# Discussion 8

1. **Sandwich Function**

   Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Suppose $g(\vec{x}) \leq f(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. Prove that there exists an affine function $h : \mathbb{R}^n \to \mathbb{R}$ such that $g(\vec{x}) \leq h(\vec{x}) \leq f(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$.

   *Hint*: Consider the interior of the epigraph of $f$ and the hypograph of $g$ and use the separating hyperplane theorem.

**2. A combinatorial problem formulated as a convex optimization problem**

In this question we will see how combinatorial optimization problems can sometimes be solved via related convex optimization problems. Consider the problem of deciding which subset of items to sell from among a collection of $n$ items. Selling the item $i$ results in revenue $s_i > 0$ and a transaction cost $c_i > 0$. There is also a fixed overall cost, which we normalize to 1, which is incurred irrespective of whether any items are sold. We are interested in maximizing the *margin*, i.e. the ratio of the total revenue to the total cost (sum of the total transaction cost and the fixed overall cost). Note that this is a combinatorial optimization problem, because the set of choices is a discrete set (here it is the set of all subsets of $\{1, \dots, n\}$). In this question we will see that it is possible to pose this combinatorial optimization problem as a convex optimization problem.

(a) Show that the original combinatorial optimization problem can be formulated as

$$\max_{\vec{x} \in \{0,1\}^n} f(\vec{x})$$

where $f(\vec{x}) := \frac{\vec{s}^\top \vec{x}}{1 + \vec{c}^\top \vec{x}}$, Here $\vec{s} \in \mathbb{R}^n$ is the column vector of revenues and $\vec{c} \in \mathbb{R}^n$ is the column vector of transaction costs associated to the individual items.

(b) Show that the combinatorial optimization problem admits a convex optimization problem formulation in the sense that the convex optimization problem

$$\begin{aligned} \min_{t} \quad & t \\ \text{subject to :} \quad & t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+, \\ & t \geq 0, \end{aligned}$$

has the same value as that of the original combinatorial optimization problem. Here $\vec{z}_+$, for a vector $\vec{z} \in \mathbb{R}^n$, denotes the vector with components $\max(0, z_i)$, $i = 1, \dots, n$. Also $\vec{\mathbf{1}}$ denotes the all-ones column vector of length $n$.

*Note*: You should also justify why the problem formulated in this part of the question is a convex optimization problem.

*Hint:* For given $t \geq 0$, express the condition "$f(\vec{x}) \leq t$ for every $\vec{x} \in \{0,1\}^n$" in simple terms. Note that this is related to the idea of introducing slack variables, as in Sec. 8.3.4.4 of the textbook of Calafiore and El Ghaoui.

 

(c) Show that the inequality constraint $t \geq \vec{\mathbf{1}}^\top (\vec{s} - t\vec{c})_+$ is active at the optimum in the convex optimization problem, namely, it will have to be satisfied with equality.

(d) How can you recover an optimal solution $\vec{x}^*$ for the original combinatorial optimization problem from an optimal value $t^*$ for the above convex optimization problem?