Homework 12

This homework is due at 11 PM on November 29, 2023.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Robust Linear Programming**

In this problem we will consider a version of linear programming under uncertainty.

Consider vector $\vec{x} \in \mathbb{R}^n$. Note that $\vec{x}^\top \vec{y} \leq \|\vec{x}\|_1$ for all $\vec{y}$ such that $\|\vec{y}\|_\infty \leq 1$. Further this inequality is tight, since it holds with equality for $\vec{y} = \operatorname{sgn}(\vec{x})$. This observation will be useful in solving the problem.

Let us focus now on a LP in inequality form:

$$\min_{\vec{x}} \quad \vec{c}^\top \vec{x} \tag{1}$$

$$\text{s.t.} \quad \vec{a}_i^\top \vec{x} \leq b_i, \ \text{for each } i = 1, ..., m. \tag{2}$$

Consider the set of linear inequalities in (2). Suppose you don't know the vectors $\vec{a}_i$ exactly. Instead you are given nominal values $\vec{\hat{a}}_i$, and you know that the actual vectors satisfy $\left\|\vec{a}_i - \vec{\hat{a}}_i\right\|_\infty \leq \rho$ for a given $\rho > 0$. In other words, the actual components a_{ij} can be anywhere in the intervals $[\hat{a}_{ij} - \rho, \hat{a}_{ij} + \rho]$. Or equivalently, each vector $\vec{a}_i$ can lie anywhere in a hypercube with corners $\vec{\hat{a}}_i + \vec{v}$ where $\vec{v} \in \{-\rho, \rho\}^n$. We desire that the set of inequalities that constrain problem (2) be satisfied for all possible values of $\vec{a}_i$; i.e., we replace these with the constraints

$$\vec{a}_i^\top \vec{x} \leq b_i \text{ for each } \vec{a}_i \in \{\vec{\hat{a}}_i + \vec{v} \ \mid \|\vec{v}\|_\infty \leq \rho\}, i = 1, ..., m. \tag{3}$$

(a) Argue why for our LP we can replace the infinite set of constraints in (3) by a finite set of $2^n m$ constraints of the form

$$\vec{\hat{a}}_i^\top \vec{x} + \vec{v}^\top \vec{x} \leq b_i \text{ for each } \vec{v} \in \{-\rho, \rho\}^n, i = 1, ..., m. \tag{4}$$

(b) Show that the constraint set in Equation (3) is in fact equivalent to the much more compact set of m nonlinear inequalities

$$\vec{\hat{a}}_i^\top \vec{x} + \rho \|\vec{x}\|_1 \leq b_i, \quad i = 1, ..., m. \tag{5}$$

Hint: The observation made at the beginning of the problem statement may be useful here.

We are interested in situations where the vectors $\vec{a}_i$ are uncertain, but satisfy bounds $\left\|\vec{a}_i - \vec{\hat{a}}_i\right\|_\infty \leq \rho$ for given $\vec{\hat{a}}_i$ and ρ. We want to minimize $\vec{c}^\top \vec{x}$ subject to the constraint that the inequalities $\vec{a}_i^\top \vec{x} \leq b_i$ are satisfied for *all* possible values of $\vec{a}_i$.
We call this a *robust LP* :

$$\min_{\vec{x}} \quad \vec{c}^\top \vec{x} \tag{6}$$

$$\text{s.t.} \quad \vec{a}_i^\top \vec{x} \leq b_i, \ \text{for each } \vec{a}_i \in \{\vec{\hat{a}}_i + \vec{v} \mid \|\vec{v}\|_\infty \leq \rho\}, \ \text{for each } i = 1, ..., m. \tag{7}$$

(c) Using the result from part (b), express the above optimization problem as an LP.

2. Formulating Problems as LPs or QPs

This problem explores what kinds of problems can be formulated as LPs or QPs. For each $j \in \{1, 2, 3, 4\}$, either formulate the problem

$$p_j^* := \min_{\vec{x} \in \mathbb{R}^n} f_j(\vec{x}),$$

for the function $f_j : \mathbb{R}^n \to \mathbb{R}$ below as a QP or LP, or explain why it cannot be done. In our formulations, we always use $\vec{x} \in \mathbb{R}^n$ as the variable, and assume that $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$.

(a) $f_1(\vec{x}) = \|A\vec{x} - \vec{y}\|_\infty + \|\vec{x}\|_1$.

(b) $f_2(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 + \|\vec{x}\|_1$.

(c) $f_3(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 - \|\vec{x}\|_1$.

(d) $f_4(\vec{x}) = \|A\vec{x} - \vec{y}\|_2^2 + \|\vec{x}\|_1^2$.

3. A Minimum Time Path Problem

This question illustrates how to formulate an optimization problem as an SOCP. The problem studied in this question arises in optics. Consider Figure 1, in which a point in 0 must move to reach point $p = [4 \;\; 2.5]^\top$, crossing three layers of fluids having different densities. In the first layer, the point must travel at speed v_1, while in the second layer and third layers it must travel at lower maximum speeds, respectively $v_2 = v_1/\eta_2$, and $v_3 = v_1/\eta_3$, with $\eta_2, \eta_3 > 1$. Assume $v_1 = 1$, $\eta_2 = 1.5$, $\eta_3 = 1.2$. Formulate a SOCP whose objective is to find the fastest (i.e., minimum time) path from 0 to p. (It is not necessary to solve this SOCP). Denote with x_1, x_2, x_3 the horizontal coordinates of points p_1, p_2 and p, respectively, and with y_1, y_2, y_3 the corresponding vertical coordinates (which are given by $y_1 = 1$, $y_2 = 2$, $y_3 = 2.5$). Define as h_1, h_2, h_3 the lengths of the horizontal projections of the three legs, that is:

$$h_1 = x_1, \quad h_2 = x_2 - x_1, \quad h_3 = x_3 - x_2.$$

Hint: You may use path leg lengths ℓ_1, ℓ_2, ℓ_3 as variables, and observe that, in this problem, equality constraints of the type "something" $= \ell_i$ can be equivalently substituted for by inequality constraints "something" $\leq \ell_i$ (explain why).

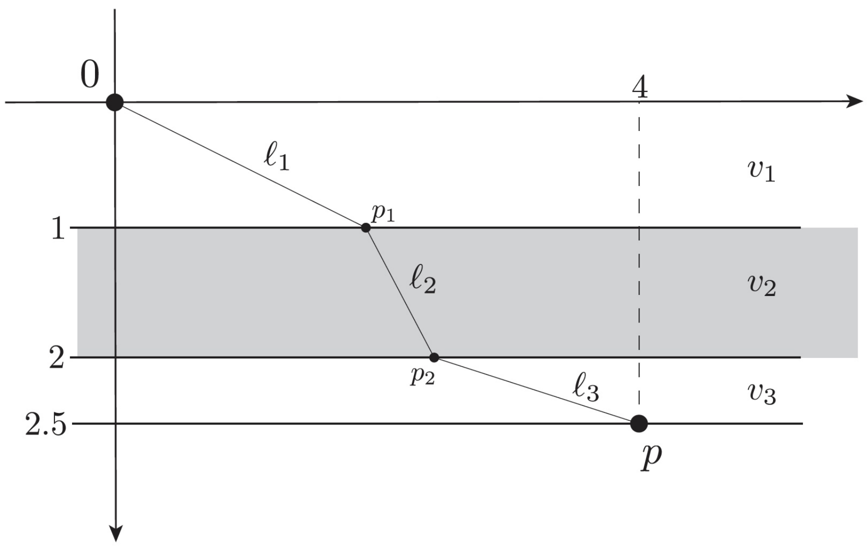

Figure 1: A minimum-time path problem.

4. A Slalom Problem

A skier must slide from left to right by going through n parallel gates of known positions (x_i, y_i) and widths c_i, $i = 1, \dots, n$. The initial position (x_0, y_0) is given, as well as the final one, (x_{n+1}, y_{n+1}). Before reaching the final position, the skier must go through gate i by passing between the points $(x_i, y_i - c_i/2)$ and $(x_i, y_i + c_i/2)$ for each $i \in \{1, \dots, n\}$. Figure 2 is a representation. Use values for (x_i, y_i, c_i) from Table 1.

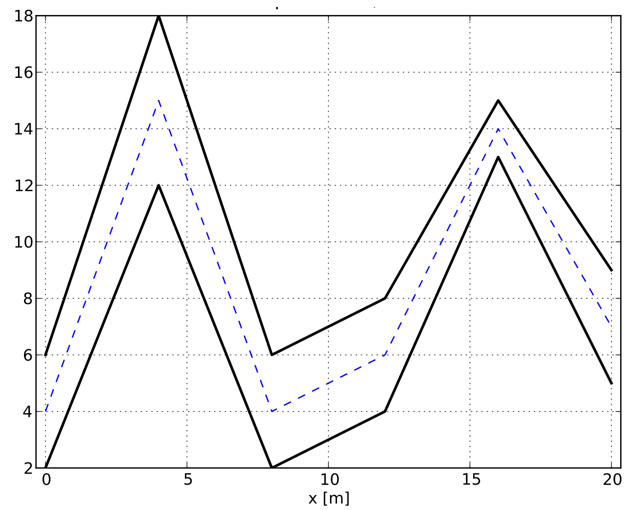

Figure 2: Slalom problem with $n = 5$ gates. The initial and final positions are fixed. The final position is not included in the figure. The skier slides from left to right. The middle path is dashed and connects the center points of gates.

Table 1: Problem data for Problem 2.

i	x_i	y_i	c_i
0	0	4	N/A
1	4	15	6
2	8	4	4
3	12	6	4
4	16	14	2
5	20	7	4
6	24	4	N/A

(a) Given the data $\{(x_i, y_i, c_i)\}_{i=0}^{n+1}$, write an optimization problem that minimizes the total length of the path. Your answer should come in the form of an SOCP.

(b) Solve the problem numerically with the data given in Table 1. You may use the starter code provided in the Jupyter notebook accompanying this HW. *HINT: You should be able to use packages such as* `cvxpy` *and* `numpy`.

5. Least Squares with Equality Constraints

Consider the least squares problem with equality constraints

$$\min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{b}\|_2^2 \ : \ G\vec{x} = \vec{h}, \tag{8}$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$, $G \in \mathbb{R}^{p \times n}$ and $\vec{h} \in \mathbb{R}^p$. For simplicity, we will assume that $\text{rank}(A) = n$ and $\text{rank}(G) = p$.

Using the KKT conditions, determine the optimal solution of this optimization problem.