



软件理论基础与实践

Induction: Proof by Induction

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多文件程序开发

- From LF Require Export Basics.
 - 从LF目录读取编译后的Basics.vo文件并导入
- LF目录
 - _CoqProject文件指明当前目录为LF
 - `-Q . LF`
- 编译得到Basics.vo文件
 - Proof General和CoqIDE会自动编译
 - VSCode不行
 - Linux下编译
 - Make Basics.vo或make
 - Windows下编译
 - `coqc -Q . LF Basics.v`



复习

- 是否能够通过simpl和reflexivity证明 $n+0=n$

```
Theorem add_0_r_firsttry : forall n:nat,  
  n + 0 = n.  
Proof. intros n.  
(*   n : nat  
  *   =====  
  *   n + 0 = n  
  *)  
  simpl. (** [Coq Proof View]  
  *   n : nat  
  *   =====  
  *   n + 0 = n  
  *)  
  (* Does nothing! *)  
Abort.
```



Destruct也不行

Theorem `add_0_r_secondtry` : `forall` `n:nat`,
`n + 0 = n`.

Proof.

```
intros n. destruct n as [| n'] eqn:E.  
- (* n = 0 *)  
  reflexivity. (* so far so good... *)  
- (* n = S n' *)  
  (* S n' + 0 = S n' *)  
  simpl.  
  (* S (n' + 0) = S n' *)
```

Abort.



结构归纳法

Structural Induction

- 数学归纳法：
 - 首先证明性质 P 对 0 成立
 - 然后证明 $P(k) \rightarrow P(k+1)$
- 结构归纳法：
 - 对于任意递归定义的类型
 - 证明 P 对该类型的每一个构造函数都成立
 - 如果构造函数接受该类型的参数，则假设 P 对参数成立
- 数学归纳法是结构归纳法在自然数上的特例



采用结构归纳法证明

```
Theorem add_0_r : forall n:nat, n + 0 = n.
(** [Coq Proof View]
 * 1 subgoal
 *
 * =====
 * forall n : nat, n + 0 = n
 *)
Proof. intros n.
(** [Coq Proof View]
 * 1 subgoal
 *
 * n : nat
 * =====
 * n + 0 = n
 *)
```



采用结构归纳法证明

```
induction n as [| n' IHn'].
(** [Coq Proof View]
 * 2 subgoals
 *
 * =====
 * 0 + 0 = 0
 *
 * subgoal 2 is:
 * S n' + 0 = S n'
 *)
```

为变量命名，可省略



采用结构归纳法证明

```
- reflexivity.  
- (** [Coq Proof View]  
* 1 subgoal  
*  
*   n' : nat  
*   IHn' : n' + 0 = n'  
*   =====  
*   S n' + 0 = S n'  
* )  
  simpl. rewrite -> IHn'. reflexivity.  
Qed.
```

归纳假设

Induction同时作用于一个变量的多个实例



Theorem minus_n_n : forall n,
 minus n n = 0.

Proof.

```
(* WORKED IN CLASS *)  
intros n. induction n as [| n' IHn'].  
- (* n = 0 *)  
  simpl. reflexivity.  
- (* n = S n' *)  
  simpl. rewrite -> IHn'. reflexivity. Qed.
```



证明子定理

```
Theorem mult_0_plus' : forall n m : nat,  
  (0 + n) * m = n * m.
```

Proof.

```
  intros n m.  
  assert (H: 0 + n = n). { reflexivity. }  
  rewrite -> H.  
  reflexivity. Qed.
```



证明子定理

```
Theorem mult_0_plus' : forall n m : nat,  
  (0 + n) * m = n * m.  
Proof. intros n m.  
(** [Coq Proof View]  
  * 1 subgoal  
  *  
  *   n, m : nat  
  *   =====  
  *   (0 + n) * m = n * m  
  *)
```



证明子定理

```
    assert (H: 0 + n = n).  
(** [Coq Proof View]  
  * 2 subgoals  
  *  
  *   n, m : nat  
  *   =====  
  *   0 + n = n  
  *  
  * subgoal 2 is:  
  *   (0 + n) * m = n * m  
  *)
```



证明子定理

```
{ reflexivity. }  
  (** [Coq Proof View]  
  * 1 subgoal  
  *  
  *   n, m : nat  
  *   H : 0 + n = n  
  *   =====  
  *   (0 + n) * m = n * m  
  *)  
rewrite -> H.  
reflexivity. Qed.
```



采用assert帮助rewrite定位

Theorem `plus_rearrange_firsttry` : `forall` n m p q : nat,
 $(n + m) + (p + q) = (m + n) + (p + q)$.

Proof. `intros` n m p q.

```
(** [Coq Proof View]
* 1 subgoal
*
*   n, m, p, q : nat
*   =====
*   n + m + (p + q) = m + n + (p + q)
*)
```



采用assert帮助rewrite定位

```
rewrite add_comm.  
(** [Coq Proof View]  
* 1 subgoal  
*  
*   n, m, p, q : nat  
*   =====  
*   p + q + (n + m) = m + n + (p + q)  
*)
```

- Rewrite只重写了最外层的加法



采用assert帮助rewrite定位

Theorem `plus_rearrange` : `forall` `n m p q` : `nat`,
 $(n + m) + (p + q) = (m + n) + (p + q)$.

Proof.

```
intros n m p q.  
assert (H: n + m = m + n).  
{ rewrite add_comm. reflexivity. }  
rewrite H. reflexivity. Qed.
```




形式化证明 vs 非形式化证明

- Coq写的是形式化证明
- 数学课上/论文里写的是非形式化证明
- 非形式化证明的作用
 - 交流——非形式化证明可以写得更简洁，更抽象，帮助交流
 - 理解——采用Coq自动证明策略很容易试出来证明，但更重要的是确保自己能理解证明



作业

- 完成Induction.v中standard非optional的4道习题
 - 请使用最新英文版教材