Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

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Outline

1. Local and quantum correlation matrices
   - Local correlation matrices
   - Quantum correlation matrices
   - The relations between quantum correlation and local correlation matrices
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
   - Gorthendieck-Tsirelson Theorem

Local correlation matrices

- Deterministic strategies of Alice and Bob correspond to vectors $\xi \in \{-1,1\}^{\mathcal{S}}$, respectively $\eta \in \{-1,1\}^{\mathcal{T}}$.

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- Their common answer is the product $\xi_i \eta_j$.
- Instead of deterministic strategies they can also answer according to the probability distribution of random variables X_i, Y_j.
- Their common answer is $\mathbb{E}[X_i Y_j]$.
- This information can be encoded in an $\mathcal{S} \times \mathcal{T}$ matrix. ◂ Motivation Quantum

Definition

Let $(X_i)_{1 \le i \le m}$ and $(Y_j)_{1 \le j \le n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \le 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses.

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$).

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- Define a Bernoulli random variable α (which is independent form $X_i^{(k)}, Y_j^{(k)}$) such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1-\beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$.

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- Define a Bernoulli random variable α (which is independent form $X_i^{(k)}, Y_j^{(k)}$) such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$.
- Then

$$\begin{aligned}
\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=1\}}] \\
&= \mathbb{E}[X_i^{(0)} Y_j^{(0)}]\mathbb{P}(\alpha = 0) + \mathbb{E}[X_i^{(1)} Y_j^{(1)}]\mathbb{P}(\alpha = 1) \\
&= \beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)}.
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- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables .$(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.

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- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$.

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- Using the same decomposition for Y we obtain

$$\begin{aligned} a_{ij} = \mathbb{E}[X_i Y_j] &= \mathbb{E}\Big[\Big(\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)} \xi_i\Big)\Big(\sum_{\eta \in \{-1,1\}^n} \lambda_\eta^{(Y)} \eta_j\Big)\Big] \\ &= \sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}\big] \xi_i \eta_j. \end{aligned}$$

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- Due to $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}\big] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$.

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Quantum correlation matrices

- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$.

Motivation Locality

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- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$.
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- They perform measurements $\{F_i^\xi\}_{\xi=\pm 1}$, respectively $\{G_j^\eta\}_{\eta=\pm 1}$.
- The probability that their response is (ξ, η) for inputs (i, j) is given by $a_{ij} = \operatorname{Tr}(\rho F_i^\xi \otimes G_j^\eta)$.
- Again we can encode this information in a matrix.

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Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\le i\le m, 1\le j\le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$$

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- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathscr{H}$ by $\beta : B^{(sa)}(\mathscr{H}) \times B^{(sa)}(\mathscr{H}) \to \mathbb{R}$ where $\beta(S, T) = \mathrm{Re}(\mathrm{Tr}\,\rho ST)$.

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- The vectors still have not the right dimension but again, we can project them onto vectors in $\mathbb{R}^m$.

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we will use the following:

Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every element is the multiple of a unitary matrix.

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Proposition

For all $n \ge 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every element is the multiple of a unitary matrix.

- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Proof.

- Define $U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)}$,
  $U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)}$ for $i = 1, \ldots n$.

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- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$.
- For $X = \sum_{i=1}^{2n} \xi_i U_i$ we can calculate

$$\begin{aligned} XX^* = XX &= \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i \neq j \leq 2n} \xi_i \xi_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j - \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i^2 I = |\xi|^2 I. \end{aligned}$$

□

- Set

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \ldots n.$$

- Moreover, taking two linear combinations $X = \sum_{i=1}^{2n} \xi_i U_i,\ Y = \sum_{i=1}^{2n} = \eta_i U_i$ and consider the trace of the product we get

$$\begin{aligned}\operatorname{Tr}(XY) &= \sum_{i=1}^{2n} \xi_i \eta_i \operatorname{Tr} I + \sum_{1 \leq i \neq j \leq\ 2n} \xi_i \eta_j \operatorname{Tr}(U_i U_j) \\ &= \sum_{i=1}^{2n} \xi_i \eta_j 2^n = 2^n \cdot \langle \xi, \eta \rangle,\end{aligned}$$

where we used that the product $U_i U_j$ is traceless for $i \neq j$.

$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \le i \le m, 1 \le j \le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$

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- Let $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i\rangle\otimes|i\rangle$ and $\rho = |\psi\rangle\langle\psi|$. Note that we can write ρ as

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{d}\sum_{1\le k,l\le d} |k\rangle\otimes|k\rangle\langle l|\otimes\langle l| = \frac{1}{d}\sum_{1\le k,l\le d} |k\rangle\langle l|\otimes|k\rangle\langle l|.$$

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- Then

$$\begin{aligned}\mathrm{Tr}\left(\rho X_i \otimes Y_j\right) &= \frac{1}{d} \sum_{1\le k,l\le d} \mathrm{Tr}\left(|k\rangle\langle l| X_i \otimes |k\rangle\langle l| Y_j\right) \\ &= \frac{1}{d} \sum_{1\le k,l\le d} \mathrm{Tr}\left(|k\rangle\langle l| X_i\right) \mathrm{Tr}\left(|k\rangle\langle l| Y_j\right) \\ &= \frac{1}{d} \mathrm{Tr}\, X_i Y_j^T = \langle x_i, y_j\rangle.\end{aligned}$$

$\square$

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- Define vectors $\tilde{x}_i := \begin{pmatrix} \sqrt{\lambda} x_i \\ \sqrt{1-\lambda}\bar{x}_i \end{pmatrix}, \tilde{y}_j := \begin{pmatrix} \sqrt{\lambda} y_j \\ \sqrt{1-\lambda}\bar{y}_j \end{pmatrix}$ for $\lambda \in [0,1]$.

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- It holds $|\tilde{x}_i| \leq \lambda \left| \begin{pmatrix} x_i \\ 0 \end{pmatrix} \right| + (1-\lambda) \left| \begin{pmatrix} 0 \\ \bar{x}_i \end{pmatrix} \right| \leq 1$ and $\langle \tilde{x}_i, \tilde{y}_j \rangle = \lambda \langle x_i, y_j \rangle + (1-\lambda) \langle \bar{x}_i, \bar{y}_j \rangle$.

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\mathrm{span}\{x_1, \ldots, x_m\}$ or $\mathrm{span}\{y_1, ..., y_n\}$, as in the proof before.

The relations between quantum correlation and local correlation matrices

- $\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$
- $\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\le i\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$
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- $\mathrm{LC}_{m,n} \subset \mathrm{QC}_{m,n}$
- Set $x_i = \xi_i |1\rangle$ and $y_j = \eta_j |1\rangle$ it immediately follows $\xi_i\eta_j = \langle x_i, y_j\rangle$. Hence, $\xi\eta^T \in \mathrm{QC}_{m,n}$ (rest follows with the convexity of $\mathrm{QC}_{m,n}$).

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- It holds

$$\mathsf{LC}_{2,2} = \operatorname{conv}\{\pm\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\}.$$

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- W can also write it as in intersections of halfspaces:

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \,|\, -1 \leq \mathrm{Tr}\, AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{1}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}), \sigma(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \,|\, \sigma \in \{\mathrm{id}, (1\ 2), (1\ 3), (1\ 4)\}\}$.

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$$\begin{aligned}\operatorname{Tr}(AM) &= \langle x_1, y_1\rangle + \langle x_1, y_2\rangle + \langle x_2, y_1\rangle - \langle x_2, y_2\rangle \\ &= \langle x_1 + x_2, y_1\rangle + \langle x_1 - x_2, y_2\rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\ &\leq |x_1 + x_2| + |x_1 - x_2|.\end{aligned}$$

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- Bound is achieved by $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$.

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{2}$$

(2) *For any $M \in \mathbb{R}^{m \times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \tag{3}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} M(\xi \eta^\top)^\top. \tag{4}$$

(3) *For any $M \in \mathbb{R}^{m \times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{i,j} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} \xi^\top M \eta. \tag{5}$$

1 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices
- Motivation: The Grothendieck-Tsirelson Theorem
- **Grothendieck's Inequality**
- Tsirelson's Theorem
- Gorthendieck-Tsirelson Theorem

Lemma (Grothendieck's identity)

Let $x, y \in S^{d-1}$. Let $r \in S^{d-1}$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\mathrm{sign}(\langle x, r\rangle) \neq \mathrm{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\mathrm{sign}(\langle x, r\rangle)\,\mathrm{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

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  - if $x = y$: $\quad \arccos(\langle x, y\rangle) = \arccos(1) = 0$.

Lemma (Grothendieck's identity)

Let $x, y \in S^{d-1}$. Let $r \in S^{d-1}$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

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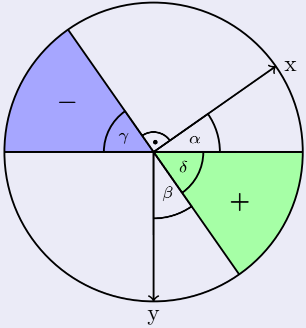
  - the normalized vector $n := s/|s|$ is uniformly distributed on the intersection of the unit sphere and $\text{span}\{x, y\}$ by the $O(d)$-invariance of the probability distribution.

Proof (cont.).

Calculation of the probability that the signs of the scalar products $\langle x, n \rangle$ and $\langle y, n \rangle$ are unlike:



$$\mathbb{P}[\text{sign}(\langle x, n \rangle) \neq \text{sign}(\langle y, n \rangle)] = 2\frac{\arccos(\langle x, y \rangle)}{2\pi} = \frac{\arccos(\langle x, y \rangle)}{\pi}$$

Proof (cont.).

We conclude with the proof of the second part of Lemma 5:

$$\begin{aligned}\mathbb{E}[\operatorname{sign}(\langle x, r\rangle)\operatorname{sign}(\langle y, r\rangle)] \\ &= 1 - 2\frac{\arccos(\langle x, y\rangle)}{\pi} \\ &= \frac{2}{\pi}\arcsin(\langle x, y\rangle),\end{aligned}$$

because $\arcsin(t) + \arccos(t) = \pi/2$. $\square$

Lemma (Krivine's trick)

Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ *be given. Furthermore, let* $r \in S^{n+m-1}$ *be a random unit vector chosen form the* $O(n+m-1)$*-invariant probability distribution on the unit sphere. Then there are* $x'_1, \ldots, x'_m, y'_1, \ldots, y'_n \in S^{m+n-1}$ *so that*

$$\mathbb{E}[\operatorname{sign}(\langle x'_i, r\rangle)\operatorname{sign}(\langle y'_j, r\rangle)] = \beta\langle x_i, y_j\rangle, \tag{6}$$

with $\beta = \frac{2}{\pi}\ln(1+\sqrt{2})$.

Definition (The k-th tensor product)

The *k-th tensor product* of $\mathbb{R}^n$ with orthonormal basis $e_1, \dots, e_n$ is denoted by $(\mathbb{R}^n)^{\otimes k}$ and it is a Euclidean vector space of dimension n^k with othonormal basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$, $i_l \in \{1, \dots, n\}$. In particular

$$\begin{aligned}\langle e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle &= \prod_{l=1}^{k} \langle e_{i_l}, e_{j_l} \rangle \\ &= \begin{cases} 1 & \text{, if } i_l = j_l \text{ for all } l = 1, \dots, n, \\ 0 & \text{, otherwise,} \end{cases} \end{aligned} \tag{7}$$

and for $v \in \mathbb{R}^n$ with $v = v_1 e_1 + \cdots + v_n e_n$ we define $v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k}$ by

$$\begin{aligned} v^{\otimes k} &:= (v_1 e_1 + \cdots + v_n e_n) \otimes \cdots \otimes (v_1 e_1 + \cdots + v_n e_n) \\ &= \sum_{i_1, \dots, i_k} v_{i_1} \cdots v_{i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \end{aligned} \tag{8}$$

Thus, for $v, w \in \mathbb{R}^n$

$$\langle v^{\otimes k}, w^{\otimes k} \rangle = \langle v, w \rangle^k. \tag{9}$$

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- Idea: To find β, x'_i, y'_j invert E:

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}}_{=:g_{2k+1}} t^{2k+1},$$

Proof (cont.).

- Define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2k+1}. \tag{11}$$

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- Define $\tilde{x}_i, \tilde{y}_j \in H$, $i = 1, \ldots, m, j = 1, \ldots, n$ componentwise:

$$(\tilde{x}_i)_k = \operatorname{sign}(g_{2k+1}) \sqrt{|g_{2k+1}| \beta^{2k+1}}\, x_i^{\otimes 2k+1} \tag{12}$$

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- Then

$$\begin{aligned}\langle \tilde{x}_i, \tilde{y}_j \rangle &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1} \langle x_i^{\otimes 2k+1}, y_j^{\otimes 2k+1} \rangle \\ &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1} \langle x_i, y_j \rangle^{2k+1} \\ &= E^{-1}(\beta \langle x_i, y_j \rangle).\end{aligned}$$

Proof (cont.).

- Hence, β is defined by the condition that the vectors $\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh(\frac{\pi}{2}\beta)$$

$$\Leftrightarrow \qquad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1 + \sqrt{2})$$

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- Problem: $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are infinite-dimensional
- Solution: the positive definite and symmetric Gram matrix G

$$G = \begin{pmatrix} \langle \tilde{x}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{x}_m \rangle & \langle \tilde{x}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{x}_m, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{x}_m \rangle & \langle \tilde{x}_m, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{y}_n \rangle \\ \langle \tilde{y}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{x}_m \rangle & \langle \tilde{y}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{y}_n, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{x}_m \rangle & \langle \tilde{y}_n, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{y}_n \rangle \end{pmatrix} \tag{14}$$

Proof (cont.).

- Due to the properties of G we can decompose G via a real orthogonal matrix Q with columns that are the eigenvectors of G and a real diagonal matrix Λ having the eigenvalues of G on the diagonal, thus

$$G = Q\Lambda Q^\top = (\underbrace{Q\Lambda^{1/2}}_{=:A})^\top (Q\Lambda^{1/2}). \tag{15}$$

□

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- The columns of A are the vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ we are looking for.

□

Definition

For $M \in \mathbb{R}^{m\times n}$ define the quadratic program

$$\|M\|_{\infty\to 1} = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\xi_i\eta_j : \xi_i^2 = 1, i = 1,\ldots,m, \eta_j^2 = 1, j = 1,\ldots,n\right\}$$
$$= \max\left\{\operatorname{Tr} M\eta\xi^\top : \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\right\}. \tag{16}$$

Definition

The SDP relaxation of $\|M\|_{\infty\to 1}$ is given via:

$$\mathrm{sdp}_{\infty\to 1}(M) = \max \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\langle x_i, y_j\rangle$$
$$x_i, y_j \in \mathbb{R}^{m+n}$$
$$|x_i| = 1, i = 1,\ldots,m$$
$$|y_j| = 1, j = 1,\ldots,n$$

Theorem (Grothendieck's inequality)

There exists a constant K such that for all $M \in \mathbb{R}^{m \times n}$:

$$\|M\|_{\infty\to 1} \leq \mathrm{sdp}_{\infty\to 1}(M) \leq K\|M\|_{\infty\to 1}. \tag{17}$$

Proof.

Use the following approximation algorithm with randomized rounding:

Algorithm 1: Approximation algorithm with randomized rounding for $\|M\|_{\infty\to 1}$

1. Solve $\mathrm{sdp}_{\infty\to 1}(M)$. Let $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$ be the optimal unit vectors.
2. Apply Krivine's trick (Lemma 8) and use vectors x_i, y_j to create new unit vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$.
3. Choose $r \in S^{m+n-1}$ randomly.
4. Round: $\xi_i = \mathrm{sign}(\langle x'_i, r\rangle)$
   $\eta_j = \mathrm{sign}(\langle y'_j, r\rangle)$

Proof (cont.).

Expected quality of the outcome:

$$\begin{aligned}\|M\|_{\infty\to 1} &\geq \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\xi_i\eta_j\right]\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\mathbb{E}[\operatorname{sign}(\langle x_i', r\rangle)\operatorname{sign}(\langle y_j', r\rangle)]\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\beta\langle x_i, y_j\rangle\\ &= \beta\,\mathrm{sdp}_{\infty\to 1}(M),\end{aligned}$$

where the second last equality follows by Krivine's trick with $\beta = \frac{2\ln(1+\sqrt{2})}{\pi}$, thus $K \leq \beta^{-1}$.

Theorem (Tsirelson)

(Hard direction) For all positive integers n, r and any $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in S^{r-1}$, there exists a positive integer $d := d(r)$, a unit vector $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $\{-1,1\}$-observables $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathbb{C}^{d \times d}$, such that for every $i, j \in \{1, \ldots, n\}$, we have

$$\langle \psi | F_i \otimes G_j | \psi \rangle = \langle x_i, y_j \rangle. \tag{18}$$

Moreover, $d \leq 2^{\lceil r/2 \rceil}$.

(Easy direction) Conversely, for all positive integers n, d, unit vectors $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $\{-1,1\}$-observables $F_1, \ldots, F_n, G_1, \ldots, G_n \in \mathbb{C}^{d \times d}$, there exist a positive integer $r := r(d)$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^{r-1}$ such that for every $i, j \in \{1, \ldots, n\}$, we have

$$\langle x_i, y_j \rangle = \langle \psi | F_i \otimes G_j | \psi \rangle. \tag{19}$$

Moreover, $r \leq 2d^2$.

Proof.

For the hard direction look at the second part of the proof of
$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \le i \le m, 1 \le j \le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$ (Lemma).
Due to the additional restriction/assumption that F_i and G_j are $\{-1, 1\}$-observables the other direction gets easier.

□

Since

1. Local and quantum correlation matrices
   - Local correlation matrices
   - Quantum correlation matrices
   - The relations between quantum correlation and local correlation matrices
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
   - Gorthendieck-Tsirelson Theorem

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{20}$$

(2) *For any $M \in \mathbb{R}^{m\times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \sum_{i,j} M_{ij}\xi_i\eta_j \tag{21}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} M(\xi\eta^\top)^\top. \tag{22}$$

(3) *For any $M \in \mathbb{R}^{m\times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{ij}\langle x_i, y_j\rangle \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} \xi^\top M\eta. \tag{23}$$

Proof.

Since (23) is a direct consequence of Grothendieck's inequality the only thing left to prove is the equivalence between (1)-(3). The equivalence of (3) and (2) (the Tsirelson's bound) is a consequence of either the proof of Lemma **??** or Tsirelsons Theorem (Theorem 1).

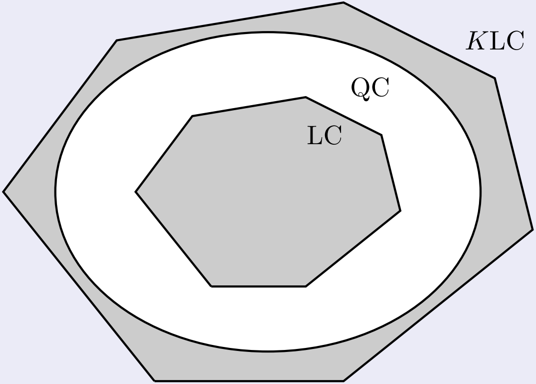

Figure: Visualization