

Nonlocal games and the Grothendieck-Tsirelson inequality

Introduction to quantum information theory

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Local correlation matrices

- Deterministic strategies of Alice and Bob correspond to vectors $\xi \in \{-1, 1\}^{\mathcal{S}}$, respectively $\eta \in \{-1, 1\}^{\mathcal{T}}$.

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- Their common answer is the product $\xi_i \eta_j$.
- Instead of deterministic strategies they can also answer according to the probability distribution of random variables X_i, Y_j .
- Their common answer is $\mathbb{E}[X_i Y_j]$.
- This information can be encoded in an $\mathcal{S} \times \mathcal{T}$ matrix.

◀ Motivation Quantum

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be families of random variables on a common probability space such that $|X_i|, |Y_j| \leq 1$ almost surely. Then $A = (a_{ij})$ is the corresponding *classical (or local) correlation matrix* if

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Lemma

$$\text{LC}_{m,n} = \text{conv}\{\xi \eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}$$

- No matter which probabilistic strategy there is a deterministic one which is at least as good as the one one chooses.

$$LC_{m,n} \supset \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}.$$

- $\xi\eta^T \in LC_{m,n}$ for all $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$ (Choose $X_i \equiv \xi_i, Y_j \equiv \eta_j$).



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- Then

$$\begin{aligned}\mathbb{E}[X_i Y_j] &= \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbf{1}_{\{\alpha=0\}}] + \mathbb{E}[X_i^{(\alpha)} Y_j^{(\alpha)} \mathbf{1}_{\{\alpha=1\}}] \\ &= \mathbb{E}[X_i^{(0)} Y_j^{(0)}] \mathbb{P}(\alpha = 0) + \mathbb{E}[X_i^{(1)} Y_j^{(1)}] \mathbb{P}(\alpha = 1) \\ &= \beta a_{ij}^{(0)} + (1 - \beta) a_{ij}^{(1)}.\end{aligned}$$



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- Set $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_n)$, then $X \in [-1, 1]^m$, $Y \in [-1, 1]^n$ almost surely.

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- Define random variables $\lambda_\xi^{(X)} : \Omega^m \rightarrow [0, 1]$ such that

$$X(\omega) = \sum_{\xi \in \{-1, 1\}^m} \lambda_\xi^{(X)}(\omega) \xi$$

almost surely and $\sum_{\xi \in \{-1, 1\}^m} \lambda_\xi^{(X)}(\omega) = 1$.

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- Using the same decomposition for Y we obtain

$$\begin{aligned} a_{ij} &= \mathbb{E}[X_i Y_j] = \mathbb{E}\left[\left(\sum_{\xi \in \{-1, 1\}^m} \lambda_\xi^{(X)} \xi_i\right) \left(\sum_{\eta \in \{-1, 1\}^n} \lambda_\eta^{(Y)} \eta_j\right)\right] \\ &= \sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}] \xi_i \eta_j. \end{aligned}$$



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- Due to $\sum_{\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n} \mathbb{E}[\lambda_\xi^{(X)} \lambda_\eta^{(Y)}] = 1$ the matrix (a_{ij}) is a convex combination of $\xi\eta^T$, $\xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n$.



Quantum correlation matrices

- Alice and Bob share a common state ρ and get inputs $i \in \mathcal{S}, j \in \mathcal{T}$.

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- They perform measurements $\{F_i^\xi\}_{\xi=\pm 1}$, respectively $\{G_j^\eta\}_{\eta=\pm 1}$.
- The probability that their response is (ξ, η) for inputs (i, j) is given by $a_{ij} = \text{Tr}(\rho F_i^\xi \otimes G_j^\eta)$.
- Again we can encode this information in a matrix.

Motivation Locality

Definition

Let $(X_i)_{1 \leq i \leq m}$ and $(Y_j)_{1 \leq j \leq n}$ be self-adjoint operators on \mathbb{C}^{d_1} , respectively \mathbb{C}^{d_2} for some positive integers d_1, d_2 , satisfying $\|X_i\|_\infty, \|Y_j\|_\infty \leq 1$. $A = (a_{ij})$ is called *quantum correlation matrix* if there exists a state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ such that

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Lemma

$$\text{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

$$\text{QC}_{m,n} \subset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

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- Define a positive semidefinite symmetric bilinear form on the space of Hermitian operators on \mathcal{H} by $\beta : B^{(sa)}(\mathcal{H}) \times B^{(sa)}(\mathcal{H}) \rightarrow \mathbb{R}$ where $\beta(S, T) = \text{Re}(\text{Tr} \rho ST)$.

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- If $\pi(y_j)$ is the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$.



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- If $\pi(y_j)$ is the projection of y_j then $\tilde{\beta}(x_i, \pi(y_j)) = \tilde{\beta}(x_i, y_j)$.
- The vectors still have not the right dimension but again, we can project them onto vectors in \mathbb{R}^m .



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For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every element is the multiple of a unitary matrix.

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For all $n \geq 1$ there is a subspace of the $2^n \times 2^n$ Hermitian matrices where every element is the multiple of a unitary matrix.

- The proof is based on n -fold tensor products of the Pauli matrices which are the three matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Proof.

- Define $U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)}$,
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- U_i 's are anti-commuting traceless Hermitian unitaries, i.e. $U_i U_j = -U_j U_i$ for $i \neq j$ and $U_i^2 = I$.
- For $X = \sum_{i=1}^{2n} \xi_i U_i$ we can calculate

$$\begin{aligned}
 XX^* &= XX = \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i \neq j \leq 2n} \xi_i \xi_j U_i U_j \\
 &= \sum_{i=1}^{2n} \xi_i^2 I + \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j - \sum_{1 \leq i < j \leq 2n} \xi_i \xi_j U_i U_j \\
 &= \sum_{i=1}^{2n} \xi_i^2 I = |\xi|^2 I.
 \end{aligned}$$



- Set

$$U_i = I^{\otimes(i-1)} \otimes X \otimes Y^{\otimes(n-i)},$$

$$U_{n+i} = I^{\otimes(i-1)} \otimes Z \otimes Y^{\otimes(n-i)}, i = 1, \dots, n.$$

- Moreover, taking two linear combinations $X = \sum_{i=1}^{2n} \xi_i U_i$, $Y = \sum_{i=1}^{2n} \eta_i U_i$ and consider the trace of the product we get

$$\begin{aligned} \text{Tr}(XY) &= \sum_{i=1}^{2n} \xi_i \eta_i \text{Tr} I + \sum_{1 \leq i \neq j \leq 2n} \xi_i \eta_j \text{Tr}(U_i U_j) \\ &= \sum_{i=1}^{2n} \xi_i \eta_i 2^n = 2^n \cdot \langle \xi, \eta \rangle, \end{aligned}$$

where we used that the product $U_i U_j$ is traceless for $i \neq j$.

$$\text{QC}_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$$

- Let $(x_i)_{1 \leq i \leq m}, (y_j)_{1 \leq j \leq n} \subset \mathbb{R}^{\min\{m,n\}}$ such that $|x_i|, |y_j| \leq 1$.

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- $\text{Tr}(X_i Y_j^T) = d \cdot \langle x_i, y_j \rangle$ and $\|X_i\|_\infty \leq 1$ since $X_i X_i^* = |x_i|^2 I$ and $|x_i|^2 \leq 1$ (the same holds for Y_j)

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- Let $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle$ and $\rho = |\psi\rangle \langle \psi|$. Note that we can write ρ as

$$\rho = |\psi\rangle \langle \psi| = \frac{1}{d} \sum_{1 \leq k, l \leq d} |k\rangle \otimes |k\rangle \langle l| \otimes \langle l| = \frac{1}{d} \sum_{1 \leq k, l \leq d} |k\rangle \langle l| \otimes |k\rangle \langle l|.$$

$$QC_{m,n} \supset \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}.$$

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- Then

$$\begin{aligned} \text{Tr}(\rho X_i \otimes Y_j) &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i \otimes |k\rangle \langle l| Y_j) \\ &= \frac{1}{d} \sum_{1 \leq k, l \leq d} \text{Tr}(|k\rangle \langle l| X_i) \text{Tr}(|k\rangle \langle l| Y_j) \\ &= \frac{1}{d} \text{Tr} X_i Y_j^T = \langle x_i, y_j \rangle. \end{aligned}$$



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- Define vectors $\tilde{x}_i := \begin{pmatrix} \sqrt{\lambda}x_i \\ \sqrt{1-\lambda}\bar{x}_i \end{pmatrix}, \tilde{y}_j := \begin{pmatrix} \sqrt{\lambda}y_j \\ \sqrt{1-\lambda}\bar{y}_j \end{pmatrix}$ for $\lambda \in [0, 1]$.

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- It holds $|\tilde{x}_i| \leq \lambda \left| \begin{pmatrix} x_i \\ 0 \end{pmatrix} \right| + (1-\lambda) \left| \begin{pmatrix} 0 \\ \bar{x}_i \end{pmatrix} \right| \leq 1$ and $\langle \tilde{x}_i, \tilde{y}_j \rangle = \lambda \langle x_i, y_j \rangle + (1-\lambda) \langle \bar{x}_i, \bar{y}_j \rangle$.

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- Right dimension is obtained by projecting $(\tilde{x}_i)_{1 \leq i \leq m}, (\tilde{y}_j)_{1 \leq j \leq n}$ on $\text{span}\{x_1, \dots, x_m\}$ or $\text{span}\{y_1, \dots, y_n\}$, as in the proof before.

The relations between quantum correlation and local correlation matrices

- $LC_{m,n} = \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n\}$
- $QC_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$
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- $LC_{m,n} \subset QC_{m,n}$
- Set $x_i = \xi_i |1\rangle$ and $y_j = \eta_j |1\rangle$ it immediately follows $\xi_i \eta_j = \langle x_i, y_j \rangle$. Hence, $\xi\eta^T \in QC_{m,n}$ (rest follows with the convexity of $QC_{m,n}$).

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- $\text{LC}_{2,2} = \text{conv}\{\xi\eta^T \mid \xi \in \{-1, 1\}^2, \eta \in \{-1, 1\}^2\}$
- It holds

$$\text{LC}_{2,2} = \text{conv}\left\{\pm \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.$$

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- We can also write it as in intersections of halfspaces:

$$\text{LC}_{2,2} = \{A \in \mathbb{R}^{2 \times 2} \mid -1 \leq \text{Tr} AM \leq 1 \text{ for all } M \in \mathcal{K}\}, \quad (1)$$

$$\text{where } \mathcal{K} = \left\{\frac{1}{2}\sigma\left(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right), \sigma\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \mid \sigma \in \{\text{id}, (1\ 2), (1\ 3), (1\ 4)\}\right\}.$$

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$$\begin{aligned} \text{Tr}(AM) &= \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle - \langle x_2, y_2 \rangle \\ &= \langle x_1 + x_2, y_1 \rangle + \langle x_1 - x_2, y_2 \rangle \leq |x_1 + x_2||y_1| + |x_1 - x_2||y_2| \\ &\leq |x_1 + x_2| + |x_1 - x_2|. \end{aligned}$$

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- $\text{Tr}(AM) \leq |x_1 + x_2| + |x_1 - x_2| \leq 2\sqrt{|x_1|^2 + |x_2|^2} \leq 2\sqrt{2}$.
- Bound is achieved by $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, induced by the vectors $x_1 = x_2 = \frac{1}{\sqrt{2}}(1, 1)$ and $y_1 = y_2 = (1, 0)$.

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2 Grothendieck-Tsirelson Theorem

- Motivation: The Grothendieck-Tsirelson Theorem
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Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n , the following three equivalent conditions hold:

(1) We have the inclusion

$$\text{QC}_{m,n} \subset K \text{LC}_{m,n}. \quad (2)$$

(2) For any $M \in \mathbb{R}^{m \times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have

$$\sum_{i,j} M_{ij} \text{Tr} \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \quad (3)$$

$$\Leftrightarrow \text{Tr} M A^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} M (\xi \eta^\top)^\top. \quad (4)$$

(3) For any $M \in \mathbb{R}^{m \times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1, |y_j| \leq 1$ we have

$$\sum_{i,j} M_{i,j} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr} \xi^\top M \eta. \quad (5)$$

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Lemma (Grothendieck's identity)

Let $x, y \in S^{d-1}$. Let $r \in S^{d-1}$ be a random unit vector chosen from $O(d)$ -invariant probability distribution on the unit sphere. Then

- i, $\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] = \frac{\arccos(\langle x, y \rangle)}{\pi}$
- ii, $\mathbb{E}[\text{sign}(\langle x, r \rangle) \text{sign}(\langle y, r \rangle)] = \frac{2}{\pi} \arcsin(\langle x, y \rangle)$.

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Proof.

- If x and y are linearly dependent, then
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Lemma (Grothendieck's identity)

Let $x, y \in S^{d-1}$. Let $r \in S^{d-1}$ be a random unit vector chosen from $O(d)$ -invariant probability distribution on the unit sphere. Then

- i, $\mathbb{P}[\text{sign}(\langle x, r \rangle) \neq \text{sign}(\langle y, r \rangle)] = \frac{\arccos(\langle x, y \rangle)}{\pi}$
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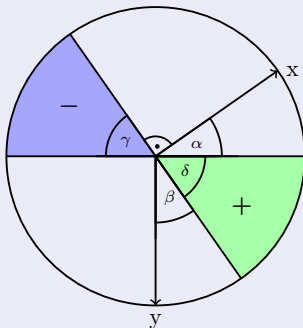
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 - ▶ project r orthogonally on $\text{span}\{x, y\}$ which gives us a vector s with $\langle x, r \rangle = \langle x, s \rangle$ and $\langle y, r \rangle = \langle y, s \rangle$,
 - ▶ the normalized vector $n := s/|s|$ is uniformly distributed on the intersection of the unit sphere and $\text{span}\{x, y\}$ by the $O(d)$ -invariance of the probability distribution.

Proof (cont.).

Calculation of the probability that the signs of the scalar products $\langle x, n \rangle$ and $\langle y, n \rangle$ are unlike:



$$\mathbb{P}[\text{sign}(\langle x, n \rangle) \neq \text{sign}(\langle y, n \rangle)] = 2 \frac{\arccos(\langle x, y \rangle)}{2\pi} = \frac{\arccos(\langle x, y \rangle)}{\pi}$$

Proof (cont.).

We conclude with the proof of the second part of Lemma 5:

$$\begin{aligned}\mathbb{E}[\text{sign}(\langle x, r \rangle) \text{sign}(\langle y, r \rangle)] \\ &= 1 - 2 \frac{\arccos(\langle x, y \rangle)}{\pi} \\ &= \frac{2}{\pi} \arcsin(\langle x, y \rangle),\end{aligned}$$

because $\arcsin(t) + \arccos(t) = \pi/2$. □

Lemma (Krivine's trick)

Let $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$ be given. Furthermore, let $r \in S^{n+m-1}$ be a random unit vector chosen from the $O(n+m-1)$ -invariant probability distribution on the unit sphere. Then there are $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ so that

$$\mathbb{E}[\text{sign}(\langle x'_i, r \rangle) \text{sign}(\langle y'_j, r \rangle)] = \beta \langle x_i, y_j \rangle, \quad (6)$$

with $\beta = \frac{2}{\pi} \ln(1 + \sqrt{2})$.

Definition (The k -th tensor product)

The k -th tensor product of \mathbb{R}^n with orthonormal basis e_1, \dots, e_n is denoted by $(\mathbb{R}^n)^{\otimes k}$ and it is a Euclidean vector space of dimension n^k with orthonormal basis $e_{i_1} \otimes \dots \otimes e_{i_k}$, $i_l \in \{1, \dots, n\}$. In particular

$$\begin{aligned} \langle e_{i_1} \otimes \dots \otimes e_{i_k}, e_{j_1} \otimes \dots \otimes e_{j_k} \rangle &= \prod_{l=1}^k \langle e_{i_l}, e_{j_l} \rangle \\ &= \begin{cases} 1 & , \text{ if } i_l = j_l \text{ for all } l = 1, \dots, k, \\ 0 & , \text{ otherwise,} \end{cases} \end{aligned} \quad (7)$$

and for $v \in \mathbb{R}^n$ with $v = v_1 e_1 + \dots + v_n e_n$ we define $v^{\otimes k} \in (\mathbb{R}^n)^{\otimes k}$ by

$$\begin{aligned} v^{\otimes k} &:= (v_1 e_1 + \dots + v_n e_n) \otimes \dots \otimes (v_1 e_1 + \dots + v_n e_n) \\ &= \sum_{i_1, \dots, i_k} v_{i_1} \dots v_{i_k} e_{i_1} \otimes \dots \otimes e_{i_k}. \end{aligned} \quad (8)$$

Thus, for $v, w \in \mathbb{R}^n$

$$\langle v^{\otimes k}, w^{\otimes k} \rangle = \langle v, w \rangle^k. \quad (9)$$

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$$\mathbb{E}[\text{sign}(\langle x'_i, r \rangle) \text{sign}(\langle y'_j, r \rangle)] = \beta \langle x_i, y_j \rangle, \quad (10)$$

with $\beta = \frac{2}{\pi} \ln(1 + \sqrt{2})$.

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- Define $E : [-1, +1] \rightarrow [-1, +1]$ by $E(t) = \frac{2}{\pi} \arcsin(t)$.

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- Idea: To find β, x'_i, y'_j invert E :

$$E^{-1}(t) = \sin(\pi/2 \cdot t) = \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^{2k+1}}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1}}_{=: g_{2k+1}} t^{2k+1},$$

Proof (cont.).

- Define the infinite-dimensional Hilbert space

$$H = \bigoplus_{r=0}^{\infty} (\mathbb{R}^{m+n})^{\otimes 2k+1}. \quad (11)$$

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- Define $\tilde{x}_i, \tilde{y}_j \in H$, $i = 1, \dots, m, j = 1, \dots, n$ componentwise:

$$(\tilde{x}_i)_k = \text{sign}(g_{2k+1}) \sqrt{|g_{2k+1}| \beta^{2k+1}} x_i^{\otimes 2k+1} \quad (12)$$

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- Then

$$\begin{aligned} \langle \tilde{x}_i, \tilde{y}_j \rangle &= \sum_{k=0}^{\infty} g_{2k+1} \beta^{2k+1} \langle x_i^{\otimes 2k+1}, y_j^{\otimes 2k+1} \rangle \\ &= \sum_{k=0}^{\infty} g_{2k+1} \beta^{2k+1} \langle x_i, y_j \rangle^{2k+1} \\ &= E^{-1}(\beta \langle x_i, y_j \rangle). \end{aligned}$$

Proof (cont.).

- Hence, β is defined by the condition that the vectors $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are unit vectors:

$$1 = \langle \tilde{x}_i, \tilde{x}_i \rangle = \langle \tilde{y}_j, \tilde{y}_j \rangle = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left(\frac{\pi}{2}\right)^{2k+1} \beta^{2k+1} = \sinh\left(\frac{\pi}{2}\beta\right)$$
$$\Leftrightarrow \quad \beta = \frac{2}{\pi} \operatorname{arcsinh}(1) = \frac{2}{\pi} \ln(1 + \sqrt{2})$$

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- Problem: $\tilde{x}_1, \dots, \tilde{x}_m, \tilde{y}_1, \dots, \tilde{y}_n$ are infinite-dimensional
- Solution: the positive definite and symmetric Gram matrix G

$$G = \begin{pmatrix} \langle \tilde{x}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{x}_m \rangle & \langle \tilde{x}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{x}_m, \tilde{x}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{x}_m \rangle & \langle \tilde{x}_m, \tilde{y}_1 \rangle & \cdots & \langle \tilde{x}_m, \tilde{y}_n \rangle \\ \langle \tilde{y}_1, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{x}_m \rangle & \langle \tilde{y}_1, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_1, \tilde{y}_n \rangle \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \tilde{y}_n, \tilde{x}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{x}_m \rangle & \langle \tilde{y}_n, \tilde{y}_1 \rangle & \cdots & \langle \tilde{y}_n, \tilde{y}_n \rangle \end{pmatrix} \quad (14)$$

Proof (cont.).

- Due to the properties of G we can decompose G via a real orthogonal matrix Q with columns that are the eigenvectors of G and a real diagonal matrix Λ having the eigenvalues of G on the diagonal, thus

$$G = Q\Lambda Q^\top = \underbrace{(Q\Lambda^{1/2})^\top}_{=:A} (Q\Lambda^{1/2}). \quad (15)$$



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- The columns of A are the vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$ we are looking for.



Definition

For $M \in \mathbb{R}^{m \times n}$ define the quadratic program

$$\begin{aligned}\|M\|_{\infty \rightarrow 1} &= \max \left\{ \sum_{i=1}^m \sum_{j=1}^n M_{ij} \xi_i \eta_j : \xi_i^2 = 1, i = 1, \dots, m, \eta_j^2 = 1, j = 1, \dots, n \right\} \\ &= \max \{ \text{Tr } M \eta \xi^\top : \xi \in \{-1, 1\}^m, \eta \in \{-1, 1\}^n \}.\end{aligned}\quad (16)$$

Definition

The SDP relaxation of $\|M\|_{\infty \rightarrow 1}$ is given via:

$$\begin{aligned}\text{sdp}_{\infty \rightarrow 1}(M) &= \max \sum_{i=1}^m \sum_{j=1}^n M_{ij} \langle x_i, y_j \rangle \\ &\quad x_i, y_j \in \mathbb{R}^{m+n} \\ &\quad |x_i| = 1, i = 1, \dots, m \\ &\quad |y_j| = 1, j = 1, \dots, n\end{aligned}$$

Theorem (Grothendieck's inequality)

There exists a constant K such that for all $M \in \mathbb{R}^{m \times n}$:

$$\|M\|_{\infty \rightarrow 1} \leq \text{sdp}_{\infty \rightarrow 1}(M) \leq K \|M\|_{\infty \rightarrow 1}. \quad (17)$$

Proof.

Use the following approximation algorithm with randomized rounding:

Algorithm 1: Approximation algorithm with randomized rounding for $\|M\|_{\infty \rightarrow 1}$

1. Solve $\text{sdp}_{\infty \rightarrow 1}(M)$. Let $x_1, \dots, x_m, y_1, \dots, y_n \in S^{m+n-1}$ be the optimal unit vectors.
2. Apply Krivine's trick (Lemma 8) and use vectors x_i, y_j to create new unit vectors $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in S^{m+n-1}$.
3. Choose $r \in S^{m+n-1}$ randomly.
4. Round: $\xi_i = \text{sign}(\langle x'_i, r \rangle)$
 $\eta_j = \text{sign}(\langle y'_j, r \rangle)$

Proof (cont.).

Expected quality of the outcome:

$$\begin{aligned}\|M\|_{\infty \rightarrow 1} &\geq \mathbb{E} \left[\sum_{i=1}^m \sum_{j=1}^n M_{ij} \xi_i \eta_j \right] \\ &= \sum_{i=1}^m \sum_{j=1}^n M_{ij} \mathbb{E}[\text{sign}(\langle x'_i, r \rangle) \text{sign}(\langle y'_j, r \rangle)] \\ &= \sum_{i=1}^m \sum_{j=1}^n M_{ij} \beta \langle x_i, y_j \rangle \\ &= \beta \text{sdp}_{\infty \rightarrow 1}(M),\end{aligned}$$

where the second last equality follows by Krivine's trick with $\beta = \frac{2 \ln(1+\sqrt{2})}{\pi}$, thus $K \leq \beta^{-1}$.

1 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices

2 Grothendieck-Tsirelson Theorem

- Motivation: The Grothendieck-Tsirelson Theorem
- Grothendieck's Inequality
- Tsirelson's Theorem
- Grothendieck-Tsirelson Theorem

Theorem (Tsirelson)

(Hard direction) For all positive integers n, r and any $x_1, \dots, x_n, y_1, \dots, y_n \in S^{r-1}$, there exists a positive integer $d := d(r)$, a unit vector $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $\{-1, 1\}$ -observables $F_1, \dots, F_n, G_1, \dots, G_n \in \mathbb{C}^{d \times d}$, such that for every $i, j \in \{1, \dots, n\}$, we have

$$\langle \psi | F_i \otimes G_j | \psi \rangle = \langle x_i, y_j \rangle. \quad (18)$$

Moreover, $d \leq 2^{\lceil r/2 \rceil}$.

(Easy direction) Conversely, for all positive integers n, d , unit vectors $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and $\{-1, 1\}$ -observables $F_1, \dots, F_n, G_1, \dots, G_n \in \mathbb{C}^{d \times d}$, there exist a positive integer $r := r(d)$ and $x_1, \dots, x_n, y_1, \dots, y_n \in S^{r-1}$ such that for every $i, j \in \{1, \dots, n\}$, we have

$$\langle x_i, y_j \rangle = \langle \psi | F_i \otimes G_j | \psi \rangle. \quad (19)$$

Moreover, $r \leq 2d^2$.

label=Tsirelsonproof.

Reminder: $\langle \psi | F_i \otimes G_j | \psi \rangle = \text{Tr} \underbrace{|\psi\rangle\langle\psi|}_{=: \rho} F_i \otimes G_j$, where ρ is a pure state.

For the hard direction look at the second part of the proof of

$\text{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| \leq 1, |y_j| \leq 1\}$ (◀ Lemma).

Due to the additional restriction that F_i and G_j are $\{-1, 1\}$ -observables the other direction gets easier:

- $F_i \otimes I |\psi\rangle$ and $I \otimes G_j |\psi\rangle$ are unit vectors in \mathbb{C}^{d^2} , since $|\psi\rangle$ has norm 1 and the observables F_i and G_j are unitary operators.

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- $F_i \otimes I|\psi\rangle$ and $I \otimes G_j|\psi\rangle$ are unit vectors in \mathbb{C}^{d^2} , since $|\psi\rangle$ has norm 1 and the observables F_i and G_j are unitary operators.
- Since F_i and G_j are Hermitian, we have that the inner product

$$\begin{aligned} \langle F_i \otimes I|\psi\rangle, I \otimes G_j|\psi\rangle &= (F_i \otimes I|\psi\rangle)^*(I \otimes G_j|\psi\rangle) = \underbrace{(|\psi\rangle^*)}_{= \langle \psi|} \underbrace{F_i^*}_{= F_i} \otimes I^*(I \otimes G_j|\psi\rangle) \\ &= \langle \psi | (F_i \otimes I)(I \otimes G_j) | \psi \rangle = \langle \psi | F_i \otimes G_j | \psi \rangle \\ &= \text{Tr} |\psi\rangle\langle\psi| F_i \otimes G_j \end{aligned}$$

is a real number.

Proof (cont.).

Thus,

$$\begin{aligned}\langle \psi | F_i \otimes G_j | \psi \rangle &= \operatorname{Re}(\langle \psi | F_i \otimes G_j | \psi \rangle) \\ &= \langle \operatorname{Re}(F_i \otimes I | \psi \rangle), \operatorname{Re}(I \otimes G_j | \psi \rangle) \rangle + \langle \operatorname{Im}(F_i \otimes I | \psi \rangle), \operatorname{Im}(I \otimes G_j | \psi \rangle) \rangle.\end{aligned}$$

Proof (cont.).

Thus,

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Define $x_i, y_j \in S^{2d^2}$ by

$$x_i = \begin{pmatrix} \operatorname{Re}(F_i \otimes I | \psi \rangle) \\ \operatorname{Im}(F_i \otimes I | \psi \rangle) \end{pmatrix}, \quad y_j = \begin{pmatrix} \operatorname{Re}(I \otimes G_j | \psi \rangle) \\ \operatorname{Im}(I \otimes G_j | \psi \rangle) \end{pmatrix}. \quad (20)$$

Proof (cont.).

Thus,

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Then,

$$\begin{aligned}\langle x_i, y_j \rangle &= \langle \operatorname{Re}(F_i \otimes I | \psi \rangle), \operatorname{Re}(I \otimes G_j | \psi \rangle) \rangle + \langle \operatorname{Im}(F_i \otimes I | \psi \rangle), \operatorname{Im}(I \otimes G_j | \psi \rangle) \rangle \\ &= \operatorname{Re}(\langle \psi | F_i \otimes G_j | \psi \rangle) \\ &= \langle \psi | F_i \otimes G_j | \psi \rangle,\end{aligned}$$

as desired. □

Due to the similarity between those two proofs, we can prove that we can replace X_i , Y_j and ρ in the definition of Quantum correlation matrices by $\{-1, 1\}$ -observables and a pure state.

The only thing we need to prove is the following:

$$\text{QC}_{m,n} = \{(\langle x_i, y_j \rangle)_{1 \leq i \leq m, 1 \leq j \leq n} \mid x_i, y_j \in \mathbb{R}^{\min\{m,n\}}, |x_i| = 1, |y_j| = 1\} \quad (21)$$

If we can find unit vectors $u, v \in \{x_i \mid 1 \leq i \leq m\}^\perp \cap \{y_j \mid 1 \leq j \leq n\}^\perp$ such that $u \perp v$, we can define unit vectors

$$\begin{aligned} x'_i &= x_i + \sqrt{1 - |x_i|^2} u \\ y'_j &= y_j + \sqrt{1 - |x_i|^2} v \end{aligned}$$

with

$$\langle x'_i, y'_j \rangle = \langle x_i, y_j \rangle.$$

Otherwise we increase the dimension to find vectors u and v and then decrease the dimension by a posteriori projecting the new vectors onto $\text{span}\{x_i : 1 \leq i \leq n\}$ or $\text{span}\{y_j : 1 \leq j \leq m\}$ according to whether $m \leq n$ or $m > n$.

1 Local and quantum correlation matrices

- Local correlation matrices
- Quantum correlation matrices
- The relations between quantum correlation and local correlation matrices

2 Grothendieck-Tsirelson Theorem

- Motivation: The Grothendieck-Tsirelson Theorem
- Grothendieck's Inequality
- Tsirelson's Theorem
- Grothendieck-Tsirelson Theorem

Theorem (Grothendieck-Tsirelson)

There exists an absolute constant $K \geq 1$ such that, for any positive integers m, n , the following three equivalent conditions hold:

(1) We have the inclusion

$$\text{QC}_{m,n} \subset \text{KLC}_{m,n}. \quad (22)$$

(2) For any $M \in \mathbb{R}^{m \times n}$ and for any ρ, X_i, Y_j verifying the conditions of Definition 4.2.1 we have

$$\sum_{i,j} M_{ij} \text{Tr } \rho(X_i \otimes Y_j) \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \sum_{i,j} M_{ij} \xi_i \eta_j \quad (23)$$

$$\Leftrightarrow \text{Tr } M A^\top \leq \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr } M (\xi \eta^\top)^\top. \quad (24)$$

(3) For any $M \in \mathbb{R}^{m \times n}$ and for any (real) Hilbert space vectors x_i, y_j with $|x_i| \leq 1, |y_j| \leq 1$ we have

$$\sum_{i,j} M_{ij} \langle x_i, y_j \rangle \leq K \max_{\xi \in \{-1,1\}^m, \eta \in \{-1,1\}^n} \text{Tr } \xi^\top M \eta. \quad (25)$$

Proof.

Since (25) is a direct consequence of Grothendieck's inequality the only thing left to prove is the equivalence between (1)-(3). The equivalence of (3) and (2) (the Tsirelson's bound) is a consequence of either the proof of Lemma ?? or Tsirelson's Theorem (Theorem 1).

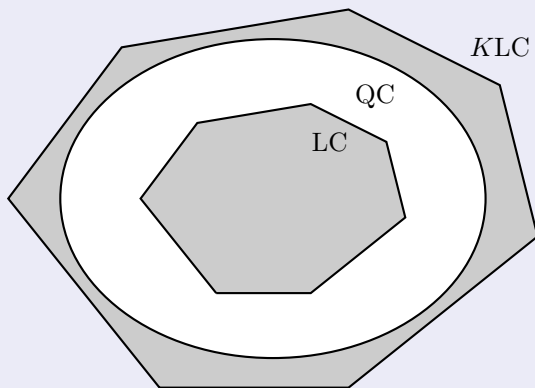


Figure: Visualization