MAT1001 Calculus I

Lecture 1 : Introduction to Calculus

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1 Introduction

What is calculus? Calculus is the study how things change. It provides us a toll to model changes that happen around us.

2 Mathematical Background: Set

2.1 Definition

A set is a collection of distinct objects; these objects are called elements of the set. Some definition and notation of sets:

- $x \in S$ means x is an element of the set S"; we say that x belongs to ,S or simply x is in S.
- $x \notin S$ means x is not an element of S".
- $S \subseteq T$ means every element of S is an element of T"; we say that S is a subset of T, or that S is contained in T.
- $S = T$ means that $S \subseteq T$ and $T \subseteq S$ both hold.
- $S \subsetneq T$ means that $S \subseteq T$ and $S \neq T$ (we say that S is a proper subset of T).
- $\emptyset$ denotes the set containing no element (called the empty set).

We use $\{x \subseteq S : P(x)\}$ or $\{x \subseteq S \mid P(x)\}$ to denote the set of all elements x in S for which $P(x)$ is true. We may simply write $\{x : P(x)\}$ when S is obvious or not important. The following symbols are frequently used:

- $\mathbb{Z} := \{..., 0, 1, 2, ...\}$ is the set of integers.
- $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 2\} = \{..., 0, 1, 2, 3, ...\}$ is the set of natural numbers.
- $\mathbb{N}_+ := \{n \in \mathbb{Z} : n > 2\} = \{..., 1, 2, 3, ...\}$ is the set of positive integers.
- $\mathbb{Q} := \{m/n : m \in Z, n \in N_+\}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers.

Note We use the symbol := to mean that the symbol on the left is defined by the object on the right.

2.2 Set Operations

Let S and T be sets. We define:

- $S \cap T := \{x : x \in S \text{ and } x \in T\}$ (Intersection of S and T)
- $S \cup T := \{x : x \in S \text{ or } x \in T\}$ (Union of S and T)
- $S \setminus T := \{x : x \in S \text{ and } x \notin T\}$ (S minus T). It is also common to write $S - T$ instead of $S \setminus T$.

$\mathbb{R} \setminus \mathbb{Q}$, the set of all real numbers that are not rational, is called the set of irrational numbers. Examples of irrational numbers include $\pi, \sqrt{2},$ and e
If $S \cap T = \emptyset$, then we say that S and T are disjoint.

2.3 Cartesian Products

Definition Let S and T be nonempty sets. The Cartesian product of S and T, denoted by $S \times T$, is the set of all ordered pairs (x, y) such that $x \in S$ and $y \in T$. That is,

$$S \times T := \{(x, y) : x \in S, y \in T\}$$

More generally, if $S_1, ..., S_n$ are nonempty sets, the Cartesian product of $S_1, ..., S_n$ is defined to be

$$S_1 \times ... \times S_n := \{(x_1, ..., x_n) : x_1 \in S_1, ..., x_n \in S_n\}$$

An object of the form $(x_1, ..., x_n)$ is called an (ordered) n-tuple.
If $S = \{1, 2, 3\}$ and $T = \{4, 8\}$ then

$$S \times T = \{(1, 4), (1, 8), (2, 4), (2, 8), (3, 4), (3, 8)\}$$

We use the symbol S_n to mean $S \times \ ... \ \times S$ (with n copies of S). In particular, $\mathbb{R}^2$ is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$$

Similarly, R^n is the set of all ordered n-tuples of real numbers.

2.4 Interval

We use the following notations for intervals, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$:

- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ (Closed intervals)
- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ (Open intervals)
- $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ (Half-open or half-closed intervals)
- $(a, b]$ is defined similarly.

We also allow a or b to be ∞ or $-\infty$ for unbounded intervals. For example:

$$(-\infty, \infty) := \mathbb{R}$$

$$[0, \infty) := \{x \in \mathbb{R} : x \geq 0\}$$

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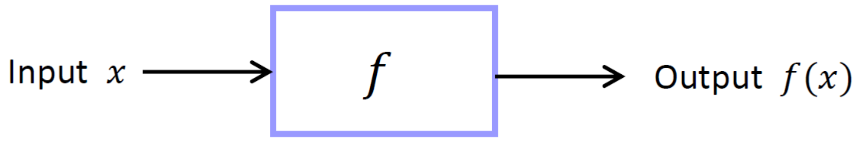

Figure 1: Function illustration

3 Function

3.1 Definition

A function is like a "machine" that takes input x which then go into a defined process resulted in output $f(x)$. A function consists of three objects, that is:

- A non empty set D, called the domain;
- A non empty set Y, called the codomain;
- A rule, f such that, for each element $x_0 \in D$, f assigns exactly one element in Y to x_0; this element is denoted by $f(x)$

Definition A function from D to Y is a rule that assigns to each element of D exactly one element of Y. We use the notation $f : D \to Y$ to mean that the function f is from D to Y, with $f(x)$ being the element of Y assigned to x. For a function $f : D \to Y$, the set D and Y are called the domain and the codomain of f, respectively, and the set $\{f(x) : x \in D\}$ is called the range of f.

Note If x and y are variables related by $y = f(x)$, x is called the independent variable and y is called the dependent variable

3.2 Piecewise Function

Definition A piecewise function is a function built from pieces of different functions over different intervals.
A function may be defined piecewisedly, for example,

$$f(x) = \begin{cases} 2x+1 & 0 \leq x \leq 3 \\ \sqrt{x+6} & 3 < x < 10 \end{cases}$$

$f(x)$ can be denoted as $f : [0, 10) \to \mathbb{R}$ with $[0, 10) = \{x : 0 \leq x < 10\}$

4 Rate of Change

4.1 Average Rate of Change

Definition For $y = f(x)$ the average rate of change of y with respect to x from $x = x_1$ to $x = x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + \epsilon) - f(x_1)}{\epsilon}$$

where $\epsilon = \Delta x = x_2 - x_1$

Geometric Representation Average rate of change is related to **secant line**. Secant line on a curve f is a line joining two points of the curve f. The slope of a secant line is the average rate of change between two point intersection of secant line and the curve.

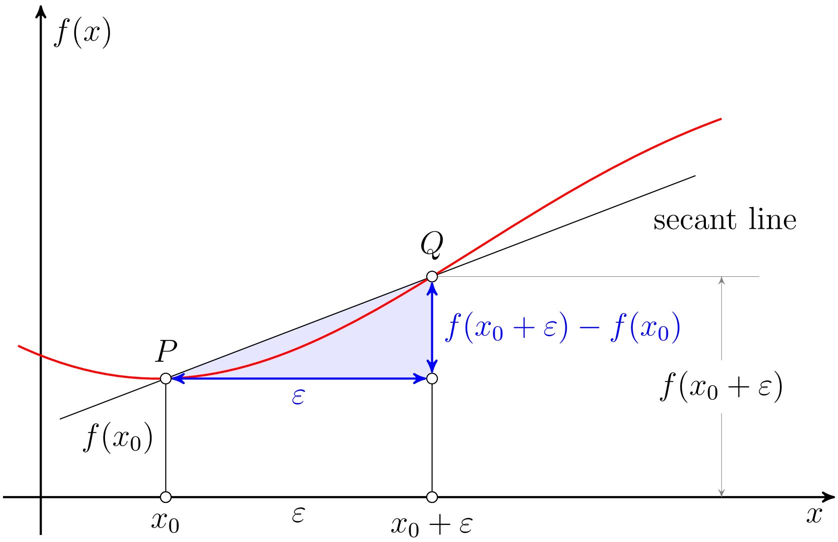

Figure 2: Illustration of secant line

The slope of the secant line through P and Q = $\Delta f(x)/\Delta x$ which is the average rate of change with respect to x over $[x_0, x_0+\epsilon]$

4.2 Instataneous Rate of Change

Definition The instantaeous rate of change is the change in the rate at a particular instant. We can find the of instantaneous rate of change at x_0 by using average of rate of change over $[x_0, x_1]$ with x_1 "very close" to x_0.
For example:

$$\text{Let } y = f(x) = t^2 \text{ , } x_0 = 1$$

x_1	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{f(x_1)-f(x_0)}{x_1-x_0}$	1.9	1.99	1.999	2.001	2.01	2.1

As x_1 approach $x_0 = 1$, the value of the average of change of y is getting closer to 2. Therefore, we can denote the instantaneous rate of change as:

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Geometric Representation While average rate of change is related to secant line, instantaneous rate of change is related to **tangent line**. Tangent line of a curve f at x_0 is a line that just "touches" the curve f at point x_0

Slope of a tangent line at point x_0 is the instantaneous rate of change at point x_0. In other words, it is the limit of the slope of the secant line as x approaches x_0.

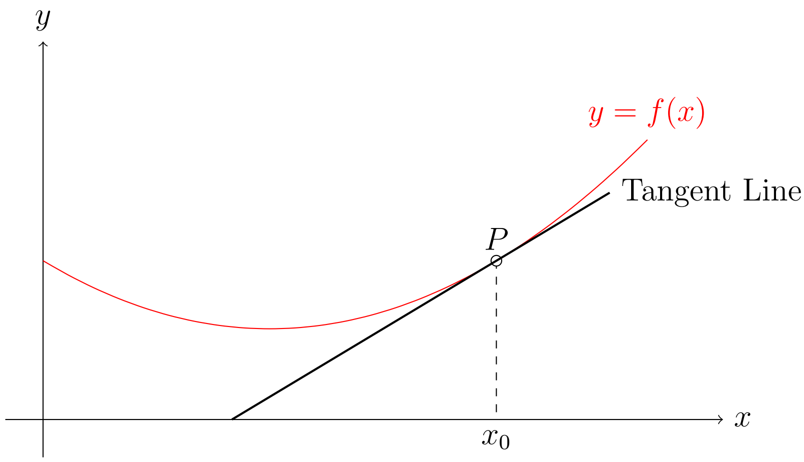

Figure 3: Illustration of tangent line

Example Let a function of the position of an object denoted as

$$f(x) = x^2$$

What is the tangent line at $x = 1$?

Let t be an arbitrarily point that is close to 1. Therefore, we have the slope of the secant line L_t

$$L_t = \frac{t^2 - 1^2}{t - 1}$$

To get the instantaneous rate of change or the slope of the tangent line, we take the limit of the slope of the secant line as t approaches 1, that is

$$\lim_{t \to 1} \frac{t^2 - 1}{t - 1} = \lim_{t \to 1} \frac{(t-1)(t+1)}{t - 1} = t + 1$$

Hence, we have the slope of the tangent line at $t = 1$ is $t + 1 = 2$.
The equation of the tangent line is

$$\frac{y - 1^2}{x - 1} = 2$$
$$y = 2x - 1$$

Note Average rate of change is the same slope of secant line, and instantaneous rate of change is the same slope of tangent line