MAT1001 Calculus I

Lecture 4 - 7 : Derivatives

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1 Derivatives

1.1 Definition

Definition: Interior Point Let $S \subseteq \mathbb{R}$ A point $c \in S$ is called an **interior point** of S if there exists $a > 0$ such that

$$(c - a, c + a) \subseteq S$$

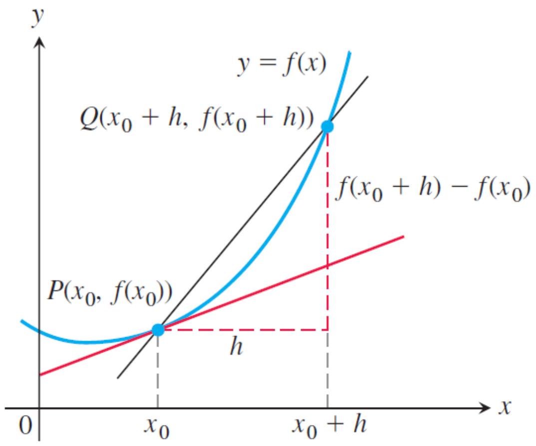


Figure 1: The slope of the tangent line at P is the derivative of $f(x)$ at P

Definition Let $f : D \to \mathbb{R}$ be a function and let x_0 be an interior point of D. Then the derivative of f at x_0, denoted by $f'(x_0)$, is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Provided that the limit exists.

For $y = f(x)$, $f'(x_0)$ can be represented as:

- The slope of the tangent line of the graph $f(x)$ at $x = x_0$
- The instantaneous rate of change of y w. r. t. x at $x = x_0$

Notations There are many ways to denote the derivative of $y = f(x)$. Some notations are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{dy}{dx} f(x) = D(f)(x) = D_x f(x)$$

We read dy/dx as "the derivative of y with respect to x," and df/dx and $(dy/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The dy/dx notations are similar to those used by Leibniz.

1.2 One-Sided Derivative

Since the definition on previous section is for interior point, it can't be applied for endpoints. For endpoints we have one-sided derivative.

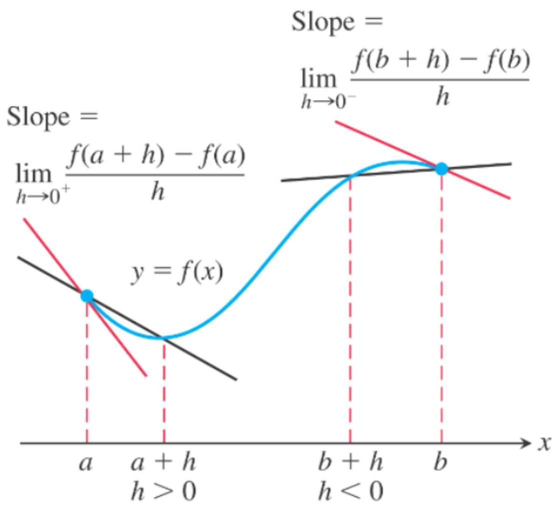


Figure 2: Derivatives of endpoints of a closed interval

Definition: Right-hand Derivative $f:[a,b]\to\mathbb{R}$ be a function. For each $x\in[a,b)$, **the right-hand derivative** of f at x_0, denoted by $f'_+(x_0)$, is defined by

$$f'_+(x_0) = \lim_{h\to0^+}\frac{f(x+h)-f(x)}{h} = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}$$

Definition: Left-hand Derivative $f:[a,b]\to\mathbb{R}$ be a function. For each $x\in(a,b]$, **the left-hand derivative** of f at x_0, denoted by $f'_-(x_0)$, is defined by

$$f'_-(x_0) = \lim_{h\to0^-}\frac{f(x+h)-f(x)}{h} = \lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$$

1.3 Differentiability

Definition A function $y=f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the right-hand derivative at a and left-hand derivative at b exists at endpoints.

Theorem 1 (Differentiability implies continuity) *If f is differentiable at c, then f is continuous at c.*

Proof Given that $f'(c)$ exists, we need to show that $\lim_{x\to c} f(x) = f(c)$

$$\begin{aligned}
\lim_{x\to c} f(x) - f(c) &= \lim_{x\to c} f(x) - \lim_{x\to c} f(c) \\
&= \lim_{x\to c} f(x) - f(c) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot (x-c) \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c) \\
&= f'(c) \cdot 0 \\
&= 0
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\lim_{x\to c} f(x) - f(c) &= 0 \\
\lim_{x\to c} f(x) &= f(c)
\end{aligned}$$

Note that, the converse of the statement is not true, that is

$$f \text{ continuous at } c \not\Rightarrow f \text{ differentiable at } c$$

Example 1 $f(x) = |x|$ is continuous but not differentiable at 0

Example 2 Show that

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous but not differentiable at 0!

To check whether $f(x)$ is continuous at 0, we need to show that $\lim_{x\to c} f(x)$ exists. Since $\sin 1/x$ is a bounded function and $\lim_{x\to 0} x = 0$ then,

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0$$

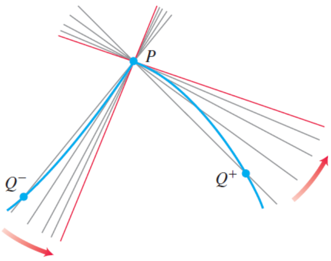
To check whether $f(x)$ is differentiable at 0, we need to show that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} &= \frac{f(h)}{h} \\
&= \lim_{h\to 0} \sin \frac{1}{h} \\
&= \text{Does Not Exist}
\end{aligned}$$

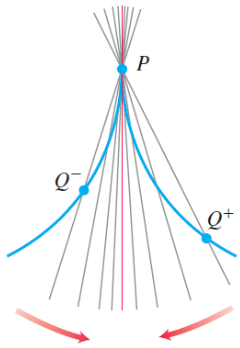
Hence, $f(x)$ is continuous but not differentiable at 0.

Non-Differentiability There are some cases that a function is not differentiable at a points, such as:

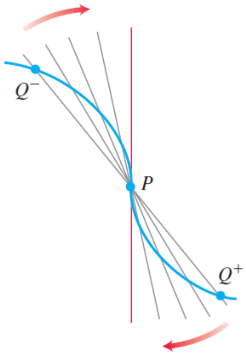
- Corner: where the one-sided derivatives differ $(f'_-(x) \neq f'_+(x))$
- Cusp : where the slope approaches ∞ from one side and $-\infty$ from the other side
- Vertical tangent : where the slope approaches either ∞ or $-\infty$from both sides.
- Discontinuity : where the function is not continuous at a point



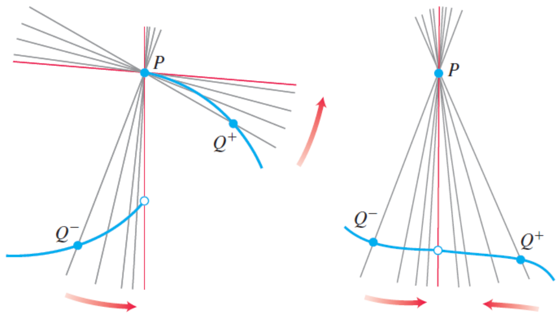
(a) corner



(b) cusp



(c) vertical tangent



(d) Discontinuity

Figure 3: Non-differentiability

2 Differentiation Rules

2.1 Constant Function

If $f(x) = c$ is a constant function, then $f'(x) = 0$, $\forall x \in D$

2.2 Rule of Linearity

The derivative of any linear combination of functions equals the same linear combination of the derivatives of the functions.
For any constants α, $\beta \in \mathbb{R}$

$$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$$

Linearity implied several rules:

Scalar Multiplication $(\alpha f(x))' = \alpha f'(x)$

Addition Rule $(f(x) + g(x))' = f'(x) + g'(x)$

Difference Rule $(f(x) + g(x))' = f'(x) - g'(x)$

2.3 Power Rule

Let $f(x) = x^n$, where $n \in \mathbb{R}$ is a constant. Then

$$f'(x) = nx^{n-1}$$

for all x where x^n and x^{n-1} are defined.

Proof Let $f(x) = x^n$, then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

From binomial expansion, we have

$$\begin{aligned}(x+h)^n &= x^n + nx^{n-1}h + \cdots + +h^n \\ &= x^n + nx^{n-1}h + O(h^2)\end{aligned}$$

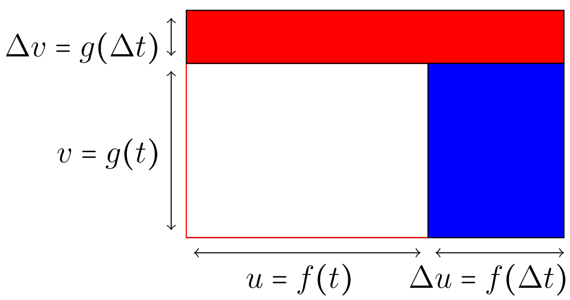
*$O(h^2)$ means all elements that contain h^n with $n >= 2$.
So,

$$\begin{aligned}f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + O(h^2) - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + O(h^2)}{h} \\ &= nx^{n-1} + 0 \\ &= nx^{n-1}\end{aligned}$$

2.4 Product Rule

if f and g are differentiable at x, then

$$(f \cdot g(x))' = f(x)g'(x) + g(x)f'(x)$$



Intuitive Proof Let $u = f(t)$ and $v = g(t)$. From the figure we can see that:

$$\begin{aligned}\text{Area of red } &= f(t+\Delta t) \cdot g(\Delta t) = f(t+\Delta t) \cdot \Delta v \\ \text{Area of blue } &= g(t) \cdot f(\Delta t) = g(t) \cdot \Delta u\end{aligned}$$

Then, we can find the changes of area Δuv :

$$\begin{aligned}\Delta(uv) &= (f \cdot g)(t+\Delta t) - (f \cdot g)(t) \\ &= f(t+\Delta t)\Delta v + g(t)\Delta u\end{aligned}$$

If we divide by Δt, we have:

$$\frac{\Delta(uv)}{\Delta t} = f(t+\Delta t)\frac{\Delta v}{\Delta t} + g(t)\frac{\Delta u}{\Delta t}$$

From the definition:

$$\begin{aligned}
\frac{d}{dt}uv = \lim_{\Delta t \to 0}\frac{\Delta(uv)}{\Delta t} &= \lim_{\Delta t \to 0} f(t+\Delta t)\frac{\Delta v}{\Delta t} + g(t)\frac{\Delta u}{\Delta t} \\
&= \left[\lim_{\Delta t \to 0} f(t+\Delta t)\right]\left[\lim_{\Delta t \to 0}\frac{\Delta v}{\Delta t}\right] + g(t)\left[\lim_{\Delta t \to 0}\frac{\Delta u}{\Delta t}\right] \\
&= f(t)g'(t) + g(t)f'(t)
\end{aligned}$$

2.5 Quotient Rule

if f and g are differentiable at x and $g(x) \neq 0$, then

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Note For formal proofs for all differentiation rules, see Chapter 3.3 of Thomas Calculus.

3 Higher Order Derivative

Definition Let $f : D \to \mathbb{R}$ be a function.

- if f' is differentiable at c then we call $f''(x)$ the second derivative of f at c, which we denote by $f''(c)$ or $f^{(2)}(c)$.
- if f'' is differentiable at c then we call $f'''(x)$ the third derivative of f at c, which we denote by $f'''(c)$ or $f^{(3)}(c)$.
- Generally, we can define the n^{th} **derivative** of f at c to be $(f^{(n-1)})'(c)$ which we denote by $f^{(n)}(c)$.

Notation The notation of higher order derivative are:

$$y^{(n)} = f^{(n)} = \frac{d^n}{dx^n}y = \frac{d^n y}{dx^n}$$

Example: the notation of second derivative are

$$y' = f' = \frac{d^2}{dx^2}y = \frac{d^2 y}{dx^2}$$

4 Application of Derivative

4.1 Mechanics

Suppose an object is moving. Its position is denoted as $s = f(t)$ at time t.

Definition

The **displacement** of the object over a time interval $[a, b]$ is

$$d = f(b) - f(a)$$

The **instantaneous velocity** of the object at time t is the derivative of position w.r.t time, we write as

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

The **speed** of the object at time t is absolute value of the velocity.

$$Speed = |v(t)| = \left|\frac{ds}{dt}\right|$$

The **acceleration** of the object at time t is the derivative of velocity w.r.t time, we write as

$$v(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

The **jerk** of the object at time t is the derivative of acceleration w.r.t time, we write as

$$v(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

When the velocity is (+) , the object moves forward. When the velocity is (-) ,the object moves backward. Meanwhile the speed is always positive.
When the acceleration is (+) , the object is speeding up (the velocity increases). When the acceleration is (-) , the object is slowing down (the velocity decreases)

4.2 Cost and Production

Suppose the cost of producing x units of good is $c(x)$. Then the marginal cost of production is the derivative of cost w.r.t. production.

$$\text{marginal cost at } x_0 \text{ unit of production} = c'(x_0) = \lim_{h \to 0} \frac{c(x_0 + h) - c(x_0)}{h}$$

or we can say "the cost needed to produce one more unit".

5 Special Derivatives

5.1 Trigonometric Function

The derivative of $\sin x$ and $\cos x$ are

$$\frac{d}{dx} \sin x = \cos x \qquad\qquad \frac{d}{dx} \cos x = -\sin x$$

Proof of the derivative of sin x First, from the definition,

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

From the trigonometry identity we have

$$\sin(x+h) = \sin x\cos h + \sin h\cos x$$

So,

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{\sin h\cos x}{h}+\lim_{h\to 0}\frac{\sin x\cos h-\sin(x)}{h}\\ &= \cos x\cdot\lim_{h\to 0}\frac{\sin h}{h}+\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}\\ &= \cos x\cdot 1+\sin x\cdot 0\\ &= \cos x\end{aligned}$$

We can also proof the derivative of $\cos x$ in the same manner but with the identity

$$\cos(x+h) = \cos x\cos h - \sin x\sin h$$

The derivative of $\tan x$ is

$$\frac{d}{dx}\tan x = \sec^2 x$$

Proof of the derivative of tan x We know that

$$\tan x = \frac{\sin x}{\cos x}$$

So, by using quotient rules

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} &= \frac{\frac{d}{dx}\sin x\cdot\cos x - \frac{d}{dx}\cos x * \sin x}{\cos^2 x}\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

The derivative of other trigonometric functions are:

$$\frac{d}{dx}\sec x = \sec x\tan x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

5.2 Natural Exponential Function

5.2.1 Euler Constant

Suppose we have 1 USD. You put in in a bank with annual interest of 100 percent. Interest is compounded n times a year. So, we have:

$$\text{Return Value } = \left(1+\frac{1}{n}\right)^n$$

As we increase n, the return value gets closer to the number, the euler constant denoted by e, where $e \approx 2.71828...$

Definition The euler constant e is defined by

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

We also have:

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

Proof: To proof that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$, We need to show that

$$\lim_{x\to 0^-}(1+x)^{\frac{1}{x}} = \lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$$

Let $x = 1/y$, so as $x \to \infty$, $y \to 0^+$. We have

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to 0^+}(1+y)^{\frac{1}{y}} = e$$

Let z satisfy

$$1+y = \frac{1}{1+z} \qquad \forall z \in [-1,0]$$

Then

$$y = \frac{1}{1+z} - 1 = -\frac{z}{1+z} \qquad \Rightarrow \qquad \frac{1}{y} = -\frac{1+z}{z}$$

So,

$$\begin{aligned}(1+y)^{\frac{1}{y}} &= \left(\frac{1}{1+z}\right)^{-\frac{1+z}{z}} \\ &= (1+z)^{\frac{1+z}{z}} \\ &= (1+z)^{\frac{1}{z}}(1+z)\end{aligned}$$

As $y \to 0^-$, $z = \frac{-y}{1+y} \to 0^+$, So

$$\begin{aligned}\lim_{y\to 0^-} &= \lim_{z\to 0^+}(1+z)^{\frac{1}{z}}(1+z) \\ &= \lim_{z\to 0^+}(1+z)^{\frac{1}{z}} \cdot \lim_{z\to 0^+}(1+z) \\ &= e\cdot 1 = e\end{aligned}$$

Compund Interest

$$\begin{aligned}P &\approx \lim_{n\to\infty} P_0\left(1+r\frac{1}{n}\right)^{nt} \\ &\approx P_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{\frac{n}{r}}\right]^{rt} \\ &\approx P_0 e^{rt}\end{aligned}$$

5.2.2 Derivative of e^x

Let $f(x) = e^x$, then $f'(x) = e^x$.

Proof First we define natural logarithm function as

$$y = e^x \Rightarrow \ln y = x$$

We have special limit forms:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

(1)

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+x)}{x} &= \lim_{x\to 0} \frac{1}{x} \cdot \ln(1+x) \\
&= \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} \\
&= \ln(\lim_{x\to 0}(1+x)^{\frac{1}{x}}) \\
&= \ln(e) = 1
\end{aligned}$$

(2)

Let $y = e^x - 1 \Rightarrow x = \ln(1+y)$ and as $x \to 0$, $y \to 0$

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1}{x} &= \lim_{y\to 0} \frac{y}{\ln(1+y)} \\
&= \frac{1}{\lim_{y\to 0} \frac{\ln(1+y)}{y}} = 1
\end{aligned}$$

$f(x) = e^x$,

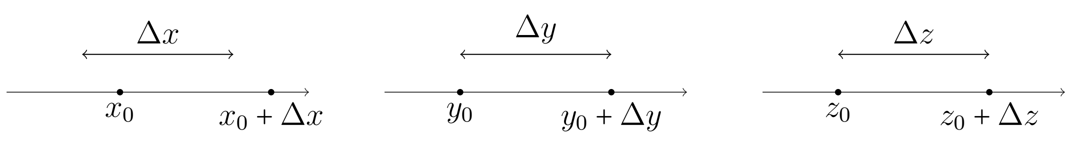
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} \\
&= e^x \cdot 1 = e^x
\end{aligned}$$

6 Chain Rule

6.1 Definition

Suppose we have three variables x, y, z, related by functions $y = f(x)$, $z = g(y)$, so $z = g(f(x))$. Then we what is dz/dx?

Intuition We have three different number lines representing each of the variables.



First fix $x = x_0$. When $\Delta x \approx 0$, we have

$$\frac{\Delta y}{\Delta x} \approx \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \left.\frac{dy}{dx}\right|_{x=x_0}$$

So, $\Delta y \approx f'(x_0) \cdot \Delta x$.

Then, when $\Delta y \approx 0$, we have

$$\frac{\Delta z}{\Delta y} \approx \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} = \left.\frac{dz}{dy}\right|_{y=y_0}$$

So,

$$\begin{aligned} \Delta z \approx g'(y_0) \cdot \Delta y = g'(f(x_0)) \cdot f'(x) \cdot \Delta x \\ \frac{\Delta z}{\Delta x} \approx g'(f(x_0)) \cdot f'(x) \end{aligned}$$

Theorem 2 (The Chain Rule) *If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differetiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example Position of an object at time T is

$$s = \cos(t^2 + 1)$$

Find the velocity!

$$\begin{aligned} v = \frac{ds}{dt} &= \frac{d}{dt} \cos(t^2 + 1) \\ &= -\sin(t^2 + 1) \cdot \frac{d}{dx} t^2 + 1 \\ &= -2t \sin(t^2 + 1) \end{aligned}$$

Proving Quotient Rule Using Chain Rule Let $u = f(x)/g(x)$, then

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx} \frac{f(x)}{g(x)} \\ &= \frac{d}{dx} f(x) \cdot \frac{1}{g(x)} \\ &= f(x) \cdot \frac{d}{dx} \frac{1}{g(x)} + f'(x) \cdot \frac{1}{g(x)} \\ &= f(x) \cdot \left(-\frac{1}{g(x)^2} \right) \cdot g'(x) + f'(x) \cdot \frac{1}{g(x)} \\ &= \frac{-f(x) \cdot g'(x)}{g(x)^2} + \frac{f'(x) \cdot g(x)}{g(x)^2} \\ &= \frac{f'(x) \cdot g(x) - g'(x) f(x)}{g(x)^2} \end{aligned}$$

6.2 Related Rates

We have from chain rule:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

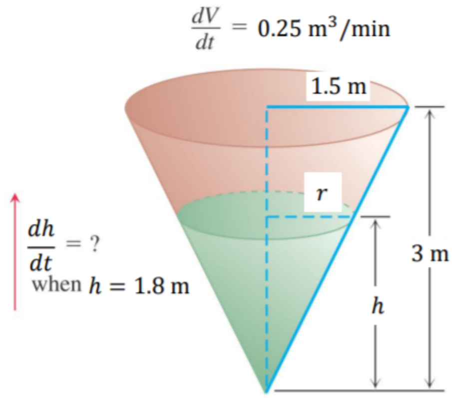


Figure 4: Illustration of example 1

Example 1 Water runs into a conical tank at the rate of 0.25 m^3/min. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

Answer : First we have

$$V = \frac{1}{3}\pi r^2 h$$

From the figure we can see that the ratio of r and h is :

$$\frac{r}{h} = \frac{1.5}{3} = \frac{1}{2}$$
$$r = \frac{1}{2}h$$

So, we have

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$V = \frac{1}{12}\pi h^3$$
$$\frac{dV}{dh} = \frac{3}{12}\pi h^2 = \frac{1}{4}\pi h^2$$

Then,

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}}$$
$$= \frac{0.25}{\frac{1}{4}\pi\cdot(1.8)^2}$$
$$= \frac{25}{81\pi}\ m/min$$

Example 2 A particle P moves clockwise at a constant rate along a circle of radius 10 m centered at the origin. The particle's initial position is (0, 10) on the y-axis, and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 s to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 m from the center of the circle?

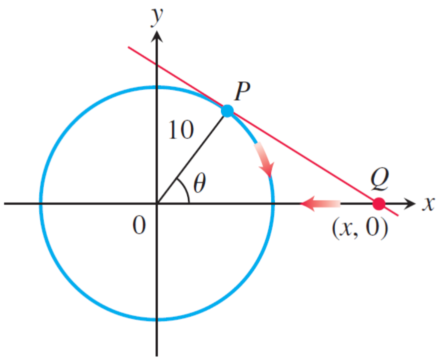


Figure 5: Illustration of example 2

Answer : From the graph we can see that

$$\begin{aligned}
\cos\theta &= \frac{10}{x} \\
\frac{d}{d\theta}\cos\theta &= \frac{d}{d\theta}\frac{10}{x} \\
-\sin\theta &= -\frac{10}{x^2}\cdot\frac{dx}{d\theta} \\
\frac{dx}{d\theta} &= \frac{\sin\theta\cdot x^2}{10} \\
&= \frac{\sqrt{x^2-100}\cdot x}{10}
\end{aligned}$$

And we also have,

$$\begin{aligned}
\frac{\pi}{2} &= \frac{d\theta}{dt}\cdot 30 \\
\frac{d\theta}{dt} &= (-)\frac{\pi}{60}
\end{aligned}$$

So,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} \\
&= \frac{\sin\theta\cdot x^2}{10}\cdot -\frac{\pi}{60} \\
&= -\frac{\sqrt{20^2-100}\cdot 20}{10}\cdot\frac{\pi}{60} \\
&= -\frac{200\sqrt{3}}{10}\cdot\frac{\pi}{60} \\
&= -\frac{1}{3}\pi\sqrt{3}\,m/s
\end{aligned}$$

7 Implicit Differentiation

7.1 Implicitly vs Explicitly Defined Equation

Explicitly defined function is a function that is defined clearly in the form of $y = ...x$. On the other hand, Implicitly defined function is a function that is expressed by an implicit equation

that relates one variable to another.
For example: $y = \sqrt{1-x^2}$ is defined explicitly, $x^2 + y^2 = 1$ is defined implicitly.

7.2 Implicit Differentiation

Let an equation defined as:

$$x^2 + y^2 = 1$$

What is slope of the tangent line at $(1/\sqrt{2}, 1/\sqrt{2})$?

There are two ways to solve the problem. First, explicit differentiation, that is we modify the equation to be an explicitly defined function.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1-x^2} \end{aligned}$$

Because $y = 1/\sqrt{2} > 0$, then $y = \sqrt{1-x^2}$. Hence,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) \\ &= \frac{-x}{\sqrt{1-x^2}} \\ \left.\frac{dy}{dx}\right|_{x=\frac{1}{\sqrt{2}}} &= \frac{-\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} = -1 \end{aligned}$$

The next method is by implicit differentiation:

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function (using chain rule)
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx

$$\begin{aligned} x^2 + y^2 &= 1 \\ 2x + 2y \cdot \frac{dy}{dx} &= 0 \\ 2y \cdot \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y} \\ \left.\frac{dy}{dx}\right|_{x=\frac{1}{\sqrt{2}}} &= \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1 \end{aligned}$$

7.3 Exception

Implicit differentiation can only be used if:

- Function doesn't have vertical tangent at the specific point
- y can be defined as function of x locally around the specific point

Example Let an equation defined as

$$x^3 + y^3 - 6xy = 0$$

At $(0,0)$, we can't find the derivative using implicit differentiation because it can't be defined as a function of x, and have a vertical tangent. However, at $(x,y) = (3,3)$, for instance, we can find the derivative:

$$\begin{aligned}
x^3 + y^3 - 6xy &= 0 \\
3x^2 + 3y^2 \cdot \frac{dy}{dx} - (6x \cdot \frac{dy}{dx} + 6y) &= 0 \\
3x^2 - 6y &= (6x - 2y) \cdot \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{3x^2 - 6y}{6x - 3y^2} \\
\left.\frac{dy}{dx}\right|_{(x,y)=(3,3)} &= \frac{3(3)^2 - 6(3)}{6(3) - 3(3)^2} = -1
\end{aligned}$$

7.4 Normal Line

For the graph of a differentiable function $y = f(x)$, normal line is a line that is perpendicular with the tangent line. The slope of the normal line at (x_0, y_0) is

$$m_{normal} = \frac{-1}{f'(x)}$$

Provided that $f'(x) \neq 0$

Example From example in 7.3, find the normal and tangent line of the graph at (3,3)

Tangent Line:

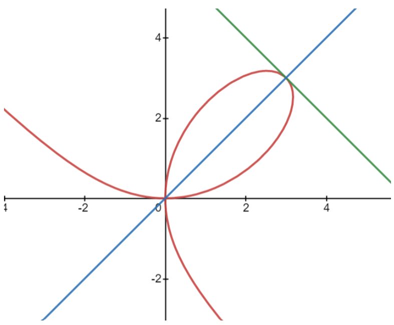


Figure 6: Graph of $x^3 + y^3 - 6xy = 0$, at $(0,0)$ it has vertical tangent. Blue line is the normal line, and the green line is the tangent line.

$$\begin{aligned}
y - 3 &= -1(x - 3) \\
y &= -x + 6
\end{aligned}$$

Normal Line:

$$
\begin{aligned}
y-3 &= \left(\frac{-1}{m_{tangent}}\right)(x-3) \\
y-3 &= \left(\frac{-1}{-1}\right)(x-3) \\
y-3 &= 1(x-3) \\
y &= x
\end{aligned}
$$

8 Linearization

8.1 Linear Approximation

Suppose we want to compute $f(x)$ for any $x \in (a-\delta, a+\delta)$, with δ close to 0. We may replace $f(x)$ with a linear function, which only provides approximation.

Definition If f is differetiable at a, for x "near" a we have

$$\frac{f(x)-f(a)}{x-a} \approx f'(a)$$

In other words,

$$\text{Slope of secant on the interval } [a-\delta, a+\delta] \approx \text{Slope of tangent line at } a$$

So, we have

$$L_a(x) = f(a) + f'(a)\cdot(x-a)$$

with

$$L_a(x) \approx f(x)$$

L is the standard linear approximation of f at a with $x = a$ as the center of approximation.

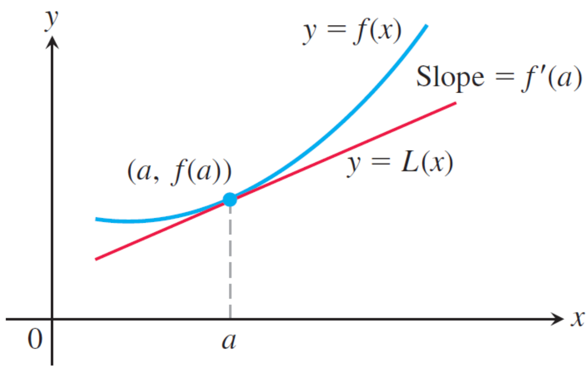


Figure 7: The tangent to the curve $y = f(x)$ at a is $L(x)$

Example 1 Approximate the numerical value of $\sqrt[3]{27.4}$ by considering linearization of $f(x) = \sqrt[3]{x}$ centered at $x = 27$

First,

$$f(27) = \sqrt[3]{27} = 3$$

Then,

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$
$$f'(27) = \frac{1}{3(9)} = \frac{1}{27}$$

So,

$$\begin{aligned} L_{27}(x) &= f(27) + f'(27) \cdot (x - 27) \\ &= 3 + \frac{1}{27} \cdot (x - 27) \\ &= 2 + \frac{x}{27} \\ L_{27}(27.4) &= 2 + \frac{27.4}{27} \approx 3.0148148 \end{aligned}$$

Example 2 Approximate $(1 + \epsilon)^k$ for small ϵ!

We have $f(x) = x^k$, we need to approximate $f(x)$ centered at $x = 1$. First:

$$f'(x) = kx^{k-1} \Rightarrow f'(1) = k$$

Hence,

$$\begin{aligned} L_1\epsilon &= f(1) + f'(1)(\epsilon) \\ &= 1 + k\epsilon \\ f(1+\epsilon) &\approx 1 + k\epsilon \end{aligned}$$

8.2 Differential

Sometimes we use the Leibniz notation dy/dx to represent the derivative of y with respect to x. However, it is not actually a ratio.

Definition Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx$$

dy depends on the value of dx and x. We call dy and dx as **differential**. They are infinitesmall, that is a "tiny tiny bit" of x and y or a very small value of x and y.

Example For $y = f(x) = x^5 + 37x$, when $x = x_0 = 1$ and $dx = 0.2$, then

$$dy = f'(x)dx = (5x^4 + 27)|_{x=1}\,(0.2) = 8.4$$

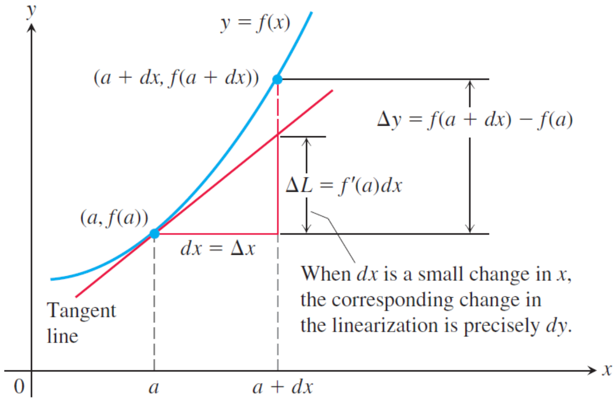


Figure 8: $\Delta L = dy$, $dy - \Delta y$ is the difference of the actual function with the linear approximation (tangent line)

Linearization using differentials Consider $L(x)$

$$\begin{aligned} L(x_0 + dx) &= f(x_0) + f'(x_0)(x_0 + dx - x_0) \\ L(x_0 + dx) &= f(x_0) + f'(x_0)(dx) \\ f'(x_0)(dx) &= L(x_0 + dx) - f(x_0) \\ dy &= L(x_0 + dx) - f(x_0) \end{aligned}$$

Hence, standard linear approximation of $f(x_0 + \Delta x)$ centered at $x = x_0$:

$$f(x_0 + dx) \approx L(x_0 + dx) = f(x_0) + dy$$

Example If the radius of a circle increases from 10 m to 10.1 m, then its change of area can be approximated:

$$A = f(r) = \pi r^2 \qquad \text{and} \qquad dr = 0.1m$$

$$\begin{aligned} f(10.1) \approx f(10) + dA &= f(10) + f'(10)dr \\ &= \pi \cdot 10^2 + 2\pi \cdot 10 \cdot 0.1 \\ &= 100\pi + 2\pi = 102\pi \end{aligned}$$

Sensitivity of Change The equation $dy = df = f'(x)$ measures how sensitive the output f is to a small change in x-value. The larger the value of f' at x, the greater the effect of a given change dx.

Example An object falls such that its position from the starting point is given by $s = 4.9x^2$. Time measurement may have an error of $\pm 0.01s$. So, the error of distance measured is:

$$ds = s'(t)dt = 9.8t \cdot dt = \pm 0.098t$$

8.3 Error of Approximation

We know that the linear approximation of $f(x)$ centered at $x = a$ is

$$L(x_0 + \Delta x) = f(x_0) + f'(x_0)(\Delta x)$$

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

Figure 9: How we describe the sensitivity of change

But what is the error of the approximation, that is

$$f(x_0 + \Delta x) - L(x_0 + \Delta x) = ?$$

First, from the definition

$$\begin{aligned} f(x_0 + \Delta x) - L(x_0 + \Delta x) &= f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x \\ &= \left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right)\Delta x \end{aligned}$$

Let $\epsilon = \left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - f'(x_0)\right)$ with $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ as the slope of secant line, and $f'(x_0)$ as the slope of tangent line. So,

$$f(x_0 + \Delta x) - L(x_0 + \Delta x) = \epsilon\Delta x$$

Since, $\epsilon = \left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - f'(x_0)\right)$, we have

$$\lim_{\Delta x \to 0} \epsilon = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0) = 0$$

Then,

$$\begin{aligned} f(x_0 + \Delta x) - L(x_0 + \Delta x) &= \epsilon\Delta x \\ \frac{f(x_0 + \Delta x) - L(x_0 + \Delta x)}{\Delta x} &= \epsilon \\ \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - L(x_0 + \Delta x)}{\Delta x} &= \lim_{\Delta x \to 0} \epsilon = 0 \end{aligned}$$

That means, as Δx become smaller and smaller, the error decreases approaching 0.

Definition If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \epsilon\Delta x$$

with Δy as the true changes, $f'(a)\Delta x = dy$ as the estimate changes, and $\epsilon\Delta x$ as the error of approximation.

8.4 Proof of Chain Rule

We want to show that $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.
First, let $y = f(x)$, $z = g(y) = g(f(x))$.

Suppose x changes by Δx. Let Δy and Δz is corresponding to the changes to y and z respectively. Then,

$$\Delta y = f'(x_0)\Delta x + \epsilon_1 \Delta x \text{ for some } \epsilon_1 \text{ with } \lim_{\Delta x \to 0} \epsilon_1 = 0$$

$$\Delta z = g'(y_0)\Delta y + \epsilon_1 \Delta y \text{ for some } \epsilon_2 \text{ with } \lim_{\Delta y \to 0} \epsilon_2 = 0$$

So,

$$\begin{aligned}
\frac{\Delta z}{\Delta x} &= g'(y_0)\frac{\Delta y}{\Delta x} + \epsilon_2 \frac{\Delta y}{\Delta x} \\
&= g'(y_0)\frac{f'(x_0)\Delta x + \epsilon_1 \Delta x}{\Delta x} + \epsilon_2 \frac{f'(x_0)\Delta x + \epsilon_1 \Delta x}{\Delta x} \\
&= (g'(y_0) + \epsilon_2)(f'(x_0) + \epsilon_1)
\end{aligned}$$

As $\Delta x \to 0$, $\epsilon_1, \epsilon_2 \to 0$, So

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} &= g'(y_0) f'(x_0) \\
\frac{dz}{dx} &= g'(f(x_0) f'(x_0)
\end{aligned}$$