MAT1001 Calculus I

Lecture 7 - 11 : Applications of Derivative

Lecture by Dr. Arjan Abeynaike

Scribe by Alfonsus Rodriques Rendy
Sep 30, 2021 - Oct 19, 2021

1 Extreme Values of Function

Definition Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c), \forall\, x \in D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c), \forall\, x \in D$$

Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are global maxima or minima.
Maxima and Minima = Extrema = Plural of Extremum / Maximum and Minimum

Theorem 1 (Extreme Values Theorem) *If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with*

$$f(x_1) = m,\ f(x_2) = M \text{ , and } m \leq f(x) \leq M \quad \forall\, x \in [a, b] \setminus x_1, x_2$$

Note The requirement of extreme values theorem is **continuous function** and **defined in a closed interval**

Example Find the absolute maximum and minimum of the graph $y = x^2$ on (a) its domain, (b) in $[0, 2]$, (c) in $(0, 2]$, (d) in $(0, 2)$!

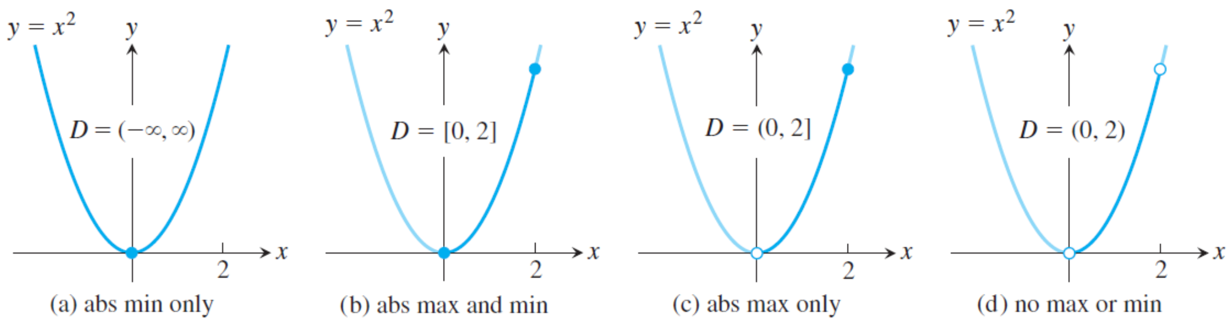


Figure 1: Absolute maximum and minimum of $f(x) = x^2$ on different interval

(a) $x \in D = \mathbb{R}$: Absolute Maximum = None, Absolute Minimum = 0 at $x = 0$
(b) $x \in [0, 2]$: Since, $f(x)$ is continuous and defined on a closed interval $[0, 2]$, the extreme values theorem guarantees that it has absolute maximum and minimum, that is:
Absolute Maximum = 4 at $x = 2$ Absolute Minimum = 0 at $x = 0$
(c) $x \in (0, 2]$: Absolute Maximum = None, Absolute Minimum = None
(d) $x \in (0, 2)$: Absolute Maximum = None, Absolute Minimum = None

Definition A function f has a **local maximum** value at a point c within its domain D if

$$f(x) \leq f(c)\ \forall\, x \in (c-\delta, c+\delta) \cap D$$

A function f has a **local minimum** value at a point c within its domain D if

$$f(x) \geq f(c)\ \forall\, x \in (c-\delta, c+\delta) \cap D$$

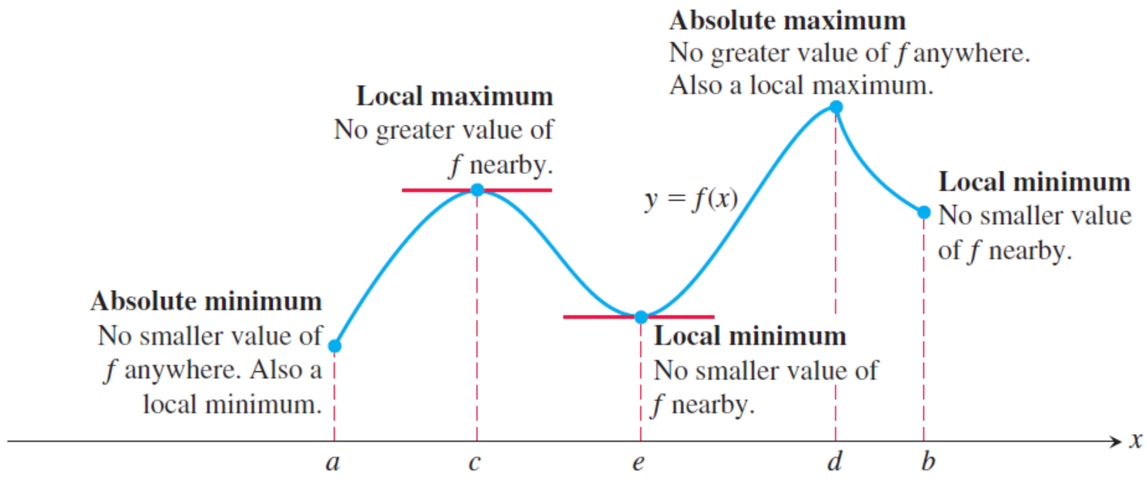


Figure 2: Illustration of local vs absolute extrema

Definition: Critical Points Let $f : D \to \mathbb{R}$, and let c be an interior point of D. Then c is a critical point of f if:

i $f'(c) = 0$

ii $f'(c)$ does not exist $(f'(c) \notin \mathbb{R})$

Example What are all the critical points of the function

$$f(x) = \begin{cases} |x| & x < 1 \\ 1 & x \geq 1 \end{cases}$$

(i) $f'(x)$ is not defined at $x = 0$, so 0 is a critical point.
(ii) $f'(x) = 0$ at $x \geq 1$, so $x \geq 1$ is critical points.
Hence, critical points $= 0 \cup [1, \infty]$

Theorem 2 (The First Derivative Theorem of Local Extrema) *If f has a local maximum or minimum value at an interior point c of its domain, and if $f'(x)$ is defined at c, then $f'(x) = 0$*

or let c be an interior point of D. If a function $f : D \to \mathbb{R}$ has a local extrema at c, then c is a critical point of f.

Note Note that the converse of theorem 2 is not true, that is critical point $\not\Rightarrow$ extrema. The counterexample is $f(x) = x^3$

Finding Local Extrema We can use theorem 1 and theorem 2 to help us find all local extrema and global extrema of $f(x)$ defined in closed interval $x \in [a, b]$, that is: (1) Evaluate all critical points and endpoints, (2) Take the largest and smallest of these value to be global extrema.

Example Find all the absolute extreme (with values and positions) of

$$f : [-2, 4] \to \mathbb{R} \qquad\qquad f(x) = 2x^3 - 3x^2 - 12x + 15$$

(1) Find all Critical values and Endpoints

$$f'(x) = 6x^2 - 6x - 12$$

First, $f'(x) = 0$:

$$6x^2 - 6x - 12 = 0$$
$$6(x-2)(x+1) = 0$$

$x = -1, 2$

Then, find x when $f'(x)$ doesn't exists. Since $f'(x) = 6x^2 - 6x - 12$ is defined on $D \in \mathbb{R}$, Then for all $x \in \mathbb{R}$, $f'(x)$ exists.

Endpoints are $x = -2, 4$, so all possible x are: $x = -2, -1, 2, 4$.

(2) Find the largest and smallest
$f(-2) = 11$, $f(-1) = 22$, $f(2) = -5$, $f(4) = 47$.
Hence, absolute maximum point = (4, 47) and absolute minimum point = (2, -5)

Proof of theorem 2 Suppose that f has a local maximum at c (the case where c gives a minimum is similar). Then

$$\exists\, a > 0 \text{ s.t. } f(c) \geq f(x)\ \forall\ x \in (c-a, c+a)$$

Let

$$g(x) = \frac{f(x) - f(c)}{x - c}$$

for

$$x \in (c-a, c+a) \setminus c$$

(i) For $x \in (c-a, c)$

$$f(x) - f(c) \leq 0 \qquad \text{and} \qquad x - c < 0 \qquad \Rightarrow \qquad g(x) \geq 0$$

Based on theorem 9 in lecture 2 (the inequality of limit), we have

$$\lim_{x \to c^-} g(x) \geq \lim_{x \to c^-} 0 = 0$$

(ii) For $x \in (c, c+a)$

$$f(x) - f(c) \leq 0 \qquad \text{and} \qquad x - c > 0 \qquad \Rightarrow \qquad g(x) \leq 0$$

Based on theorem 9 in lecture 2 (the inequality of limit), we have

$$\lim_{x \to c^+} g(x) \leq \lim_{x \to c^+} 0 = 0$$

Since $f'(x)$ is defined on c $f'_-(x) = f'_+(x) = f'(x)$, So

$$0 \leq \lim_{x \to c^-} g(x) = f'_-(x) = f'(x) = f'_+(x) = \lim_{x \to c^+} g(x) \leq 0$$

Hence, $f'(x) = 0$

2 Rolle's Theorem

Theorem 3 (Rolle's Theorem) *Suppose that a function f is continuous on $[a,b]$ and differentiable at every point in (a,b), and it satisfies $f(a) = f(b)$. Then there exists c in (a,b)such that $f'(c) = 0$.*

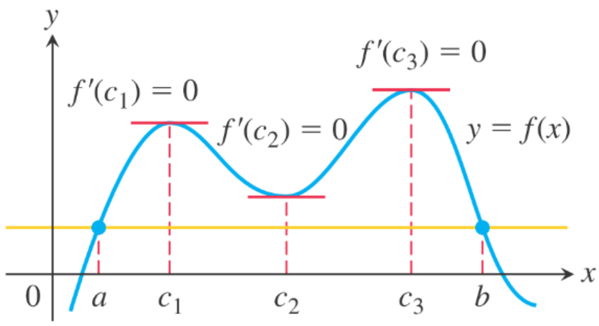


Figure 3: Illustration of Rolle's theorem.

Proof By Extreme Value Theorem, there must exists $c_1, c_2 \in [a,b]$ such that $f(c_1) = m$ and $f(c_2) = M$, where m and M are absolute minimum and maximum respectively.

(i) If $m = M$, then $f(x) = M\ \forall x \in [a,b]$, so $f'(c) = 0$ for any $c \in (a,b)$.
(ii) Suppose $m < M$ then either $f(a) = f(b) \neq m$ or $f(a) = f(b) \neq M$. One or both must holds true. So:
    (1) $f(a) = f(b) \neq m \Rightarrow\ \exists\, c_1 \in (a,b)$ s.t. $f(c_1) = m$
    (2) $f(a) = f(b) \neq M \Rightarrow\ \exists\, c_2 \in (a,b)$ s.t. $f(c_2) = M$

Since $f(c_1)$ and $f(c_2)$ is defined in both cases, then by theorem 2, $f'(c_1)$ and $f'(c_2) = 0$. Hence, rolle's theorem holds on both cases with $c = c_1$ and $c = c_2$ respectively.

Intuition If an object moves and the final position on $t = b$ is the same as the start position on $t = a$, then we can conclude that there must exists $t \in (a,b)$ such that the velocity of the object is 0.

Example Show that the equation $x^3 + 3x + 1 = 0$ has exac

Solution Let $f(x) = x^3 + 3x + 1$.

$$f(-1) = -3 \qquad \text{and} \qquad f(0) = 1$$

Since $f(x)$ is continuous and $f(x) \in [-3,1]$ for $x \in [-1,0]$, there must exists $f(c) = 0$ with $c \in (-1,0)$.

$$f'(x) = 3x^2 + 3$$

Since $f'(x) > 0$ for all $x \in \mathbb{R}$ then the $f(x)$ is strictly increasing (no critical point). So, there is only one solution of $x^3 + 3x + 1$.

3 The Mean Value Theorem

3.1 Theorem and Proof

Theorem 4 (Mean Value Theorem) *Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists c in (a, b) such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

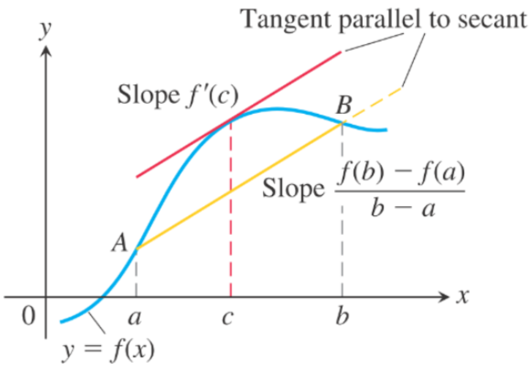


Figure 4: There exists a tangent line in (a, b) that have the same gradient with the secant line on (a, b)

Proof First we define $h(x)$ on $[a, b]$ as

$$h(x) = f(x) - ((f(a) + \frac{f(b) - f(a)}{b - a}(x - a)))$$

Hence, $h(a) = h(b) = 0$ (see from the graph).

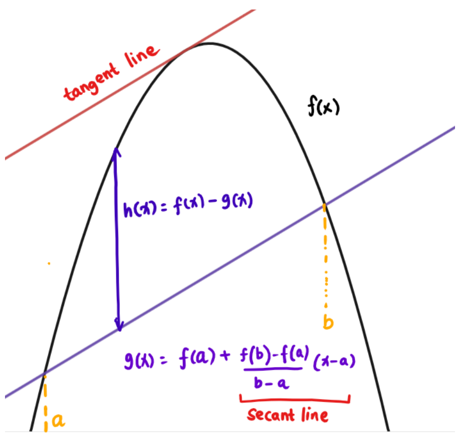


Figure 5: Illustration of $h(x)$

Since h is continuous on $[a, b]$ and differentiable on (a, b), then by Rolle's theorem there exists $c \in (a, b)$ such that $h'(c) = 0$

$$
\begin{aligned}
h'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \forall x \in (a, b) \\
0 &= f'(c) - \frac{f(b) - f(a)}{b - a} \\
f'(c) &= \frac{f(b) - f(a)}{b - a}
\end{aligned}
$$

3.2 Consequences and Corollary

Physical Consequence Suppose that $f(t)$ represents the distance travelled until time t.Then the mean value theorem implies that if we pick two moments $t = a$ and $t = b$, there has to be some moment $t = c$ in between at which the instantaneous speed is equal to the average speed between $t = a$ and $t = b$.

Corollary 1 (MVT) *If $f'(x) = 0$ at each point of x of an open interval (a, b) then $f(x) = C$ for all $x \in (a, b)$ where C is a constant.*

Proof Let $x_1, x_2 \in (a, b)$. From MVT we know that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

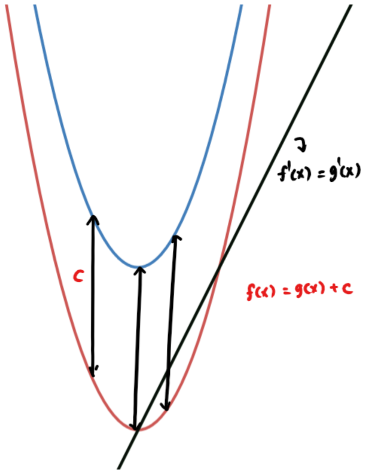
with $c \in (a, b)$. Then, if $f'(c) = 0$

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Hence, if $f'(x) = 0 \; \forall \, x \in (a, b)$:

$$f(x_2) = f(x_1) = C \qquad \forall x \in (a, b)$$

Corollary 2 (MVT) *If $f'(x) = g'(x)$ at each point c of an open interval (a, b) then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$, or $f - g$ is a constant for all $x \in (a, b)$.*



Proof Let $h(x) = f(x) - g(x)$ then

$$h'(x) = f'(x) - g'(x) = 0$$

By corollary MVT 1:

$$h(x) = f(x) - g(x) = C \qquad \forall\, x \in (a, b)$$

Example 1 If $f'(x) = \sin x$ then what is $f(x)$?

We know that if $g(x) = -\cos x$ then $g'(x) = \sin(x) = f'(x)\ \forall x$. Hence $f(x) = -\cos x + C$ with C is a constant.

Note This concept is called antiderivative and will be covered later.

Example 2 Prove that

$$|\sin(x) - \sin(y)| \le |x - y|\ \forall\, x, y \in \mathbb{R}$$

(i) For the case $x = y$ then both sides = 0, hence it's proven
(ii) For the case $x \neq y$, let $y < x$.
Let $f(z) = \sin(z)$. Since $\sin z$ is differentiable and continuous for all $x \in \mathbb{R}$, by MVT there exists:

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$
$$\cos(z) = \frac{f(x) - f(y)}{x - y}$$

Since $-1 \le \cos z \le 1$, then

$$\left|\frac{\sin x - \sin y)}{x - y}\right| \le 1$$
$$|\sin x - \sin y\,| \le |x - y|$$

4 Monotonicity

4.1 Definition and Corollary

Definition Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$ then f is said to be **increasing**
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then f is said to be **decreasing**

A function that is increasing or decreasing on I is said to be **monotonic** on I.

Corollary 3 (Monotonicity) *Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$*
2. *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$*

Proof (1) Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by MVT there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_2 - x_1 > 0$, and $f'(c) > 0$ then we have

$$f(x_2) - f(x_1) > 0$$

Hence, $f(x_2) > f(x_1)$ with $x_1 < x_2$, and by definition f is increasing on $[a, b]$

Proof (2) Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), by MVT there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_2 - x_1 > 0$, and $f'(c) < 0$ then we have

$$f(x_2) - f(x_1) < 0$$

Hence, $f(x_2) < f(x_1)$ with $x_1 < x_2$, and by definition f is decreasing on $[a, b]$

Example Prove that $f(x) = \sqrt(x)$ is increasing on $(0, \infty)$.
First,

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Since $f'(x) > 0 \ \forall \, x \in [0, \infty]$ then by corollary 3 (monotonicity), $f(x)$ is increasing on $[0, b]$ with $b \in \mathbb{R}$. Hence, f is increasing on $[0, \infty)$.

4.2 Interval of Monotonicity

Definition Suppose that $x_1, x_2, ..., x_n$ are all critical points of a function f, with $x_1 < x_2 < ... < x_n$. Let x_i and x_{i+1} be two consecutive critical points. If f' is continuous on $[x_i, x_{i+1}]$, then f' is either entirely positive or entirely negative on (x_i, x_{i+1}). Hence, by corollary monotonicity (3):

1. f is increasing on $[x_i, x_{i+1}]$ if $f'(c) > 0$ for some $c \in (x_i, x_{i+1})$
2. f is decreasing on $[x_i, x_{i+1}]$ if $f'(c) < 0$ for some $c \in (x_i, x_{i+1})$

Similar statements can be made about the intervals $(-\infty, x_1)$ and (x_n, ∞).

Example Let $f(x) = x^3 - 12x + 5$, determine the intervals of monotonicity!

$$f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2)$$

We have critical points $c = \{-2, 2\}$ Hence, the intervals of monotonicity is $(-\infty, -2), (-2, 2), (2, \infty)$

5 Derivative Test

5.1 First Derivative Test

Suppose c is a critical point of a continuous function f that is differentiable at every point in an interval containing c except possibly at c, then if we move from left to right of the function:

1. If f' changes from negative to positive at c, then c is a local minimum.
2. If f' changes from positive to negative at c, then c is a local maximum.
3. If f' doesn't change then f has no extremum at c.

In other words:

1. $\exists\, a > 0 : f'(x) < 0\ \forall\, x \in (c-a, c)$ and $f'(x) > 0\ \forall x \in (c, c+a)$, c is a local minimum
2. $\exists\, a > 0 : f'(x) > 0\ \forall\, x \in (c-a, c)$ and $f'(x) < 0\ \forall x \in (c, c+a)$, c is a local maximum
3. $\exists\, a > 0 : f'(x) > 0$ or $f'(x) < 0\ \forall\, x \in (c-a, c+1)$

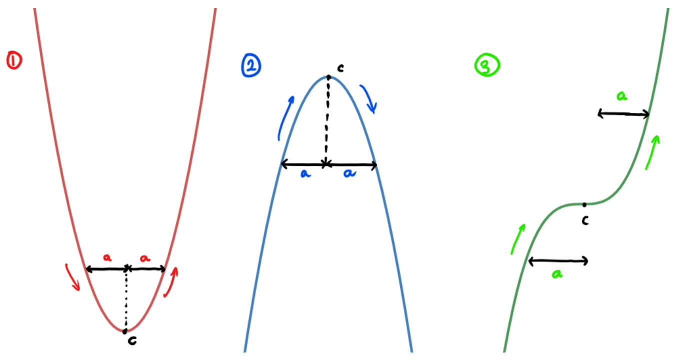


Figure 6: First derivative test (1) Minimum point, (2) Maximum point (3) Neither

Proof (1) By corollary (3) of monotonicity, f is decreasing on $[c-a, c]$ and increasing on $(c, c+a]$ which means $f(c) < f(x)\ \forall\, x \in [c-a, c)$ and $f(c) < f(x)\ \forall\, x \in (c-a, c+a) \setminus \{c\}$. Hence, by definition f has a local minimum at c.

Example Find all absolute and local extrema for $f(x) = x^{4/3} - 4x^{1/3}$

$$f'(x) = \frac{4}{3}x^{-\frac{2}{3}}(x-1)$$

$f'(x) = 0$ when $x = 1$ and $x = 0$ For $x = 0$

$$\lim_{h\to 0} \frac{f(x) - f(0)}{h} = \frac{h^{\frac{4}{3}} - 4h^{-\frac{1}{3}}}{h} = h^{\frac{1}{3}} - 4h^{-\frac{2}{3}} = DNE$$

Hence, $f'(x)$ is not defined. Then we have - - - —0—- - - —1— +++. By first derivative test, $f(x)$ has a local minimum point at $x = 1$. Since $x \to \pm\infty$ f(x) is increasing, then $x = 1$ is also an absolute minimum.

Note Note that first derivative test **does not always apply**. For example:

$$f(x) = \begin{cases} x^2(1+\sin\frac{\pi}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0} h(1+\sin\frac{\pi}{h}) = 0$$

Hence, $x = 0$ is a critical point. However,

$$f'(x) = 2x(\sin\frac{\pi}{x}) - \pi(\cos\frac{\pi}{x})$$

which no matter what value of $a > 0$ is, in the interval $(0, a)$ there exist x_1 and x_2 such that $f'(x_1) > 0$ and $f'(x_2) < 0$. So f is NEVER always increasing or always decreasing near 0, on the right. Hence none of the conditions in the first derivative test is satisfied.

However, $f(0)$ is still a local and absolute minimum because $f(0) = 0$, and $f(x) > 0\ \forall\, x \neq 0$

5.2 Concavity and Inflection Point

Definition - Concavity The graph of a differentiable function $y = f(x)$ is:

1. **Concave up** on an interval I if $f'(x)$ is increasing.
2. **Concave down** on an interval I if $f'(x)$ is decreasing.

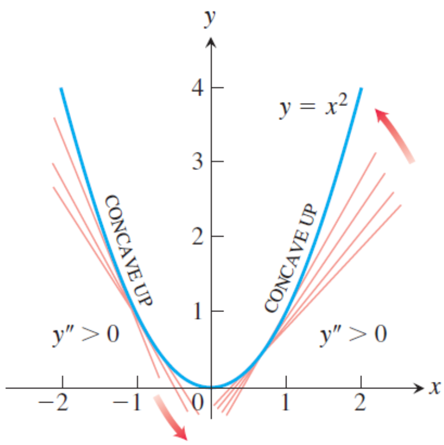


Figure 7: Function Concavity

Second Derivative Test of Concavity Suppose $y = f(x)$ is a twice differentiable function on an interval I then

1. If $f''(x) > 0$ on I, the graph f on I is concave up
2. If $f''(x) < 0$ on I, the graph f on I is concave down

Proof (1) Suppose $f''(x) > 0\ \forall\, x \in I$. Then, by corollary (3) $f'(x)$ is increasing on I. Hence, f is concave up.

Example Find the concavity of $y = x^3$

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

So, $f''(x) > 0$ if $x > 0$ and $f''(x) < 0$ if $x < 0$.

Definition - Inflection Point is a point $(c, f(c))$ where $f(x)$ has a tangent line and where the concavity changes. If c is an inflection point, then either $f''(c) = 0$ or $f''(x)$ doesn't exist. However, $f''(c) = 0$ doesn't imply that c is an inflection point.

Note In Thomas' Calculus, an inflection point need to have **a tangent line** or differentiable at c.

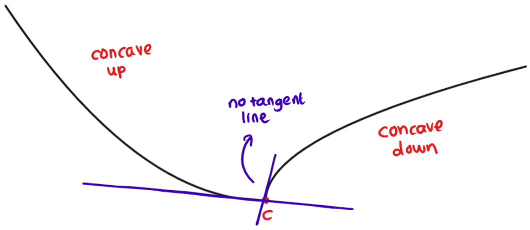


Figure 8: Where inflection point fails, at c, $f(x)$ doesn't have a tangent line

Example Find the inflection point of $f(x) = x^{1/3}$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$

At $x = 0$, $f'(x)$ and $f''(x)$ doesn't exist.

$$\lim_{x\to 0} f'(x) = \frac{1}{3\sqrt[3]{x^2}} = +\infty$$

Hence, $f(x)$ has a tangent line at $x = 0$.

$$f''(x) \begin{cases} > 0 \; x < 0 \\ < 0 \; x > 0 \end{cases}$$

Hence, (0,0) is a point of inflection.

5.3 Second Derivative Test

Second derivative test on local extrema Suppose f'' is continuous on an interval containing c.

1. If $f'(x) = 0$ and $f''(x) < 0$, then f has a local maximum at c
2. If $f'(x) = 0$ and $f''(x) > 0$, then f has a local minimum at c
3. If $f'(x) = 0$ and $f''(x) > 0$, can be local maximum, local minimum or neither

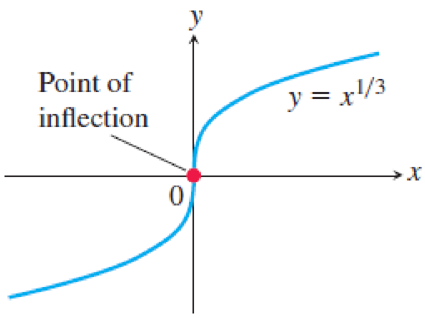
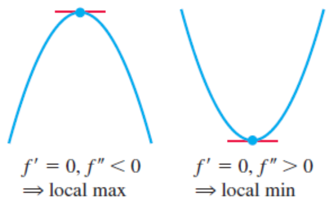


Figure 9: Graph of $y = x^{1/3}$



Proof (Statement 2) Suppose $f'(c) = 0$ and $f''(x) > 0$. Then,

$$0 < f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{x - c}$$

Let $L = f''(c)$ for $\epsilon = L/2$, $\exists\, \delta > 0$ such that:

$$\begin{aligned} \frac{f'(x)}{x-c} &\in (L - \epsilon, L + \epsilon) \qquad \forall\, x \in (c - \epsilon, c + \epsilon) \\ \frac{f'(x)}{x-c} &\in (\frac{L}{2}, \frac{3L}{2}) \qquad \forall\, x \in (c - \epsilon, c + \epsilon) \\ \Rightarrow \frac{f'(x)}{x-c} &> \frac{L}{2} > 0 \qquad \forall\, x \in (c - \delta, c + \delta) \end{aligned}$$

So,

$$\begin{aligned} x - c < 0 \Rightarrow f'(x) < 0 \\ x - c > 0 \Rightarrow f'(x) > 0 \end{aligned}$$

Hence,

$$\begin{aligned} f'(x) < 0 \text{ for } (c - \delta, c) \\ f'(x) > 0 \text{ for } (c, c + \delta) \end{aligned}$$

By corolarry (3) f is decreasing on $[c - \delta, c]$ and increasing on $[c, c + \delta]$. Hence, f has a local minimum at c.

Statement 1 can be proved by the same method.

Proof (Statement 3) Let $f(x) = x^3$, $g(x) = x^4$, $h(x) = -x^4$
We have:

$$\begin{aligned} f'(x) &= 3x^2 \qquad & g'(x) &= 4x^3 \qquad & h'(x) &= -4x^3 \\ f''(x) &= 6x \qquad & g''(x) &= 12x^2 \qquad & h''(x) &= -12x^2 \end{aligned}$$

At $x = 0$:

$$f'(x) = f''(x) = g'(x) = g''(x) = h'(x) = h''(x)$$

But, at $x = 0$ (1) f doesn't have a local extremum, (2) g has a local minimum, (3) h has a local maximum. Hence, when $f''(c) = 0$, c can be an local maximum, minimum, or neither.

Example Find all local extrema of $f(x) = x^4 - 4x^3 + 10$!

- **Step 1** Find all values of x such that $f'(x) = 0$
- **Step 2** Find all values of $f''(x)$ when $f'(x) = 0$
- **Step 3** Apply second derivative test to identify the concavity
- **Step 4** Find $f(x)$ at the local extrema

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) \qquad f'(x) = 0 \Rightarrow x = \{0, 3\}$$
$$f''(x) = 12x^2 - 24x \qquad f''(0) = 0 \;\; f''(3) = 36 > 0$$

So, at $x = 0$, $f(x)$ fails the second derivative test, at $x = 3$, $f''(x) > 0$, and $f(x)$ has a local minimum at $x = 3$.

At $x = 0$, consider $a = -1$ and $b = 1$. $f'(1) < 0$ and $f'(-1) < 0$, so, it is not a local extrema.

5.4 Concavity and Secant Line

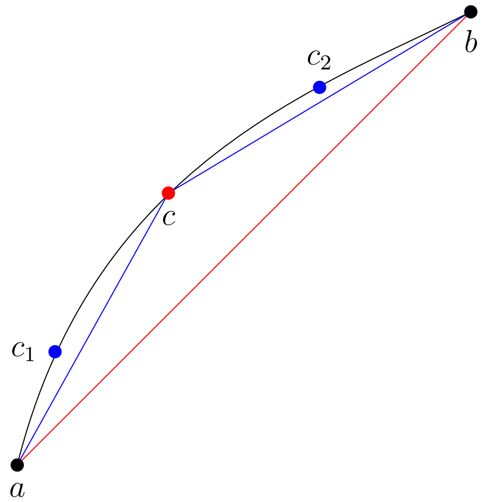
Theorem 5 (Concavity and Secant Lines) *Let f be continuous on $[a, b]$ and differentiable on (a, b).*

1. *If f is **concave down** on (a, b), then the graph of f lies **above** the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a, b).*
2. *If f is **concave up** on (a, b), then the graph of f lies **below** the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a, b).*

(a) Concave down

(b) Concave up

Figure 10: Concavity and Secant Line



Proof (Statement 1) We have the secant line of a graph

$$y = g(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}(x - a)\right)$$

We will show that $f(x) > g(x)$, $\forall\, x \in (a,b)$, given that $f(x)$ is concave down on (a,b). Let $x \in (a,b)$. By MVT,

$$\begin{aligned} f(x) &= f(a) + f'(c_1)(x-a) && \text{for some } c_1 \in (a,x) \\ f(b) &= f(x) + f'(c_2)(b-x) && \text{for some } c_2 \in (x,b) \end{aligned}$$

Then,

$$f(b) = f(a) + f'(c_1)(x-a) + f'(c_2)(b-x)$$

Since f is concave down on (a,b), then $f'(c_1) > f'(c_2)$, So,

$$\begin{aligned} f(b) &< f(a) + f'(c_1)(x-a) + f'(c_1)(b-x) \\ f(b) &< f(a) + f'(c_1)(b-a) \\ \frac{f(b)-f(a)}{b-a} &< f'(c_1) \end{aligned}$$

Therefore we have:

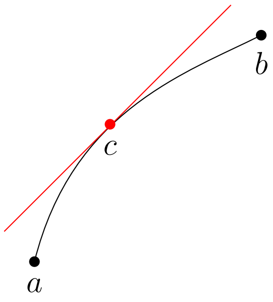
$$\begin{aligned} g(x) = f(a) + \left(\frac{f(b) - f(a)}{b - a}(x - a)\right) \\ f(x) = f(a) + f'(c_1)(x-a) \\ \frac{f(b)-f(a)}{b-a} < f'(c_1) \end{aligned}$$

From the above we have $f(x) > g(x)$

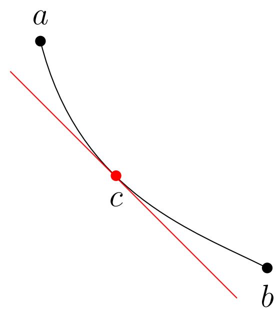
5.5 Concavity and Tangent Line

Theorem 6 (Concavity and Tangent Lines) *Let f be continuous on $[a,b]$ and differentiable on (a,b).*

1. *If f is **concave down** on (a,b), then $\forall\, c \in (a,b)$, the tangent line of f at c lies **above** the graph of $y = f(x)$.*
2. *If f is **concave up** on (a,b), then $\forall\, c \in (a,b)$, the tangent line of f at c lies **below** the graph of $y = f(x)$.*



(a) Concave down



(b) Concave up

Figure 11: Concavity and Tangent Line

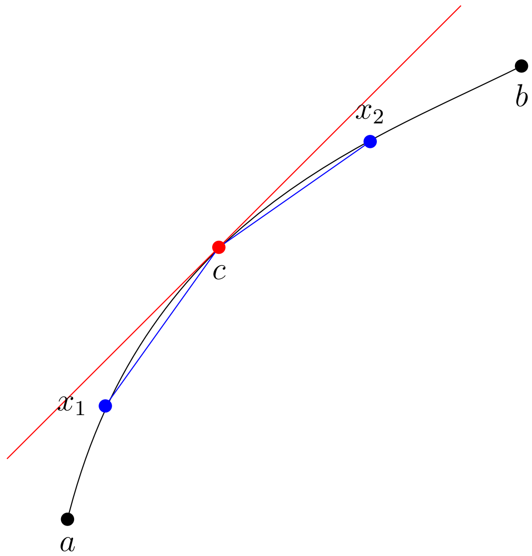
Proof (Statement 1) Let $g(x)$ as a tangent line at c

$$g(x) = f(c) + f'(c)(x - c)$$

Let $x \in [a, b] \setminus \{c\}$

$$\begin{aligned} g(x) > f(x) &\Leftrightarrow f(c) + f'(c)(x - c) > f(x) \\ &\Leftrightarrow f'(c)(x - c) > f(x) - f(c) \\ &\Leftrightarrow f'(c) \begin{cases} > \frac{f(x)-f(c)}{x-c} & \text{if } x > c \Rightarrow x - c > 0 \\ < \frac{f(x)-f(c)}{x-c} & \text{if } x < c \Rightarrow x - c < 0 \end{cases} \end{aligned}$$

Then, we will show that the last equation is true (eq. 1), that is:



$$f'(c) \begin{cases} > \frac{f(x)-f(c)}{x-c} & \forall\, x \in (c, b] \\ < \frac{f(x)-f(c)}{x-c} & \forall\, x \in [a, c) \end{cases}$$

First, assume $x \in (c, b]$. By MVT,

$$f'(c_1) = \frac{f(x) - f(c)}{x - c} \qquad \text{for some } c_1 \in (c, x)$$

Since f is concave down on (a, b), then $f'(c_1) < f'(c)$, So,

$$\frac{f(x) - f(c)}{x - c} < f'(c)$$

Which prove the first case of eq. 1. the second case can be proven in the same manner. Hence, eq. 1 is proven.

6 Graph Sketching

Important things to sketch a graph: domain, symmetry, critical points, interval of monotonicity, points of inflection, intervals of concavity, asymptotes, and x-y intercepts.

Example Graph the function

$$f(x) = \frac{x^2 + 4}{2x}$$

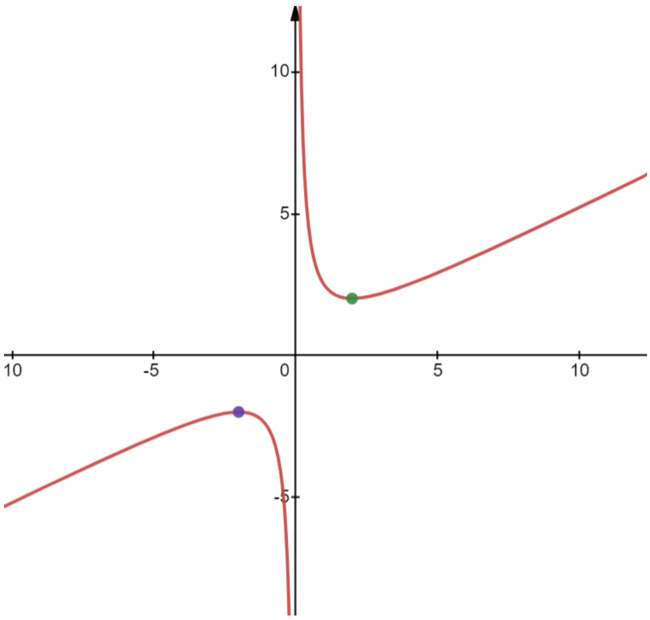


Figure 12: Graph of $f(x)$

(1) Domain $= (-\infty, 0) \cup (0, \infty)$
(2) Symmetry

- Odd function/ symmetry on origin if $f(-x) = -f(x)$
- Even function / symmetry along y axis if $f(-x) = f(x)$

$$f'(x) = \frac{(-x)^2 + 4}{-2x} = -(\frac{x^2 + 4}{2x}) = -f(x)$$

Hence $f(x)$ is an odd function

(3) Critical points

$$f'(x) = \frac{x^2 - 4}{2x^2} \qquad f'(x) = 0 \Rightarrow x = -2, 2$$

So, $+++|-2|---|0|---|2|+++$, Minimum point at 2, and maximum point at -2.

(4) Point of Inflection

$$f''(x) = 4x^{-3} \qquad f''(x) > 0 \ \forall x > 0 \text{ and } f''(x) < 0 \ \forall x < 0$$

Hence, f concave down on $(-\infty, 0)$ f concave up on $(0, \infty)$ and no point of inflection.

(5) Asymptotes
Oblique asymptote: $y = Ax + B$

$$A = \lim_{x\to\infty} \frac{f(x)}{x} = \frac{x^2+4}{2x^2} = \frac{1}{2}$$

$$B = \lim_{x\to\infty} f(x) - A(x) = \frac{x^2+4-x^2}{2x} = 0$$

So, $y = 1/2x$ is an oblique asymptote

Vertical Asymptote :

$$\lim_{x\to 0^+} f(x) = \infty$$

Hence, $x = 0$ is a vertical asymptote.

(6) x and y intercepts
No x and y intercept since $x, y \neq 0$

7 Applied Optimization

Example 1 Rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

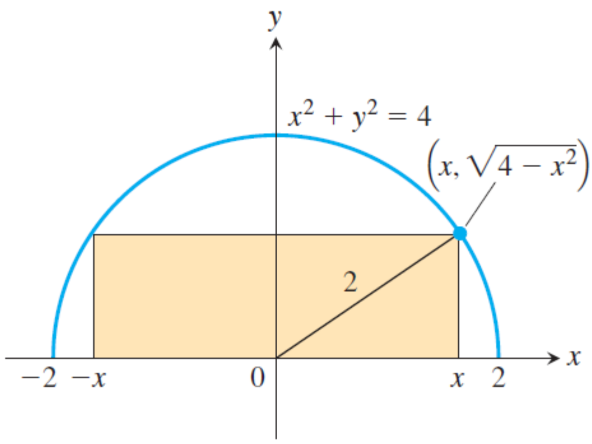


Figure 13: Illustration of Example 1

Solution From the circle equation we have:

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \sqrt{4 - x^2}$$

The area of the rectangle can be denoted as

$$A = 2x \cdot y$$
$$A = 2x\sqrt{4-x^2}$$

To maximize the area, then $A' = 0$

$$
\begin{aligned}
A' &= 2\sqrt{4-x^2} + \frac{-2x^2}{\sqrt{4-x^2}} \\
0 &= \frac{2(4-x^2)-2x^2}{\sqrt{4-x^2}} \\
2x^2 &= 8-2x^2 \\
x^2 &= 2 \\
x &= \pm\sqrt{2}
\end{aligned}
$$

$$y = \sqrt{4-2} = \sqrt{2}$$

We have $A(0) = 0$, $A(\sqrt{2}) = 4$, $A(2) = 0$. Hence, absolute maximum at $x = \sqrt{2}$, and the dimension is $(\sqrt{2}, \sqrt{2})$

Example 2 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see figure). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible.)

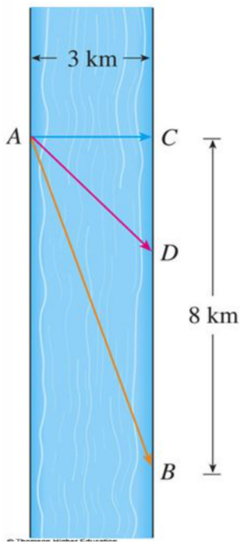


Figure 14: Illustration of Example 2

Solution

$$t = \frac{s}{v} = \frac{|AD|}{6} + \frac{|DB|}{8}$$

From the figure we can see that

$$|AD| = \sqrt{(8-|DB|)^2 + 9}$$

Let $|DB| = x$, then

$$\begin{aligned} t &= \frac{s}{v} = \frac{\sqrt{(8-x)^2+9}}{6} + \frac{x}{8} \\ t' &= \frac{1}{6} \cdot \frac{-2(8-x)}{2\sqrt{(8-x)^2+9}} + \frac{1}{8} \\ t' &= \frac{x-8}{6\sqrt{(8-x)^2+9}} + \frac{1}{8} \end{aligned}$$

To minimze t, $t' = 0$, so,

$$\begin{aligned} -\frac{1}{8} &= \frac{x-8}{6\sqrt{(8-x)^2+9}} \\ 6\sqrt{(8-x)^2+9} &= 8(8-x) \\ 36((8-x^2)+9) &= 64(8-x)^2 \\ 36(8-x^2)+324 &= 64(8-x)^2 \\ 28(8-x)^2 &= 324 \\ (8-x)^2 &= \frac{324}{28} \\ 8-x &= \sqrt{\frac{324}{28}} \\ x = 8 - \sqrt{\frac{324}{28}} &\approx 4.598 \end{aligned}$$

So, $|CD| = 8 - 4.598 = 3.402$ km.

Example 3 Suppose x = number of video game consoles, million units
Cost : $C(x) = x^3 - 6x^2 + 15x$
Revenue : $R(x) = 9x$
Profit : $P(x) = R(x) - C(x)$
Find x that maximizes profit, if any.

Solution

$$\begin{aligned} P(x) &= 9x - (x^3 - 6x^2 + 15x) \\ P'(x) &= 9 - 3x^2 + 12x - 15 \\ P'(x) &= 3(-x^2 + 4x - 6) \end{aligned}$$

To maximizes $P(x)$, $P'(x) = 0$

$$\begin{aligned} P'(x) &= 3(-x^2 + 4x - 2) \\ 0 &= x^2 - 4x - 2 \\ x &= \frac{4 \pm \sqrt{16 - 4(1)(-2)}}{2} \\ x &= 2 \pm \sqrt{2} \end{aligned}$$

Check the local min/max

$$P''(x) = -6x + 12$$
$$P''(2+\sqrt{2}) = -6(2+\sqrt{2}) + 12 = -6\sqrt{2} < 0$$
$$P''(2-\sqrt{2}) = -6(2-\sqrt{2}) + 12 = 6\sqrt{2} > 0$$

Since $P''(2-\sqrt{2}) < 0$ it's a local minimum, and $P''((2+\sqrt{2})) > 0$ then it's a local maximum

$$P(2+\sqrt{2}) = x^3 - 6x^2 - 6x = (2+\sqrt{2})^3 - 6(2+\sqrt{2})^2 - 6((2+\sqrt{2})) > 0$$

Hence, profit maximizes at $x = 2+\sqrt{2}$ or 3.14 million units

8 Newton's Method

8.1 Introduction

Newton's method (a.k.a. Newton Raphson method) is a numerical method for finding approximate solution to a function especially those which is difficult/impossible to be solved analytically.

Example Consider a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

What value is x when $f(x) = 0$?

For $n = 1, 2$ we can solve it easily (using linear eq. or quadratic formula). For $n = 3, 4$, it's a bit hard but can be solved using general formula. But, for $n > 4$, there's no general formula.

8.2 Method

Procedure for Newton's method is:

1. Start with a point x_0 near root a root r.
2. Let L_i be the tangent line to $y = f(x)$ at $x = x_i$, with $L_i = f(x_i) + f'(x_1)(x - x_i)$
3. If $f'(x_i) \neq 0$, L_i will intersect the x-axis at some point x_{i+1}.
4. Repeat above step until $f(x_i) \approx 0$.

Hence,

$$x_{i+1} = x_1 - \frac{f'(x_i)}{f(x_i)} \qquad \text{if } f'(x_1) \neq 0$$

Example Consider $f(x) = x^3 - 2x - 5$.
Since $f(2) = -1$ and $f(3) = 11 > 0$, there by IVT, exists $r \in (2,3)$ such that $f(r) = 0$. Consider $x_0 = 2$.

$$f'(x) = 3x^2 - 2$$

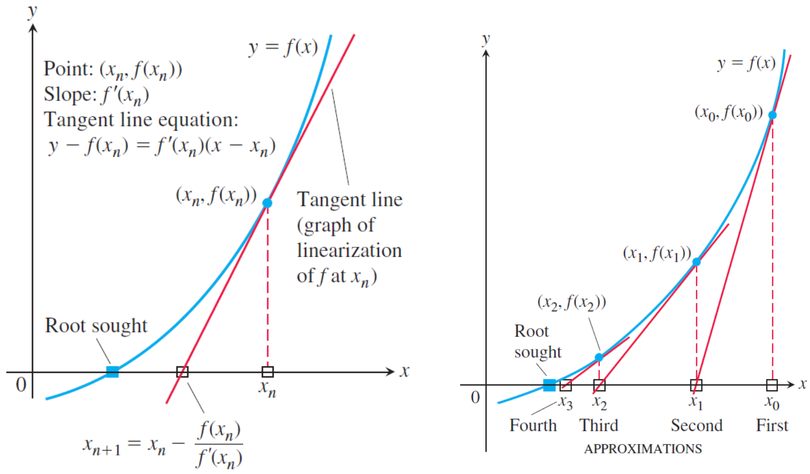


Figure 15: How Newton's method works

So,

$$x_1 = 2 - \frac{8-4-5}{3(4)-2} = 2.1$$
$$x_2 = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.094568$$
$$x_3 = 2.094568 - \frac{(2.094568)^3 - 2(2.094568) - 5}{3(2.094568)^2 - 2} = 2.094551$$

We take $x = 2.094568$ as r, $f(r) = -5.357 \times 10^{-6} \approx 0$.
$x_0 = 2$ is an arbitrary choice. We can choose $x_0 = 3$ and x_n converges to the same r.

8.3 Newton's Method Failure

Sometimes the method doesn't work because x_n doesn't converge to a root r, or it can converge to a root we're not interested in (e.g. negative root).

Example

$$f(x) = \begin{cases} \sqrt{x-r} & \text{if } x \geq r \\ -\sqrt{r-x} & \text{if } x < r \end{cases}$$

If we pick $x_0 = r - h$, then we have $x_1 = r + h$, $x_2 = r - h$, $x_3 = r + h$. Hence, x_n does not converge to any single number, and so does not converge to r (oscillating between $r - h$ and $r + h$).

Note There are some conditions when newton's method work. One possible condition is:
Suppose f is continuous on $[a, b]$ and differentiable on (a, b) and $f(r) = 0$ where $r \in (a, b)$. If $f(r) = 0$, then there exists $\delta > 0$ such that, with any starting point, $x_0 \in (r - \delta, r + \delta)$, the sequence x_n converges to r.