Discussion 12

1. Median versus average

This question illustrates the connection between the median and the mean of a finite set of real numbers.

For a given vector $\vec{v} \in \mathbb{R}^n$, the average can be found as the solution to the optimization problem

$$\min_{x \in \mathbb{R}} \|\vec{v} - x\vec{1}\|_2^2, \tag{1}$$

where $\vec{1}$ denotes the vector of ones in $\mathbb{R}^n$. Similarly, it turns out that the median (a median is any value x such that it is possible to partition the values at x (if any) as being either just below x, at x, or just above x, in such a way that there is an equal number of values in $\vec{v}$ above and below x) can be found via the optimization problem

$$\min_{x \in \mathbb{R}} \|\vec{v} - x\vec{1}\|_1. \tag{2}$$

We consider a robust version of problem (1) of finding the average, i.e.

$$\min_{x} \max_{\vec{u} : \|\vec{u}\|_\infty \leq \lambda} \|\vec{v} + \vec{u} - x\vec{1}\|_2^2, \tag{3}$$

in which we assume that the components of $\vec{v}$ can be independently perturbed by a vector $\vec{u}$ each of whose components has magnitude bounded by a given number $\lambda \geq 0$.

(a) Is the robust problem (3) convex? You should be able to justify your answer based on the expression (3), without having to do any manipulations,

(b) Show that problem (3) can be expressed as

$$\min_{x \in \mathbb{R}} \sum_{i=1}^{n} \left(|v_i - x| + \lambda\right)^2.$$

(c) Express problem (2) as an LP. State precisely the variables and the constraints if any.

(d) Express problem (3) as a QP. State precisely the variables and the constraints, if any.

(e) Show that when λ is large the solution set of the problem in (3) approaches that of the median problem (2).

(f) It is often said that the median is a more robust notion of "middle" value of a finite set of real numbers than the average, when noise is present in the observations. Based on the previous part of this question, justify this statement.

2. A matrix problem with strong duality

This problem discusses a convex optimization problem arising from the perturbation analysis of dynamical systems.

Consider the problem

$$p^* \doteq \min_{\Delta} \vec{c}^\top (A+\Delta)^{-1}\vec{b} \ : \ \|\Delta\| \leq 1,$$

where $A \in \mathbb{R}^{n\times n}$, with smallest singular value $\sigma_{\min}(A)$ strictly greater than one, and $\vec{b}, \vec{c} \in \mathbb{R}^n$ with $\vec{b}, \vec{c} \neq 0$. Here, $\|\cdot\|$ stands for the largest singular value norm, i.e. the spectral norm. This problem arises in the study of equilibrium states of a dynamical system subject to perturbations.

(a) Show that the objective function is well-defined everywhere on the feasible set.

Hint: You can show that a square matrix is invertible if it has no singular values equal to 0.

(b) Is the problem, as stated, convex? Give a proof or a counter-example.

(c) Show that the problem can be expressed as

$$\min_{\vec{x}} \vec{c}^T \vec{x} \ : \ \|A\vec{x} - \vec{b}\|_2^2 \leq \|\vec{x}\|_2^2.$$

(d) Let $K := A^T A - I$. Since $\sigma_{\min}(A) > 1$, we know that K is invertible. Prove that

$$AK^{-1}A^T - I = (AA^T - I)^{-1}.$$

(e) Show that the feasible set of the formulation in (c) is an ellipsoid, expressing it in terms of the matrix $K := A^T A - I$, the vector $\vec{x}_0 := K^{-1} A^T \vec{b}$, and the scalar $\gamma := \vec{x}_0^\top K \vec{x}_0 - \vec{b}^T \vec{b}$. Explain why the above problem (which we called the new problem) is convex.

(f) Form a Lagrange dual to the problem. Does strong duality hold?

(g) Show that the optimal value can be written

$$p^* = \vec{c}^\top (A^\top A - I)^{-1} A^\top \vec{b} - \|(AA^\top - I)^{-1/2} \vec{b}\|_2 \cdot \|(A^\top A - I)^{-1/2} \vec{c}\|_2.$$