Homework 11

This homework is due at 11 PM on November 15, 2023.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Optimization over a Polytope**

   Consider the optimization problem:

$$\begin{aligned} \min_{x,y\in\mathbb{R}}. \quad & ax + by + c \\ \text{s.t.} \quad & x \geq 0, \\ & y \geq 0, \\ & x + 2y \leq 6, \\ & 5x + 2y \leq 10. \end{aligned}$$

   The feasible set is the closed bounded set enclosed by the red lines as given in Fig. 1. Find the optimal solutions for objectives (i) $-2x + 3y + 5$, (ii) $-x - 2y + 5$ and (iii) 5 by finding the values at the vertices of the feasible set and then using the `cvxpy` package in the optimization_over_polytope.ipynb file. Please refer to `https://www.cvxpy.org/tutorial/index.html` to gain some familiarity with `cvxpy`, a widely used package for convex optimization.

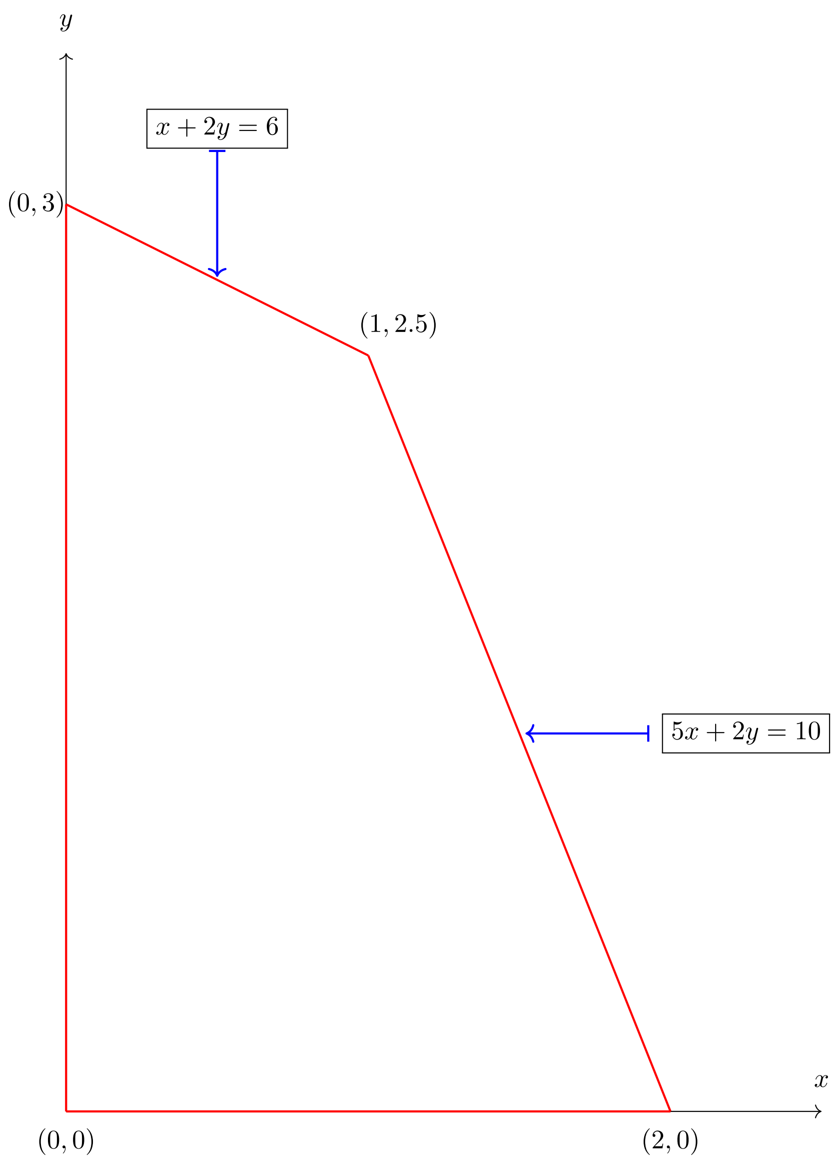

Figure 1: The feasible set for Problem 2 is the polytope bounded by the red lines.

2. LP Duality, Part 1

This problem explores basic features of linear programming duality. Consider the following linear programming problem

$$
\begin{aligned}
p^* := \min_{x_1, x_2, x_3 \in \mathbb{R}} \quad & x_1 + x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 5, \\
& x_1 + 2x_3 = 6, \\
& x_1 \geq 0, \\
& x_2 \geq 0, \\
& x_3 \geq 0.
\end{aligned}
$$

(a) Show that the problem is feasible and that the feasible set is a polytope. (Recall that a polytope is a bounded polyhedron, so what this problem is asking you to show is that the feasible set is nonempty, is a polyhedron, and is bounded.)

(b) Determine the vertices of the feasible polytope, show that the optimal primal value is $p^* = 3$, and find all the optimal feasible points.

(c) Write the Lagrangian for traditional Lagrange duality for this problem. There will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

(d) Find the dual objective function and show that the dual problem can be simplified to read

$$
\begin{aligned}
\max_{u,v} \quad & -5u - 6v \\
\text{s.t.} \quad & u \geq 0, \\
& v \geq -\frac{1}{2}.
\end{aligned}
$$

Note that this is also a linear program.

(e) Verify that the optimal dual value is $d^* = 3$ and so strong duality holds.

(f) Is Slater's condition satisfied in this problem?

3. LP Duality, Part 2

This problem continues to explore basic features of linear programming duality with a variation on the preceding problem. Consider the following linear programming problem

$$
\begin{aligned}
p^* := \min_{\vec{x} \in \mathbb{R}^3} \quad & x_1 + x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq -5, \\
& x_1 + 2x_3 = 6, \\
& x_1 \geq 0, \\
& x_2 \geq 0, \\
& x_3 \geq 0.
\end{aligned}
$$

(a) Show that the problem is infeasible and therefore the optimal primal value is $p^* = \infty$.

(b) Write the Lagrangian for traditional Lagrange duality for this problem. There will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

(c) Find the dual objective function and show that the dual problem can be simplified to a linear program involving two variables.

(d) Show that the optimal dual value is $d^* = \infty$ and so strong duality once again holds.

(e) Is Slater's condition satisfied in this problem?

4. Dual Norms and SOCP

Consider the problem

$$p^{\star} = \min_{\vec{x} \in \mathbb{R}^n} \left\| A\vec{x} - \vec{y} \right\|_1 + \mu \left\| \vec{x} \right\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$, and $\mu > 0$.

(a) Express this (primal) problem in standard SOCP form.

(b) Find a dual to the problem and express it in standard SOCP form.

HINT: Recall that for every vector $\vec{z}$, the following dual norm equalities hold:

$$\left\| \vec{z} \right\|_2 = \max_{\vec{u} \,:\, \left\| \vec{u} \right\|_2 \le 1} \vec{u}^\top \vec{z}, \qquad \left\| \vec{z} \right\|_1 = \max_{\vec{u} \,:\, \left\| \vec{u} \right\|_\infty \le 1} \vec{u}^\top \vec{z}. \tag{2}$$

Thus, we can rewrite the objective function of the original problem as

$$\left\| A\vec{x} - \vec{y} \right\|_1 + \mu \left\| \vec{x} \right\|_2 = \max_{\vec{u} : \left\| \vec{u} \right\|_\infty \le 1} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \max_{\vec{v} : \left\| \vec{v} \right\|_2 \le 1} \vec{v}^\top \vec{x}. \tag{3}$$

We can then express the original (primal) problem as

$$p^{\star} = \min_{\vec{x}} \max_{\vec{u}, \vec{v} : \left\| \vec{u} \right\|_\infty \le 1, \left\| \vec{v} \right\|_2 \le 1} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \vec{v}^\top \vec{x}. \tag{4}$$

To form the dual, we reverse the order of min and max.

(c) Assume strong duality holds[1] and that $m = 100$ and $n = 10^6$, i.e., A is 100×10^6. Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

[1] In fact, you can show that strong duality holds using Sion's theorem, a generalization of the minimax theorem that is beyond the scope of this class.

5. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}.$$

1) Consider the minimization problem

$$\begin{aligned} p^* = \min_{x \in \mathbb{R}}. \quad & f_0(x) \\ \text{s.t.} \quad & -1 \leq x, \\ & x \leq 1. \end{aligned} \tag{5}$$

(a) Show that $f_0(x)$ is differentiable everywhere and compute its derivative.

(b) Show that $p^* = 2$ and that the set of optimizers is $\mathcal{X}^* = \{-1, 1\}$.

(c) Show that the dual problem can be represented as

$$d^* = \max_{\lambda_1, \lambda_2 \geq 0} \; g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\},$$

with

$$\begin{aligned} g_1(\vec{\lambda}) &= \min_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1), \\ g_2(\vec{\lambda}) &= \min_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1). \end{aligned}$$

(d) Next, show that

$$\begin{aligned} g_1(\vec{\lambda}) &\leq -3\lambda_1 + \lambda_2, \\ g_2(\vec{\lambda}) &\leq \lambda_1 - 3\lambda_2. \end{aligned}$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

(e) Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

(f) Does strong duality hold?

2) Now, consider a problem equivalent to the minimization in (5):

$$p^* = \min_{x \in \mathbb{R}} \; f_0(x) \tag{6}$$

$$\text{s.t. } x^2 \leq 1. \tag{7}$$

Observe that $p^* = 2$ and the set of optimizers is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part 1).

(a) Show that the dual problem can be represented as

$$d^* = \max_{\lambda \geq 0} \; g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$g_1(\lambda) = \min_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1),$$
$$g_2(\lambda) = \min_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).$$

(b) Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3, \\ -\frac{4}{27}(3-\lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

(c) Conclude that $d^* = 2$ and $\lambda^\star = \frac{3}{2}$.

(d) Does strong duality hold?