Discussion 8

1. Sandwich Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Suppose $g(\vec{x}) \leq f(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. Prove that there exists an affine function $h : \mathbb{R}^n \to \mathbb{R}$ such that $g(\vec{x}) \leq h(\vec{x}) \leq f(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.

Hint: Consider the interior of the epigraph of f and the hypograph of g and use the separating hyperplane theorem.

Solution: Note that int(epi(f)) and hypo(g) are two disjoint, nonempty, convex sets in $\mathbb{R}^{n+1}$, where int(epi(f)) denotes the interior of the epigraph of f, namely

$$\text{int}(\text{epi}(f)) = \{(\vec{x}, t) \in \mathbb{R}^n \times \mathbb{R} : t > f(\vec{x})\}.$$

and hypo(g) denotes the hypograph of g, namely

$$\text{hypo}(g) = \{(\vec{x}, t) \in \mathbb{R}^n \times \mathbb{R} : g(\vec{x}) \geq t\}.$$

Therefore, by the separating hyperplane theorem in $\mathbb{R}^{n+1}$, there is a hyperplane separating the two sets, i.e. there exists a nonzero vector $\vec{a} \in \mathbb{R}^n$ and scalars $b, c \in \mathbb{R}$ such that

$$\vec{a}^\top \vec{x} + bt \geq c \geq \vec{a}^\top \vec{y} + bv, \ \forall \vec{x}, \vec{y} \in \mathbb{R}^n, t > f(\vec{x}), v \leq g(\vec{y}).$$

b should be nonzero, since otherwise $\vec{a}^\top \vec{x} \geq \vec{a}^\top \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, but this could happen only if $\vec{a} = \vec{0}$. Also b should be positive, since if we plug in $\vec{x} = \vec{y}$, then $t > v$ and $b(t - v) \geq 0$. Together with $b \neq 0$ this implies that $b > 0$.

Now, for all $\vec{x} \in \mathbb{R}^n$, setting $\vec{y} = \vec{x}$ and $v = g(\vec{x})$, we have as a special case of the preceding displayed equation that

$$\vec{a}^\top \vec{x} + bt \geq c \geq \vec{a}^\top \vec{x} + bg(\vec{x}).$$

Subtracting $\vec{a}^\top \vec{x}$ throughout and then dividing by b (which is possible because $b \neq 0$, and does not change the directions of the inequalities, because $b > 0$), we get

$$t \geq -\frac{\vec{a}^\top \vec{x}}{b} + \frac{c}{b} \geq g(\vec{x}), \ \forall t > f(\vec{x}).$$

Therefore, by letting t decrease to $f(\vec{x})$, we see that for all $\vec{x} \in \mathbb{R}^n$, we have

$$f(\vec{x}) \geq -\frac{\vec{a}^\top \vec{x}}{b} + \frac{c}{b} \geq g(\vec{x}).$$

If we define $h(\vec{x}) := -\frac{\vec{a}^\top \vec{x}}{b} + \frac{c}{b}$ then h will be an affine function of the desired type.

2. A combinatorial problem formulated as a convex optimization problem

In this question we will see how combinatorial optimization problems can sometimes be solved via related convex optimization problems. Consider the problem of deciding which subset of items to sell from among a collection of n items. Selling the item i results in revenue $s_i > 0$ and a transaction cost $c_i > 0$. There is also a fixed overall cost, which we normalize to 1, which is incurred irrespective of whether any items are sold. We are interested in maximizing the *margin*, i.e. the ratio of the total revenue to the total cost (sum of the total transaction cost and the fixed overall cost). Note that this is a combinatorial optimization problem, because the set of choices is a discrete set (here it is the set of all subsets of $\{1, \ldots, n\}$). In this question we will see that it is possible to pose this combinatorial optimization problem as a convex optimization problem.

(a) Show that the original combinatorial optimization problem can be formulated as

$$\max_{\vec{x} \in \{0,1\}^n} f(\vec{x})$$

where $f(\vec{x}) := \frac{\vec{s}^\top \vec{x}}{1+\vec{c}^\top \vec{x}}$, Here $\vec{s} \in \mathbb{R}^n$ is the column vector of revenues and $\vec{c} \in \mathbb{R}^n$ is the column vector of transaction costs associated to the individual items.

Solution: Let us parametrize our decision with a Boolean vector $\vec{x} \in \{0,1\}^n$, with $x_i = 1$ if item i is sold, $x_i = 0$ if not. The total revenue is then $\vec{s}^\top \vec{x}$, while the total cost is $1 + \vec{c}^\top \vec{x}$. This proves the claim.

(b) Show that the combinatorial optimization problem admits a convex optimization problem formulation in the sense that the convex optimization problem

$$\begin{aligned} \min_{t} \quad & t \\ \text{subject to :} \quad & t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+, \\ & t \geq 0, \end{aligned}$$

has the same value as that of the original combinatorial optimization problem. Here $\vec{z}_+$, for a vector $\vec{z} \in \mathbb{R}^n$, denotes the vector with components $\max(0, z_i)$, $i = 1, \ldots, n$. Also $\vec{\mathbf{1}}$ denotes the all-ones column vector of length n.

Note: You should also justify why the problem formulated in this part of the question is a convex optimization problem.

Hint: For given $t \geq 0$, express the condition "$f(\vec{x}) \leq t$ for every $\vec{x} \in \{0,1\}^n$" in simple terms. Note that this is related to the idea of introducing slack variables, as in Sec. 8.3.4.4 of the textbook of Calafiore and El Ghaoui.

Solution: For given $t \geq 0$, the condition

$$f(\vec{x}) \leq t \text{ for every } \vec{x} \in \{0,1\}^n,$$

is equivalent to

$$\forall\, \vec{x} \in \{0,1\}^n \text{ we have } \vec{s}^\top \vec{x} \leq t(1 + \vec{c}^\top \vec{x}),$$

which in turn is equivalent to

$$t \geq \max_{\vec{x} \in \{0,1\}^n} (\vec{s} - t\vec{c})^\top \vec{x} = \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+.$$

It follows that the newly formulated problem results in the same value as the combinatorial optimization problem.

To check that the new problem formulation is a convex optimization problem, first note that there is only a single variable, namely t. The objective t is a convex function of t. The inequality constraint $t \geq 0$ can be written as $-t \leq 0$, and $-t$ is a convex function of t.The inequality constraint $t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ can be written as

$$-t + \sum_{i=1}^{n} \max(0, s_i - tc_i) \leq 0,$$

and $-t + \sum_{i=1}^{n} \max(0, s_i - tc_i)$, being a sum of convex functions of t, is a convex function of t. (To clarify, also note that for each i the function $\max(0, s_i - tc_i)$, being the maximum of convex functions of t, is a convex function of t.)

(c) Show that the inequality constraint $t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ is active at the optimum in the convex optimization problem, namely, it will have to be satisfied with equality.

Solution: Suppose we had $t > \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ at a feasible value of t (i.e. one that satisfies the constraints). Then we would have $t > 0$, since $\vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ is nonnegative. But this means that we cannot be at an optimum value of t, because we could decrease t slightly while still remaining in the feasible set and strictly decreasing the objective. Hence, if we are at an optimum, then we must have $t = \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$, i.e. the inequality constraint $t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ must be active.

(d) How can you recover an optimal solution $\vec{x}^*$ for the original combinatorial optimization problem from an optimal value t^* for the above convex optimization problem?

Solution: As we have seen in the previous part of the problem, at an optimum the inequality constraint $t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ is active. Hence we have

$$t^* = \max_{\vec{x} \in \{0,1\}^n} (\vec{s} - t^*\vec{c})^\top \vec{x}.$$

Therefore there exists a vector $\vec{x}^* \in \{0, 1\}^n$ such that

$$\max_{\vec{x} \in \{0,1\}^n} (\vec{s} - t^*\vec{c})^\top \vec{x} = (\vec{s} - t^*\vec{c})^\top \vec{x}^* = t^*.$$

By construction, $\vec{x}^*$ achieves the optimal value:

$$t^* = \frac{\vec{s}^\top \vec{x}^*}{1 + \vec{c}^\top \vec{x}^*}.$$

Since $\vec{x}^*$ is feasible for the original combinatorial optimization problem, and achieves the value in that problem, it is optimal for that problem.

We can identify $\vec{x}^*$ more precisely, setting

$$\forall i = 1, \ldots, m \; : \; x_i^* = \begin{cases} 1 & \text{if } s_i > t^* c_i, \\ 0 & \text{otherwise.} \end{cases}$$

There is some freedom in the choice of $\vec{x}^*$ in that nothing prevents us from setting $x_i^* = 1$ if $s_i = t^* c_i$. However, if $s_i < t^* c_i$ we have to set $x_i^* = 0$.