Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

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Outline

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- We should be familiar with the *Dirac notation*
  - $|\psi\rangle$ is a vector in $\mathbb{C}^n$ and $\langle\psi|$ is its conjugate transpose

Quantum systems

- A quantum system is a portion of the whole universe. For example a set electrons.
- A quantum system X is associated with a copy of $\mathbb{C}^k$
- It may consist of subsystems $X_1, \ldots, X_N$ each of which is associated with a copy of $\mathbb{C}^{n_i}$. In this case $k = n_1 \ldots n_N$

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- This can be achieved:

Definition

Measurement

- We define a measurement by a set of psd matrices $\{F^a\}_{a \in \mathcal{A}} \subseteq \mathbb{C}^{n \times n}$ that sum up to the identity matrix, i.e. $\sum_{a \in \mathcal{A}} F^a = I$
- The outcome of a measurement is a random variable χ with probability distribution: $\mathbb{P}[\chi = a] = \text{Tr}(\rho F^a)$
- A projective measurement is defined by psd matrices that satisfy $F^a F^b = \delta_{ab} F^a \; \forall a, b \in \mathcal{A}$

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- if we consider projective measurements we have

$$(F^+ - F^-)^2 = \underbrace{F^{+2}}_{=F^+} - \underbrace{F^+ F^-}_{\delta_{+-}=0} + \underbrace{F^{-2}}_{F^-} = F^+ + F^- = I$$

- i.e. a $\{-1, 1\}$-valued observable is both unitary an Hermitian

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- Every party may perform a measurement on their subsystem X_i, i.e. there are N sets of psd matrices $\{F^{a_1}\}_{a_1 \in \mathscr{A}_1} \in \mathbb{C}^{n_1 \times n_1}, \ldots, \{F^{a_N}\}_{a_N \in \mathscr{A}_N} \in \mathbb{C}^{n_N \times n_N}$

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The joint probability distribution of the N measurement outcomes $\chi_1, \ldots, \chi_N$ is

$$\mathbb{P}[\chi_1 = a_1, \chi_2 = a_2, \ldots, \chi_N = a_N] = \mathsf{Tr}(\rho F_1^{a_1} \otimes \cdots \otimes F_N^{a_N})$$

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Calculation

$$
\begin{aligned}
\mathrm{Tr}(|\psi\rangle\langle\psi|F^a \otimes G^b) &= \langle\psi|F^a \otimes G^b|\psi\rangle \\
&= (\langle\psi_A| \otimes \langle\psi_B|)(F^a \otimes G^b)(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= ((\langle\psi_A|F^a) \otimes (\langle\psi_B|G^b))(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= \langle\psi_A|F^a|\psi_A\rangle \otimes \langle\psi_B|G^b|\psi_B\rangle \\
&= \langle\psi_A|F^a|\psi_A\rangle\langle\psi_B|G^b|\psi_B\rangle
\end{aligned}
$$

This is equal to the product of the probabilities of Alice measuring a and Bob measuring b, i.e. the outcome do not correlate.

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- Alice sends answer a and Bob sends answer b back to the referee, who then decides whether both win or both lose

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- They win if $V(s, t, a, b) = 1$ and lose otherwise
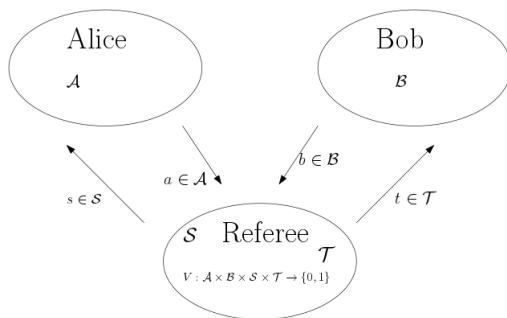
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- The winning probability then is:

$$\mathbb{E}_{(s,t) \sim \pi} \left[V(a(s), b(t), s, t) \right]$$

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- If the state is entangled measurements might give correlated measurement outcomes
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- Answering according to measurement outcomes could increase winning probability

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- Their winning probability is:

$$\mathbb{E}_{(s,t) \sim \pi} \left[\sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \mathsf{Tr}(\rho F_s^a \otimes G_t^b) V(a, b, s, t) \right]$$

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- A truth table for $a \oplus b$ looks like this

$\oplus$	0	1
0	0	1
1	1	0

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- Since $\frac{1}{2} + \gamma - (1 - \frac{1}{2} - \gamma) = 2\gamma$
- The violation ratio is defined as $\frac{\beta^*(G)}{\beta(G)}$

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Signs and observables

- It is convenient to use the $\{-1, 1\}$-basis instead of the $\{0, 1\}$-basis for boolean valued objects
- Let $a : \mathcal{S} \to \{0, 1\}$ and $b : \mathcal{T} \to \{0, 1\}$ be classical strategies and π the probability distribution the referee uses to pick s, t
- The bias is given by the probability under π that $a(s) \oplus b(t) = f(s, t)$ minus the probability under π that $a(s) \oplus b(t) \neq f(s, t)$

This means the bias can be written as:

$$\mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{[a(s)\oplus b(t)=f(s,t)]}\right] =$$

$$= \mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{a(s)\oplus b(t)+f(s,t)}\right] =$$

$$= \mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{a(s)}(-1)^{b(t)}(-1)^{f(s,t)}\right]$$

And we can define the sign matrix $\Sigma_{s,t} = (-1)^{f(s,t)}$ and functions $\chi(s) = (-1)^{a(s)}$ and $\psi(t) = (-1)^{b(t)}$. So the bias is

$$\mathbb{E}_{(s,t)\sim\pi}\left[\chi(s)\psi(t)\Sigma_{st}\right]$$

- The outcomes in an XOR game are $\{0, 1\}$
- Alice and Bob have measurements $\{F_s^0, F_s^1\}$ and $\{G_t^0, G_t^1\}$ and share an entangled state
- The probability of Alice and Bob answering with a, b upon receiving s, t respectively is $\langle \psi | F_s^a \otimes G_t^b | \psi \rangle$
- Lets calculate the expected value of $(-1)^{a \oplus b}$

$$(1) \cdot \mathbb{P}[a = b] + (-1) \cdot \mathbb{P}[a \neq b] =$$
$$= \langle \psi | F_s^0 \otimes G_t^0 | \psi \rangle + \langle \psi | F_s^1 \otimes G_t^1 | \psi \rangle$$
$$- \langle \psi | F_s^1 \otimes G_t^0 | \psi \rangle - \langle \psi | F_s^0 \otimes G_t^1 | \psi \rangle$$
$$= \langle \psi | (F_s^0 - F_s^1) \otimes (G_t^0 - G_t^1) | \psi \rangle$$

- Define $\{-1, 1\}$-observables $F_s = F_s^0 - F_s^1$ and $G_t = G_t^0 - G_t^1$ with the property that its difference squared is the identity matrix
- Using this strategy the bias becomes

$$\mathbb{E}_{(s,t)\sim\pi}\left[\langle\psi|F_s \otimes G_t|\psi\rangle\Sigma_{s,t}\right]$$

More generally speaking

- For any XOR game the bias is defined as the difference of the probabilities of winning and loosing
- Which is, if considering the $\{-1, 1\}$ basis, the expected value
- We are looking to maximize this quantity

Classical strategies

When using classical strategies this is

$$\max\{\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{st}\chi(s)\psi(t)\right] : \chi : \mathcal{S} \to \{-1, 1\},$$
$$\psi : \mathcal{T} \to \{-1, 1\}\}$$

Entangled strategies

When using entangled strategies the winning probability might increase indefinitely with the dimensions, so we use the $\sup_{n \in \mathbb{N}}$

$$\sup_{n \in \mathbb{N}} \{ \mathbb{E}_{(s,t) \sim \pi} [\Sigma_{st} \langle \psi | F_s \otimes G_t | \psi \rangle] : |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n,$$
$$F_s, G_t \in O(\mathbb{C}^n) \}$$

The CHSH game

- The CHSH game (Clauser, Horner, Shimony, Holt) is a two player XOR game with $\mathcal{A} = \mathcal{B} = \mathcal{S} = \mathcal{T} = \{0,1\}$ and π being the uniform distribution
- $f(s,t) = s \wedge t$, i.e. $f(1,1) = 1$ and $f(0,0) = f(0,1) = f(1,0) = 0$
- Alice and Bob can win $\frac{3}{4}$ of the games by using deterministic strategies $(0,0),(1,0)$ or $(0,1)$

Quantum strategy

- Let Alice and Bob share an EPR state
- Define

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $XY + YX = 0$ and $X^2 = Y^2 = I$
- For Alice define the observable for question 0 by $F_0 = X$ and for question 1 by $F_1 = Y$
- Bobs observables are going to be $G_0 = (X - Y)/\sqrt{2}$ for question 0 and $G_1 = (X + Y)/\sqrt{2}$ for question 1

The following auxiliary calculations will be helpful later:

$$\langle \text{EPR}|X \otimes X|\text{EPR}\rangle = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{2}{2} = 1$$

$$\langle\mathsf{EPR}|Y\otimes Y|\mathsf{EPR}\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -1$$

$$\langle \text{EPR}|X \otimes Y|\text{EPR}\rangle = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} i & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\langle \text{EPR}|Y \otimes X|\text{EPR}\rangle = 0$$

Lets calculate the expected values of the sign $a \oplus b$:

$$\langle \text{EPR}|F_0 \otimes G_0|\text{EPR}\rangle = \langle \text{EPR}|X \otimes \frac{1}{\sqrt{2}}(X - Y)|\text{EPR}\rangle$$

$$= \langle \text{EPR}|X \otimes \frac{1}{\sqrt{2}}X|\text{EPR}\rangle - \langle \text{EPR}|X \otimes \frac{1}{\sqrt{2}}Y|\text{EPR}\rangle$$

$$= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

$$\langle \text{EPR}|F_1 \otimes G_1|\text{EPR}\rangle = \langle \text{EPR}|Y \otimes \frac{1}{\sqrt{2}}(X + Y)|\text{EPR}\rangle$$

$$= \langle \text{EPR}|Y \otimes \frac{1}{\sqrt{2}}X|\text{EPR}\rangle + \langle \text{EPR}|Y \otimes \frac{1}{\sqrt{2}}Y|\text{EPR}\rangle$$

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$$= 0 - (-\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$$

Thus, we have

$$\langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle = \begin{cases} \frac{1}{\sqrt{2}}, (0,0), (1,0), (0,1) \\ -\frac{1}{\sqrt{2}}, (1.1) \end{cases}$$

which is equivalent to

$$\langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle = \frac{(-1)^{s \wedge t}}{\sqrt{2}}, s, t \in \{0, 1\}$$

The bias of the entangled strategy equals

$$\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{s,t}\langle\psi|F_s\otimes G_t|\psi\rangle\right] =$$

$$= \frac{1}{4}\sum_{s,t=0}^{1}(-1)^{s\wedge t}\langle\mathsf{EPR}|F_s\otimes G_t|\mathsf{EPR}\rangle$$

$$= \frac{1}{4}\cdot\frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

The bias is $\frac{1}{\sqrt{2}}$ from which follows that the winning probability is by definition:

$$\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \cos(\pi/8) \approx 0.85\ldots$$

Outline

Local correlation matrices

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- Their common answer is $\mathbb{E}[X_i Y_j]$
- This information can be encoded in an $\mathcal{S} \times \mathcal{T}$ matrix ◀ Motivation Quantum

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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$$\mathsf{LC}_{m,n} = \operatorname{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

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- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

$\mathsf{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}.$

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

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- Then

$$\mathbb{E}[X_i Y_j] = \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(1)}] \mathbb{1}_{\{\alpha=1\}}]$$
$$= \beta\mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1-\beta)\mathbb{E}[X_i^{(1)} Y_j^{(1)}]$$

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- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.

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- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

  almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$

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- Using the same decomposition for Y we obtain

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\begin{aligned}
a_{ij} = \mathbb{E}[X_i Y_j] &= \mathbb{E}\Big[\Big(\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}\xi_i\Big)\Big(\sum_{\eta \in \{-1,1\}^n} \lambda_\eta^{(Y)}\eta_j\Big)\Big] \\
&= \sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)}\lambda_\eta^{(Y)}\big]\xi_i\eta_j \\
&= \Big(\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}]\Big)\xi_i\eta_j
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\end{aligned}$$

- Due to $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$

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- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$

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Motivation Locality

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  $a_{ij} = \text{Tr}\left(\rho F_s^\xi \otimes G_t^\eta\right)$
- Again we can encode this information in a matrix

Motivation Locality

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$) such that

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Lemma

$$QC_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\},$$

$QC_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

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- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \mathrm{Re}(\mathrm{Tr}\, \rho S T)$.

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- Obtain an inner product space $U := B^{sa}(\mathcal{H})/\ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

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$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

- Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\text{Tr}\,(\rho X_i \otimes Y_j) = a_{ij}$$

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Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

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Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)}, \ i = 1, \dots n$$

- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$

- For $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} = \eta_i U_i$ we can calculate

$$
\begin{aligned}
XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i,j \leq \leq 2n} \xi_i \eta_j U_i U_j \\
&= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\
&= \langle \xi, \eta \rangle I.
\end{aligned}
$$

- The result follows by setting $X = Y$.

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- Let $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ and $\rho = |\phi\rangle \langle \rho|$. Note that we can write ρ as

$$\rho = |\phi\rangle \langle \phi| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |kk\rangle \langle ll| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |k\rangle \langle l| \otimes |k\rangle \langle l|$$

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- Then

$$\begin{aligned}
\text{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i \otimes |k\rangle \langle l| Y_j) \\
&= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i) \text{Tr}(|k\rangle \langle l| Y_j) \\
&= \frac{1}{d} \text{Tr} X_i Y_j^T = \langle x_i, y_j \rangle.
\end{aligned}$$

$\square$

Maxi Brandstetter, Felix Kirschner, Arne Hei‌Nonlocal games and the Grothendieck-Tsirels‌ September 19, 2018 47 / 54

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- Let $a_{ij} = \langle x_i, y_j \rangle$ and $\bar{a}_{ij} = \langle \bar{x}_i, \bar{y}_j \rangle$ for $x_i, y_j, \bar{x}_i, \bar{y}_j \in \mathbb{R}^{\min\{m,n\}}$ such that $|x_i|, |y_j|, |\bar{x}_i|, |\bar{y}_j| \leq 1$.

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\mathrm{span}\{x_1, \ldots, x_m\}$ or $\mathrm{span}\{y_1, ..., y_n\}$, as in the proof before.

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- Inclusion is strict in general

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$$LC_{2,2} = \text{conv}\left\{\pm\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$

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- W can also write it as in intersections of halfspaces:

$$LC_{2,2} = \{A \in \mathbb{R}^{2\times2} \,|\, -1 \leq \text{Tr}\, AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \qquad (1)$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}), \sigma(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \,|\, \sigma \in \{\text{id}, (1\ 2), (1\ 3), (1\ 4)\}\}.$

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\operatorname{Tr}(AM) &= \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle - \langle x_2, y_2 \rangle \\
&= \langle x_1 + x_2, y_1 \rangle + \langle x_1 - x_2, y_2 \rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\
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- Bound is achieved by $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1,1)$ and $y_1 = y_2 = (1,0)$

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] = \frac{\arccos(\langle x, y \rangle)}{\pi}$

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    - the normalized vector $n := s/\|s\|$ is uniformly distributed on the intersection of the unit sphere and $\text{span}\{x, y\}$ by the $O(d)$-invariance of the probability distribution

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