Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

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   - Nonlocal games
   - A special case of nonlocal games
   - A specific example

2. Local and quantum correlation matrices
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   - Quantum correlation matrices
   - The relations between quantum correlation and local correlation matrices
   - Motivation: The Grothendieck-Tsirelson Theorem
   - Grothendieck's Inequality
   - Tsirelson's Theorem
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Quantum systems

- A quantum system is a portion of the whole universe. For example a set electrons.
- A quantum system X is associated with a copy of $\mathbb{C}^k$
- It may consist of subsystems $X_1, \dots, X_N$ each of which is associated with a copy of $\mathbb{C}^{n_i}$. In this case $k = n_1 \dots n_N$

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- This can be achieved:

Definition

Measurement

- We define a measurement by a set of psd matrices $\{F^a\}_{a\in\mathcal{A}} \subseteq \mathbb{C}^{n\times n}$ that sum up to the identity matrix, i.e. $\sum_{a\in\mathcal{A}} F^a = I$
- The outcome of a measurement is a random variable χ with probability distribution: $\mathbb{P}[\chi = a] = \mathrm{Tr}(\rho F^a)$
- A projective measurement is defined by psd matrices that satisfy $F^a F^b = \delta_{ab} F^a \;\forall a, b \in \mathcal{A}$

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- if we consider projective measurements we have

$$(F^+ - F^-)^2 = \underbrace{{F^+}^2}_{=F^+} - \underbrace{F^+ F^-}_{\delta_{+-}=0} + \underbrace{{F^-}^2}_{F^-} = F^+ + F^- = I$$

- i.e. a $\{-1, 1\}$-valued observable is both unitary an Hermitian

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- Every party may perform a measurement on their subsystem X_i, i.e. there are N sets of psd matrices $\{F^{a_1}\}_{a_1 \in \mathcal{A}_1} \in \mathbb{C}^{n_1 \times n_1}, \dots, \{F^{a_N}\}_{a_N \in \mathcal{A}_N} \in \mathbb{C}^{n_N \times n_N}$

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The joint probability distribution of the N measurement outcomes $\chi_1, \dots, \chi_N$ is

$$\mathbb{P}\left[\chi_1 = a_1, \chi_2 = a_2, \dots, \chi_N = a_N\right] = \mathrm{Tr}(\rho F_1^{a_1} \otimes \cdots \otimes F_N^{a_N})$$

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Calculation

$$\begin{aligned}
\mathrm{Tr}(|\psi\rangle\langle\psi| F^a \otimes G^b) &= \langle\psi|F^a \otimes G^b|\psi\rangle \\
&= (\langle\psi_A| \otimes \langle\psi_B|)(F^a \otimes G^b)(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= ((\langle\psi_A|F^a) \otimes (\langle\psi_B|G^b))(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= \langle\psi_A|F^a|\psi_A\rangle \otimes \langle\psi_B|G^b|\psi_B\rangle \\
&= \langle\psi_A|F^a|\psi_A\rangle\langle\psi_B|G^b|\psi_B\rangle
\end{aligned}$$

This is equal to the product of the probabilities of Alice measuring a and Bob measuring b, i.e. the outcomes do not correlate.

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- Alice sends answer a and Bob sends answer b back to the referee, who then decides whether both win or both lose

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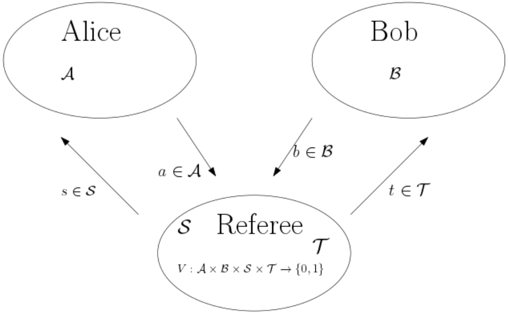
- They win if $V(s,t,a,b) = 1$ and lose otherwise



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- The winning probability then is:

$$\mathbb{E}_{(s,t)\sim\pi}\left[V(a(s), b(t), s, t)\right]$$

Quantum case

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- If the state is entangled measurements might give correlated measurement outcomes
- Alice and Bob may gain information by performing measurements
- Answering according to measurement outcomes could increase winning probability

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- They send their measurement outcome as their answer to the referee
- Their winning probability is:

$$\mathbb{E}_{(s,t)\sim\pi}\left[\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \mathrm{Tr}(\rho F_s^a \otimes G_t^b) V(a,b,s,t)\right]$$

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- The predicate V is defined as $V(a,b,s,t) = [a \oplus b = f(s,t)]$, where $f : \mathcal{S} \times \mathcal{T} \to \{0,1\}$
- A truth table for $a \oplus b$ looks like this

$\oplus$	0	1
0	0	1
1	1	0

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- Since $\frac{1}{2} + \gamma - (1 - \frac{1}{2} - \gamma) = 2\gamma$
- The violation ratio is defined as $\frac{\beta^*(G)}{\beta(G)}$

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- Let $a : \mathcal{S} \to \{0, 1\}$ and $b : \mathcal{T} \to \{0, 1\}$ be classical strategies and π the probability distribution the referee uses to pick s, t
- The bias is given by the probability under π that $a(s) \oplus b(t) = f(s, t)$ minus the probability under π that $a(s) \oplus b(t) \neq f(s, t)$

This means the bias can be written as:

$$\mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{[a(s)\oplus b(t)=f(s,t)]}\right] =$$

$$= \mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{a(s)\oplus b(t)+f(s,t)}\right] =$$

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Define the sign matrix $\Sigma_{s,t} = (-1)^{f(s,t)}$ and functions $\chi(s) = (-1)^{a(s)}$ and $\psi(t) = (-1)^{b(t)}$.
The expected value becomes

$$\mathbb{E}_{(s,t)\sim\pi}\left[\chi(s)\psi(t)\Sigma_{st}\right]$$

Recap

- The measurement outcomes in an XOR game are $\{0, 1\}$
- Alice and Bob have measurements $\{F_s^0, F_s^1\}$ and $\{G_t^0, G_t^1\}$ and share an entangled state
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$$\begin{aligned}(1) \cdot \mathbb{P}[a = b] + (-1) \cdot \mathbb{P}[a \neq b] &= \\ = \langle\psi|F_s^0 \otimes G_t^0|\psi\rangle + \langle\psi|F_s^1 \otimes G_t^1|\psi\rangle \\ -\langle\psi|F_s^1 \otimes G_t^0|\psi\rangle - \langle\psi|F_s^0 \otimes G_t^1|\psi\rangle \\ = \langle\psi|(F_s^0 - F_s^1) \otimes (G_t^0 - G_t^1)|\psi\rangle\end{aligned}$$

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- For any XOR game the bias is defined as the difference of the probabilities of winning and loosing
- Which is, if considering the $\{-1,1\}$ basis, the expected value above

- Define $\{-1, 1\}$-observables $F_s = F_s^0 - F_s^1$ and $G_t = G_t^0 - G_t^1$ with the property that their square is the identity matrix
- Using this strategy the bias becomes

$$\mathbb{E}_{(s,t)\sim\pi}\left[\langle\psi|F_s \otimes G_t|\psi\rangle \Sigma_{s,t}\right]$$

- For any XOR game the bias is defined as the difference of the probabilities of winning and loosing
- Which is, if considering the $\{-1, 1\}$ basis, the expected value above
- We are looking to maximize this quantity

Classical strategies

When using classical strategies this is

$$\max\{\mathbb{E}_{(s,t)\sim\pi}\left[\chi(s)\psi(t)\Sigma_{st}\right] : \chi : \mathcal{S} \to \{-1,1\},\\ \psi : \mathcal{T} \to \{-1,1\}\}$$

Entangled strategies

When using entangled strategies the winning probability might increase indefinitely with the dimensions, so we use the $\sup_{n\in\mathbb{N}}$

$$\sup_{n\in\mathbb{N}}\{\mathbb{E}_{(s,t)\sim\pi}\left[\langle\psi|F_s \otimes G_t|\psi\rangle\Sigma_{st}\right] : |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n, \\ F_s, G_t \in \mathscr{W}(\mathbb{C}^n)\}$$

with $\mathscr{W}$ being the set of all $\{-1, 1\}$-observables in $\mathbb{C}^{n\times n}$.

The CHSH game

- The CHSH game (Clauser, Horner, Shimony, Holt) is a two player XOR game with $\mathscr{A} = \mathscr{B} = \mathscr{S} = \mathscr{T} = \{0, 1\}$ and π being the uniform distribution
- $f(s, t) = s \wedge t$, i.e. $f(1, 1) = 1$ and $f(0, 0) = f(0, 1) = f(1, 0) = 0$
- Alice and Bob can win $\frac{3}{4}$ of the games by using deterministic strategies $(1, 0)$ or $(0, 1)$

Quantum strategy

- Let Alice and Bob share an EPR state
- Define

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $XY + YX = 0$ and $X^2 = Y^2 = I$
- For Alice define the observable for question 0 by $F_0 = X$ and for question 1 by $F_1 = Y$
- Bobs observables are going to be $G_0 = (X - Y)/\sqrt{2}$ for question 0 and $G_1 = (X + Y)/\sqrt{2}$ for question 1

The bias of the entangled strategy equals

$$\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{s,t}\langle\psi|F_s\otimes G_t|\psi\rangle\right] = \frac{1}{4}\sum_{s,t=0}^{1}(-1)^{s\wedge t}\langle \mathrm{EPR}|F_s\otimes G_t|\mathrm{EPR}\rangle$$

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Straight forward calculation shows

$$\langle \mathrm{EPR}|F_s \otimes G_t|\mathrm{EPR}\rangle = \begin{cases} \frac{1}{\sqrt{2}}, (0,0), (1,0), (0,1) \\ -\frac{1}{\sqrt{2}}, (1,1) \end{cases}$$

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Thus,

$$\frac{1}{4}\left((-1)^{0\wedge 0}\frac{1}{\sqrt{2}} + (-1)^{1\wedge 0}\frac{1}{\sqrt{2}} + (-1)^{0\wedge 1}\frac{1}{\sqrt{2}} + (-1)^{1\wedge 1}(-\frac{1}{\sqrt{2}})\right) = \frac{1}{4}\frac{4}{\sqrt{2}}$$

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Note

$$\frac{1}{\sqrt{2}} = \beta^*(G) = \underbrace{\frac{1}{2} + \gamma}_{\text{winning probability}} -(1 - \tfrac{1}{2} - \gamma) = 2\gamma \Rightarrow \gamma = \frac{1}{2\sqrt{2}}$$

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The winning probability follows directly

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \approx 0.85\ldots$$

Outline

Local correlation matrices

- Deterministic strategies of Alice and Bob correspond to vectors $\xi \in \{-1,1\}^{\mathcal{S}}$, respectively $\eta \in \{-1,1\}^{\mathcal{T}}$.

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- Instead of deterministic strategies they can also answer according to the probability distribution of random variables X_i, Y_j.
- Their common answer is $\mathbb{E}[X_i Y_j]$.
- This information can be encoded in an $\mathcal{S} \times \mathcal{T}$ matrix. Motivation Quantum

Definition

Let $(X_i)_{1 \le i \le m}$ and $(Y_j)_{1 \le j \le n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \le 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

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Lemma

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- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses.

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$).

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- Then

$$\begin{aligned}\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=1\}}] \\ &= \mathbb{E}[X_i^{(0)} Y_j^{(0)}]\mathbb{P}(\alpha = 0) + \mathbb{E}[X_i^{(1)} Y_j^{(1)}]\mathbb{P}(\alpha = 1) \\ &= \beta a_{ij}^{(0)} + (1-\beta)a_{ij}^{(1)}.\end{aligned}$$

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- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$.

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- Using the same decomposition for Y we obtain

$$\begin{aligned} a_{ij} = \mathbb{E}[X_i Y_j] &= \mathbb{E}\Big[\Big(\sum_{\xi\in\{-1,1\}^m} \lambda_\xi^{(X)}\xi_i\Big)\Big(\sum_{\eta\in\{-1,1\}^n} \lambda_\eta^{(Y)}\eta_j\Big)\Big] \\ &= \sum_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)}\lambda_\eta^{(Y)}\big]\xi_i\eta_j. \end{aligned}$$

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- Due to $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}\big] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$.

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Quantum correlation matrices

- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$.

Motivation Locality

Quantum correlation matrices

- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$.
- They perform measurements $\{F_i^{\xi}\}_{\xi=\pm 1}$, respectively $\{G_j^{\eta}\}_{\eta=\pm 1}$.

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- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}$, $j \in \mathcal{T}$.
- They perform measurements $\{F_i^\xi\}_{\xi=\pm 1}$, respectively $\{G_j^\eta\}_{\eta=\pm 1}$.
- The probability that their response is (ξ, η) for inputs (i, j) is given by $a_{ij} = \mathrm{Tr}\,(\rho F_i^\xi \otimes G_j^\eta)$.
- Again we can encode this information in a matrix.

Motivation Locality

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ such that

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Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\leq i\leq m, 1\leq j\leq n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1\le i\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$

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- Obtain an inner product vector space $U := B^{sa}(\mathcal{H}) / \ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

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- Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i \otimes I, I \otimes Y_j) = \mathrm{Re}\mathrm{Tr}\,(\rho X_i \otimes Y_j) = a_{ij}.$$

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- The vectors still have not the right dimension but again, we can project them onto vectors in $\mathbb{R}^m$.

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- In order to show

$$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \le i \le m, 1 \le j \le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$$

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Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every element is the multiple of a unitary matrix.

- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Proof.

- Define $U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)}$,
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- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$.
- For $X = \sum_{i=1}^{2n} \xi_i U_i$ we can calculate

$$
\begin{aligned}
XX^* = XX &= \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i \neq j \leq 2n} \xi_i \xi_j U_i U_j \\
&= \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j - \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j \\
&= \sum_{i=1}^{2n} \xi_i^2 I = |\xi|^2 I.
\end{aligned}
$$

$\square$

- Set

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \ldots n.$$

- Moreover, taking two linear combinations $X = \sum_{i=1}^{2n} \xi_i U_i,\ Y = \sum_{i=1}^{2n} = \eta_i U_i$ and consider the trace of the product we get

$$\begin{aligned} \operatorname{Tr}(XY) &= \sum_{i=1}^{2n} \xi_i \eta_i \operatorname{Tr} I + \sum_{1 \le i \ne j \le\ 2n} \xi_i \eta_j \operatorname{Tr}(U_i U_j) \\ &= \sum_{i=1}^{2n} \xi_i \eta_j 2^n = 2^n \cdot \langle \xi, \eta \rangle, \end{aligned}$$

where we used that the product $U_i U_j$ is traceless for $i \neq j$.

$QC_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1\le i\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$

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- Let $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$ and $\rho = |\psi\rangle\langle\psi|$. Note that we can write ρ as

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{d} \sum_{1\le k,l\le d} |k\rangle \otimes |k\rangle \langle l| \otimes \langle l| = \frac{1}{d} \sum_{1\le k,l\le d} |k\rangle\langle l| \otimes |k\rangle\langle l|.$$

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- Then

$$\begin{aligned}
\operatorname{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1\le k,l\le d} \operatorname{Tr}\left(|k\rangle\langle l| X_i \otimes |k\rangle\langle l| Y_j\right) \\
&= \frac{1}{d} \sum_{1\le k,l\le d} \operatorname{Tr}\left(|k\rangle\langle l| X_i\right) \operatorname{Tr}\left(|k\rangle\langle l| Y_j\right) \\
&= \frac{1}{d} \operatorname{Tr} X_i Y_j^T = \langle x_i, y_j \rangle.
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- It holds $|\tilde{x}_i| \leq \lambda \left| \begin{pmatrix} x_i \\ 0 \end{pmatrix} \right| + (1-\lambda) \left| \begin{pmatrix} 0 \\ \bar{x}_i \end{pmatrix} \right| \leq 1$ and $\langle \tilde{x}_i, \tilde{y}_j \rangle = \lambda \langle x_i, y_j \rangle + (1-\lambda) \langle \bar{x}_i, \bar{y}_j \rangle$.

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\operatorname{span}\{x_1, \ldots, x_m\}$ or $\operatorname{span}\{y_1, ..., y_n\}$, as in the proof before.

The relations between quantum correlation and local correlation matrices

- $\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$
- $\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\le i\le m, 1\le j\le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$
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- W can also write it as in intersections of halfspaces:

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \mid -1 \leq \mathrm{Tr}\, AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{1}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}), \sigma(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}) \mid \sigma \in \{\mathrm{id}, (1\ 2), (1\ 3), (1\ 4)\}\}$.

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- $(|x_1 + x_2| + |x_1 - x_2|)^2 \leq 4(|x_1|^2 + |x_2|^2)$

- $\max\{\mathrm{Tr}\,(AM) \mid A \in \mathrm{LC}_{2,2}\} = 2$
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- $A \in \mathrm{QC}_{2,2}$ we obtain, by Cauchy-Schwarz and $|y_i| \leq 1$,

$$\begin{aligned}\mathrm{Tr}\,(AM) &= \langle x_1, y_1\rangle + \langle x_1, y_2\rangle + \langle x_2, y_1\rangle - \langle x_2, y_2\rangle \\ &= \langle x_1 + x_2, y_1\rangle + \langle x_1 - x_2, y_2\rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\ &\leq |x_1 + x_2| + |x_1 - x_2|.\end{aligned}$$

- $(|x_1 + x_2| + |x_1 - x_2|)^2 \leq 4(|x_1|^2 + |x_2|^2)$
- $\mathrm{Tr}\,(AM) \leq |x_1 + x_2| + |x_1 - x_2| \leq 2\sqrt{|x_1|^2 + |x_2|^2} \leq 2\sqrt{2}$.

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- Bound is achieved by $A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$.

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{2}$$

(2) *For any $M \in \mathbb{R}^{m\times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \sum_{i,j} M_{ij}\xi_i\eta_j \tag{3}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} M(\xi\eta^\top)^\top. \tag{4}$$

(3) *For any $M \in \mathbb{R}^{m\times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{i,j}\langle x_i, y_j\rangle \leq K \max_{\xi\in\{-1,1\}^m, \eta\in\{-1,1\}^n} \operatorname{Tr} \xi^\top M\eta. \tag{5}$$

Lemma (Grothendieck's identity)

Let $x, y \in S^{d-1}$. Let $r \in S^{d-1}$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

i, $\mathbb{P}[\text{sign}(\langle x, r\rangle) \neq \text{sign}(\langle y, r\rangle)] = \frac{\arccos(\langle x,y\rangle)}{\pi}$

ii, $\mathbb{E}[\text{sign}(\langle x, r\rangle)\,\text{sign}(\langle y, r\rangle)] = \frac{2}{\pi}\arcsin(\langle x, y\rangle)$.

Proof.

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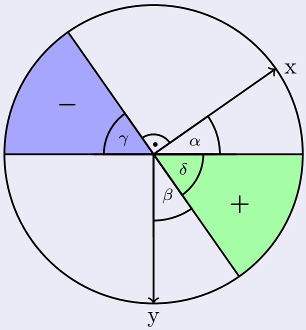
- If x and y are linearly independent, then
  - project r orthogonally on span$\{x, y\}$ which gives us a vector s with $\langle x, r\rangle = \langle x, s\rangle$ and $\langle y, r\rangle = \langle y, s\rangle$,
  - the normalized vector $n := s/|s|$ is uniformly distributed on the intersection of the unit sphere and span$\{x, y\}$ by the $O(d)$-invariance of the probability distribution.

Proof (cont.).

Calculation of the probability that the signs of the scalar products $\langle x, n\rangle$ and $\langle y, n\rangle$ are unlike:



$$\mathbb{P}[\text{sign}(\langle x, n\rangle) \neq \text{sign}(\langle y, n\rangle)] = 2\frac{\arccos(\langle x, y\rangle)}{2\pi} = \frac{\arccos(\langle x, y\rangle)}{\pi}$$

Proof (cont.).

We conclude with the proof of the second part of Lemma 6:

$$\begin{aligned}\mathbb{E}[\operatorname{sign}(\langle x, r\rangle)\operatorname{sign}(\langle y, r\rangle)] \\ = 1 - 2\frac{\arccos(\langle x, y\rangle)}{\pi} \\ = \frac{2}{\pi}\arcsin(\langle x, y\rangle),\end{aligned}$$

because $\arcsin(t) + \arccos(t) = \pi/2$. □

Lemma (Krivine's trick)

Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ be given. Furthermore, let $r \in S^{n+m-1}$ be a random unit vector chosen form the $O(n+m-1)$-invariant probability distribution on the unit sphere. Then there are $x'_1, \ldots, x'_m, y'_1, \ldots, y'_n \in S^{m+n-1}$ so that

$$\mathbb{E}[\operatorname{sign}(\langle x'_i, r\rangle)\operatorname{sign}(\langle y'_j, r\rangle)] = \beta\langle x_i, y_j\rangle, \tag{6}$$

with $\beta = \frac{2}{\pi}\ln(1+\sqrt{2})$.

Definition (The k-th tensor product)

The *k-th tensor product* of $\mathbb{R}^n$ with orthonormal basis $e_1, \ldots, e_n$ is denoted by $(\mathbb{R}^n)^{\otimes k}$ and it is a Euclidean vector space of dimension n^k with othonormal basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$, $i_l \in \{1, \ldots, n\}$. In particular

$$\begin{aligned}\langle e_{i_1} \otimes \cdots \otimes e_{i_k}, e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle &= \prod_{l=1}^{k} \langle e_{i_l}, e_{j_l} \rangle \\ &= \begin{cases} 1 & \text{, if } i_l = j_l \text{ for all } l = 1, \ldots, n, \\ 0 & \text{, otherwise,} \end{cases} \end{aligned} \tag{7}$$

and for $v \in \mathbb{R}^n$ with $v = v_1 e_1 + \cdots + v_n e_n$ we define $v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k}$ by

$$\begin{aligned} v^{\otimes k} &:= (v_1 e_1 + \cdots + v_n e_n) \otimes \cdots \otimes (v_1 e_1 + \cdots + v_n e_n) \\ &= \sum_{i_1, \ldots, i_k} v_{i_1} \cdots v_{i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}. \end{aligned} \tag{8}$$

Thus, for $v, w \in \mathbb{R}^n$

$$\langle v^{\otimes k}, w^{\otimes k} \rangle = \langle v, w \rangle^k. \tag{9}$$

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Proof of Krivine's trick.

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- Idea: To find β, x'_i, y'_j invert E:

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}}_{=:g_{\mathbf{2k+1}}} t^{2k+1},$$

Proof (cont.).

- Define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2k+1}. \tag{11}$$

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- Define $\tilde{x}_i, \tilde{y}_j \in H$, $i = 1, \dots, m, j = 1, \dots, n$ componentwise:

$$(\tilde{x}_i)_k = \operatorname{sign}(g_{2k+1}) \sqrt{|g_{2k+1}|\beta^{2k+1}}\, x_i^{\otimes 2k+1} \tag{12}$$

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- Then

$$\begin{aligned} \langle \tilde{x}_i, \tilde{y}_j \rangle &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1}\langle x_i^{\otimes 2k+1}, y_j^{\otimes 2k+1}\rangle \\ &= \sum_{k=0}^{\infty} g_{2k+1}\beta^{2k+1}\langle x_i, y_j\rangle^{2k+1} \\ &= E^{-1}(\beta\langle x_i, y_j\rangle). \end{aligned}$$

Proof (cont.).

- Hence, β is defined by the condition that the vectors $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh(\frac{\pi}{2}\beta)$$

$$\Leftrightarrow \qquad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1+\sqrt{2})$$

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- Problem: $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are infinite-dimensional
- Solution: the positive definite and symmetric Gram matrix G

$$G = \begin{pmatrix} \langle \tilde{x}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{x}_m \rangle & \langle \tilde{x}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{x}_m, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{x}_m \rangle & \langle \tilde{x}_m, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{y}_n \rangle \\ \langle \tilde{y}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{x}_m \rangle & \langle \tilde{y}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{y}_n, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{x}_m \rangle & \langle \tilde{y}_n, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{y}_n \rangle \end{pmatrix} \tag{14}$$

Proof (cont.).

- Due to the properties of G we can decompose G via a real orthogonal matrix Q with columns that are the eigenvectors of G and a real diagonal matrix Λ having the eigenvalues of G on the diagonal, thus

$$G = Q\Lambda Q^\top = (\underbrace{Q\Lambda^{1/2}}_{=:A})^\top (Q\Lambda^{1/2}). \tag{15}$$

□

Proof (cont.).

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$$G = Q\Lambda Q^\top = (\underbrace{Q\Lambda^{1/2}}_{=:A})^\top (Q\Lambda^{1/2}). \tag{15}$$

- The columns of A are the vectors $x'_1, \ldots, x'_m, y'_1, \ldots, y'_n \in S^{m+n-1}$ we are looking for.

$\square$

Definition

For $M \in \mathbb{R}^{m\times n}$ define the quadratic program

$$\|M\|_{\infty\to 1} = \max\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\xi_i\eta_j : \xi_i^2 = 1, i = 1,\dots,m, \eta_j^2 = 1, j = 1,\dots,n\right\}$$
$$= \max\left\{\operatorname{Tr} M\eta\xi^\top : \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\right\}. \quad (16)$$

Definition

The SDP relaxation of $\|M\|_{\infty\to 1}$ is given via:

$$\mathrm{sdp}_{\infty\to 1}(M) = \max \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\langle x_i, y_j\rangle$$
$$x_i, y_j \in \mathbb{R}^{m+n}$$
$$|x_i| = 1, i = 1,\dots,m$$
$$|y_j| = 1, j = 1,\dots,n$$

Theorem (Grothendieck's inequality)

There exists a constant K such that for all $M \in \mathbb{R}^{m \times n}$:

$$\|M\|_{\infty\to 1} \leq \mathrm{sdp}_{\infty\to 1}(M) \leq K\|M\|_{\infty\to 1}. \tag{17}$$

Proof.

Use the following approximation algorithm with randomized rounding:

Algorithm 1: Approximation algorithm with randomized rounding for $\|M\|_{\infty\to 1}$

1. Solve $\mathrm{sdp}_{\infty\to 1}(M)$. Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in S^{m+n-1}$ be the optimal unit vectors.
2. Apply Krivine's trick (Lemma 9) and use vectors x_i, y_j to create new unit vectors $x'_1, \ldots, x'_m, y'_1, \ldots, y'_n \in S^{m+n-1}$.
3. Choose $r \in S^{m+n-1}$ randomly.
4. Round: $\xi_i = \mathrm{sign}(\langle x'_i, r\rangle)$
   $\eta_j = \mathrm{sign}(\langle y'_j, r\rangle)$

Proof (cont.).

Expected quality of the outcome:

$$\begin{aligned}
\|M\|_{\infty\to 1} &\geq \mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\xi_i\eta_j\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\mathbb{E}[\operatorname{sign}(\langle x_i', r\rangle)\operatorname{sign}(\langle y_j', r\rangle)] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\beta\langle x_i, y_j\rangle \\
&= \beta\,\mathrm{sdp}_{\infty\to 1}(M),
\end{aligned}$$

where the last equality follows by Krivine's trick with $\beta = \frac{2\ln(1+\sqrt{2})}{\pi}$, thus $K \leq \beta^{-1}$.

Theorem (Tsirelson)

(Hard direction) For all positive integers n, r *and any* $x_1, \ldots, x_n,$ $y_1, \ldots, y_n \in S^{r-1}$*, there exists a positive integer* $d := d(r)$*, a state* $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ *and* $\{-1,1\}$*-observables* $F_1, \ldots, F_n, G_1, \ldots, G_n \in O(\mathbb{C}^d)$*, such that for every* $i, j \in \{1, \ldots, n\}$*, we have*

$$\langle\psi|F_i \otimes G_j|\psi\rangle = \langle x_i, y_j\rangle. \tag{18}$$

Moreover, $d \leq 2^{\lceil r/2 \rceil}$.

(Easy direction) Conversely, for all positive integers n, d*, state* $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ *and* $\{-1,1\}$*-observables* $F_1, \ldots, F_n, G_1, \ldots, G_n \in O(\mathbb{C}^d)$*, there exist a positive integer* $r := r(d)$ *and* $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^{r-1}$ *such that for every* $i, j \in \{1, \ldots, n\}$*, we have*

$$\langle x_i, y_j\rangle = \langle\psi|F_i \otimes G_j|\psi\rangle. \tag{19}$$

Moreover, $r \leq 2d^2$.

Proof.

For the hard direction look at the second part of the proof of $QC_{m,n} = \{(\langle x_i, y_j\rangle)_{1\leq i\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$ (Lemma).
Due to the additional restriction/assumption that F_i and G_j are $\{-1, 1\}$-observables the other direction gets easier.

$\square$

Since

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n, the following three equivalent conditions hold:

(1) *We have the inclusion*

$$\mathrm{QC}_{m,n} \subset K\mathrm{LC}_{m,n}. \tag{20}$$

(2) *For any $M \in \mathbb{R}^{m \times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have*

$$\sum_{i,j} M_{ij} \operatorname{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \tag{21}$$

$$\Leftrightarrow \qquad \operatorname{Tr} MA^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} M(\xi \eta^\top)^\top. \tag{22}$$

(3) *For any $M \in \mathbb{R}^{m \times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1$, $|y_j| \leq 1$ we have*

$$\sum_{i,j} M_{ij} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \operatorname{Tr} \xi^\top M \eta. \tag{23}$$

Proof.

Since (23) is a direct consequence of Grothendieck's inequality the only thing left to prove is the equivalence between (1)-(3). The equivalence of (3) and (2) (the Tsirelson's bound) is a consequence of either the proof of Lemma **??** or Tsirelsons Theorem (Theorem 1).

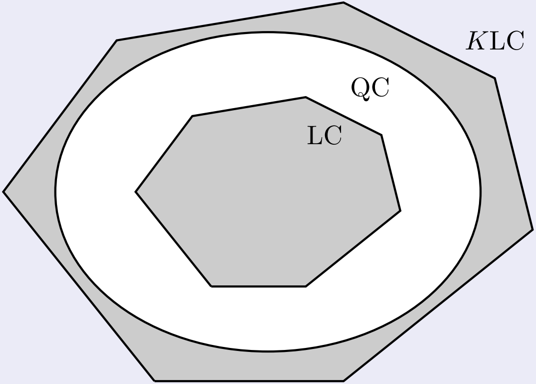

Figure: Visualization