MAT1001 Calculus I

Lecture 15 - 17 : Application of Integrals

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1 Volumes using Cross-Sections

1.1 Cross-Sections

Intuition Let S be a solid in the three dimensional space, lying between the planes $x = a$ and $x = b$. For $c \in [a, b]$, let $A(c)$ be the area of the cross section obtained by intersecting S with the plane $x = c$. Consider a partition $P = x_0, x_1, ..., x_n$ of $[a, b]$. When Δx_k is small, the volume of the solid lying between $x = x_{k-1}$ and $x = x_k$ is approximately $A(x_k)\Delta x_k$. When $||P||$ is small, the volume of S is approximately

$$\sum_{k=1}^{n} A(x_k)\Delta x_k$$

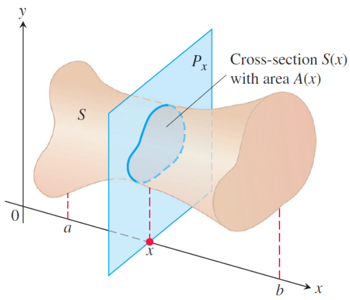


Figure 1: Cross-section Volume

Definition Let S be a solid that lies between the planes $x = a$ and $x = b$. The volume V of S is defined by

$$V = \int_a^b A(x)dx$$

provided that the cross-section area function $A(x)$ is integrable.

Example A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

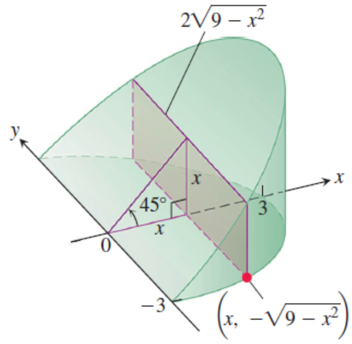


Figure 2: Illustration of example 1

Answer The equation of the base circle is

$$x^2 + y^2 = 9$$

So,

$$y = \pm\sqrt{9 - x^2}$$

Since the angle of the cut its 45 degree, then the height of the cross-section is x Then, the cross-section area is

$$A(x) = 2x\sqrt{9 - x^2}$$

$$\begin{aligned} V &= \int_0^3 2x\sqrt{9-x^2}\,dx \\ &\text{Let } u = 9 - x^2 \text{ then } du = -2xdx \\ &= \int_9^0 \sqrt{u}\,(-du) \\ &= \int_0^9 \sqrt{u}\,du \\ &= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9 \\ &= 18 \end{aligned}$$

1.2 Cavalieri's Principle

Cavalieri's principle says that solid with equal altitudes and identical cross-sectional areas at each height have the same volume. This follows from the definition of the volume, because the cross-sectional area $A(x)$ and the interval $[a, b]$ are the same for both solids.

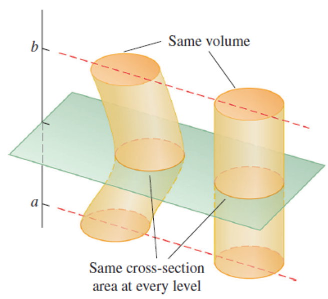


Figure 3: Cavalieri's Principle

2 Solids of Revolution

2.1 The Disk Method

Intuition If the solid S is generated by rotating the region

$$\{(x, y) : 0 \le y \le R(x), a \le x \le b\}$$

around the x-axis, then the cross-sections of S are discs with radii $R(x)$. Consequently

$$A(x) = \pi R(x)^2$$

Definition: Rotating around x-axis Volume by disks for rotation about x-axis with the radius of $R(x)$ is

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx$$

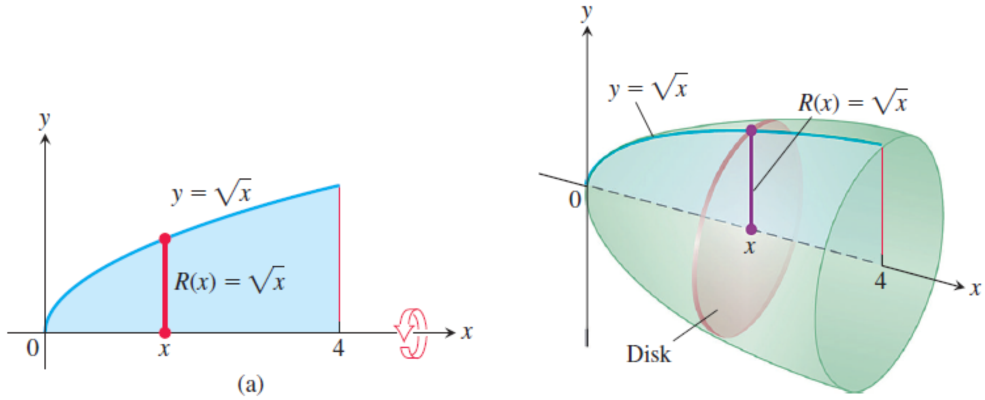


Figure 4: Solid of rotation about x-axis

Definition: Rotating around y-axis Volume by disks for rotation about y-axis with the radius of $R(x)$ is

$$V = \int_a^b A(y)dy = \int_a^b \pi[R(y)]^2 dy$$

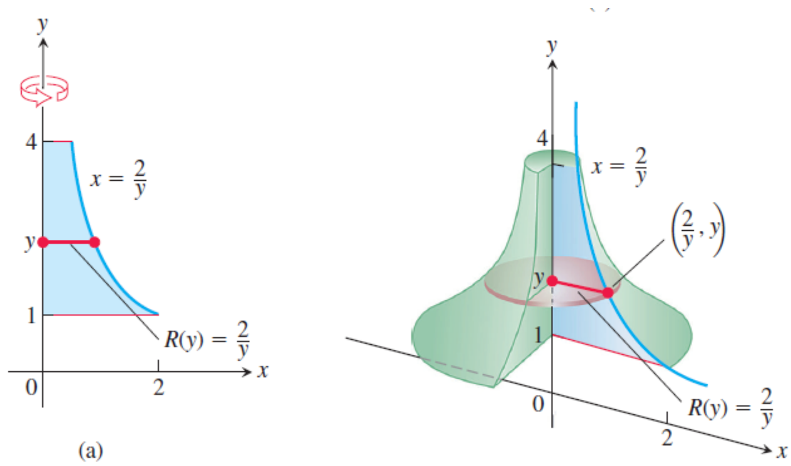


Figure 5: Solid of rotation about y-axis

Example 1 What is the volume of $y = \sqrt{x}$ rotated about x-axis?

$$\begin{aligned} V &= \int_0^4 \pi (R(x))^2 \, dx \\ &= \int_0^4 \pi x \, dx \\ &= \left[\frac{\pi}{2} x^2\right]_0^4 \\ &= 8\pi \end{aligned}$$

Example 2 The circle $x^2 + y^2 = a^2$ is rotated about x-axis to generate a sphere. Find its volume

$$R(x) = y = \sqrt{a^2 - x^2}$$

$$\begin{aligned} V &= \int_{-a}^{a} \pi (R(x))^2 \, dx \\ &= \int_{-a}^{a} \pi (a^2 - x^2) \, dx \\ &= 2\pi \int_0^a \pi (a^2 - x^2) \, dx \\ &= 2\pi \left[a^2 x - \frac{1}{3} x^3\right]_0^a \\ &= \frac{4}{3} \pi a^3 \end{aligned}$$

2.2 The Washer Method

Definition If the solid S is generated by rotating the region

$$\{(x, y) : 0 \le r(x) \le y \le R(x), a \le x \le b\}$$

around the x-axis, then similarly,

$$V = \int_a^b \pi (R(x)^2 - r(x)^2) \, dx$$

for rotation around y-axis of

$$\{(x, y) : 0 \le r(y) \le x \le R(y), a \le y \le b\}$$

the volume is

$$V = \int_a^b \pi (R(y)^2 - r(y)^2) \, dy$$

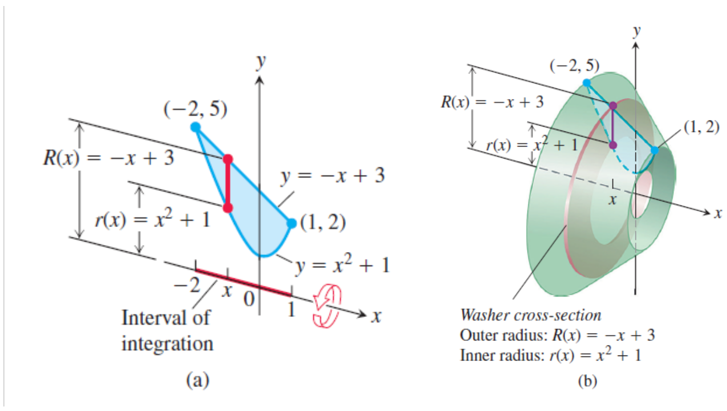


Figure 6: The Washer Method

Example 1 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

First, find the limits of integration in the x-domain,

$$x^2 + 1 = -x + 3$$

We found $x = -2, y = 5$ and $x = 1, y = 2$.

$$V = \int_a^b \pi(R(x)^2 - r(x)^2)\,dx$$

$R(x) = -x + 3$ and $r(x) = x^2 + 1$

$$\begin{aligned} V &= \pi \int_{-2}^{1} x^2 - 6x + 9 - (x^4 + 2x^2 + 1)\,dx \\ &= \pi \int_{-2}^{1} -x^4 - x^2 - 6x + 8\,dx \\ &= \pi \left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x \right]_{-2}^{1} \\ &= \frac{117}{5}\pi \end{aligned}$$

Example 2 Find the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ aboutthe line $x = -1$.

Since the rotation is about the line $x = -1$, we can create an equivalent revolution around the y-axis by shifting the plots to the right by 1 unit, i.e. add 1 to each function.

$$\begin{aligned} x &= y + 1 \Rightarrow x = y + 2 \\ x &= \frac{y^2}{4} + 1 \Rightarrow x = \frac{y^2}{4} + 2 \end{aligned}$$

Then find the limits of integration in the y-domain where $x = y + 2$ and $x = \frac{y^2}{4} + 2$ intersect.

$$\begin{aligned} y + 2 &= \frac{y^2}{4} + 2 \\ 0 &= y(y - 4) \end{aligned}$$

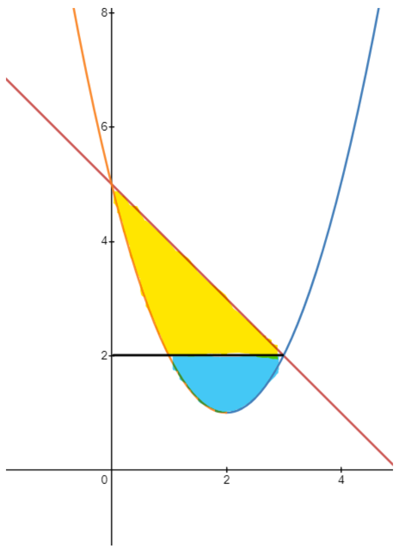
$y = 0, x = 2$ or $y = 4, x = 6$

$$\begin{aligned} V &= \pi \int_0^4 (y+2)^2 - (\frac{y^2}{4} + 2)^2 dy \\ &= \pi \int_0^4 4y - \frac{y^4}{16} dy \\ &= \pi \left[2y^2 - \frac{y^5}{80} \right]_0^4 \\ &= \frac{96\pi}{5} \end{aligned}$$

Example 3 A region is bounded by $y = (x-2)^2 + 1$ and $y = 5 - x$. Find the volume of the solid generated by revolving the region around the y-axis.

Find the intersection points between the two functions.

$$(x-2)^2 + 1 = 5 - x$$



Find the volume:

$$V = \pi \int_2^5 (5-y)^2 - (-\sqrt{y-1}+2)^2 dy + \pi \int_1^2 (\sqrt{y-1}+2)^2 - (-\sqrt{y-1}+2)^2 dy$$
$$= \frac{27\pi}{2}$$

3 Cylindrical Shells

Intuition Consider revolving the region in blue about the y-axis to generate a solid. Its volume can be computed by adding the volumes of all the "cylindrical shells".

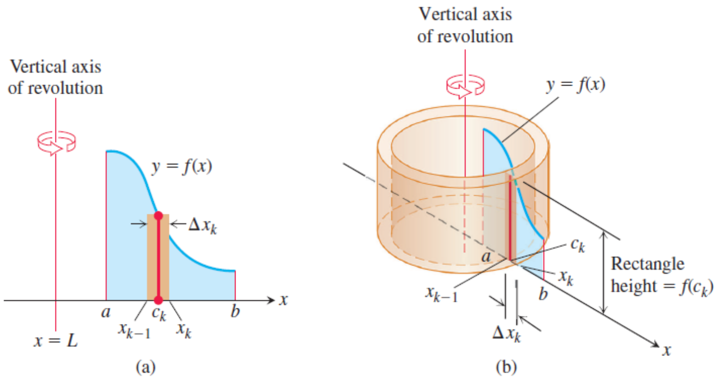
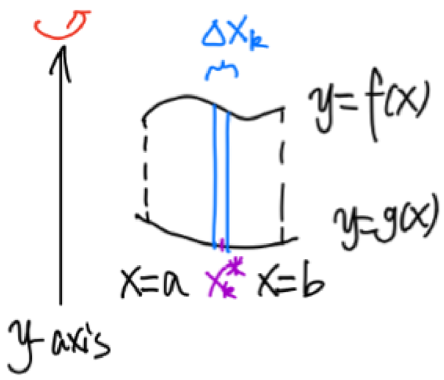


Figure 7: Volume with cylindrical shells

Volume of a "thin cylindrical shell" or "thin annullus" is approximately the volume of cuboids of heigh $h(x)$ and the length $2\pi x$. The height of cylindrical shell can be denoted as

$$h(x) = f(x) - g(x)$$



The volume of the cylindrical shell at x is $V_k \approx 2\pi x_k^* h(x_k)^* \Delta x_k$. Hence, the total volume

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* h(x_k)^* \Delta x_k$$

Definition Given the solid S generated bu revolving the region

$$\{(x, y) : g(x) \le y \le f(x), a \le x \le b\}$$

about y-axis, let $h(x) = f(x) - g(x)$ be the height of the region at x. Then the volume V of S can be computed by

$$V = \int_a^b 2\pi x h(x)\, dx$$

Example 1 A region is bounded by $y = (x-2)^2 + 1$ and $y = 5 - x$. Find the volume of the solid generated by revolving the region around the y-axis.

$$\begin{aligned}
V &= \int_0^3 2\pi x(5 - x - (x-2)^2 - 1)dx \\
&= 2\pi \int_0^3 x(5 - x - (x^2 - 4x + 4) - 1)dx \\
&= 2\pi \int_0^3 x(-x^2 + 3x)dx \\
&= 2\pi \left[-\frac{1}{4}x^4 + x^3 \right]_0^3 \\
&= \frac{27\pi}{2}
\end{aligned}$$

Example 2 Find the volume of the solid generated by revolving the region between $y = \sqrt{x}$ and $y = 0$ from $x = 0$ to $x = 1$ around the x-axis in two different ways.

(1) Using solid of revolution

$$\begin{aligned}
V &= \pi \int_0^1 x dx \\
&= \pi \left[\frac{1}{2}x^2 \right]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}$$

(2) Using cylindrical shells

$$
\begin{aligned}
V &= 2\pi \int_0^1 y(1-y^2)dy \\
&= 2\pi \int_0^1 y - y^3 dy \\
&= 2\pi \left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 \\
&= \frac{\pi}{2}
\end{aligned}
$$

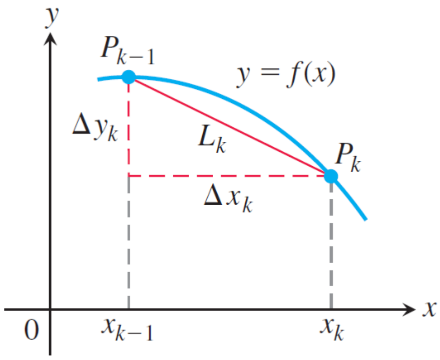
4 Arc Length

Intuition Consider a curve given by a continuous function $y = f(x)$ defined on the interval $[a,b]$ and let P be a partition of $[a,b]$.

If $y_k = f(x_k)$ and $\Delta y_k = y_k - y_{k-1}$, then the length of the curve between the points (x_{k-1}, y_{k-1}) and $(x_k, y+k)$ is approximately

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\Delta x_k$$

The definition of arc length is obtained by taking the limit of $\sum_{k=1}^{n} L_k$ as $||P|| \to 0$



Definition Let f be a function such that f' is continuous on $[a,b]$. The length/arc length L of the curve $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is defined by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example 1 Compute the length of the curve given by $y = x^{3/2}$, $0 \le x \le 3$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

$$
\begin{aligned}
L &= \int_0^3 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)} dx \\
&= \int_0^3 \sqrt{1 + \frac{9}{4}x\, dx}
\end{aligned}
$$

Let $u = 1 + 9/4x$, $du/dx = 9/4, dx = 4/9du$

$$L = \int_1^{\frac{31}{4}} \sqrt{u}\frac{4}{9}\,du$$
$$= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{\frac{31}{4}}$$
$$= \frac{8}{27}\left(\left(\frac{31}{2}\right)^{\frac{3}{2}} - 1\right)$$

Note If the curve is given by $x = g(y)$, $c \le y \le d$, and g' is continuous, then the arc length is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy = \int_c^d \sqrt{1 + (g'(y))^2}dy$$

5 Areas of Surfaces of Revolution

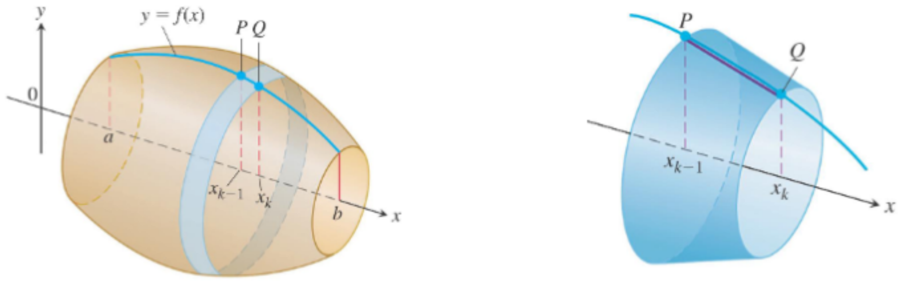


Figure 8: Cylindrical surface

Intuition We want to determine the area of a surface generated by revolving about the x-axis a curve $y = f(x)$, where f is non-negative, for $x \in [a, b]$ Consider a cylindrical surface, generated by revolving a horizontal line around the x-axis.

$$A = 2\pi y\Delta x$$

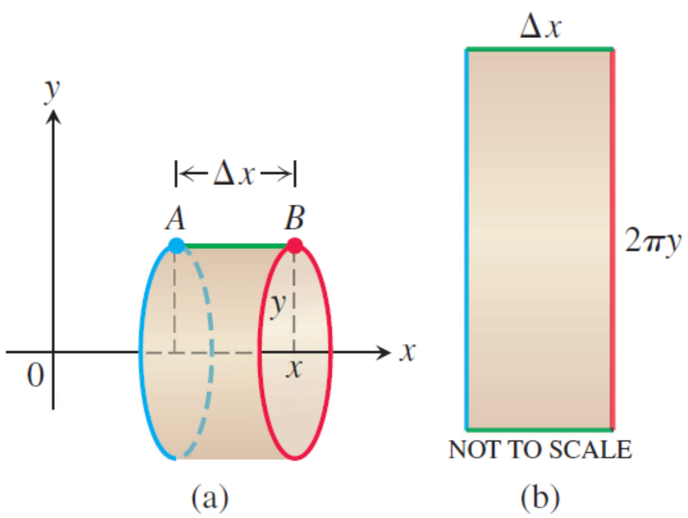


Figure 9: Cylindrical surface

Consider a conical frustum generated bu revolving a straight line around x-axis.

$$A = 2\pi y^* L$$
$$y^* = \frac{y_1 + y_2}{2}$$
$$A = \pi(y_1 + y_2)L$$

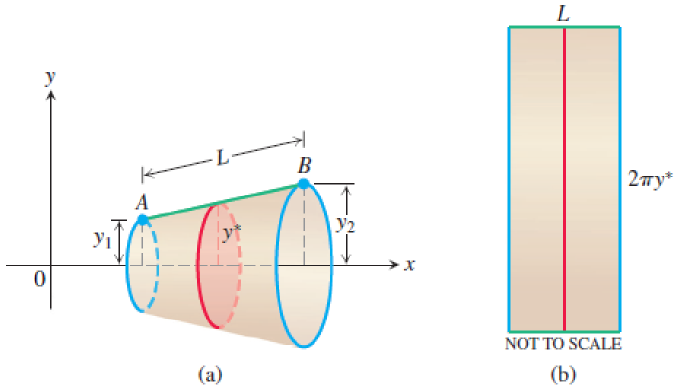


Figure 10: Conic Fustrum surface

For area of a surface of revolution about the x-axis in general, we make a partition $[a, b]$. The k portion of the curve has the length $\approx \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k$ The area is approximately

$$A \approx \pi(f(x_{k-1})) + f(x_k))\sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k$$

Definition If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

and if the function $g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2}\, dy$$

Example The curve $y = x^{1/3}$, $0 \leq x \leq 1$ is revolved about the y-axis to generate a surface. The area is?

$x = y^3$, $0 \leq y \leq 1$, $dx/dy = 3y^2$

$$S = \int_0^1 2\pi y^3 \sqrt{1 + 9y^4}\, dy$$

Let $u = 1 + 9y^4$, $du/dy = 36y^3$

$$\begin{aligned} S &= 2\pi \int_1^1 0\sqrt{u}\frac{1}{36}\,du \\ &= \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^1 0 \\ &= \frac{\pi}{27}(10^{\frac{2}{3}} - 1) \end{aligned}$$

6 Work

Intuition If a constant force F moves an object along a straight line for a distance d, then the work done by F to the object is $W = Fd$ If a variable force F moves an object the x-axis, and $F = F(x)$ depends on the position and F is continuous on $[a, b]$, we make a partition $[a, b]$. Work done from x_{k-1} to $x_k \approx F(x_k)\Delta x_k$. The total work then is approximately

$$\sum_{k=1}^{n} F(x_k)\Delta x_k$$

Definition The work done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ and $x = b$ is

$$W = \int_a^b F(x)\,dx$$

Example 1 **Hooke's Law** states that the force required to stretch or compress a spring is directly proportional to its distance x away from the natural position of the spring

$$F(x) = kx$$

What is the work required to compress a spring from its natural length of 30 cm to a length of 20 cm?

$$\begin{aligned} W = \int_0^0 .1F(x)\,dx &= \int_0^0 .1kx\,dx \\ &= \left[\frac{1}{2}kx^2\right]_0^0 .1 \\ &= 0.005kJ \end{aligned}$$

Example 2 The conical tank is filled to within 2 m of the top with olive oil weighing 0.9 g/cm^3 or 8820 N/m^3. How much work does it take to pump the oil to the rim?

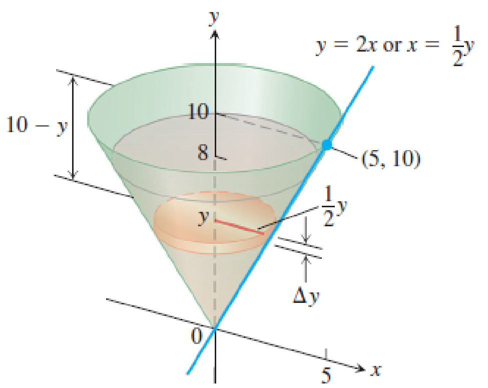


Figure 11: Illustration of the conical tank

Answer The typical volume of the slab has the volume

$$\Delta V = \pi \left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4} y^2 \Delta y$$

The force $F(y)$ is

$$F(y) = 8820\Delta V = \frac{8820\pi}{4} y^2 \Delta y$$

Hence, the work needed is

$$\Delta W = F(y)s = \frac{8820\pi}{4}(10 - y)y^2 \Delta y$$

$$\begin{aligned} W &= \int_0^8 \frac{8820\pi}{4}(10 - y)y^2 dy \\ &= \frac{8820\pi}{4} \int_0^8 10y^2 - y^3\, dy \\ &= \frac{8820\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \\ &\approx 4.73 \times 10^6 J \end{aligned}$$

7 Fluid Forces

Pressure Pressure is the force that a fluid exerts on a surface divided by the surface's area.

$$p = \frac{F}{A} \qquad \Rightarrow \qquad p = \frac{dF}{dA}$$

For a static liquid the pressure p at depth h is given by

$$p = wh$$

where w is the weight-density, or ρg.

For a container with a horizontal base, the total force applied by the fluid to the base is $F = pA = whA$ where A is the base area.

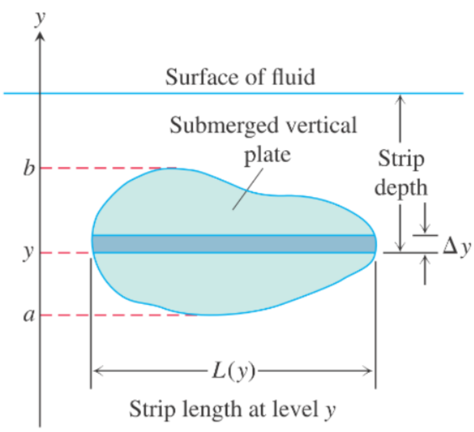


Figure 12: The force exerted by a fluid against one side of a thin horizontal strip

If a flat plate is submerged vertically, the pressure against it depends on the depth of the portion of the plate. First, we divide the plate into horizontal thin slices S_k with the width of Δy_k, depth of h_k^*, length of $L(y_k^*)$ and area of $L(y_k^*)\Delta y_k$. The force exerted on S_k is

$$F_k = whA \approx wh_k^* L(y_k^*)\Delta y_k$$

Then the total force exerted on plate:

$$F \approx \sum_{k=1}^{n} wh_k^* L(y_k^*)\Delta y_k$$

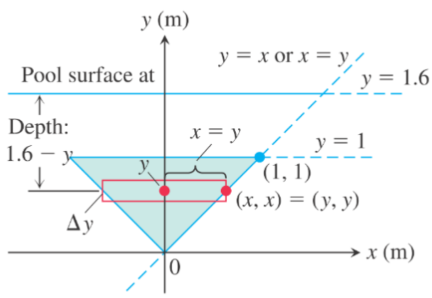
The total force then is:

$$F = \int_a^b wh(y)L(y)\, dy$$

Definition Suppose that a plate submerged vertically in the fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y and $h(y)$ is the strip depth measured from the top of the fluid, then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b wh(y)L(y)\, dy$$

Example A flat isosceles right-triangular plate with base 2m and height 1m is submerged vertically, base up, 0.6 m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.



Answer

$$\begin{aligned}
F &= \int_a^b wh(y)L(y)\, dy \\
&= \int_0^1 9800(1.6 - y)2y\, dy \\
&= 19600 \int_0^1 (1.6y - y^2)\, dy \\
&= 19600 \left[0.8y^2 - \frac{y^3}{3}\right]_0^1 \approx 9147N
\end{aligned}$$