Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

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Outline

1. Local and quantum correlation matrices
   - Local correlation matrices
   - Quantum correlation matrices
   - The relations between quantum correlation and local correlation matrices

Local correlation matrices

- Deterministic strategies of Alice and Bob correspond to vectors $\xi \in \{-1,1\}^{\mathcal{S}}$, respectively $\eta \in \{-1,1\}^{\mathcal{T}}$

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- Their common answer is the product $\xi_i \eta_j$
- Instead of deterministic strategies they can also answer according to the probability distribution of random variables X_i, Y_j
- Their common answer is $\mathbb{E}[X_i Y_j]$
- This information can be encoded in an $\mathcal{S} \times \mathcal{T}$ matrix Motivation Quantum

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$.

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)

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- Define a Bernoulli random variable α (which is independent form $X_i^{(k)}, Y_j^{(k)}$) such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}, Y_j = Y_j^{(\alpha)}$

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- Then

$$\begin{aligned}
\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)}] \mathbb{1}_{\{\alpha=1\}}] \\
&= \mathbb{E}[X_i^{(0)} Y_j^{(0)}] \mathbb{P}(\alpha = 0) + \mathbb{E}[X_i^{(1)} Y_j^{(1)}] \mathbb{P}(\alpha = 1) \\
&= \beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)}
\end{aligned}$$

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- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely.

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- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$

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- Using the same decomposition for Y we obtain

$$\begin{aligned} a_{ij} = \mathbb{E}[X_i Y_j] &= \mathbb{E}\Big[\Big(\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)} \xi_i\Big)\Big(\sum_{\eta \in \{-1,1\}^n} \lambda_\eta^{(Y)} \eta_j\Big)\Big] \\ &= \sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\Big[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}\Big] \xi_i \eta_j \\ &= \sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}] \xi_i \eta_j \end{aligned}$$

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- Due to $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$

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- Again we can encode this information in a matrix

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Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

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Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\leq i\leq m, 1\leq j\leq n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\leq i\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

- $a_{ij} = \mathrm{Tr}\,(\rho X_i \otimes Y_j)$, state ρ on a Hilbert space $\mathscr{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and Hermitian operators $(X_i)_{1\geq m}$, $(Y_j)_{1\geq n}$ on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$

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- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \mathrm{Re}(\mathrm{Tr}\,\rho ST)$.

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- Obtain an inner product space $U := B^{sa}(\mathcal{H}) / \ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

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- Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\mathrm{Tr}\,(\rho X_i \otimes Y_j) = a_{ij}$$

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- $\beta(X \otimes I, X \otimes I), \beta(I \otimes Y, I \otimes Y) \le 1$
(this can be shown by using a *Schmidt-decomposition* of ρ and using $\|X_i\|_\infty, \|Y_j\|_\infty \le 1$)

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- If $\pi(y_j)$ is the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$

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- If $\pi(y_j)$ is the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$
- The vectors still have not the right dimension but again, we can project them onto vectors in $\mathbb{R}^m$

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- In order to show

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we will use the following

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- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \dots n$$

- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$
- For $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} = \eta_i U_i$ we can calculate

$$\begin{aligned} XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i,j \leq\ 2n} \xi_i \eta_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\ &= \langle \xi, \eta \rangle I. \end{aligned}$$

- The result follows by setting $Y = X^*$.

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- Let $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle \otimes |i\rangle$ and $\rho = |\psi\rangle\langle\psi|$. Note that we can write ρ as

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{d}\sum_{1\le k,l\le d} |k\rangle \otimes |k\rangle\langle l| \otimes \langle l| = \frac{1}{d}\sum_{1\le k,l\le d} |k\rangle\langle l| \otimes |k\rangle\langle l|$$

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- Then

$$\begin{aligned}
\mathrm{Tr}\left(\rho X_i \otimes Y_j\right) &= \frac{1}{d} \sum_{1\leq k,l\leq d} \mathrm{Tr}\left(|k\rangle\langle l| X_i \otimes |k\rangle\langle l| Y_j\right) \\
&= \frac{1}{d} \sum_{1\leq k,l\leq d} \mathrm{Tr}\left(|k\rangle\langle l| X_i\right) \mathrm{Tr}\left(|k\rangle\langle l| Y_j\right) \\
&= \frac{1}{d} \mathrm{Tr}\, X_i Y_j^T = \langle x_i, y_j\rangle.
\end{aligned}$$

$\square$

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- define vectors $\tilde{x}_i := \begin{pmatrix} \sqrt{\lambda} x_i \\ \sqrt{1-\lambda}\bar{x}_i \end{pmatrix}, \tilde{y}_j := \begin{pmatrix} \sqrt{\lambda} y_j \\ \sqrt{1-\lambda}\bar{y}_j \end{pmatrix}$ for $\lambda \in [0, 1]$

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- it holds $|\tilde{x}_i| \leq \lambda \left| \begin{pmatrix} x_i \\ 0 \end{pmatrix} \right| + (1-\lambda) \left| \begin{pmatrix} 0 \\ \bar{x}_i \end{pmatrix} \right| \leq 1$ and $\langle \tilde{x}_i, \tilde{y}_j \rangle = \lambda \langle x_i, y_j \rangle + (1-\lambda) \langle \tilde{x}_i, \tilde{y}_j \rangle$.

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\mathrm{span}\{x_1, \ldots, x_m\}$ or $\mathrm{span}\{y_1, ..., y_n\}$, as in the proof before.

The relations between quantum correlation and local correlation matrices

- $\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$
- $\mathrm{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \le i \le m, 1 \le j \le n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$
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- Inclusion is strict in general

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- It holds

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- W can also write it as in intersections of halfspaces:

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \,|\, -1 \le \mathrm{Tr}\, AM \le 1 \text{ for all } M \in \mathcal{K}\}, \tag{1}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}), \sigma(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \,|\, \sigma \in \{\mathrm{id}, (1\ 2), (1\ 3), (1\ 4)\}\}$.

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\mathrm{Tr}\,(AM) &= \langle x_1, y_1\rangle + \langle x_1, y_2\rangle + \langle x_2, y_1\rangle - \langle x_2, y_2\rangle \\
&= \langle x_1 + x_2, y_1\rangle + \langle x_1 - x_2, y_2\rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\
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- $(|x_1 + x_2| + |x_1 - x_2|)^2 \leq 4(|x_1|^2 + |x_2|^2)$
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- Bound is achieved by $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$