MAT1001 Calculus I

Lecture 2 - 3: Limits

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1 Limits

1.1 Definition

Definition Let $f : D \to \mathbb{R}$ be a function "near" $x = c$, we write

$$\lim_{x \to c} f(x) = L$$

if $f(x)$ can get arbitrarily close to L for all x close enough to c.

Limits Existence $\lim_{x \to c} f(x)$ exists if and only if:

1. $\lim_{x \to c^+} f(x)$ exists
2. $\lim_{x \to c^-} f(x)$ exists
3. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$, such that $L \in \mathbb{R}$

In other words, as x approaches c from the left, the value of $f(x)$ is approaching the same value of $L \in \mathbb{R}$ that $f(x)$ is approaching when x approaches c from the right. Note that that $\pm\infty$ do not count as a limit here (because they are not real numbers).

Delta-Epsilon Definition Let $f : D \to \mathbb{R}$ be a function defined on an open iterval containing c, except possibly at c itself. Let $L \in \mathbb{R}$ so $L \neq \pm\infty$, then we write:

$$\lim_{x \to c} f(x) = L$$

$$\text{If, for all } \epsilon > 0 \text{, there exists } \delta > 0$$
$$\text{such that for all } x \in D \text{ , we have } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Example Use the formal definition of limits to show that

$$\lim_{x \to 2} x^2 = 4$$

Let $\epsilon > 0$. We want find $\delta > 0$ such that $|x^2 - 4| < \epsilon$ given that $0 < |x - 2| < \delta$ for all $x \in D$.
Consider

$$|x^2 - 4| < \epsilon \to |x - 2| \cdot |x + 2| < \epsilon \to |x - 2| < \frac{\epsilon}{|x + 2|}$$

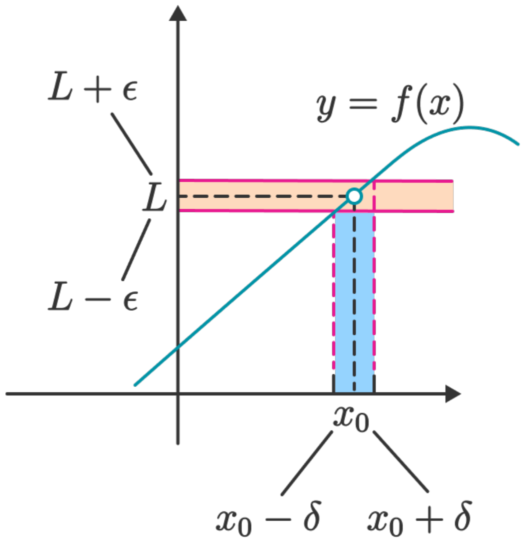


Figure 1: Illustration of the precise definition of limit

Note that ϵ is a very small number, so δ is also very small. We can assume that $\delta < 1$, then $|x-2| < 1 \to 1 < x < 3$
We have

$$\frac{1}{5} < \frac{1}{|x-2|} < \frac{1}{3}$$
$$\frac{1}{5} < \frac{1}{|x-2|}$$
$$\frac{\epsilon}{5} < \frac{\epsilon}{|x-2|}$$

If we assume that $\delta \leq \frac{\epsilon}{5}$ and $|x-2| < \delta$:

$$|x-2| < \delta$$
$$|x-2| < \frac{\epsilon}{5}$$
$$|x-2| \cdot |x+2| < |x+2| \cdot \frac{\epsilon}{5} < |x+2| \cdot \frac{\epsilon}{|x+2|} = \epsilon$$

Hence, given $\epsilon > 0$, set $\delta \leq \epsilon/5$. Then $|x-2| < \delta$ implies that $|x^2-4| < \epsilon$ for all $x \in D$. This shows that $\lim_{x\to 2} x^2 = 4$

Note that the limit of $f(x)$ as x approaching c has no relation with the value of f at c. In fact, even if $f(c)$ is undefined, f may still have a limit at c. We can write

$$\lim_{x\to c} f(x) \neq f(c)$$

1.2 Properties of limit

Theorem 1 (Properties of Limits) *Let L, M, c, $k \in \mathbb{R}$, and f and g be functions such that*

$$\lim_{x\to c} f(x) = L \textit{ and } \lim_{x\to c} g(x) = M$$

Then:

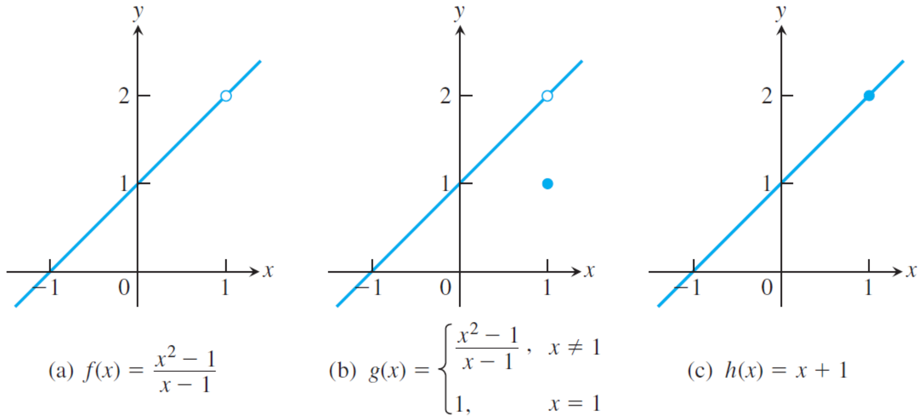


Figure 2: Limit of $f(x)$, $g(x)$, and $h(x)$ all equal to 2 as x approaches 1, but only $h(x)$ have $h(1) = 2$

- $\lim_{x\to c}(f(x) \pm g(x)) = L + M$
- $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, *provided that* $M \neq 0$.
- $\lim_{x\to c} k.f(x) = k.L$
- $\lim_{x\to c}[f(x)]^n = L^n$, n *is a positive integer*
- $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n *is a positive integer,*
  If n *is even then we assume that* f *is positive in some open interval containing* c.

Theorem 2 (Limits of Polynomials) *If* $P(x)$ *is a polynomial function, that is if there exists real numbers* a_1, ..., a_n *such that*

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

then

$$\lim_{x\to c} P(x) = p(c) \text{ for any polynomial function } p.$$

Theorem 3 (Limits of Rational Functions) *If* p *and* q *are polynomial functions on* $\mathbb{R}$*, then for any* $c \in \mathbb{R}$ *such that* $q(c) \neq 0$*, then*

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

1.3 One-sided Limits

If f is defined on some interval $(c, c+a)$, then we have a **right-hand limit**:

$$\lim_{x\to c^+} f(x)$$

We say "limit of $f(x)$ as x approaches c from the right".
If f is defined on some interval $(c-a, c)$, then we have a **left-hand limit**:

$$\lim_{x\to c^-} f(x)$$

We say "limit of $f(x)$ as x approaches c from the left".

Theorem 4 (Equalled One-Sided Limits) *Let $c \in R$ and $L \in R$, and let f be a function defined on some open interval D containing c, except possibly at c. Then*

$$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

2 Continuity

2.1 Definition

Definition Let $f : D \to \mathbb{R}$ be a function defined on an open interval containing c, we say f is continuous at c if

$$\lim_{x\to c} f(x) = f(c)$$

Precise definition: Let $f : D \to \mathbb{R}$ be a function, where D is an interval, and let $c \in D$. Then f is said to be continuous at c if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \text{ for all } x \in D \text{ satisfying } |x - c| < \delta$$

If f is not continuous at $c \in D$, then we say that f is discontinuous at c. We say that f is continuous on D if it is continuous at every point in D.

Functions that are continuous at every $c \in \mathbb{R}$ are:

- Constant function: $f(x) = k$, where k is constant, $D \in \mathbb{R}$.
- Identity function: $f(x) = x$, $D \in \mathbb{R}$.
- Absolute value function: $f(x) = |x|$, $D \in \mathbb{R}$.
- Natural exponential function: $f(x) = e^x$, $D \in \mathbb{R}$.
- Natural logarithmic function: $f(x) = \ln x$, $D = \{x \in \mathbb{R} : x > 0\}$.
- Basic trigonometric functions: $f(x) = \sin x$, $g(x) = \cos x$, $D \in \mathbb{R}$.

2.2 One-sided Continuity

Definition Let $f : D \to \mathbb{R}$ be a function, where D is an interval, and let $c \in D$. We say that f is **right-continuous** at c if $\lim_{x\to c^+} f(x) = f(c)$. We say that f is **left-continuous** at c if $\lim_{x\to c^-} f(x) = f(c)$.

From the definition of one-sided continuity, we can define that a function $f : [a,b] \to \mathbb{R}$ is continuous if f is continuous at every $c \in (a,b)$, left-continuous at b, and right-continuous at a.

2.3 Properties

Theorem 5 (Combinations of Continuous Functions) *Let f and g be functions. If both f and g are continuous at a point c, then so are the functions $f + g$, $f - g$, $f \cdot g$ and f/g. (For f/g, we assume that $g(c) \neq 0$.)*

Theorem 6 (Limit of Continuous Functions) *Let f and g be functions, such that $g \circ f$ is defined. Suppose that g is continuous at a point b and $\lim_{x\to c} f(x) = b$. Then*

$$\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x)) = g(b)$$

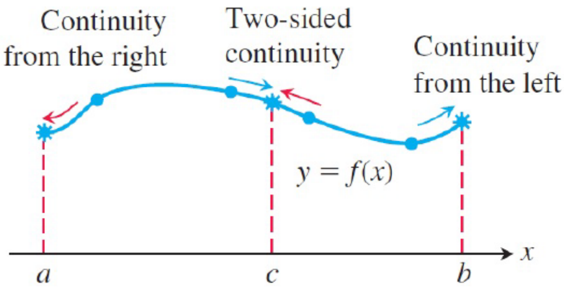


Figure 3: Continuity at point a, b, and c

Example

$$g(x) = \sin x$$
$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Since g is continuous at every $c \in \mathbb{R}$ and we have $\lim_{x\to 0} f(x) = 0$, then

$$\lim_{x\to 1} g \circ f = g(\lim_{x\to 1} f(x)) = g(0) = \sin 0 = 0$$

Theorem 7 (Compositions of Continuous Functions) *Let f and g be functions, such that $g \circ f$ is defined. Suppose that f is continuous at c, and g is continuous at $f(c)$. Then $g \circ f$ is continuous at c.*

Proof of Theorem 7 To proof theorem 7, we can use the formal definition of limits, that is for any given $\epsilon = 0$, there exists $\delta > 0$ such that for all x with $|x - c| < \delta$ we have

$$|g(f(x)) - g(f(c))| < \epsilon$$

By definition of limits

$$\lim_{x\to c} g(f(x)) = g(f(c))$$

Therefore we have

$$\lim_{x\to c} g \circ f = g \circ f$$

Hence, by the definition of continuity, $g \circ f$ is continuous at c.

2.4 Discontinuity

Let $f : D \to \mathbb{R}$ be a function, where D is an interval containing c. Suppose that f is not continuous at c. If c is an interior point of D, then one of the following is true:

i The function f has a **removable discontinuity** at c, which means that $\lim_{x\to c} f(x) = L$ for some $L \in R$, but $L \neq f(c)$. It can be made continuous at c by redefining $f(c)$ to be L.

ii The function f has a **jump discontinuity** at c, which means that both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^-} f(x)$ exists, but not equal.

iii The function f has a **essential discontinuity** at c, which means that at least one of both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^-} f(x)$ doesn't exist.

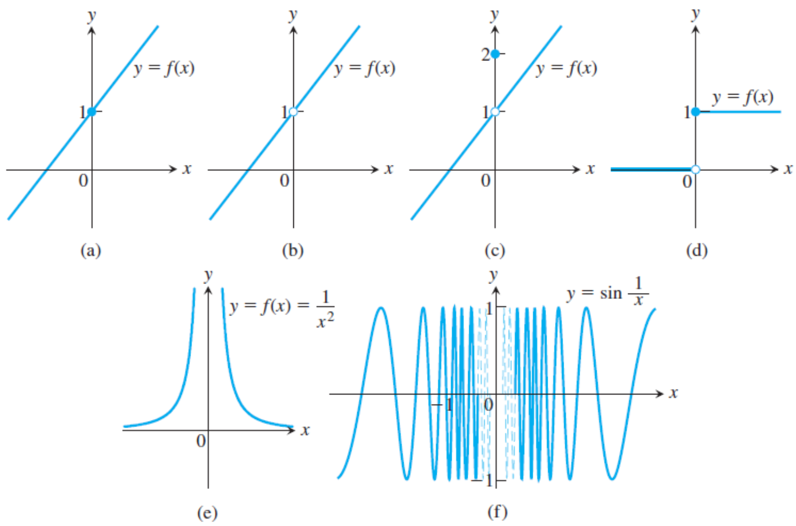


Figure 4: (a) is continous at 0, (b) and (c) have a removable discontinuity, (d) has a jump discontinuity, (e) and (f) has an essential discontinuity

If c is the left endpoint of D, then one of the following is true:

i The function f has a **jump discontinuity** at c, which means that $\lim_{x\to c^+} f(x) = L$ but $L \neq f(c)$. It can be made continuous at c by redefining $f(c)$ to be L.

ii The function f has a **essential discontinuity** at c, which means that $\lim_{x\to c^+} f(x)$ doesn't exist.

The discontinuity is defined similarly if c is the left endpoint of D.

3 Sandwich Theorem/Squeeze Theorem

Theorem 8 (The Sandwich Theorem) *Suppose that g(x) $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and suppose we have that*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L$$

Example

$$\lim_{x\to 0} x^2 \sin\frac{1}{x}$$

Note that $-1 \leq \sin x \leq 1$ for all x with $x \neq 0$. So,

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \text{ for } x \neq 0$$

Since

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$$

by sandwich theorem we have

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

Theorem 9 *If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then*

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

4 Intermediate Value Theorem (IVT)

Theorem 10 (Intermediate Value Theorem) *If f is a continuous function on a closed interval $[a, b]$ and if y_0 is any value between $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some c in $[a, b]$*

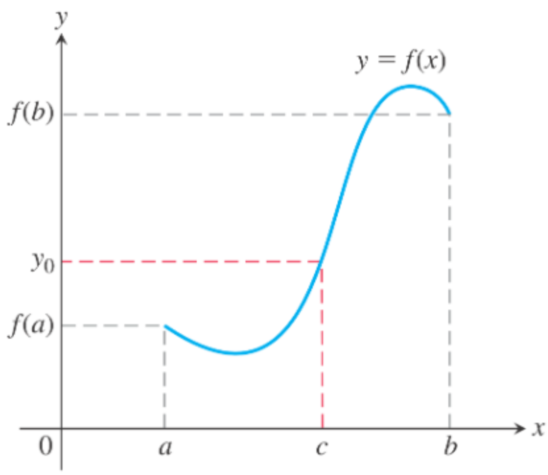


Figure 5: Intermediate Value Theorem

Example Show that $x^3 - x = 7$ has a solution

To find whether $x^3 - x = 7$ has a solution, it is suffice to show that $x^3 - x - 7 = 0$

Let $f(x) = x^3 - x - 7$, which is a polynomial function and is continuous for all $x \in \mathbb{R}$.

Take two arbitrary point, that is $a = 0$ and $b = 3$. Since $f(0) = -7$ and $f(3) = 17$, $-7 < 0 < 17$, by IVT $\exists\, x_0 \in (0, 3)$ such that $f(x_0) = 0$.

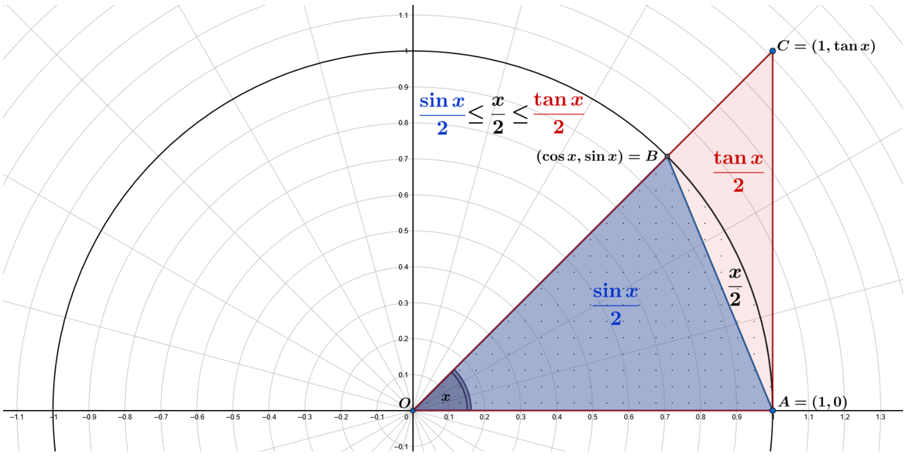
Hence, $x_0^3 - x_0 = 7$ has a solution.

5 Special Limits (Trigonometry)

There are several special limits in trigonometric function. To solve limits problem for trigonometric function, we usually use there special limits (try to change the form of the function using these special limits).

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

To prove this form of limit, we use geometric approach. First we draw a unit circle and try to use sandwich theorem to proof the limit.



$$
\begin{aligned}
\text{Area of } \triangle OAB &= \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot \sin x \\
\text{Area of } \triangledown OAB &= \frac{x}{2\pi} \cdot \pi \cdot r^2 = \frac{x}{2} \\
\text{Area of } \triangle OAC &= \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot \tan x
\end{aligned}
$$

From the figure we can see that $\triangle OAB \le \triangledown OAB \le \triangle OAC$ Therefore,

$$
\begin{gathered}
\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2} \\
\sin x \le x \le \tan x \\
1 \le \frac{x}{\sin x} \le \frac{1}{\cos x} \\
1 \ge \frac{\sin x}{x} \ge \cos x
\end{gathered}
$$

Since

$$
\lim_{x \to 0} 1 = 1 \qquad \text{and} \qquad \lim_{x \to 0} \cos x = 1
$$

By sandwich theorem, $\lim_{x \to 0} \frac{\sin x}{x} = 1$

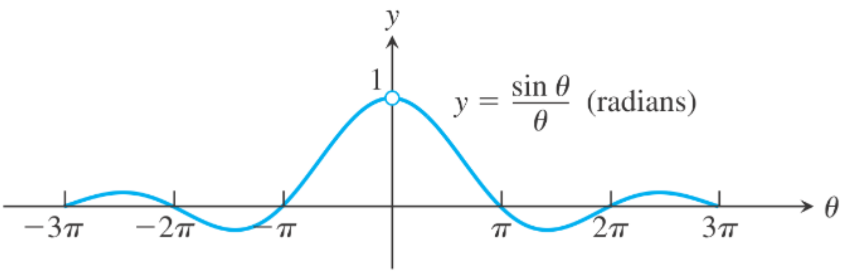


Figure 6: Graph of $\frac{\sin x}{x}$

6 Limits Involving Bounded Functions

Definition A function f is said to be bounded on S if $\exists\, M \in \mathbb{R}$ such that $f \leq M\ \forall x \in S$.

Example of bounded function is $\sin x$ and $\cos x$ in its domain $D \in \mathbb{R}$ because for all $x \in D$, $f(x) \in [-1, 1]$.

Theorem 11 (Limits Involving Bounded Function) *Suppose that f and g are functions defined on some open interval D containing c (except possibly at c). If $lim_{x\to c} f(x) = 0$ and $g(x)$ is bounded on $D \smallsetminus \{c\}$, then*

$$\lim_{x\to c} f(x)g(x) = 0$$

Proof on Theorem 11 g is bounded on $D \smallsetminus \{c\}$, So

$$\begin{aligned} &\Rightarrow \exists\, M \in \mathbb{R} : |\, g(x) \,|\leq M, && \forall x \in D \smallsetminus \{c\} \\ &\Rightarrow 0 \leq |\, f(x) \,||\, g(x) \,| \leq M \,|\, f(x) \,|, && \forall x \in D \smallsetminus \{c\} \end{aligned}$$

Since

$$\lim_{x\to c} M|f(x)| = M \lim_{x\to c} |f(x)| = 0$$

by sandwich theorem

$$\lim_{x\to c} |f(x)||g(x)| = 0$$

Since

$$-|f(x)||g(x)| \leq f(x)g(x) \leq |f(x)||g(x)|, \forall x \in D \smallsetminus \{c\}$$

by sandwich theorem

$$\lim_{x\to c} f(x)g(x) = 0$$

Example $|\cos x\,| \leq 1\ \forall x \in \mathbb{R}$. Find $\lim_{x\to 5}(x^3 - 25x)\cos(\log|x-5|)$

Since $\cos(\log|x-5|)$ is bounded for all $x \in \mathbb{R} \smallsetminus \{5\}$
and $\lim_{x\to 5}(x^3 - 25x) = 0$, then by theorem 11

$$\lim_{x\to 5}(x^3 - 25x)\cos(\log|x-5|) = 0$$

7 Limits Involving Infinity

Infinity (∞) does not represent a real number. However, we use ∞ to describe the behavior of a function when the value in its domain / range outgrows all finite bound that is can't be represented by a single real value.

7.1 Finite Limits as $x \to \pm\infty$

Definition First, we say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

Second, we say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

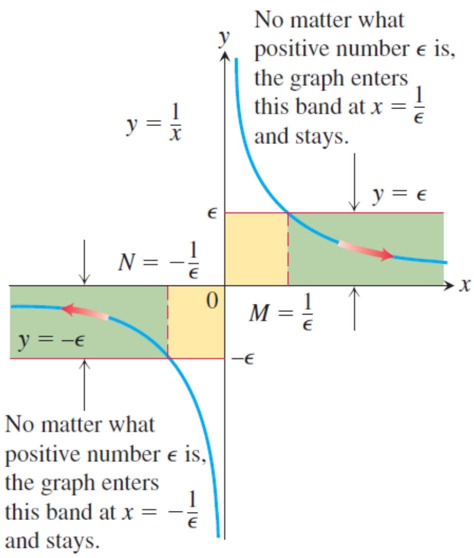


Figure 7: Geometry illustration of the limit on infinity definition

Example 1 Show that $\lim_{x\to\infty} \frac{1}{x} = 0$ Let $\epsilon > 0$ be given and let $M = \frac{1}{\epsilon}$.

If $x > M = \frac{1}{\epsilon}$, then

$$\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

Hence, by definition

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Example 2 Limit at infinity of rational function
Find

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

To solve this, we can use theorem 3 for rational function and combine it with the general properties of limit in theorem 1. First, we divide the denominator and the numerator with the biggest degree of x, that is in this case x^2

$$= \lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$$

$$= \frac{5 + 8(0) + 3(0)}{3 + 2(0)}$$

$$= \frac{5}{3}$$

Geometric Representation: Definition If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an **asymptote** of the graph.

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \textit{or} \qquad \lim_{x\to-\infty} f(x) = b$$

Example Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 + 2}{|x|^3 + 1}$$

For all $x \geq 0$:

$$\lim_{x\to\infty} \frac{x^3 + 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 + 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 + \frac{2}{x^3}}{1 + \frac{1}{x^3}} = 1$$

For all $x < 0$:

$$\lim_{x\to\infty} \frac{x^3 + 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 + 2}{(-x)^3 + 1} = \lim_{x\to\infty} \frac{1 + \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = -1$$

Hence, there are two horizontal asymtote for $f(x)$, that is

$$y = -1 \text{ and } y = 1$$

We can verify this with the graph shown below.

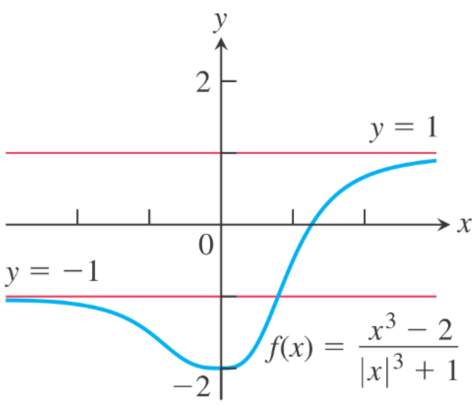
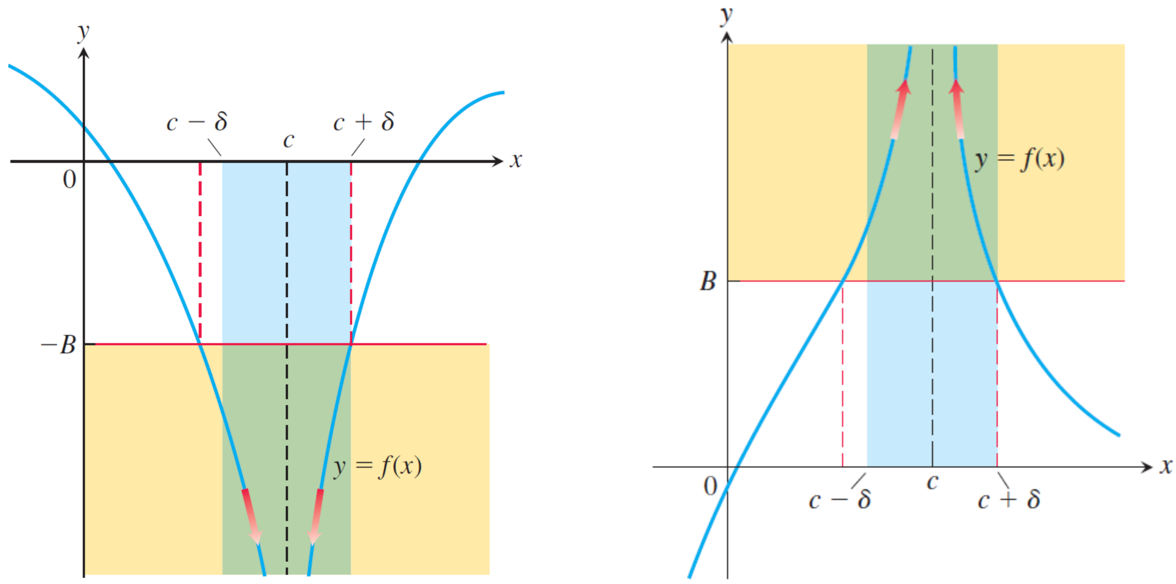


Figure 8: Graph of $\frac{x^3+2}{|x|^3+1}$

7.2 Infinite Limits



(a) Limit approaches negative infinity

(b) Limit approaches positive infinity

Figure 9: Infinite Limits

Definition First, We say that $f(x)$ approaches infinity as x approaches c, and write

$$\lim_{x \to c} f(x) = \infty$$

if, for every number positive real number B , there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \qquad \Rightarrow \qquad f(x) > B$$

Second, We say that $f(x)$ approaches minus infinity as x approaches c, and write

$$\lim_{x \to c} f(x) = -\infty$$

if, for every number negative real number $-B$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \qquad \Rightarrow \qquad f(x) < -B$$

Note Note that, when we say a limit $= \pm\infty$ limit laws still axis, that is the limit exists if the left-side limit = right-side limit.

Example 1 Find

$$\lim_{x \to 2} \frac{x-3}{x^2-4}$$

$$\lim_{x \to 2^-} \frac{x-3}{x^2-4} = \frac{-}{-} = +\infty$$
$$\lim_{x \to 2^+} \frac{x-3}{x^2-4} = \frac{-}{+} = -\infty$$

Since $\lim_{x \to 2^-} \frac{x-3}{x^2-4} \neq \lim_{x \to 2^+} \frac{x-3}{x^2-4}$, the limit does not exist.

Example 2 Find

$$\lim_{x\to-\infty} \frac{2x^5-6x^4+1}{3x^2+x-7}$$

$$\begin{aligned}\lim_{x\to-\infty} \frac{2x^5-6x^4+1}{3x^2+x-7} &= \lim_{x\to-\infty} \frac{2x^5-6x^4+1}{3x^2+x-7}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to-\infty} \frac{2x^3-6x^2+\frac{1}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}\\ &= \frac{1}{3}\lim_{x\to-\infty} 2x^2(x-3)\\ &= -\infty\end{aligned}$$

Geometric Representation: Definition While limit at infinity is related to horizontal asymtote, infinite limit is related to vertical asymptote.

A line $x=a$ is a **vertical asymptote** of the graph of a function $y=f(x)$ if either

$$\lim_{x\to a^+} f(x)=\pm\infty \qquad \textit{or} \qquad \lim_{x\to a^-} f(x)=\pm\infty$$

7.3 Dominant Terms

Definition When considering functions made up of the sums, differences, products or quotients of different sorts of functions (polynomials, exponentials and logarithms), or different powers of the same sort of function we say that one function dominates the other and is called **dominant terms**.

Dominant terms is useful for us to predict the function's behavior in certain condition of x. In polynomial functions, the dominant term is the function that has the highest degree. For example. $f(x)=3x^4+x^3-2x^2+7x-5$ has $3x^4$ as the dominant term. In rational functions we can find the dominant term by modify the form of the function as polynomial/quadratic/linear function plus the reminder. For example:

$$f(x)=\frac{x^2+3}{2x-4}=\left(\frac{x}{2}+1\right)+\left(\frac{1}{2x+4}\right)$$

For $|x|$ large, the dominant term is $x/2+1$ and for x close to 2, the dominant term is $1/2x+4$

7.4 Oblique Asymptote

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an oblique or slant line asymptote.

Definition The line given by $y=Ax+B$ is called an oblique asymptote of the graph of a function $y=f(x)$ if

$$\lim_{x\to\infty}(f(x)-Ax-B)=0 \qquad \text{or} \qquad \lim_{x\to-\infty}(f(x)-Ax-B)=0$$

Suppose that $y = f(x)$ has an oblique asymptote $y = Ax+B$ as $x \to \infty$ then $f(x)-Ax-B \to 0$ as $x \to \infty$. So

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x) - Ax - B}{x} &= 0 \\
\lim_{x\to\infty} \frac{f(x)}{x} - \lim_{x\to\infty} \frac{Ax}{x} + \lim_{x\to\infty} \frac{B}{x} &= 0 \\
A = \lim_{x\to\infty} \frac{f(x)}{x}
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lim_{x\to\infty} (f(x) - Ax - B) &= 0 \\
\lim_{x\to\infty} f(x) - Ax - \lim_{x\to\infty} B &= 0 \\
B = \lim_{x\to\infty} f(x) - Ax
\end{aligned}$$

Hence,

$$A = \lim_{x\to\infty} \frac{f(x)}{x} \qquad \text{and} \qquad B = \lim_{x\to\infty} f(x) - Ax$$

Example Find the oblique asymptote of

$$y = \frac{x^2 - 3}{2x - 4}$$

Let $y = Ax + B$ as the oblique asymptote of y, then we have

$$\begin{aligned}
A &= \lim_{x\to\infty} \frac{f(x)}{x} \\
&= \lim_{x\to\infty} \frac{x - \frac{3}{x}}{2x - 4} \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{1 - \frac{3}{x^2}}{2 - \frac{4}{x}} \\
&= \frac{1}{2}
\end{aligned}$$

Then we calculate B

$$\begin{aligned}
B &= \lim_{x\to\infty} f(x) - Ax \\
&= \lim_{x\to\infty} \frac{x^2 - 3}{2x - 4} \cdot \frac{x}{2} \\
&= \lim_{x\to\infty} \frac{x^2 - 3 - \frac{x}{2}(2x - 4)}{2x - 4} \qquad = \lim_{x\to\infty} \frac{2x - 3}{2x - 4} \\
&= 1
\end{aligned}$$

Hence, the oblique asymptote is $x/2 + 1$

We can also find the oblique asymptote by changing the form of the function to linear function plus the reminder using long division.

$$\begin{aligned} y &= \frac{x^2-3}{2x-4} \\ &= \left(\frac{x}{2}+1\right)+\left(\frac{1}{2x+4}\right) \end{aligned}$$

As $x \to \pm\infty$ the value of $f(x)$ will resemble the dominant term of the linear function $\left(\frac{x}{2}+1\right)$

8 Evaluating Limits

Methods of Solving Limits To solve limits, there are several methods we can use, such as:

Substitution Substitute c into x directly. This applies if $f(x)$ is continuous at c. For example:

$$\lim_{x\to 2} x^2 = 2^2 = 4$$

Factorization This method is used to solve limit of rational functions, that is in the form of $p(x)/q(x)$ with $p(x)$ and $q(x)$ are polynomials. First factorize the polynomials, then eliminate common factors from zero denominators. For example:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4$$

Multiply with conjugate root This method is used to solve limit of rational functions that have root function. Multiply the form with the conjugate of the root so that we can get a form where we can eliminate common factors from zero denominators. For example:

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0}\frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)} \\ &= \lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+100}+10)} \\ &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10} \\ &= \frac{1}{20} \end{aligned}$$

Using sandwich theorem This method is used to solve limit with a special condition that satisfy the sandwich theorem For example:

$$\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor$$

Let t $= 1/x$ so that

$$\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor = \lim_{t\to\infty}\frac{1}{t}\lfloor t\rfloor$$

Since, $\lfloor t\rfloor$ is a floor function that by definition:

$$t-1 \le \lfloor t\rfloor \le t$$

which gives,

$$1-\frac{1}{t} \le \frac{\lfloor t\rfloor}{t} \le 1$$

Since

$$\lim_{t\to\infty} 1 - \frac{1}{t} = 1 \text{ and } \lim_{t\to\infty} 1 = 1$$

by sandwich theorem, $\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor = 1$

Substitute the variable When we do substitution in a limit, we have to change all appearances of the original variable, which also includes the place under "lim". There we simply change the name of the variable, but the limit point itself must change according to the basic substitution equality.
For example:

$$\lim_{x\to\infty} x \sin\frac{1}{x}$$

Let $y = \frac{1}{x}$, so $y \to 0^+$ as $x \to \infty$

$$\begin{aligned}\lim_{x\to\infty} x\sin\frac{1}{x} &= \lim_{y\to 0^+} \frac{1}{y}\sin y \\ &= \lim_{y\to 0^+} \frac{\sin y}{y} \\ &= 1\end{aligned}$$

Divide with highest degree denominator This method can be used for limit as $x \to \pm\infty$. We can divide the numerator and denominator with the highest degree denominator and we can see the function's behavior as x approaches $\pm\infty$. For example,

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3} &= \lim_{x\to\infty} \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x\to\infty} \frac{3x^4 + \frac{5}{x} - \frac{1}{x^3}}{6 - \frac{7}{x^2} + \frac{3}{x^3}} \\ &= \infty\end{aligned}$$