Homework 11

Self grades are due at 11 PM on November 30, 2023.

1. **Optimization over a Polytope**

Consider the optimization problem:

$$\begin{aligned} \min_{x,y\in\mathbb{R}}. \quad & ax + by + c \\ \text{s.t.} \quad & x \geq 0, \\ & y \geq 0, \\ & x + 2y \leq 6, \\ & 5x + 2y \leq 10. \end{aligned}$$

The feasible set is the closed bounded set enclosed by the red lines as given in Fig. 1. Find the optimal solutions for objectives (i) $-2x + 3y + 5$, (ii) $-x - 2y + 5$ and (iii) 5 by finding the values at the vertices of the feasible set and then using the `cvxpy` package in the optimization_over_polytope.ipynb file. Please refer to `https://www.cvxpy.org/tutorial/index.html` to gain some familiarity with `cvxpy`, a widely used package for convex optimization.

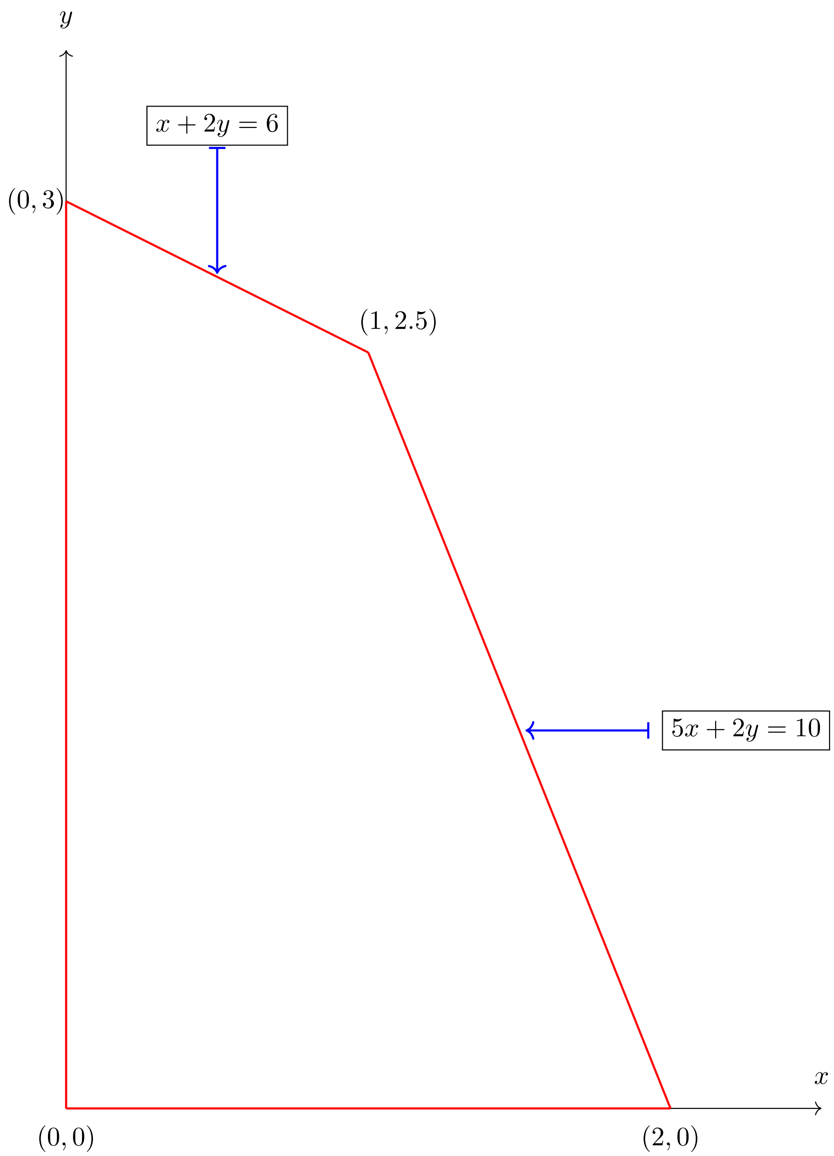

Figure 1: The feasible set for Problem 2 is the polytope bounded by the red lines.

2. LP Duality, Part 1

This problem explores basic features of linear programming duality. Consider the following linear programming problem

$$\begin{aligned} p^* := \min_{x_1,x_2,x_3\in\mathbb{R}} \quad & x_1 + x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 5, \\ & x_1 + 2x_3 = 6, \\ & x_1 \ge 0, \\ & x_2 \ge 0, \\ & x_3 \ge 0. \end{aligned}$$

(a) Show that the problem is feasible and that the feasible set is a polytope. (Recall that a polytope is a bounded polyhedron, so what this problem is asking you to show is that the feasible set is nonempty, is a polyhedron, and is bounded.)

Solution: It suffices to find at least one feasible point to demonstrate feasibility. $(x_1, x_2, x_3) = (0, 0, 3)$ is one such feasible point.

The feasible set of any LP is a polyhedron. To show that the feasible set is a polytope, we need to argue that it is bounded. We already have the constraints $x_1 \ge 0$, $x_2 \ge 0$ and $x_3 \ge 0$. The inequality constraint $x_1 + 2x_2 \le 5$ tells us that $x_1 \le 5$ and $x_2 \le 2.5$. The equality constraint $x_1 + 2x_3 = 6$, together with the constraint $x_1 \ge 0$, tells us that $x_3 \le 3$. This shows that the feasible set is bounded and hence is a polytope.

(b) Determine the vertices of the feasible polytope, show that the optimal primal value is $p^* = 3$, and find all the optimal feasible points.

Solution: Since the objective is linear, the optimal primal value can be found by minimizing the objective function, $x_1 + x_3$ over the vertices of the feasible polytope. This gives $p^* = 3$.

Since the feasible set is a polytope, the set of optimal feasible points is the convex hull of the optimal vertices, i.e. the set of optimal feasible points is

$$\{(x_1, x_2, x_3) = (0, \alpha, 3) : 0 \le \alpha \le 2.5\}.$$

(c) Write the Lagrangian for traditional Lagrange duality for this problem. There will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

Solution: The Lagrangian reads

$$\mathcal{L}(\vec{x}, \vec{\lambda}, \nu) = x_1 + x_3 - \sum_{i=1}^{3} \lambda_i x_i + \lambda_4(x_1 + 2x_2 - 5) + \nu(x_1 + 2x_3 - 6).$$

The domain of the Lagrangian is $\mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}$ because the intersection of the domains of the objective function and the constraints functions, which we denote by $\mathcal{D}$, is given as $\mathcal{D} = \mathbb{R}^3$. We are only interested in the Lagrangian for $\lambda \ge 0$, this inequality being interpreted coordinatewise in $\mathbb{R}^4$.

(d) Find the dual objective function and show that the dual problem can be simplified to read

$$\begin{aligned} \max_{u,v} \quad & -5u - 6v \\ \text{s.t.} \quad & u \ge 0, \\ & v \ge -\frac{1}{2}. \end{aligned}$$

Note that this is also a linear program.

Solution: The dual objective function is given by

$$
\begin{aligned}
g(\vec{\lambda}, \nu) &:= \min_{\vec{x} \in \mathbb{R}^3} \mathcal{L}(\vec{x}, \vec{\lambda}, \nu) \\
&= \min_{\vec{x} \in \mathbb{R}^3} x_1(1 - \lambda_1 + \lambda_4 + \nu) + x_2(-\lambda_2 + 2\lambda_4) + x_3(1 - \lambda_3 + 2\nu) - 5\lambda_4 - 6\nu \\
&= \begin{cases} -5\lambda_4 - 6\nu & \text{if } \ 1 - \lambda_1 + \lambda_4 + \nu = 0, -\lambda_2 + 2\lambda_4 = 0 \text{ and } 1 - \lambda_3 + 2\nu = 0, \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

Since the dual problem reads

$$
\begin{aligned}
\max_{\vec{\lambda} \in \mathbb{R}^4, \nu \in \mathbb{R}} . \quad & g(\vec{\lambda}, \nu) \\
\text{s.t.} \quad & \lambda_i \geq 0, \qquad \text{for each } i \in \{1, 2, 3, 4\},
\end{aligned}
$$

and since $g(\vec{\lambda}, \nu)$ depends only on λ_4 and ν, we ought to be able to simplify the dual problem so that it is expressed only in terms of these two variables. Those $(\vec{\lambda}, \nu)$ for which $g(\vec{\lambda}, \nu) = -\infty$ will not be competitive for the maximum, so we can assume that the conditions

$$
\begin{aligned}
1 - \lambda_1 + \lambda_4 + \nu &= 0, \\
-\lambda_2 + 2\lambda_4 &= 0, \\
1 - \lambda_3 + 2\nu &= 0
\end{aligned}
$$

hold. This determines λ_1, λ_2 and λ_3 in terms of (λ_4, ν). However, in the dual problem we have the constraints $\lambda_i \geq 0$ for $1 \leq i \leq 4$, and these constraints for $1 \leq i \leq 3$ reduce to the constraints

$$
\begin{aligned}
\lambda_4 + \nu &\geq -1, \\
\lambda_4 &\geq 0, \\
\nu &\geq -\frac{1}{2},
\end{aligned}
$$

for the variables λ_4 and ν that we have chosen to work with.

Renaming λ_4 as u and ν as v gives the dual problem in the simplified form

$$
\begin{aligned}
\max_{u,v} \quad & -5u - 6v \\
\text{s.t.} \quad & u \geq 0, \\
& v \geq -\frac{1}{2}.
\end{aligned}
$$

because the constraint $u + v \geq -1$ is automatically satisfied when one has the constraints $u \geq 0$ and $v \geq -\frac{1}{2}$.

(e) Verify that the optimal dual value is $d^* = 3$ and so strong duality holds.

Solution: The dual feasible set is $\{(u, v) : u \geq 0, v \geq -\frac{1}{2}\}$ and the maximum of the dual objective $-5u - 6v$ over the dual feasible set will therefore occur at $(u, v) = (0, -\frac{1}{2})$. The dual optimal value can be found by evaluating $-5u - 6v$ at this point, which gives $d^* = 3$. Since $d^* = p^*$, strong duality holds.

(f) Is Slater's condition satisfied in this problem?

Solution: As we saw earlier, the intersection of the domains of the objective function and the constraint functions in the primal problem, which we denote by $\mathcal{D}$, equals $\mathbb{R}^3$ and so its relative interior is also $\mathbb{R}^3$. Any point in the primal feasible set will therefore be in the relative interior of $\mathcal{D}$, and since the primal feasible set is nonempty, there are such points. To check if the simple form of Slater's condition is satisfied we therefore need now to ask if there are feasible points at which the inequality constraints in the primal problem are strictly satisfied. $(x_1, x_2, x_3) = (2, 1.5, 2)$ is one such point. Hence the simple form of Slater's condition is satisfied.

However we could instead have just checked whether the stronger form of Slater's condition is satisfied. Here all the inequality conditions are defined by affine functions, so the stronger form of Slater's condition will be satisfied as long as there is a point in the relative interior of $\mathcal{D}$ which is feasible. This is true in our problem, so this simpler check suffices to verify Slater's condition.

3. LP Duality, Part 2

This problem continues to explore basic features of linear programming duality with a variation on the preceding problem. Consider the following linear programming problem

$$
\begin{aligned}
p^* := \min_{\vec{x}\in\mathbb{R}^3} \quad & x_1 + x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 \le -5, \\
& x_1 + 2x_3 = 6, \\
& x_1 \ge 0, \\
& x_2 \ge 0, \\
& x_3 \ge 0.
\end{aligned}
$$

(a) Show that the problem is infeasible and therefore the optimal primal value is $p^* = \infty$.

Solution: The inequalities $x_1 \ge 0$ and $x_2 \ge 0$ are incompatible with $x_1 + 2x_2 \le -5$. Hence the feasible set is empty, i.e. the problem is infeasible. Since the infimum over an empty set is ∞, this means that $p^* = \infty$.

(b) Write the Lagrangian for traditional Lagrange duality for this problem. There will be five dual variables, one for each of the four inequality constraints, and one for the equality constraint.

Solution: The Lagrangian can now be written as

$$\mathcal{L}(\vec{x}, \vec{\lambda}, \nu) = x_1 + x_3 - \sum_{i=1}^{3} \lambda_i x_i + \lambda_4(x_1 + 2x_2 + 5) + \nu(x_1 + 2x_3 - 6).$$

As before, the domain of the Lagrangian is $\mathbb{R}^3 \times \mathbb{R}^4 \times \mathbb{R}$ because the intersection of the domains of the objective function and the constraints functions is $\mathcal{D} = \mathbb{R}^3$. We are only interested in the Lagrangian for values of $\vec{\lambda}$ satisfying $\lambda_i \ge 0$ for each $i \in \{1, 2, 3, 4\}$.

(c) Find the dual objective function and show that the dual problem can be simplified to a linear program involving two variables.

Solution: The dual objective function is given by

$$
\begin{aligned}
g(\vec{\lambda}, \nu) \quad &:= \quad \min_{\vec{x}\in\mathbb{R}^3} \mathcal{L}(\vec{x}, \vec{\lambda}, \nu) \\
&= \quad \min_{\vec{x}\in\mathbb{R}^3} x_1(1 - \lambda_1 + \lambda_4 + \nu) + x_2(-\lambda_2 + 2\lambda_4) + x_3(1 - \lambda_3 + 2\nu) + 5\lambda_4 - 6\nu \\
&= \quad \begin{cases} 5\lambda_4 - 6\nu & \text{if } 1 - \lambda_1 + \lambda_4 + \nu = 0,\ -\lambda_2 + 2\lambda_4 = 0 \text{ and } 1 - \lambda_3 + 2\nu = 0, \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

Since the dual problem reads

$$
\begin{aligned}
\max_{\vec{\lambda}\in\mathbb{R}^4, \nu\in\mathbb{R}} \quad & g(\vec{\lambda}, \nu) \\
\text{s.t.} \quad & \lambda_i \ge 0, \qquad \text{for each } i \in \{1, \cdots, 4\},
\end{aligned}
$$

and since $g(\vec{\lambda}, \nu)$ depends only on λ_4 and ν, we can simplify the dual problem as before so that it is expressed only in terms of these two variables. Renaming λ_4 as u and ν as v gives the dual problem in the simplified form

$$\max_{u,v\in\mathbb{R}} \quad 5u - 6v$$

$$\text{s.t.} \quad u \geq 0,$$
$$v \geq -\frac{1}{2}.$$

(d) Show that the optimal dual value is $d^* = \infty$ and so strong duality once again holds.

Solution: The dual feasible set is $\{(u, v) : u \geq 0, v \geq -\frac{1}{2}\}$ as it was for the problem in part (a) of this question. The dual objective $5u - 6v$ now takes on arbitrarily large values over the dual feasible set and so $d^* = \infty$. Since $d^* = p^*$, strong duality holds.

(e) Is Slater's condition satisfied in this problem?

Solution: The feasible set of the primal problem is empty, so Slater's condition cannot hold.

4. Dual Norms and SOCP

Consider the problem

$$p^\star = \min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{y}\|_1 + \mu \|\vec{x}\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$, $\vec{y} \in \mathbb{R}^m$, and $\mu > 0$.

(a) Express this (primal) problem in standard SOCP form.

Solution: Introducing slack variables $\vec{z} \in \mathbb{R}^m$, $t \in \mathbb{R}$, we can write the problem as

$$\min_{\vec{x},\vec{z},t} \vec{z}^\top \vec{1} + \mu t \tag{2}$$

$$\text{s.t. } |(A\vec{x} - \vec{y})_i| \leq z_i, \quad i = 1, \ldots, m \tag{3}$$

$$\|\vec{x}\|_2 \leq t, \tag{4}$$

which can be written as

$$\min_{\vec{x},\vec{z},t} \vec{z}^\top \vec{1} + \mu t \tag{5}$$

$$\text{s.t. } \left|\vec{a}_i^\top \vec{x} - y_i\right| \leq z_i, \quad i = 1, \ldots, m \tag{6}$$

$$\|\vec{x}\|_2 \leq t, \tag{7}$$

where $\vec{a}_i^\top$ are the rows of A and y_i are the entries of $\vec{y}$.

This expression now satisfies our definition of an SOCP: the objective is linear, and all constraints are SOC constraints, of the form

$$\left\| \begin{bmatrix} A_i & B_i & C_i \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{z} \\ t \end{bmatrix} - \vec{d}_i \right\|_2 \leq \begin{bmatrix} \vec{f}_i \\ \vec{g}_i \\ h_i \end{bmatrix}^\top \begin{bmatrix} \vec{x} \\ \vec{z} \\ t \end{bmatrix} + k_i. \tag{8}$$

For the first m constraints, we just take $A_i = \vec{a}_i^\top$, $B_i = 0$, $C_i = 0$, $\vec{d}_i = y_i$, $\vec{f}_i = \vec{0}$, $\vec{g}_i = \vec{e}_i$ (i.e., the i^{th} standard basis vector), $h_i = 0$, and $k_i = 0$. For the last constraint, we take $A_i = I$, $B_i = 0$, $C_i = 0$, $\vec{d}_i = \vec{0}$, $\vec{f}_i = \vec{0}$, $\vec{g}_i = \vec{0}$, $h_i = 1$, and $k_i = 0$.

(b) Find a dual to the problem and express it in standard SOCP form.

HINT: Recall that for every vector $\vec{z}$, the following dual norm equalities hold:

$$\|\vec{z}\|_2 = \max_{\vec{u}\,:\,\|\vec{u}\|_2 \leq 1} \vec{u}^\top \vec{z}, \qquad \|\vec{z}\|_1 = \max_{\vec{u}\,:\,\|\vec{u}\|_\infty \leq 1} \vec{u}^\top \vec{z}. \tag{9}$$

Thus, we can rewrite the objective function of the original problem as

$$\|A\vec{x} - \vec{y}\|_1 + \mu \|\vec{x}\|_2 = \max_{\vec{u}:\|\vec{u}\|_\infty \leq 1} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \max_{\vec{v}:\|\vec{v}\|_2 \leq 1} \vec{v}^\top \vec{x}. \tag{10}$$

We can then express the original (primal) problem as

$$p^\star = \min_{\vec{x}} \max_{\vec{u},\vec{v}:\|\vec{u}\|_\infty \leq 1, \|\vec{v}\|_2 \leq 1} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \vec{v}^\top \vec{x}. \tag{11}$$

To form the dual, we reverse the order of min and max.

Solution: Following the hint, we can write the dual problem as

$$d^\star = \max_{\vec{u},\vec{v}:\|\vec{u}\|_\infty \leq 1, \|\vec{v}\|_2 \leq 1} \min_{\vec{x}} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \vec{v}^\top \vec{x} \tag{12}$$

$$\doteq \max_{\vec{u},\vec{v}:\|\vec{u}\|_\infty \le 1, \|\vec{v}\|_2 \le 1} g(\vec{u},\vec{v}), \tag{13}$$

where g is defined as

$$g(\vec{u},\vec{v}) \doteq \min_{\vec{x}} \vec{u}^\top (A\vec{x} - \vec{y}) + \mu \vec{v}^\top \vec{x} \tag{14}$$

$$= \min_{\vec{x}} (\vec{u}^\top A + \mu \vec{v}^\top)\vec{x} - \vec{u}^\top \vec{y} \tag{15}$$

$$= \begin{cases} -\vec{u}^\top \vec{y} & \text{if } A^\top \vec{u} + \mu \vec{v} = \vec{0}, \\ -\infty & \text{otherwise.} \end{cases} \tag{16}$$

We can thus rewrite the dual problem as

$$d^\star = \max_{\vec{u},\vec{v}} \quad -\vec{u}^\top \vec{y} \tag{17}$$

$$\text{s.t.} \quad A^\top \vec{u} + \mu \vec{v} = \vec{0} \tag{18}$$

$$\|\vec{u}\|_\infty \le 1, \quad \|\vec{v}\|_2 \le 1. \tag{19}$$

Noting that the first constraint fully restricts the value of $\vec{v}$ — rewriting it, $\vec{v} = -\frac{A^\top \vec{u}}{\mu}$ — we can plug this value into the third constraint and eliminate $\vec{v}$ from our optimization altogether:

$$d^\star = \max_{\vec{u}} \quad -\vec{u}^\top \vec{y} \tag{20}$$

$$\text{s.t.} \quad \left\|A^\top \vec{u}\right\|_2 \le \mu \tag{21}$$

$$\|\vec{u}\|_\infty \le 1, \tag{22}$$

generating our final SOCP dual. If desired, we can further rewrite the final constraint as $\|\vec{u}\|_\infty = \max_i |u_i| \le 1 \Leftrightarrow |u_i| \le 1,\ i = 1,\ldots,m \Leftrightarrow u_i \le 1$ and $u_i \ge -1,\ i = 1,\ldots,m$ to make the linearity of that constraint more explicit.

(c) Assume strong duality holds[1] and that $m = 100$ and $n = 10^6$, i.e., A is 100×10^6. Which problem would you choose to solve using a numerical solver: the primal or the dual? Justify your answer.

Solution: To determine the rough computational complexity of each problem, we examine the number of variables and the number of constraints in each problem. The primal SOCP has $\sim 10^6$ variables and $m + 1 = 101$ constraints, while the dual has 100 variables and $m + 1 = 101$ constraints. The dual problem is thus much more efficient to solve.

[1] In fact, you can show that strong duality holds using Sion's theorem, a generalization of the minimax theorem that is beyond the scope of this class.

5. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}.$$

1) Consider the minimization problem

$$\begin{aligned} p^* = \min_{x \in \mathbb{R}}. \quad & f_0(x) \\ \text{s.t.} \quad & -1 \leq x, \\ & x \leq 1. \end{aligned} \tag{23}$$

(a) Show that $f_0(x)$ is differentiable everywhere and compute its derivative.

Solution: By the differentiability of polynomials, $f_0(x)$ is differentiable everywhere except possibly at $x = 0$. We show that $f_0(x)$ is in fact differentiable everywhere by taking the right and left derivatives at $x = 0$ and showing that they are equal.

The right derivative at $x = 0$ is given by

$$\begin{aligned} &\lim_{h \to 0^+} \frac{f_0(0+h) - f_0(0)}{h} \\ &= \lim_{h \to 0^+} \frac{(h^3 - 3h^2 + 4) - 4}{h} \\ &= 0. \end{aligned}$$

The left derivative at $x = 0$ is given by

$$\begin{aligned} &\lim_{h \to 0^-} \frac{f_0(0+h) - f_0(0)}{h} \\ &= \lim_{h \to 0^-} \frac{(-h^3 - 3h^2 + 4) - 4}{h} \\ &= 0. \end{aligned}$$

Thus f_0 is differentiable everywhere. The derivative is

$$\frac{df_0(x)}{dx} = \begin{cases} 3x^2 - 6x, & x \geq 0 \\ -3x^2 - 6x, & x < 0 \end{cases}.$$

(b) Show that $p^* = 2$ and that the set of optimizers is $\mathcal{X}^* = \{-1, 1\}$.

Solution: Since, $f_0(x)$ is differentiable everywhere, the minimum must be achieved at a boundary point or when the derivative is 0. From the previous part, we showed that $f_0'(0) = 0$ so, $x = 0$ is one of the critical points.

Next, we calculate the remaining critical points at which the derivative is zero:

$$\frac{d}{dx} f_0(x) = \begin{cases} 3x^2 - 6x, & x \geq 0 \\ -3x^2 - 6x, & x < 0 \end{cases} = 0 \Rightarrow x \in \{0, \pm 2\}$$

Since 2 and -2 are not in the feasible set and the boundary points of the feasible set are 1 and -1, we examine the function at these 3 points:

$$f_0(1) = f_0(-1) = 2$$

$$f_0(0) = 4.$$

Thus, $p^* = 2$ and $\mathcal{X}^* = \{-1, 1\}$.

(c) Show that the dual problem can be represented as

$$d^* = \max_{\lambda_1, \lambda_2 \geq 0} g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\},$$

with

$$g_1(\vec{\lambda}) = \min_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1),$$
$$g_2(\vec{\lambda}) = \min_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1).$$

Solution:

The Lagrangian is given by

$$\mathcal{L}(x, \vec{\lambda}) = f_0(x) + \lambda_1(-x-1) + \lambda_2(x-1).$$

The dual function $g(\vec{\lambda})$ is then given by

$$\begin{aligned} g(\vec{\lambda}) &= \min_x \mathcal{L}(x, \vec{\lambda}) \\ &= \min \left\{ \min_{x \geq 0} \mathcal{L}(x, \vec{\lambda}),\ \min_{x < 0} \mathcal{L}(x, \vec{\lambda}) \right\} \\ &= \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\} \end{aligned}$$

for the given $g_1(\vec{\lambda})$ and $g_2(\vec{\lambda})$, as desired.

(d) Next, show that

$$g_1(\vec{\lambda}) \leq -3\lambda_1 + \lambda_2,$$
$$g_2(\vec{\lambda}) \leq \lambda_1 - 3\lambda_2.$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

Solution: Because $g_1(\vec{\lambda})$ is the minimum over all $x \geq 0$ of $\mathcal{L}(x, \vec{\lambda})$, it is less than or equal to any instantiation of $\mathcal{L}(x, \vec{\lambda})$ at a particular value of $x \geq 0$. In particular, by instantiating at $x = 2$, we obtain

$$\begin{aligned} g_1(\vec{\lambda}) &= \min_{x \geq 0} \mathcal{L}(x, \vec{\lambda}) \\ &\leq \mathcal{L}(2, \vec{\lambda}) \\ &= -3\lambda_1 + \lambda_2, \end{aligned}$$

as desired. Analogously, we can instantiate $g_2(\vec{\lambda})$ at $x = -2$ and write

$$\begin{aligned} g_2(\vec{\lambda}) &= \min_{x < 0} \mathcal{L}(x, \vec{\lambda}) \\ &\leq \mathcal{L}(-2, \vec{\lambda}) \end{aligned}$$

$$= \lambda_1 - 3\lambda_2,$$

giving us the two desired inequalities.

We now use these inequalities to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$. Since $g(\vec{\lambda})$ is the minimization of $g_1(\vec{\lambda})$ and $g_2(\vec{\lambda})$, we can use the upper bounds we just established to write

$$\begin{aligned} g(\vec{\lambda}) &= \min\left\{g_1(\vec{\lambda}), g_2(\vec{\lambda})\right\} \\ &\leq \min\{-3\lambda_1 + \lambda_2, \lambda_1 - 3\lambda_2\} \\ &\leq 0. \end{aligned}$$

The last inequality follows because either $\lambda_1 - 3\lambda_2 \leq 0$, which immediately yields the last inequality, or $\lambda_1 - 3\lambda_2 \geq 0$, in which case $-3\lambda_1 + \lambda_2 \leq -3\lambda_1 + \frac{1}{3}\lambda_1 \leq 0$, which also yields the last inequality.

(e) Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

Solution: In part (d), we proved that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$. Since d^* is the maximum over all feasible values of $g(\vec{\lambda})$, it is sufficient to show that there exists a $\vec{\lambda}$ for which this upper bound is attained. With this in mind, consider g at $\vec{\lambda} = 0$:

$$\begin{aligned} g(\vec{0}) &= \min\{g_1(\vec{0}), g_2(\vec{0})\} \\ &= \min\left\{\min_{x\geq 0} x^3 - 3x^2 + 4,\ \min_{x<0} -x^3 - 3x^2 + 4\right\} \\ &= \min\{0, 0\} \\ &= 0. \end{aligned}$$

Note that the third equality can be shown by examining the critical points of each objective function, which are the same as those of the unconstrained primal function in part (b); this minimum is achieved at $x = \pm 2$.

We can now conclude that the maximum possible value of the dual (i.e., zero) is attained for $\vec{\lambda} = \vec{0}$, and thus $d^* = 0$ as desired.

(f) Does strong duality hold?

Solution: Since $d^* = 0 < 2 = p^*$, strong duality does not hold. This is not surprising, since the objective function $f_0(x)$ is non-convex.

2) Now, consider a problem equivalent to the minimization in (23):

$$p^* = \min_{x\in\mathbb{R}} \quad f_0(x) \tag{24}$$

$$\text{s.t. } x^2 \leq 1. \tag{25}$$

Observe that $p^* = 2$ and the set of optimizers is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part 1).

(a) Show that the dual problem can be represented as

$$d^* = \max_{\lambda\geq 0} \quad g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$g_1(\lambda) = \min_{x\geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1),$$

$$g_2(\lambda) = \min_{x<0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).$$

Solution: The Lagrangian is given by

$$\mathcal{L}(x, \lambda) = f_0(x) + \lambda(x^2 - 1).$$

The dual function $g(\lambda)$ is then given by

$$\begin{aligned} g(\lambda) &= \min_x \mathcal{L}(x, \lambda) \\ &= \min \left\{ \min_{x \geq 0} \mathcal{L}(x, \lambda), \min_{x<0} \mathcal{L}(x, \lambda) \right\} \\ &= \min \{g_1(\lambda), g_2(\lambda)\} \end{aligned}$$

for the given $g_1(\lambda)$ and $g_2(\lambda)$, as desired.

(b) Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3, \\ -\frac{4}{27}(3-\lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

Solution: We first show that $g_2(\lambda) = g_1(\lambda)$:

$$\begin{aligned} g_2(\lambda) &= \min_{x<0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ &= \min_{-x>0} (-x)^3 - 3(-x)^2 + 4 + \lambda((-x)^2 - 1) \\ &= \min_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ &= g_1(\lambda). \end{aligned}$$

Next, let us compute $g_1(\lambda)$ directly. Differentiating $L(x, \lambda)$ in x for fixed λ and setting the derivative to 0, we seek the solutions of

$$3x^2 - 2(3 - \lambda)x = 0,$$

so $x = 0$ or $x = \frac{2}{3}(3 - \lambda)$. Hence, if $\lambda \geq 3$ then to find $g_1(\lambda)$, we just compute $L(0, \lambda)$, which is $4 - \lambda$, while if $0 \leq \lambda < 3$, we compute the minimum of $L(0, \lambda)$ and $L\big(\frac{2}{3}(3 - \lambda), \lambda\big)$, which is $-\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda$.

(c) Conclude that $d^* = 2$ and $\lambda^\star = \frac{3}{2}$.

Solution: Since $g_1(\lambda) = g_2(\lambda)$, we have $g(\lambda) = g_1(\lambda) = g_2(\lambda)$. We examine each range of possible λ values in turn to determine the maximum.

For $\lambda \geq 3$, the maximum value of $g(\lambda) = 1$ is achieved at $\lambda = 3$.

For $0 \leq \lambda < 3$, the maximum of $g(\lambda)$ is computed as follows. First, we set the derivative of $g(\lambda)$ with respect to λ to 0:

$$\frac{12}{27}(3 - \lambda)^2 - 1 = 0 \implies (3 - \lambda)^2 = \frac{9}{4} \implies \lambda = \frac{3}{2} \text{ or } \lambda = \frac{9}{2}.$$

Since the expression is valid only for $0 \leq \lambda < 3$, we examine values at $\lambda \in \{0, 3\}$ (boundary points) and at the computed $\lambda = \frac{3}{2}$. We observe that the maximum is achieved at $\lambda = \frac{3}{2}$ with $g(\frac{3}{2}) = 2$.

Finally, we note that the overall maximum occurs in the second case, at $\lambda^\star = \frac{3}{2}$, and thus $d^* = 2$ as desired.

(d) Does strong duality hold?

Solution: In this case, $p^* = d^* = 2$, so strong duality holds.

