Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 7, 2018

Outline

1. Local and quantum correlation matrices
   - Local correlation matrices
   - Quantum correlation matrices
   - The relations between quantum correlation and local correlation matrices

Nice slide to draw the connection between the games an LC

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

[

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$]

$\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$
(Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

Suffices to show that $\mathrm{LC}_{m,n}$ is convex.

Let $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$ for $k \in \{0,1\}$
Find $(X_i),(Y_j)$ with $|X_i|,|Y_j| \leq 1$ almost surely such that

$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for $\beta \in [0,1]$

Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1-\beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$

Then

$$\mathbb{E}[X_i Y_j] = \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(1)}]\mathbb{1}_{\{\alpha=1\}}]$$
$$= \beta\mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1-\beta)\mathbb{E}[X_i^{(1)} Y_j^{(1)}]$$

$\mathrm{LC}_{m,n} \subset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.
pause

Set $X = (X_1, ..., X_m)$ and $Y = (Y_1, ..., Y_n)$, then
$X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.

Hypercube description by its vertices:
$[-1,1]^d = \mathrm{conv}\{\xi \mid \xi \in \{-1,1\}^d\}$

Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$
Using the same decomposition for Y we obtain

$$a_{ij} = \mathbb{E}[X_i Y_j] = \mathbb{E}\big[(\sum \lambda_\xi^{(X)}\xi_i)(\sum \lambda_\eta^{(Y)}\eta_j)\big]$$

Some nice frame to connect QCs to the games

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$.
$A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state
Introduce a symbol zo define operators form one space to another
$\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

$$a_{ij} = \operatorname{Tr} \rho(X_i \otimes Y_j).$$

Definition

Let $(X_i)_{1\le i\le m}$ and $(Y_j)_{1\le j\le n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \le 1$.
$A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state
Introduce a symbol zo define operators form one space to another
$\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

$$a_{ij} = \operatorname{Tr} \rho(X_i \otimes Y_j).$$

Set of all quantum correlation matrices denoted by $\mathrm{QC}_{m,n}$

Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\le 1\le m, 1\le j\le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\},$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\leq 1\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\le 1\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$

$a_{ij} = \operatorname{Tr} \rho X_i \otimes Y_j$, sate ρ on a Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and Hermitian operators $(X_i)_{1\ge m}$, $(Y_j)_{1\ge n}$ on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \le 1$

Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \operatorname{Re}(\operatorname{Tr} \rho ST)$.

perhaps verification of at least some of these properties

Obtain an inner product space $U := B^{sa}(\mathcal{H}) / \ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\operatorname{Tr}(\rho X_i \otimes Y_j) = a_{ij}$$

$QC_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1\leq 1\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\mathrm{Tr}\left(\rho X_i \otimes Y_j\right) = a_{ij}$$

$\beta(X \otimes I, X \otimes I), \beta(I \otimes Y, I \otimes Y) \leq 1$
(this can be shown by using a *Schmidt-decomposition* of ρ and using $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$)
Project the y_j's orthogonally onto span$\{x_1, ..., x_m\}$ (wlog $m \leq n$)
$\pi(y_j)$ the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$
Let $\{a_1, ..., a_r\}$ be an orthonormal basis of span$\{x_1, ..., x_m\}$ with respect to β and $x_i = \sum_{k=1}^r \alpha_k^{(i)} a_k$ and $\pi(y_j) = \sum_{k=1}^r \gamma_k^{(j)} a_k$ for $\alpha^{(i)}, \gamma^{(j)} \in \mathbb{R}^r$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\le 1\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$

Let $\{a_1, \ldots, a_r\}$ be an orthonormal basis of $\mathrm{span}\{x_1, \ldots, x_m\}$ with respect to β and $x_i = \sum_{k=1}^r \alpha_k^{(i)} a_k$ and $\pi(y_j) = \sum_{k=1}^r \gamma_k^{(j)} a_k$ for $\alpha^{(i)}, \gamma^{(j)} \in \mathbb{R}^r$

$$a_{ij} = \tilde{\beta}(x_i, y_j) = \tilde{\beta}(x, \pi(y)) = \sum_{1\le k,l\le r} \alpha_k^{(i)} \gamma_l^{(j)} \tilde{\beta}(a_k, a_l)$$
$$= \sum_{k=1}^r \alpha_k^{(i)} \gamma_k^{(j)} = \langle \alpha^{(i)}, \gamma^{(j)} \rangle.$$

$\left|\alpha^{(i)}\right|, \left|\gamma^{(j)}\right| \le 1$ due to $\tilde{\beta}(x_i), \tilde{\beta}(y_j) \le 1$

$$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof.

Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \dots n$$

U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$
For $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} = \eta_i U_i$ we can calculate

$$XY = \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i,j \leq \leq 2n} \xi_i \eta_j U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq \leq 2n} \xi_i \eta_j U_i U_j -$$

$$2n$$

$QC_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

Proof.

Let $(x_i)_{1 \leq i \leq m}$, $(y_j)_{1 \leq j \leq n} \subset \mathbb{R}^{\min\{m,n\}}$ such that $|x_i|$,$|y_j| \leq 1$.

Let $X_i = \sum_{k=1}^{\min\{m,n\}} x_i(k) U_i$ and $Y_j^T = \sum_{k=1}^{\min\{m,n\}} y_j(k) U_k$ where the U_i's are $d \times d$ matrices with $d = 2^{\lceil \min\{m,n\}/2 \rceil}$

$\mathrm{Tr}\left(X_i Y_j^T\right) = d \cdot \langle x_i, y_j \rangle$ and $\|X_i\|_\infty \leq 1$ since $X_i X_i^* = |x_i|^2 I$ and $|x_i|^2 \leq 1$ (the same holds for Y_j)

Let $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ and $\rho = |\phi\rangle \langle \rho|$. Note that we can write ρ as

$$\rho = |\phi\rangle \langle \phi| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |kk\rangle \langle ll| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |k\rangle \langle l| \otimes |k\rangle \langle l|$$

Then

$$\mathrm{Tr}\left(\rho X_i \otimes Y_j\right) = \frac{1}{d} \sum_{1 \leq k,l \leq d} \mathrm{Tr}\left(|k\rangle \langle l| X_i \otimes |k\rangle \langle l| Y_j \right) = \frac{1}{d} \sum_{1 \leq k,l \leq d} \mathrm{Tr}\left(|k\rangle \right.$$

$\mathrm{QC}_{m,n}$ convex

$a_{ij} = \langle x_i, y_j \rangle$ and $\bar{a}_{ij} = \langle \bar{x}_i, \bar{y}_j \rangle$ for $x_i, y_j, \bar{x}_i, \bar{y}_j \in \mathbb{R}^{\min\{m,n\}}$ such that $|x_i|, |y_j|, |\bar{x}_i|, |\bar{y}_j| \le 1$.

define vectors $\tilde{x}_i := (\sqrt{\lambda} x_i, \sqrt{1-\lambda}\bar{x}_i)$, $\tilde{y}_j := (\sqrt{\lambda} y_j, \sqrt{1-\lambda}\bar{y}_j)$ for $\lambda \in [0,1]$

it holds $|\tilde{x}_i| \le \lambda |(x_i, 0)| + (1-\lambda)|(0, \bar{x}_i)| \le 1$ and $\langle \tilde{x}_i, \tilde{y}_j \rangle = \lambda \langle x_i, y_j \rangle + (1-\lambda)\langle \tilde{x}_i, \tilde{y}_j \rangle$.

Right dimension is obtained by projecting on $\mathrm{span}\{x_1, \dots, x_m\}$ or $\mathrm{span}\{y_1, ..., y_n\}$, as in the proof before.

$\mathrm{LC}_{m,n} \subset \mathrm{QC}_{m,n}$

Set $x_i = \xi_i |0\rangle$ and $y_j = \eta_j |0\rangle$ it immediately follows $\xi_i \eta_j = \langle x_i, y_j \rangle$.
Hence, $\xi \eta^T \in \mathrm{QC}_{m,n}$ (rest follows with the convexity of $\mathrm{QC}_{m,n}$)

Inclusion is strict in general

Let us consider the case $n = m = 2$.

$$\mathrm{LC}_{2,2} = \mathrm{conv}\{\pm \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\}.$$

Claim

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \mid -1 \leq \mathrm{Tr}\, AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{3}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}), \sigma(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \mid \sigma \in \{\mathrm{id}(1\ 2), (1\ 3), (1\ 4)\}\}$.

Claim

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \mid -1 \leq \operatorname{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{4}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}), \sigma(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \mid \sigma \in \{\mathrm{id}(1\ 2), (1\ 3), (1\ 4)\}\}$.

$\mathrm{LC}_{2,2}$ affinely isomorphic to the cross polytope scaled by 2, i.e. $\mathrm{LC}_{2,2} \cong 2CP_4 :=:= \mathrm{conv}\{\pm e_i \mid i = 1, \ldots, 4\}$

$(CP_n)^o = [-1, 1]^n$

Both points implies that the face lattice of $\mathrm{LC}_{2,2}$ is isomorphic to the opposite lattice of $[-1, 1]^4$

$\mathrm{LC}_{2,2}$ has as many facets as $[-1, 1]^4$ vertices, hence 2^4

Equation 4 is a non-redundant hyperplane description of $\mathrm{LC}_{2,2}$, all linear constraints define facets.

$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\max\{\operatorname{Tr}(AM) \mid A \in \mathrm{LC}_{2,2}\} = 2$

$A \in \mathrm{QC}_{2,2}$ we obtain, by Cauchy-Schwarz and $|y_i| \leq 1$,

$$\begin{aligned} \operatorname{Tr}(AM) &= \langle x_1, y_1\rangle + \langle x_1, y_2\rangle + \langle x_2, y_1\rangle - \langle x_2, y_2\rangle \\ &= \langle x_1 + x_2, y_1\rangle + \langle x_1 - x_2, y_2\rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\ &\leq |x_1 + x_2| + |x_1 - x_2|. \end{aligned}$$

$(|x_1 + x_2| + |x_1 - x_2|)^2 \leq 4(|x_1|^2 + |x_2|^2)$

$\operatorname{Tr}(AM) \leq |x_1 + x_2| + |x_1 - x_2| \leq 2\sqrt{|x_1|^2 + |x_2|^2} \leq 2\sqrt{2}.$

Bound is achieved by $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$