrm{Tr}(|\psi\rangle\langle\psi|F^a \otimes G^b) &= \langle\psi|F^a \otimes G^b|\psi\rangle \\ &= (\langle\psi_A| \otimes \langle\psi_B|)(F^a \otimes G^b)(|\psi_A\rangle \otimes |\psi_B\rangle) \\ &= ((\langle\psi_A|F^a) \otimes (\langle\psi_B|G^b))(|\psi_A\rangle \otimes |\psi_B\rangle) \\ &= \langle\psi_A|F^a|\psi_A\rangle \otimes \langle\psi_B|G^b|\psi_B\rangle \\ &= \langle\psi_A|F^a|\psi_A\rangle\langle\psi_B|G^b|\psi_B\rangle\end{aligned}$$

This is equal to the product of the probabilities of Alice measuring  $a$  and Bob measuring  $b$ , i.e. the outcome do not correlate.

# Nonlocal games

- Three participants: Alice, Bob and a referee
- Referee does out a question  $s$  to Alice and a question  $t$  to Bob
- Alice and Bob are assumed to be located anywhere in the universe respectively
- Alice and Bob must not communicate
- Alice sends answer  $a$  Bob sends answer  $b$  back to the referee, who then decides whether both win or both lose

## Nonlocal games



# Mathematically speaking

- Four finite sets  $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$
- probability distribution  $\pi$  over  $\mathcal{S} \times \mathcal{T}$   
 $\pi : \mathcal{S} \times \mathcal{T} \rightarrow [0, 1]$
- The referee sends with probability  $\pi(s, t)$   $s$  to Alice and  $t$  to Bob
- They answer with an element  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  respectively
- A map  $V : \mathcal{S} \times \mathcal{T} \times \mathcal{A} \times \mathcal{B} \rightarrow \{0, 1\}$
- They win if  $V(s, t, a, b) = 1$  and lose otherwise

# Classical strategies

- All players know  $\pi$  and  $V$  and the information they received but not what the other players received
- They are allowed to agree on a strategy beforehand



# Classical strategies

- All players know  $\pi$  and  $V$  and the information they received but not what the other players received
- They are allowed to agree on a strategy beforehand but must not communicate once the game started
- A deterministic strategy is a map  $a : \mathcal{S} \rightarrow \mathcal{A}$  for Alice and  $b : \mathcal{T} \rightarrow \mathcal{B}$  for Bob The winning probability then is:

$$\mathbb{E}_{s,t \sim \pi} [V(a(s), b(t), s, t)]$$

# Quantum case

- Suppose Alice and Bob have a subsystem  $X_A, X_B$  of a quantum system  $X$  which is in state  $\rho$ , i.e. Alice and Bob share state  $\rho$
- If the state is entangled measurements can give correlated measurement outcomes
- Alice and Bob may gain information by performing measurements
- Answering according to measurement outcomes could increase winning probability

# Mathematically speaking

- A quantum system  $X$  consisting of two  $n$ -dimensional subsystems  $X_A, X_B$  in some entangled state  $\rho$
- Alice performs a measurement  $\{F_s^a\}_{a \in \mathcal{A}} \subseteq \mathbb{C}^{n \times n}$  on her subsystem  $X_A$  and Bob performs a measurement  $\{G_t^b\}_{b \in \mathcal{B}} \subseteq \mathbb{C}^{n \times n}$  on his subsystem  $X_B$
- They send their measurement outcome as their answer to the referee
- Their winning probability is:

$$\mathbb{E}_{s,t \sim \pi} \left[ \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \text{Tr}(\rho F_s^a \otimes G_t^b) V(a, b, s, t) \right]$$

- Since states are convex combinations of pure states and the trace function is linear we only need to consider pure entangled states

## Two player XOR games

- Let the sets  $\mathcal{A}$  and  $\mathcal{B}$  be  $\{0, 1\}$ , so Alice and Bob both answer either with 1 or 0
- The predicate  $V$  is defined as  $V(a, b, s, t) = [a \oplus b = f(s, t)]$ , where  $f : \mathcal{S} \times \mathcal{T} \rightarrow \{0, 1\}$
- A truth table for  $a \oplus b$  looks like this

$\oplus$	0	1
0	0	1
1	1	0

# Bias and violation ratio

- Alice and Bob can always win with probability  $\frac{1}{2}$  by flipping an unbiased coin
- The classical bias of an XOR game  $G$  is defined as the difference of the probabilities of winning and losing for an optimal strategy and denoted by  $\beta(G)$
- The bias  $\beta^*(G)$  of entangled strategies is calculated the same way
- It is twice the amount by which the maximal winning probability exceeds  $\frac{1}{2}$
- The violation ratio is defined as  $\frac{\beta^*(G)}{\beta(G)}$

# Signs and observables

- It is convenient to use the  $\{-1, 1\}$ -basis instead of the  $\{0, 1\}$ -basis for boolean valued objects.
- Let  $a : \mathcal{S} \rightarrow \{0, 1\}$  and  $b : \mathcal{T} \rightarrow \{0, 1\}$  be classical strategies and  $\pi$  the probability distribution the referee uses to pick  $s, t$
- The bias is given by the probability under  $\pi$  that  $a(s) \oplus b(t) = f(s, t)$  minus the probability under  $\pi$  that  $a(s) \oplus b(t) \neq f(s, t)$

This means the bias can be written as:

$$\begin{aligned} & \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{[a(s) \oplus b(t) = f(s,t)]} \right] = \\ &= \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{a(s) \oplus b(t) + f(s,t)} \right] = \\ &= \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{a(s)} (-1)^{b(t)} (-1)^{f(s,t)} \right] \end{aligned}$$

And we can define the sign matrix  $\Sigma_{s,t} = (-1)^{f(s,t)}$  and functions  $\chi(s) = (-1)^{a(s)}$  and  $\psi(t) = (-1)^{b(t)}$ . So the bias is

$$\mathbb{E}_{(s,t) \sim \pi} [\chi(s) \psi(t) \Sigma_{st}]$$



- The outcomes in an XOR game are  $\{0, 1\}$
- Alice and Bob have measurements  $\{F_s^0, F_s^1\}$  and  $\{G_t^0, G_t^1\}$  and share an entangled state
- The probability of Alice and Bob answering with  $a, b$  upon receiving  $s, t$  respectively is  $\langle \psi | F_s^a \otimes G_t^b | \psi \rangle$
- Lets calculate the expected value of  $(-1)^{a \oplus b}$

$$\begin{aligned}
 & (1) \cdot \mathbb{P}[a = b] + (-1) \cdot \mathbb{P}[a \neq b] = \\
 & = \langle \psi | F_s^0 \otimes G_t^0 | \psi \rangle + \langle \psi | F_s^1 \otimes G_t^1 | \psi \rangle \\
 & \quad - \langle \psi | F_s^1 \otimes G_t^0 | \psi \rangle - \langle \psi | F_s^0 \otimes G_t^1 | \psi \rangle \\
 & = \langle \psi | (F_s^0 - F_s^1) \otimes (G_t^0 - G_t^1) | \psi \rangle
 \end{aligned}$$

- Define  $\{-1, 1\}$ -observables  $F_s = F_s^0 - F_s^1$  and  $G_t = G_t^0 - G_t^1$  with the property that its difference squared is the identity matrix
- Using this strategy the bias becomes

$$\mathbb{E}_{(s,t) \sim \pi} [\langle \psi | F_s \otimes G_t | \psi \rangle \Sigma_{s,t}]$$

## More generally speaking

- For any XOR game the bias is defined as the difference of the probabilities of winning and losing
- Which is, if considering the  $\{-1, 1\}$  basis, the expected value
- We are looking to maximize this quantity

# Classical strategies

When using classical strategies this is

$$\max\{\mathbb{E}_{(s,t)\sim\pi} [\Sigma_{st}\chi(s)\psi(t)] : \chi : \mathcal{S} \rightarrow \{-1, 1\}, \\ \psi : \mathcal{T} \rightarrow \{-1, 1\}\}$$

# Entangled strategies

When using entangled strategies the winning probability might increase indefinitely with the dimensions, so we use the  $\sup_{n \in \mathbb{N}}$

$$\sup_{n \in \mathbb{N}} \{ \mathbb{E}_{(s,t) \sim \pi} [\sum_{st} \langle \psi | F_s \otimes G_t | \psi \rangle] : |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n, \\ F_s, G_t \in O(\mathbb{C}^n) \}$$

# The CHSH game

- The CHSH game (Clauser, Horner, Shimony, Holt) is a two player XOR game with  $\mathcal{A} = \mathcal{B} = \mathcal{S} = \mathcal{T} = \{0, 1\}$  and  $\pi$  being the uniform distribution
- $f(s, t) = s \wedge t$ , i.e.  $f(1, 1) = 1$  and  $f(0, 0) = f(0, 1) = f(1, 0) = 0$
- Alice and Bob can win  $\frac{3}{4}$  of the games by using deterministic strategies  $(0, 0)$ ,  $(1, 0)$  or  $(0, 1)$

# Quantum strategy

- Let Alice and Bob share an EPR state
- Define

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $XY + YX = 0$  and  $X^2 = Y^2 = I$
- For Alice define the observable for question 0 by  $F_0 = X$  and for question 1 by  $F_1 = Y$
- Bobs observables are going to be  $G_0 = (X - Y)/\sqrt{2}$  for question 0 and  $G_1 = (X + Y)/\sqrt{2}$  for question 1

The following auxiliary calculations will be helpful later:

$$\begin{aligned}\langle \text{EPR} | X \otimes X | \text{EPR} \rangle &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{2}{2} = 1\end{aligned}$$



$$\begin{aligned}
 \langle \text{EPR} | Y \otimes Y | \text{EPR} \rangle &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -1
 \end{aligned}$$

$$\begin{aligned}
 \langle \text{EPR} | X \otimes Y | \text{EPR} \rangle &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} i & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0
 \end{aligned}$$

$$\langle \text{EPR} | Y \otimes X | \text{EPR} \rangle = 0$$

Lets calculate the expected values of the sign  $a \oplus b$ :

$$\begin{aligned}\langle \text{EPR} | F_0 \otimes G_0 | \text{EPR} \rangle &= \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}}(X - Y) | \text{EPR} \rangle \\ &= \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}}X | \text{EPR} \rangle - \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}}Y | \text{EPR} \rangle \\ &= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}
 \langle \text{EPR} | F_1 \otimes G_1 | \text{EPR} \rangle &= \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} (X + Y) | \text{EPR} \rangle \\
 &= \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} X | \text{EPR} \rangle + \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} Y | \text{EPR} \rangle \\
 &= 0 - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
\langle \text{EPR} | F_0 \otimes G_1 | \text{EPR} \rangle &= \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}} (X + Y) | \text{EPR} \rangle \\
&= \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}} X | \text{EPR} \rangle + \langle \text{EPR} | X \otimes \frac{1}{\sqrt{2}} Y | \text{EPR} \rangle \\
&= \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\langle \text{EPR} | F_1 \otimes G_0 | \text{EPR} \rangle &= \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} (X - Y) | \text{EPR} \rangle \\
&= \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} X | \text{EPR} \rangle - \langle \text{EPR} | Y \otimes \frac{1}{\sqrt{2}} Y | \text{EPR} \rangle \\
&= 0 - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}
\end{aligned}$$

Thus, we have

$$\langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle = \begin{cases} \frac{1}{\sqrt{2}}, & (0, 0), (1, 0), (0, 1) \\ -\frac{1}{\sqrt{2}}, & (1, 1) \end{cases}$$

which is equivalent to

$$\langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle = \frac{(-1)^{s \wedge t}}{\sqrt{2}}, s, t \in \{0, 1\}$$

The bias of the entangled strategy equals

$$\begin{aligned}\mathbb{E}_{(s,t) \sim \pi} [\Sigma_{s,t} \langle \psi | F_s \otimes G_t | \psi \rangle] &= \\ &= \frac{1}{4} \sum_{s,t=0}^1 (-1)^{s \wedge t} \langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle \\ &= \frac{1}{4} \cdot \frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}}\end{aligned}$$

The bias is  $\frac{1}{\sqrt{2}}$  from which follows that the winning probability is by definition:

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \cos(\pi/8) \approx 0.85 \dots$$



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Nice slide to draw the connection between the games an LC

## Definition

Let  $(X_i)_{1 \leq i \leq m}$  and  $(Y_j)_{1 \leq j \leq n}$  be families of random variables on a common probability space such that  $|X_i|, |Y_j| \leq 1$  almost surely. Then  $A = (a_{ij})$  is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all  $1 \leq i \leq m, 1 \leq j \leq n$ .

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- Set of all local correlation matrices:  $\text{LC}_{m,n}$

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## Lemma

$$\text{LC}_{m,n} = \text{conv}\{\xi \eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}$$

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## Lemma

$$\text{LC}_{m,n} = \text{conv}\{\xi \eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}$$

- No matter which probabilistic strategy there is a deterministic one which is at least as good as the one one chooses

$$LC_{m,n} \supset \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}.$$

- $\xi\eta^T \in LC_{m,n}$  for all  $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$  (Choose  $X_i \equiv \xi_i, Y_j \equiv \eta_j$ )



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- Suffices to show that  $LC_{m,n}$  is convex.
- Let  $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$  for  $k \in \{0, 1\}$



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- Suffices to show that  $LC_{m,n}$  is convex.
- Let  $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$  for  $k \in \{0, 1\}$
- Find  $(X_i), (Y_j)$  with  $|X_i|, |Y_j| \leq 1$  almost surely such that

$$\beta a_{ij}^{(0)} + (1 - \beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for  $\beta \in [0, 1]$



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$$\beta a_{ij}^{(0)} + (1 - \beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for  $\beta \in [0, 1]$

- Define a Bernoulli random variable  $\alpha$  such that  $\mathbb{P}(\alpha = 0) = \beta$ ,  $\mathbb{P}(\alpha = 1) = 1 - \beta$  and set  $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$



$$LC_{m,n} \supset \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}.$$

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- Then

$$\begin{aligned} \mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=1\}}] \\ &= \beta \mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1 - \beta) \mathbb{E}[X_i^{(1)} Y_j^{(1)}] \end{aligned}$$



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$$\begin{aligned} a_{ij} &= \mathbb{E}[X_i Y_j] = \mathbb{E}\left[\left(\sum_{\xi \in \{-1, 1\}^m} \lambda_\xi^{(X)} \xi_i\right) \left(\sum_{\eta \in \{-1, 1\}^n} \lambda_\eta^{(Y)} \eta_j\right)\right] \\ &= \sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}] \xi_i \eta_j \\ &= \left(\sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)}] \mathbb{E}[\lambda_\eta^{(Y)}]\right) \xi_i \eta_j \end{aligned}$$



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- $\sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)}] \mathbb{E}[\lambda_\eta^{(Y)}] = 1$  the matrix  $(a_{ij})$  is a convex combination of  $\xi\eta^T$ ,  $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$



Some nice frame to connect QCs to the games

## Definition

Let  $(X_i)_{1 \leq i \leq m}$  and  $(Y_j)_{1 \leq j \leq n}$  be self-adjoint operators on  $\mathbb{C}^{d_1}$ , respectively  $\mathbb{C}^{d_2}$  for some positive integers  $d_1, d_2$ , satisfying  $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$ .  $A = (a_{ij})$  is called *quantum correlation matrix* if there exists a state on  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$  such that

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## Lemma

$$\text{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\},$$

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- The vectors have still not the right dimension but again, we can project them onto vectors in  $\mathbb{R}^m$



- In order to show

$$\text{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

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### Proposition

For all  $n \geq 1$  there is a subspace of the  $2^n \times 2^n$  Hermitian matrices where every vector is the multiple of a unitary matrix.

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### Proposition

For all  $n \geq 1$  there is a subspace of the  $2^n \times 2^n$  Hermitian matrices where every vector is the multiple of a unitary matrix.

- The proof is based on  $n$ -fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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## Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$

$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)}, \quad i = 1, \dots, n$$

- $U_i$ 's are anti-commuting traceless Hermitian unitaries, i.e.  $U_i U_j = -U_j U_i$  for  $i \neq j$  and  $U_i^2 = I$
- For  $X = \sum_{i=1}^{2n} \xi_i U_i$ ,  $Y = \sum_{i=1}^{2n} \eta_i U_i$  we can calculate

$$\begin{aligned} XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i, j \leq 2n} \xi_i \eta_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\ &= \langle \xi, \eta \rangle I. \end{aligned}$$

- The result follows by setting  $X = Y$ .

$$QC_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

- Let  $(x_i)_{1 \leq i \leq m}, (y_j)_{1 \leq j \leq n} \subset \mathbb{R}^{\min\{m,n\}}$  such that  $|x_i|, |y_j| \leq 1$ .

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- $\text{Tr}(X_i Y_j^T) = d \cdot \langle x_i, y_j \rangle$  and  $\|X_i\|_\infty \leq 1$  since  $X_i X_i^* = |x_i|^2 I$  and  $|x_i|^2 \leq 1$  (the same holds for  $Y_j$ )



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- Let  $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$  and  $\rho = |\phi\rangle \langle \phi|$ . Note that we can write  $\rho$  as

$$\rho = |\phi\rangle \langle \phi| = \frac{1}{d} \sum_{1 \leq k, l \leq d} |kk\rangle \langle ll| = \frac{1}{d} \sum_{1 \leq k, l \leq d} |k\rangle \langle l| \otimes |k\rangle \langle l|$$

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$$\begin{aligned} \text{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i \otimes |k\rangle \langle l| Y_j) = \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i) \text{Tr}(|k\rangle \langle l| Y_j) \\ &= \frac{1}{d} \text{Tr} X_i Y_j^T = \langle x_i, y_j \rangle. \end{aligned}$$



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- Right dimension is obtained by projecting  $(\tilde{x}_i)_{1 \leq i \leq m}$ ,  $(\tilde{y}_j)_{1 \leq j \leq n}$  on  $\text{span}\{x_1, \dots, x_m\}$  or  $\text{span}\{y_1, \dots, y_n\}$ , as in the proof before.



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- We can also write it as in intersections of halfspaces:

$$\text{LC}_{2,2} = \{A \in \mathbb{R}^{2 \times 2} \mid -1 \leq \text{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \quad (1)$$

$$\text{where } \mathcal{K} = \left\{\frac{1}{2}\sigma\left(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right), \sigma\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \mid \sigma \in \{\text{id}, (1\ 2), (1\ 3), (1\ 4)\}\right\}.$$

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- Bound is achieved by  $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , induced by the vectors  $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$  and  $y_1 = y_2 = (1, 0)$



## Lemma (Grothendieck's identity)

Let  $x, y \in \mathbb{R}^d$  be unit vectors. Let  $r \in \mathbb{R}^d$  be a random unit vector chosen from  $O(d)$ -invariant probability distribution on the unit sphere. Then

- i,  $\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] = \frac{\arccos(\langle x, y \rangle)}{\pi}$
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  - ▶ the normalized vector  $n := s/\|s\|$  is uniformly distributed on the intersection of the unit sphere and  $\text{span}\{x, y\}$  by the  $O(d)$ -invariance of the probability distribution

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