Discussion 6

1. Convexity of Sets

Definition. A set C is convex if and only if the line segment between any two points in C lies in C:

$$C \text{ is convex} \iff \forall \vec{x}_1, \vec{x}_2 \in C,\ \forall \theta \in [0,1],\ \theta\vec{x}_1 + (1-\theta)\vec{x}_2 \in C \tag{1}$$

(a) Show that the following sets are convex:

i. **(OPTIONAL)** A vector subspace of $\mathbb{R}^n$.

ii. **(OPTIONAL)** A hyperplane, $\mathcal{L} = \{\vec{x} \mid \vec{a}^\top \vec{x} = b\}$.

iii. A halfspace, $\mathcal{H} = \{\vec{x} \mid \vec{a}^\top \vec{x} \le b\}$.

(b) Show that the **intersection of convex sets is convex**:

$$C_1, C_2 \text{ are convex} \implies C = C_1 \cap C_2 \text{ is convex} \tag{2}$$

Definition. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is the sum of a linear function and a constant,

$$f(\vec{x}) = A\vec{x} + \vec{b}, \tag{3}$$

for $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$.

(c) **(OPTIONAL)** Prove that if $S \subseteq \mathbb{R}^n$ is convex, then the image of S under an affine function f,

$$f(S) = \{f(\vec{x}) \mid \vec{x} \in S\}, \tag{4}$$

is convex.

2. Convexity of Functions

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is a nonempty convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and $\theta \in [0, 1]$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \leq \theta f(\vec{x}) + (1-\theta) f(\vec{y}). \tag{5}$$

The function f is strictly convex if the inequality is strict whenever $\vec{x} \neq \vec{y}$ and $\theta \notin \{0, 1\}$.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if $\mathrm{dom}(f)$ is a nonempty convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and θ with $0 \leq \theta \leq 1$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \geq \theta f(\vec{x}) + (1-\theta) f(\vec{y}). \tag{6}$$

The function f is strictly concave if the inequality is strict whenever $\vec{x} \neq \vec{y}$ and $\theta \notin \{0, 1\}$.

Property. A function f is concave if and only if $-f$ is convex. An affine function is both convex and concave.

Property: Jensen's inequality. The inequality in Equation (5) is known as **Jensen's Inequality**. This can be extended to convex combinations of more than one point. If f is convex, and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathrm{dom}(f)$, and $\theta_1, \theta_2, \ldots, \theta_k \geq 0$ with $\sum_{i=1}^k \theta_i = 1$ then,

$$f(\theta_1\vec{x}_1 + \theta_2\vec{x}_2 + \cdots + \theta_k\vec{x}_k) \leq \theta_1 f(\vec{x}_1) + \theta_2 f(\vec{x}_2) + \cdots + \theta_k f(\vec{x}_k). \tag{7}$$

Property: first order condition. Suppose dom(f) is a nonempty open set and f is differentiable. Then f is convex if and only if

$$f(\vec{y}) \geq f(\vec{x}) + \nabla f(\vec{x})^\top (\vec{y} - \vec{x}), \tag{8}$$

for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$.

Property: Second order condition. Suppose dom(f) is a nonempty open set and f is twice differentiable. Then f is convex if and only if the Hessian of f, $\nabla^2 f(\vec{x})$, is positive semi-definite for all $\vec{x} \in \mathrm{dom}(f)$.

(a) **Point-wise maximum**.

Show that if f_1 and f_2 are convex functions then their pointwise maximum f, defined by

$$f(\vec{x}) = \max(f_1(\vec{x}), f_2(\vec{x})), \tag{9}$$

with $\mathrm{dom}(f) = \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, is also convex, when $\mathrm{dom}(f) \neq \emptyset$.

(b) **Restriction to a line**.

Show that a function f is convex if and only if for all $\vec{x} \in \operatorname{dom}(f)$ and all $\vec{v}$, the function $g : \operatorname{dom}(g) \to \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ is convex for $\operatorname{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \operatorname{dom}(f)\}$.

(c) **Non-negative weighted sum**.

Show that the non-negative weighted sum of convex functions is convex: i.e. if $f_1, \ldots, f_n$ are n convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $w_1, \ldots, w_n \in \mathbb{R}_+$ are n positive scalars, then the function:

$$f = \sum_{i=1}^{n} w_i f_i \tag{10}$$

is convex. To make the question easier, you can assume that the functions $f_1, \ldots, f_n$ are twice-differentiable.

3. Convexity of Constraint Sets

Let $f_1, \ldots, f_m, h_1, \ldots, h_p \colon \mathbb{R}^n \to \mathbb{R}$ be functions. Let $S \subseteq \mathbb{R}^n$ be defined as

$$S \doteq \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} f_i(\vec{x}) \leq 0 & \forall i = 1, \ldots, m \\ h_j(\vec{x}) = 0 & \forall j = 1, \ldots, p \end{array} \right\}. \tag{11}$$

Show that if $f_1, \ldots, f_m$ are convex functions, and $h_1, \ldots, h_p$ are affine functions, then S is a convex set.

4. Properties of Convex Functions

In this exercise, we examine convexity and what it represents graphically.

(a) In what region between $[0, 2\pi]$ is $\sin(x)$ a convex function? In what region between $[0, 2\pi]$ is $\sin(x)$ a concave function? Give a region between $[0, 2\pi]$ where $\sin(x)$ is neither convex nor concave.

(b) Plot $\sin(x)$ between $[0, 2\pi]$. For each of the 3 intervals defined above in part (a), draw a chord to illustrate graphically on what regions the function is convex, concave, and neither convex nor concave.

(c) Show that for all $x \in [0, \frac{\pi}{2}]$,

$$\frac{2}{\pi}x \leq \sin x \leq x. \tag{12}$$