Homework 7

This homework is due at 11 PM on Wednesday, October 18, 2023.

1. **Symmetric Matrices**

   Recall that $\mathbb{R}^{n\times n}$ can be thought of as the vector space of all $n \times n$ matrices. As a vector space, $\mathbb{R}^{n\times n}$ has dimension n^2. Let $\mathbb{S}^n \subseteq \mathbb{R}^{n\times n}$ denote the set of symmetric matrices $n \times n$ matrices. Let $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ denote the set of positive semidefinite $n \times n$ matrices. Let $\mathbb{S}^n_{++} \subseteq \mathbb{S}^n_+$ denote the set of positive definite $n \times n$ matrices.

   (a) Show that $\mathbb{S}^n$ is a subspace of $\mathbb{R}^{n\times n}$ of dimension $\binom{n+1}{2}$.

   (b) Show that $\mathbb{S}^n_+$ is a convex subset of $\mathbb{R}^{n\times n}$.

   (c) Show that the affine hull of $\mathbb{S}^n_+$ is $\mathbb{S}^n$.

   Recall that the affine hull of a subset A of a vector space V is the smallest subspace of V that contains A. It can be characterized as the set of all linear combinations of the form $\sum_{i=1}^k \theta_i \vec{x}_i$, where $k \geq 1$ is arbitrary, $\vec{x}_1, \ldots, \vec{x}_k$ are vectors in A, and $\theta_1, \ldots, \theta_k$ are arbitrary real numbers satisfying $\sum_{i=1}^k \theta_i = 1$. Note that, in contrast to the definition of the convex hull of A, the θ_i are allowed to be negative.

   HINT: Every symmetric matrix is conjugate to a diagonal matrix by an orthogonal change of basis.

   (d) Show that $\mathbb{S}^n_{++}$ is a convex subset of $\mathbb{R}^{n\times n}$.

   (e) Show that $\mathbb{S}^n_{++}$ is the relative interior of $\mathbb{S}^n_+$. For this problem, to define distances in $\mathbb{R}^{n\times n}$, it does not matter whether you use the Frobenius norm or the induced 2-norm, but use the induced 2-norm.

   Recall that the relative interior of a subset A of a vector space V is the interior of A when A is viewed as a subset of its affine hull.

   (f) Show that if $n > 1$ then the interior of $\mathbb{S}^n_+$ is empty. Here again, to define distances in $\mathbb{R}^{n\times n}$, it does not matter whether you use the Frobenius norm or the induced 2-norm, but use the induced 2-norm.

2. Distance between polytopes as a quadratic program

Let $\vec{p}^{(1)}, \ldots, \vec{p}^{(r)}$ and $\vec{q}^{(1)}, \ldots, \vec{q}^{(s)}$ be vectors in $\mathbb{R}^d$, where $r, s \geq 1$. Let $\mathcal{P}$ denote the polytope defined as the convex hull of $\{\vec{p}^{(1)}, \ldots, \vec{p}^{(r)}\}$, and $\mathcal{Q}$ the polytope defined as the convex hull of $\{\vec{q}^{(1)}, \ldots, \vec{q}^{(s)}\}$. Thus every point in $\mathcal{P}$ can be written as $\sum_{i=1}^{r} x_i \vec{p}^{(i)}$ for some $x_i \geq 0$, $1 \leq i \leq r$ such that $\sum_{i=1}^{r} x_i = 1$, and every point in $\mathcal{Q}$ can be written as $\sum_{j=1}^{s} x_{r+j} \vec{q}^{(j)}$ for some $x_j \geq 0$, $r+1 \leq j \leq r+s$ such that $\sum_{j=r+1}^{r+s} x_j = 1$. Let us define $n = r + s$.

Define the matrix $C \in \mathbb{R}^{d \times n}$ whose i-th column is $\vec{p}^{(i)}$, $1 \leq i \leq r$ and whose $r+j$-th column is $-\vec{q}^{(j)}$, $1 \leq j \leq s$.

Pose the problem of finding the minimum squared ℓ_2 distance between points in $\mathcal{P}$ and points in $\mathcal{Q}$ as a quadratic program with objective function $\|C\vec{x}\|_2^2$, viewed as a function on $\mathbb{R}^n$.

NOTE: A quadratic program is a convex optimization problem where the objective function is a quadratic function and the constraints are linear equalities and inequalities. Recall that a quadratic convex function on $\mathbb{R}^n$ is one of the form $\vec{x}^T H \vec{x} + \vec{a}^T \vec{x} + \vec{b}$ where $b \in \mathbb{R}$, $\vec{a} \in \mathbb{R}^n$, and H is a positive semidefinite matrix in $\mathbb{R}^{n \times n}$ (i.e. $H \in \mathbb{S}_+^n$).