

# Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

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# Outline

## 1 Introduction

- Basics
- Nonlocal games
- A special case of nonlocal games
- A specific example

## 2 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices
- Motivation: The Grothendieck-Tsirelson Theorem
- Grothendieck's Inequality
- Tsirelson's Theorem
- Grothendieck-Tsirelson Theorem

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- We should be familiar with the *Dirac notation*
  - ▶  $|\psi\rangle$  is a vector in  $\mathbb{C}^n$  and  $\langle\psi|$  is its conjugate transpose

## Quantum systems

- A quantum system is a portion of the whole universe. For example a set of electrons.
- A quantum system  $X$  is associated with a copy of  $\mathbb{C}^k$
- It may consist of subsystems  $X_1, \dots, X_N$  each of which is associated with a copy of  $\mathbb{C}^{n_i}$ . In this case  $k = n_1 \dots n_N$

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- This can be achieved:

## Definition

### Measurement

- We define a measurement by a set of psd matrices  $\{F^a\}_{a \in \mathcal{A}} \subseteq \mathbb{C}^{n \times n}$  that sum up to the identity matrix, i.e.  $\sum_{a \in \mathcal{A}} F^a = I$
- The outcome of a measurement is a random variable  $\chi$  with probability distribution:  $\mathbb{P}[\chi = a] = \text{Tr}(\rho F^a)$
- A projective measurement is defined by psd matrices that satisfy  $F^a F^b = \delta_{ab} F^a \forall a, b \in \mathcal{A}$

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- if we consider projective measurements we have

$$(F^+ - F^-)^2 = \underbrace{F^{+2}}_{=F^+} - \underbrace{F^+ F^-}_{\delta_{+-}=0} + \underbrace{F^{-2}}_{F^-} = F^+ + F^- = I$$

- i.e. a  $\{-1, 1\}$ -valued observable is both unitary and Hermitian

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- Every party may perform a measurement on their subsystem  $X_i$ , i.e. there are  $N$  sets of psd matrices  $\{F^{a_1}\}_{a_1 \in \mathcal{A}_1} \in \mathbb{C}^{n_1 \times n_1}, \dots, \{F^{a_N}\}_{a_N \in \mathcal{A}_N} \in \mathbb{C}^{n_N \times n_N}$

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The joint probability distribution of the  $N$  measurement outcomes  $\chi_1, \dots, \chi_N$  is

$$\mathbb{P}[\chi_1 = a_1, \chi_2 = a_2, \dots, \chi_N = a_N] = \text{Tr}(\rho F_1^{a_1} \otimes \dots \otimes F_N^{a_N})$$

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- A state that is not a product state is called entangled

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## Calculation

$$\begin{aligned}\text{Tr}(|\psi\rangle\langle\psi|F^a \otimes G^b) &= \langle\psi|F^a \otimes G^b|\psi\rangle \\ &= (\langle\psi_A| \otimes \langle\psi_B|)(F^a \otimes G^b)(|\psi_A\rangle \otimes |\psi_B\rangle) \\ &= ((\langle\psi_A|F^a) \otimes (\langle\psi_B|G^b))(|\psi_A\rangle \otimes |\psi_B\rangle) \\ &= \langle\psi_A|F^a|\psi_A\rangle \otimes \langle\psi_B|G^b|\psi_B\rangle \\ &= \langle\psi_A|F^a|\psi_A\rangle\langle\psi_B|G^b|\psi_B\rangle\end{aligned}$$

This is equal to the product of the probabilities of Alice measuring  $a$  and Bob measuring  $b$ , i.e. the outcome do not correlate.

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- Alice sends answer  $a$  and Bob sends answer  $b$  back to the referee, who then decides whether both win or both lose

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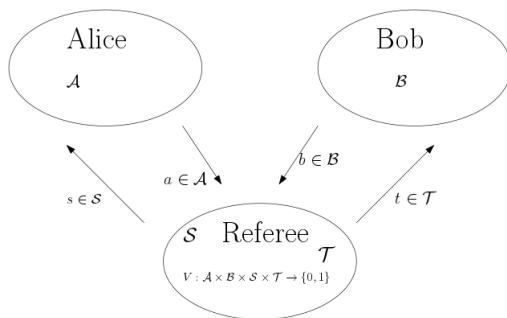
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- A map  $V : \mathcal{S} \times \mathcal{T} \times \mathcal{A} \times \mathcal{B} \rightarrow \{0, 1\}$
- They win if  $V(s, t, a, b) = 1$  and lose otherwise



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- The winning probability then is:

$$\mathbb{E}_{(s,t) \sim \pi} [V(a(s), b(t), s, t)]$$

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- Answering according to measurement outcomes could increase winning probability

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- They send their measurement outcome as their answer to the referee
- Their winning probability is:

$$\mathbb{E}_{(s,t) \sim \pi} \left[ \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \text{Tr}(\rho F_s^a \otimes G_t^b) V(a, b, s, t) \right]$$

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- A truth table for  $a \oplus b$  looks like this

$\oplus$	0	1
0	0	1
1	1	0



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- Since  $\frac{1}{2} + \gamma - (1 - \frac{1}{2} - \gamma) = 2\gamma$
- The violation ratio is defined as  $\frac{\beta^*(G)}{\beta(G)}$

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- The bias is given by the probability under  $\pi$  that  $a(s) \oplus b(t) = f(s, t)$  minus the probability under  $\pi$  that  $a(s) \oplus b(t) \neq f(s, t)$

This means the bias can be written as:

$$\begin{aligned} & \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{[a(s) \oplus b(t) = f(s,t)]} \right] = \\ &= \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{a(s) \oplus b(t) + f(s,t)} \right] = \\ &= \mathbb{E}_{(s,t) \sim \pi} \left[ (-1)^{a(s)} (-1)^{b(t)} (-1)^{f(s,t)} \right] \end{aligned}$$

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Define the sign matrix  $\Sigma_{s,t} = (-1)^{f(s,t)}$  and functions  $\chi(s) = (-1)^{a(s)}$  and  $\psi(t) = (-1)^{b(t)}$ . The expected value becomes

$$\mathbb{E}_{(s,t) \sim \pi} [\chi(s) \psi(t) \Sigma_{st}]$$

## Recap

- The outcomes in an XOR game are  $\{0, 1\}$
- Alice and Bob have measurements  $\{F_s^0, F_s^1\}$  and  $\{G_t^0, G_t^1\}$  and share an entangled state
- The probability of Alice and Bob answering with  $a, b$  upon receiving  $s, t$  respectively is  $\langle \psi | F_s^a \otimes G_t^b | \psi \rangle$

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- The probability of Alice and Bob answering with  $a, b$  upon receiving  $s, t$  respectively is  $\langle \psi | F_s^a \otimes G_t^b | \psi \rangle$

Lets calculate the expected value of  $(-1)^{a \oplus b}$

$$\begin{aligned} & (1) \cdot \mathbb{P}[a = b] + (-1) \cdot \mathbb{P}[a \neq b] = \\ &= \langle \psi | F_s^0 \otimes G_t^0 | \psi \rangle + \langle \psi | F_s^1 \otimes G_t^1 | \psi \rangle \\ & \quad - \langle \psi | F_s^1 \otimes G_t^0 | \psi \rangle - \langle \psi | F_s^0 \otimes G_t^1 | \psi \rangle \\ &= \langle \psi | (F_s^0 - F_s^1) \otimes (G_t^0 - G_t^1) | \psi \rangle \end{aligned}$$

- Define  $\{-1, 1\}$ -observables  $F_s = F_s^0 - F_s^1$  and  $G_t = G_t^0 - G_t^1$  with the property that its difference squared is the identity matrix

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- Using this strategy the bias becomes

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- For any XOR game the bias is defined as the difference of the probabilities of winning and loosing
- Which is, if considering the  $\{-1, 1\}$  basis, the expected value above
- We are looking to maximize this quantity

# Classical strategies

When using classical strategies this is

$$\max\{\mathbb{E}_{(s,t)\sim\pi} [\chi(s)\psi(t)\Sigma_{st}] : \chi : \mathcal{S} \rightarrow \{-1, 1\}, \\ \psi : \mathcal{T} \rightarrow \{-1, 1\}\}$$

# Entangled strategies

When using entangled strategies the winning probability might increase indefinitely with the dimensions, so we use the  $\sup_{n \in \mathbb{N}}$

$$\sup_{n \in \mathbb{N}} \{ \mathbb{E}_{(s,t) \sim \pi} [\langle \psi | F_s \otimes G_t | \psi \rangle \Sigma_{st}] : |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n, \\ F_s, G_t \in \mathcal{W}(\mathbb{C}^n) \}$$

with  $\mathcal{W}$  being the set of all  $\{-1, 1\}$ -observables in  $\mathbb{C}^{n \times n}$ .

# The CHSH game

- The CHSH game (Clauser, Horner, Shimony, Holt) is a two player XOR game with  $\mathcal{A} = \mathcal{B} = \mathcal{S} = \mathcal{T} = \{0, 1\}$  and  $\pi$  being the uniform distribution
- $f(s, t) = s \wedge t$ , i.e.  $f(1, 1) = 1$  and  $f(0, 0) = f(0, 1) = f(1, 0) = 0$
- Alice and Bob can win  $\frac{3}{4}$  of the games by using deterministic strategies  $(0, 0)$ ,  $(1, 0)$  or  $(0, 1)$

## Quantum strategy

- Let Alice and Bob share an EPR state
- Define

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- $XY + YX = 0$  and  $X^2 = Y^2 = I$
- For Alice define the observable for question 0 by  $F_0 = X$  and for question 1 by  $F_1 = Y$
- Bobs observables are going to be  $G_0 = (X - Y)/\sqrt{2}$  for question 0 and  $G_1 = (X + Y)/\sqrt{2}$  for question 1

The bias of the entangled strategy equals

$$\mathbb{E}_{(s,t) \sim \pi} [\Sigma_{s,t} \langle \psi | F_s \otimes G_t | \psi \rangle] = \frac{1}{4} \sum_{s,t=0}^1 (-1)^{s \wedge t} \langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle$$

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Straight forward calculation shows

$$\langle \text{EPR} | F_s \otimes G_t | \text{EPR} \rangle = \begin{cases} \frac{1}{\sqrt{2}}, & (0,0), (1,0), (0,1) \\ -\frac{1}{\sqrt{2}}, & (1,1) \end{cases}$$

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Thus,

$$\frac{1}{4} \left( (-1)^{0 \wedge 0} \frac{1}{\sqrt{2}} + (-1)^{1 \wedge 0} \frac{1}{\sqrt{2}} + (-1)^{0 \wedge 1} \frac{1}{\sqrt{2}} + (-1)^{1 \wedge 1} \left( -\frac{1}{\sqrt{2}} \right) \right) = \frac{1}{4} \frac{4}{\sqrt{2}}$$

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The winning probability follows directly

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \approx 0.85 \dots$$

# Outline

## 1 Introduction

- Basics
- Nonlocal games
- A special case of nonlocal games
- A specific example

## 2 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices
- Motivation: The Grothendieck-Tsirelson Theorem
- Grothendieck's Inequality
- Tsirelson's Theorem
- Grothendieck-Tsirelson Theorem

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- Instead of deterministic strategies they can also answer according to the probability distribution of random variables  $X_i, Y_j$
- Their common answer is  $\mathbb{E}[X_i Y_j]$
- This information can be encoded in an  $\mathcal{S} \times \mathcal{T}$  matrix

◀ Motivation Quantum

## Definition

Let  $(X_i)_{1 \leq i \leq m}$  and  $(Y_j)_{1 \leq j \leq n}$  be families of random variables on a common probability space such that  $|X_i|, |Y_j| \leq 1$  almost surely. Then  $A = (a_{ij})$  is the corresponding *classical (or local) correlation matrix* if

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- No matter which probabilistic strategy there is a deterministic one which is at least as good as the one one chooses

$$LC_{m,n} \supset \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}.$$

- $\xi\eta^T \in LC_{m,n}$  for all  $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$  (Choose  $X_i \equiv \xi_i, Y_j \equiv \eta_j$ )



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- Then

$$\begin{aligned}\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbb{1}_{\{\alpha=1\}}] \\ &= \beta \mathbb{E}[X_i^{(0)} Y_j^{(0)}] + (1 - \beta) \mathbb{E}[X_i^{(1)} Y_j^{(1)}]\end{aligned}$$



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$$\begin{aligned} a_{ij} &= \mathbb{E}[X_i Y_j] = \mathbb{E}\left[\left(\sum_{\xi \in \{-1, 1\}^m} \lambda_\xi^{(X)} \xi_i\right) \left(\sum_{\eta \in \{-1, 1\}^n} \lambda_\eta^{(Y)} \eta_j\right)\right] \\ &= \sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}] \xi_i \eta_j \\ &= \left(\sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)}] \mathbb{E}[\lambda_\eta^{(Y)}]\right) \xi_i \eta_j \end{aligned}$$



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- Due to  $\sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)}] \mathbb{E}[\lambda_\eta^{(Y)}] = 1$  the matrix  $(a_{ij})$  is a convex combination of  $\xi\eta^T$ ,  $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$



# Quantum correlation matrices

- Alice and Bob share a common state  $\rho$  and get inputs  $i \in \mathcal{S}, j \in \mathcal{T}$

Motivation Locality

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- Again we can encode this information in a matrix

Motivation Locality

## Definition

Let  $(X_i)_{1 \leq i \leq m}$  and  $(Y_j)_{1 \leq j \leq n}$  be self-adjoint operators on  $\mathbb{C}^{d_1}$ , respectively  $\mathbb{C}^{d_2}$  for some positive integers  $d_1, d_2$ , satisfying  $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$ .  $A = (a_{ij})$  is called *quantum correlation matrix* if there exists a state on  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$  such that

$$a_{ij} = \text{Tr } \rho(X_i \otimes Y_j).$$



## Definition

Let  $(X_i)_{1 \leq i \leq m}$  and  $(Y_j)_{1 \leq j \leq n}$  be self-adjoint operators on  $\mathbb{C}^{d_1}$ , respectively  $\mathbb{C}^{d_2}$  for some positive integers  $d_1, d_2$ , satisfying  $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$ .  $A = (a_{ij})$  is called *quantum correlation matrix* if there exists a state on  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$  such that

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## Lemma

$$\text{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\},$$

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- The vectors have still not the right dimension but again, we can project them onto vectors in  $\mathbb{R}^m$



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### Proposition

For all  $n \geq 1$  there is a subspace of the  $2^n \times 2^n$  Hermitian matrices where every vector is the multiple of a unitary matrix.

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### Proposition

For all  $n \geq 1$  there is a subspace of the  $2^n \times 2^n$  Hermitian matrices where every vector is the multiple of a unitary matrix.

- The proof is based on  $n$ -fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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## Proof.

- Define

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$

$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)}, \quad i = 1, \dots, n$$

- $U_i$ 's are anti-commuting traceless Hermitian unitaries, i.e.  $U_i U_j = -U_j U_i$  for  $i \neq j$  and  $U_i^2 = I$
- For  $X = \sum_{i=1}^{2n} \xi_i U_i$ ,  $Y = \sum_{i=1}^{2n} \eta_i U_i$  we can calculate

$$\begin{aligned} XY &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i, j \leq 2n} \xi_i \eta_j U_i U_j \\ &= \sum_{i=1}^{2n} \xi_i \eta_i I + \sum_{1 \leq i < j \leq 2n} \xi_i \eta_j U_i U_j - \sum_{1 \leq i < j \leq 2n} U_i U_j = \sum_{i=1}^{2n} \xi_i \eta_i I \\ &= \langle \xi, \eta \rangle I. \end{aligned}$$

- The result follows by setting  $X = Y$ .



$$\text{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

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- Let  $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$  and  $\rho = |\phi\rangle \langle \phi|$ . Note that we can write  $\rho$  as

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- Then

$$\begin{aligned} \text{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i \otimes |k\rangle \langle l| Y_j) \\ &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i) \text{Tr}(|k\rangle \langle l| Y_j) \\ &= \frac{1}{d} \text{Tr} X_i Y_j^T = \langle x_i, y_j \rangle. \end{aligned}$$



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- Right dimension is obtained by projecting  $(\tilde{x}_i)_{1 \leq i \leq m}$ ,  $(\tilde{y}_j)_{1 \leq j \leq n}$  on  $\text{span}\{x_1, \dots, x_m\}$  or  $\text{span}\{y_1, \dots, y_n\}$ , as in the proof before.

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- It holds

$$\text{LC}_{2,2} = \text{conv}\left\{\pm \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$



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- Recall  $\text{LC}_{2,2} = \{\xi\eta^T \mid \xi \in \{-1, 1\}^2, \eta \in \{-1, 1\}^2\}$
- It holds

$$\text{LC}_{2,2} = \text{conv}\left\{\pm \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$

- We can also write it as in intersections of halfspaces:

$$\text{LC}_{2,2} = \{A \in \mathbb{R}^{2 \times 2} \mid -1 \leq \text{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \quad (1)$$

$$\text{where } \mathcal{K} = \left\{\frac{1}{2}\sigma\left(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right), \sigma\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \mid \sigma \in \{\text{id}, (1\ 2), (1\ 3), (1\ 4)\}\right\}.$$

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$$\begin{aligned} \text{Tr}(AM) &= \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle - \langle x_2, y_2 \rangle \\ &= \langle x_1 + x_2, y_1 \rangle + \langle x_1 - x_2, y_2 \rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\ &\leq |x_1 + x_2| + |x_1 - x_2|. \end{aligned}$$



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- Bound is achieved by  $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , induced by the vectors  $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$  and  $y_1 = y_2 = (1, 0)$

# Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

Maxi Brandstetter, Felix Kirschner, Arne Heimendahl

University of Cologne

September 19, 2018

## 1 Introduction

- Basics
- Nonlocal games
- A special case of nonlocal games
- A specific example

## 2 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices
- Motivation: The Grothendieck-Tsirelson Theorem
- Grothendieck's Inequality
- Tsirelson's Theorem
- Grothendieck-Tsirelson Theorem

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## Theorem (Grothendieck-Tsirelson)

There exists an absolute constant  $K \geq 1$  such that, for any positive integers  $m, n$ , the following three equivalent conditions hold:

(1) We have the inclusion

$$\text{QC}_{m,n} \subset K \text{LC}_{m,n}. \quad (2)$$

(2) For any  $M \in \mathbb{R}^{m \times n}$  and for any  $\rho, X_i, Y_j$  verifying the conditions of Definition 4.2.1 we have

$$\sum_{i,j} M_{ij} \text{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \quad (3)$$

$$\Leftrightarrow \text{Tr} M A^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} M (\xi \eta^\top)^\top. \quad (4)$$

(3) For any  $M \in \mathbb{R}^{m \times n}$  and for any (real) Hilbert space vectors  $x_i, y_j$  with  $|x_i| \leq 1, |y_j| \leq 1$  we have

$$\sum_{i,j} M_{i,j} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} \xi^\top M \eta. \quad (5)$$

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## Lemma (Grothendieck's identity)

Let  $x, y \in \mathbb{R}^d$  be unit vectors. Let  $r \in \mathbb{R}^d$  be a random unit vector chosen from  $O(d)$ -invariant probability distribution on the unit sphere. Then

- i,  $\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] = \frac{\arccos(\langle x, y \rangle)}{\pi}$
- ii,  $\mathbb{E}[\text{sign}(\langle x, r \rangle) \text{sign}(\langle y, r \rangle)] = \frac{2}{\pi} \arcsin(\langle x, y \rangle)$ .

## Proof.

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  - ▶ project  $r$  orthogonally on  $\text{span}\{x, y\}$  which gives us a vector  $s$  with  $\langle x, r \rangle = \langle x, s \rangle$  and  $\langle y, r \rangle = \langle y, s \rangle$

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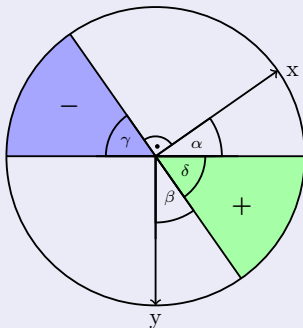
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  - ▶ the normalized vector  $n := s/\|s\|$  is uniformly distributed on the intersection of the unit sphere and  $\text{span}\{x, y\}$  by the  $O(d)$ -invariance of the probability distribution

## Proof (cont.).

Calculation of the probability that the signs of the scalar products  $\langle x, n \rangle$  and  $\langle y, n \rangle$  are unlike:



$$\mathbb{P}[\text{sign}(\langle x, n \rangle) \neq \text{sign}(\langle y, n \rangle)] = 2 \frac{\frac{\pi}{2} + \alpha}{2\pi} = \frac{\arccos(\langle x, y \rangle)}{\pi}$$

## Proof (cont.).

We conclude with the proof of the second part of Lemma 6:

$$\begin{aligned}\mathbb{E}[\text{sign}(\langle x, r \rangle) \text{sign}(\langle y, r \rangle)] &= 1 \cdot \mathbb{P}[\text{sign}(\langle x, r \rangle) = \text{sign}(\langle y, r \rangle)] - 1 \cdot \mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] \\ &= 1 - 2\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] \\ &= 1 - 2 \frac{\arccos(\langle x, y \rangle)}{\pi} \\ &= \frac{2}{\pi} \arcsin(\langle x, y \rangle),\end{aligned}$$

because  $\arcsin(t) + \arccos(t) = \pi/2$ . □



## Lemma (Krivine's trick)

Let  $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$  be given. Furthermore, let  $r \in \mathbb{R}^d$  be a random unit vector chosen from the  $O(d)$ -invariant probability distribution on the unit sphere. Then there are  $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$  so that

$$\mathbb{E}[\text{sign}(\langle x'_i, r \rangle) \text{sign}(\langle y'_j, r \rangle)] = \beta \langle x_i, y_j \rangle, \quad (6)$$

with  $\beta = \frac{2}{\pi} \ln(1 + \sqrt{2})$ .

## Definition (The $k$ -th tensor product)

The  $k$ -th tensor product of  $\mathbb{R}^n$  with orthonormal basis  $e_1, \dots, e_n$  is denoted by  $(\mathbb{R}^n)^{\otimes k}$  and it is a Euclidean vector space of dimension  $n^k$  with orthonormal basis  $e_{i_1} \otimes \dots \otimes e_{i_k}$ ,  $i_j \in \{1, \dots, n\}$ . In particular

$$\begin{aligned} \langle e_{i_1} \otimes \dots \otimes e_{i_k}, e_{j_1} \otimes \dots \otimes e_{j_k} \rangle &= \prod_{l=1}^k \langle e_{i_l}, e_{j_l} \rangle \\ &= \begin{cases} 1 & , \text{ if } i_l = j_l \text{ for all } l = 1, \dots, k, \\ 0 & , \text{ otherwise,} \end{cases} \end{aligned} \quad (7)$$

and for  $v \in \mathbb{R}^n$  with  $v = v_1 e_1 + \dots + v_n e_n$  we define  $v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k}$  by

$$\begin{aligned} v^{\otimes k} &:= (v_1 e_1 + \dots + v_n e_n) \otimes \dots \otimes (v_1 e_1 + \dots + v_n e_n) \\ &= \sum_{i_1, \dots, i_k} v_{i_1} \dots v_{i_k} e_{i_1} \otimes \dots \otimes e_{i_k}. \end{aligned} \quad (8)$$

Thus, for  $v, w \in \mathbb{R}^n$

$$\langle v^{\otimes k}, w^{\otimes k} \rangle = \langle v, w \rangle^k. \quad (9)$$

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- idea: To find  $\beta, x'_i, y'_j$  invert  $E$ :

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1}}_{=: g_{2k+1}} t^{2k+1}$$

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- define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2k+1}. \quad (10)$$

## Proof (cont.).

- define  $\tilde{x}_i, \tilde{y}_j \in H$ ,  $i = 1, \dots, m, j = 1, \dots, n$  componentwise:

$$(\tilde{x}_i)_k = \text{sign}(g_{2k+1}) \sqrt{|g_{2k+1}| \beta^{2k+1}} x_i^{\otimes 2k+1} \quad (11)$$

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- then

$$\begin{aligned} \langle \tilde{x}_i, \tilde{y}_j \rangle &= \sum_{k=0}^{\infty} g_{2k+1} \beta^{2k+1} \langle x_i^{\otimes 2k+1}, y_j^{\otimes 2k+1} \rangle \\ &= \sum_{k=0}^{\infty} g_{2k+1} \beta^{2k+1} \langle x_i, y_j \rangle^{2k+1} \\ &= E^{-1}(\beta \langle x_i, y_j \rangle). \end{aligned}$$



## Proof (cont.).

- hence,  $\beta$  is defined by the condition that the vectors  $\tilde{x}_i, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$  are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh\left(\frac{\pi}{2}\beta\right)$$
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- problem:  $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$  are infinite-dimensional
- solution: the positive definite and symmetric Gram matrix  $G$

$$G = \begin{pmatrix} \langle \tilde{x}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{x}_m \rangle & \langle \tilde{x}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{x}_m, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{x}_m \rangle & \langle \tilde{x}_m, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{y}_n \rangle \\ \langle \tilde{y}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{x}_m \rangle & \langle \tilde{y}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{y}_n, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{x}_m \rangle & \langle \tilde{y}_n, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{y}_n \rangle \end{pmatrix} \quad (13)$$

## Proof (cont.).

- due to the properties of  $G$  we can decompose  $G$  via a real orthogonal matrix  $Q$  with columns that are the eigenvectors of  $G$  and a real diagonal matrix  $\Lambda$  having the eigenvalues of  $G$  on the diagonal, thus

$$G = Q\Lambda Q^{\top} = \underbrace{(Q\Lambda^{1/2})^{\top}}_{=:A} (Q\Lambda^{1/2}) \quad (14)$$



## Proof (cont.).

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- the columns of  $A$  are the vectors  $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$  we are looking for



## Definition

For  $M \in \mathbb{R}^{m \times n}$  define the quadratic program

$$\begin{aligned}\|M\|_{\infty \rightarrow 1} &= \max \left\{ \sum_{i=1}^m \sum_{j=1}^n M_{ij} \xi_i \eta_j : \xi_i^2 = 1, i = 1, \dots, m, \eta_j^2 = 1, j = 1, \dots, n \right\} \\ &= \max \{ \text{Tr } M \eta \xi^\top : \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n \}.\end{aligned}\quad (15)$$

## Definition

The SDP relaxation of  $\|M\|_{\infty \rightarrow 1}$  is given via:

$$\begin{aligned}\text{sdp}_{\infty \rightarrow 1}(M) &= \max \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle \\ &\quad x_i, y_j \in \mathbb{R}^{m+n} \\ &\quad \|x_i\| = 1, i = 1, \dots, m \\ &\quad \|y_j\| = 1, j = 1, \dots, n\end{aligned}$$

## Theorem (Grothendieck's inequality)

There exists a constant  $K$  such that for all  $M \in \mathbb{R}^{m \times n}$ :

$$\|M\|_{\infty \rightarrow 1} \leq \text{sdp}_{\infty \rightarrow 1}(M) \leq K \|M\|_{\infty \rightarrow 1}. \quad (16)$$

### Proof.

Use the following approximation algorithm with randomized rounding:

**Algorithm 1:** Approximation algorithm with randomized rounding for  $\|M\|_{\infty \rightarrow 1}$

1. Solve  $\text{sdp}_{\infty \rightarrow 1}(M)$ . Let  $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$  be the optimal unit vectors
2. Apply Krivine's trick (Lemma 7) and use vectors  $x_i, y_j$  to create new unit vectors  $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ .
3. Choose  $r \in S^{m+n-1}$  randomly
4. Round:  $u_i = \text{sign}(\langle x'_i, r \rangle)$   
 $v_j = \text{sign}(\langle y'_j, r \rangle)$

## Proof (cont.).

Expected quality of the outcome:

$$\begin{aligned}\|M\|_{\infty \rightarrow 1} &\geq \mathbb{E} \left[ \sum_{i=1}^m \sum_{j=1}^n M_{ij} u_i v_j \right] \\ &= \sum_{i=1}^m \sum_{j=1}^n M_{ij} \mathbb{E}[\text{sign}(\langle x'_i, r \rangle) \text{sign}(\langle y'_j, r \rangle)] \\ &= \sum_{i=1}^m \sum_{j=1}^n M_{ij} \beta \langle x_i, y_j \rangle \\ &= \beta \text{sdp}_{\infty \rightarrow 1}(M),\end{aligned}$$

where the last equality follows by Krivine's trick with  $\beta = \frac{2 \ln(1+\sqrt{2})}{\pi}$ , thus  $K \leq \beta^{-1}$ .



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- Local correlation matrices
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- Grothendieck's Inequality
- **Tsirelson's Theorem**
- Grothendieck-Tsirelson Theorem

## Theorem (Tsirelson)

(Hard direction) For all positive integers  $n, r$  and any  $x_1, \dots, x_n, y_1, \dots, y_n \in S^r$ , there exists a positive integer  $d := d(r)$ , a state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  and  $\{-1, 1\}$ -observables  $F_1, \dots, F_n, G_1, \dots, G_n \in O(\mathbb{C}^d)$ , such that for every  $i, j \in \{1, \dots, n\}$ , we have

$$\langle \psi | F_i \otimes G_j | \psi \rangle = \langle x_i, y_j \rangle. \quad (17)$$

Moreover,  $d \leq 2^{\lceil r/2 \rceil}$ .

(Easy direction) Conversely, for all positive integers  $n, d$ , state  $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$  and  $\{-1, 1\}$ -observables  $F_1, \dots, F_n, G_1, \dots, G_n \in O(\mathbb{C}^d)$ , there exist a positive integer  $r := r(d)$  and  $x_1, \dots, x_n, y_1, \dots, y_n \in S^r$  such that for every  $i, j \in \{1, \dots, n\}$ , we have

$$\langle x_i, y_j \rangle = \langle \psi | F_i \otimes G_j | \psi \rangle. \quad (18)$$

Moreover,  $r \leq 2d^2$ .

Since

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## Theorem (Grothendieck-Tsirelson)

There exists an absolute constant  $K \geq 1$  such that, for any positive integers  $m, n$ , the following three equivalent conditions hold:

(1) We have the inclusion

$$\text{QC}_{m,n} \subset \text{KLC}_{m,n}. \quad (19)$$

(2) For any  $M \in \mathbb{R}^{m \times n}$  and for any  $\rho, X_i, Y_j$  verifying the conditions of Definition 4.2.1 we have

$$\sum_{i,j} M_{ij} \text{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \quad (20)$$

$$\Leftrightarrow \text{Tr} M A^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} M (\xi \eta^\top)^\top. \quad (21)$$

(3) For any  $M \in \mathbb{R}^{m \times n}$  and for any (real) Hilbert space vectors  $x_i, y_j$  with  $|x_i| \leq 1, |y_j| \leq 1$  we have

$$\sum_{i,j} M_{i,j} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} \xi^\top M \eta. \quad (22)$$

## Proof.

Since (22) is a direct consequence of Grothendieck's inequality the only thing left to prove is the equivalence between (1)-(3). The equivalence of (3) and (2) (the Tsirelson's bound) is a consequence of either the proof of Lemma ?? or Tsirelson's Theorem (Theorem 1).

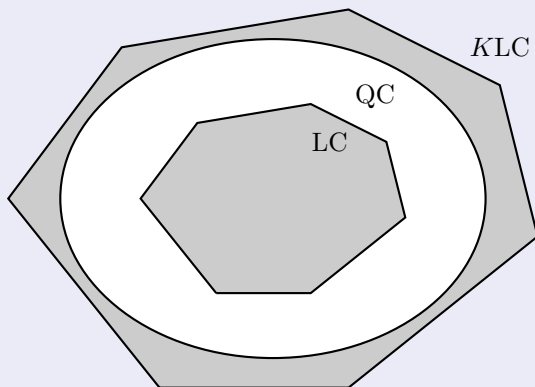


Figure: Visualization