MAT1001 Calculus I

Additional Notes : Lecture 1

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1 Function

1.1 Injectivity, Surjectivity and Bijectivity

Definition Let $f : D \to Y$ be a function.

- We say that f is one-to-one (or **injective**) if $f(x_1) \neq f(x_2)$ for all distinct x_1 and x_2 in D (that is, $x_1 \neq x_2$).
- We say that f is onto (or **surjective**) if, for every $y \in Y$, there exists $x \in D$ such that $f(x) = y$.
- We say that f is **bijective** if it is both one-to-one and onto. A bijective function is called a bijection.

Example

- The rule $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) := 2x$ is a one-to-one (injective) function. However, the rule $f(x) := 2x$ de

  nes a bijection between $\mathbb{Z}$ and the set of all even integers.
- The rule $f : \mathbb{R} \to \{3\}$ defined by $f(x) := 3$ (for all x in $\mathbb{R}$) is an onto (surjective) function. It is common to write $f(x) = 3$ for such a constant function.
- The function $f : \mathbb{R} \to [0, 1)$ with $f(x) := x^2$ is onto, while the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := x^2$) is not. Hence, whether a function is onto depends on its codomain and by definition any function is automatically onto its range.

1.2 Monotone Function

Definition Let $f : D \to \mathbb{R}$ be a function.

- If $f(x_1) \leq f(x_2)$ for all x_1 and x_2 in D with $x_1 < x_2$, then f is said to be **nondecreasing or weakly increasing**.
- If $f(x_1) \geq f(x_2)$ for all x_1 and x_2 in D with $x_1 < x_2$, then f is said to be **nonincreasing or weakly decreasing**.
- If $f(x_1) < f(x_2)$ for all x_1 and x_2 in D with $x_1 < x_2$, then f is said to be **increasing or strictly increasing**.
- If $f(x_1) > f(x_2)$ for all x_1 and x_2 in D with $x_1 < x_2$, then f is said to be **decreasing or strictly decreasing**.
- If f is nondecreasing or nonincreasing, then f is said to be **monotone or monotonic**.
- If f is (strictly) decreasing or increasing, then f is said to be **strictly monotone or strictly monotonic**.

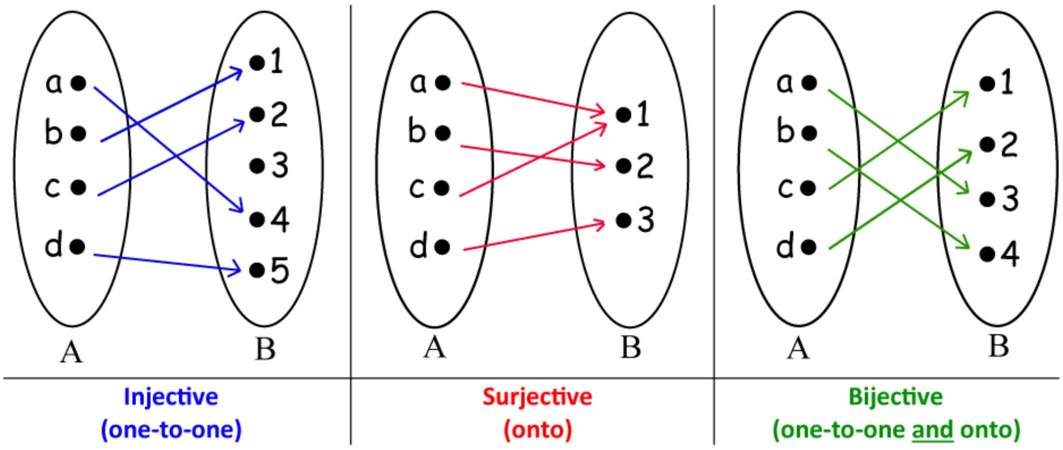

Figure 1: Examples of injective, surjective and bijective

1.3 Even/Odd Function

Definition Let $f : D \to \mathbb{R}$ be a function, where D is a subset of $\mathbb{R}$ that is symmetric about the origin. Then

- f is called an even function if $f(-x) = f(x)$ for every $x \in D$.
- f is called an odd function if $f(-x) = -f(x)$ for every $x \in D$.

Example The function $f(x) := x^2$ is even, while $f(x) := \sin x$ is odd (with domain $\mathbb{R}$).

2 Extreme Value

2.1 Maxima and Minima

Definition Let S be a nonempty subset of $\mathbb{R}$, and let $y \in S$.
We say that y is a maximum of S if $y \geq s$ for all $s \in S$.
We say that y is a minimum of S if $y \leq s$ for all $s \in S$.

Note that not every set has an extremum. However, every nonempty finite set has a maximum and a minimum.

Example The interval [a, b) has minimum a but no maximum, and any open interval (a, b) has neither a maximum nor a minimum.

2.2 Boundedness

Definition Let S be a nonempty subset of $\mathbb{R}$.

- The set S is said to be bounded above if there exists $u \in \mathbb{R}$ such that $x \leq u$ for all $x \in S$. Any such number u is called an upper bound of S.
- The set S is said to be bounded below if there exists $l \in \mathbb{R}$ such that $x \geq l$ for all $x \in S$. Any such number l is called a lower bound of S.

- A set is said to be bounded if it is both bounded above and bounded below, and it is said to be unbounded otherwise.

Note that if S has one upper bound u, then it has infinitely many upper bounds, e.g., u + 1, u + 2, ...

2.3 Suprema and Infirma

Definition Let S be a nonempty subset of $\mathbb{R}$. A real number x is called the supremum of S (or the least upper bound of S) and write $x = \sup S$, if

i x is an upper bound of S, and;

ii Let u is a set of upper bound of S then for every u, $x \leq u$.

Similarly, we call x the infirmum of S (or the greatest lower bound of S), and write $x = \inf S$, if

i x is an lower bound of S, and;

ii Let l is a set of lower bound of S then for every l, $x \geq l$.

By definition, there cannot be two different suprema or two different infirma for the same set.

The supremum and the infirmum of S do not have to belong to S even if they exist. For example, if $S = \{1/n : n \in \mathbb{N}^+\}$, then $\sup S = 1$ and $\inf S = 0$, so S contains its supremum but not its infirmum.

If $\sup S \in S$, then $\sup S = \max S$. Similarly, if $\inf S \in S$, then $\inf S = \min S$.

Theorem 1 (Least-Upper-Bound Property of $\mathbb{R}$) *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Theorem 2 (Greatest-Lower-Bound Property of $\mathbb{R}$) *Every nonempty set of real numbers that is bounded below has a greatest lower bound.*