Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 6, 2018

Outline

1. Local and quantum correlation matrices
   - Local correlation matrices
   - Quantum correlation matrices

Nice slide to draw the connection between the games an LC

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Definition

Let $(X_i)_{1\le i\le m}$ and $(Y_j)_{1\le j\le n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \le 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \le i \le m, 1 \le j \le n$.

Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

[

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}]$

$\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$
(Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

Suffices to show that $\mathrm{LC}_{m,n}$ is convex.

Let $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$ for $k \in \{0,1\}$
Find $(X_i), (Y_j)$ with $|X_i|, |Y_j| \leq 1$ almost surely such that

$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for $\beta \in [0,1]$

Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$

Then

$$\mathbb{E}[X_i Y_j] = \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(1)}]\mathbb{1}_{\{\alpha=1\}}]$$
$$= \beta\mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1-\beta)\mathbb{E}[X_i^{(1)} Y_j^{(1)}]$$

$\mathrm{LC}_{m,n} \subset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.
pause

Set $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.

Hypercube description by its vertices:
$[-1,1]^d = \mathrm{conv}\{\xi \mid \xi \in \{-1,1\}^d\}$

Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$
Using the same decomposition for Y we obtain

$$a_{ij} = \mathbb{E}[X_i Y_j] = \mathbb{E}\big[\big(\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}\xi_i\big)\big(\sum_{\eta \in \{-1,1\}^n} \lambda_\eta^{(Y)}\eta_j\big)\big]$$

Some nice frame to connect QCs to the games

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$.
$A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state
Introduce a symbol zo define operators form one space to another
$\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

$$a_{ij} = \operatorname{Tr} \rho(X_i \otimes Y_j).$$

Definition

Let $(X_i)_{1\le i\le m}$ and $(Y_j)_{1\le j\le n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \le 1$.
$A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state
Introduce a symbol zo define operators form one space to another
$\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

$$a_{ij} = \operatorname{Tr} \rho(X_i \otimes Y_j).$$

Set of all quantum correlation matrices denoted by $\mathrm{QC}_{m,n}$

Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1\le 1\le m, 1\le j\le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\},$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

$a_{ij} = \operatorname{Tr} \rho X_i \otimes Y_j$, sate ρ on a Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and Hermitian operators $(X_i)_{1 \geq m}$, $(Y_j)_{1 \geq n}$ on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$

Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \operatorname{Re}(\operatorname{Tr} \rho ST)$.

perhaps verification of at least some of these properties

Obtain an inner product space $U := B^{sa}(\mathcal{H}) / \ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\operatorname{Tr}(\rho X_i \otimes Y_j) = a_{ij}$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\leq 1\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\mathrm{Tr}\left(\rho X_i \otimes Y_j\right) = a_{ij}$$

$\beta(X \otimes I, X \otimes I), \beta(I \otimes Y, I \otimes Y) \leq 1$
(this can be shown by using a *Schmidt-decomposition* of ρ and using $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$)

Project the y_j's orthogonally onto $\mathrm{span}\{x_1, ..., x_m\}$ (wlog $m \leq n$)

$\pi(y_j)$ the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$

Let $\{a_1, ..., a_r\}$ be an orthonormal basis of $\mathrm{span}\{x_1, ..., x_m\}$ with respect to β and $x_i = \sum_{k=1}^r \alpha_k^{(i)} a_k$ and $\pi(y_j) = \sum_{k=1}^r \gamma_k^{(j)} a_k$ for $\alpha^{(i)}, \gamma^{(j)} \in \mathbb{R}^r$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\leq 1\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

Let $\{a_1, ..., a_r\}$ be an orthonormal basis of $\operatorname{span}\{x_1, ..., x_m\}$ with respect to β and $x_i = \sum_{k=1}^r \alpha_k^{(i)} a_k$ and $\pi(y_j) = \sum_{k=1}^r \gamma_k^{(j)} a_k$ for $\alpha^{(i)}, \gamma^{(j)} \in \mathbb{R}^r$

$$\begin{aligned} a_{ij} = \tilde{\beta}(x_i, y_j) = \tilde{\beta}(x, \pi(y)) &= \sum_{1\leq k,l\leq r} \alpha_k^{(i)} \gamma_l^{(j)} \tilde{\beta}(a_k, a_l) \\ &= \sum_{k=1}^r \alpha_k^{(i)} \gamma_k^{(j)} = \langle \alpha^{(i)}, \gamma^{(j)} \rangle. \end{aligned}$$

$\left|\alpha^{(i)}\right|, \left|\gamma^{(j)}\right| \leq 1$ due to $\tilde{\beta}(x_i), \tilde{\beta}(y_j) \leq 1$

$$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$