Recommenders in the Wild

...a tutorial

RecSys 2023 | Singapore

Morten Arngren
Lead Data Scientist

Kim Falk
Principal Data Scientist

Morten Arngren

PhD in Machine Learning in 2011
ex-Nokia | ex-Issuu | ex-Adform
(My first Rec. Engine)

Lead Data Scientist at

WUNDERMAN

Kim Falk

Author: "Practical Recommender Systems"
ex-Adform | ex-Shopify

Principal Data Scientist at ViSenze

Recommenders in the Wild

Its a big jungle out there.....

Prepare you for entering the wild

Understanding of what A/B test is...to prepare you

Own experience

Practical angle...

What is a Recommender in the Wild

Developing the Recommender

Business considerations

Beyond accuracy

Personalisation - the myth of the long sessions

RecSysOps

14:10 Kim Falk 15:20

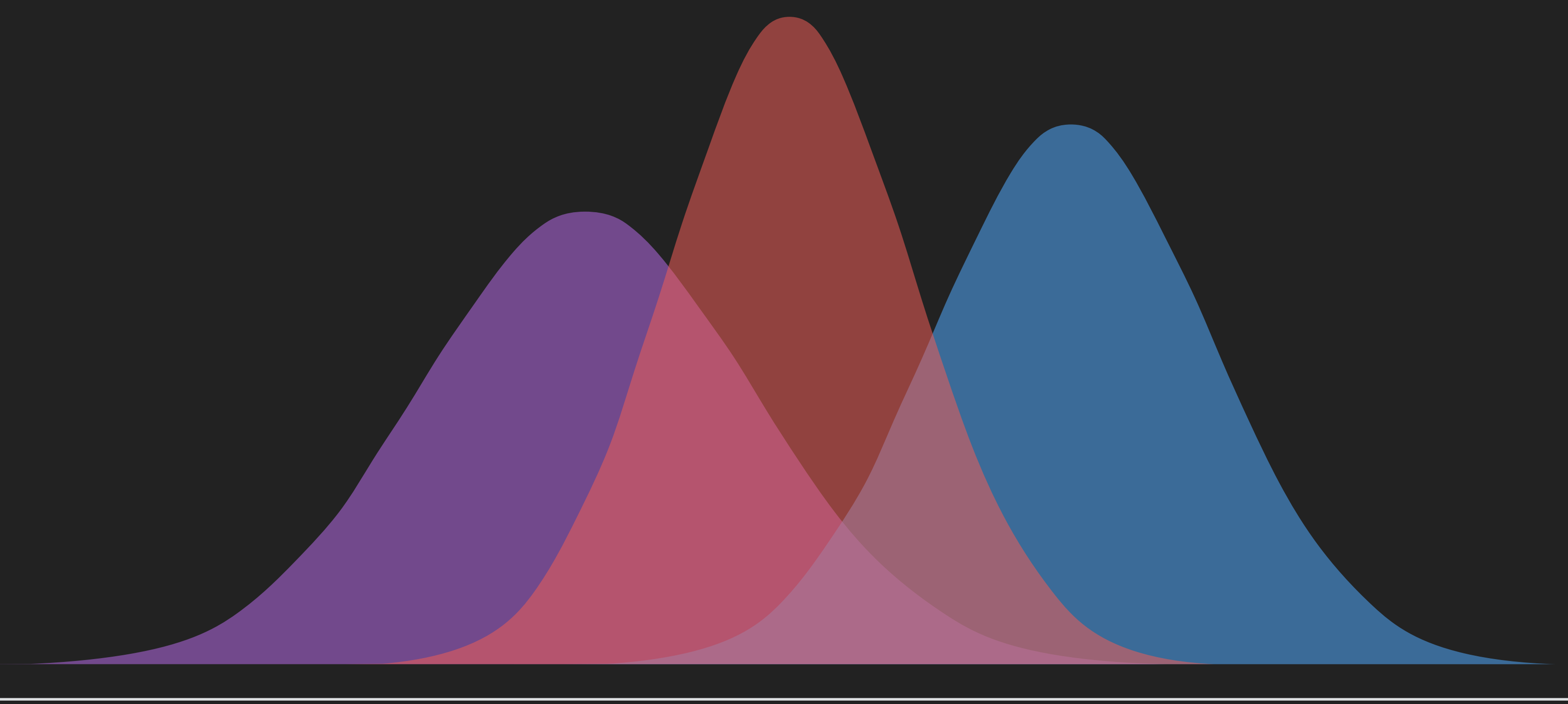
A/B test deep dive from a practical angle

Classic A/B test

Bayesian A/B test

Bandit A/B test

16:05 Morten Arngren 17:35



A/B-Testing
......Bayesian
....and Bandits

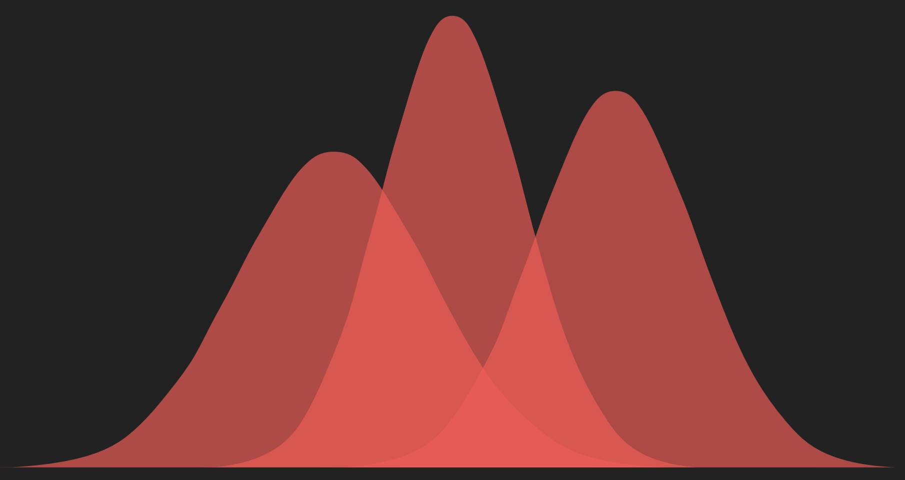
RecSys 2023
Singapore

Morten Arngren
Lead Data Scientist
WUNDERMAN

Classic & Complex A/B testing

Hypothesis testing and the infamous

p-value



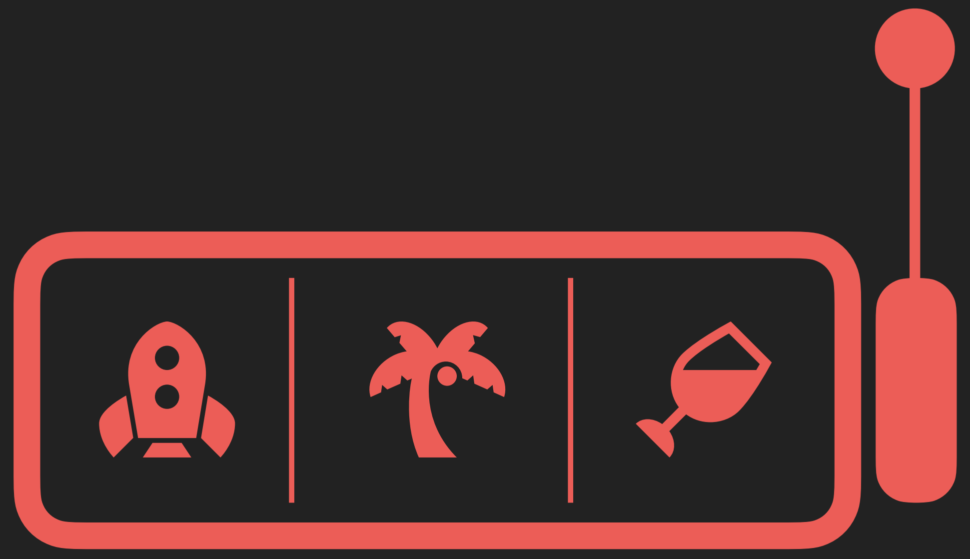
Bayesian A/B testing

A/A-testing

Bayesian evaluation

Campaign

Simulation



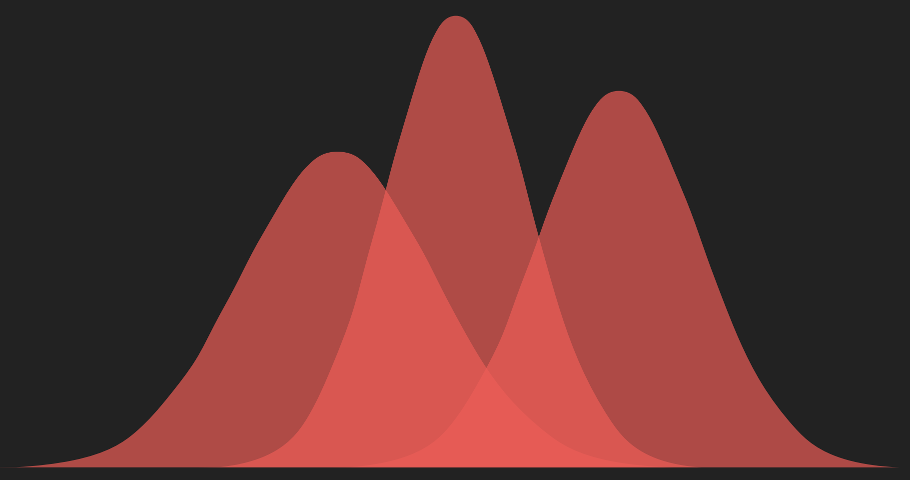
Bandit testing

Dynamic A/B-testing

Classic & Complex A/B testing

Hypothesis testing and the infamous

p-value



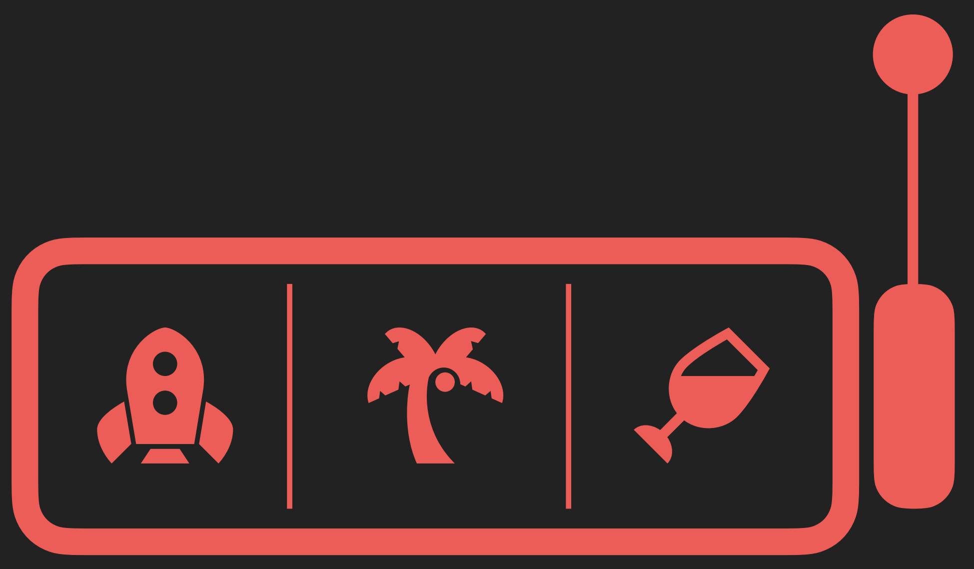
Bayesian A/B testing

A/A-testing

Bayesian evaluation

Campaign

Simulation



Bandit testing

Dynamic A/B-testing

Code and keynote available on GitHub

https://github.com/Arngren

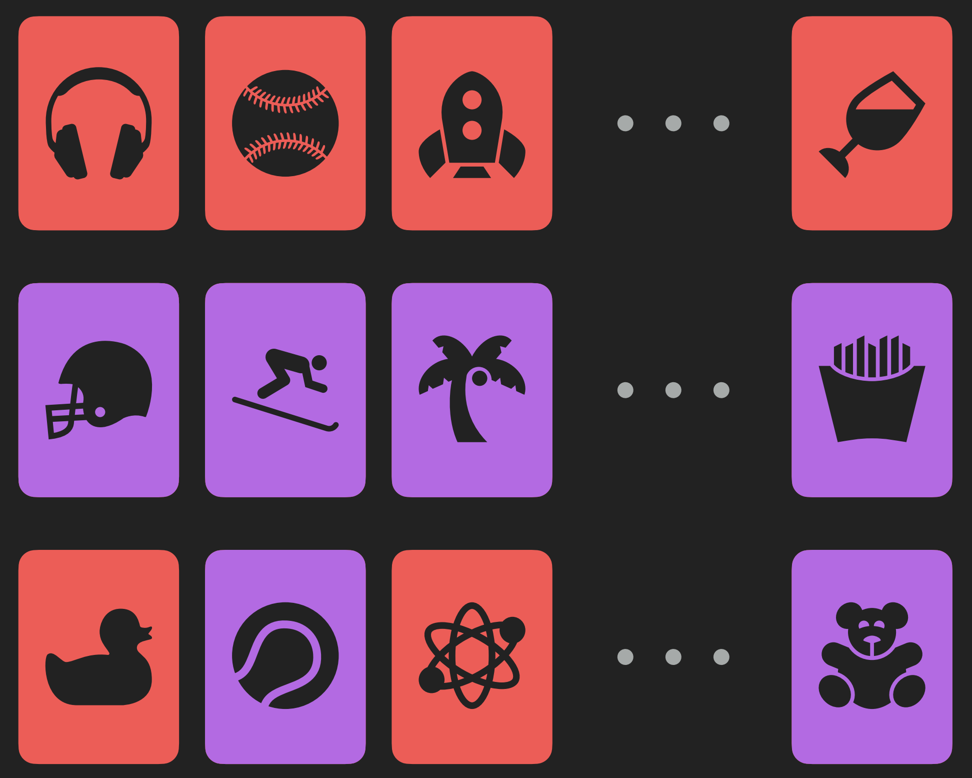
The stage

The stage

Two Rec. Engines

Movie Streaming

Recommendations



Slates

A

Running in production

New developed Rec. Engine

B

which has
the best
performance
?

News site

Advertisement

Advertisements

A/B testing

Classic approach

A

B

state null & alternative hypothesis

H_0 H_1

define metric CTR, CVR, CpC?

define experiment

assume specific data distribution

which test to use?

calc. sample size to achieve significance

....

activate

execute campaigns

statistical test

p-value < 5%?

reject H_0?

or

bayesian test

p(B>A)
Loss?

B

decide

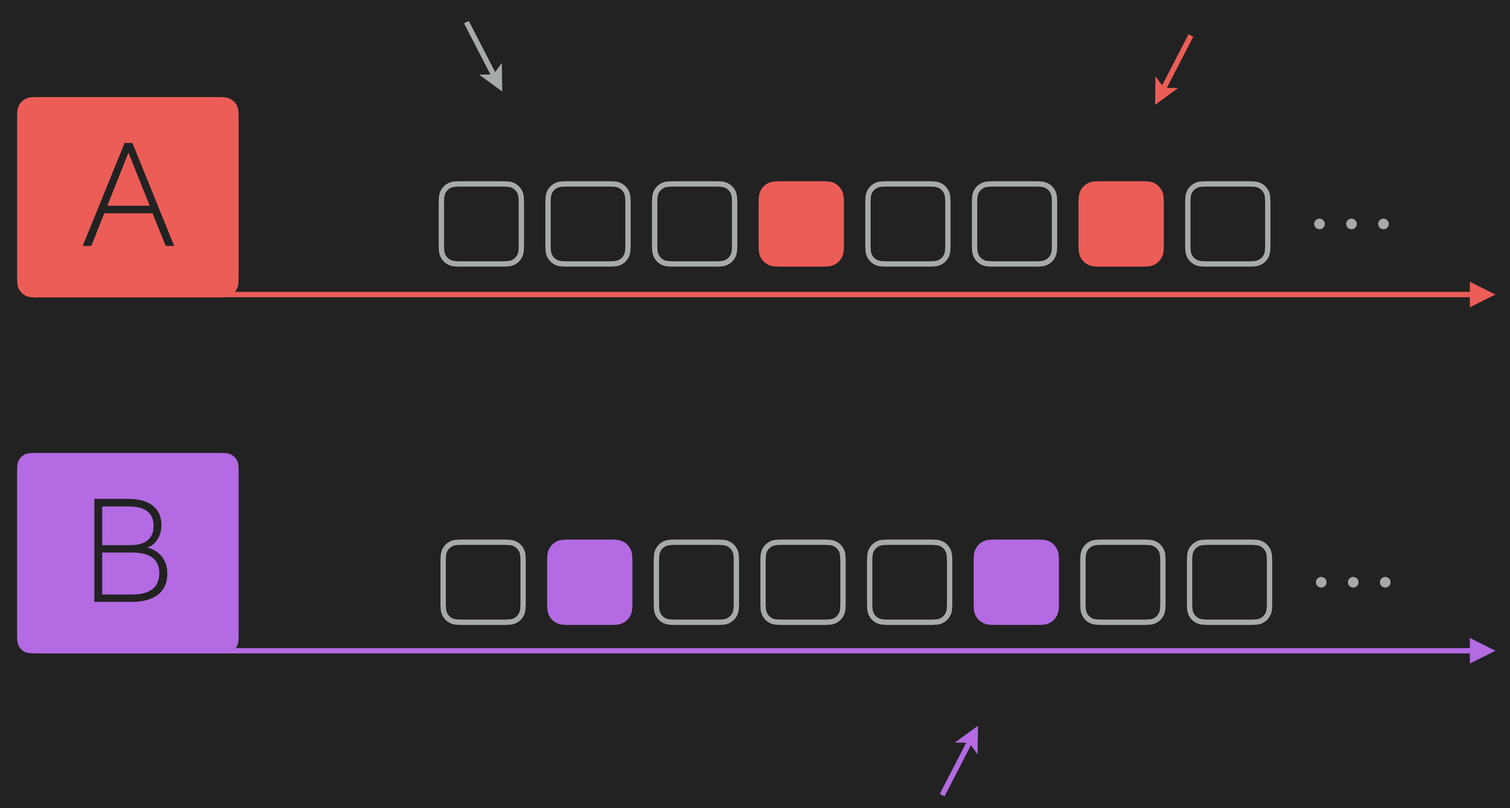
...and go home

The stage

....

logs

impressions clicks / conversions



clicks / conversions

business metrics

CTR | Click-Through-Rate
Fraction of clicks? [%]

CVR | Conversion Rate
Product buys [%]

CpC | Cost-per-Click
Price of a click [$]

??? | Webshops margins
Earnings [$]

⋮

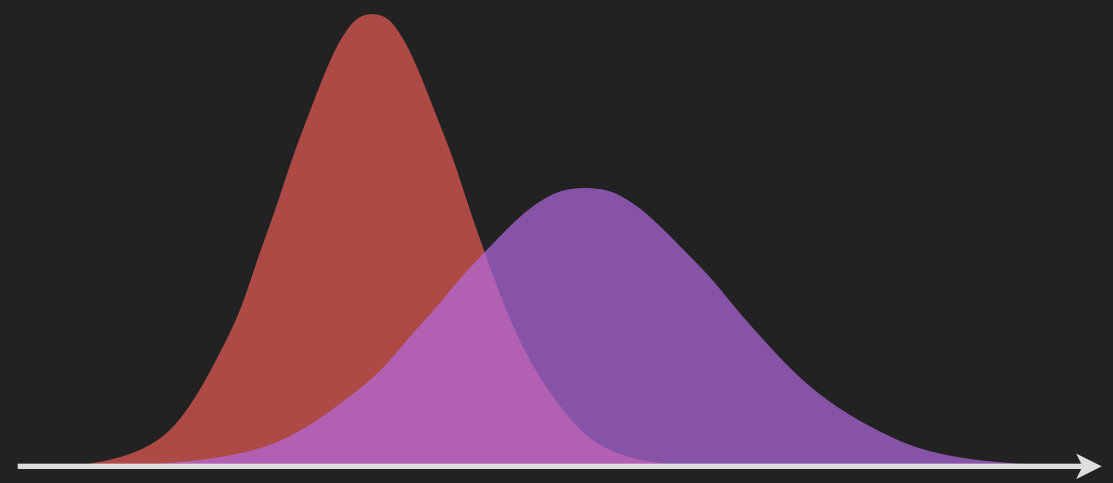
evaluation

Hypothesis testing

p-value < 5%?

or

Bayesian testing

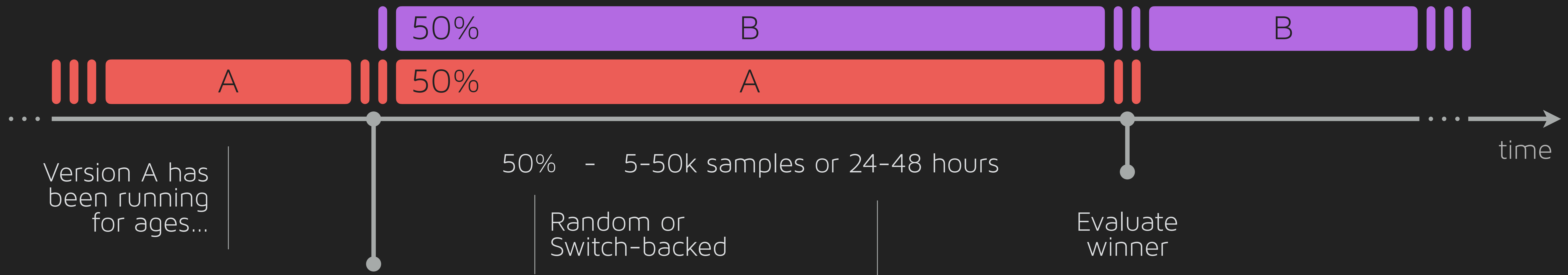


performance probability

Introduction

Classic A/B-test

Timeline...



Version A has been running for ages...

Start test

A lot of planning / assumptions...

How many versions / samples?

What / how to measure?

50% - 5-50k samples or 24-48 hours

Random or Switch-backed

Both A and B are put in production and we wait some time...

Evaluate winner

only evaluate when time is up...

hypothesis testing p-values...

decide p-value < 5%

Complex A/B-test

...or the more right way...

~20%

Reserve

don't spend it all on testing,
what if B sucks....

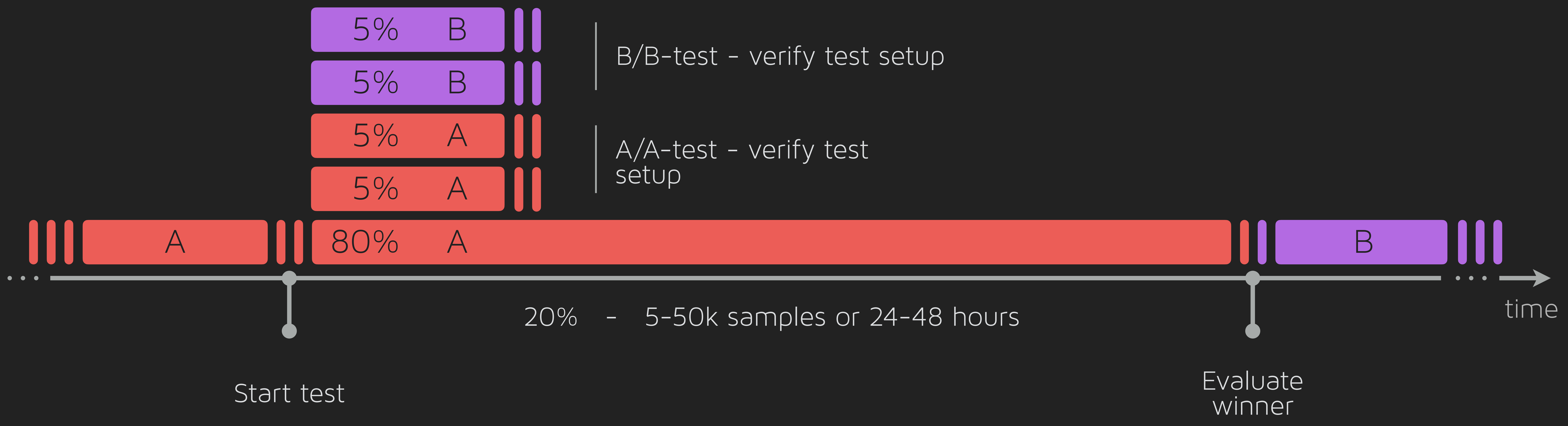
consider business requirements
and price...!

A/A test

Verify

split into several smaller tracks

conducting an A/A test to verify
the whole test setup



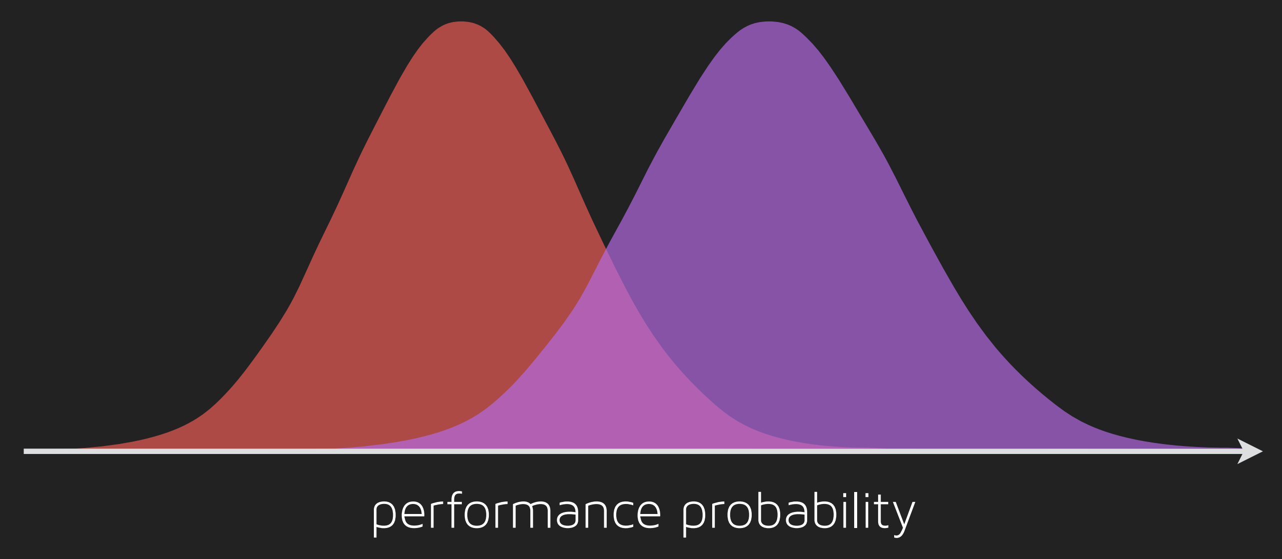
Bayesian A/B-test

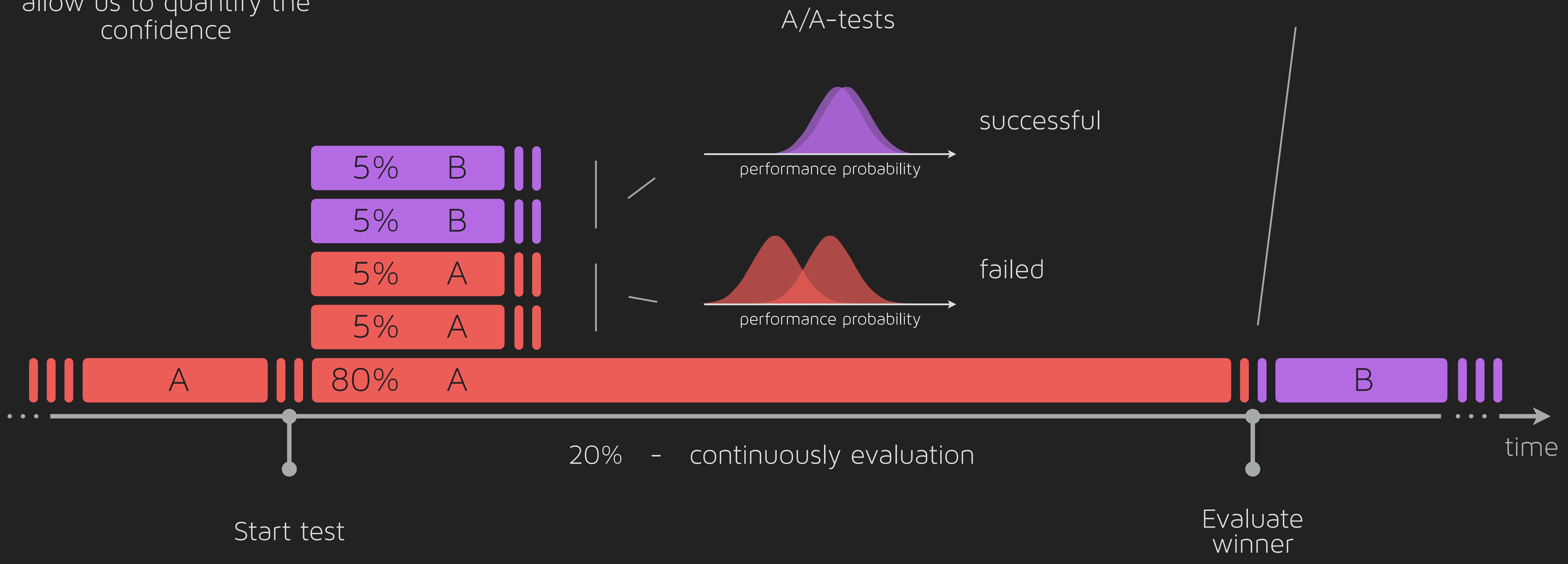
Bayesian evaluating - probability of winner

models performance as probability distributions

captures the epistemic uncertainty

allow us to quantify the confidence





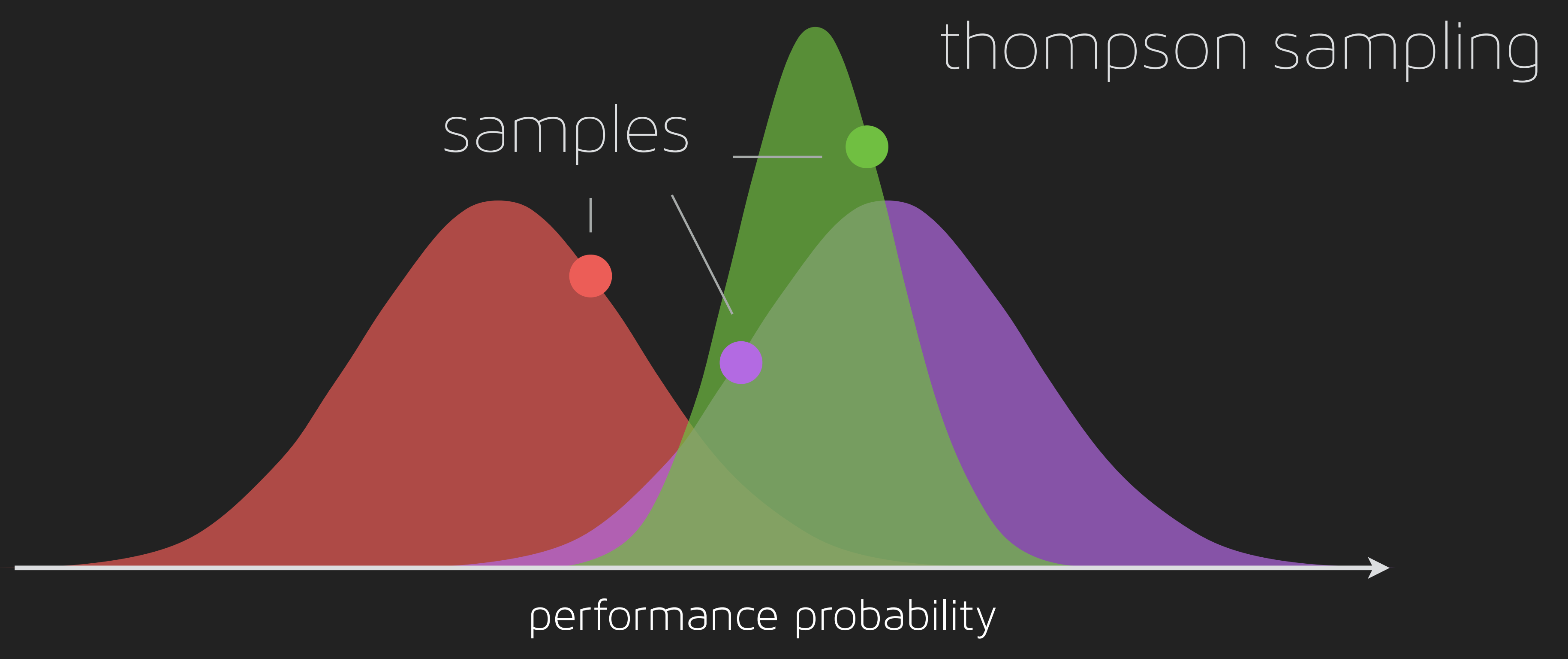
Bandit A/B-test

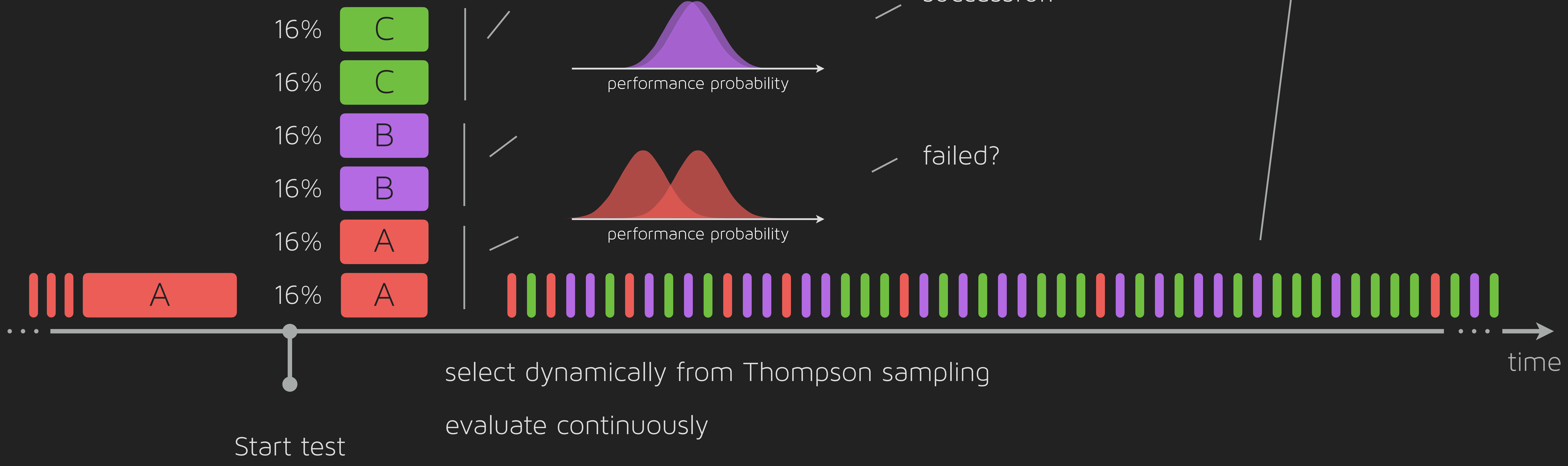
Dynamic architecture - runs self-contained - drop in new versions as cold start

models performance as probability distributions

captures the epistemic uncertainty

allow us to quantify the confidence





select dynamically from Thompson sampling

evaluate continuously

can run until sufficient confidence......and A/A test has passed

What we really want to know…

λ Performance – eg. CTR / CVR / CpC

$$P(\lambda_B > \lambda_A)$$

Probability of B
being better than A

Hypothesis test answers
another question

Bayesian approach
answers this question

Hypothesis Testing

Hypothesis significance testing

What's specific to it...?

A

B

state null & alternative hypothesis

H_0 H_1

define metric
CTR, CVR, CpC?

define experiment

assume specific data distribution

which test to use?

calc. sample size to achieve significance

....

activate

execute campaigns

statistical test

p-value < 5%?

reject H_0 ?

simple
accept / reject,
not size

B

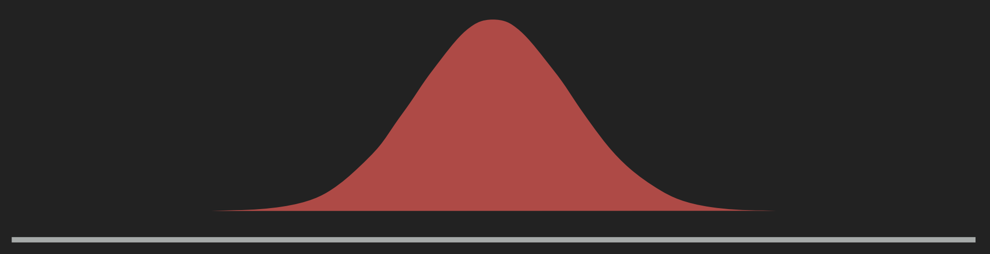
decide

...and go home

Hypothesis significance testing

Popular hypothesis tests...

Students T-Test



Assumptions

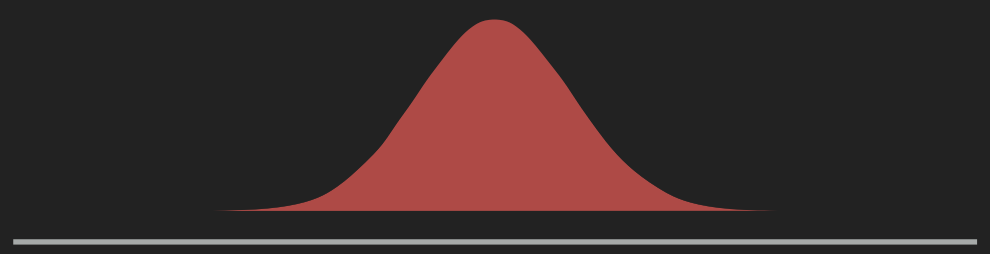
Student-t distributions

Very few samples

Mean of two identical distributions

https://en.wikipedia.org/wiki/Student%27s_t-test

Z-test



Assumptions

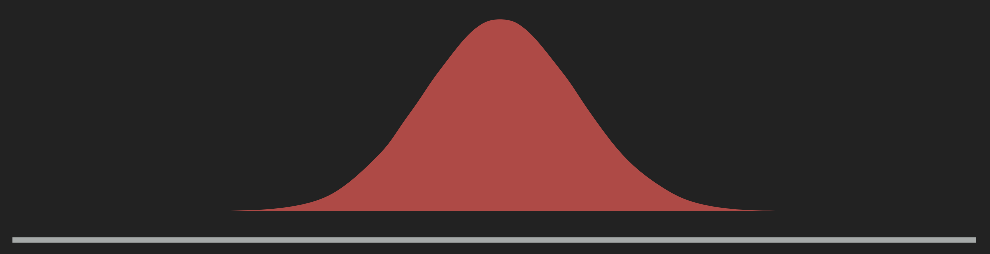
Gaussian distributions

Homogeneity of variance

Mean of two identical Gaussian distributions

https://en.wikipedia.org/wiki/Z-test

Fisher's exact test



Assumptions

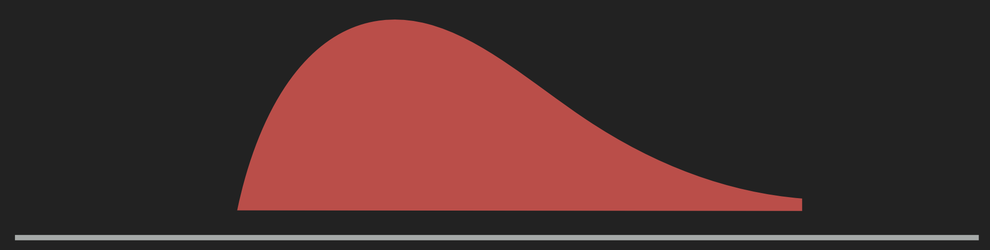
Contingency tables

Small sample sizes

Count data evaluating ratios

https://en.wikipedia.org/wiki/Fisher%27s_exact_test

Chi-squared test



Assumptions

Contingency table

Large sample sizes

Count data evaluating ratios

Chi-squared distribution

Generalisation of Fischer test

https://en.wikipedia.org/wiki/Chi-squared_test

Hypothesis significance testing

Classic approach

Test up against alternative hypothesis

Prove by rejecting null-hypothesis
being responsible for the observation

p-value

Probability of seeing the observed if the null hypothesis is true...

Reject null-hypothesis if p-value <5%

R. Fischer – "Lady tasting tea" - 1935

Milk or tea first?

Dr. Murial Bristol claimed she could...

The null hypothesis is that she has no ability to distinguish the teas.

Dr. Murial Bristol nailed all of them...

4 successes has 1 chance out of 70 (≈ 1.4% < 5%)

= p-value

https://en.wikipedia.org/wiki/Lady_tasting_tea

Hypothesis significance testing

Power Analysis - how many samples?

Sample size to detect a significant difference?

$$n = \frac{(Z_\alpha + Z_\beta)^2 \cdot (p_1 \cdot [1 - p_1] + p_2 \cdot [1 - p_2])}{(p_2 - p_1)^2}$$

have to assume a certain lift - what if we are wrong in our assumptions?

Significance	$Z_\alpha = 1.96$ $Z_\beta = 0.84$	Reliability of 5% Power of 20% (80% to reject H_0 if false)
Expected CTR values	$p_1 = 0.1$ $p_2 = 0.11$	CTR of 10% for A - expected... CTR of 11% for B - expected...
Sample size	$n = 14.752$	Samples required

https://splitmetrics.com/blog/mobile-a-b-testing-sample-size/
https://vwo.com/blog/how-to-calculate-ab-test-sample-size/

Single-sided or double sided test?

What are we testing for?

H_1 A < B

H_1 A <> B

Hypothesis testing challenges

General

can't really interpret probabilities of belief

if p-value > 5%, then whole campaign is thrown out

blackbox to most....

tells nothing of the difference between A and B

Assumptions

data distribution? Gaussian, t-student? etc.

one-sided or double sided?

a lot of assumptions - MUST be correct

Sample Size

hypothesis testing requires to estimate sample size first

must guess the performance of each variant. Not easy for B!

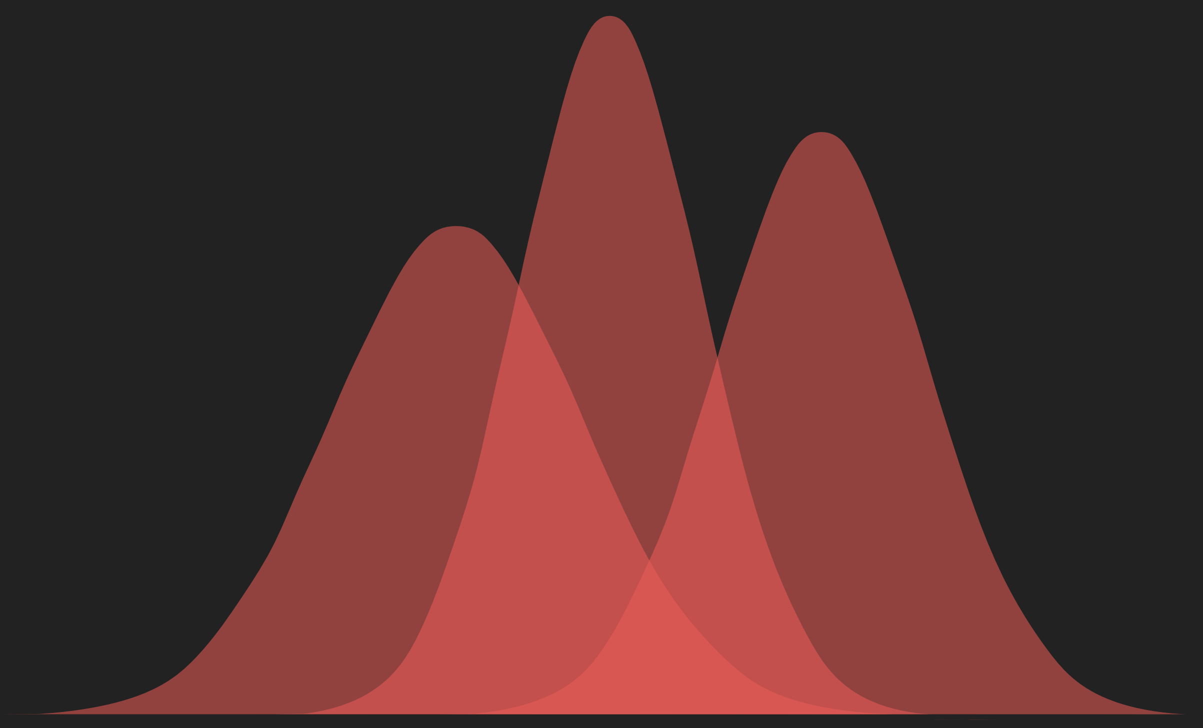
relies on estimating the significance when samples is reached

Peeking

"...checking A/B results before the test is over..."

very tempting....

significance assessment only occurs when samples size is reached....



Bayesian A/B testing

Types of Uncertainty

two main types

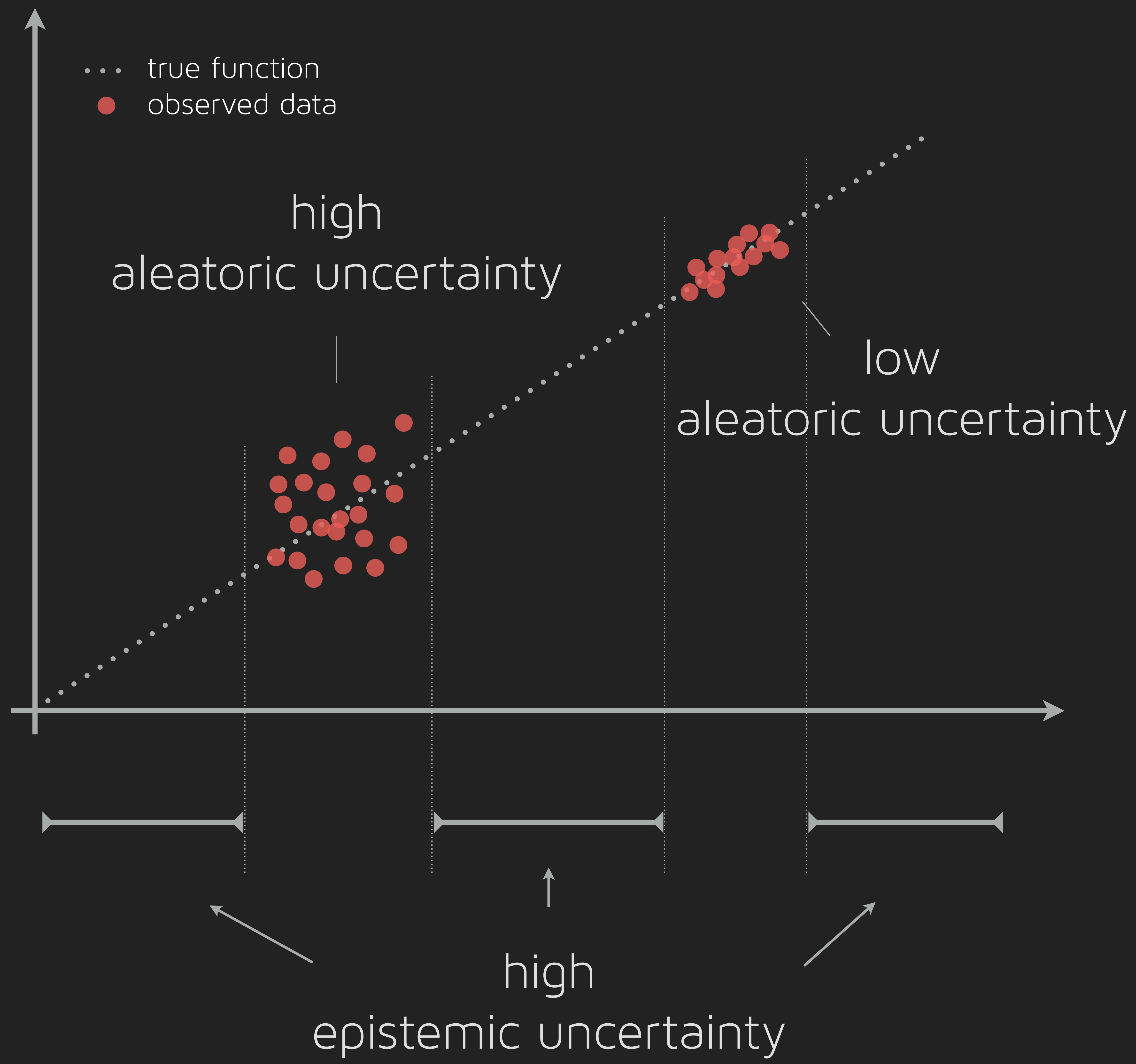
aleatoric uncertainty

- natural stochasticity in observations (noise)
- can't be reduced with more data

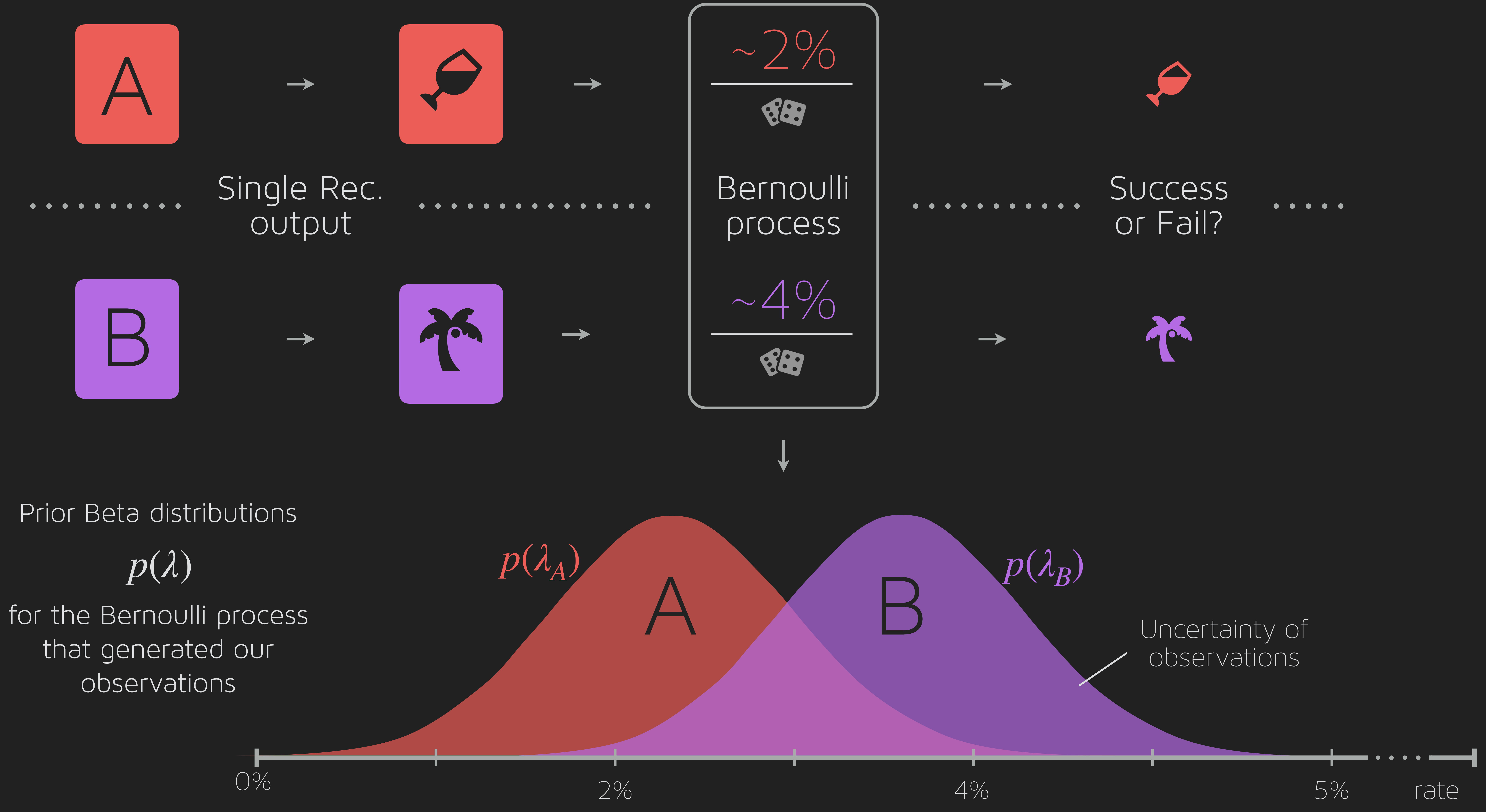
epistemic uncertainty

- uncertainty around too few data points
- out-of-distribution outliers

example



Stochastic process behind the scenes...



Bayesian stochastic modeling / testing

Model events as stochastic Bernoulli engagement events.

The engagement rate for each candidate can then be modelled as a probability distribution to capture the uncertainty.

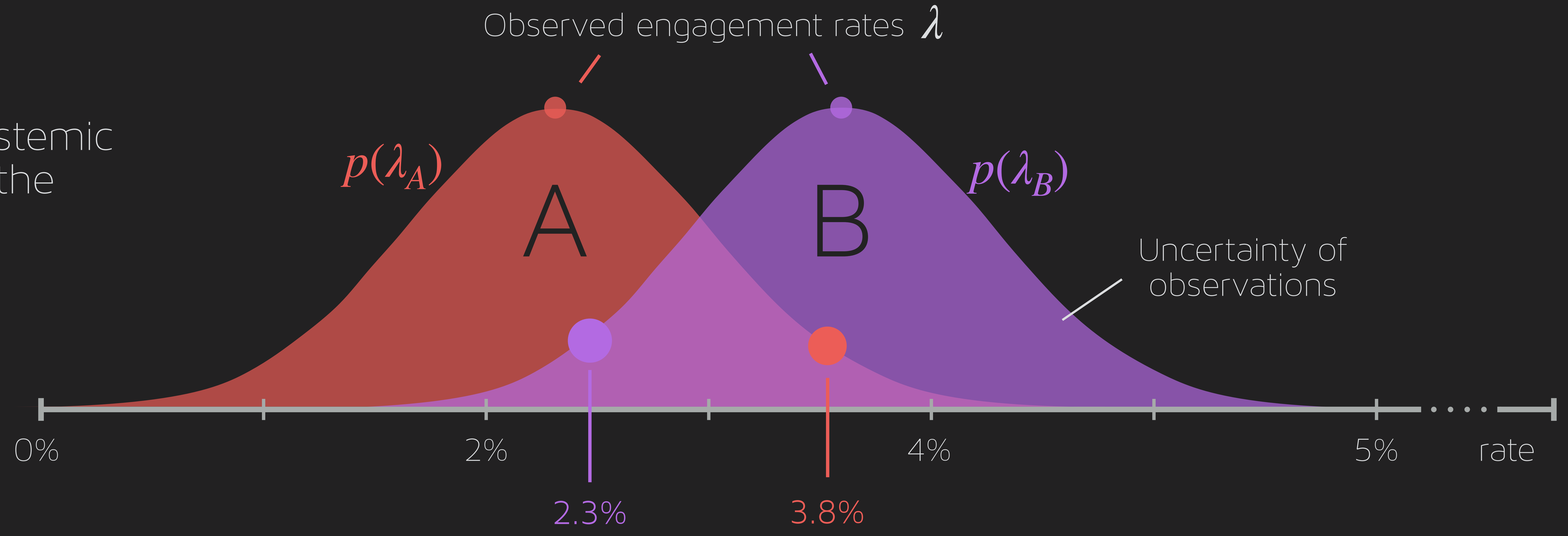
Example of two candidates and their engagement rates with uncertainty.

Captures the epistemic uncertainty, not the aleatoric in this case.

The uncertainty allows us to quantify

how much do we believe B is in fact the winner

...and evaluate the likelihood of many other scenarios.



Example: The unlikely case, where the true engagement rate dictates A is better than B. These probabilities can be quantified....

Bayesian stochastic modeling / testing

Practical probability distributions to model engagement

clicks / impressions

beta distribution

$$p(x\,|\,\alpha,\beta) = x^{\alpha-1}(1-x)^{\beta-1}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

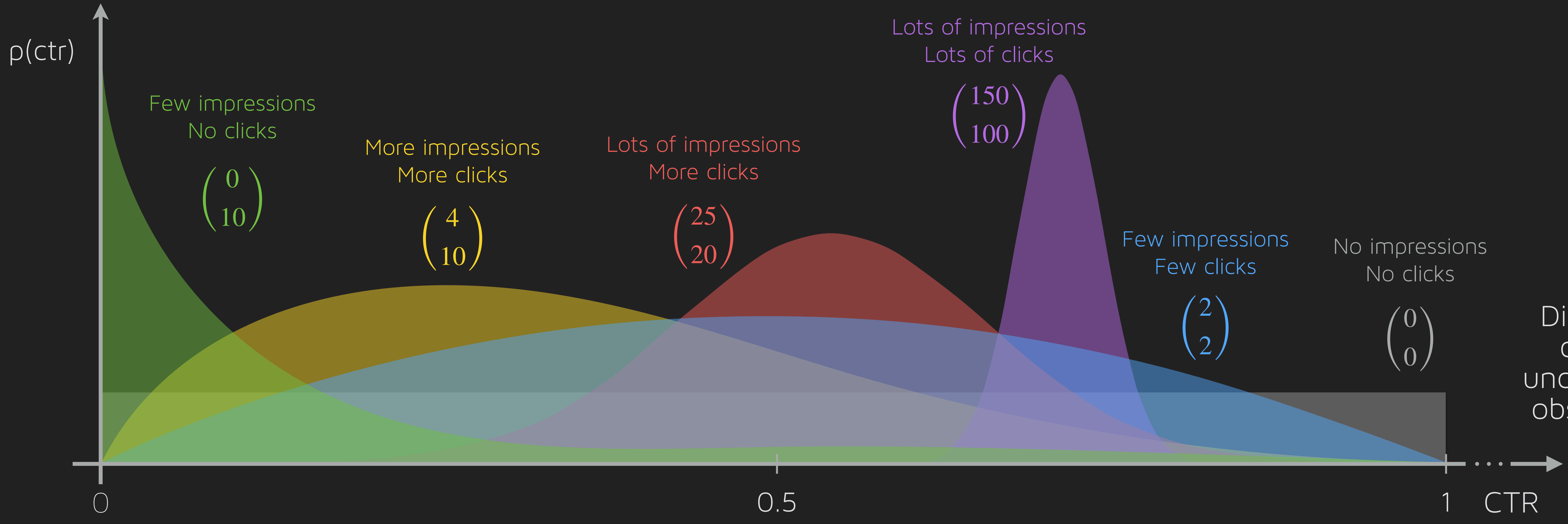
$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}clicks\\ impressions - clicks\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}$$

cost / click

gamma distribution

$$p(x\,|\,\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}cost\\ clicks^{-1}\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}$$

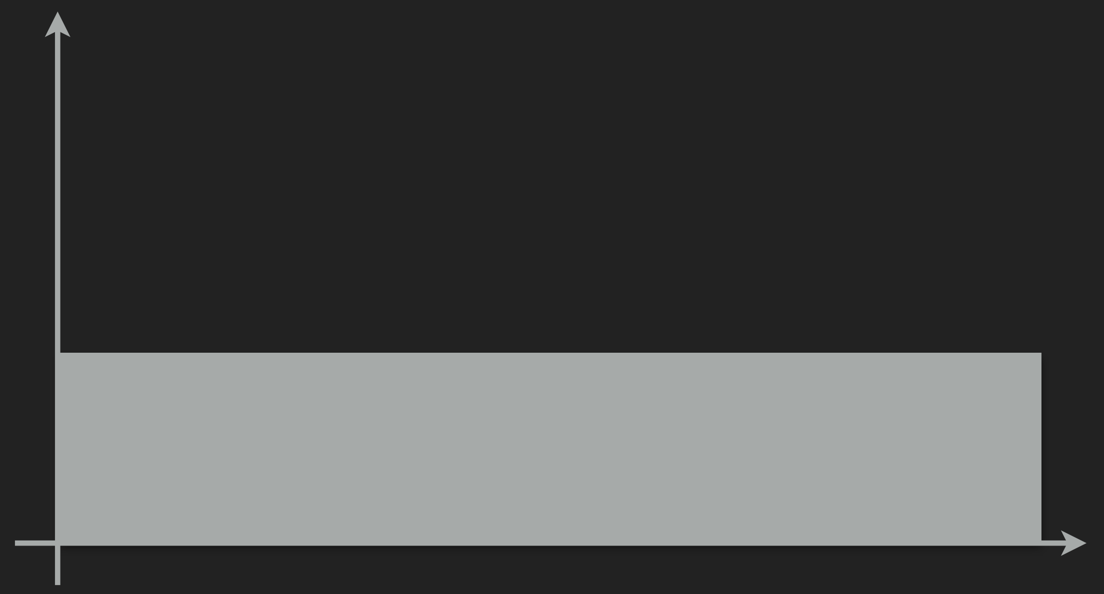


Distribution captures uncertainty of observations

Bayesian stochastic modeling / testing

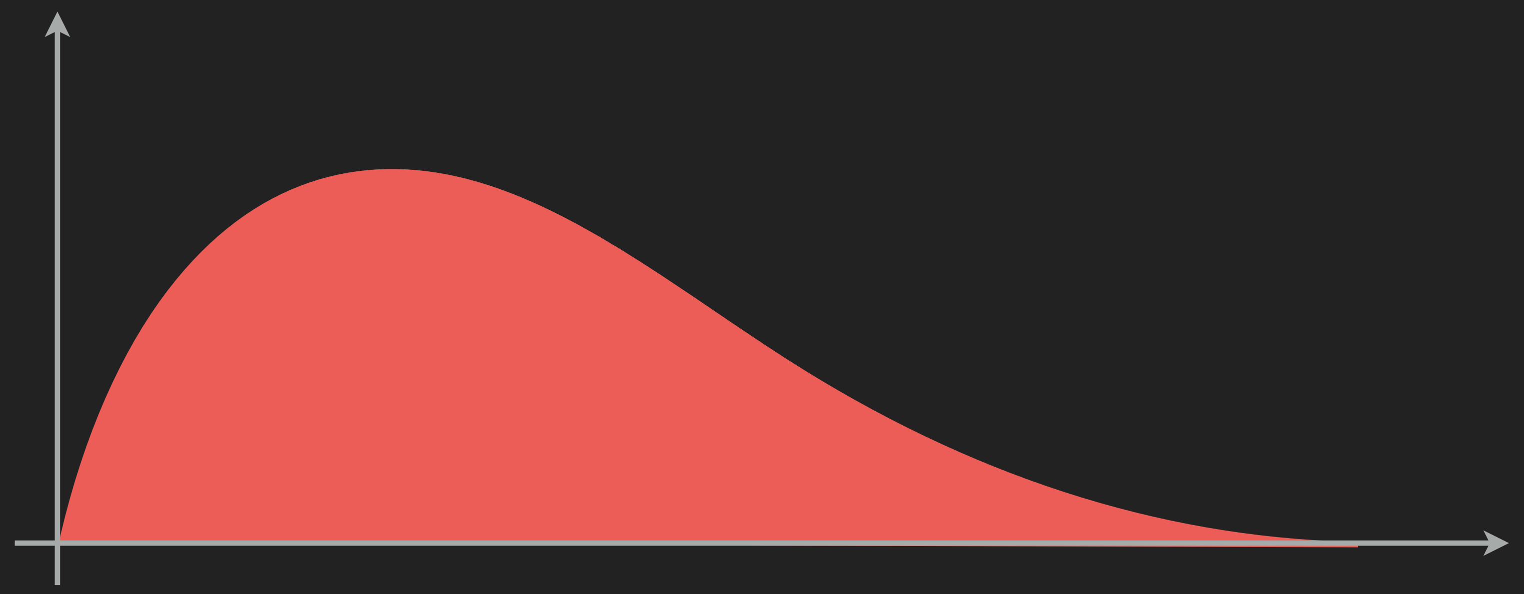
Why Gauss modelling is often not optimal....

Know nothing



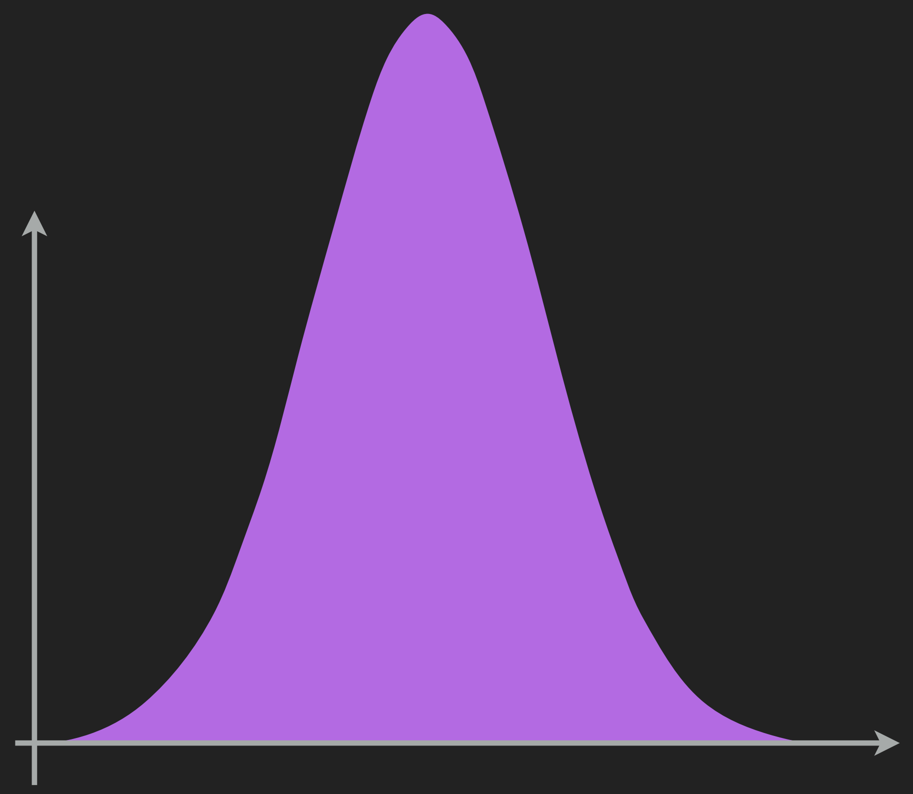
Uniform

Know a little bit



Beta

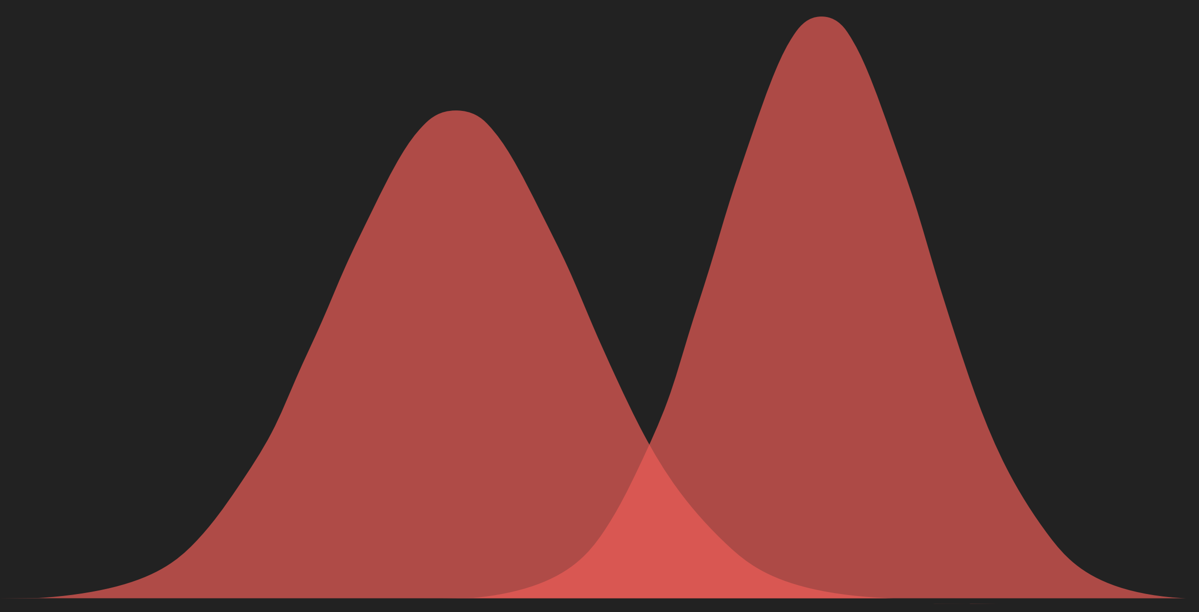
Know everything



Gaussian

Because it is never gaussian, when you have few samples, when you want to evaluate.

Time matters – to save impressions and $$$...



Bayesian Metrics
A/A testing

Bayesian same/same test

Are A/A the same distribution?

Selected metrics for measuring differences between two probability distributions

Kolmogorov-Smirnov

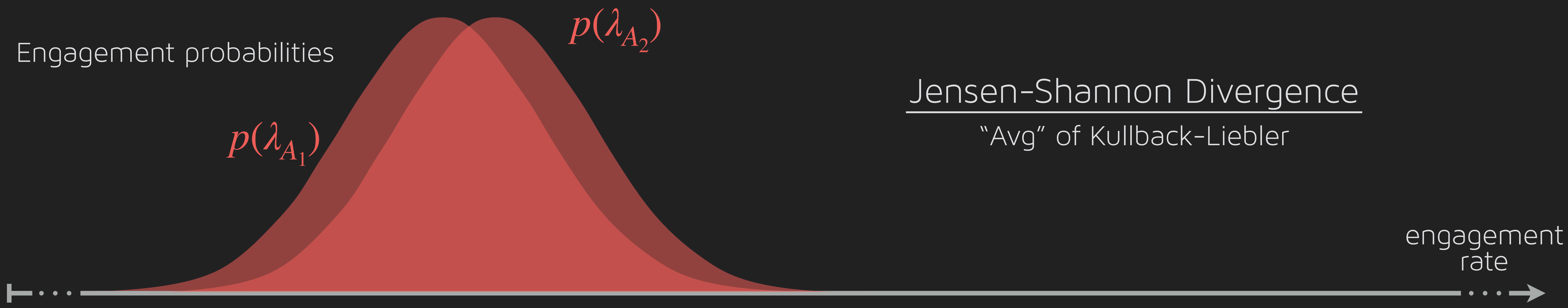
How much are the distribution overlapping?

Wasserstein

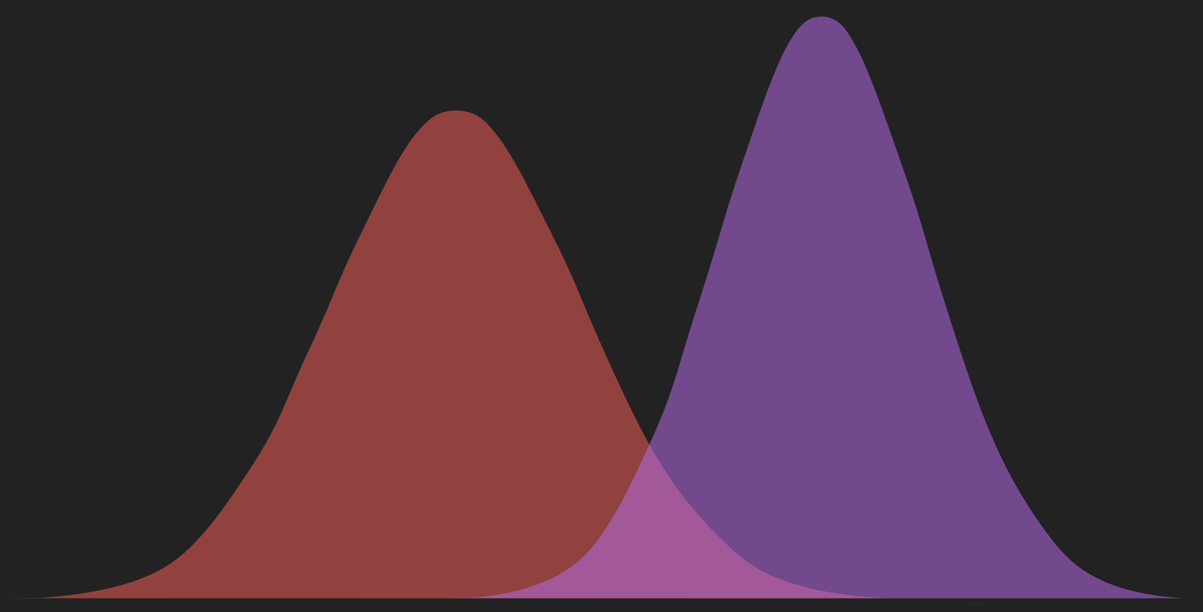
How much mass is moved?
(1 - wasserstein)

Jensen-Shannon Divergence

"Avg" of Kullback-Liebler



Example: A1 and A2 are close enough to each other....or what?



Bayesian Metrics
A/B testing

Bayesian Metrics - Probability of winner

Bayesian testing by calculating

the probability of selecting the right winner

ie. confidence score / certainty

This can be deducted from the joint probability $p(\lambda_A, \lambda_B)$
...and corresponds to the area, where $p_{\lambda_B} > p_{\lambda_A}$

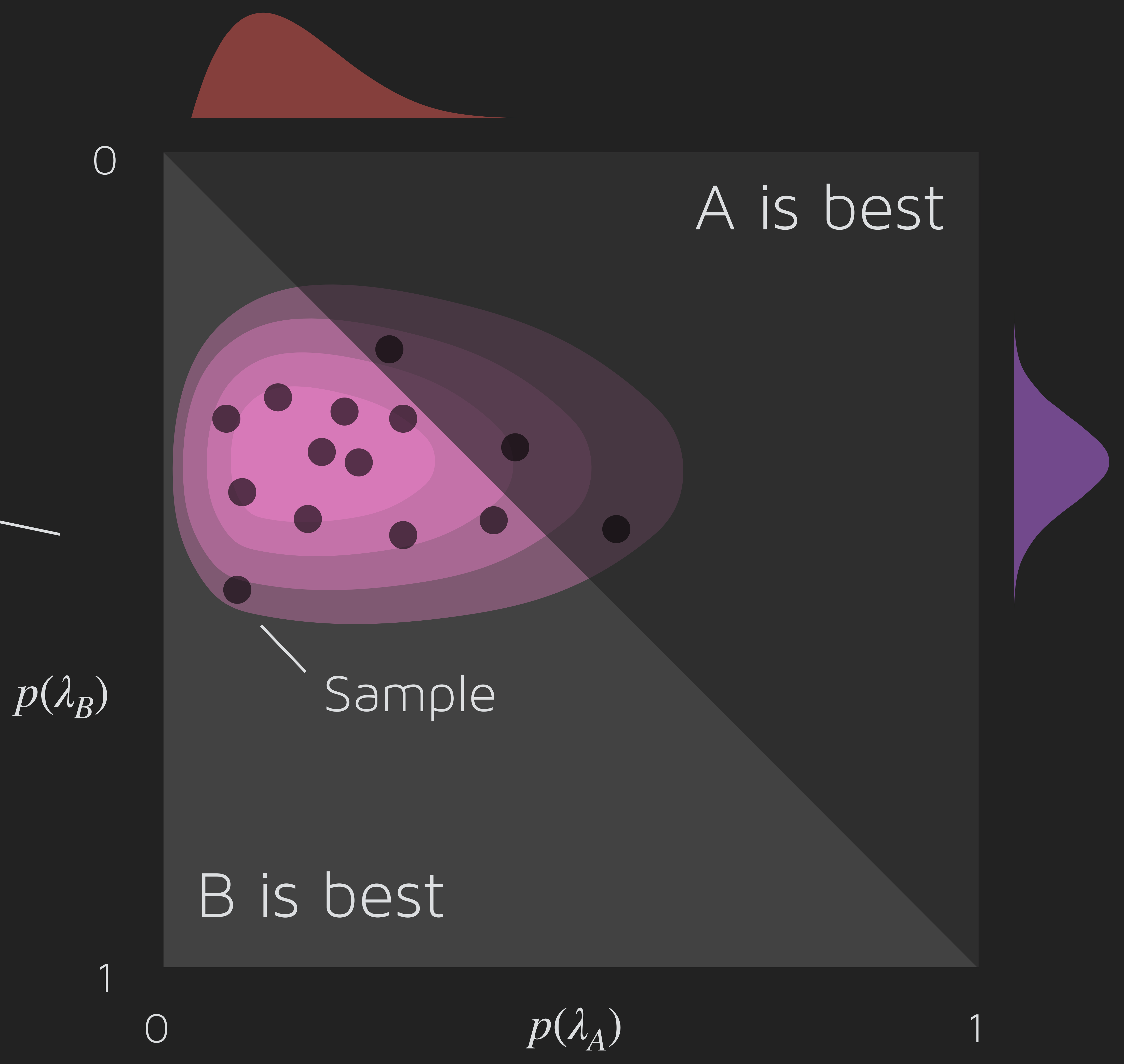
This marginalisation can be estimated via
Monte Carlo sampling

....or calculated analytically from for the
Beta distribution

$$P(p_B > p_A) = \int_0^1 \int_0^{\lambda_A} P(\lambda_A, \lambda_B)\ \mathrm{d}\lambda_B \mathrm{d}\lambda_A$$

$$= \sum_{i=0}^{\alpha_B - 1} \frac{B(\alpha_A + i, \beta_A + \beta_B)}{(\beta_B + i) B(1 + i, \beta_B) B(\alpha_A, \beta_A)}$$

Joint probability of engagement-rates



https://www.evanmiller.org/bayesian-ab-testing.html
https://www.chrisstucchio.com/blog/2014/bayesian_ab_decision_rule.html

Bayesian Metrics - Expected Loss

Average performance loss for wrong choice

Evaluation of loss if choosing wrong

$$\mathscr{L}(\lambda_A, \lambda_B, A) = max(\lambda_B - \lambda_A, 0)$$

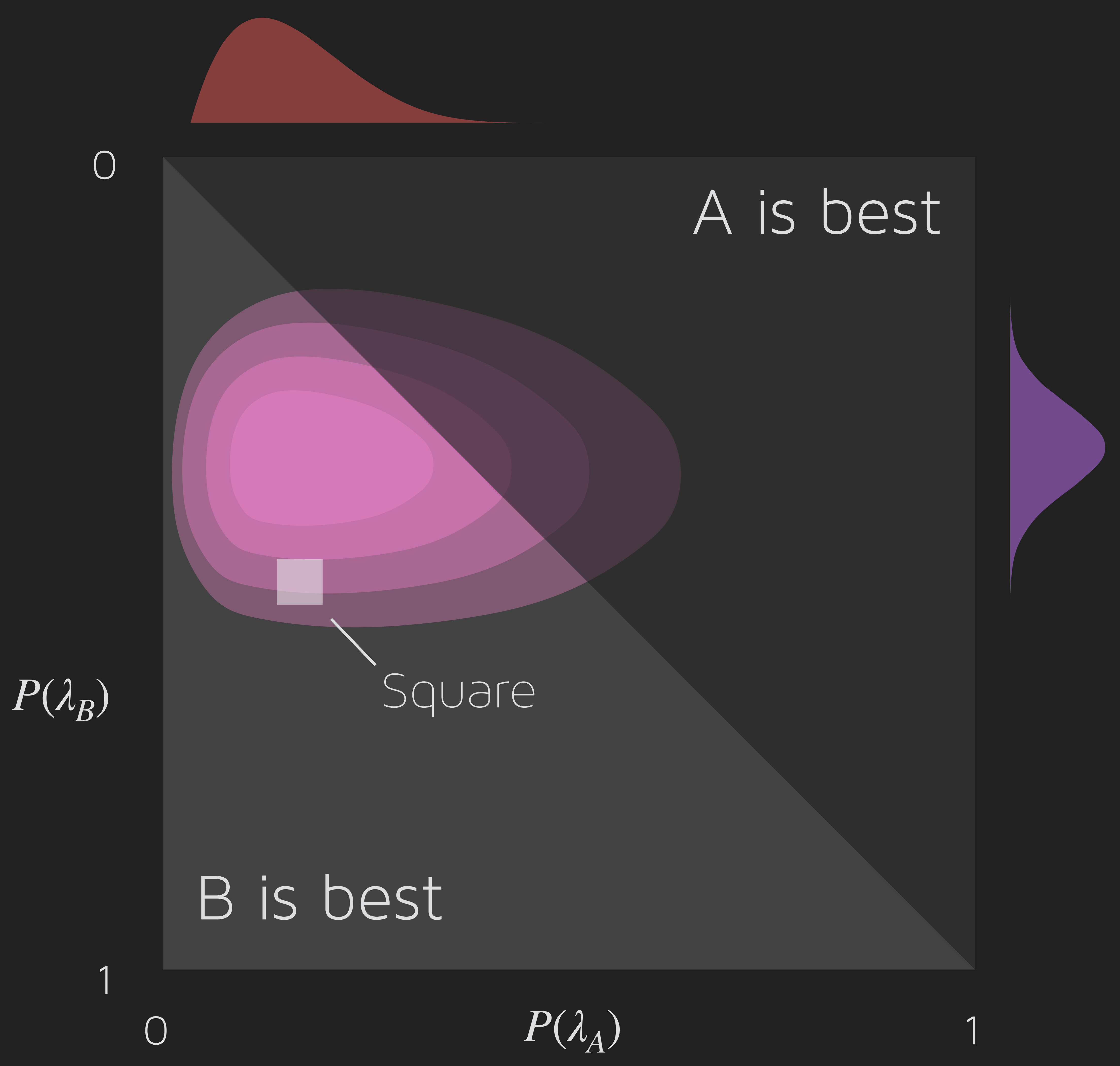
$$\mathscr{L}(\lambda_A, \lambda_B, B) = max(\lambda_A - \lambda_B, 0)$$

Expectation of loss by integrating across joint distribution

$$E[\mathscr{L}](?) = \int_0^1 \int_0^1 P(\lambda_A \lambda_B)\ \mathscr{L}(\lambda_A, \lambda_B, ?)\ \partial\lambda_B\lambda_A$$

Unit will be λ - in some cases close to $, as we shall see...

Joint probability of engagement-rates



https://www.chrisstucchio.com/blog/2014/bayesian_ab_decision_rule.html
https://medium.com/vptech/introduction-to-bayesian-a-b-testing-in-python-df81a9b3f5fd

Single Example

Example....

Observations

A	1000 / impressions	100 / clicks	$11 / Cost
B	1000 / impressions	120 / clicks	$19 / Cost

Hypothesis testing - p-values....

Let's see what the numbers are after 1000 impressions...

Example

Impressions	1000
Clicks A	100
Clicks B	120

$$\lambda_A = \frac{100}{1000} = 10\,\%$$

$$\lambda_B = \frac{120}{1000} = 12\,\%$$

$\longrightarrow$ $\frac{\text{B}}{\text{winner}}$?

$\frac{1000}{\#\text{samples}}$ $\frac{12\%}{\text{p-value}}$

Uuh, this is not good....
....Peeking

$\frac{3840}{\#\text{samples actually required}}$ $\frac{0.2\%}{\text{p-value}}$

Aaaah, much better.....

Bayesian Metrics - Probability of winner & lift analysis

The uncertainty allows for reasoning and exploring the probability of various scenarios.

Example

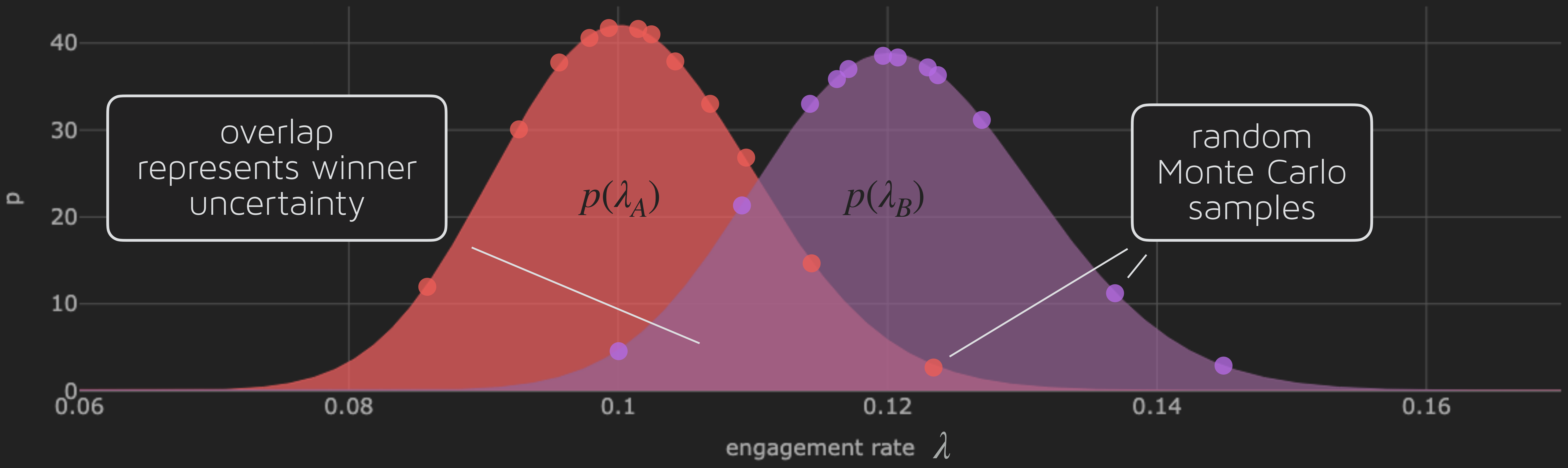
Impressions	1000
Clicks A	100
Clicks B	120

$$\lambda_A = \frac{100}{1000} = 10\,\%$$

$$\lambda_B = \frac{120}{1000} = 12\,\%$$

$\longrightarrow$ $\frac{B}{\text{winner}}$?

Beta distributions of CTR



Bayesian Metrics - Probability of winner & lift analysis

The uncertainty allows for reasoning and exploring the probability of various scenarios.

Example

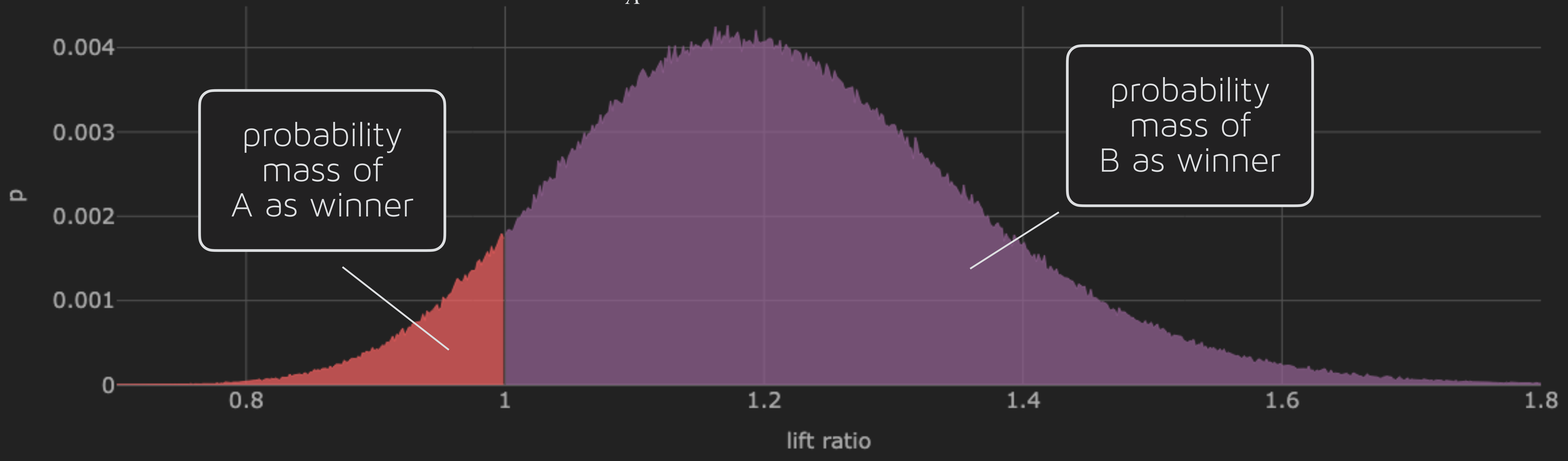
Impressions	1000
Clicks A	100
Clicks B	120

$$\lambda_A = \frac{100}{1000} = 10\,\%$$

$$\lambda_B = \frac{120}{1000} = 12\,\%$$

$\longrightarrow$ $\frac{\text{B}}{\text{winner}}$?

Probability distribution of lift ratio $\frac{\lambda_B}{\lambda_A}$



Bayesian Metrics - Probability of winner & lift analysis

The uncertainty allows for reasoning and exploring the probability of various scenarios.

Example

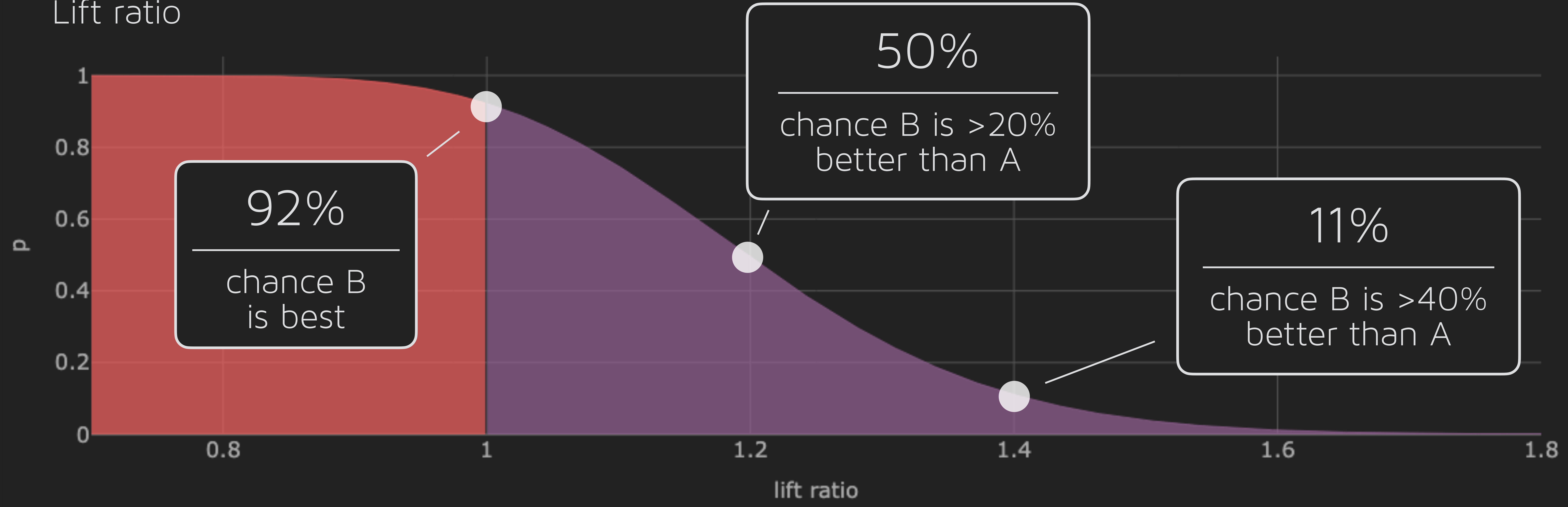
Impressions	1000
Clicks A	100
Clicks B	120

$$\lambda_A = \frac{100}{1000} = 10\,\%$$

$$\lambda_B = \frac{120}{1000} = 12\,\%$$

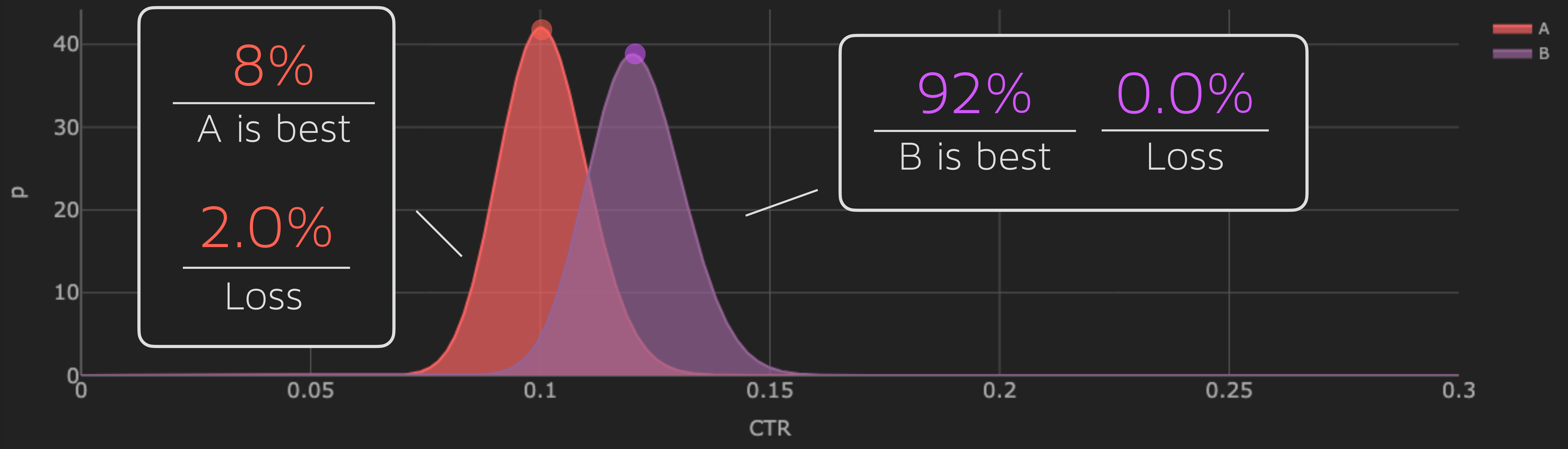
$\longrightarrow \frac{\text{B}}{\text{winner}}$?

Lift ratio



Bayesian A/B Testing - Decision

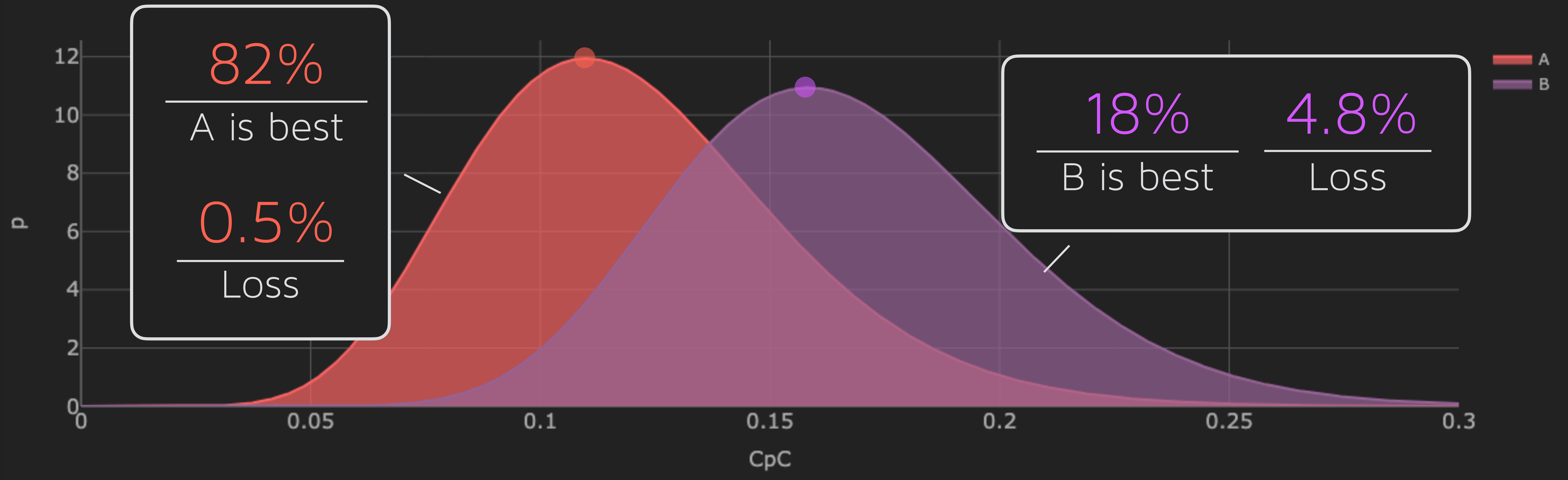
Click-Through-Rate



After 1000 impressions

Cost-per-Click

Lower is better



Campaign

A/B Testing - Campaign simulation

...

A

10%

CTR A

$11

CPM A

B

12%

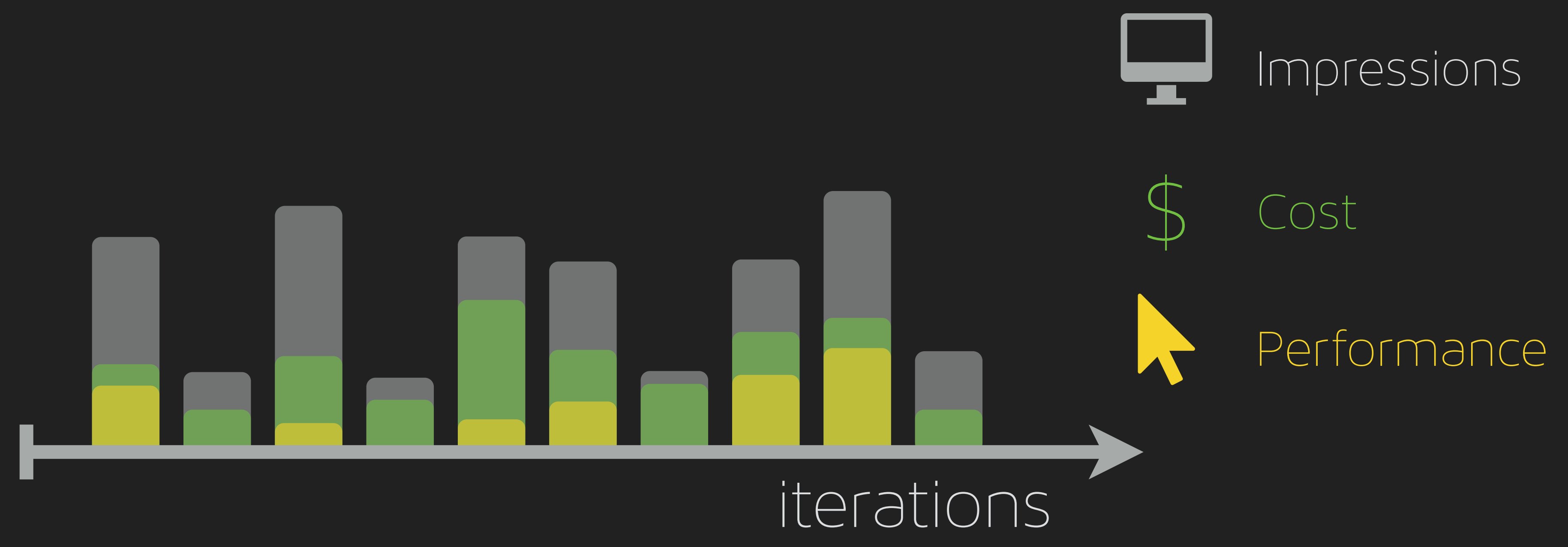
CTR B

$19

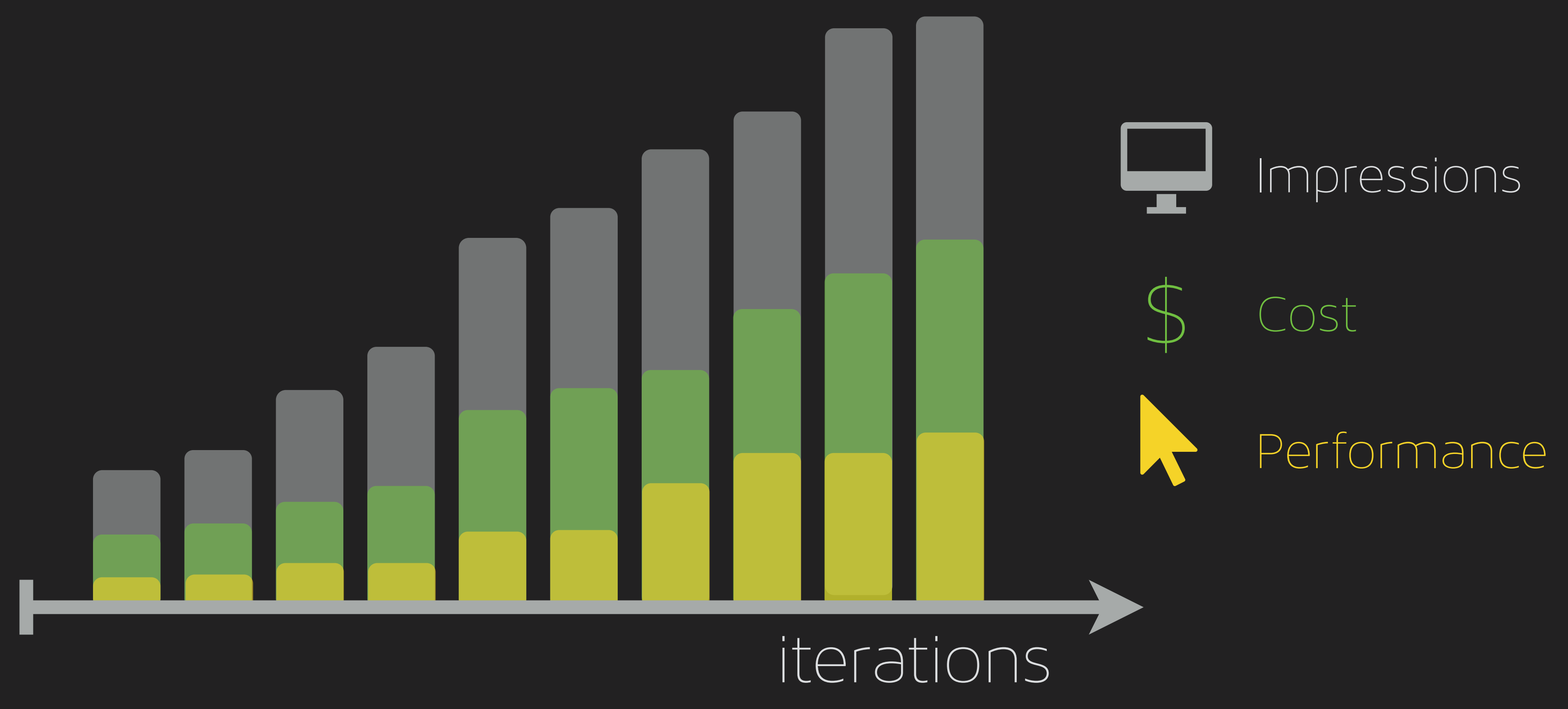
CPM B

Simulate impression,
clicks and cost

logs



accumulated



Campaign simulation

Observations and performance

A

10%

CTR A

$11

CPM A

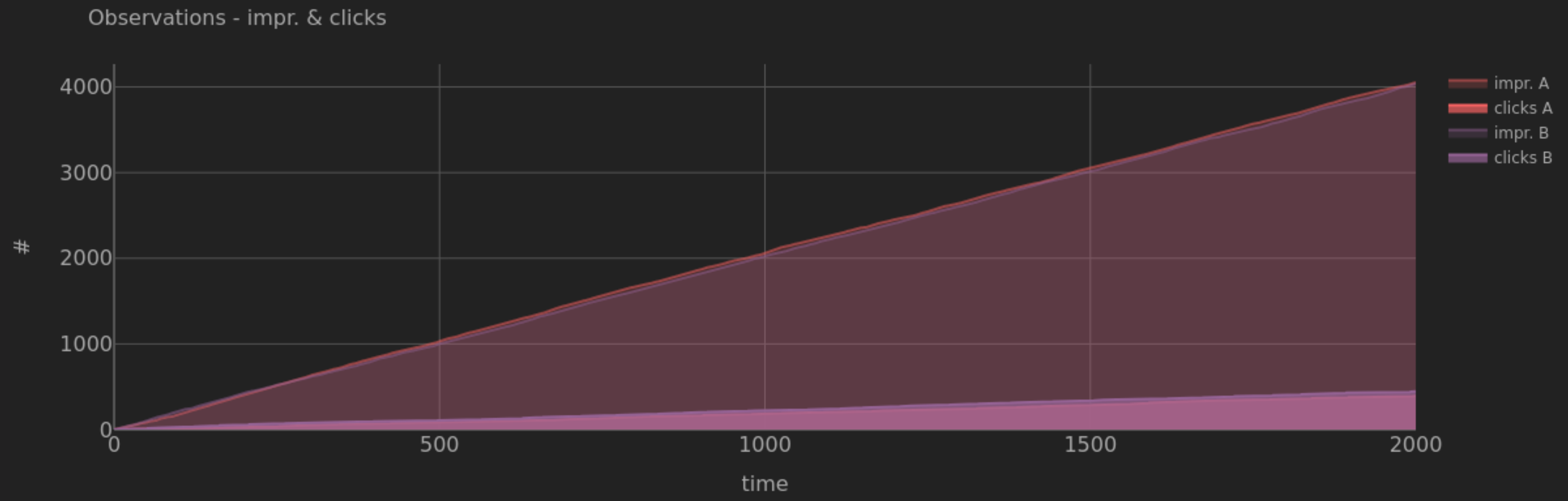
B

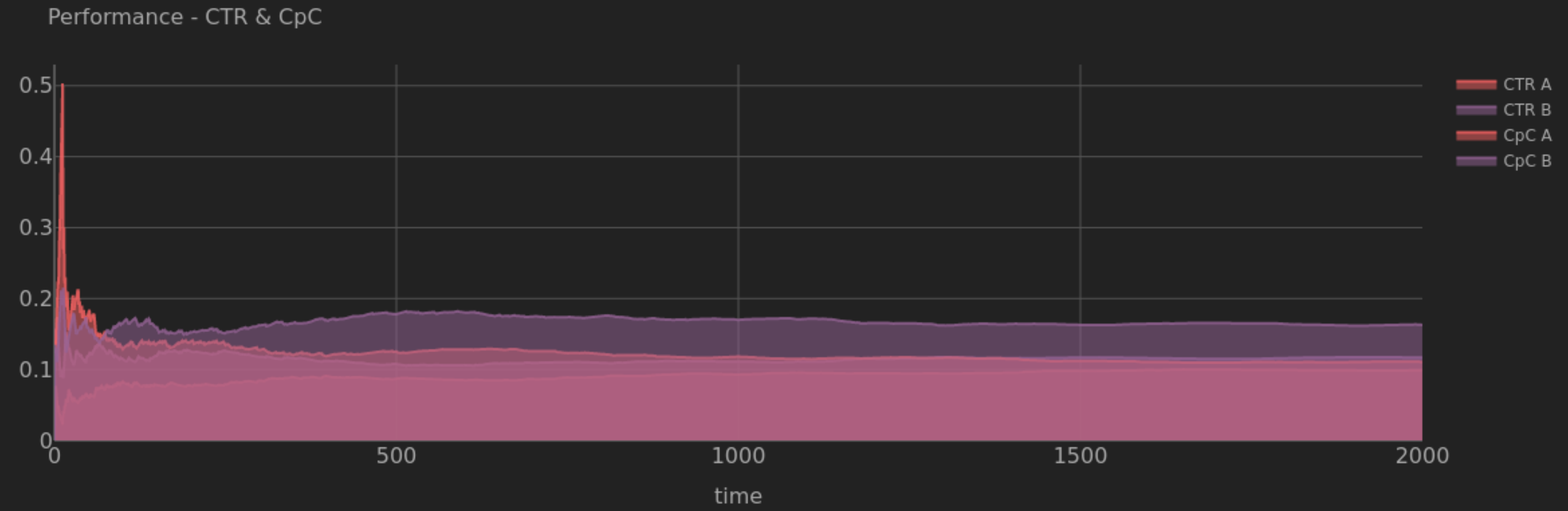
12%

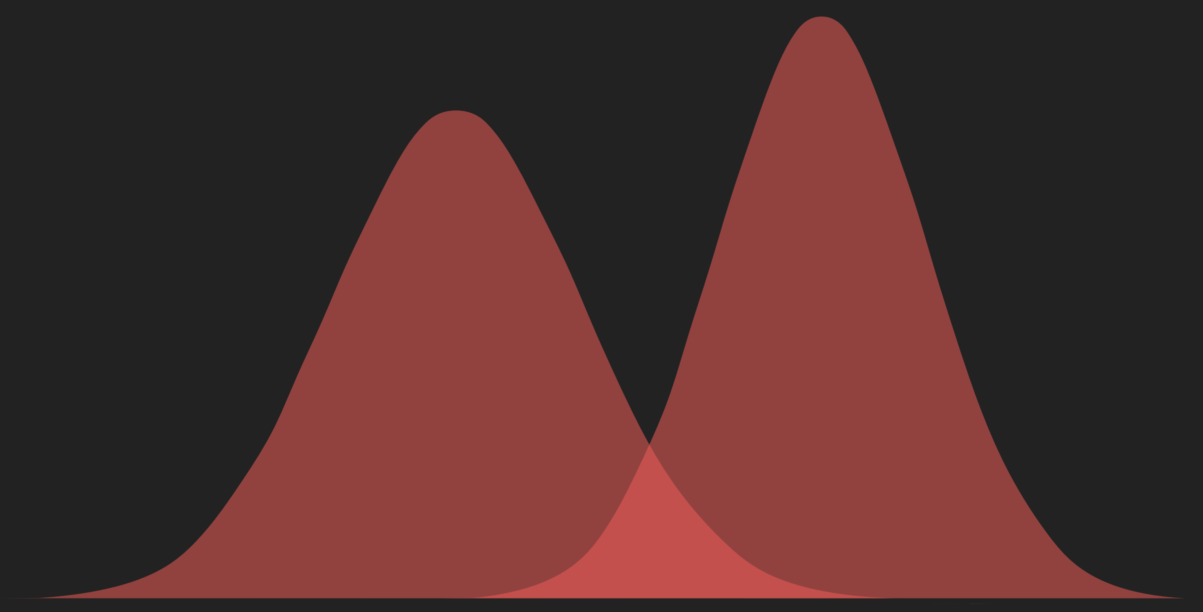
CTR B

$19

CPM B







Bayesian A/A testing

Hypothesis A/A Testing - Campaign simulation

Same / Same test - CTR

A

10%

CTR A

$11

CPM A

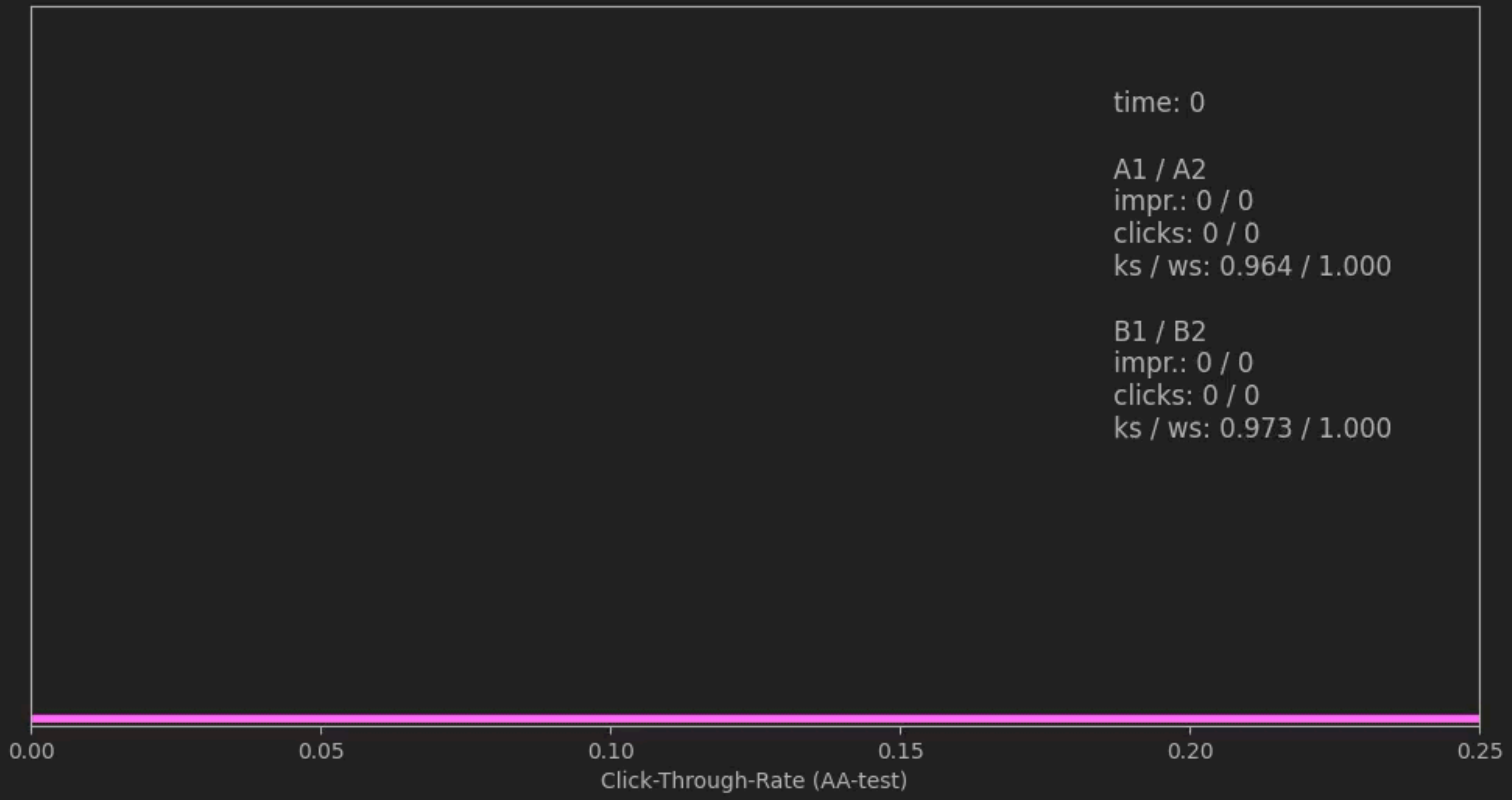
B

12%

CTR B

$19

CPM B



Hypothesis A/A Testing - Campaign simulation

Same / Same test

A

10%

CTR A

$11

CPM A

4054 / 4078

impressions

393 / 410

clicks

45 / 45

cost

B

12%

CTR B

$19

CPM B

4040 / 4088

impressions

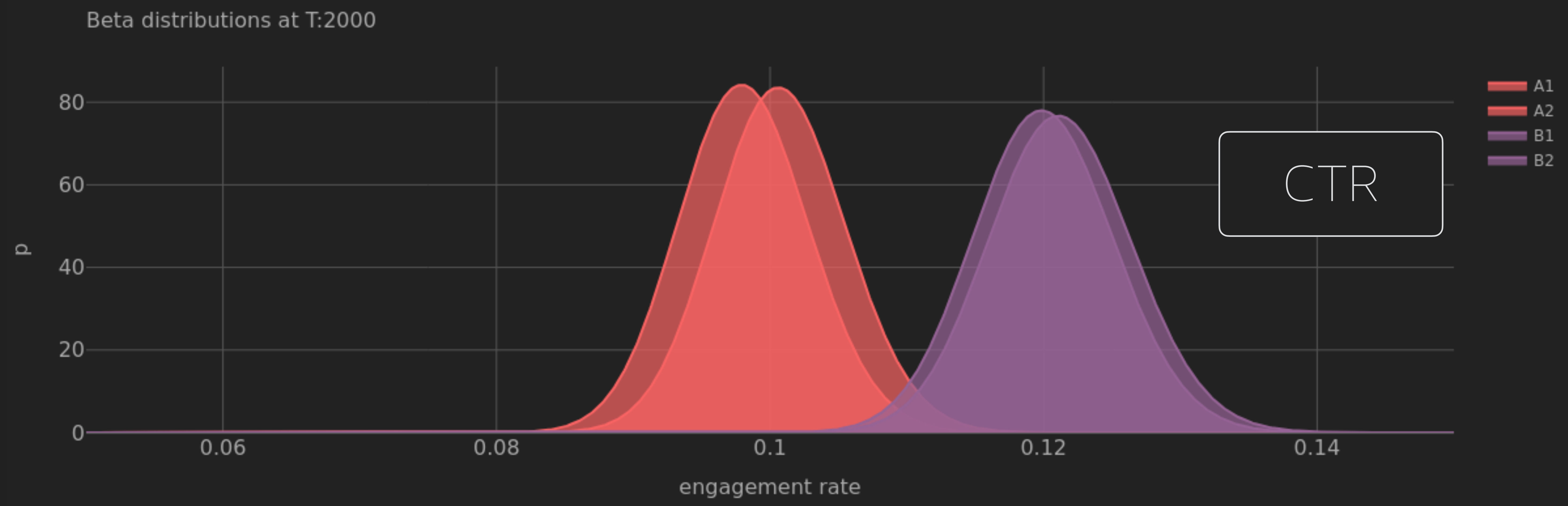
447 / 503

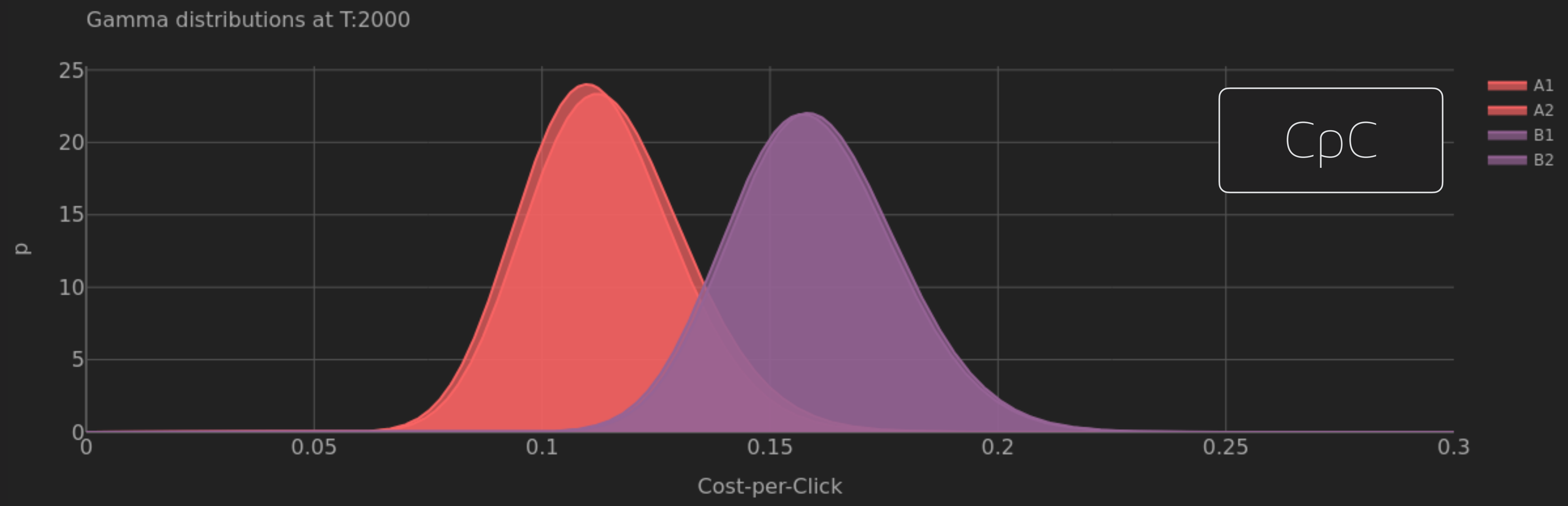
clicks

77 / 78

cost

End-of-simulation





Hypothesis A/A Testing - Campaign simulation

Same / Same test - CTR

A

10%

CTR A

$11

CPM A

B

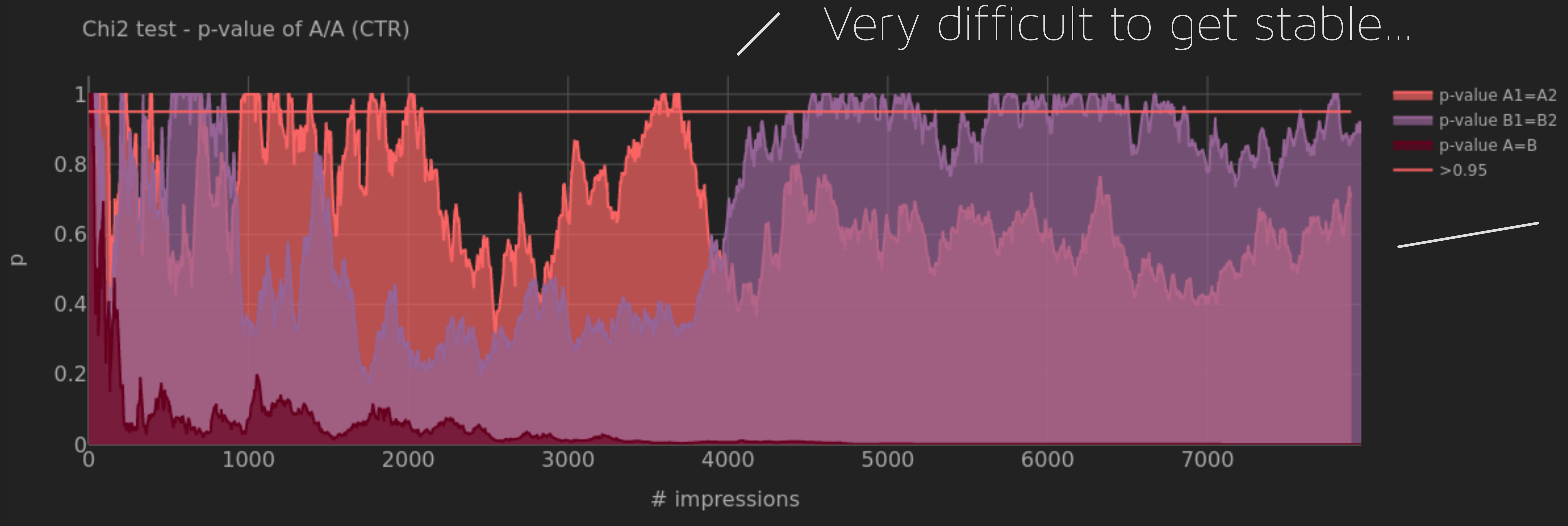
12%

CTR B

$19

CPM B

Aleatoric uncertainty
Very difficult to get stable...



A didn't make it...

65%

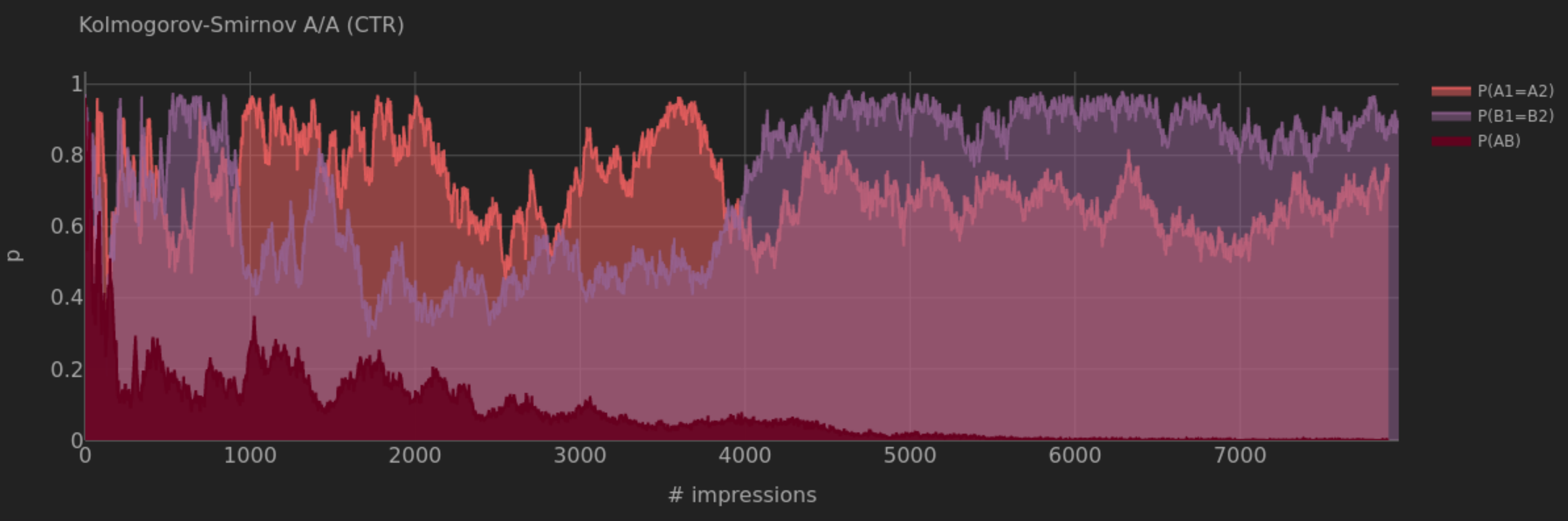
p-value

97%

p-value

Does it make sense to use hyp. test for this?

AA & BB are more similar than AB



Hypothesis A/A Testing - Campaign simulation

Same / Same test - CpC

A

10%
CTR A

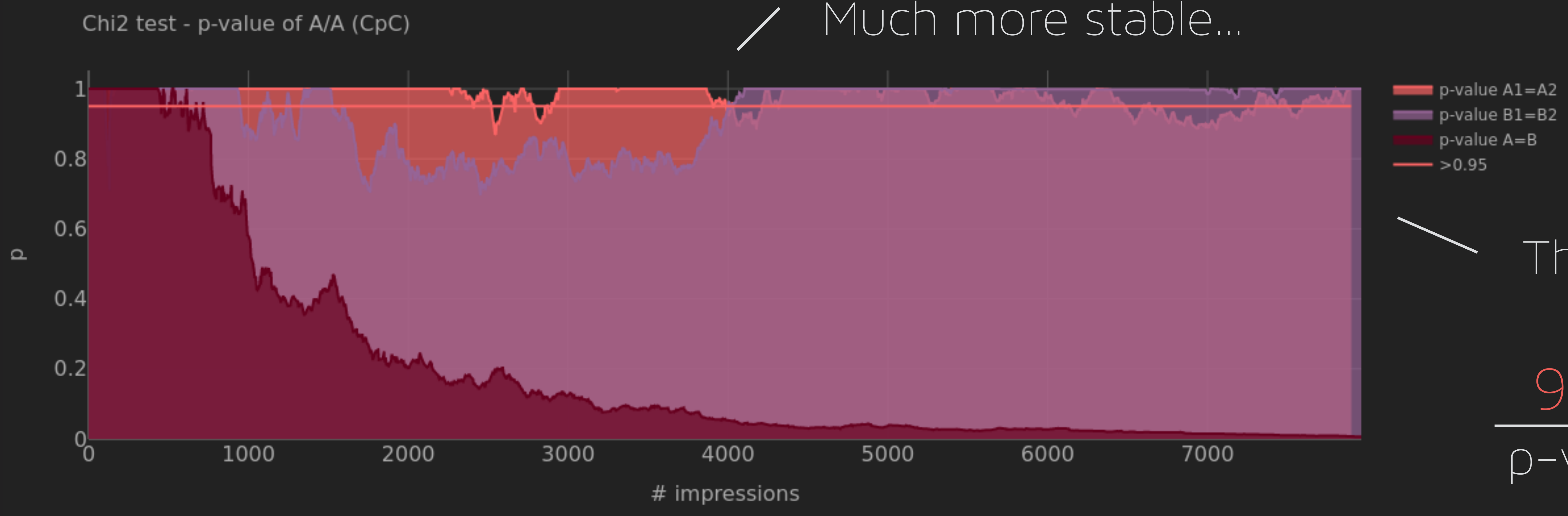
$11
CPM A

B

12%
CTR B

$19
CPM B

Aleatoric uncertainty
Much more stable...

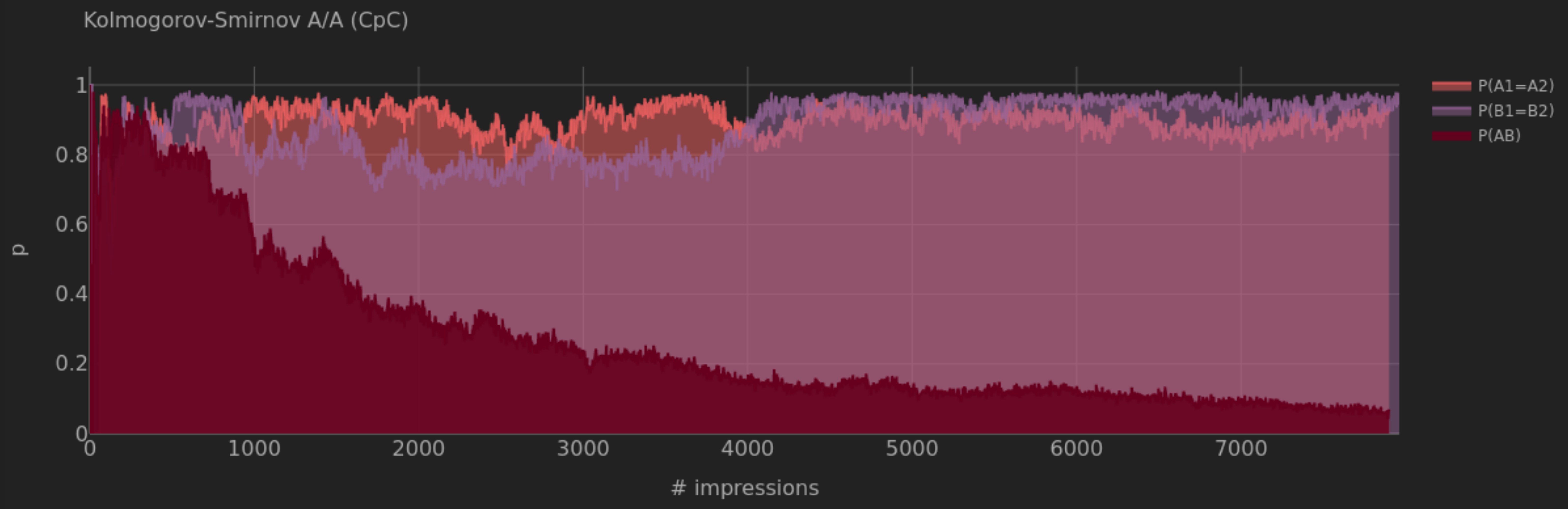


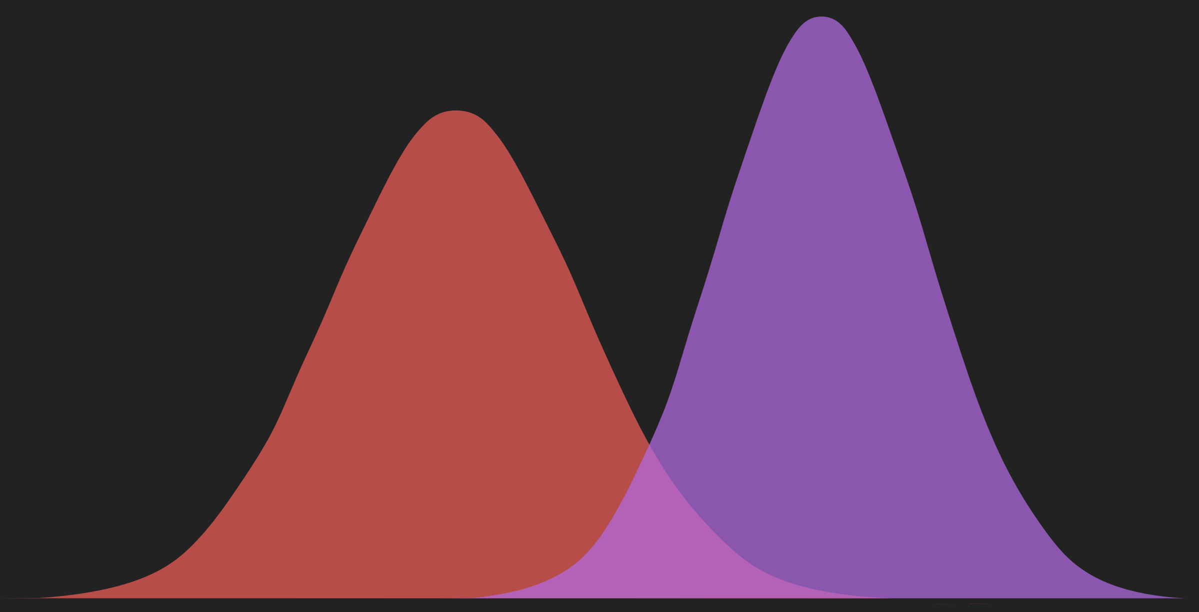
They made it...

98%
p-value

100%
p-value

AA & BB are more similar than AB





Bayesian A/B testing

Bayesian A/B Testing - Campaign simulation

A/B-test - CTR

A

10%

CTR A

$11

CPM A

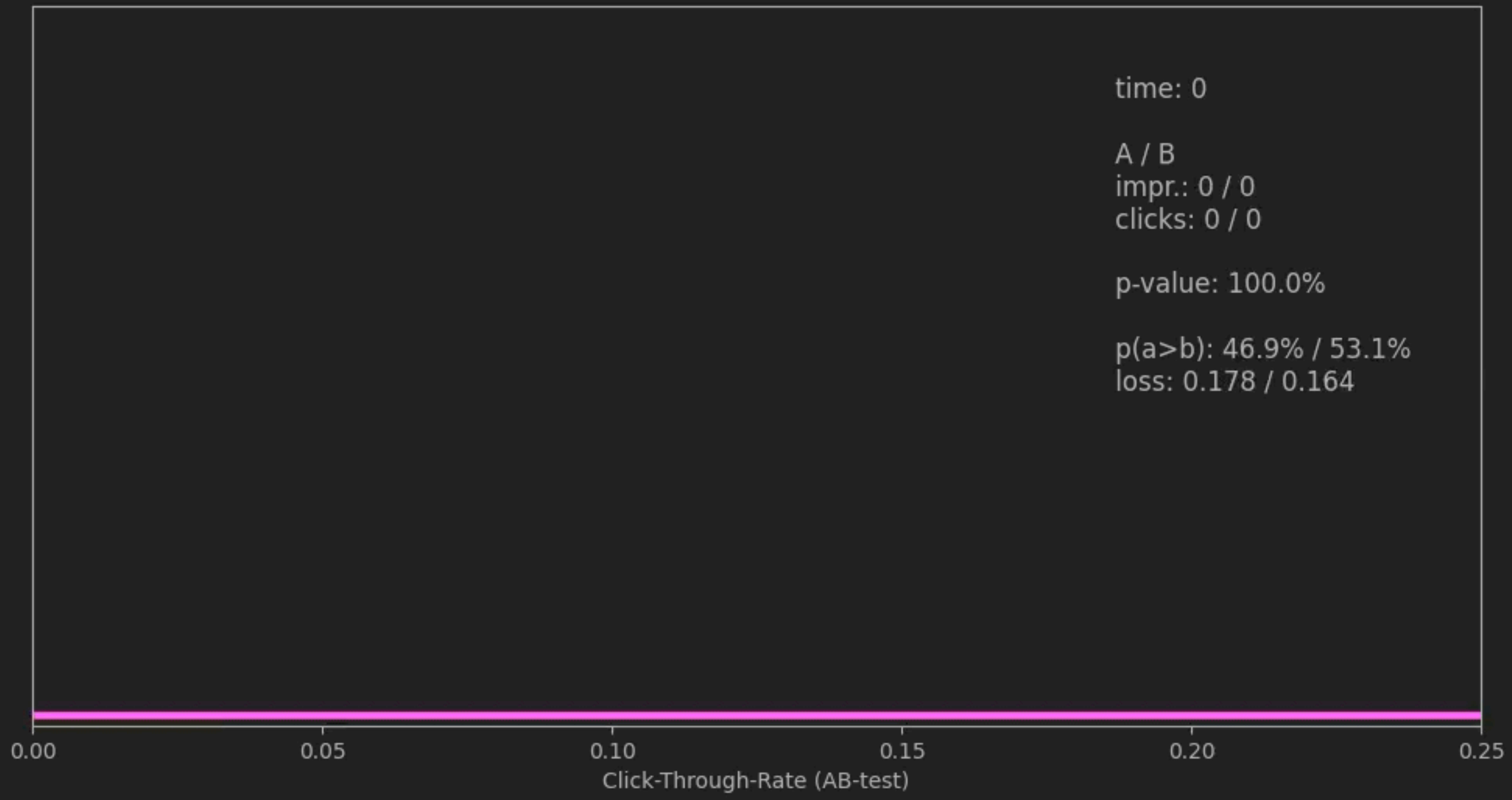
B

12%

CTR B

$19

CPM B



Hypothesis A/B Testing - Campaign simulation

A/B-test

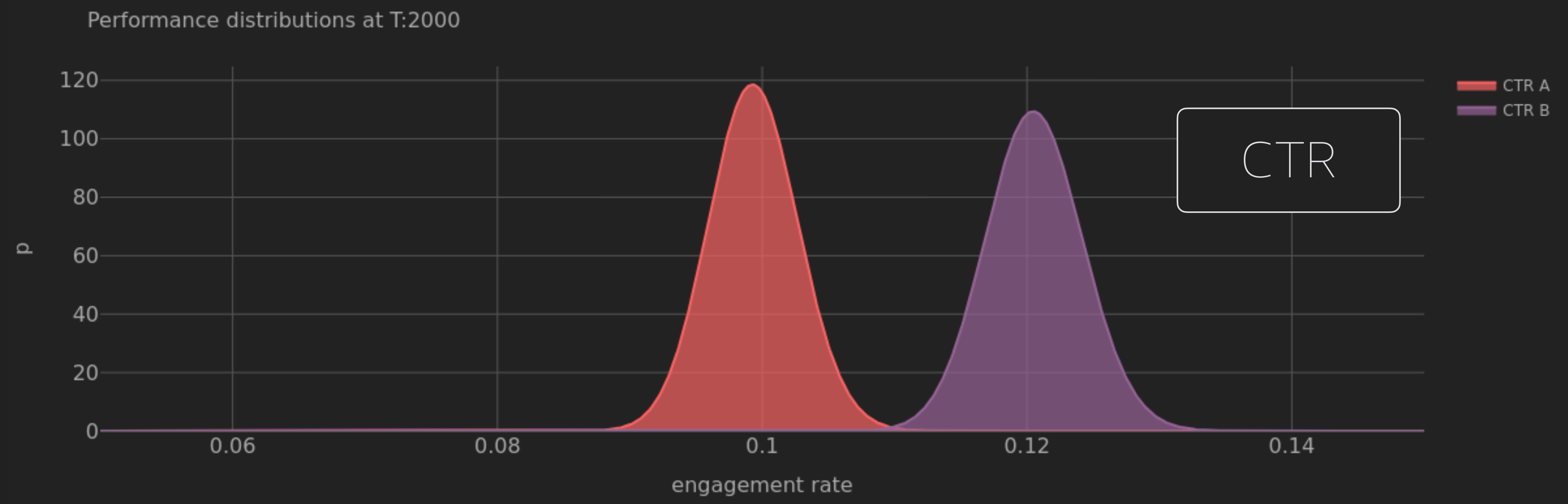
A

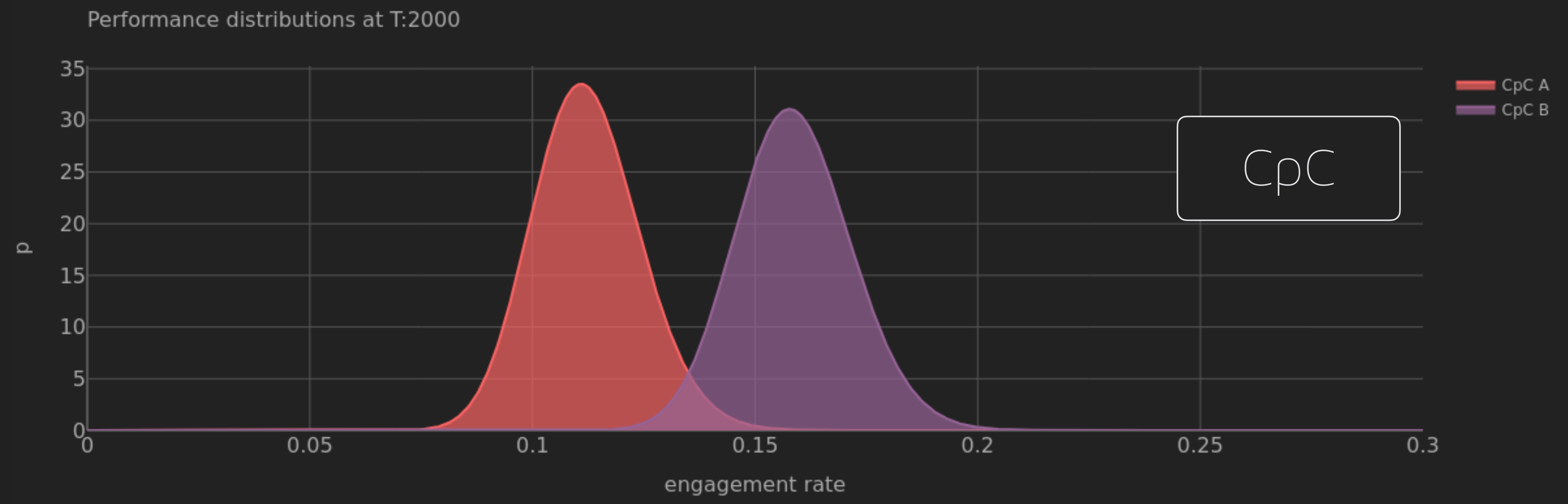
10%	CTR A
$11	CPM A
7898	impressions
784	clicks
87	cost

B

12%	CTR B
$19	CPM B
7961	impressions
959	clicks
151	cost

End-of-simulation





Hypothesis A/B Testing - Campaign simulation

A/B test

A

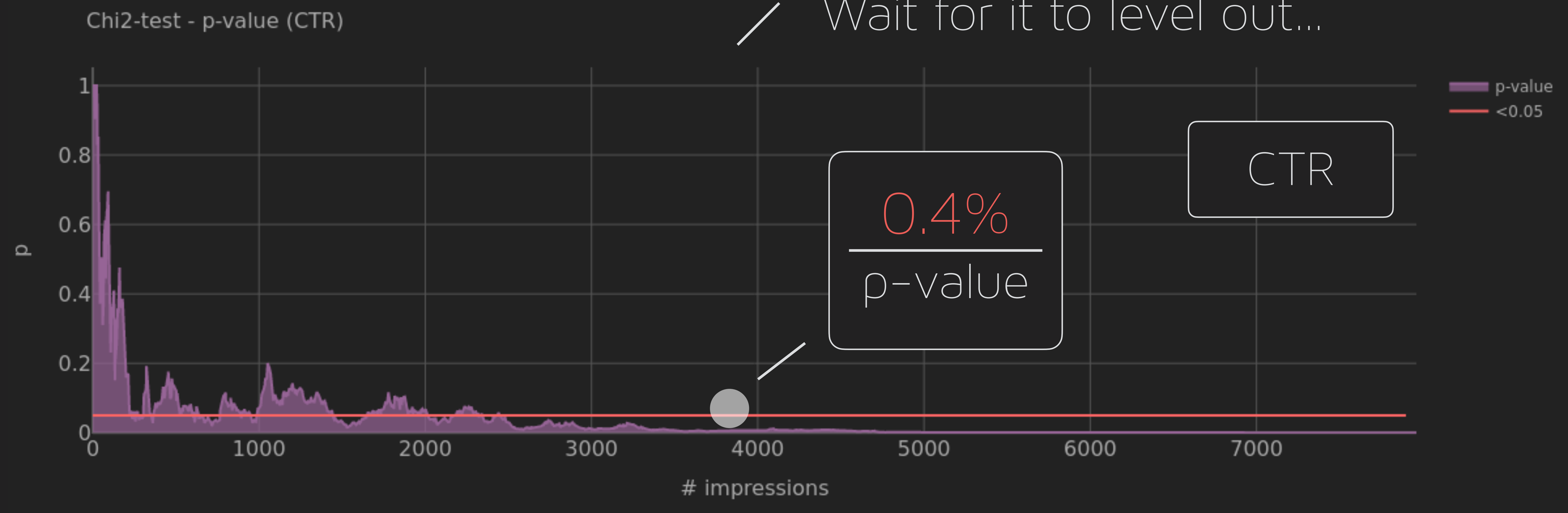
10%

CTR A

$11

CPM A

Aleatoric uncertainty
Wait for it to level out...



Required number of samples

3840

#samples

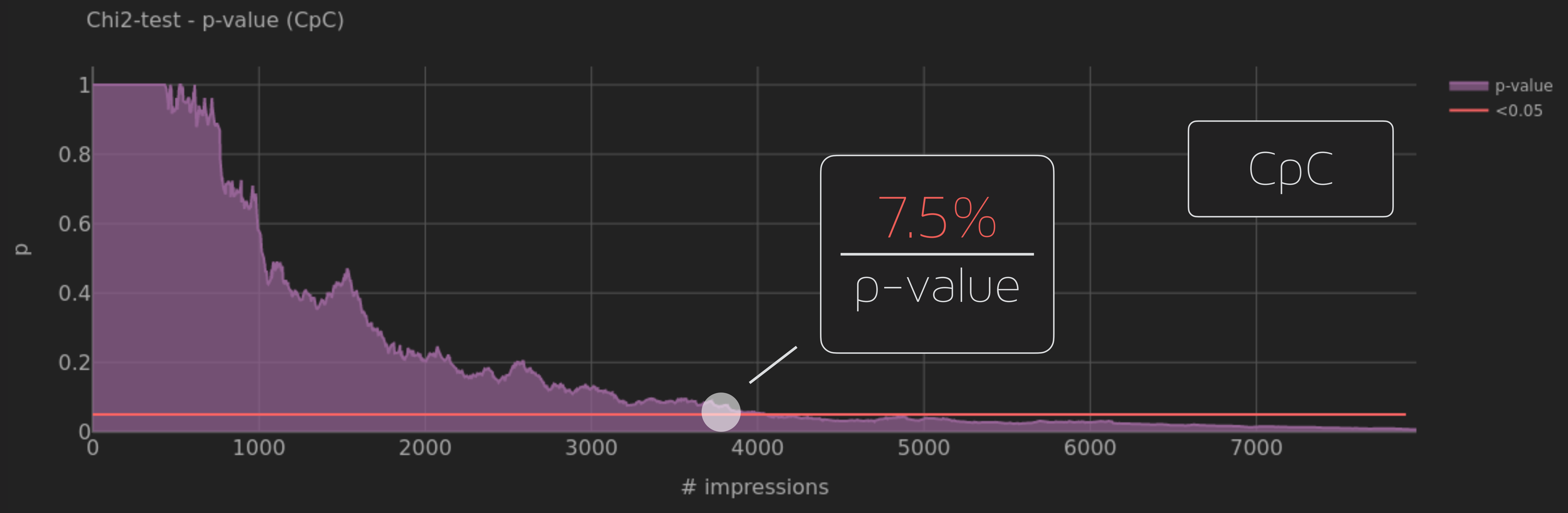
B

12%

CTR B

$19

CPM B



Hmmm, B didn't pass...

Higher cost did not outweigh better click rate...

Bayesian A/B Testing – Campaign simulation

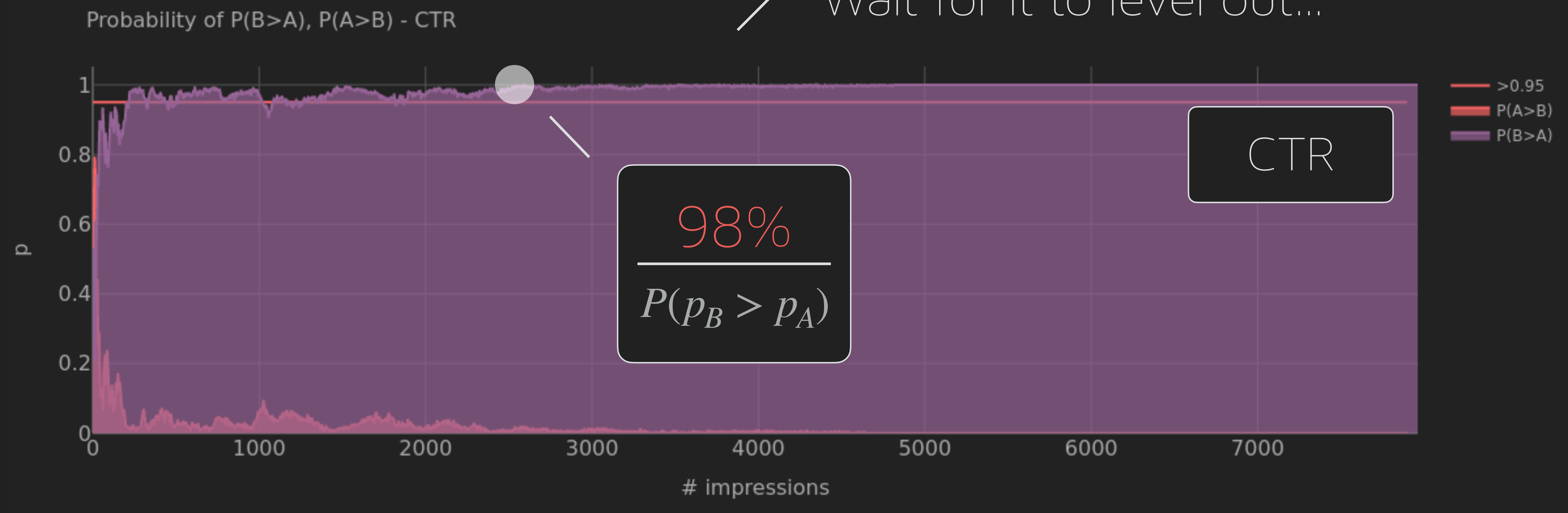
A/B test – P(B>A)

A

10%
CTR A

$11
CPM A

Aleatoric uncertainty
Wait for it to level out...



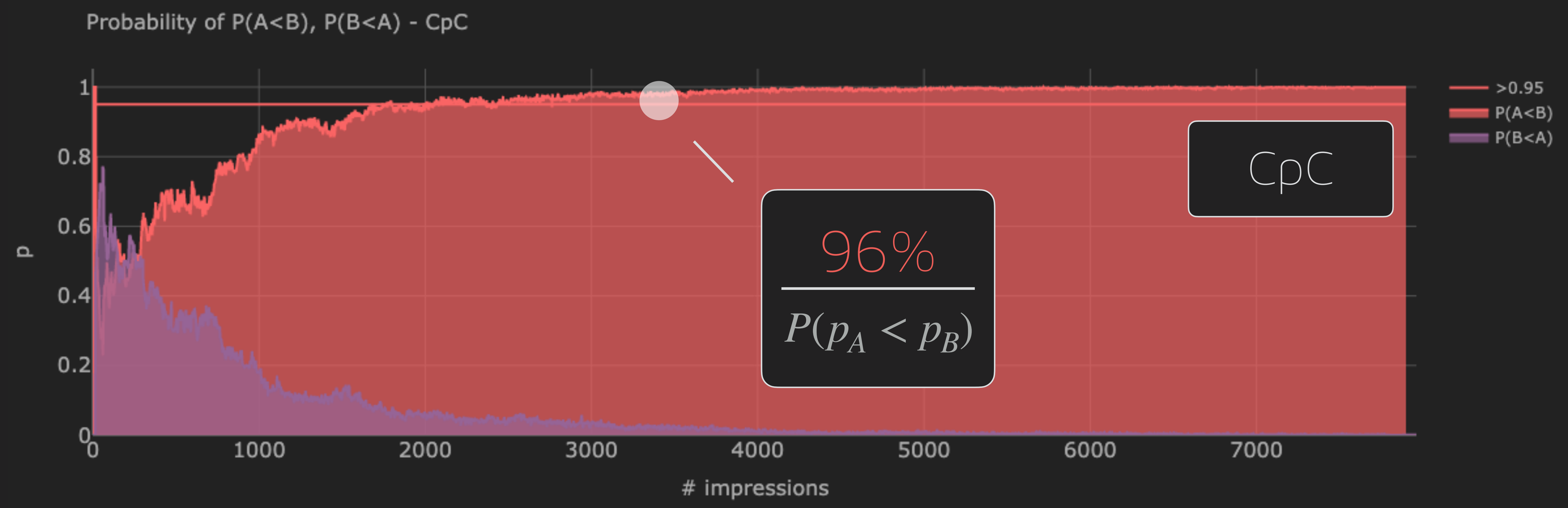
Could we get results earlier?

~500-2500
#samples

B

12%
CTR B

$19
CPM B



Aleatoric uncertainty messes with us...

Bayesian A/B Testing - Campaign simulation

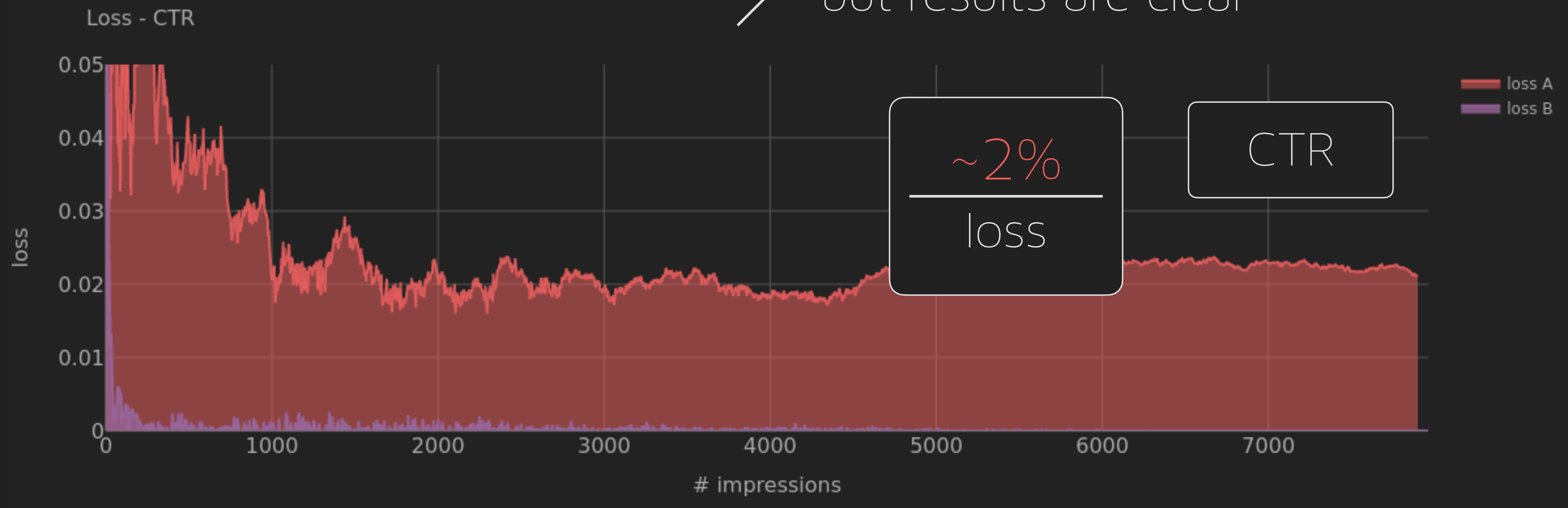
A/B test - Losses

A

10%
CTR A

$11
CPM A

Aleatoric uncertainty not stable, but results are clear

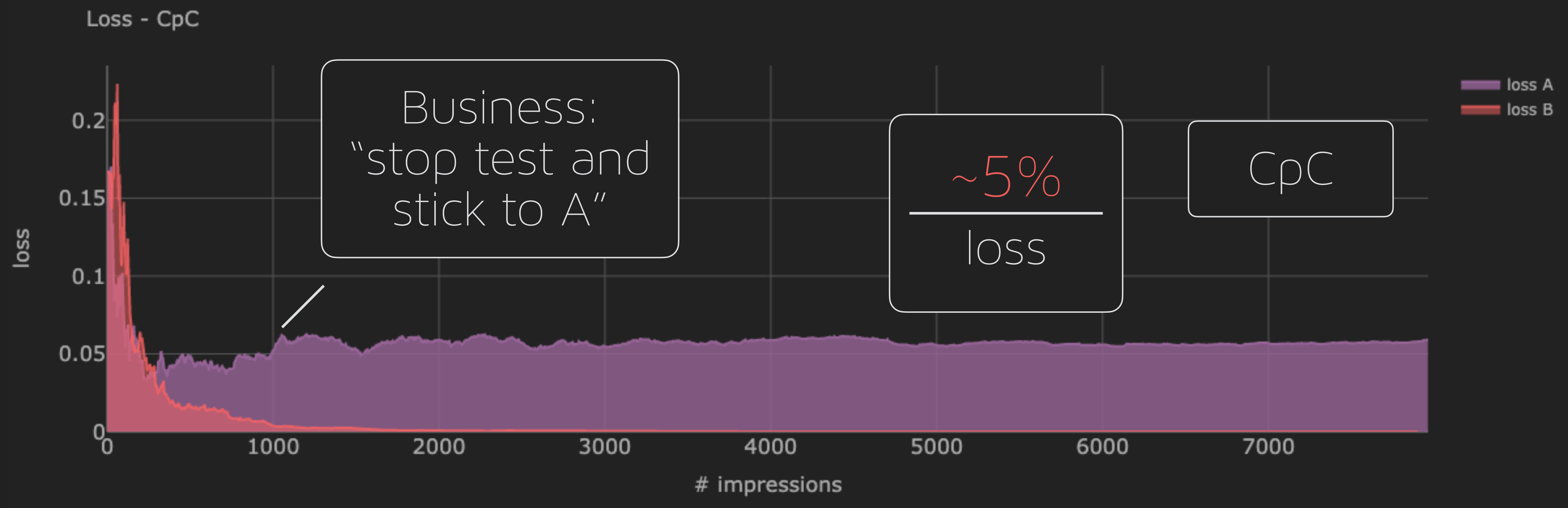


Choosing A is pretty expensive....

B

12%
CTR B

$19
CPM B

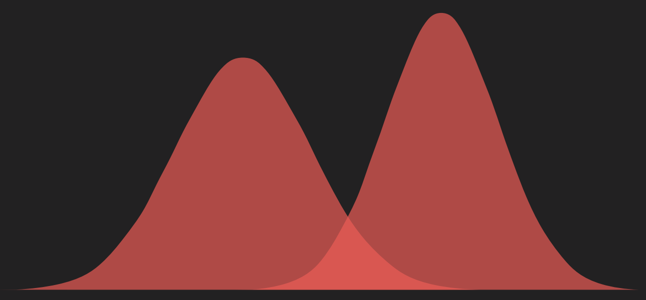


Choosing B is pretty expensive....

This time in real $$$

Bayesian End-of-Level summary

Uncertainty is important for practical evaluations

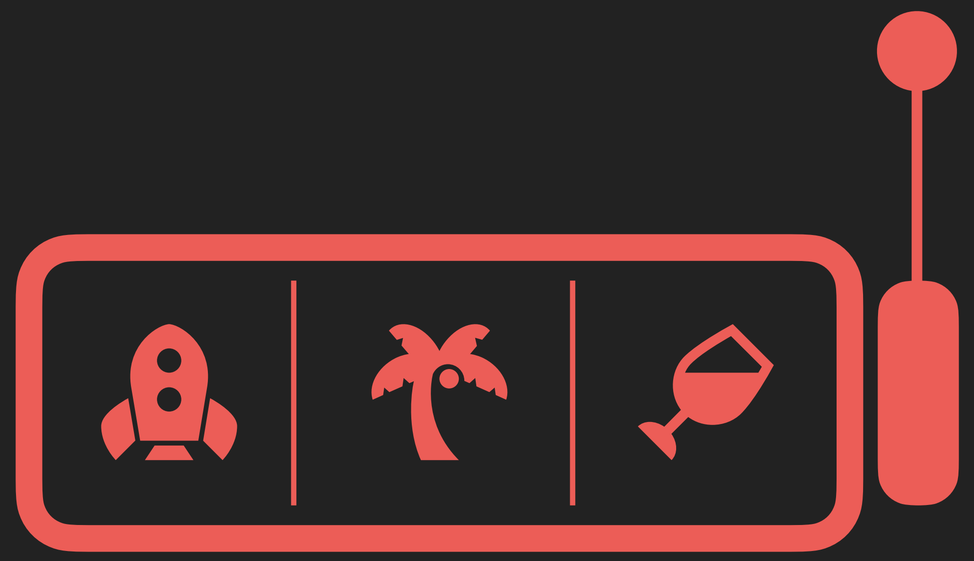


A/A-test - don't forget
Simple tools to evaluate - but not so simple to decide…?

Bayesian A/B-testing - white box - easier to understand
Interpretable metrics - probabilities & losses

Peeking is "allowed", but be carefull with the aleatoric uncertainty - it will mess with you….

Always be critical



Bayesian Bandit Testing

A/B-test challenges….

$$$

Suboptimal for 23.5 hours

Test keeps running even after winner is found…

non-stationary environment

Fails test on Monday / Tuesday

But better on weekends

Bayesian Bandits

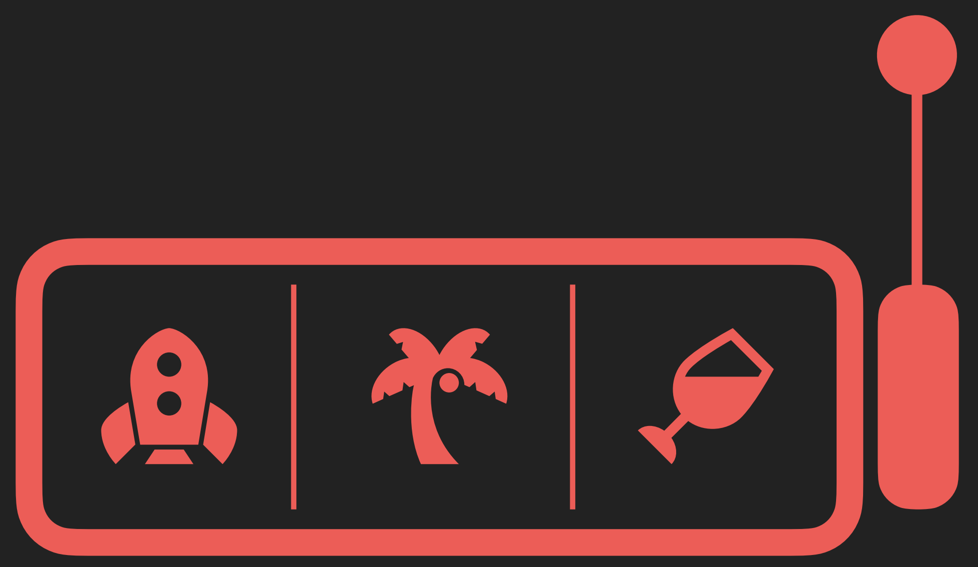
A bandit test system can handle soft activation / deactivation of all variants

….while you sleep :-)

Just put it in and let it run….

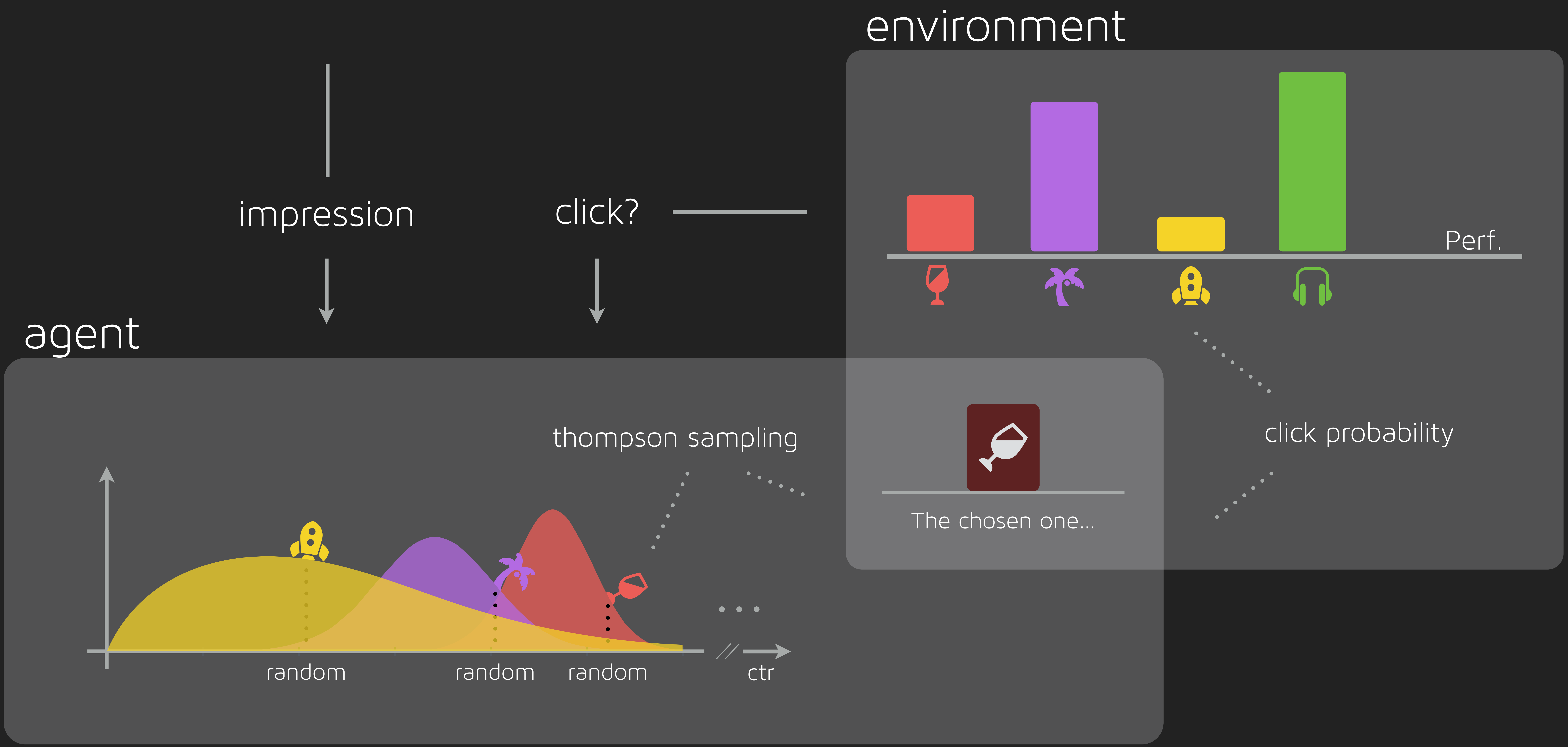
Opportunity to rule out losers…

Barraza-Urbina and Glowacka
RecSys 2020 Tutorial: Introduction to Bandits in Recommender Systems
https://www.youtube.com/watch?v=rDjCfQJ_sYY

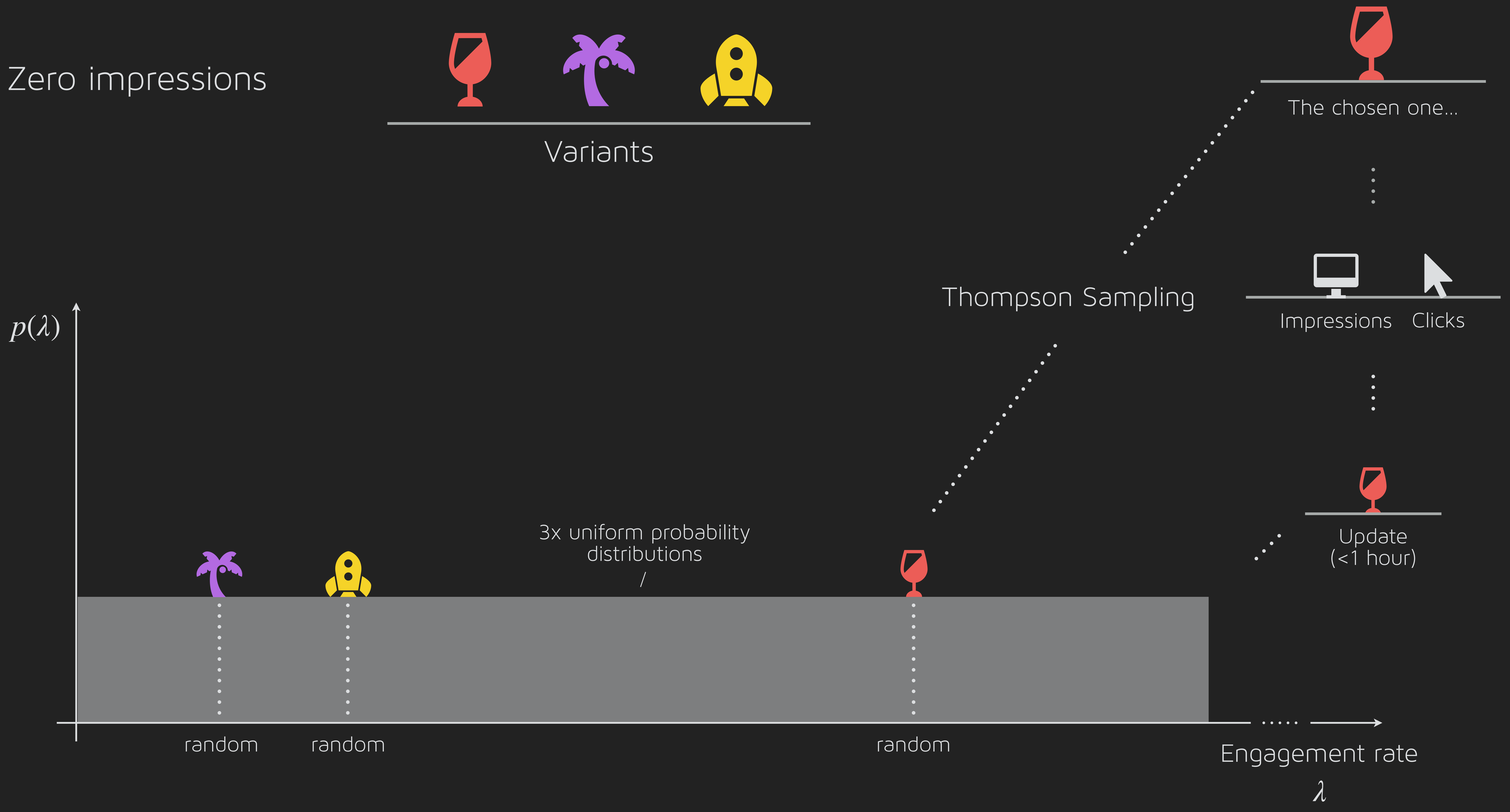


The Process

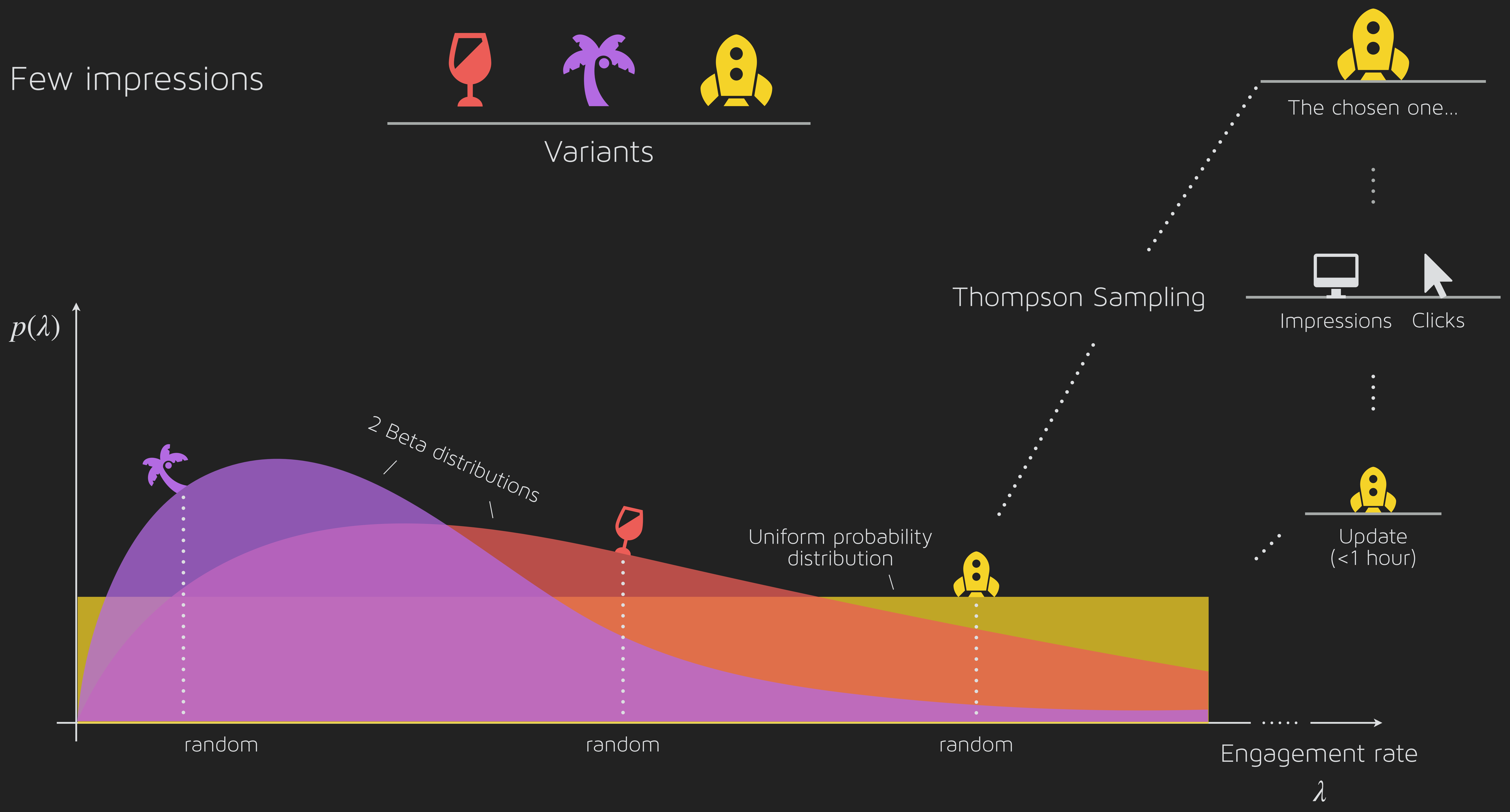
Bandit Architecture



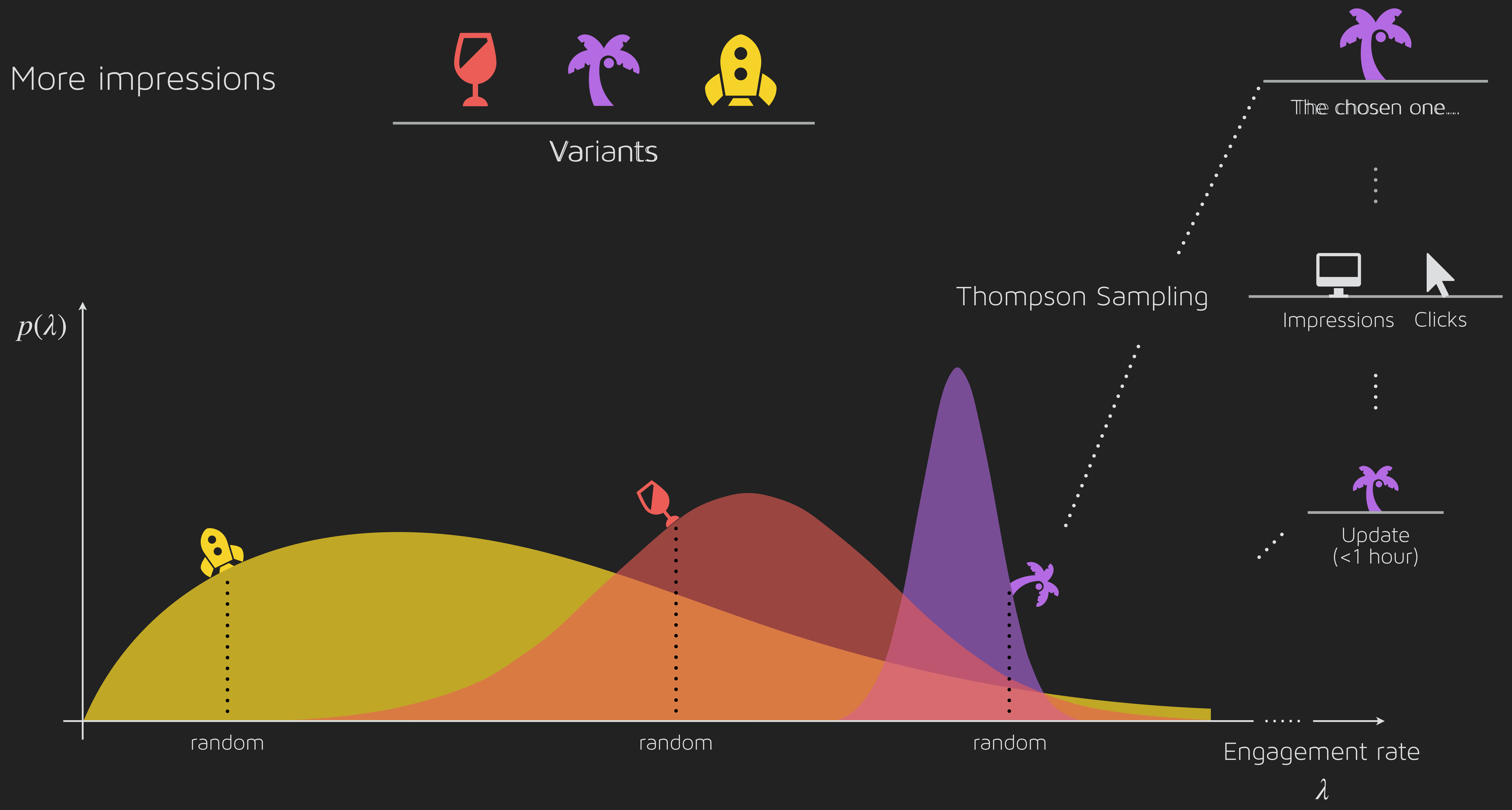
Bayesian Beta Bandits



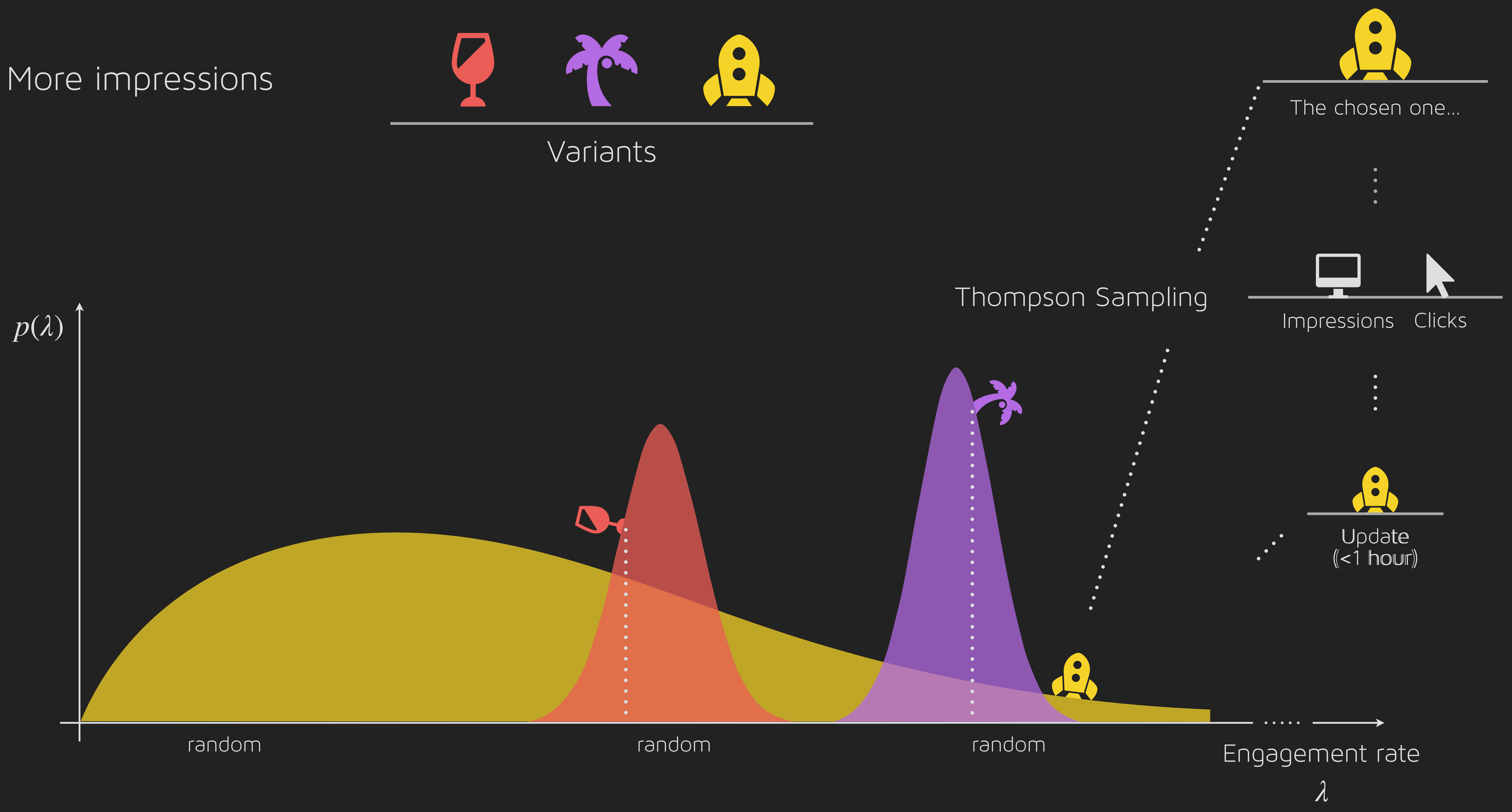
Bayesian Beta Bandits



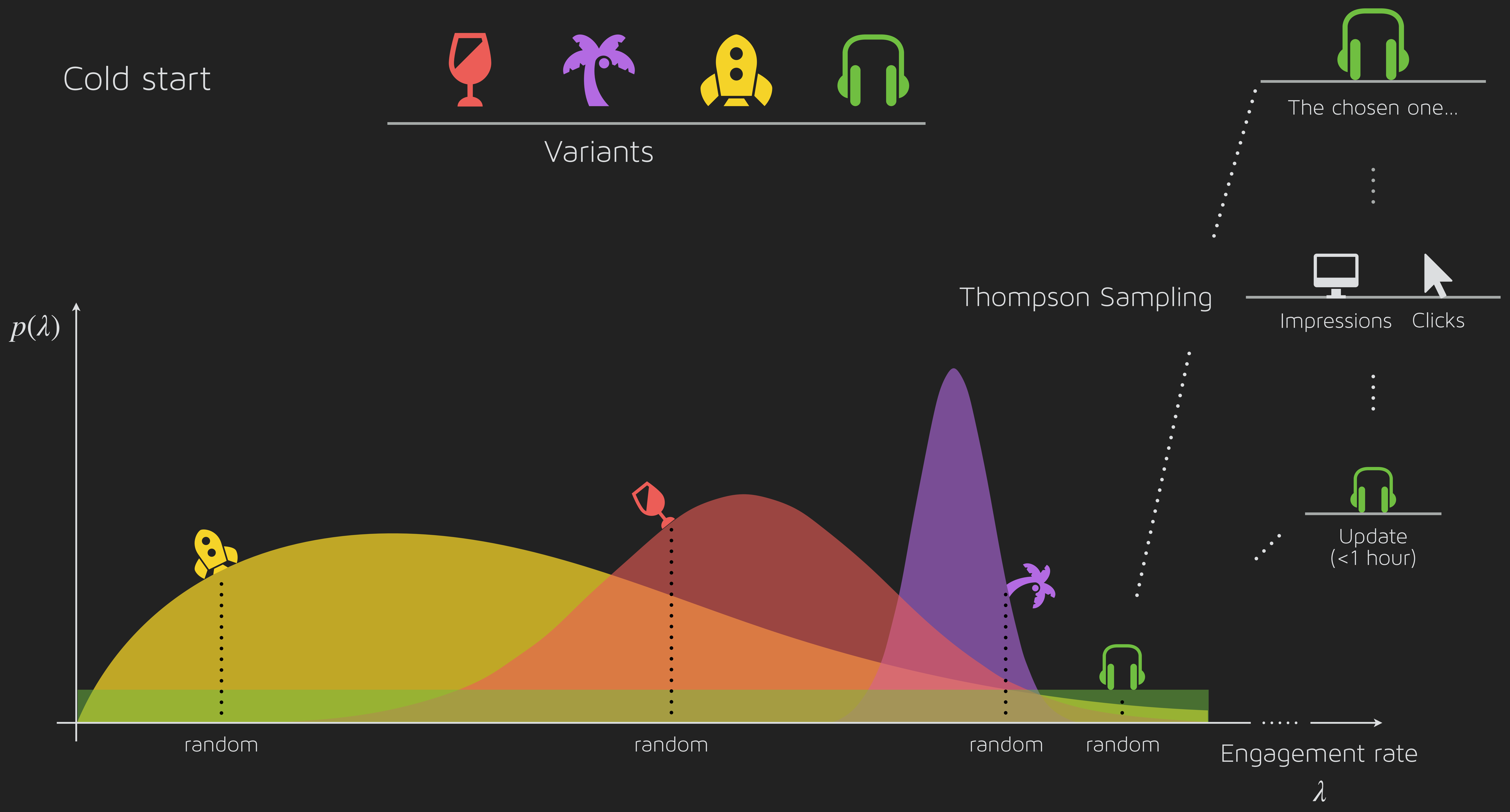
Bayesian Beta Bandits



Bayesian Beta Bandits

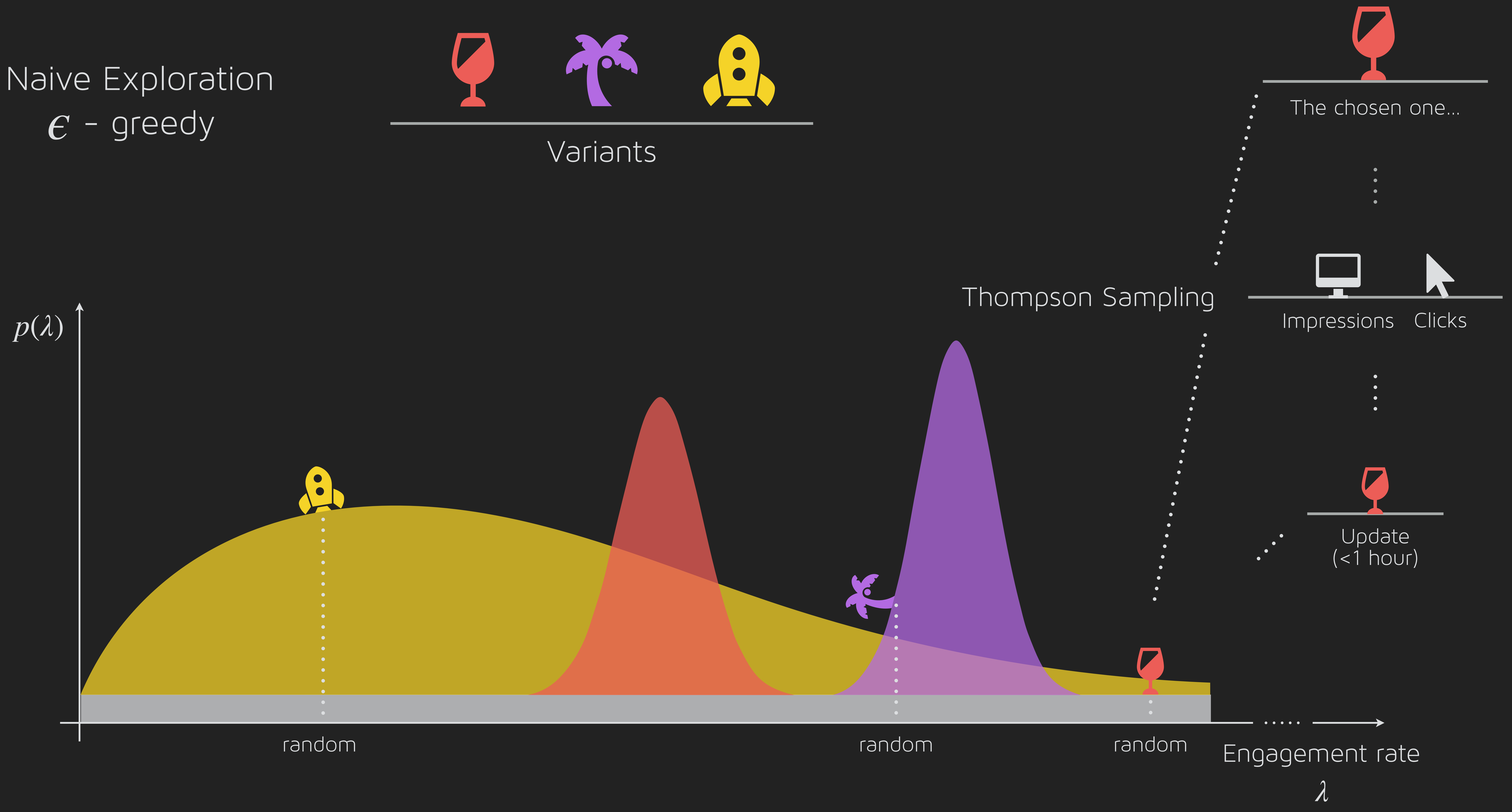


Bayesian Beta Bandits

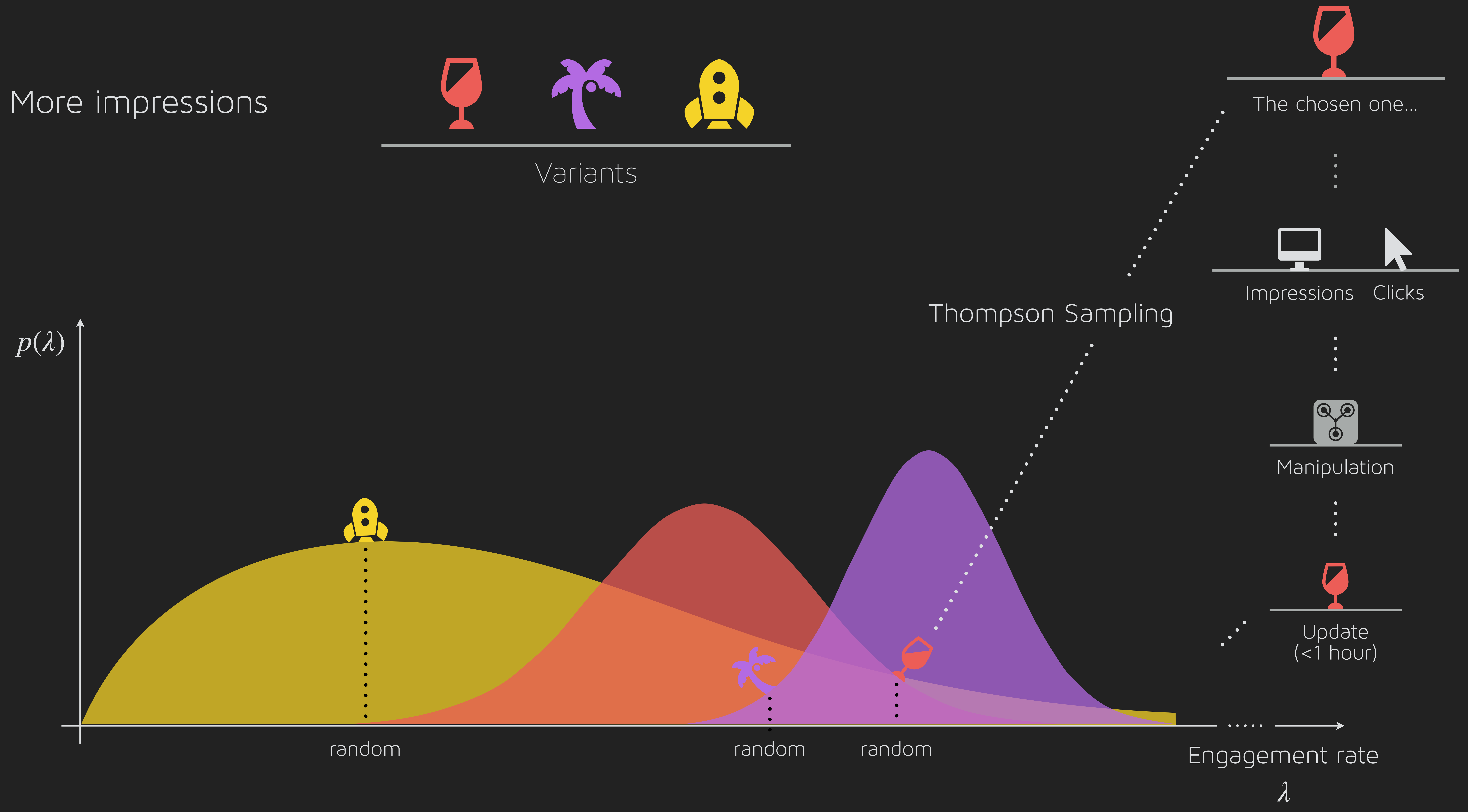


Exploration | vs. | Exploitation

Bayesian Beta Bandits



Bayesian Beta Bandits



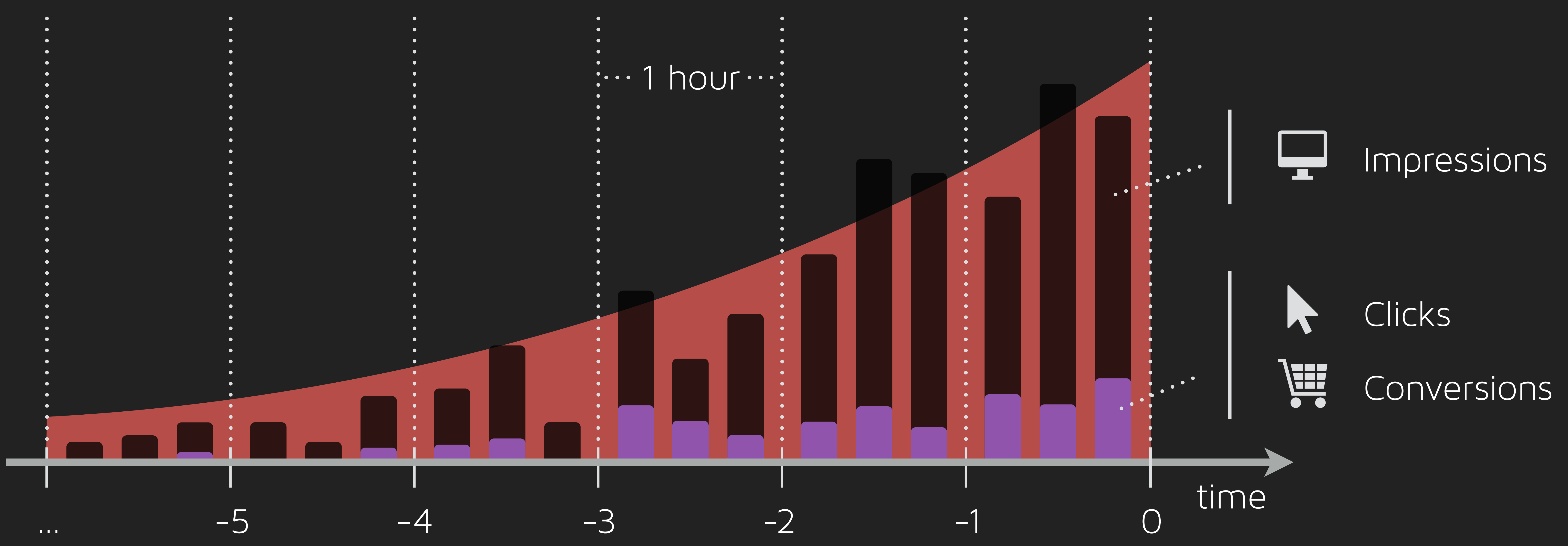
Recency

forgetting the past...

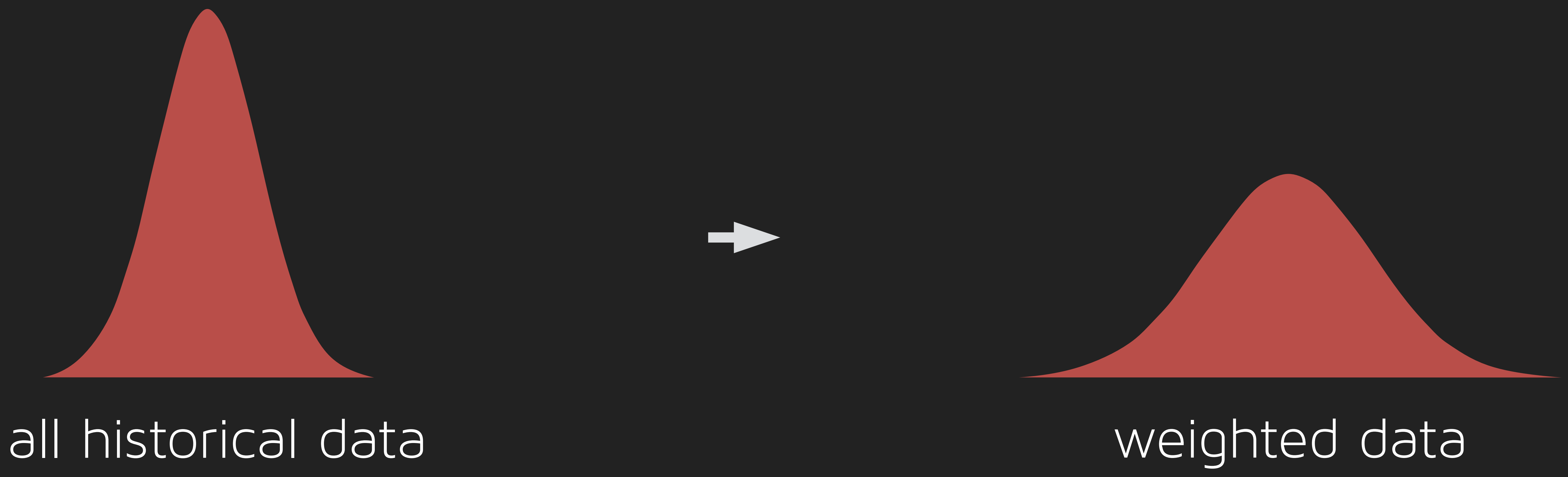
Recency – forgetting the past

Weight all counts too limit the influence of historical data

Increases the epistemic uncertainty



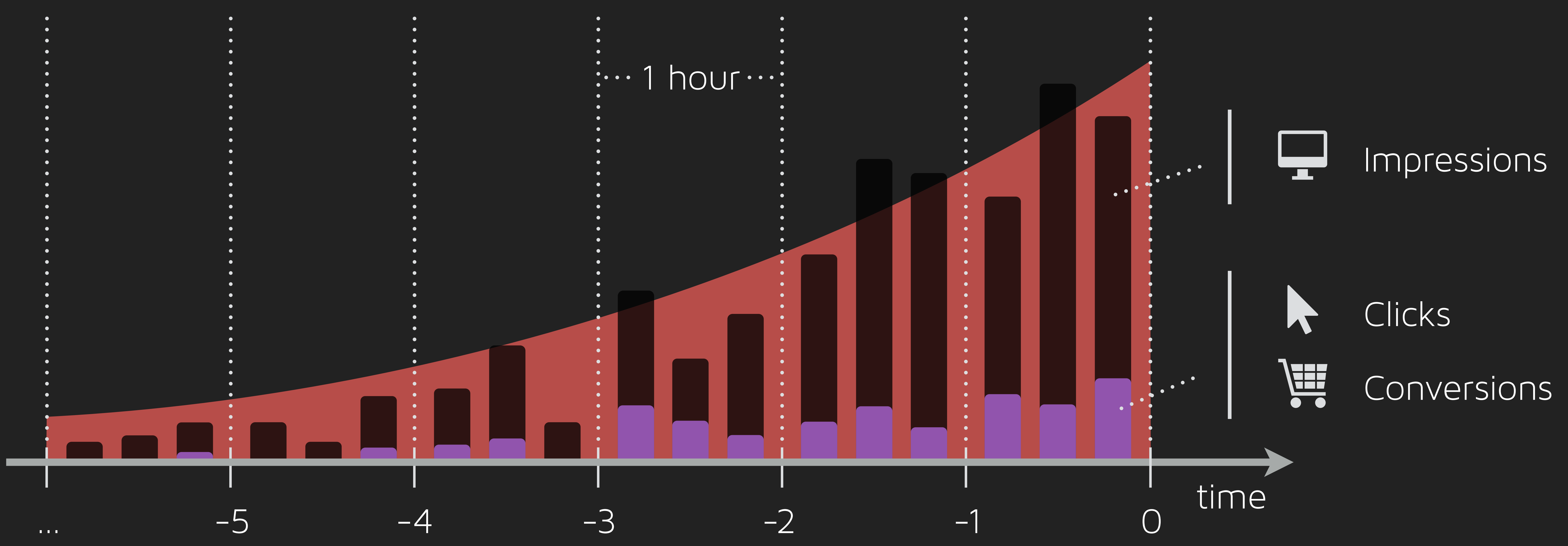
weighted counts are used to define our knowledge



Recency – forgetting the past

Weight all counts too limit the influence of historical data

Increases the epistemic uncertainty



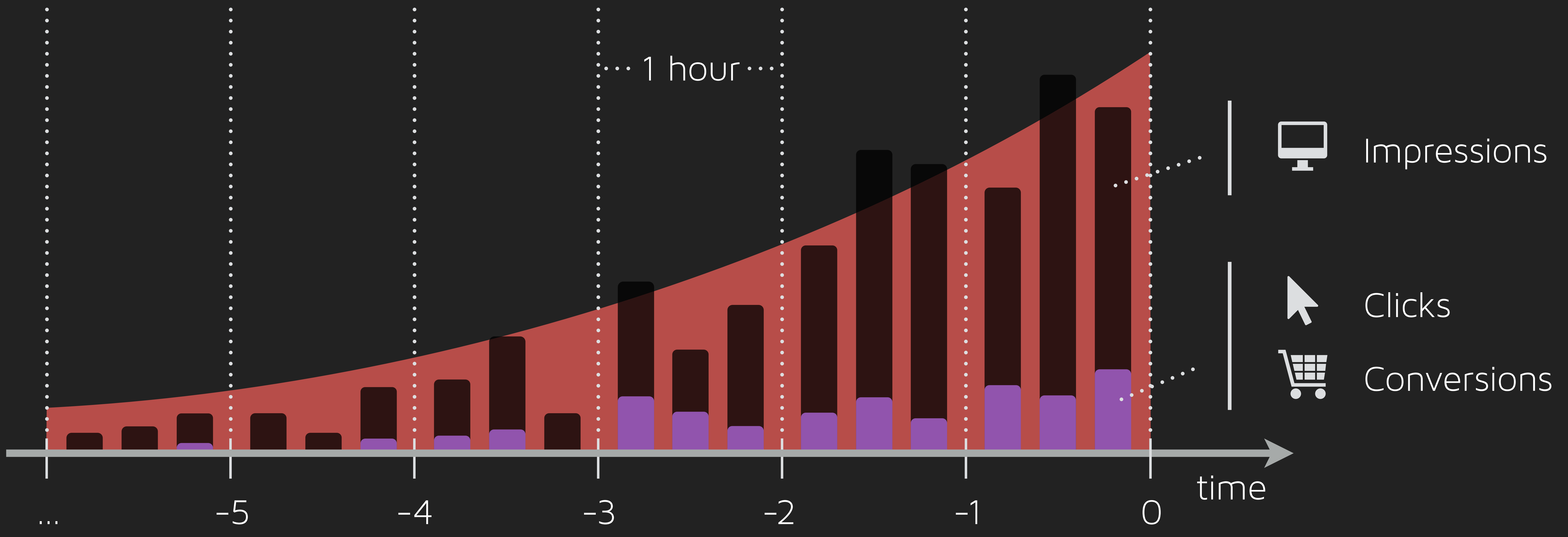
decay function

$$f(t) = \alpha^t, \quad \text{where} \quad 0 \leq \alpha \leq 1$$

Explore / Exploit trade-off

Weight all counts too limit the influence of historical data

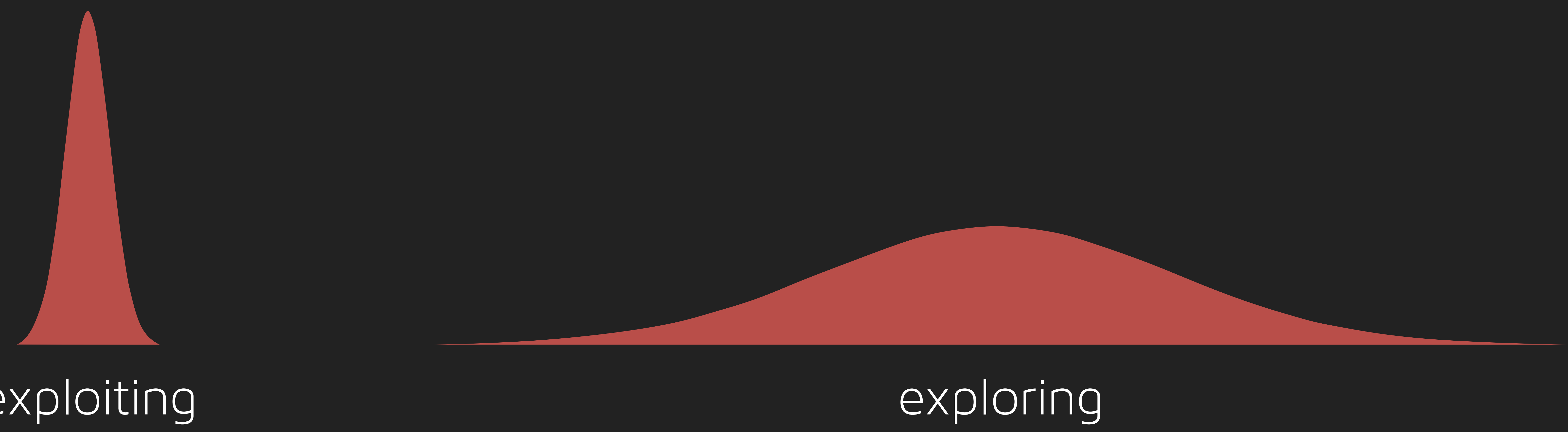
Increases the epistemic uncertainty

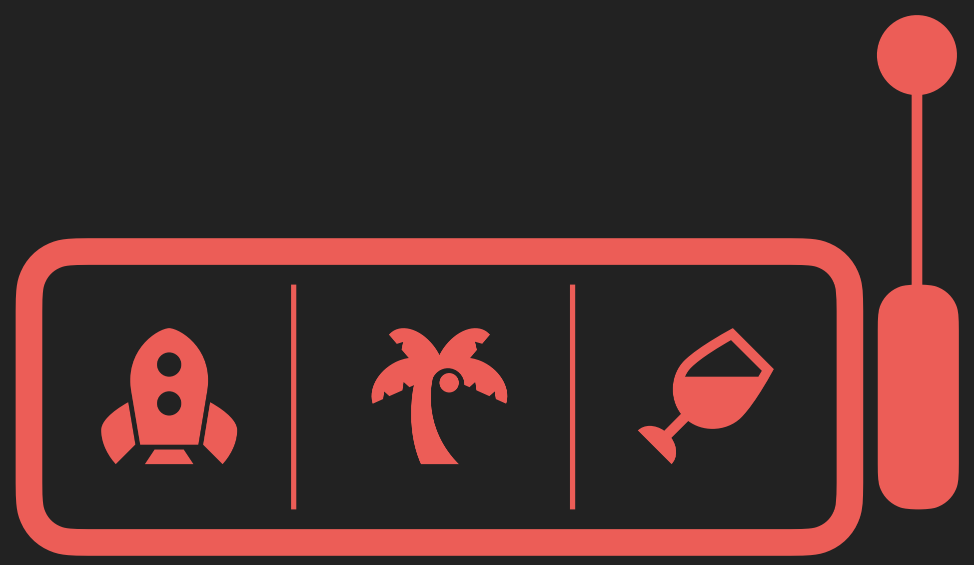


balance between adapting to the environment

or

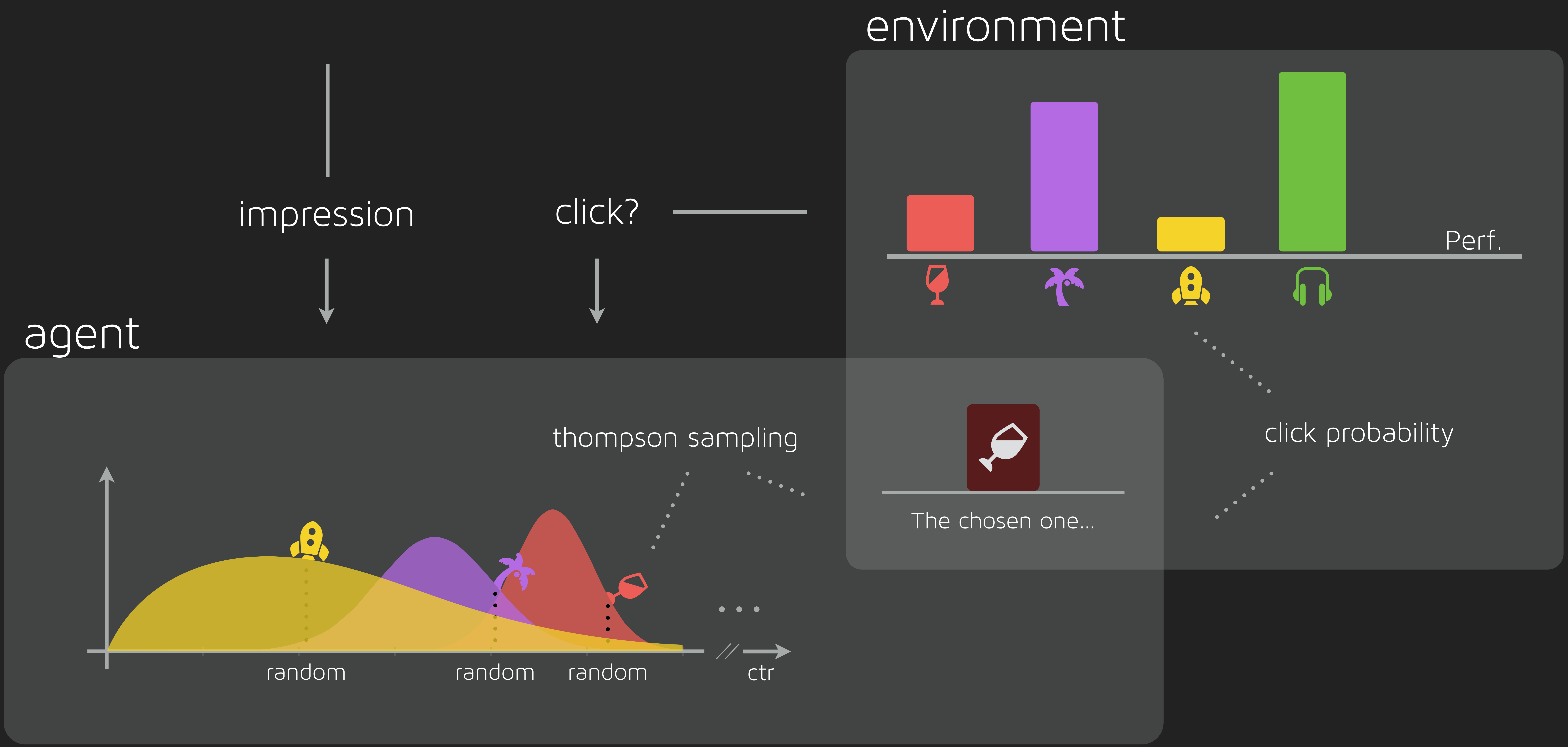
just going for it...



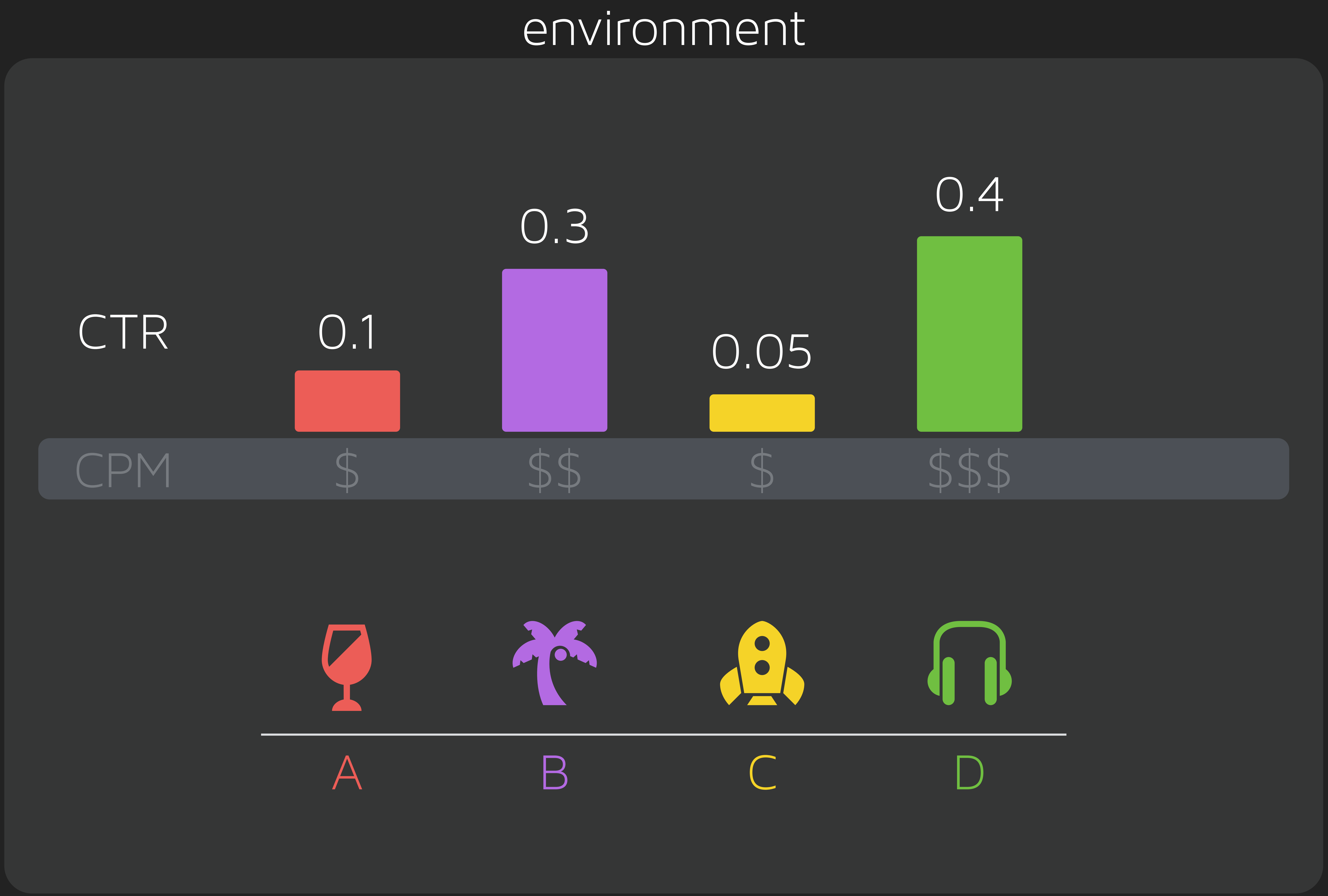


Simulation

Bandit Architecture



Bandit Architecture



metric

regret - average

in words

Average number of times the optimal ad was not selected.

lower is better

in math

$$\rho = T\mu^* - \sum_{t=1}^{T} \hat{r}_t$$

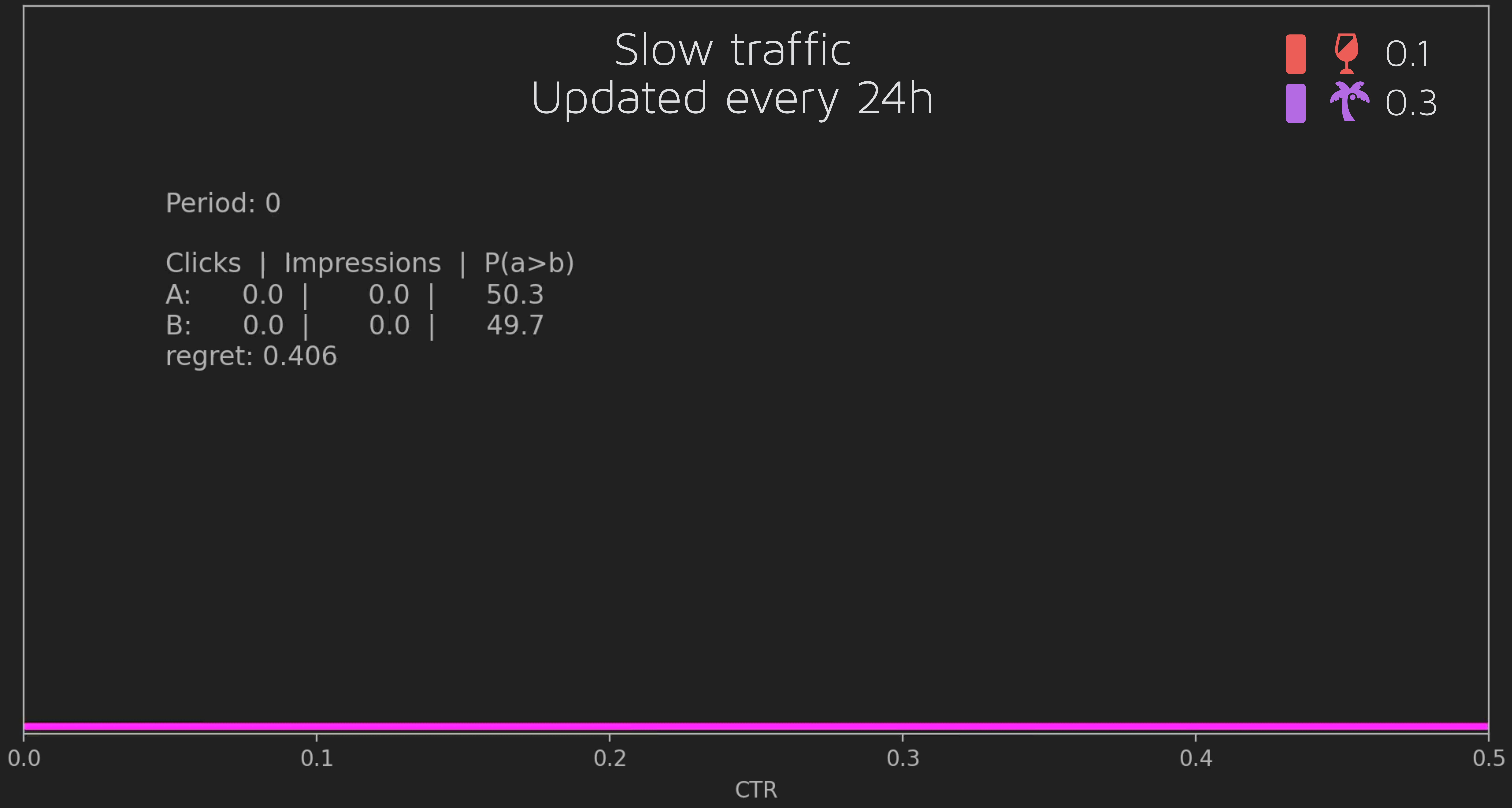
$$\mu^* = \max_k(\mu_k)$$

in code

```
(max_reward_mean - n_optimal_selections_b) / imp_per_period
```

Bayesian Bandit A/B testing

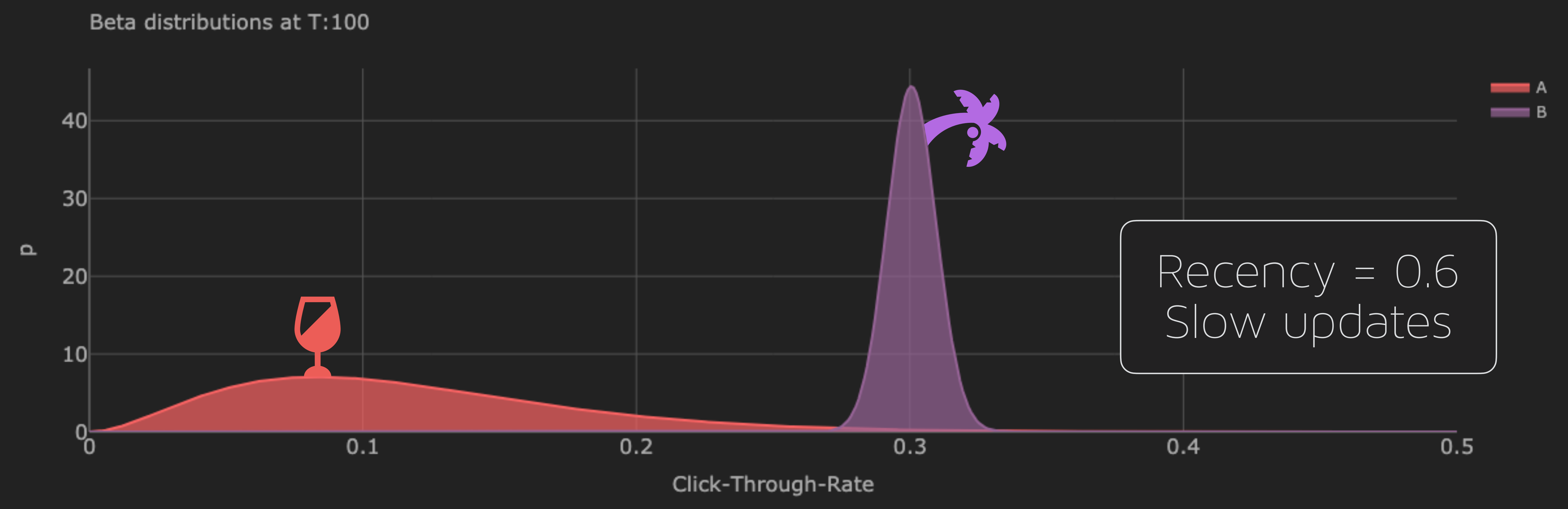
Bandit A/B-test



Bandit A/B-test

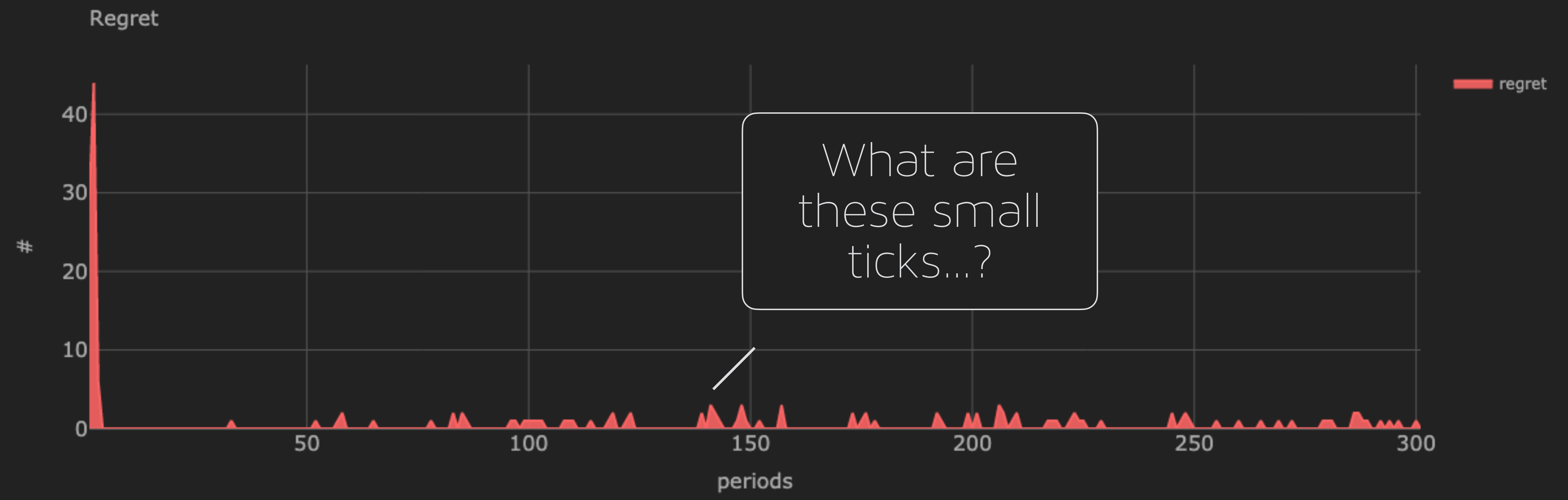
Beta Distributions

B is clearly in front and we know very little of A

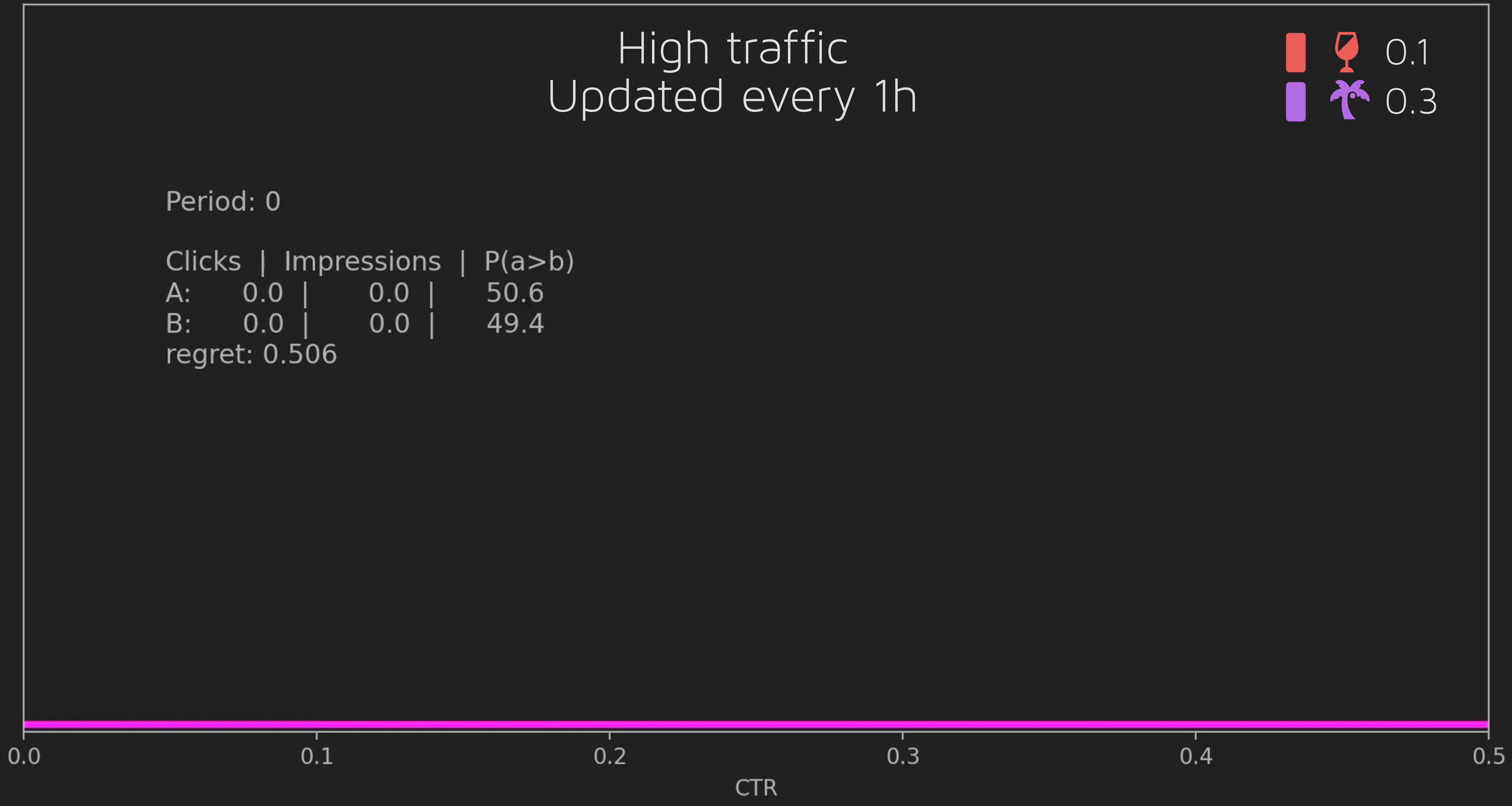


Regret

Average number of times the optimal ad was not selected.



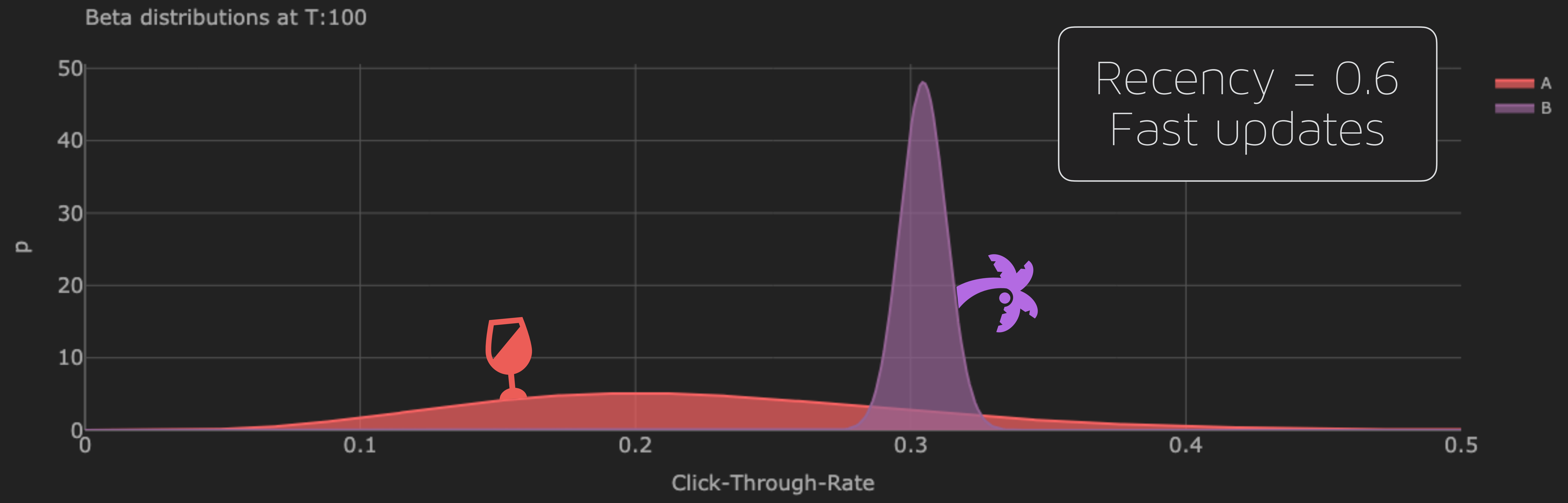
Bandit A/B-test



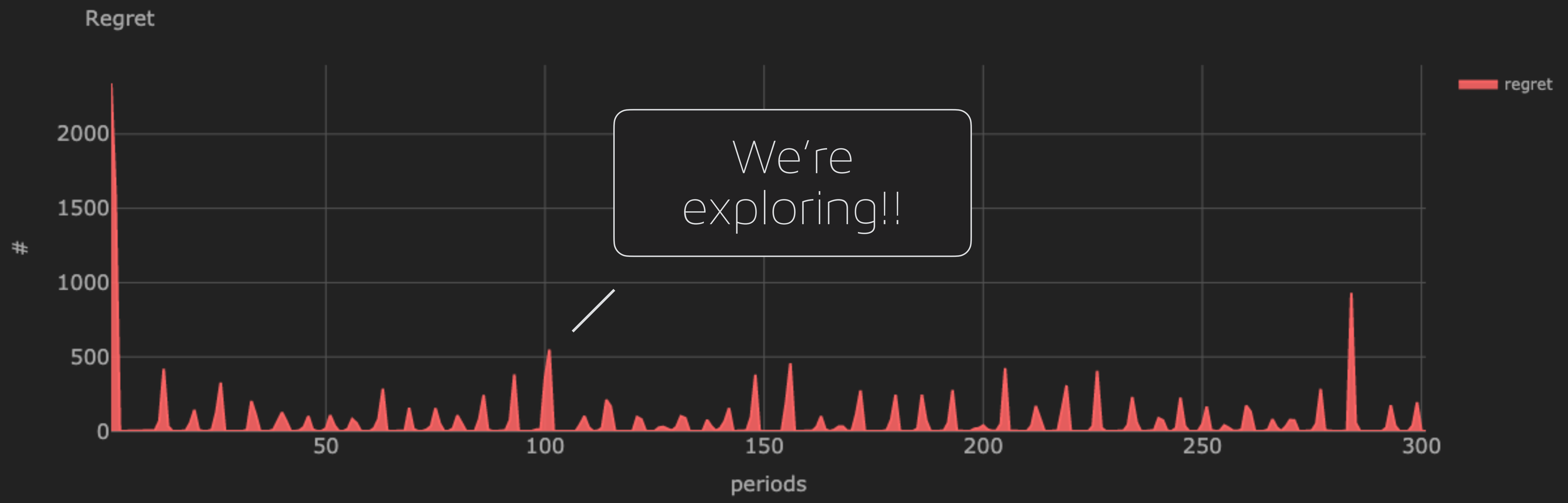
Bandit A/B-test

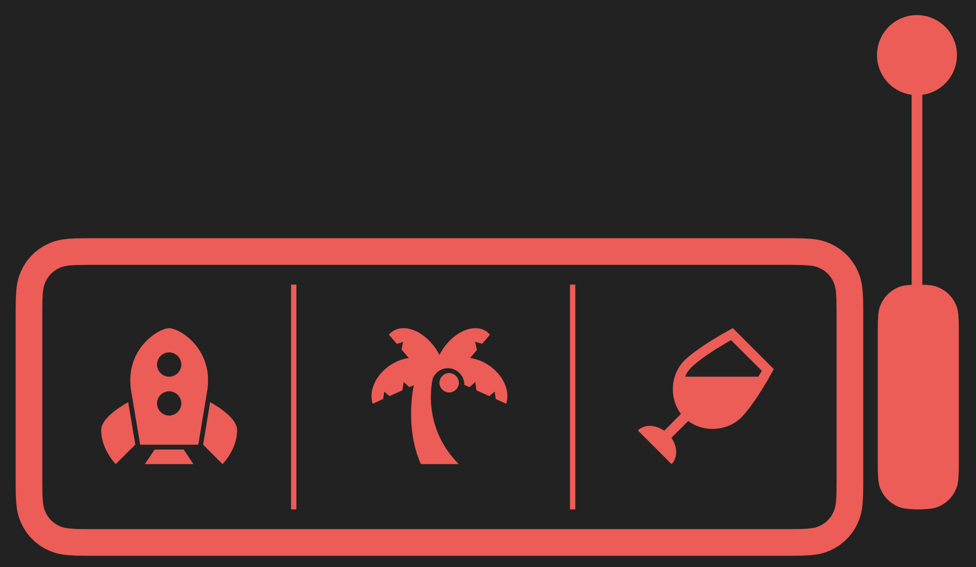
Beta Distributions

Example of the recency making A wide



This makes A get selected more often and we're exploring...

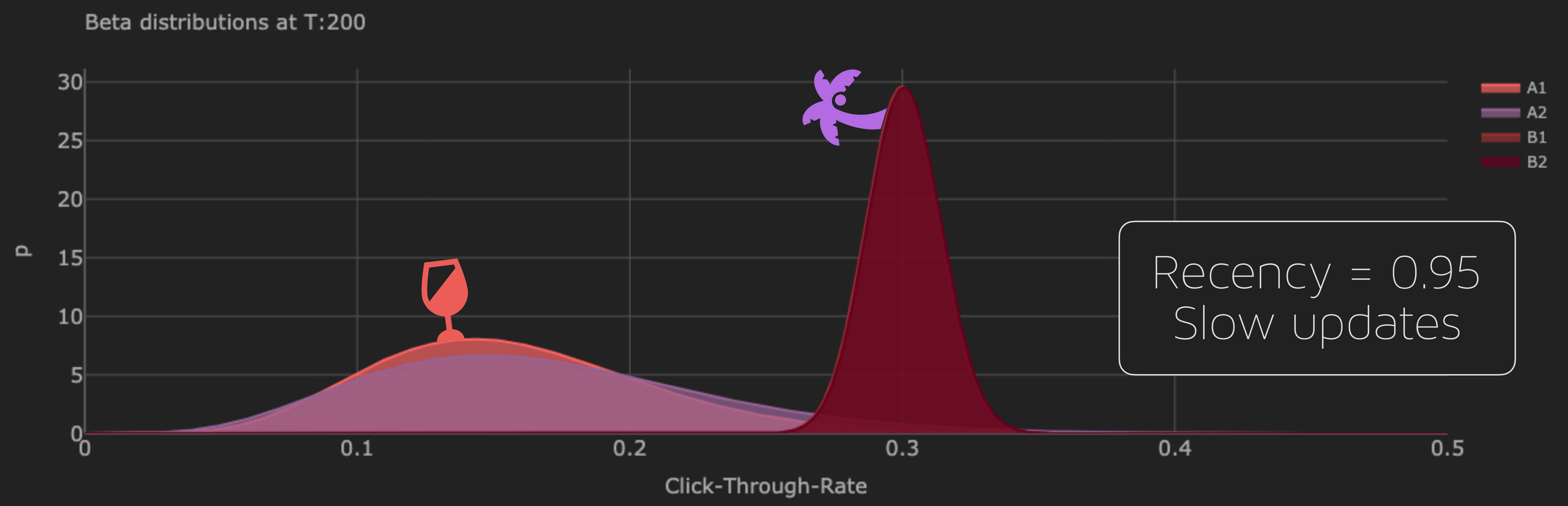




Bayesian A/A testing

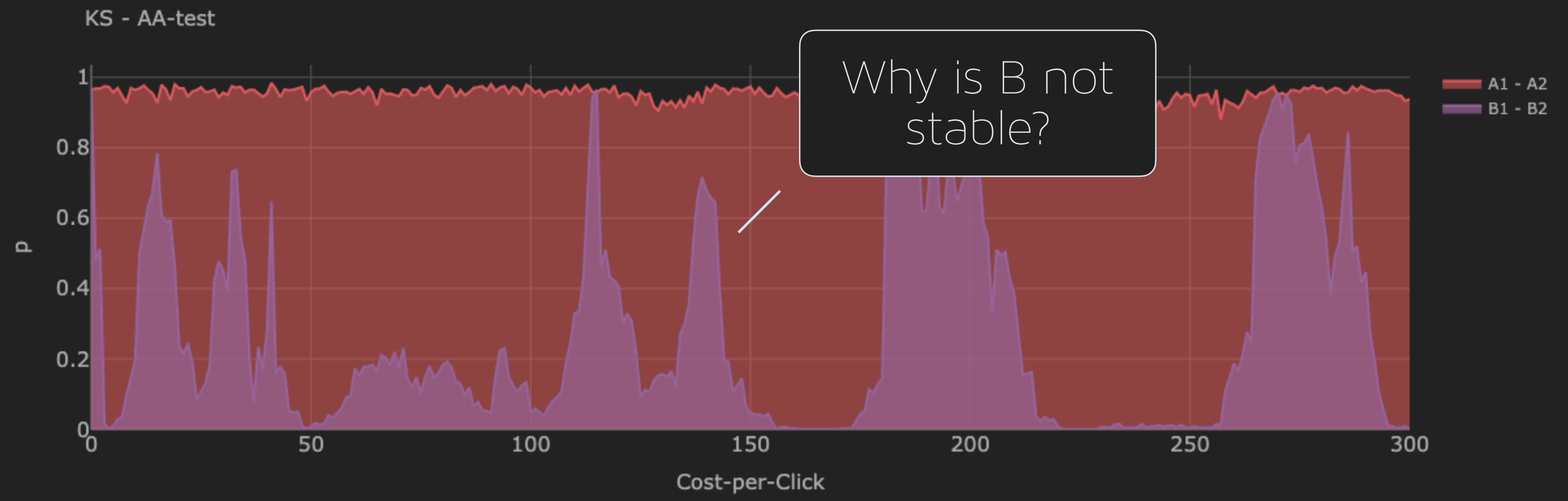
Bandit A/A-test

Both A/A and B/B seems similar

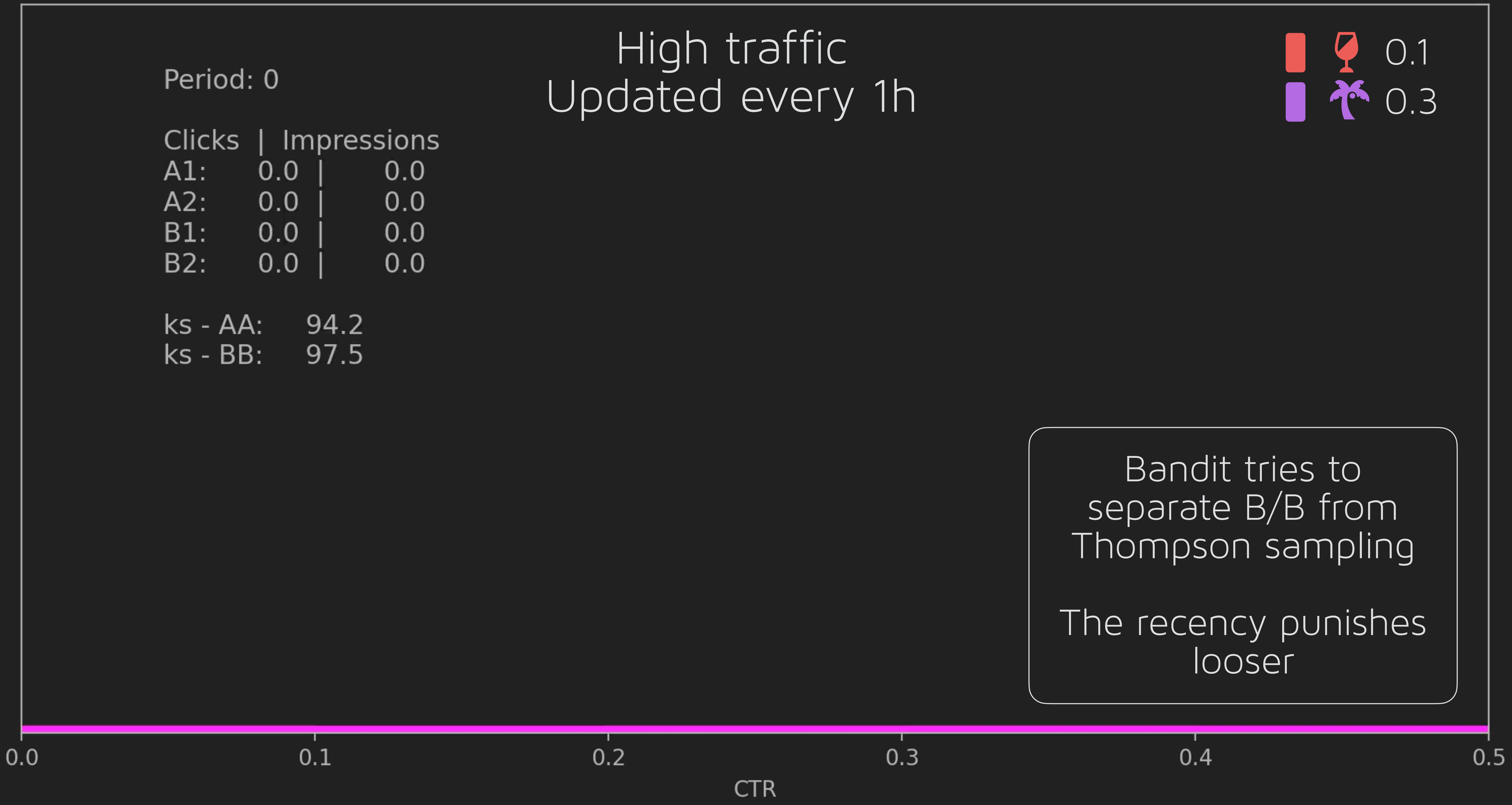


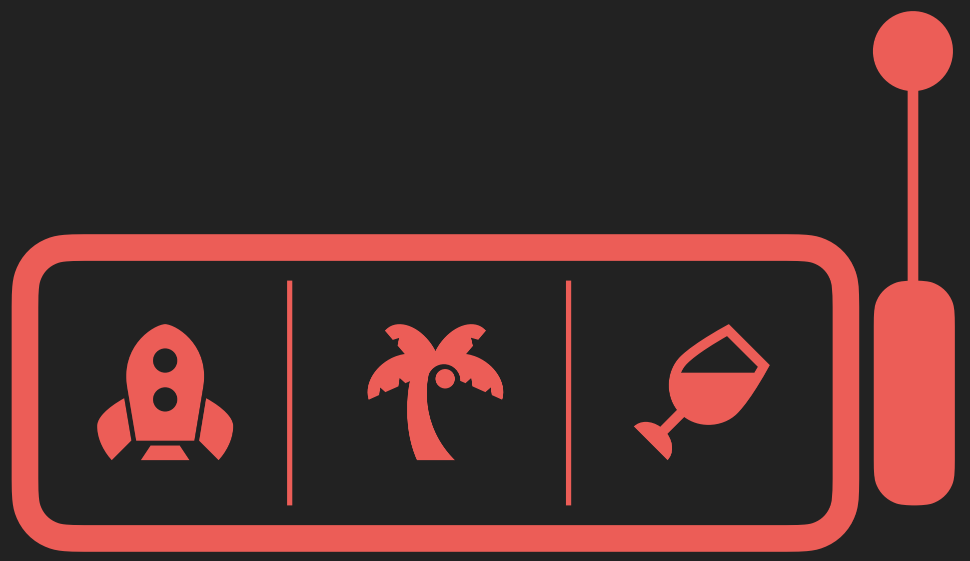
Kolmogorov-Smirnof

Similarity score



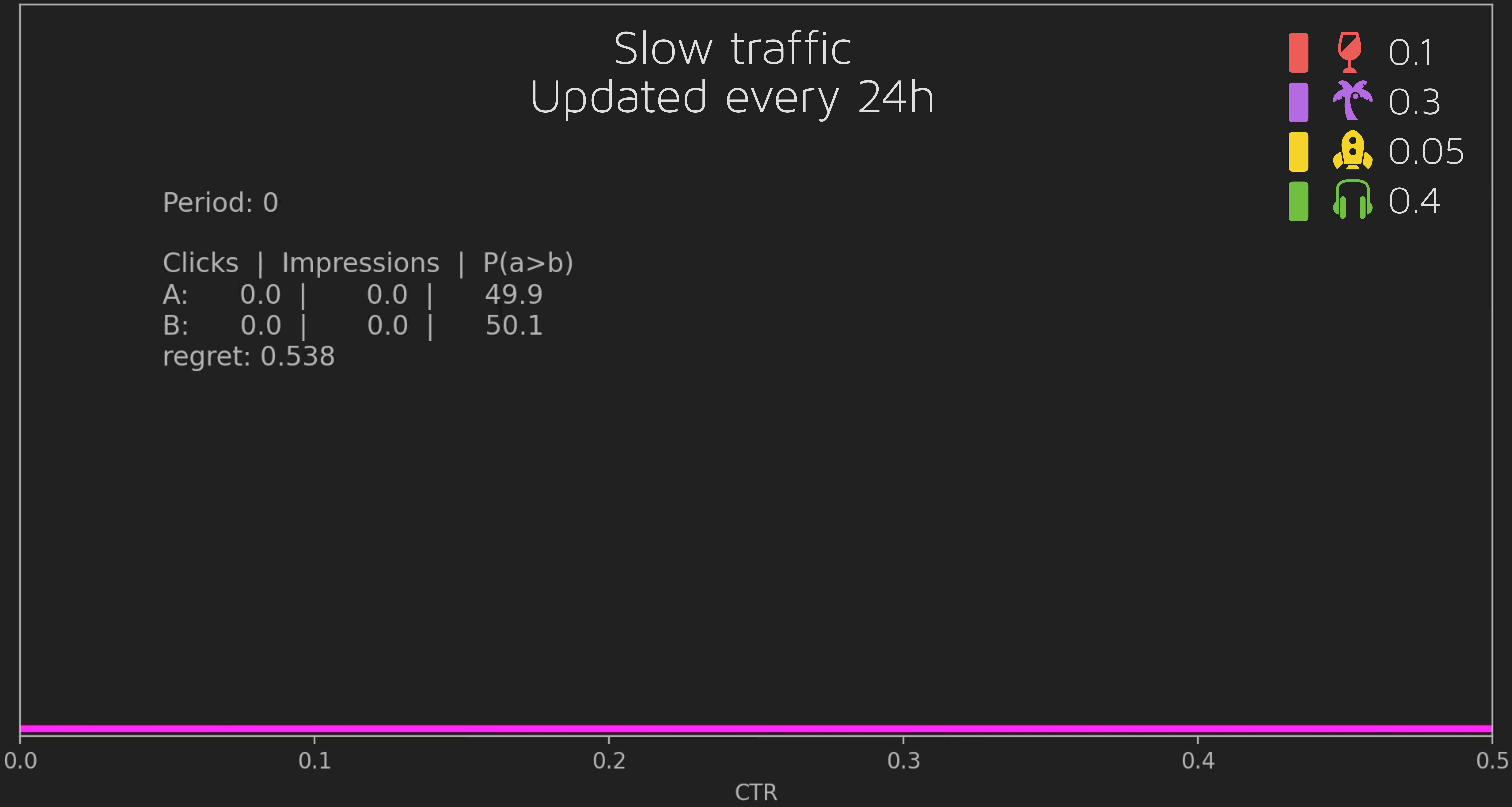
Bandit A/A-test





Cold starting...

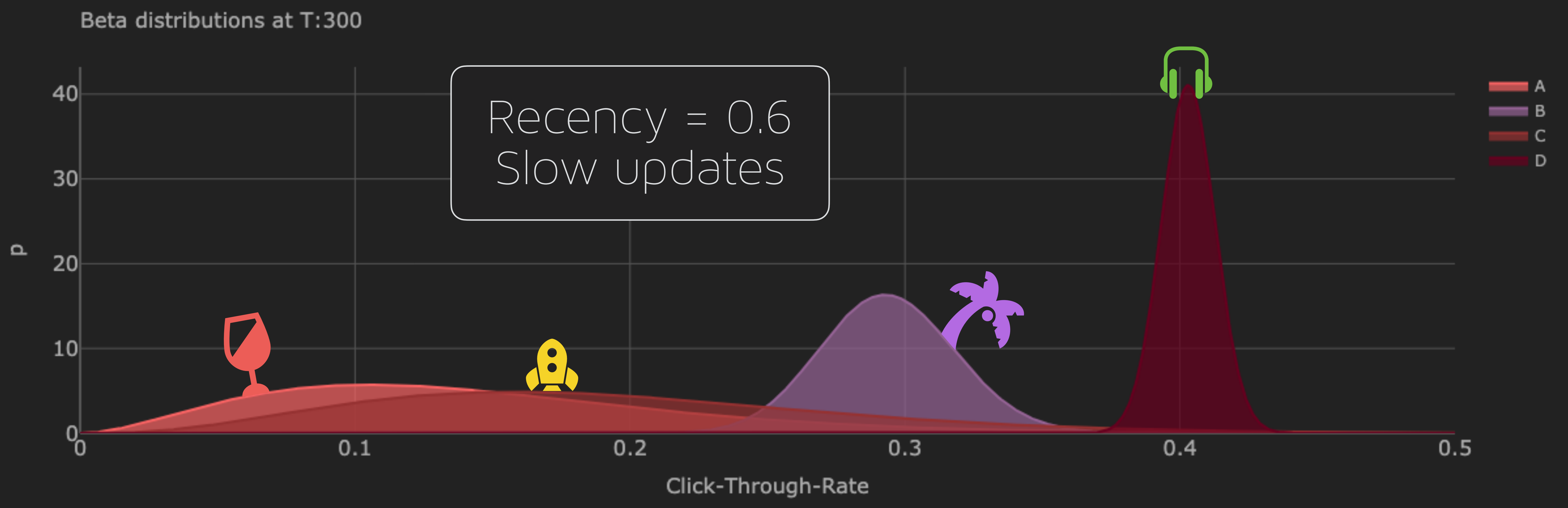
Bandit A/B/C/D-test



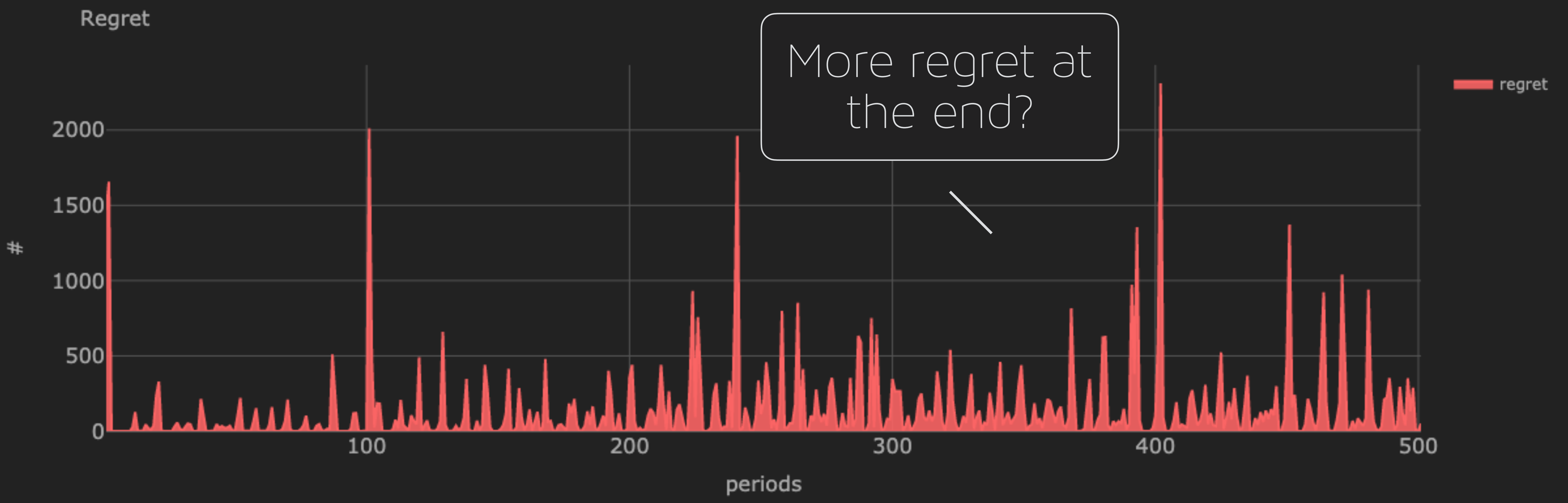
Bandit A/B/C/D-test

Beta Distributions

Example of the recency making A wide

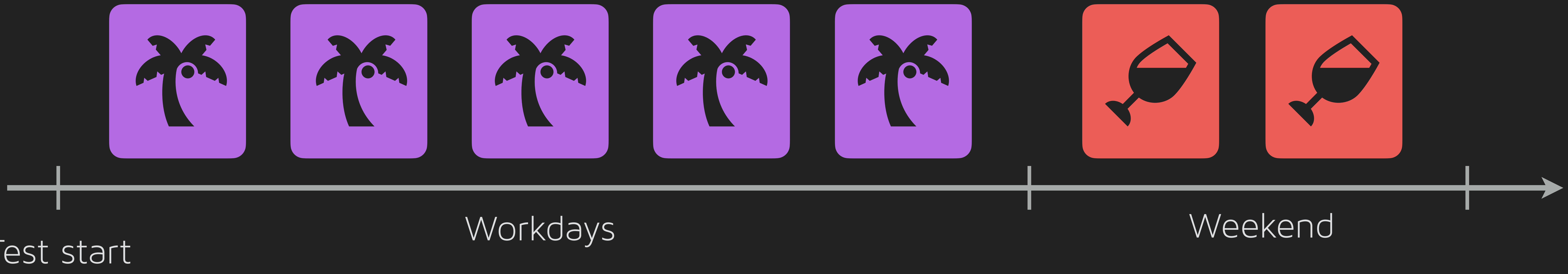


This makes A get selected more often and we're exploring...



Bandit A/B/C/D-test

Look for losers instead...



Run bandit test with all three variants

Purple Palmtree won all workdays...

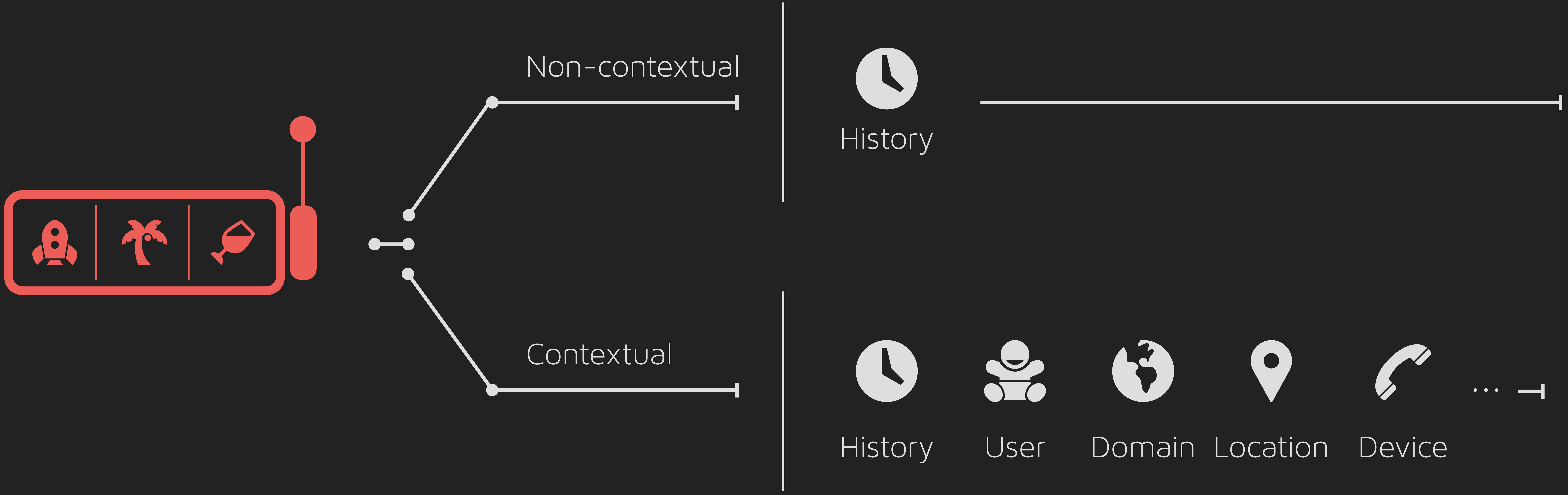
Red wineglas was best in weekends...

Let them ride!

Take out yellow rocket

In a normal A/B test - expensive to let it ride

Things I didn't cover...

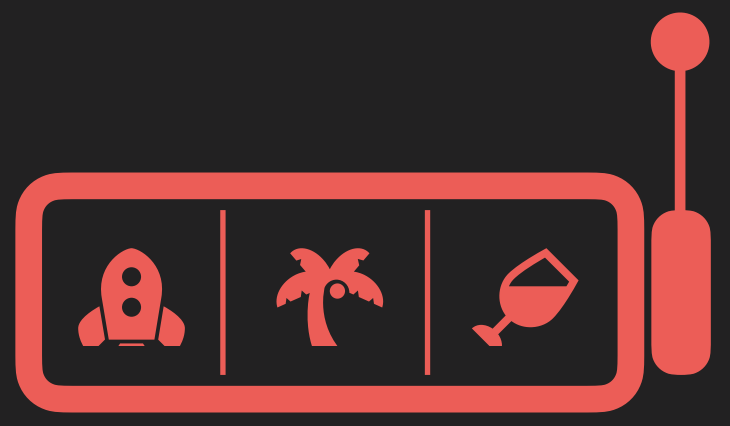


Other Metrics — Bayes Factor | False Detection Rate | ....

Stopping Criteria — ROPE metrics -

Bayesian Bandit End-of-Level summary

Uncertainty is important for practical evaluations



Bayesian Bandit A/B-testing - white box
More dynamic - harder to interpret and yet not...

Let it ride - not necessarily one winner.
Don’t have to use it as an A/B-test....

Always be critical

References

References

Calculating Sample Size for A/B Testing: Formulas, Examples & Errors
https://splitmetrics.com/blog/mobile-a-b-testing-sample-size/ (paywall)

Kees Schippers
How to Calculate A/B Testing Sample Sizes?
https://vwo.com/blog/how-to-calculate-ab-test-sample-size/

Cameron Davidson-Pilon
Probabilistic Programming and Bayesian Methods for Hackers
https://github.com/CamDavidsonPilon/Probabilistic-Programming-and-Bayesian-Methods-for-Hackers

David J. C. MacKay
Information Theory, Inference, and Learning Algorithms
https://www.inference.org.uk/itprnn/book.pdf

Alicia A. Johnson, Miles Q. Ott, Mine Dogucu
Bayes Rules! An Introduction to Applied Bayesian Modeling
https://www.bayesrulesbook.com

Evan Miller
Bayesian A/B-Testing
https://www.evanmiller.org/bayesian-ab-testing.html

John Kruschke
Rejecting or Accepting Parameter Values in Bayesian Estimation
https://jkkweb.sitehost.iu.edu/articles/Kruschke2018RejectingOrAcceptingParameterValuesWithSupplement.pdf

Michael Frasco
The Power of Bayesian A/B Testing
https://medium.com/convoy-tech/the-power-of-bayesian-a-b-testing-f859d2219d5

Urteaga et al.
Bayesian bandits: balancing the exploration-exploitation tradeoff via double sampling
https://arxiv.org/pdf/1709.03162.pdf

John Cook
Exact Calculation of Beta Inequalities
https://www.johndcook.com/exact_beta_inequalities.pdf

Chris Stucchio
Bayesian A/B Testing at VWO
https://vwo.com/downloads/VWO_SmartStats_technical_whitepaper.pdf

kruschke
Bayesian Estimation Supersedes the t Test
https://jkkweb.sitehost.iu.edu/articles/Kruschke2013JEPG.pdf

Hass et al.
Exact Bayesian Inference for A/B testing
https://gist.github.com/lucidyan/89eee0db8ce353d91e6bfbc51b5dcf19

Sureshkumar
Bayesian experimentation methods for products
https://towardsdatascience.com/bayesian-experimentation-methods-for-products-636514951e43


The end...