Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 10, 2018

Outline

Let's get on the same page

- We should know what a *state* is
- We should know what the *tensor product* does
- We should be familiar with the *Dirac notation*

Quantum systems

- A quantum system is a portion of the whole universe. For example a set electrons.
- A quantum system X is associated with a copy of $\mathbb{C}^k$
- It may consist of subsystems $X_1, \dots, X_N$ each of which is associated with a copy of $\mathbb{C}^{n_i}$. In this case $k = n_1 \dots n_N$

Measurements

- A measurement can be performed on a system X that is in state ρ
- Let $\mathscr{A}$ be a finite set of outcomes of the measurement
- The measurement itself is defined by a set of psd matrices $\{F^a\}_{a\in\mathscr{A}} \subseteq \mathbb{C}^{n\times n}$ that sum up to the identity matrix, i.e. $\sum_{a\in\mathscr{A}} F^a = I$

Measurements

- A projective measurement is defined by psd matrices that satisfy $F^a F^b = \delta_{ab} F^a \; \forall a, b \in \mathcal{A}$
- The outcome of a measurement is a random variable χ with probability distribution: $\mathbb{P}[\chi = a] = \mathrm{Tr}(\rho F^a)$
- To define an expected value we define outcomes in $\mathcal{A}$ as real numbers

Measurements

- $\mathbb{E}[\chi] = \sum_{a\in\mathcal{A}} a\mathrm{Tr}(\rho F^a) = \mathrm{Tr}(\rho(\sum_{a\in\mathcal{A}} aF^a))$
- $\sum_{a\in\mathcal{A}} aF^a$ is called observable
- A simple case we will use later are $\{-1,1\}$-valued observables
- if we consider projective measurements we have

$$(F^+ - F^-)^2 = \underbrace{{F^+}^2}_{=F^+} - \underbrace{F^+F^-}_{\delta_{+-}=0} + \underbrace{{F^-}^2}_{F^-} = F^+ + F^- = I$$

- i.e. a $\{-1,1\}$-valued observable is both unitary an Hermitian

Doling out subsystems

- Consider a system X consisting of subsystems $X_1, \dots X_N$ which we distribute among N parties, which may be located anywhere in the universe
- The parties *share* the state X is in
- Every party may perform a measurement on their subsystem X_i, i.e. there are N sets of psd matrices $\{F^{a_1}\}_{a_1 \in \mathcal{A}_1} \in \mathbb{C}^{n_1 \times n_1}, \dots, \{F^{a_N}\}_{a_N \in \mathcal{A}_N} \in \mathbb{C}^{n_N \times n_N}$

The joint probability distribution of the N measurement outcomes $\chi_1, \ldots, \chi_N$ is

$$\mathbb{P}\left[\chi_1 = a_1, \chi_2 = a_2, \ldots, \chi_N = a_N\right] = \mathrm{Tr}(\rho F_1^{a_1} \otimes \cdots \otimes F_N^{a_N})$$

Entanglement

- We will only consider pure states meaning states that they have rank 1 and therefore can be written as $\rho = |\psi\rangle\langle\psi|$
- A state is called product state if it can be written as $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle \ldots |\psi_N\rangle$
- When a vector $|\psi\rangle$ is referred to as a state we mean the matrix $|\psi\rangle\langle\psi|$
- A state that is not a product state is called entangled

Example

- Let $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle$ be a system and give $|\psi_A\rangle$ to Alice and $|\psi_B\rangle$ to Bob
- Let them perform measurements $\{G^b\}_{b\in\mathcal{B}}$ and $\{F^a\}_{a\in\mathcal{A}}$ on their respective quantum systems
- What is the probability of Alice getting measurement outcome $\chi_A = a$ and Bob getting $\chi_B = b$?

Example

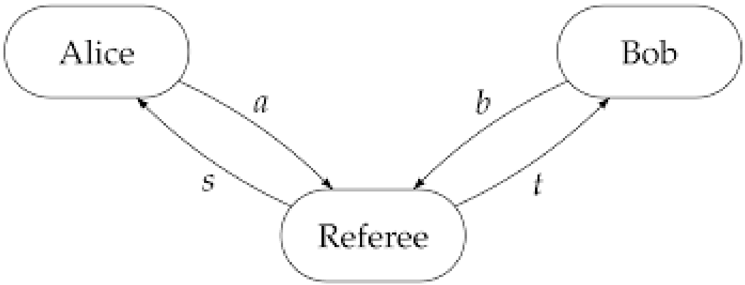
$$
\begin{aligned}
\mathrm{Tr}(|\psi\rangle\langle\psi|F^a \otimes G^b) &= \langle\psi|F^a \otimes G^b|\psi\rangle \\
&= (\langle\psi_A| \otimes \langle\psi_B|)(F^a \otimes G^b)(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= ((\langle\psi_A|F^a) \otimes (\langle\psi_B|G^b))(|\psi_A\rangle \otimes |\psi_B\rangle) \\
&= \langle\psi_A|F^a|\psi_A\rangle \otimes \langle\psi_B|G^b|\psi_B\rangle \\
&= \langle\psi_A|F^a|\psi_A\rangle\langle\psi_B|G^b|\psi_B\rangle
\end{aligned}
$$

This is equal to the product of the probabilities of Alice measuring a and Bob measuring b, i.e. the outcome do not correlate.

Nonlocal games

- Three participants: Alice, Bob and a referee
- Referee doles out a question s to Alice and a question t to Bob
- Alice and Bob are assumed to be located anywhere in the universe respectively
- Alice and Bob must not communicate
- Alice sends answer a Bob sends answer b back to the referee, who then decides whether both win or both lose

Nonlocal games



Mathematically speaking

- Four finite sets $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$
- probability distribution π over $\mathcal{S} \times \mathcal{T}$
  $\pi : \mathcal{S} \times \mathcal{T} \to [0, 1]$
- The referee sends with probability $\pi(s, t)$ s to Alice and t to Bob
- They answer with an element $a \in \mathcal{A}$ and $b \in \mathcal{B}$ respectively
- A map $V : \mathcal{S} \times \mathcal{T} \times \mathcal{A} \times \mathcal{B} \to \{0, 1\}$
- They win if $V(s, t, a, b) = 1$ and lose otherwise

Classical strategies

- All players know π and V and the information they received but not what the other players received
- They are allowed to agree on a strategy beforehand

Classical strategies

- All players know π and V and the information they received but not what the other players received
- They are allowed to agree on a strategy beforehand but must not communicate once the game started
- A deterministic strategy is a map $a : \mathcal{S} \rightarrow \mathcal{A}$ for Alice and $b : \mathcal{T} \rightarrow \mathcal{B}$ for Bob The winning probability then is:

$$\mathbb{E}_{s,t\sim\pi}\left[V(a(s), b(t), s, t)\right]$$

Quantum case

- Suppose Alice and Bob have a subsystem X_A, X_B of a quantum system X which is in state ρ, i.e. Alice and Bob share state ρ
- If the state is entangled measurements can give correlated measurement outcomes
- Alice and Bob may gain information by performing measurements
- Answering according to measurement outcomes could increase winning probability

Mathematically speaking

- A quantum system X consisting of two n-dimensional subsystems X_A, X_B in some entangled state ρ
- Alice performs a measurement $\{F_s^a\}_{a\in\mathcal{A}} \subseteq \mathbb{C}^{n\times n}$ on her subsystem X_A and Bob performs a measurement $\{G_t^b\}_{b\in\mathcal{B}} \subseteq \mathbb{C}^{n\times n}$ on his subsystem X_B
- They send their measurement outcome as their answer to the referee
- Their winning probability is:

$$\mathbb{E}_{s,t\sim\pi}\left[\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \mathrm{Tr}(\rho F_s^a \otimes G_t^b) V(a,b,s,t)\right]$$

- Since states are convex combinations of pure states and the trace function is linear we only need to consider pure entangled states

Two player XOR games

- Let the sets $\mathscr{A}$ and $\mathscr{B}$ be $\{0,1\}$, so Alice and Bob both answer either with 1 or 0
- The predicate V is defined as $V(a,b,s,t) = [a \oplus b = f(s,t)]$, where $f : \mathscr{S} \times \mathscr{T} \to \{0,1\}$
- A truth table for $a \oplus b$ looks like this

$\oplus$	0	1
0	0	1
1	1	0

Bias and violation ratio

- Alice and Bob can always win with probability $\frac{1}{2}$ by flipping an unbiased coin
- The classical bias of an XOR game G is defined as the difference of the probabilities of winning and losing for an optimal strategy and denoted by $\beta(G)$
- The bias $\beta^*(G)$of entangled strategies is calculated the same way
- It is twice the amount by which the maximal winning probability exceeds $\frac{1}{2}$
- The violation ratio is defined as $\frac{\beta^*(G)}{\beta(G)}$

Signs and observables

- It is convenient to use the $\{-1, 1\}$-basis instead of the $\{0, 1\}$-basis for boolean valued objects.
- Let $a : \mathcal{S} \rightarrow \{0, 1\}$ and $b : \mathcal{T} \rightarrow \{0, 1\}$ be classical strategies and π the probability distribution the referee uses to pick s, t
- The bias is given by the probability under π that $a(s) \oplus b(t) = f(s, t)$ minus the probability under π that $a(s) \oplus b(t) \neq f(s, t)$

This means the bias can be written as:

$$\mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{[a(s)\oplus b(t)=f(s,t)]}\right] =$$
$$= \mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{a(s)\oplus b(t)+f(s,t)}\right] =$$
$$= \mathbb{E}_{(s,t)\sim\pi}\left[(-1)^{a(s)}(-1)^{b(t)}(-1)^{f(s,t)}\right]$$

And we can define the sign matrix $\Sigma_{s,t} = (-1)^{f(s,t)}$ and functions $\chi(s) = (-1)^{a(s)}$ and $\psi(t) = (-1)^{b(t)}$. So the bias is

$$\mathbb{E}_{(s,t)\sim\pi}\left[\chi(s)\psi(t)\Sigma_{st}\right]$$

- The outcomes in an XOR game are $\{0,1\}$
- Alice and Bob have measurements $\{F_s^0, F_s^1\}$ and $\{G_t^0, G_t^1\}$ and share an entangled state
- The probability of Alice and Bob answering with a, b upon receiving s, t respectively is $\langle\psi|F_s^a \otimes G_t^b|\psi\rangle$
- Lets calculate the expected value of $(-1)^{a\oplus b}$

$$\begin{aligned}(1)\cdot\mathbb{P}[a=b]+(-1)\cdot\mathbb{P}[a\neq b] &= \\ =\langle\psi|F_s^0\otimes G_t^0|\psi\rangle+\langle\psi|F_s^1\otimes G_t^1|\psi\rangle \\ -\langle\psi|F_s^1\otimes G_t^0|\psi\rangle-\langle\psi|F_s^0\otimes G_t^1|\psi\rangle \\ =\langle\psi|(F_s^0-F_s^1)\otimes(G_t^0-G_t^1)|\psi\rangle\end{aligned}$$

- Define $\{-1, 1\}$-observables $F_s = F_s^0 - F_s^1$ and $G_t = G_t^0 - G_t^1$ with the property that its difference squared is the identity matrix
- Using this strategy the bias becomes

$$\mathbb{E}_{(s,t)\sim\pi}\left[\langle\psi|F_s \otimes G_t|\psi\rangle\Sigma_{s,t}\right]$$

More generally speaking

- For any XOR game the bias is defined as the difference of the probabilities of winning and loosing
- Which is, if considering the $\{-1, 1\}$ basis, the expected value
- We are looking to maximize this quantity

Classical strategies

When using classical strategies this is

$$\max\{\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{st}\chi(s)\psi(t)\right] : \chi : \mathcal{S} \to \{-1,1\},\\ \psi : \mathcal{T} \to \{-1,1\}\}$$

Entangled strategies

When using entangled strategies the winning probability might increase indefinitely with the dimensions, so we use the $\sup_{n\in\mathbb{N}}$

$$\sup_{n\in\mathbb{N}}\{\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{st}\langle\psi|F_s\otimes G_t|\psi\rangle\right] : |\psi\rangle\in\mathbb{C}^n\otimes\mathbb{C}^n,$$

$$F_s, G_t \in O(\mathbb{C}^n)\}$$

The CHSH game

- The CHSH game (Clauser, Horner, Shimony, Holt) is a two player XOR game with $\mathcal{A} = \mathcal{B} = \mathcal{S} = \mathcal{T} = \{0, 1\}$ and π being the uniform distribution
- $f(s, t) = s \wedge t$, i.e. $f(1, 1) = 1$ and $f(0, 0) = f(0, 1) = f(1, 0) = 0$
- Alice and Bob can win $\frac{3}{4}$ of the games by using deterministic strategies $(0, 0), (1, 0)$ or $(0, 1)$

Quantum strategy

- Let Alice and Bob share an EPR state
- Define

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $XY + YX = 0$ and $X^2 = Y^2 = I$
- For Alice define the observable for question 0 by $F_0 = X$ and for question 1 by $F_1 = Y$
- Bobs observables are going to be $G_0 = (X - Y)/\sqrt{2}$ for question 0 and $G_1 = (X + Y)/\sqrt{2}$ for question 1

The following auxiliary calculations will be helpful later:

$$\langle \mathrm{EPR}|X \otimes X|\mathrm{EPR}\rangle = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{2}{2} = 1$$

$$\langle \mathrm{EPR}|Y \otimes Y|\mathrm{EPR}\rangle = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}-1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = -1$$

$$\langle \mathrm{EPR}| X \otimes Y |\mathrm{EPR}\rangle = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}i & 0 & 0 & -i\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = 0$$

$$\langle \mathrm{EPR}| Y \otimes X |\mathrm{EPR}\rangle = 0$$

Lets calculate the expected values of the sign $a \oplus b$:

$$
\begin{aligned}
\langle \mathrm{EPR} | F_0 \otimes G_0 | \mathrm{EPR} \rangle &= \langle \mathrm{EPR} | X \otimes \frac{1}{\sqrt{2}}(X - Y) | \mathrm{EPR} \rangle \\
&= \langle \mathrm{EPR} | X \otimes \frac{1}{\sqrt{2}} X | \mathrm{EPR} \rangle - \langle \mathrm{EPR} | X \otimes \frac{1}{\sqrt{2}} Y | \mathrm{EPR} \rangle \\
&= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathrm{EPR}|F_1 \otimes G_1|\mathrm{EPR}\rangle &= \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}(X+Y)|\mathrm{EPR}\rangle \\
&= \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}X|\mathrm{EPR}\rangle + \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}Y|\mathrm{EPR}\rangle \\
&= 0 - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}
\end{aligned}
$$

$$\langle \mathrm{EPR}|F_0 \otimes G_1|\mathrm{EPR}\rangle = \langle \mathrm{EPR}|X \otimes \frac{1}{\sqrt{2}}(X+Y)|\mathrm{EPR}\rangle$$

$$= \langle \mathrm{EPR}|X \otimes \frac{1}{\sqrt{2}}X|\mathrm{EPR}\rangle + \langle \mathrm{EPR}|X \otimes \frac{1}{\sqrt{2}}Y|\mathrm{EPR}\rangle$$

$$= \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

$$
\begin{aligned}
\langle \mathrm{EPR}|F_1 \otimes G_0|\mathrm{EPR}\rangle &= \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}(X - Y)|\mathrm{EPR}\rangle \\
&= \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}X|\mathrm{EPR}\rangle - \langle \mathrm{EPR}|Y \otimes \frac{1}{\sqrt{2}}Y|\mathrm{EPR}\rangle \\
&= 0 - (-\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Thus, we have

$$\langle \mathrm{EPR}|F_s \otimes G_t|\mathrm{EPR}\rangle = \begin{cases} \frac{1}{\sqrt{2}}, (0,0),(1,0),(0,1) \\ -\frac{1}{\sqrt{2}}, (1.1) \end{cases}$$

which is equivalent to

$$\langle \mathrm{EPR}|F_s \otimes G_t|\mathrm{EPR}\rangle = \frac{(-1)^{s\wedge t}}{\sqrt{2}}, s,t \in \{0,1\}$$

The bias of the entangled strategy equals

$$\mathbb{E}_{(s,t)\sim\pi}\left[\Sigma_{s,t}\langle\psi|F_s \otimes G_t|\psi\rangle\right] =$$

$$= \frac{1}{4}\sum_{s,t=0}^{1}(-1)^{s\wedge t}\langle \mathrm{EPR}|F_s \otimes G_t|\mathrm{EPR}\rangle$$

$$= \frac{1}{4}\cdot\frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

The bias is $\frac{1}{\sqrt{2}}$ from which follows that the winning probability is by definition:

$$\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \cos(\pi/8) \approx 0.85\dots$$

Outline

Nice slide to draw the connection between the games an LC

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

$$a_{ij} = \mathbb{E}[X_i Y_j]$$

for all $1 \leq i \leq m, 1 \leq j \leq n$.

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- Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

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Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

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Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

$\square$

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- Let $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$ for $k \in \{0,1\}$

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- Find $(X_i), (Y_j)$ with $|X_i|, |Y_j| \leq 1$ almost surely such that

$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

for $\beta \in [0,1]$

$\square$

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$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

  for $\beta \in [0,1]$
- Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$

□

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- Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}$, $Y_j = Y_j^{(\alpha)}$
- Then

$$\begin{aligned}\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(1)}]\mathbb{1}_{\{\alpha=1\}}] \\ &= \beta \mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1-\beta)\mathbb{E}[X_i^{(1)} Y_j^{(1)}]\end{aligned}$$

□

$\mathrm{LC}_{m,n} \subset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$

- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.

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- Set $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.

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- Set $X = (X_1, ..., X_m)$ and $Y = (Y_1, ..., Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.
- Hypercube description by its vertices: $[-1,1]^d = \mathrm{conv}\{\xi \mid \xi \in \{-1,1\}^d\}$

$\mathrm{LC}_{m,n} \subset \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$

- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.
- Set $X = (X_1, ..., X_m)$ and $Y = (Y_1, ..., Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.
- Hypercube description by its vertices: $[-1,1]^d = \mathrm{conv}\{\xi \,|\, \xi \in \{-1,1\}^d\}$
- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

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- $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$

$\square$

Some nice frame to connect QCs to the games

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state **Introduce a symbol zo define operators form one space to another** $\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

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- Set of all quantum correlation matrices denoted by $\mathrm{QC}_{m,n}$

Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\le 1\le m, 1\le j\le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\},$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

- $a_{ij} = \operatorname{Tr} \rho X_i \otimes Y_j$, sate ρ on a Hilbert space $\mathscr{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and Hermitian operators $(X_i)_{1 \geq m}$, $(Y_j)_{1 \geq n}$ on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$

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- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \operatorname{Re}(\operatorname{Tr} \rho S T)$.

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- $\left|\alpha^{(i)}\right|, \left|\gamma^{(j)}\right| \leq 1$ due to $\tilde{\beta}(x_i), \tilde{\beta}(y_j) \leq 1$

□

- In order to show

$$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \le 1 \le m, 1 \le j \le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$$

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Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

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we will use the following

Proposition

For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \dots n$$

- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$
- For $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} = \eta_i U_i$ we can calculate

$$\begin{aligned} XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i,j \leq \leq 2n} \xi_i \eta_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\ &= \langle \xi, \eta \rangle I. \end{aligned}$$

- The result follows by setting $X = Y$.

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- $\mathrm{Tr}\,(X_i Y_j^T) = d \cdot \langle x_i, y_j\rangle$ and $\|X_i\|_\infty \leq 1$ since $X_i X_i^* = |x_i|^2 I$ and $|x_i|^2 \leq 1$ (the same holds for Y_j)

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- Let $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ and $\rho = |\phi\rangle\langle\rho|$. Note that we can write ρ as

$$\rho = |\phi\rangle\langle\phi| = \frac{1}{d}\sum_{1\leq k,l\leq d} |kk\rangle\langle ll| = \frac{1}{d}\sum_{1\leq k,l\leq d} |k\rangle\langle l| \otimes |k\rangle\langle l|$$

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- Then

$$\mathrm{Tr}\,(\rho X_i \otimes Y_j) = \frac{1}{d} \sum_{1\leq k,l\leq d} \mathrm{Tr}\,\big(|k\rangle\langle l| X_i \otimes |k\rangle\langle l| Y_j\big) = \frac{1}{d} \sum_{1\leq k,l\leq d} \mathrm{Tr}\,\big(|k\rangle\langle l| X_i\big)\mathrm{T}$$

$$= \frac{1}{d} \mathrm{Tr}\, X_i Y_j^T = \langle x_i, y_j \rangle.$$

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\text{span}\{x_1, \ldots, x_m\}$ or $\text{span}\{y_1, ..., y_n\}$, as in the proof before.

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- It holds

$$\mathrm{LC}_{2,2} = \mathrm{conv}\{\pm\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\}.$$

Claim

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \mid -1 \leq \operatorname{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{1}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}), \sigma(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}) \mid \sigma \in \{\mathrm{id}(1\ 2), (1\ 3), (1\ 4)\}\}$.

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- Equation 2 is a non-redundant hyperplane description of $\mathrm{LC}_{2,2}$, hence all linear constraints define facets.

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- Bound is achieved by $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1,1)$ and $y_1 = y_2 = (1,0)$

Lemma (Grothendieck's identity)

Let $x, y \in \mathbb{R}^d$ be unit vectors. Let $r \in \mathbb{R}^d$ be a random unit vector chosen from $O(d)$-invariant probability distribution on the unit sphere. Then

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  - the normalized vector $n := s/\|s\|$ is uniformly distributed on the intersection of the unit sphere and $\mathrm{span}\{x, y\}$ by the $O(d)$-invariance of the probability distribution

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