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Outline

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

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Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

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$\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

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Suffices to show that $\mathrm{LC}_{m,n}$ is convex.

$a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$ for $k \in \{0,1\}$

Find $(X_i), (Y_j)$ with $|X_i|, |Y_j| \leq 1$ almost surely such that

$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for $\beta \in [0,1]$

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Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$