Note: Matrix Manipulation Tricks

This note contains many ways to manipulate block matrices. Since each fact in here is something you can derive yourself using definitions (e.g. of matrix multiplication), you may use any of them without proof.

1 Transposes of Block Matrices

$$\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}^\top = \begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix}^\top \tag{2}$$

$$\begin{bmatrix} A & B \end{bmatrix}^\top = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}^\top = \begin{bmatrix} A^\top & B^\top \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\top = \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}. \tag{5}$$

2 Block Matrix Products

In the following, $\vec{e}_i$ is the i^{th} standard basis vector – it has a 1 in the i^{th} coordinate and 0 in all other coordinates.

$$\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix} \begin{bmatrix} \vec{y}_1^\top \\ \vdots \\ \vec{y}_n^\top \end{bmatrix} = \sum_{i=1}^{n} \vec{x}_i \vec{y}_i^\top \tag{6}$$

$$\begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} \begin{bmatrix} \vec{y}_1 & \cdots & \vec{y}_n \end{bmatrix} = \begin{bmatrix} \vec{x}_1^\top \vec{y}_1 & \cdots & \vec{x}_1^\top \vec{y}_n \\ \vdots & \ddots & \vdots \\ \vec{x}_n^\top \vec{y}_1 & \cdots & \vec{x}_n^\top \vec{y}_n \end{bmatrix} \tag{7}$$

$$\vec{e}_i^\top \begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} = \vec{x}_i^\top \tag{8}$$

$$\begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix} \vec{e}_i = \vec{x}_i \tag{9}$$

$$A \begin{bmatrix} \vec{x}_1 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & \cdots & A\vec{x}_n \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} \vec{x}_1^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} A = \begin{bmatrix} \vec{x}_1^\top A \\ \vdots \\ \vec{x}_n^\top A \end{bmatrix} \tag{11}$$

$$A \begin{bmatrix} B & C \end{bmatrix} = \begin{bmatrix} AB & AC \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} C = \begin{bmatrix} AC \\ BC \end{bmatrix}. \tag{13}$$

3 Block Diagonal Matrices

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} \vec{x}_1^\top \\ \vec{x}_2^\top \\ \vdots \\ \vec{x}_n^\top \end{bmatrix} = \begin{bmatrix} d_1 \vec{x}_1^\top \\ \vdots \\ d_n \vec{x}_n^\top \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} A_1 B_1 \\ A_2 B_2 \\ \vdots \\ A_n B_n \end{bmatrix} \tag{15}$$

4 Quadratic Forms

$$\vec{x}^\top A \vec{y} = \sum_i \sum_j A_{ij} x_i y_j \tag{16}$$

$$\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}^\top \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix} = \vec{x}^\top A \vec{x} + \vec{x}^\top B \vec{y} + \vec{y}^\top C \vec{x} + \vec{y}^\top D \vec{y}. \tag{17}$$

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