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Outline

1. Local and quantum correlation matrices
    - Local correlation matrices
    - Quantum correlation matrices
    - The relations between quantum correlation and local correlation matrices

Nice slide to draw the connection between the games an LC

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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- Set of all local correlation matrices: $\mathrm{LC}_{m,n}$

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Lemma

$$\mathrm{LC}_{m,n} = \mathrm{conv}\{\xi\eta^T \,|\, \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$$

- No matter which probabilistic strategy there is a deterministic one which as at least as good as the one one chooses

$\mathrm{LC}_{m,n} \supset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$ (Choose $X_i \equiv \xi_i,\ Y_j \equiv \eta_j$)
- Suffices to show that $\mathrm{LC}_{m,n}$ is convex.
- Let $a_{ij}^{(k)} = \mathbb{E}[X_i^{(k)} Y_j^{(k)}]$ for $k \in \{0,1\}$
- Find $(X_i), (Y_j)$ with $|X_i|, |Y_j| \leq 1$ almost surely such that

$$\beta a_{ij}^{(0)} + (1-\beta) a_{ij}^{(1)} = \mathbb{E}[X_i Y_j]$$

  for $\beta \in [0,1]$
- Define a Bernoulli random variable α such that $\mathbb{P}(\alpha = 0) = \beta$, $\mathbb{P}(\alpha = 1) = 1 - \beta$ and set $X_i = X_i^{(\alpha)}$, $Y_j = Y_j^{(\alpha)}$
- Then

$$\begin{aligned}\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(1)}]\mathbb{1}_{\{\alpha=1\}}] \\ &= \beta\mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1-\beta)\mathbb{E}[X_i^{(1)} Y_j^{(1)}]\end{aligned}$$

$\mathrm{LC}_{m,n} \subset \mathrm{conv}\{\xi\eta^T \mid \xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n\}$

- Let $a_{ij} = \mathbb{E}[X_i Y_j]$ for $\mathbb{R}$-valued random variables $(X_i), (Y_j)$ defined on a common probability space Ω with $|X_i|, |Y_j| \leq 1$ almost surely. pause

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- Set $X = (X_1, ..., X_m)$ and $Y = (Y_1, ..., Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.

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- Hypercube description by its vertices: $[-1,1]^d = \mathrm{conv}\{\xi \,|\, \xi \in \{-1,1\}^d\}$

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- Set $X = (X_1, ..., X_m)$ and $Y = (Y_1, ..., Y_n)$, then $X \in [-1,1]^m$, $Y \in [-1,1]^n$ almost surely.
- Hypercube description by its vertices: $[-1,1]^d = \mathrm{conv}\{\xi \mid \xi \in \{-1,1\}^d\}$
- Define random variables $\lambda_\xi^{(X)} : \Omega^m \to [0,1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega)\xi$$

almost surely and $\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)}(\omega) = 1$

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- Using the same decomposition for Y we obtain

$$\begin{aligned} a_{ij} = \mathbb{E}[X_i Y_j] &= \mathbb{E}\Big[\Big(\sum_{\xi \in \{-1,1\}^m} \lambda_\xi^{(X)} \xi_i\Big)\Big(\sum_{\eta \in \{-1,1\}^n} \lambda_\eta^{(Y)} \eta_j\Big)\Big] \\ &= \sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}\big[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}\big] \xi_i \eta_j \\ &= \Big(\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}]\Big) \xi_i \eta_j \end{aligned}$$

$\square$

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- $\sum_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \mathbb{E}[\lambda_\xi^{(X)}]\mathbb{E}[\lambda_\eta^{(Y)}] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n$

□

Some nice frame to connect QCs to the games

Definition

Let $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ be self-adjoint operators on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ for some positive integers d_1, d_2, satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state **Introduce a symbol zo define operators form one space to another** $\rho \in D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ such that

$$a_{ij} = \operatorname{Tr} \rho(X_i \otimes Y_j).$$

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- Set of all quantum correlation matrices denoted by $\mathrm{QC}_{m,n}$

Lemma

$$\mathrm{QC}_{m,n} = \{(\langle x_i, y_j\rangle)_{1\leq 1\leq m, 1\leq j\leq n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\},$$

$\mathrm{QC}_{m,n} \subset \{(\langle x_i, y_j\rangle)_{1\leq 1\leq m, 1\leq j\leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$

- $a_{ij} = \operatorname{Tr} \rho X_i \otimes Y_j$, sate ρ on a Hilbert space $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and Hermitian operators $(X_i)_{1\geq m}$, $(Y_j)_{1\geq n}$ on $\mathbb{C}^{d_1}$, respectively $\mathbb{C}^{d_2}$ satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$
- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on $\mathcal{H}$ by $\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ where $\beta(S, T) = \mathrm{Re}(\operatorname{Tr} \rho ST)$.
- perhaps verification of at least some of these properties
- Obtain an inner product space $U := B^{sa}(\mathcal{H})/\ker \beta$ equipped with the inner product

$$\tilde{\beta}([S], [T]) = \beta(S, T).$$

- Identify $X_i \otimes I, I \otimes Y_j$ with vectors x_i, y_j in U, then

$$\tilde{\beta}(x_i, y_j) = \beta(X_i, Y_j) = Re\operatorname{Tr}(\rho X_i \otimes Y_j) = a_{ij}$$

$QC_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1\leq 1 \leq m, 1\leq j \leq n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}.$

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- $\beta(X \otimes I, X \otimes I), \beta(I \otimes Y, I \otimes Y) \leq 1$
  (this can be shown by using a *Schmidt-decomposition* of ρ and using $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$)

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- $\pi(y_j)$ the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$

□

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- Let $\{a_1, ..., a_r\}$ be an orthonormal basis of $\mathrm{span}\{x_1, ..., x_m\}$ with respect to β and $x_i = \sum_{k=1}^r \alpha_k^{(i)} a_k$ and $\pi(y_j) = \sum_{k=1}^r \gamma_k^{(j)} a_k$ for $\alpha^{(i)}, \gamma^{(j)} \in \mathbb{R}^r$

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- Then $a_{ij} = \tilde{\beta}(x_i, y_j) = \tilde{\beta}(x, \pi(y)) = \sum_{1\leq k,l\leq r} \alpha_k^{(i)}\gamma_l^{(j)}\tilde{\beta}(a_k, a_l) = \sum_{k=1}^r \alpha_k^{(i)}\gamma_k^{(j)} = \langle \alpha^{(i)}, \gamma^{(j)}\rangle.$

□

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- $\left|\alpha^{(i)}\right|, \left|\gamma^{(j)}\right| \le 1$ due to $\tilde{\beta}(x_i), \tilde{\beta}(y_j) \le 1$

$\square$

- In order to show

$$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \le 1 \le m, 1 \le j \le n} \,|\, x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \le 1, |y_j| \le 1\}$$

we will use the following

Proposition

For all $n \ge 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every vector is the multiple of a unitary matrix.

- The proof is based on $n-$fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$
$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)},\ i = 1, \dots n$$

- U_i's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$
- For $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} = \eta_i U_i$ we can calculate

$$\begin{aligned} XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i,j \leq \leq 2n} \xi_i \eta_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\ &= \langle \xi, \eta \rangle I. \end{aligned}$$

- The result follows by setting $X = Y$.

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- $\mathrm{Tr}\,(X_i Y_j^T) = d \cdot \langle x_i, y_j\rangle$ and $\|X_i\|_\infty \leq 1$ since $X_i X_i^* = |x_i|^2 I$ and $|x_i|^2 \leq 1$ (the same holds for Y_j)

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- Let $X_i = \sum_{k=1}^{\min\{m,n\}} x_i(k) U_i$ and $Y_j^T = \sum_{k=1}^{\min\{m,n\}} y_j(k) U_k$ where the U_i's are $d \times d$ matrices with $d = 2^{\lceil \min\{m,n\}/2 \rceil}$
- $\mathrm{Tr}\left(X_i Y_j^T\right) = d \cdot \langle x_i, y_j \rangle$ and $\|X_i\|_\infty \leq 1$ since $X_i X_i^* = |x_i|^2 I$ and $|x_i|^2 \leq 1$ (the same holds for Y_j)
- Let $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ and $\rho = |\phi\rangle \langle \rho|$. Note that we can write ρ as

$$\rho = |\phi\rangle \langle \phi| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |kk\rangle \langle ll| = \frac{1}{d} \sum_{1 \leq k,l \leq d} |k\rangle \langle l| \otimes |k\rangle \langle l|$$

$\mathrm{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq 1 \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$.

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- Then

$$\begin{aligned} \operatorname{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1\leq k,l\leq d} \operatorname{Tr}\left(|k\rangle\langle l| X_i \otimes |k\rangle\langle l| Y_j\right) = \frac{1}{d} \sum_{1\leq k,l\leq d} \operatorname{Tr}\left(|k\rangle\langle l| X_i\right)\mathrm{T} \\ &= \frac{1}{d} \operatorname{Tr} X_i Y_j^T = \langle x_i, y_j\rangle. \end{aligned}$$

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- define vectors $\tilde{x}_i := (\sqrt{\lambda} x_i, \sqrt{1-\lambda}\bar{x}_i)$, $\tilde{y}_j := (\sqrt{\lambda} y_j, \sqrt{1-\lambda}\bar{y}_j)$ for $\lambda \in [0,1]$

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}$, $(\tilde{y}_j)_{1 \leq i \leq n}$ on $\mathrm{span}\{x_1, \ldots, x_m\}$ or $\mathrm{span}\{y_1, ..., y_n\}$, as in the proof before.

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Claim

$$\mathrm{LC}_{2,2} = \{A \in \mathbb{R}^{2\times 2} \mid -1 \leq \operatorname{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \tag{1}$$

where $\mathcal{K} = \{\frac{1}{2}\sigma(\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}), \sigma(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}) \mid \sigma \in \{\mathrm{id}(1\ 2), (1\ 3), (1\ 4)\}\}$.

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- Equation 2 is a non-redundant hyperplane description of $\mathrm{LC}_{2,2}$, all linear constraints define facets.

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- Bound is achieved by $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$