Ungleichungen und ähnlich verwirrende Konzepte

Coffee

Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 19, 2018

Ungleichungen und ähnlich verwirrende Konzepte

Coffee

Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 19, 2018

1. Grothendieck-Tsirelson Theorem
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
   - Gorthendieck-Tsirelson Theorem

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{1}$$

(2) *For any $M \in \mathbb{R}^{m \times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \tag{2}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} M(\xi \eta^\top)^\top. \tag{3}$$

(3) *For any $M \in \mathbb{R}^{m \times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{i,j} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} \xi^\top M \eta. \tag{4}$$

1. Grothendieck-Tsirelson Theorem
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
   - Gorthendieck-Tsirelson Theorem

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\operatorname{sign}(\langle x, r\rangle) \neq \operatorname{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\operatorname{sign}(\langle x, r\rangle)\operatorname{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\text{sign}(\langle x, r\rangle) \neq \text{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\text{sign}(\langle x, r\rangle)\,\text{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then
  - if $x = y$: $\arccos(\langle x, y\rangle) = \arccos(1) = 0$

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\text{sign}(\langle x, r\rangle) \neq \text{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\text{sign}(\langle x, r\rangle)\,\text{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then
  - if $x = y$: $\arccos(\langle x, y\rangle) = \arccos(1) = 0$
  - if $x = -y$: $\arccos(\langle x, y\rangle) = \arccos(-1) = \pi$

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\mathrm{sign}(\langle x, r\rangle) \neq \mathrm{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\mathrm{sign}(\langle x, r\rangle)\,\mathrm{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then
  - if $x = y$: $\arccos(\langle x, y\rangle) = \arccos(1) = 0$
  - if $x = -y$: $\arccos(\langle x, y\rangle) = \arccos(-1) = \pi$
- if x and y are linearly independent, then

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\text{sign}(\langle x, r\rangle) \neq \text{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\text{sign}(\langle x, r\rangle)\,\text{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then
  - if $x = y$: $\arccos(\langle x, y\rangle) = \arccos(1) = 0$
  - if $x = -y$: $\arccos(\langle x, y\rangle) = \arccos(-1) = \pi$
- if x and y are linearly independent, then
  - project r orthogonally on $\text{span}\{x, y\}$ which gives us a vector s with $\langle x, r\rangle = \langle x, s\rangle$ and $\langle y, r\rangle = \langle y, s\rangle$

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i. $\mathbb{P}[\mathrm{sign}(\langle x, r\rangle) \neq \mathrm{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

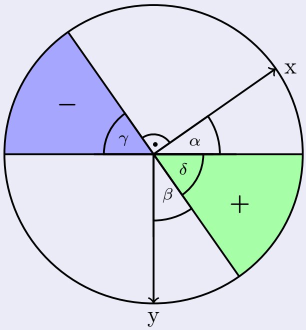
ii. $\mathbb{E}[\mathrm{sign}(\langle x, r\rangle)\,\mathrm{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

- if x and y are linearly dependent, then
  - if $x = y$: $\arccos(\langle x, y\rangle) = \arccos(1) = 0$
  - if $x = -y$: $\arccos(\langle x, y\rangle) = \arccos(-1) = \pi$
- if x and y are linearly independent, then
  - project r orthogonally on $\mathrm{span}\{x, y\}$ which gives us a vector s with $\langle x, r\rangle = \langle x, s\rangle$ and $\langle y, r\rangle = \langle y, s\rangle$
  - the normalized vector $n := s/\|s\|$ is uniformly distributed on the intersection of the unit sphere and $\mathrm{span}\{x, y\}$ by the $O(d)$-invariance of the probability distribution

Proof (cont.).

Calculation of the probability that the signs of the scalar products $\langle x, n \rangle$ and $\langle y, n \rangle$ are unlike:



$$\mathbb{P}[\operatorname{sign}(\langle x, n \rangle) \neq \operatorname{sign}(\langle y, n \rangle)] = 2\,\frac{\frac{\pi}{2} + \alpha}{2\pi} = \frac{\arccos(\langle x, y \rangle)}{\pi}$$

Proof (cont.).

We conclude with the proof of the second part of Lemma 1:

$$
\begin{aligned}
&\mathbb{E}[\operatorname{sign}(\langle x, r\rangle) \operatorname{sign}(\langle y, r\rangle)] \\
&\qquad = 1 \cdot \mathbb{P}[\operatorname{sign}(\langle x, r\rangle) = \operatorname{sign}(\langle y, r\rangle)] - 1 \cdot \mathbb{P}[\operatorname{sign}(\langle x, r\rangle) \neq \operatorname{sign}(\langle y, r\rangle)] \\
&\qquad = 1 - 2\mathbb{P}[\operatorname{sign}(\langle x, r\rangle) \neq \operatorname{sign}(\langle y, r\rangle)] \\
&\qquad = 1 - 2\frac{\arccos(\langle x, y\rangle)}{\pi} \\
&\qquad = \frac{2}{\pi}\arcsin(\langle x, y\rangle),
\end{aligned}
$$

because $\arcsin(t) + \arccos(t) = \pi/2$. □

Lemma (Krivine's trick)

Let $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$ *be given. Furthermore, let* $r \in \mathbb{R}^d$ *be a random unit vector chosen form the* $O(d)$*-invariant probability distribution on the unit sphere. Then there are* $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ *so that*

$$\mathbb{E}[\operatorname{sign}(\langle x'_i, r\rangle)\operatorname{sign}(\langle y'_j, r\rangle)] = \beta\langle x_i, y_j\rangle, \tag{5}$$

with $\beta = \frac{2}{\pi}\ln(1+\sqrt{2})$.

Definition (The k-th tensor product)

The *k-th tensor product* of $\mathbb{R}^n$ with orthonormal basis $e_1, \dots, e_n$ is denoted by $(\mathbb{R}^n)^{\otimes k}$ and it is a Euclidean vector space of dimension n^k with othonormal basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$, $i_j \in \{1, \dots, n\}$. In particular

$$\begin{aligned} \langle e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle &= \prod_{l=1}^{k} \langle e_{i_l}, e_{j_l} \rangle \\ &= \begin{cases} 1 & , \text{ if } i_l = j_l \text{ for all } l = 1, \dots, n, \\ 0 & , \text{ otherwise}, \end{cases} \end{aligned} \tag{6}$$

and for $v \in \mathbb{R}^n$ with $v = v_1 e_1 + \cdots + v_n e_n$ we define $v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k}$ by

$$\begin{aligned} v^{\otimes k} &:= (v_1 e_1 + \cdots + v_n e_n) \otimes \cdots \otimes (v_1 e_1 + \cdots + v_n e_n) \\ &= \sum_{i_1, \dots, i_k} v_{i_1} \cdots v_{i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \end{aligned} \tag{7}$$

Thus, for $v, w \in \mathbb{R}^n$

$$\langle v^{\otimes k}, w^{\otimes k} \rangle = \langle v, w \rangle^k. \tag{8}$$

Proof of Krivine's trick.

- define $E : [-1, +1] \to [-1, +1]$ by $E(t) = \frac{2}{\pi} \arcsin(t)$

Proof of Krivine's trick.

- define $E : [-1, +1] \to [-1, +1]$ by $E(t) = \frac{2}{\pi} \arcsin(t)$
- $E(\langle x_i', y_j' \rangle) = \mathbb{E}[\mathrm{sign}(\langle x_i', r \rangle)\,\mathrm{sign}(\langle y_j', r \rangle)] \stackrel{!}{=} \beta \langle x_i, y_j \rangle$ by Grothendieck's identity

Proof of Krivine's trick.

- define $E : [-1,+1] \to [-1,+1]$ by $E(t) = \frac{2}{\pi}\arcsin(t)$
- $E(\langle x_i', y_j' \rangle) = \mathbb{E}[\mathrm{sign}(\langle x_i', r\rangle)\,\mathrm{sign}(\langle y_j', r\rangle)] \overset{!}{=} \beta\langle x_i, y_j\rangle$ by Grothendieck's identity
- idea: To find β, x_i', y_j' invert E:

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}}_{=:g_{2k+1}} t^{2k+1}$$

Proof of Krivine's trick.

- define $E : [-1,+1] \to [-1,+1]$ by $E(t) = \frac{2}{\pi} \arcsin(t)$
- $E(\langle x_i', y_j' \rangle) = \mathbb{E}[\mathrm{sign}(\langle x_i', r\rangle)\,\mathrm{sign}(\langle y_j', r\rangle)] \overset{!}{=} \beta\langle x_i, y_j\rangle$ by Grothendieck's identity
- idea: To find β, x_i', y_j' invert E:

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1}}_{=:g_{2k+1}} t^{2k+1}$$

- define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2k+1}. \tag{9}$$

Proof (cont.).

- define $\tilde{x}_i, \tilde{y}_j \in H$, $i = 1, \dots, m, j = 1, \dots, n$ componentwise:

$$(\tilde{x}_i)_k = \operatorname{sign}(g_{2k+1})\sqrt{|g_{2k+1}|\beta^{2k+1}}\, x_i^{\otimes 2k+1} \tag{10}$$

$$(\tilde{y}_j)_k = \sqrt{|g_{2k+1}|\beta^{2k+1}}\, y_j^{\otimes 2k+1} \tag{11}$$

Proof (cont.).

- define $\tilde{x}_i, \tilde{y}_j \in H$, $i = 1, \dots, m, j = 1, \dots, n$ componentwise:

$$(\tilde{x}_i)_k = \operatorname{sign}(g_{2k+1})\sqrt{|g_{2k+1}|\beta^{2k+1}}\, x_i^{\otimes 2k+1} \tag{10}$$

$$(\tilde{y}_j)_k = \sqrt{|g_{2k+1}|\beta^{2k+1}}\, y_j^{\otimes 2k+1} \tag{11}$$

- then

$$\begin{aligned}\langle \tilde{x}_i, \tilde{y}_j \rangle &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1}\langle x_i^{\otimes 2k+1}, y_j^{\otimes 2k+1}\rangle \\ &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1}\langle x_i, y_j\rangle^{2k+1} \\ &= E^{-1}(\beta\langle x_i, y_j\rangle).\end{aligned}$$

Proof (cont.).

- hence, β is defined by the condition that the vectors $\tilde{x}_i, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh(\frac{\pi}{2}\beta)$$

$$\Leftrightarrow \qquad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1 + \sqrt(2))$$

Proof (cont.).

- hence, β is defined by the condition that the vectors $\tilde{x}_i, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh(\frac{\pi}{2}\beta)$$

$$\Leftrightarrow \qquad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1 + \sqrt{(}2))$$

- problem: $\tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_n$ are infinite-dimensional

Proof (cont.).

- hence, β is defined by the condition that the vectors $\tilde{x}_i, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh(\frac{\pi}{2}\beta)$$

$$\Leftrightarrow \qquad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1 + \sqrt(2))$$

- problem: $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are infinite-dimensional
- solution: the positive definite and symmetric Gram matrix G

$$G = \begin{pmatrix} \langle \tilde{x}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{x}_m \rangle & \langle \tilde{x}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{x}_m, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{x}_m \rangle & \langle \tilde{x}_m, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{y}_n \rangle \\ \langle \tilde{y}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{x}_m \rangle & \langle \tilde{y}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{y}_n, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{x}_m \rangle & \langle \tilde{y}_n, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{y}_n \rangle \end{pmatrix} \tag{12}$$

Proof (cont.).

- due to the properties of G we can decompose G via a real orthogonal matrix Q with columns that are the eigenvectors of G and a real diagonal matrix Λ having the eigenvalues of G on the diagonal, thus

$$G = Q\Lambda Q^\top = (\underbrace{Q\Lambda^{1/2}}_{=:A})^\top (Q\Lambda^{1/2}) \tag{13}$$

$\square$

Proof (cont.).

- due to the properties of G we can decompose G via a real orthogonal matrix Q with columns that are the eigenvectors of G and a real diagonal matrix Λ having the eigenvalues of G on the diagonal, thus

$$G = Q\Lambda Q^\top = (\underbrace{Q\Lambda^{1/2}}_{=:A})^\top (Q\Lambda^{1/2}) \tag{13}$$

- the columns of A are the vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ we are looking for

$\square$

Definition

For $M \in \mathbb{R}^{m\times n}$ define the quadratic program

$$\|M\|_{\infty\to 1} = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\xi_i\eta_j : \xi_i^2 = 1, i = 1,\dots,m, \eta_j^2 = 1, j = 1,\dots,n\right\}$$
$$= \max\left\{\operatorname{Tr} M\eta\xi^\top : \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\right\}. \tag{14}$$

Definition

The SDP relaxation of $\|M\|_{\infty\to 1}$ is given via:

$$\begin{aligned}
\mathrm{sdp}_{\infty\to 1}(M) = \max \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\langle x_i, y_j\rangle \\
x_i, y_j \in \mathbb{R}^{m+n} \\
\|x_i\| = 1, i = 1,\dots,m \\
\|y_j\| = 1, j = 1,\dots,n
\end{aligned}$$

Theorem (Grothendieck's inequality)

There exists a constant K such that for all $M \in \mathbb{R}^{m \times n}$:

$$\|M\|_{\infty\to1} \leq \mathrm{sdp}_{\infty\to1}(M) \leq K\|M\|_{\infty\to1}. \tag{15}$$

Proof.

Use the following approximation algorithm with randomized rounding:

Algorithm 1: Approximation algorithm with randomized rounding for $\|M\|_{\infty\to1}$

1. Solve $\mathrm{sdp}_{\infty\to1}(M)$. Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ be the optimal unit vectors
2. Apply Krivine's trick (Lemma 2) and use vectors x_i, y_j to create new unit vectors $x'_1, \ldots, x'_m, y'_1, \ldots, y'_n \in S^{m+n-1}$.
3. Choose $r \in S^{m+n-1}$ randomly
4. Round: $u_i = \mathrm{sign}(\langle x'_i, r\rangle)$
   $v_j = \mathrm{sign}(\langle y'_j, r\rangle)$

Proof (cont.).

Expected quality of the outcome:

$$\begin{aligned}\|M\|_{\infty\to1} &\geq \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}u_i v_j\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\mathbb{E}[\operatorname{sign}(\langle x_i', r\rangle)\operatorname{sign}(\langle y_j', r\rangle)] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\beta\langle x_i, y_j\rangle \\ &= \beta\,\mathrm{sdp}_{\infty\to1}(M),\end{aligned}$$

where the last equality follows by Krivine's trick with $\beta = \frac{2\ln(1+\sqrt{2})}{\pi}$, thus $K \leq \beta^{-1}$.

Theorem (Tsirelson)

(Hard direction) For all positive integers n, r *and any* $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^r$, *there exists a positive integer* $d := d(r)$, *a state* $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ *and* $\{-1,1\}$*-observables* $F_1, \ldots, F_n, G_1, \ldots, G_n \in O(\mathbb{C}^d)$, *such that for every* $i, j \in \{1, \ldots, n\}$, *we have*

$$\langle\psi|F_i \otimes G_j|\psi\rangle = \langle x_i, y_j\rangle. \tag{16}$$

Moreover, $d \leq 2^{\lceil r/2 \rceil}$.
(Easy direction) Conversely, for all positive integers n, d, *state* $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ *and* $\{-1,1\}$*-observables* $F_1, \ldots, F_n, G_1, \ldots, G_n \in O(\mathbb{C}^d)$, *there exist a positive integer* $r := r(d)$ *and* $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^r$ *such that for every* $i, j \in \{1, \ldots, n\}$, *we have*

$$\langle x_i, y_j\rangle = \langle\psi|F_i \otimes G_j|\psi\rangle. \tag{17}$$

Moreover, $r \leq 2d^2$.

Since

1. Grothendieck-Tsirelson Theorem
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
   - Gorthendieck-Tsirelson Theorem

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{18}$$

(2) *For any $M \in \mathbb{R}^{m\times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \sum_{i,j} M_{ij}\xi_i\eta_j \tag{19}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} M(\xi\eta^\top)^\top. \tag{20}$$

(3) *For any $M \in \mathbb{R}^{m\times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{i,j}\langle x_i, y_j\rangle \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} \xi^\top M\eta. \tag{21}$$

Proof.

Since (21) is a direct consequence of Grothendieck's inequality the only thing left to prove is the equivalence between (1)-(3). The equivalence of (3) and (2) (the Tsirelson's bound) is a consequence of either the proof of Lemma **??** or Tsirelsons Theorem (Theorem 1).

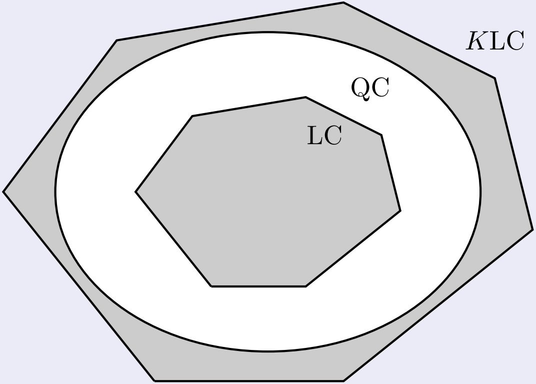


Figure: Visualization