Discussion 14

1. Sum of Squares

Given a polynomial $p(t)$ in a single variable t, we are interested in knowing if we can write $p(t)$ as a sum of squares of polynomials, i.e. whether we can write

$$p(t) = \sum_{j=1}^{k} (q_j(t))^2,$$

for some $k \geq 1$ and some polynomials $q_1(t), \cdots, q_k(t)$. This is an interesting question in many contexts, because if we could do this then we would know that $p(t)$ is nonnegative for all values of t. [1]

First observe that if $p(t)$ can be written as a sum of squares of polynomials then it must have even degree. We will therefore assume that $p(t)$ has degree $2d$ for some integer $d \geq 1$ (the case $d = 0$ corresponds to $p(t)$ being a constant).

Let $\vec{z} := \begin{bmatrix} 1 & t & \cdots & t^d \end{bmatrix}^\top$. Note that $\vec{z}$ is a $(d+1)$-dimensional vector whose entries are polynomials in t.

(a) Show that the polynomial $p(t)$ of degree $2d$ can be written as a sum of squares of polynomials if and only if there is a symmetric positive semidefinite matrix Q such that

$$p(t) = \vec{z}^\top Q \vec{z}.$$

(The equality here is an equality between polynomials in t.)

Hint: Every symmetric positive semidefinite matrix Q can be written as a sum of dyads, i.e. $Q = \sum_{i=1}^{n} \vec{u}_i \vec{u}_i^\top$ if $Q \in \mathbb{S}_+^n$.

[1] In fact, it is known that if a polynomial $p(t)$ in a single variable is nonnegative for all values of t then it can be written as a sum of squares of two polynomials, i.e. $p(t) = r(t)^2 + s(t)^2$ for some polynomials $r(t)$ and $s(t)$, but we do not need this fact.

(b) Show that we can pose the question of whether a given polynomial $p(t)$ of degree $2d$ can be written as a sum of squares of polynomials as a feasibility question for an SDP in standard form.

Remark: Recall that an SDP in standard form looks like:

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \quad & \text{trace}(CX) \\ \text{s.t.} \quad & \text{trace}(A_i X) = b_i, \text{ for each } i \in \{1, \cdots, m\}, \\ & X \succeq 0. \end{aligned}$$

Here the minimization is over matrices $X \in \mathbb{S}^n$. The matrices $C, A_1, \cdots, A_m \in \mathbb{S}^n$ as well as the scalars $b_1, \cdots, b_m \in \mathbb{R}$ are given. The constraint $X \succeq 0$ is the constraint that X should be symmetric positive semidefinite.

Also recall that to pose a minimization problem as a feasibility problem, we can just take the objective to be the constant 0 (so the question then just becomes whether the value of the problem is 0, in which case the problem is feasible, or ∞, in which case the problem is infeasible). For an SDP in standard form to be a feasibility problem, therefore, we could just take C to be the zero matrix.

2. SDP Duality

Consider the following SDP in inequality form:

$$\begin{aligned} \min_{(x,y)\in\mathbb{R}^2} \quad & x \\ \text{s.t.} \quad & \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \succeq 0. \end{aligned} \tag{1}$$

(a) Draw the feasible set. Is it convex?

(b) Write the conic dual SDP in standard form.

(c) Is the primal SDP feasible? Is it strictly feasible?

Remark: The SDP in inequality form

$$\begin{aligned} \min_{\vec{x} \in \mathbb{R}^m} \quad & \vec{c}^\top \vec{x} \\ \text{s.t.} \quad & F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0, \end{aligned}$$

where $F_0, F_1, \cdots, F_m \in \mathbb{S}^n$, $\vec{c} \in \mathbb{R}^n$, is said to be strictly feasible if there is some $\vec{x} \in \mathbb{R}^n$ such that $F(\vec{x}) \in \mathbb{S}^n_{++}$, i.e. $F(\vec{x})$ is symmetric positive definite. Here $F(\vec{x})$ denotes $F_0 + \sum_{i=1}^{m} x_i F_i$.

(d) Is the dual SDP feasible? Is it strictly feasible?

Remark: The SDP in standard form

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{trace}(CX) \\ \text{s.t.} \quad & \text{trace}(A_i X) = b_i, \text{ for each } i \in \{1, \cdots, m\}, \\ & X \succeq 0, \end{aligned}$$

where $C, A_1, \cdots, A_m \in \mathbb{S}^m$, $b_1, \cdots, b_m \in \mathbb{R}$, is said to be strictly feasible if there is some $X \in \mathbb{S}^n_{++}$ (symmetric positive definite) satisfying the equality constraints $\text{trace}(A_i X) = b_i$ for each $i \in \{1, \cdots, m\}$.

(e) Find the optimal primal value p^* and the optimal dual value d^*. Does strong duality hold?

