

5. Der Integralsatz von Stokes

Definition

Sei $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ eine Fläche mit Parameterbereich $B \subseteq \mathbb{R}^2, D \subseteq \mathbb{R}^2$ offen, $B \subseteq D, \Phi \in C^1(D, \mathbb{R}^3)$ und $S = \Phi(B)$.

Für $f : S \rightarrow \mathbb{R}$ stetig und $F : S \rightarrow \mathbb{R}^3$ stetig:

$$\left. \begin{aligned} \int_{\Phi} f \, d\sigma &:= \int_B f(\Phi(u, v)) \cdot \|N(u, v)\| \, d(u, v) \\ \int_{\Phi} F \cdot n \, d\sigma &:= \int_B F(\Phi(u, v)) \cdot N(u, v) \, d(u, v) \end{aligned} \right\} \text{Oberflächenintegrale}$$

Beispiele 5.1

(1) Für $f \equiv 1 : \int_{\Phi} 1 \, d\sigma =: \int_{\Phi} d\sigma = I(\Phi)$

(2) Sei $B := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$, $\Phi(u, v) := (u, v, u^2 + v^2)$, $F(x, y, z) = (x, y, z)$

Bekannt: $N(u, v) = (-2u, -2v, 1)$, $F(\Phi(u, v)) = (u, v, u^2 + v^2) \Rightarrow \int_{\Phi} F \cdot n \, d\sigma =$

$$\int_B (u, v, u^2 + v^2) \cdot (-2u, -2v, 1) \, d(u, v) = - \int_B (u^2 + v^2) \, d(u, v) \stackrel{u=r \cos \varphi, v=r \sin \varphi}{=} - \int_0^{2\pi} \left(\int_0^1 r^3 \, dr \right) d\varphi = - \frac{\pi}{2}$$

Satz 5.2 (Integralsatz von Stokes)

B, D, Φ seien wie oben. B sei zulässig, $\partial B = \Gamma\gamma$, wobei $\gamma = (\gamma_1, \gamma_2)$ wie in §2. Es sei $\Phi \in C^2(D, \mathbb{R}^3)$, $G \subseteq \mathbb{R}^3$ sei offen, $F \subseteq G$ und $F = (F_1, F_2, F_3) \in C^1(G, \mathbb{R}^3)$. Dann:

$$\underbrace{\int_{\Phi} \text{rot } F \cdot n \, d\sigma}_{\text{Oberflächenintegral}} = \underbrace{\int_{\Phi \circ \gamma} F(x, y, z) \, d(x, y, z)}_{\text{Wegintegral}}$$

Beweis

$\varphi := \Phi \circ \gamma, \varphi = (\varphi_1, \varphi_2, \varphi_3)$, also: $\varphi_j = \Phi_j \circ \gamma \quad (j = 1, 2, 3)$

Zu zeigen: $\int_{\Phi} \text{rot } F \cdot n \, d\sigma = \int_0^{2\pi} F(\varphi(t)) \cdot \varphi'(t) \, dt = \sum_{j=1}^3 \int_0^{2\pi} F_j(\varphi(t)) \cdot \varphi'_j(t) \, dt$

Es ist $\int_{\Phi} \text{rot } F \cdot n \, d\sigma = \int_B \underbrace{(\text{rot } F)(\Phi(x, y)) \cdot (\Phi_x(x, y) \times \Phi_y(x, y))}_{=:g(x,y)} \, d(x, y)$

Für $j = 1, 2, 3 : g_j(x, y) := \underbrace{(F_j(\Phi(x, y))) \frac{\partial \Phi_j}{\partial y}(x, y)}_{=:u_j(x,y)}, \underbrace{-F_j(\Phi(x, y)) \frac{\partial \Phi_j}{\partial x}(x, y)}_{=:v_j(x,y)}, (x, y) \in D$

$F \in C^1, \Phi \in C^2 \Rightarrow g_j \in C^1(D, \mathbb{R}^2)$

Nachrechnen: $g = \text{div } g_1 + \text{div } g_2 + \text{div } g_3 \Rightarrow \int_{\Phi} \text{rot } F \cdot n \, d\sigma = \sum_{j=1}^3 \int_B \text{div } g_j(x, y) \, d(x, y)$

$$\begin{aligned} \int_B \text{div } g_j(x, y) \, d(x, y) &\stackrel{2.1}{=} \int_{\gamma} (u_j \, dy - v_j \, dx) = \int_0^{2\pi} (u_j(\gamma(t)) \cdot \gamma'_2(t) - v_j(\gamma(t)) \cdot \gamma'_1(t)) \, dt = \\ &\int_0^{2\pi} (F_j(\varphi(t)) \frac{\partial \Phi_j}{\partial y}(\gamma(t)) \gamma'_2(t) + F_j(\varphi(t)) \frac{\partial \Phi_j}{\partial x}(\gamma(t)) \gamma'_1(t)) \, dt = \int_0^{2\pi} F_j(\varphi(t)) \cdot \varphi'_j(t) \, dt \Rightarrow \int_{\Phi} \text{rot } F \cdot n \, d\sigma = \\ &\sum_{j=1}^3 \int_B \text{div } g_j(x, y) \, d(x, y) = \sum_{j=1}^3 \int_0^{2\pi} F_j(\varphi(t)) \cdot \varphi'_j(t) \, dt \quad \blacksquare \end{aligned}$$

Beispiel

B, Φ, F seien wie in Beispiel 4.1.(2). $\gamma(t) = (\cos t, \sin t), t \in [0, 2\pi]$. Verifiziere 4.2

Hier: $\operatorname{rot} F = 0$, also $\int_{\Phi} \operatorname{rot} F \cdot n d\sigma = 0$. $(\Phi \circ \gamma)(t) = (\cos t, \sin t, 1) \Rightarrow \int_{\Phi \circ \gamma} F(x, y, z) d(x, y, z) = \int_0^{2\pi} (\cos t, \sin t, 1) \cdot (-\sin t, \cos t, 0) dt = 0$