MAT1001 Calculus I

Lecture 17 - 20 : Transcendental Functions

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1 Differentiation of Inverse Functions

1.1 Type of Functions

Definition Let $f : D \to Y$ be a function.

- We say that f is one-to-one (or **injective**) if $f(x_1) \neq f(x_2)$ for all distinct x_1 and x_2 in D (that is, $x_1 \neq x_2$).
- We say that f is onto (or **surjective**) if, for every $y \in Y$, there exists $x \in D$ such that $f(x) = y$.
- We say that f is **bijective** if it is both one-to-one and onto. A bijective function is called a bijection.

Example

- $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ is one-to-one function
- $f : [-1, 1] \to \mathbb{R}$, $f(x) = x^2$ is not one-to-one nor onto function
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$ is bijective function
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is bijective function
- $f : \mathbb{R} \to [0, \infty]$, $f(x) = x^2$ is onto function

Remarks $f : D \to range(f)$ is always onto / surjective where range$(f) = \{f(x) : x \in D\}$. By definition, every function is onto its range. Therefore, $f : D \to range(f)$ is bijective if only if it is one-to-one, since f is always onto.

1.2 Inverse Function

Let $f : D \to range(f)$ be one-to-one (bijective). The inverse function of f is the function $f^{-1} : range(f) \to D$ defined by

$$f^{-1} = x_0 \qquad \text{where} \qquad f(x_0) = y_0$$

Inverse of f is only defined when $f : D \to range(f)$ is one-to-one (injective). Since $\forall\, x_0 \in D, \exists\, y_0 \in range(f)$ such that $f^{-1}(y_0) = x_0$, we have $range(f^{-1}) = D$.

By definition:

$$\forall x_0 \in D, \qquad (f^{-1} \circ f)(x_0) = f^{-1}(f(x_0)) = x_0$$

and

$$\forall y_0 \in range(f), \qquad (f \circ f^{-1})(y_0) = f(f^{-1}(y_0)) = y_0$$

So $(f^{-1} \circ f)$ and $(f \circ f^{-1})$ are both identity function (a function that maps an element back to itself)

If f is monotonic on D then f is one-to-one on D, so $f^{-1} : range(f) \to D$ must exist.

Example 1 $f : [0,2] \to [0,4], f(x) = x^2$ has inverse $f^{-1} : [0,4] \to [0,2], f^{-1} = \sqrt{y}$

Example 2 $f : R \setminus \{0\} \to R \setminus \{0\}, f(x) = \frac{1}{x}$ is the inverse of itself $(f^{-1} = f)$ f is not monotonic on $R \setminus \{0\}$ but is monotonic on $(-\infty, 0)$ and on $(0, \infty)$ separately.

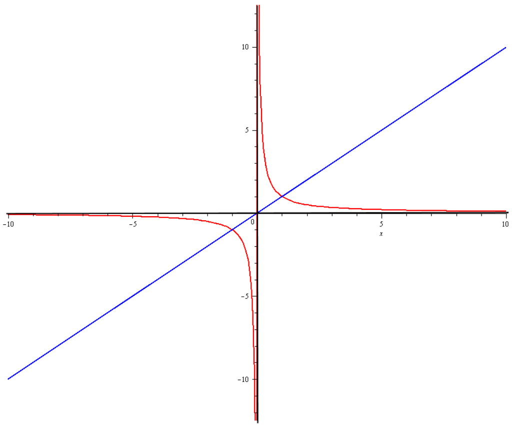

Figure 1: The graph of $1/x$, it is the inverse function of itself

Several facts we will take for granted (no proof given)

- If f is continuous and f^{-1} exists, then f^{-1} is also continuous.
- If f is continuous and its domain is an interval, then $range(f)$ is also an interval
- If f is one-to-one and continuous on an interval I then f is monotonic on I

Theorem 1 (Derivative rule for inverses) *Let $f : I \to Y$ be bijective where I is an interval and $Y = range(f)$. If f is differentiable and f' is never zero on I then f^{-1} is differentiable and*

$$(f^{-1}(y_0)) = \frac{1}{f'(f^{-1}(y_0))}$$

for all $y_0 \in Y$, or if $f(x_0) = y_0$,

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

Proof By fact 2, for $f : I \to Y$ where I is an interval, Y is also an interval.
Let $y_0 \in Y$ and let $x_0 \in I$ be such that $f(x_0) = y_0$.
For any $y_0 + h \in Y$ there is a unique $x_0 + k \in I$ such that

$$f(x_0 + k) = y_0 + h$$

Note that

$$\lim_{h \to 0} \frac{f^{-1}(y_0 + h) - f^{-1}(y_0)}{h} = \lim_{h \to 0} \frac{x_0 + k - x_0}{f(x_0 + k) - y_0}$$

Since $f(x_0 + k) = y_0 + h$, we have

$$x_0 + k = f^{-1}(y_0 + h)$$

By the fact 1, f^{-1} is also continuous, So

$$\lim_{h\to 0} f^{-1}(y_0+h) = f^{-1}(\lim_{h\to 0} y_0+h) = f^{-1}(y_0) = x_0$$

This means as $h \to 0$, $x_0 + k \to x_0$ so $k \to 0$. Hence, we have

$$(f^{-1})'(y_0) = \lim_{k\to 0} \frac{k}{f(x_0+k) - f(x_0)}$$

By definition

$$f'(x_0) = \lim_{k\to 0} \frac{f(x_0+k) - f(x_0)}{k}$$

So,

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

If we assume the differentiability of f^{-1} then the formula can be easily derived from the chain rule:

$$\begin{aligned} \because f^{-1}(f(x)) &= x \qquad \forall\, x \in I \\ \therefore (f^{-1})'(f(x_0)) \cdot f'(x_0) &= 1 \\ \Rightarrow f^{-1}(y_0) = \frac{1}{f'(x_0)} &= \frac{1}{f'(f^{-1}(y_0))} \end{aligned}$$

Remark If y_0 is an endpoint of Y, then we may consider one-sided derivative instead.

1.3 Inverse Function Examples

Inverse of sine function Let $y_0 \in (-1,1)$ and $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be such that

$$\sin x_0 = y_0$$

Then by inverse differentiation

$$\arcsin'(y_0) = \frac{1}{\sin'(x_0)} = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1-\sin^2 x_0}} = \frac{1}{1-y_0^2}$$

If $y_0 = \pm 1$, then $x_0 = \pm\frac{\pi}{2}$ so $\sin'(x_0) = \cos x_0 = 0$ and the rule does not apply as the denominator is 0. from geometric perspective, there are vertical tangent lines at $y_0 = \pm 1$ so the derivative do not exist as real numbers.

Inverse of ln function If $exp : \mathbb{R} \to (0,\infty)$ is the function $exp(x) = e^x$, then it has an inverse function

$$\ln : (0,\infty) \to \mathbb{R}$$

that is called the natural logarithmic function. Let $y_0 \in (0,\infty)$. By inverse differentiation, if $\exp(x_0) = y_0$ then

$$\ln'(y_0) = \frac{1}{\exp'(x_0)} = \frac{1}{\exp(x_0)} = \frac{1}{y_0}$$

Hence

$$\frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \forall\, x \in (0,\infty)$$

1.4 Inverse Trigonometric Function

Since trigonometric functions are not one-to-one, so its domain must be restricted to an interval I.

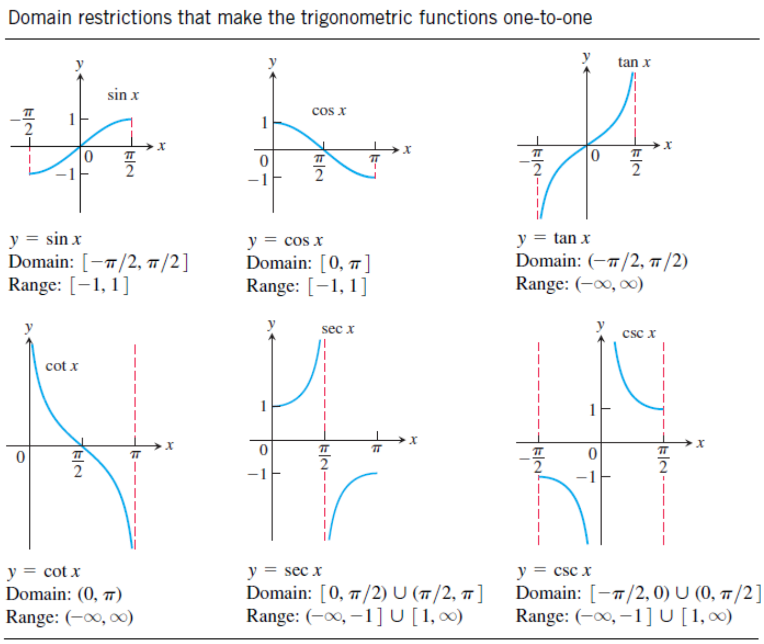

Figure 2: Trigonometric Functions with Restricted Domain

The inverse function can be obtained by reflecting the graph through the line $y = x$

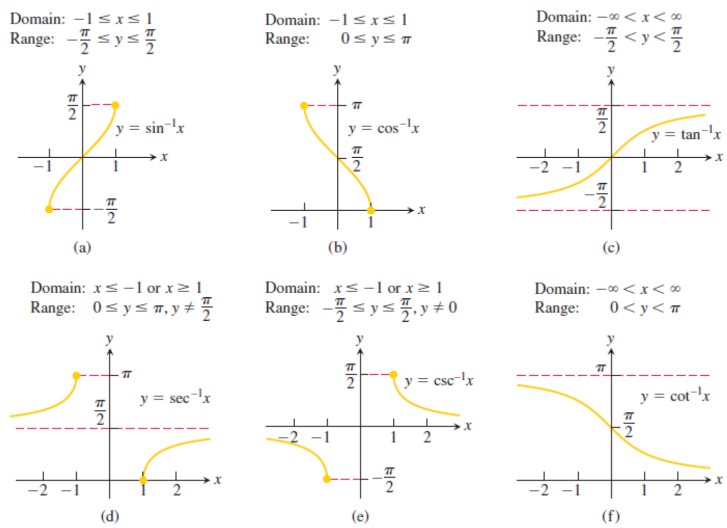

Figure 3: Graph of inverse trigonometric functions

Inverse of $\tan x$ $\quad \tan : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $\arctan : \mathbb{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
If $\arctan x_0 = y_0$ then $\tan y_0 = x_0$. By derivative rule for inverse function, we have

$$\arctan'(x_0) = \frac{1}{\tan' y_0} = \frac{1}{\sec^2 y_0} = \frac{1}{1 + \tan^2 y_0} = \frac{1}{1 + x_0^2}$$

Inverse of $\sec x$ $\quad \sec : \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \to [1, \infty) \cup (-\infty, -1]$ $\operatorname{arcsec} : [1, \infty) \cup (-\infty, -1] \to \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

If $\operatorname{arcsec} x_0 = y_0$, $x_0 \neq \pm 1$, then

$$\operatorname{arcsec}'(x_0) = \frac{1}{\sec' y_0} = \frac{1}{\sec y_0 \tan y_0} = \frac{1}{x_0 \tan y_0}$$

Since $\sec^2 y_0 = 1 + \tan^2 y_0$

$$\tan y_0 = \begin{cases} \sqrt{\sec^2 y_0 - 1} & y_0 \in \left(0, \frac{\pi}{2}\right) \\ -\sqrt{\sec^2 y_0 - 1} & y_0 \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

hence

$$\tan y_0 = \begin{cases} \sqrt{x_0^2 - 1} & x_0 > 1 \\ -\sqrt{x_0^2 - 1} & x_0 < -1 \end{cases}$$

Then,

$$\operatorname{arcsec}' x_0 = \begin{cases} \frac{1}{x_0\sqrt{x_0^2-1}} & x_0 > 1 \\ \frac{1}{-x_0\sqrt{x_0^2-1}} & x_0 < -1 \end{cases} = \frac{1}{|x_0|\sqrt{x_0^2 - 1}}$$

For other trigonometric function. We can derive it from the trigonometric identity: Since $\sin y = \cos(y - \pi/2) = \cos(\pi/2 - y)$. If $x = \sin y$ then

$$\arcsin x = y \qquad \text{and} \qquad \arccos x = \frac{\pi}{2} - y$$

Then

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

We can also show other identity:

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{\pi}{2}$$

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

Figure 4: Derivatives of inverse trigonometric functions

The following formulas hold for any constant $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Figure 5: Integral of inverse trigonometric functions

2 Natural Logarithmic Function

2.1 Defining e

Definition We define ln with domain $(0, \infty)$ by

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

Properties of natural logarithm

- $\ln 1 = 0$ since $\int_1^1 \frac{1}{t}\, dt = 0$
- $\frac{d}{dx}\ln(x) = \frac{1}{x}$ (by FTC 1)

By those properties, ln is differentiable and hence continous on $(0, \infty)$. Since $\ln'(x) = \frac{1}{x} > 0 \;\forall\, x \in (0, \infty)$ ln is increasing on $(0, \infty)$.
Note that

$$\ln 4 = \int_1^4 \frac{1}{t}\, dt = \int_1^2 \frac{1}{t}\, dt \int_2^3 \frac{1}{t}\, dt + \int_3^4 \frac{1}{t}\, dt \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$$

since $1/2, 1/3, 1/4$ is the minimum value of $\frac{1}{x}$ in the interval (by min-max inequality of integral). Hence, by IVT, since ln is increasing, $\ln(1) = 0$ and $\ln(4) > 1$, there exists x_0 such that $\ln(x_0) = 0$ (with $x_0 \in (1, 4)$)

Definition : Euler's Number We define e to be the unique number in $(0, \infty)$ such that $\ln(e) = 1$

2.2 Algebraic Properties

Theorem 2 (Algebraic properties of ln) *For any $b \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{>0}$*

1. $\ln(bx) = \ln(b) + \ln(x)$
2. $\ln(\frac{b}{x}) = \ln(b) - \ln(x)$
3. $\ln(\frac{1}{x}) = -\ln(x)$
4. $\ln(x^r) = r\ln(x)$ *for any* $r \in \mathbb{Q}$

Proof 1 Let $f(x) = \ln(bx)$, defined on $(0, \infty)$. Then

$$f'(x) = \frac{b}{bx} = \frac{1}{x} = \ln'(x)$$

So, $f(x) = \ln(x) + C$ for some constant C, $\forall\, x \in (0, \infty)$. Since it must holds for all x, let $x = 1$

$$\ln(b) = f(1) = \ln(1) + C = C$$

Hence, $f(x) = \ln(bx) = \ln(x) + \ln(b)$

Proof 4 Let $f(x) = \ln(x^r)$, defined on $(0, \infty)$ then

$$f'(x) = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d}{dx} r\ln(x)$$

So, $f(x) = r\ln(x) + C$ for some constant C, $\forall\, x \in (0, \infty)$ Since it must holds for all x, let $x = 1$

$$f(1) = \ln(1^r) = 0 = r\ln(1) + C = C$$

So, $C = 0$, and $\ln(x^r) = f(x) = r\ln(x)$

Proof 2 Let $f(x) = \ln(b/x)$

$$\begin{aligned}\ln(\frac{b}{x}) &= \ln(bx^{-1}) \\ &= \ln(b) + \ln(x^{-1}) && \text{by prop. 1} \\ &= \ln(b) - \ln(x) && \text{by prop. 4}\end{aligned}$$

Proof 3 Let $f(x) = \ln(1/x)$

$$\begin{aligned}\ln(\frac{1}{x}) &= \ln(1) - \ln(x) && \text{by prop. 2} \\ &= -\ln(1)\end{aligned}$$

2.3 Graph and Range

Range

$$\ln(2) = \int_1^2 \frac{1}{t}\, dt > (2-1)(\frac{1}{2}) = \frac{1}{2}$$

For any $n \in \mathbb{Z}_+$,

$$\ln(2^n) = n\ln(2) > \frac{n}{2}$$

Since ln is increasing,

$$\ln(x) \geq \ln(x^n) > \frac{n}{2} \qquad \forall\, x \in [2^n, \infty)$$

So, as $n \to \infty$, $\ln(2^n) \to \infty$ and $\lim x \to \infty \ln(x) = \infty$ And for $x \to 0^+$

$$\lim_{x\to 0^+} \stackrel{x=\frac{1}{u}}{=} \lim_{u\to\infty} \ln\left(\frac{1}{u}\right) = \lim_{u\to\infty}(-\ln(u)) = -\infty$$

Now let y_0 be any fixed real number. By two limits above, there exist x_1 and x_2 in $(0, \infty)$ such that

$$\ln(x_1) < y_0 \qquad \text{and} \qquad \ln(x_2) > y_0$$

By IVT, there exists $c \in [x_1, x_2]$ such that $\ln c = y_0$. Hence, $range(\ln) = \mathbb{R}$

Concavity Since

$$\ln''(x) = \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} < 0 \qquad \forall\, x \in (0,\infty)$$

the curve $y = \ln x$ is concave down.

Limits of derivatives Since

$$\lim_{x\to 0^+} \ln'(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$$

and

$$\lim_{x\to\infty} \ln'(x) = \lim_{x\to\infty} \frac{1}{x} = 0$$

$y = \ln x$ tends to have a "flat-ish" tangent line as x grows big, yet it is not bounded above.

2.4 Composite function of $\ln(x)$

Let $g : D \to R_{>0}$ be differentiable on D. Then:

$$(\ln \circ g)'(x) = \ln'(g(x))g'(x) = \frac{g'(x)}{g(x)}$$

If we take $g(x) = |x|$ with $D = \mathbb{R} \setminus 0$, then

$$g'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

so,

$$g'(x) = \frac{|x|}{x}$$

By the formula above, we have:

$$\frac{d}{dx}\ln|x| = \frac{g'(x)}{g(x)} = \frac{|x|}{x}\cdot\frac{1}{|x|} = \frac{1}{x}$$

This means that for $x \in R \setminus 0$, $\ln|x|$ is an antiderivative of $1/x$,

The Integral of 1/x If x is a differentiable function, and is never zero:

$$\int \frac{1}{x} = \ln|x| + C$$

More generally if $g(x) = |f(x)|$ where f is differentiable and never zero, then by the chain rule

$$g'(x) = \frac{|f(x)|}{f(x)} \cdot f'(x)$$

So,

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{|f(x)|}\cdot\frac{d}{dx}|f(x)| = \frac{f'(x)}{f(x)}$$

The Composite Integral of f'(x)/f(x) If $f(x)$ is differentiable and $f(x) \neq 0$

$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$$

Example 1 $\int \tan x\, dx$ and $\int \sec x\, dx$

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x} \\ &= -\int \frac{-\sin x}{\cos x}\, dx \\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

valids on where $\sec x \neq 0$

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \frac{\sec x \tan x}{\sec x \tan x} \\ &= -\int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x}\, dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

valids on where $\sec x + \tan x \neq 0, \in \mathbb{R}$.

Integrals of tangent, cotangent, secant and cosecant

- $\int \tan u\, du = \ln|\sec u| + C$
- $\int \cot u\, du = \ln|\sin u| + C$
- $\int \sec u\, du = \ln|\sec u + \tan u| + C$
- $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

Example 2

$$\int_0^{\frac{\pi}{6}} \tan 2x\, dx$$

Let $u = 2x$

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \tan 2x\, dx &= \int_0^{\frac{\pi}{3}} \tan u \frac{1}{2}\, du \\ &= \frac{1}{2}\left[\ln|\sec u|\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{2}\ln\left|\frac{\sec\frac{\pi}{3}}{\sec 0}\right| \\ &= \frac{1}{2}\ln 2\end{aligned}$$

2.5 Logarithmic differentiation

Suppose $F(x)$ involves complicated products, quotients and powers, e.g.

$$F(x) = \frac{f_1(x)^{m_1} f_2(x)^{m_2}}{f_3(x)^{m_3}}$$

Takin ln on both sides we get:

$$\ln F(x) = m_1 \ln f_1(x) + m_2 \ln f_2(x) - m_3 \ln f_3(x)$$

Differentiating both sides yields:

$$\frac{F'(x)}{F(x)} = m_1 \frac{f_1'(x)}{f_1(x)} + m_2 \frac{f_2'(x)}{f_2(x)} - m_3 f_3'(x)/f \frac{f_3'(x)}{f_3(x)}$$

Example Find y' where

$$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$$

$$\begin{aligned}
\ln y &= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2) \\
\frac{y'}{y} &= \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2} \\
y' &= \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)
\end{aligned}$$

3 Natural Exponential Function

3.1 Definition

Since $\ln : (0,\infty) \to \mathbb{R}$ is bijective, it has an inverse function

$$\exp : \mathbb{R} \to (0,\infty)$$

called the natural exponential function.
Since $\ln e = 1$ by definition, we have $e = \exp(1)$. Since $e > 0$, e^r is defined for any rational power r, we have

$$\exp(r) = e^r, \qquad \forall\, r \in \mathbb{Q}$$

For irrational power x we simply define e^x as follows:

$$e^x = \exp x, \qquad \forall\, x \in \mathbb{R} \setminus \mathbb{Q}$$

Consequently:

$$e^{\ln x} \qquad \forall\, x \in (0,\infty)$$

$$\ln(e^x) = x, \qquad \forall\, x \in \mathbb{R}$$

Since ln is differentiable and $\ln'$ is never zero, by derivative rule for inverse function, exp is also differentiable. Let $y = \exp x$ then $\ln y = x$

$$\begin{aligned}
\frac{1}{y}\cdot\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= y = \exp(x)
\end{aligned}$$

Integral of e^x

$$\exp'(x) = \exp(x) \qquad \text{or} \qquad \frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \int e^x = e^x + C$$

Example 1 $\frac{d}{dx}e^{\sqrt{3x+1}}$

$$\begin{aligned}
\frac{d}{dx}e^{\sqrt{3x+1}} &= e^{\sqrt{3x+1}}\cdot\frac{d}{dx}\sqrt{3x+1} \\
&= e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{\frac{1}{2}}\cdot 3
\end{aligned}$$

Example 2 Find

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$$

Let $u = \sin x$,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx &= \int_0^1 e^u \, du \\ &= e^1 - e^0 = e - 1\end{aligned}$$

3.2 Algebraic Properties

Theorem 3 (Algebraic Properties of e^x) *For any real number x_1, x_2 and x:*

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = 1/e^x$
3. $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$
4. $(e^x)^r = e^{rx}, \forall\, r \in \mathbb{Q}$

Proof 1 Let $y_1 = e^{x_1}, y_2 = e^{x_2}$. then

$$\begin{aligned}e^{x_1+x_2} &= e^{\ln y_1 + \ln y_2} \\ &= e^{\ln y_1 y_2} \\ &= y_1 y_2 = e^{x_1} e^{x_2}\end{aligned}$$

Proof 4 Let $y = e^x$. Then

$$e^{rx} = e^{r \ln y} = e^{\ln y^r} = y^r = (e^x)^r$$

Proof 3

$$\begin{aligned}e^{x_1 - x_2} &= e^{x_1} e^{-x_2} \\ &= e^{x_1} (e^{x_2})^{-1} \\ &= \frac{e^{x_1}}{e^{x_2}}\end{aligned}$$

Proof 2

$$\begin{aligned}e^{-x} &= e^{0-x} \\ &= \frac{e^0}{e^x} = \frac{1}{e^x}\end{aligned}$$

4 General Power and Exponential Functions

With e^x defined for all $x \in \mathbb{R}$, we can now define a^x for any $a \in (0, \infty)$ and $x \in \mathbb{R}$

Definition For any $a \in (0, \infty)$ and $x \in \mathbb{R}$ we define

$$a^x = e^{x \ln a}$$

When $a = e$, we have $a^x = e^{x \ln a} = e^{x \ln e} = e^x$

Definition: General Exponential Function with base a Fix $a \in (0, \infty)$. Define $f(x) = a^x,\ D = \mathbb{R}$

Definition: General Power Function Fix $a \in \mathbb{R}$. Define $f(x) = x^a,\ D = (0, \infty)$.

$$x^a = e^{a \ln x}$$

The algebraic propety for ln and exp also holds for irrational values of r:

1. $\ln x^a = a \ln x,\ \forall\, x \in (0, \infty),\ \forall\, a \in \mathbb{R}$
2. $(e^x)^a = e^{ax},\ \forall\, x \in \mathbb{R},\ \forall\, a \in \mathbb{R}$

(1) holds because $e^{\ln(x^a)} = x^a = e^{a \ln x}$ (2) can be shown using (1), $x^a = \ln(e^x)^a = a \ln(e^x) = ax$

4.1 Algebraic Properties

Theorem 4 (Algebraic Properties of a^x) *For $a \in (0, \infty)$ and anyreal number x_1, x_2, x, and r:*

1. $a^{x_1} e^{x_2} = a^{x_1 + x_2}$
2. $a^{-x} = 1/a^x$
3. $a^{x_1}/a^{x_2} = a^{x_1 - x_2}$
4. $(a^x)^r = a^{rx}$

Proof 1

$$\begin{aligned} a^{x_1 + x_2} &= e^{(x_1 + x_2) \ln a} \\ &= e^{x_1 \ln a + x_2 \ln a} \\ &= e^{x_1 \ln a} e^{x_2 \ln a} \\ &= a^{x_1} a^{x_2} \end{aligned}$$

4.2 General Power Rule

Theorem 5 (General Power Rule for Differentiation)

$$\frac{d}{dx} x^a = a x^{a-1}, \qquad \forall\, x \in (0, \infty)$$

Proof

$$\begin{aligned} \frac{d}{dx} x^a &= \frac{d}{dx} e^{a \ln x} \\ &= e^{a \ln x} a \frac{d}{dx} \ln(x) \\ &= e^{a \ln x} \frac{a}{x} \\ &= x^a \cdot \frac{a}{x} = a x^{a-1} \end{aligned}$$

Example 1 If $f(x) = x^x \; \forall \, x \in (0, \infty)$, what is $f'(x)$?

$$f(x) = x^x = e^{x \ln x}$$

So,

$$\begin{aligned} f'(x) &= e^{x \ln x}\left(x \cdot \frac{1}{x} + 1 \ln x\right) \\ &= e^{x \ln x}(1 + \ln x) \\ &= x^x(1 + \ln x) \end{aligned}$$

4.3 Finding e

Theorem 6 (Limit of e)

$$e = \lim_{x \to 0}(1 + x)^{\frac{1}{x}}$$

Proof

$$(1 + x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}, \qquad \forall \, x \in (-1, 1) \setminus \{0\}$$

So, by continuity of exp function, we have

$$\lim_{x \to 0}(1 + x)^{\frac{1}{x}} = \lim_{x \to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim_{x \to 0} \frac{1}{x}\ln(1+x)}$$

Now the value of the limit is:

$$\lim_{x \to 0} \frac{1}{x}\ln(1 + x) = \lim_{x \to 0} \frac{(\ln(1 + x) - \ln(1))}{x} = \ln'(1) = 1$$

Hence

$$\lim_{x \to 0}(1 + x)^{\frac{1}{x}} = e^1 = e$$

By looking at the value of $(1 + x)^{1/x}$, we get $e \approx 2.718281823845$

4.4 Derivative and Graphs

For a fixed $a \in (0, \infty)$, let $f(x) = a^x$, then $f(x) = e^{x \ln a}$, so

$$f'(x) = e^{x \ln a} \ln a = a^x \ln a$$

Derivative of a^x For a fixed $a \in (0, \infty)$ we have

$$\frac{d}{dx}a^x = a^x \ln a$$

Integral of a^x Provided that $a \neq 1$, we have

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Note that

$$\frac{d}{dx}a^x \begin{cases} > 0 \; \forall \, x & a > 1 \\ < 0 \; \forall \, x & 0 < a < 1 \end{cases}$$

So $f(x) = a^x$ is increasing on $\mathbb{R}$ if $a > 1$ and decreasing on $\mathbb{R}$ if $0 < a < 1$

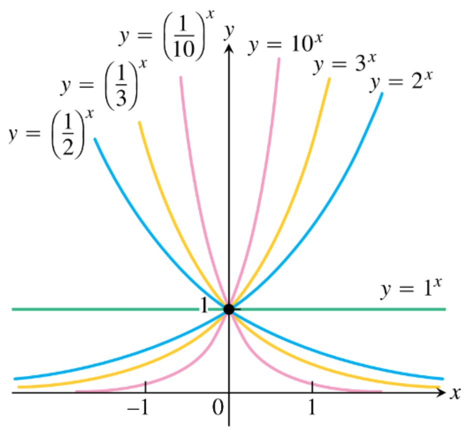

Figure 6: Graph of a^x

Since $f''(x) = (\ln a)^2 a^x > 0$, $y = a^x$ is always concave up on $\mathbb{R}$

5 General Logarithmic Functions

5.1 Algebraic Properties

Fix $a \in (0, \infty) \setminus \{1\}$. The function f_a given $f_a(x) = a^x$ is monotonic, so it is one-to-one domain $\mathbb{R}$. From the definition $a^x = e^{x \ln a}$ and the fact that range(exp) $= (0, \infty)$, it follows that range of f_a is also $(0, \infty)$. Hence, $f_a : \mathbb{R} \to (0, \infty)$ has an inverse function $\log_a : (0, \infty) \to \mathbb{R}$ called the logarithmic function with base a.
By definition,

$$a^{\log_a x} = x, \qquad \forall\, x \in (0, \infty)$$

and

$$\log_a a^x = x, \qquad \forall\, x \in \mathbb{R}$$

Since $a^{\log_a x} = x$,

$$\ln a^{\log_a x} = \ln x$$

So,

$$\log_a x = \frac{\ln x}{\ln a}$$

Hence, with that identity, we can show that the algebraic properties for ln also hold for $\log_a$

Property 1

$$\begin{aligned}\log_a xy &= \frac{\ln xy}{\ln a} \\ &= \frac{\ln x + \ln y}{\ln a} \\ &= \log_a x + \log_a y\end{aligned}$$

5.2 Derivatives

From the identity $\log_a x = \ln x / \ln a$, we can see that

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

6 L'hospital's Rule

6.1 Rule and Examples

In some cases, limit can takes the indeterminate forms, such as 0/0 and ∞/∞, for example:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} \qquad \lim_{x\to\infty}\frac{\ln x}{x^2}$$

Theorem 7 (L'Hospital's Rule) *Let $c\in\mathbb{R}$, suppose that f and g are differentiable on $D=(c-a,c+a)\setminus\{c\}$ for some $a>0$ and that $g'(x)\neq 0$ for all $x\in D$. Suppose that one of the two following conditions holds:*

- $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$
- $\lim_{x\to c} f(x) \in \{-\infty,\infty\}$ *and* $\lim_{x\to c} g(x)\in\{-\infty,\infty\}$

Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (or is ∞ or $-\infty$)

Remarks L'Hospital's rule is also valid if we replace it with one-sided limit. If D is changed to an unbounded interval, then L'Hospital's rule is also valid if we replace it the limit $x\to c$ with $x\to\infty$ or $x\to-\infty$

Example 1 Show that

$$\lim_{x\to 1}\frac{x^3+3x-4}{x^4-6x^2+5} = \frac{-3}{4}$$

Since the limit take the form 0/0, we can apply L'Hospital's Rule.

$$\begin{aligned}\lim_{x\to 1}\frac{x^3+3x-4}{x^4-6x^2+5} &= \lim_{x\to 1}\frac{3x^2+3}{4x^3-12x}\\ &= \frac{3+3}{4-12} = \frac{-3}{4}\end{aligned}$$

Example 2 Show that

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}$$

Since the limit take the form 0/0, we can apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{\sin x}{6x}$$

Since it takes the form 0/0 again.

$$\lim_{x\to 0}\frac{\sin x}{6x} = \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

Example 3 Show that

$$\lim_{x\to 0}\frac{\ln^3(x+1)}{e^x-x-1}=0$$

Since the limit take the form 0/0, we can apply L'Hospital's Rule.

$$\lim_{x\to 0}\frac{\ln^3(x+1)}{e^x-x-1}=\lim_{x\to 0}\frac{3\ln^2(x+1)}{(x+1)(e^x-1)}$$

Since it takes the form 0/0 again.

$$\lim_{x\to 0}\frac{3\ln^2(x+1)}{(e^x-1)}=\lim_{x\to 0}\frac{6\ln(x+1)}{e^x(x+1)}=\frac{0}{1}=0$$

Example 4 Show that

$$\lim_{x\to 2}\frac{\sin(\pi x)}{(x-2)^2}$$

does not exist. Since the limit take the form 0/0, we can apply L'Hospital's Rule.

$$\lim_{x\to 2}\frac{\sin(\pi x)}{(x-2)^2}=\lim_{x\to 2}\frac{\pi\cos(\pi x)}{2(x-2)}$$

Taking the left-hand limit we get $-\infty$ and the right hand limit is ∞, hence the limit does not exist.

Example 4 Show that

$$\lim_{x\to\infty}\frac{\ln x}{x^\pi}=0$$

Since the limit take the form ∞/∞, we can apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{\ln x}{x^\pi}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\pi x^{\pi-1}}=\lim_{x\to\infty}\frac{1}{\pi x^\pi}=0$$

There are other indeterminate forms such as $\infty\cdot 0$, 1^∞, and $\infty-\infty$. By transforming such forms into $0/0$ or $\pm\infty/\pm\infty$, L'Hospital's rule can be used.

Example 5 Show that

$$\lim_{x\to\infty}x^2e^{-\sqrt{x}}=0$$

The form is $\infty\cdot 0$

$$\begin{aligned}
\lim_{x\to\infty}x^2e^{-\sqrt{x}} &= \lim_{x\to\infty}\frac{x^2}{e^{\sqrt{x}}}\\
&= \lim_{x\to\infty}\frac{2x}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{4x^{\frac{3}{2}}}{e^{\sqrt{x}}}\\
&= \lim_{x\to\infty}\frac{6x^{\frac{1}{2}}}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{12x}{e^{\sqrt{x}}}\\
&= \lim_{x\to\infty}\frac{12}{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{24\sqrt{x}}{e^{\sqrt{x}}}\\
&= \lim_{x\to\infty}\frac{12}{\sqrt{x}e^{\sqrt{x}}\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{24}{e^{\sqrt{x}}}=\frac{24}{\infty}=0
\end{aligned}$$

Example 5 Show that

$$\lim_{x\to 0}(1-2x)^{\frac{3}{x}} = e^{-6}$$

The form is 1^∞

$$\begin{aligned}\lim_{x\to 0}(1-2x)^{\frac{3}{x}} &= e^{\frac{3}{x}\ln(1-2x)} \\ &= e^{\lim_{x\to 0}\frac{3}{x}\ln(1-2x)}\end{aligned}$$

$$\lim_{x\to 0}\frac{3}{x}\ln(1-2x) = \lim_{x\to 0}\frac{3}{1-2x}(-2) = -6$$

Hence,

$$\lim_{x\to 0}(1-2x)^{\frac{3}{x}} = e^{-6}$$

Example 5 Show that

$$\lim_{x\to 0}\frac{1}{x} - \frac{1}{\sin x} = 0$$

The form is $\infty - \infty$

$$\lim_{x\to 0}\frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} \qquad = \lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0}\frac{-\sin x}{\cos x + \cos x - \sin x} = \frac{0}{2} = 0$$

6.2 L'Hospital's Limitations

To apply L'Hospital's rule, make sure to stop at the right step, that is when the indeterminate forms do not occur anymore. For example

$$\lim_{x\to 0}\frac{x^2}{x^2+\sin x} = \lim_{x\to 0}\frac{2x}{2x+\cos x} = \lim_{x\to 0}\frac{2}{2-\sin x} = \frac{2}{2} = 1$$

The second equality fails because the second limit does not satisfy the assumption of L'Hospital's rule. L'Hospital's rule has its limitation. For some limit, it will only go back to the original form. For example:

$$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2} \qquad \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

6.3 Cauchy's Mean Value Theorem

Theorem 8 (Cauchy's Mean Value Theorem) *Suppose f and g are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $g'(x) \neq 0$ throughout (a,b). Then there exists $c \in (a,b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

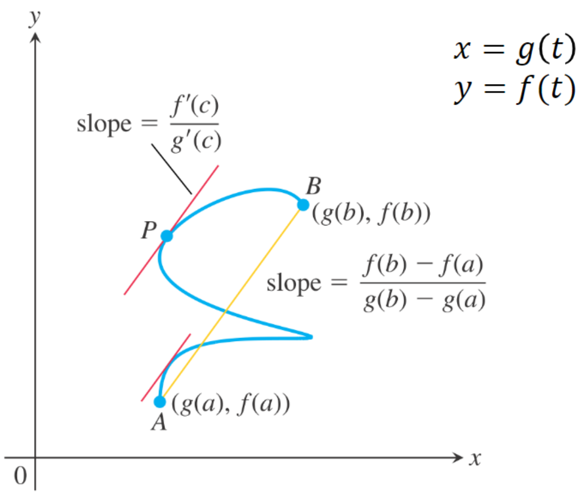

Figure 7: There is at least one point P on curve C for which the slope of the tangent is the same as the secant joining A and B

Proof Note that $g(b) \neq g(a)$, since by MVT

$$g(b) = g(a) + g'(c_0)(b-a), \qquad \text{for some } c_0 \in (a,b)$$

and we assume $g'(c_0) \neq 0$ (Rolle's Theorem).
Now define h by

$$h(t) = f(a) + \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(t)-g(a)) - f(t)$$

and note that $h(a) = 0 = h(b)$. Since h is continuous on $[a,b]$ and differentiable on (a,b), by Rolle's Theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$ means:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

6.4 Proof of L'Hospital's Rule

We will prove L'Hospital's rule for special case where: f and g are differentiable on $(c-a, c+a)$ for some $a > 0$, $g'(x) \neq 0$, $\forall\, x \in (c-a, c+a) \setminus \{c\}$ and $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$
First we will show for $\lim_{x\to c^+}$ Pick any $x \in (c-a, c+a)$ and consider Cauchy's MVT on $[c,x]$, there exists $x_0 \in (c,x)$ such that

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x)-f(c)}{g(x)-g(c)}$$

Since f and g are differentiable at c, they are also continuous at c, So

$$f(c) = \lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x) = g(c)$$

Hence,

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f(x)}{g(x)}$$

Hence, as $x \to c^+, x_0 \to c^+$ as well, so

$$\begin{aligned}
\lim_{x \to c^+} \frac{f(x)}{g(x)} &= \lim_{x \to c^+} \frac{f'(x_0)}{g'(x_0)} \\
&= \lim_{x_0 \to c^+} \frac{f'(x_0)}{g'(x_0)} \\
&= \lim_{x \to c^+} \frac{f'(x)}{g'(x)}
\end{aligned}$$

Proof for $\lim_{x \to c^-}$ is similar. Hence the rule holds for $\lim_{x \to c}$

7 Relative Rates of growth

7.1 Search Algorithm

Suppose we have n numbers $x_1, x_2, ..., x_n$ listed in increasing order. We want to find a target number T and we know $T = x_k$ for some $k \in \{1, 2, ..., n\}$.

Linear Search / Sequential Search

1: $i \leftarrow 1$
2: **if** $x_i = T$ **then**
3:     return i
4: **else**
5:     $i \leftarrow i + 1$
6:     go to step 2
7: **end if**

The worst case scenario, where $x_n = T$, n loops of this algorithm are required with each loop containing two steps.

Binary Search

1: $L \leftarrow 1$ and $R \leftarrow n$
2: $M \leftarrow \lfloor (L + R)/2 \rfloor$
3: **if** $x_M = T$ **then**
4:     return M
5: **else**
6:     **if** $x_M < T$ **then**
7:         $L \leftarrow M + 1$
8:         go to step 2
9:     **end if**
10:     **if** $x_M > T$ **then**
11:         $R \leftarrow M - 1$
12:         go to step 2
13:     **end if**
14: **end if**

In general, for a list of n numbers, the binary search requires at most $\lfloor \log_2 n \rfloor$ times a constant number of steps to find T.
In the theory of computational complexity **linear search** is said to have class $O(n)$ and **binary search** is said to have class $O(\log n)$

7.2 Rates of Growth

Definition Let $f(x)$ and $g(x)$ be positive for x sufficiently large.
f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

We also say that g grows slower than f as $x \to \infty$
f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

Example 1 If k is a positive constant, then

$$\lim_{x\to\infty} \frac{kf(x)}{f(x)} = k$$

so kf and f always grow at the same rate.

Example 2 x^x and b^x: for any fixed $b \in (0, \infty)$

$$\lim_{x\to\infty} \frac{x^x}{b^x} = \lim_{x\to\infty} \left(\frac{x}{b}\right)^x = \lim_{x\to\infty} e^{x \ln(\frac{x}{b})} = \infty$$

so x^x grows faster than b^x

Example 3 b^x vs a^x if $b > a > 0$ then

$$\lim_{x\to\infty} \frac{b^x}{a^x} = \lim_{x\to\infty} \left(\frac{b}{a}\right)^x = \lim_{x\to\infty} e^{x \ln(\frac{b}{a})} = \infty$$

so b^x grows faster than a^x for $b > a > 0$

Example 4 a^x vs x^n if $a > 1$ and $n \in \mathbb{Z}_+$

$$\begin{aligned}\lim_{x\to\infty} \frac{a^x}{x^n} &= \lim_{x\to\infty} \frac{a^x \ln a}{nx^{n-1}} \\ &= \lim_{x\to\infty} \frac{a^x (\ln a)^2}{n(n-1)x^{n-2}} = \cdots = \lim_{x\to\infty} \frac{a^x (\ln a)^n}{n!} = \infty\end{aligned}$$

So, a^x grows faster than x^n for $a > 1$ and $n \in \mathbb{Z}_+$

Example 5 x^n vs $\ln x$ for $n \in \mathbb{Z}_+$

$$\lim_{x\to\infty} \frac{x^n}{\ln x} = \lim_{x\to\infty} \frac{nx^{n-1}}{\frac{1}{x}} = \lim_{x\to\infty} nx^n = \infty$$

So, x^n grows faster than $\ln x$ for $n \in \mathbb{Z}_+$

Example 6 $\log_a$ and $\log_b$ for $a > 1$ and $b > 1$

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln a}}{\frac{\ln x}{\ln b}} = \frac{\ln b}{\ln a} > 0$$

So, log functions with base $a, b > 1$ all grow at the same rate

Transitive Relation If f and g grow at the same rate and g and h also grow at the same rate, the so do f and h

Proof

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L_1 > 0 \qquad \lim_{x\to\infty} \frac{g(x)}{h(x)} = L_2 > 0$$

Hence

$$\lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} \right) = L_1 L_2 > 0$$

Example $\sqrt{x^2 + 2021}$ and $(98\sqrt{x} - 1)^2$ grow at the same rate since they both grow at the same rate as $f(x) = x$

7.3 Big-Oh and Little-oh Notation

Definition: Little-oh A function f is of smaller order that g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

We denote this as $f = o(g)$ or f is little-oh of g.

Definition: Big-oh Let $f(x)$ and $g(x)$ be positive for x sufficiently large, then f is of at most the order of g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M$$

for x sufficiently large. We indicate this by writing $f = O(g)$ or f is big-oh of g

Example 1 Show that $\log_2 x = o(x)$

$$\lim_{x\to\infty} \frac{\log_2 x}{x} = \lim_{x\to\infty} \frac{1}{x \ln 2} = 0$$

Example 2 Show that $x = o(x^2)$

$$\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Example 3 Show that $x = O(x^2)$ is also true

$$\frac{x}{x^2} = \frac{1}{x} \leq 1 \ \forall\, x \geq 1$$

For x sufficiently large. Hence $x = O(x^2)$

8 Limits of Product and Quotients

Suppose that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$, then

$$\lim_{x\to a} f(x)h(x) = \lim_{x\to a} g(x)\frac{f(x)}{g(x)}h(x) = L\lim_{x\to a} g(x)h(x)$$

provided that the limts exist. Also,

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{g(x)}{h(x)} \cdot \frac{f(x)}{g(x)} = L\lim_{x\to a} \frac{g(x)}{h(x)}$$

Replacing $f(x)$ with $Lg(x)$ in a product or quotient will keep the limit unchanged. It also applies to infinite limite (i.e. $x \to \pm\infty$). If $\lim_{x\to a} f(x)/g(x) = 1$, then we may write $f(x) \approx g(x)$ as $x \to a$

Example 1

$$\sin x \approx x \text{ as } x \to 0 \because \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1$$

Example 2

$$\tan x \approx x \text{ as } x \to 0 \because \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \sec^2 x = 1$$

Example 3

$$e^x - 1 \approx x \text{ as } x \to 0 \because \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} e^x = e^0 = 1$$

Example 4

$$\ln(1+x) \approx x \text{ as } x \to 0 \because \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$$

Example 5 Compute

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x - x\cos x}{x^3 \cos x} = \lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3 \cos x}$$

Since

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

we may replace $1 - \cos x$ with $1/2x^2$ in limits of products an quotients as $x \to 0$

$$\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3 \cos x} = \lim_{x\to 0} \frac{x(\frac{1}{2}x^2)}{x^3 \cos x} = \lim_{x\to 0} \frac{1}{2\cos x} = \frac{1}{2}$$

Note We cannot replace $\tan x$ and $\sin x$ with x above because the limit does not exists. Hence we cannot make a replacement in a sum or difference.