MAT1001 Calculus I

Lecture 12 - 15 : Integrals

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1 Antiderivatives

Definition If $F'(x) = f(x)$ for all x in an interval I, then F is said to be an antiderivative of f on I.
By corollary (2) of MVT, we have

Theorem 1 (Antiderivative) *If F is an antiderivative of f on an interval I, the the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

	Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$
3.	cos kx	$\frac{1}{k}\sin kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

Figure 1: General antiderivative formula

Example 1 Find the antiderivative F of $f(x) = 3\sqrt{2} + \sin 2x$ that satisfies $F(0) = 1$!

$$f(x) = 3\sqrt{2} + \sin 2x$$
$$F(x) = 3(\frac{2}{3})x^{\frac{3}{2}} - \frac{1}{2}\cos 2x + C$$
$$F(0) = 1 \Rightarrow 1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$
$$F(x) = 2x^{\frac{3}{2}} - \frac{1}{2}\cos 2x + \frac{3}{2}$$

Definition : Indefinite Integral The set of all antiderivatives of f is called the **indefinite integrals** of f. If F is one antiderivative of f, we write

$$\int f(x)\,dx = F(x) + C$$

$\int f(x)dx$ is called the indefinite integral of f with respect to x
$f(x)$ is the integrand.
dx is called the variable of integration.

Example 1 Solve

$$\int (x^2 + x)dx$$

Answer

$$\int (x^2 + x)dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Example 2 Solve

$$\int (x^2 - 2x + 5)dx$$

Answer

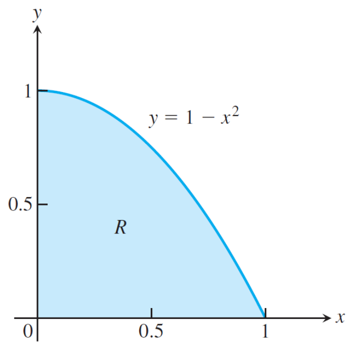
$$\int (x^2 - 2x + 5)dx = \frac{1}{3}x^3 + x^2 + 5x + C$$

Note Antiderivatives also apply linearity rule, that is

$$\int \alpha f(x) \pm \beta g(x)\, dx = \alpha \int f(x)\, dx \pm \beta \int g(x)\, dx$$

2 Finite Sums Estimations

Consider finding the area R under the graph of the function $y = 1 - x^2$, above x-axis, between the vertical lines $x = 0$ and $x = 1$.



We may approximate R by summing areas of rectangles with the following procedure:

- Divide $[0, 1]$ into sub-intervals with equal length and construct rectangles using the function values of the left/right endpoints.
- The sum areas the these rectangles is an approximation of R

First we divide $[a, b]$ into n sub-intervals of the length $\Delta x = \frac{(b-a)}{n}$, that is with $[0, 1]$, $\Delta x = \frac{1}{n}$.

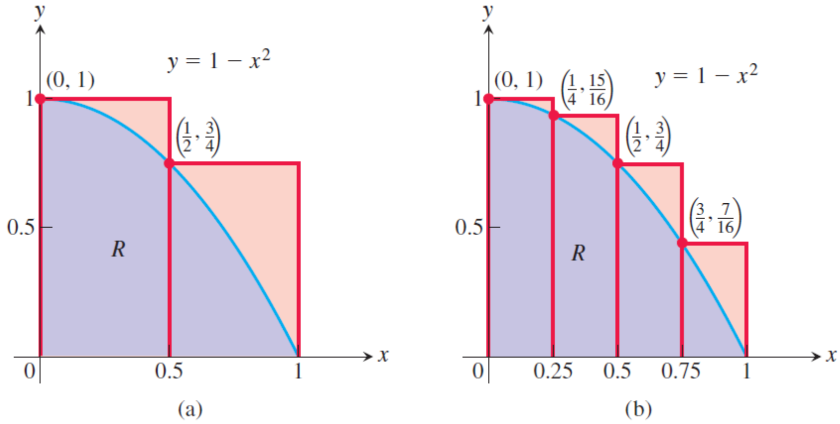


Figure 2: (a) left endpoint sum, $n = 2$ (b) left endpoint sum, $n = 4$

2.1 Left Endpoint Sum

Now, suppose we choose the left endpoint function value.
We have $x_1, x_2, ...x_n$, with x_i as

$$x_i = a + (i - 1)\frac{b - a}{n}$$

So, in the case of $[0, 1]$ with $n = 2$, $x_1 = 0$, $x_2 = 1/2$. Hence,

$$Area = \frac{1}{2} \cdot \left(f(0) + f\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(1 + \frac{3}{4}\right) = \frac{7}{8}$$

With $n = 4$, $x_1 = 0$, $x_2 = 1/4$, $x_3 = 1/2$, $x_4 = 7/16$. Hence,

$$Area = \frac{1}{2} \cdot \left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(1 + \frac{15}{16} + \frac{3}{4} + \frac{7}{16}\right) = \frac{25}{32}$$

2.2 Right Endpoint Sum

Instead of choosing the right endpoint function value, now suppose we choose the right endpoint function value.

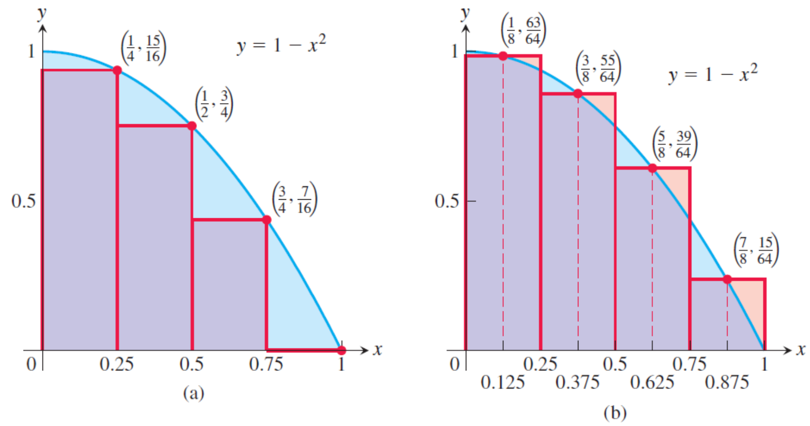


Figure 3: (a) right endpoint sum, $n = 4$ (b) mid-point sum, $n = 4$

We have $x_1, x_2, ...x_n$, with x_i as

$$x_i = a + i\left(\frac{b - a}{n}\right)$$

So, in the case of $[0,1]$ with $n = 4$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$, and $x_4 = 1$. Hence,

$$Area = \frac{1}{4} \cdot \left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) \right) = \frac{1}{4}\left(\frac{15}{16} + \frac{3}{4} + \frac{7}{16} + 0\right) = \frac{17}{32}$$

2.3 Area Approximation

Let n as the number of sub-intervals, then the length of the sub-intervals is $\Delta x = \frac{b-a}{n}$.We have $x_i = a + i(\Delta x) = a + i\left(\frac{b-a}{n}\right)$. The approximated area of $f(x)$ in the interval of $[a,b]$ is

$$\frac{b-a}{n}\sum_{i=1}^{n} f(c_i)$$

With $c_i \in [x_{i-1}, x_i]$

- For left endpoint sum, $c_i = x_{i-1}$
- For right endpoint sum, $c_i = x_i$
- For mid point sum, $c_i = \frac{x_{i-1}+x_i}{2}$

2.4 Area Sums and Concavity

Suppose $y = f(x)$ is **concave down** on (a,b) and $f(x) \geq 0$, $\forall \in [a,b]$

- If we approximate the area between the curve and the x-axis, for $[a,b]$, using a midpoint sum S, the sum will always over-estimate the area.

Suppose $y = f(x)$ is **concave up** on (a,b) and $f(x) \geq 0$, $\forall \in [a,b]$

- If we approximate the area between the curve and the x-axis, for $[a,b]$, using a midpoint sum S, the sum will always under-estimate the area.

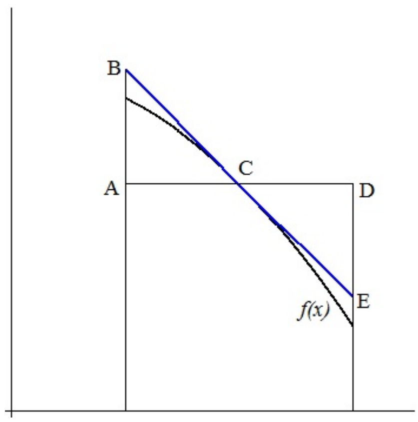


Figure 4: $|AB| = |DE|$, so, $A_{sq} = A_{\triangle}$

Proof (Concave down - Midpoint) The area under the tangent line = The area under the rectangle. Since on a concave down function, tangent line is always above the function, the area under the tangent line > the area of the graph. So, the approximation will always over-estimate the area.

2.5 Finite Sums

Notation for finite sum is

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

The variable i is a "dummy variable" meaning it can be changed to a different symbol without changing the meaning.

Properties Finite sums satisfy linearity, that is

$$\sum_{i=1}^{n}(k \cdot a_i + t \cdot b_i) = k\sum_{i=1}^{n} a_i + t\sum_{i=1}^{n} b_i$$

There are some identities for specific finite sum, some of which are:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2}{4}(n+1)^2$$

For $\sum_{i=1}^{n} i^m$ we can find the formula by starting with n^{m+1} and using the formula for smaller m.

$m = 1 \quad \sum_{i=1}^{n} i$
Start with n^2, Let $S_n = \sum_{k=1}^{n} k$

$$n^2 = (n^2 - (n-1)^2) + ((n-1)^2 - (n-2)^2) + \ldots + (2^2 - 1^2) + (1^2 - 0^2)$$

Then we have

$$\begin{aligned}
n^2 &= \sum_{k=1}^{n}(k^2 - (k-1)^2) \\
&= \sum_{k=1}^{n}(2k-1) \\
&= 2\sum_{k=1}^{n} k - n \\
\sum_{k=1}^{n} k &= \frac{n^2+n}{2} \\
S_n &= \frac{n}{2}(n+1)
\end{aligned}$$

$m = 2 \quad \sum_{i=1}^{n} i^2$
Start with n^3, Let $S_n = \sum_{k=1}^{n} k^2$

$$n^3 = (n^3 - (n-1)^3) + ((n-1)^3 - (n-2)^3) + \ldots + (2^3 - 1^3) + (1^3 - 0^3)$$

Then we have

$$
\begin{aligned}
n^3 &= \sum_{k=1}^{n}(k^3-(k-1)^3) \\
&= \sum_{k=1}^{n}(k^3-k^3+3k^2-3k+1) \\
&= \sum_{k=1}^{n}(3k^2-3k+1) \\
&= 3\sum_{k=1}^{n}k^2-3\sum_{k=1}^{n}k+n \\
&= 3S_n-\frac{3n}{2}(n+1)+n \\
S_n &= \frac{1}{3}(n^3+\frac{3n}{2}(n+1)-n) \\
S_n &= \frac{1}{6}(2n^3+3n^2+n) \\
S_n &= \frac{1}{6}(n)(n+1)(2n+1)
\end{aligned}
$$

3 Riemann Sums

3.1 Definition

Definition : Partition A partition of the interval $[a, b]$ is a set

$$P = x_0, x_1, \ldots, x_{n-1}, x_n$$

such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

Definition : Riemann Sums Given a function $f : [a, b] \Rightarrow \mathbb{R}$ with a partition P of $[a, b]$, a Riemann sum of f (w.r.t. P) is a sum of the form

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + \ldots + f(c_n)\Delta x_n$$

where $c_k \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$ for each $k \in \{1, ..., n\}$

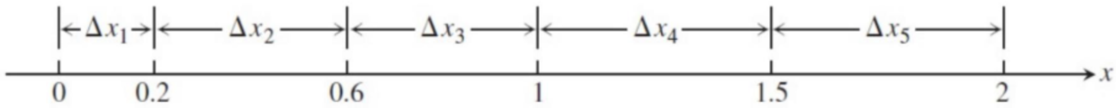
There are many Riemann sums for a function. It depends on the partition P and the points c chosen from the subintervals. The left-endpoint, midpoint and right-endpoint sums are all special cases of Riemann sum.

Definition : Norm of Partition Let $P = x_0, x_1, x_2, ..., x_n$ be a partition of $[a, b]$. The **norm** of P denoted by $||P||$ is defined by

$$||P|| = \max_{k:1\le k\le n} \Delta x_n$$

That is, $||P||$ is the length of the largest subinterval given by P.

Example The partition of $P = [0, 2]$ represented below has norm $||P|| = 0.5$



4 Definite Integrals

4.1 Definition

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the definite integral of f over $[a, b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = x_0, x_1, ..., x_n$ of $[a, b]$ with $||P|| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \epsilon$$

By definition, we can write the above definition as

$$\lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J$$

If the limit J exists, we say that f is integrable on $[a, b]$ and write the limit J as

$$\int_a^b f(x)dx$$

It's called the **definite integral** or Riemann integral of f over $[a, b]$

4.2 Integrability

Theorem 2 (Integrability of Continuous Functions) *If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump or removable discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$*

Proof For each $[x_{k-1}, x_k]$ we define:

- $M_k = \max\{f_{x_k^*} : x_k^* \in [x_{k-1}, x_k]\}$
- $m_k = \min\{f_{x_k^*} : x_k^* \in [x_{k-1}, x_k]\}$

Or we say M_k as the maximum value of $f(c)$ with $c \in [x_{k-1}, x_k]$ and m_k as the minimum value of $f(c)$ with $c \in [x_{k-1}, x_k]$

From that, we have

$$U_p(f) = \sum_{k=1}^{n} M_k\Delta x_k \qquad \text{Upper sum}$$
$$L_p(f) = \sum_{k=1}^{n} m_k\Delta x_k \qquad \text{Lower sum}$$

Thus for each $c_k \in [x_{k-1}, x_k]$, $m_k \leq f(c_k) \leq M_k$, So

$$L_p = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} f(c_k) \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k = U_p$$

As $||P|| \to 0$, $U_p - L_p \to 0$. Or, for any given ϵ, choose all Δx_k small enough such that $M_k - m_k < \frac{\epsilon}{b-a}, \forall\, k$. Then,

$$U_p(f) - L_p(f) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k < \frac{\epsilon}{b-a} \sum_{k-1}^{n} \Delta x_k$$

Since

$$\frac{\epsilon}{b-a} \sum_{k-1}^{n} \Delta x_k = \frac{\epsilon}{b-a}(b-a) = \epsilon$$

We have

$$U_p(f) - L_p(f) < \epsilon$$

Hence

$$\lim_{||P|| \to 0} (U_p(f) - L_p(f)) = 0$$

$$\lim_{||P|| \to 0} U_p(f) = \lim_{||P|| \to 0} L_p(f))$$

Since all Riemann sums $S_p = S_p(f)$ satisfy

$$L_p(f) \leq S_p(f) \leq U_p(f)$$

then

$$\lim_{||P|| \to 0} S_p(f) = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k$$

must exists.

4.3 Computation Riemann Integrals

Suppose we know that f is integrable on $[a, b]$. Then we can compute the limit of Riemann sums by choosing any sequence of partitions P such that $||P|| \to 0$.

In particular, we may choose $\Delta x_k = \Delta x = \frac{b-a}{n}$ or P divides $[a, b]$ into n subintervals of equal length.

Then we can compute the definite integral as:

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x$$

With $c_k \in [x_{k-1}, x_k]$ can be chosen in any way.

Example Evaluate $\int_0^1 x^2 dx$ using Riemann sums.

$$\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n \to \infty} \sum_{k=1}^{n} x_k^2 \Delta x = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{x}{k}\right)^2 \cdot \frac{1}{n} \\
&= \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\
&= \lim_{n \to \infty} \frac{1}{n^3} (\frac{1}{6} n(n+1)(2n+1)) \\
&= \frac{1}{3}
\end{aligned}$$

Choosing the right endpoint as c_k gives the similar result.

4.4 Nonintegrability

$\int_a^b f(x)dx$ does not exist when the upper sum and the lower sum do not converge to the same number J. In other words, here exists $\epsilon > 0$ such that no matter how small a given $\delta > 0$ is, there is a partition P of $[a, b]$ with $\|P\| < \delta$ such that $U_p(f) - L_p(f) > \epsilon$. ϵ is gap between $L_p(f)$ and $U_p(f)$.

Example Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $\int_a^b f(x)dx$ does not exist for all a, b with $a < b$.

Proof For any interval $[x_{k-1}, x_k]$ we have some values $c_1, c_2 \in [x_{k-1}, x_k]$ such that $f(c_1) = 1$ and $f(c_2) = 0$. hen for any partition P of $[a, b]$ with $\|P\| < \delta$,

$$L_p = \sum_{k=1}^{n} m_k \, \Delta x_k = \sum_{k=1}^{n} 0 \, \Delta x_k = 0$$
$$U_p = \sum_{k=1}^{n} M_k \, \Delta x_k = \sum_{k=1}^{n} 1 \, \Delta x_k = b - a$$

Hence,

$$U_p(f) - L_p(f) = b - a > \epsilon$$

and integral does not exists

5 Properties of Integral

Definition For $a < b$ we define:

$$\int_a^b f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Other rules of integral:

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

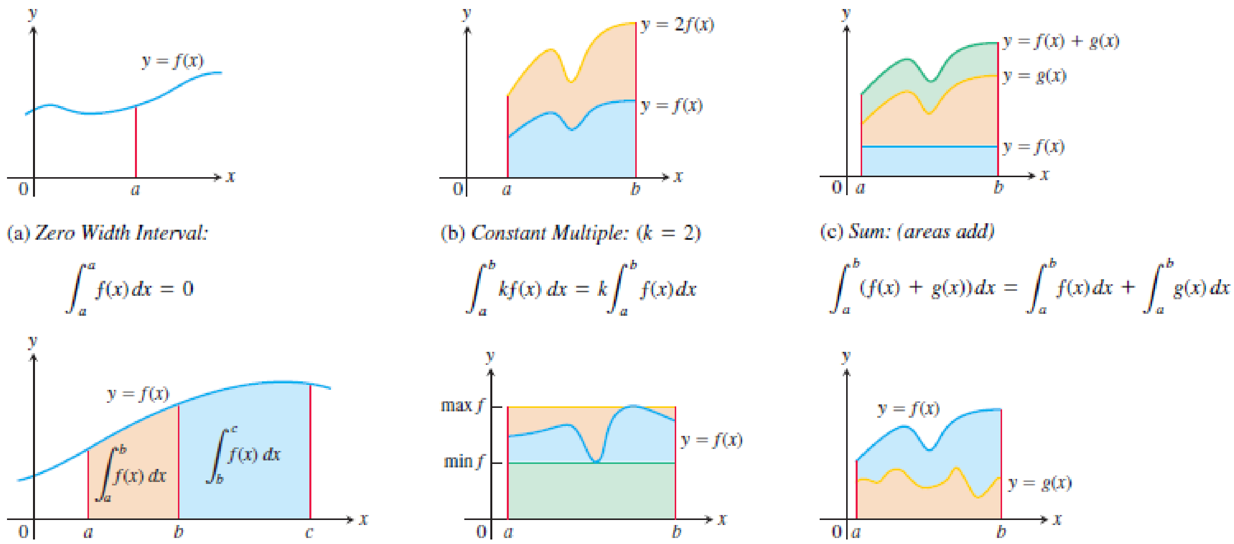
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special case)

Figure 5: Other rules of definite integrals



(a) *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$

(b) *Constant Multiple:* $(k = 2)$ $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

(c) *Sum: (areas add)* $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(d) *Additivity for Definite Integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:* $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$

(f) *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

Figure 6: Geometric Illustration of rules of definite integrals

Note Additivity (property 5) works for $b \in [a, c]$ even if $b \notin [a, c]$. For example:

$$\int_3^5 f(x)dx + \int_5^6 f(x)dx = \int_3^6 f(x)dx$$

then

$$\int_3^5 f(x)dx = \int_3^6 f(x)dx - \int_5^6 f(x)dx$$
$$\int_3^5 f(x)dx = \int_3^6 f(x)dx + \int_6^5 f(x)dx$$

Note All properties (3-6 and 7(i)) work even if there exists $f < 0$ on $[a, b]$ and even if f is not continuous

Proof of Property 6 Suppose $\int_a^b f(x)dx$ exists. Let

$$M = \max_{x\in[a,b]} f(x) \qquad \text{and} \qquad m = \min_{x\in[a,b]} f(x)$$

Then every Riemann sum satisties

$$m\sum_{k=1}^{n}\Delta x_n = \sum_{k=1}^{n} m\Delta x_n \le \sum_{k=1}^{n} f(c_k)\Delta x_n \le \sum_{k=1}^{n} M\Delta x_n = M\sum_{k=1}^{n}\Delta x_n$$

So,

$$m\lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x_n \le \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_n \le M\lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x_n$$

Hence

$$m\int_a^b dx \le \int_a^b f(x)dx \le M\int_a^b dx$$

Note that $\int_a^b dx = b - a$ Then:

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Example Since $0 \le \sqrt{1+\cos x} \le \sqrt{2}$ for all x, by domination:

$$\int_0^1 0\,dx \le \int_0^1 \sqrt{1+\cos x}\,dx \le \int_a^b \sqrt{2}\,dx$$

Hence

$$0 \le \int_0^1 \sqrt{1+\cos x}\,dx \le \sqrt{2}$$

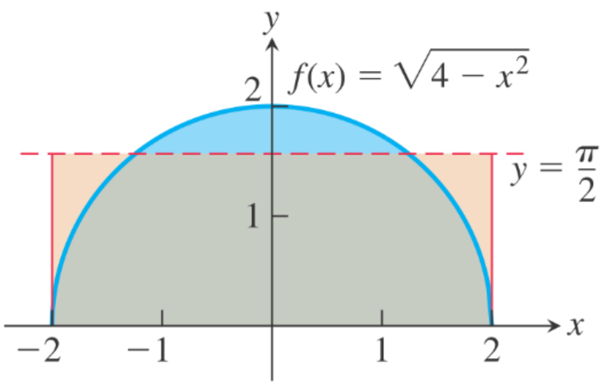
6 Average Value of Function

Let $v(t)$ be the velocity of an object at time t, where $v(t) \ge 0$ for all $t \in [a, b]$
Then $\sum_{k=1}^{n} v(c_k)\Delta t_k$ is an approximated distance travelled over time interval $[a, b]$
$\int_a^b v(t)$ is the total distance travelled over $[a, b]$
Hence: $\frac{1}{b-a}\int_a^b v(t)dt$ is average velocity

Definition If f is integrable on $[a, b]$, then its average value on $[a, b]$, also called its mean, is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example What is the average value of $f(x) = \sqrt{4-x^2}$ with $x \in [-2, 2]$



Answer

$$\begin{aligned} av(f) &= \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2}dx \\ &= \frac{1}{4} \cdot \frac{1}{2}\pi(2)^2 \\ &= \frac{\pi}{2} \end{aligned}$$

7 MVT of Definite Integrals

Theorem 3 (MVT for Definite Integrals) *If f is continuous on $[a,b]$ then at some point $c \in [a,b]$*

$$f(c) = \frac{1}{b-a} \int_{a}^{b} f(x)dx$$

Geometric Representation If f is continuous, then there is a point $c \in [a,b]$ such that $f(c)$ is the average height

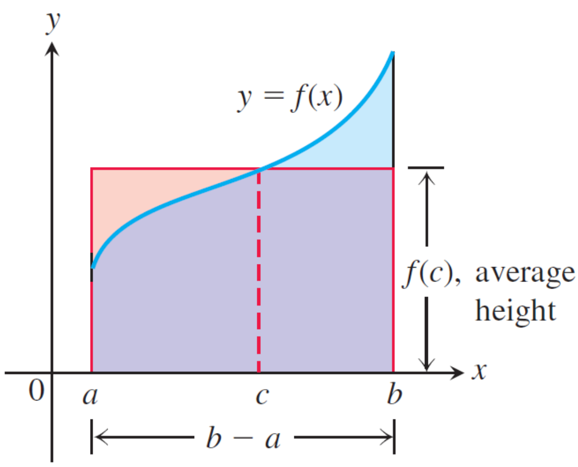


Figure 7: $f(c)$ is the average height of f on $[a,b]$

Proof Since f is continuous, there exist x_1 and x_2 in $[a,b]$ such that

$$f(x_1) = m = \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad f(x_2) = M = \max_{x \in [a,b]} f(x)$$

Let assume that $x_1 \neq x_2$. By min-max inequality, we have

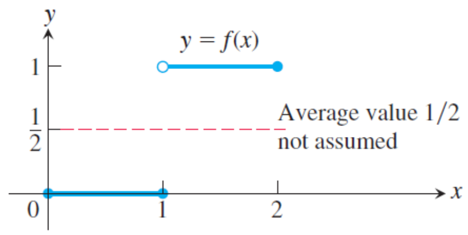
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

By IVT

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

for some c between x_1 and x_2. So, $c \in [a,b]$

Note The condition of the theorem applies is **continuity**.



Example Since $f(x)$ is not continuous at $x = 1$:

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = 0 + 1 = 1$$

Hence,

$$av(f) = \frac{1}{2}$$

However, the average value is not equal to the function at any point on $[0,2]$. So, **the average value need not to be assumed**.

Consequence If f is continuous on $[a,b]$, $a \neq b$ and if $\int_a^b f(x)dx = 0$ then $f(x) = 0$ at least once in $[a,b]$

8 The Fundamental Theorem of Calculus

8.1 FTC Part 1

Intuition Let $v(t)$ be the velocity of an object at time t, where $v(t) \geq 0$ for all $t = [1,8]$. Then:

- $\int_1^5 v(t)dt$ represent the distance travelled from $t = 1$ to $t = 5$
- If $F(t) = \int_1^x v(t)$ then $F(x)$ represent the distance travelled between $t = 1$ and $t = x$
- Then $F'(x)$ is the instantenous velocity at $t = x$

Theorem 4 (The Fundamental Theorem of Calculus Part 1) *If f is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$*

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

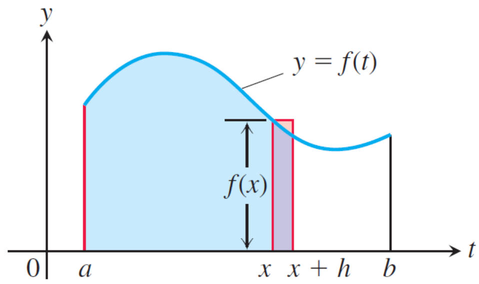


Figure 8: Illustration of FTC 1

Proof FTC Part 1 Suppose f is continuous on $[a,b]$. Let $F:[a,b]\to\mathbb{R}$, $F(x)=\int_a^x f(t)\,dt$

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}\\
&= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt+\int_x^a f(t)\,dt}{h}\\
&= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}
\end{aligned}$$

For $h>0$, by MVT for integrals,

$$\frac{1}{h}\int_x^{x+h} f(t)dt = f(c)$$

for some $c\in[x,x+h]$ As $h\to 0^+$, $c\to x^+$, so

$$F'_+(x)=\lim_{h\to 0^+} f(c)=\lim_{c\to x^+} f(c)=f(x)$$

For $h<0$, by MVT for integrals,

$$\frac{1}{h}\int_x^{x+h} f(t)dt=\frac{1}{-h}\int_x^{x+h} f(t)dt=f(c)$$

for some $c\in[x+h,x]$ As $h\to 0^-$, $c\to x^-$, so

$$F'_-(x)=\lim_{h\to 0^-} f(c)=\lim_{c\to x^-} f(c)=f(x)$$

Hence,

$$F'(x)=f(x) \qquad \forall\, x\in(a,b)$$

Remark We have showed that

$$F'(x)=f(x) \qquad \forall\, x\in(a,b)$$

Hence, F is differentiable on interval (a,b) and continuous on interval (a,b). The same argument can also be used to show That

$$F'_+(a)=f(a) \qquad \text{and} \qquad F'_-(b)=f(b)$$

Hence, F is one-sided differentiable at $x=a$ and $x=b$. Therefore, F is continuous on $[a,b]$.

Example 1 $y = \int_a^x (t^3 + 1)\, dt$
Let $F(x) = y = \int_a^x (t^3 + 1)\, dt$ and $f(t) = t^3 + 1$ By FTC 1,

$$\frac{dy}{dx} = F'(x) = f(x) = x^3 + 1$$

Example 2 $y = \int_x^5 (3t \sin t)\, dt$
Let $F(x) = y = \int_5^x (3t \sin t)\, dt$ and $f(t) = 3t \sin t$.
By FTC 1,

$$\frac{dy}{dx} = -F'(x) = -f(x) = -3x \sin x$$

Example 3 $y = \int_1^{x^2} (\cos t)\, dt$
Let $F(x) = y = \int_1^u (\cos t)\, dt$ and $u = x^2$. So, $y = F(u)$
By FTC 1,

$$\frac{dy}{du} = F'(u) = \cos u = \cos(x^2)$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2x \cos(x^2)$$

Example 4 $y = \int_{1+3x^2}^4 (\frac{1}{2+t})\, dt$
Let $F(x) = y = \int_4^u (\cos t)\, dt$ and $u = 1 + 3x^2$. So, $y = -F(u)$
By FTC 1,

$$\frac{dy}{du} = -F'(u) = -\frac{1}{2+u} = -\frac{1}{3+3x^2}$$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{6x}{3+3x^2} = -\frac{2x}{1+x^2}$$

Remark If $F(u) = \int_a^u f(t)\, dt$ then

$$\int_a^{g(x)} f(t)\, dt = (F \circ g)(x) = F(g(x))$$

By chain rule,

$$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt = (F \circ g)(x) = F'(g(x))g'(x)$$

By FTC 1,

$$F'(g(x))g'(x) = f(g(x))g'(x)$$

Hence,

$$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt = f(g(x))g'(x)$$

8.2 FTC Part 2

Theorem 5 *If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, Then*

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Proof FTC Part 2 Let $G(x) = \int_a^x f(t)dt$. By FTC 1, G is an antiderivative of f on (a,b). Since F is also an antiderivative of f, $\exists\, c$ such that

$$G(x) = F(x) + c \qquad \forall\, x \in (a,b)$$

G and F are continuous at a, since

$$G(a) = \lim_{x\to a^+} G(x) = \lim_{x\to a^+} F(x) + c = F(a) + c$$

ans similarly

$$G(b) = F(b) + c$$

From FTC 1,

$$\int_a^b f(x)dx = G(b)$$

Since

$$G(a) = \int_a^a f(t)dt = 0$$

We have

$$\int_a^b f(x)dx = G(b) - 0 = G(b) - G(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a)$$

We can write:

$$\int_a^b f(x)dx = F(x)|_{x=a}^b = F(x)|_a^b = [F(x)]_a^b$$

Consequence By FTC 2, we can find $\int_a^b f(x)dx$ only by finding the antiderivative F of f.

Example 1

$$\int_0^\pi \cos x\, dx = \sin x|_0^\pi = \sin\pi - \sin 0 = 0$$

Example 2

$$\int_0^1 x^2\, dx = [\frac{1}{3}x^2]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

8.3 Note on FTC

$\int_a^x f(t)dt = \int_a^x f(s)ds$ since t and s are only "dummy variables" which has no particular meaning. However, we cannot use $\int_a^x f(x)dx$ since x is already an independent variable, and it doesn't make sense to use it for the "dummy variable".

FTC 1 gives a form of an antiderivative of a continuous function. Some elementary functions have antiderivatives not expressible in terms of an elementary function, such as $f(x) = \frac{\sin x}{x}$. However, we know an antiderivative

$$F(x) = \int_a^x \frac{\sin t}{t} dt$$

where $a \neq 0$ is a constant and x has the same sign as a

8.4 Application of FTC

Economics If $C(x)$ is the total cost for producing x units of goods, then by FTC2:

$$M(x) = \int_a^b C'(x)\,dx = C(b) - C(a)$$

$M(x)$ is the extra cost for increasing production from a units to b units.

Physics If $s(t)$ is position on the s-axis, then its velocity $v(t) = s'(t)$. By FTC 2

$$d(t) = \int_b^a v(t)dt = s(b) - s(a)$$

$d(t)$ is the displacement over the time interval $[a, b]$

Mathematical consequence The average slope of all the tangent lines to the curve $y = f(x)$ over the interval $[a, b]$ can be denoted as

$$av(f'(x)) = \frac{\int_a^b f'(x)\,dx}{b-a} = \frac{f(b) - f(a)}{b-a}$$

by FTC 2. It is the same as the slope of the secant from $x = a$ to $x = b$. So, the average of the slopes of the tangents to the curve between a and b is the slope of the secant line. Hence, the average rate of change = average of all instantaneous rates of change.

8.5 Relation of Differentiation and Integration

By FTC 1, we have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

and by FTC 2 we have

$$\int_a^x f(t)dt = F(x) - F(a)$$

which means $f(x)$ **is shifted by a constant** a. Hence applying integration and differentiation to a continuous function f, or vice versa, gives f back (only subject to a difference by constant). Or,

$$F(x) = \int_a^x f(t)dt \qquad \text{and} \qquad F'(x) = f(x)$$

9 Area of Curves

9.1 Area under Curves

Definition If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$ then the area uder the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b, that is

$$A = \int_a^b f(x)dx$$

Note that if $f(x) < 0$ for some $x \in [a, b]$, then the definition does not hold

9.2 Area between Curves

Definition Let f and g be functions that are integrable on $[a, b]$. Then, the area A between the graph of $y = f(x)$ and the graph $y = g(x)$ from $x = a$ to $x = b$ is defined By

$$A = \int_a^b |\, f(x) - g(x) \,| \; dx$$

For the area between $y = f(x)$ and the x-axis, take $g(x) = 0$, So

$$A = \int_a^b |\, f(x) \,| \; dx$$

If f is not negative then,

$$A = \int_a^b f(x)\, dx$$

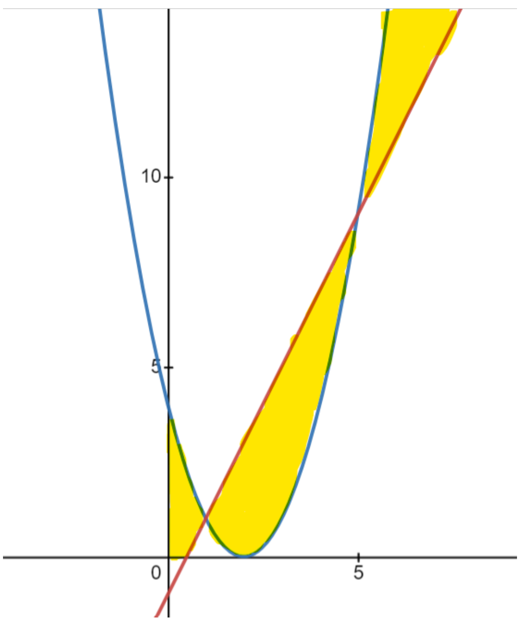
which is the same as the previous definition (definition area under curves)

Example 1 Find the area A between the graph of $y = f(x) = \sin x$ and the x-axis from $x = a = 0$ to $x = b = 2\pi$.

Answer

$$\begin{aligned} A &= \int_0^{2\pi} |\sin x| \; dx \\ &= \int_0^{\pi} |\sin x| \; dx + \int_{\pi}^{2\pi} |\sin x| \; dx \\ &= \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx \\ &= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\ &= \cos 0 - \cos \pi + \cos 2\pi - \cos \pi \\ &= 4 \end{aligned}$$

Example 1 Find the area A between the graph of $y = f(x) = (x-2)^2$ and $g(x) = 2x - 1$ from $x = a = 0$ to $x = b = 8$.



Answer

$$\begin{aligned} f(x) - g(x) &= x^2 - 4x + 4 - 2x + 1 \\ &= x^2 - 6x + 5 \\ &= (x-5)(x-1) \end{aligned}$$

$$|\,0\,| - - (+) - - |\,1\,| - - (-) - - |\,5\,| - - (+) - - |\,8\,|$$

Hence,

$$\begin{aligned} A &= \int_0^8 |\, f(x) - g(x) \,|\ dx \\ &= \int_0^8 |\, x^2 - 6x + 5 \,|\ dx \\ &= \int_0^1 x^2 - 6x + 5 \,|\ dx - \int_1^5 x^2 - 6x + 5 \,|\ dx + \int_5^8 x^2 - 6x + 5 \,|\ dx \\ &= 40 \end{aligned}$$

Remark Area A between curves $x = f(y)$ and $x = g(y)$, from $y = a$ to $y = b$ can be defined similarly:

$$A = \int_a^b |\, f(y) - g(y) \,|\ dy$$

10 Substitution Method

Theorem 6 (The Substitution Rule) *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, Then*

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Proof Since f is continuous, by FTC1 it has an antiderivative F. Let $f(x) = F'(x)$

$$\begin{aligned} \int f(g(x))g'(x)\,dx &= \int F'(g(x))g'(x)\,dx \\ &= \int (F \circ g)'(x)\,dx \\ &= (F \circ g)(x) + C \\ &= F(g(x)) + C \\ &= F(u) + C \\ &= \int F'(u)du = \int f(u)du \end{aligned}$$

Example 1 What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}\,dx$?

Answer Let $u = \sqrt{x}$, $du/dx = 1/2\sqrt{x}$.

$$\int \frac{1}{\sqrt{x}} \cos\sqrt{x}\,dx = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$$

Hence, $F(x) = 2\sin\sqrt{x}$

Example 2 Find $\int \sin^3 x\, dx$
Let $u = \cos x$, $dx = \frac{-1}{\sin x} du$

$$\begin{aligned}
\int \sin^3 x\, dx &= \int (1-\cos^2 x)\sin x\, dx \\
&= \int (1-\cos^2 x)\sin x\, dx \\
&= \int -(1-u^2)\, du \\
&= \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3(x) - \cos(x) + C
\end{aligned}$$

or, we may write $d(g(x))$ instead of du if $u = g(x)$

$$\begin{aligned}
\int \sin^3 x\, dx &= \int (1-\cos^2 x)\sin x\, dx \\
&= \int (1-\cos^2 x)\sin x\, dx \\
&= \int (\cos^2 -1)(-\sin x) dx \\
&= \int (\cos^2 -1) d(\cos x) \\
&= \frac{1}{3}\cos^3(x) - \cos(x) + C
\end{aligned}$$

Example 3 Find $\int x\sqrt{2x+1}\, dx$
Let $u = 2x+1 \Rightarrow x = 1/2(u-1)$. Then $\frac{du}{dx} = 2 \Rightarrow dx = 1/2du$

$$\begin{aligned}
\int x\sqrt{2x+1}\, dx &= \int \frac{u-1}{2}\sqrt{u}\,\frac{1}{2}dx \\
&= \frac{1}{4}\int u\sqrt{u} - \sqrt{u}\, du \\
&= \frac{1}{4}(\frac{2}{5}u^{\frac{5}{2}}) - \frac{2}{3}u^{\frac{3}{2}}) + C \\
&= \frac{1}{4}(\frac{2}{5}(2x+1)^{\frac{5}{2}}) - \frac{2}{3}(@x+1)^{\frac{3}{2}}) + C
\end{aligned}$$

Remark If f is continuous on an interval I and $F' = f$ on I, then

$$\int f(Ax+B)\, dx = \int f(u)\frac{1}{A}\, du = \frac{1}{A}F(u) + C = \frac{1}{A}F(Ax+B) + C$$

Example $\int \sec^2(5x+1)\, dx = \frac{1}{5}\tan(5x+1) + C$

Theorem 7 (Substitution in Definite Integrals) *If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then*

$$\int_a^b f(g(x))\cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof Let F be an antiderivative of f on range(g). Then

$$\begin{aligned}\int_{g(a)}^{g(b)} f(u)\,du &= F(g(b)) - F(g(a)) \\ &= (F\circ g)(b) - (F\circ g)(a) \\ &= \int_a^b (F\circ g)'(x)\,dx \\ &= \int_a^b F'(g(x))g'(x)\,dx\end{aligned}$$

Example $I = \int_{-1}^{1} 3x^2\sqrt{x^2+1}\,dx$. Find I

Method 1 Let $u = x^3+1$, $du = 3x^2dx$. $x=-1 \Rightarrow u=0$ and $x=1 \Rightarrow u=2$

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \int_{u=0}^{u=2} \sqrt{u}\,du = \frac{4}{3}\sqrt{2}$$

Method 2 Find the antiderivative first:

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\,d(x^3+1) = \frac{2}{3}(x^3+1)^{\frac{3}{2}} + C$$

Apply FTC2:

$$\int_{x=-1}^{x=1} 3x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{3}(x^3+1)^{\frac{3}{2}}\right]_{-1}^{1} = \frac{2}{3}\left(\sqrt{8}-0\right) = \frac{4}{3}\sqrt{2}$$

11 Even and Odd Function

Definition A function $f : D \to \mathbb{R}$ is called

- an even function, if $f(x) = f(-x)$ for all $x \in D$
- an odd function, if $f(x) = -f(-x)$ for all $x \in D$

Theorem 8 (Integrals of Symmetric Functions) *Let $f : [-a,a] \to \mathbb{R}$ be an integrable function*

- *if f is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$*
- *if f is an odd function, $\int_{-a}^{a} f(x)\,dx = 0$*

Proof If f is even, then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \\ &= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx\end{aligned}$$

Let $u=-x$, $dx = -du$

$$\begin{aligned}-\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx &= -\int_0^a f(-u)\ -du + \int_0^a f(x)\,dx \\ &= \int_0^a f(-u)\,du + \int_0^a f(x)\,dx \\ &= \int_0^a f(u)\,du + \int_0^a f(x)\,dx \qquad \text{since } f(-u) = f(u) \\ &= 2\int_0^a f(x)\,dx\end{aligned}$$

If f is odd, then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\\ &= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx\end{aligned}$$

Let $u = -x$, $dx = -du$

$$\begin{aligned}-\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx &= -\int_{0}^{a} f(-u)\ -du + \int_{0}^{a} f(x)\,dx\\ &= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx\\ &= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad \text{since } f(-u) = -f(u)\\ &= 0\end{aligned}$$

Example 1 Show that $I = \int_{-\sqrt{2}}^{\sqrt{2}} (15x^4 - 4x^3 + 6x^2 + 7x)\,dx = 32\sqrt{2}$

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} (15x^4 + 6x^2)\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} (-4x^3 + 7x)\,dx$$

Since $15x^4 + 6x^2$ is an even function and $-4x^3 + 7x$ is an odd function, then

$$I = 2\int_{0}^{\sqrt{2}} (15x^4 + 6x^2)\,dx = 32\sqrt{2}$$

Example 2 Show that $I = \int_{-1}^{3} (x+1)^2(x-3)^2\,dx = 512/15$
First we will show that the function is symmetrical about $x = 1$

$$\begin{gathered}f(x_0 + \delta) = f(x_0 - \delta) \qquad \forall\delta > 0\\ (x_0 + \delta + 1)^2(x_0 + \delta - 3)^2 = (x_0 - \delta + 1)^2(x_0 - \delta - 3)^2\end{gathered}$$

Let $\delta = 1$

$$\begin{gathered}(x_0 + 2)^2(x_0 - 2)^2 = (x_0)^2(x_0 - 4)^2\\ x_0 = 1\end{gathered}$$

Let $u = x - 1$, then $du = dx$

$$\begin{aligned}I &= \int_{-2}^{2} (u+2)^2(u-2)^2\,du\\ &= \int_{-2}^{2} ((u+2)(u-2))^2\,du\\ &= \int_{-2}^{2} (u^2 - 4)^2\,du\\ &= \int_{-2}^{2} (u^4 - 8u^2 + 16)\,du\\ &= 2\int_{0}^{2} (u^4 - 8u^2 + 16)\,du\\ &= 2\left[\frac{1}{5}u^5 - \frac{8}{3}u^3 + 16u\right]_0^2 = \frac{512}{15}\end{aligned}$$