Discussion 10

1. Simple Constrained Optimization Problem with Duality

Consider the optimization problem

$$\min_{x_1, x_2 \in \mathbb{R}} \quad f(x_1, x_2) \tag{1}$$

$$\text{s.t.} \quad 2x_1 + x_2 \geq 1 \tag{2}$$

$$x_1 + 3x_2 \geq 1 \tag{3}$$

$$x_1 \geq 0, \tag{4}$$

$$x_2 \geq 0 \tag{5}$$

(a) Express the Lagrangian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for each fixed (x_1, x_2).

(c) Express the dual function of the problem, and show that it is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2) for each fixed $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4_+$.

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

(f) Conclude that $\min_{(x_1,x_2)\in\mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2)\in\mathcal{X}} f(x_1, x_2)$.

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ as a function of ∇f and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4_+$ to solve:

$$\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{7}$$

2. Lagrangian Dual of a QP

Consider the general form of a convex quadratic program, where $Q \succ 0$ is a positive definite $n \times n$ matrix, $A \in \mathbb{R}^{m \times n}$, and $\vec{b} \in \mathbb{R}^m$:

$$\min_{\vec{x}} \quad \frac{1}{2}\vec{x}^\top Q \vec{x} \tag{8}$$

$$\text{s.t.} \quad A\vec{x} \leq \vec{b} \tag{9}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?