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SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

INFORMATION THEORY -Line of Sight MIMO

Tesi di Laurea Magistrale in
Telecommunication Engineering

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Academic Year: 2025-07

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1 | Line of Sight MIMO

1.1. Theoretical Foundations of LoS MIMO

Scenario: We consider a point-to-point microwave link in which the transmitter and receiver towers are directly visible to each other. There is no scattering or reflection, and the dominant propagation path is a direct line-of-sight (LoS) path.

Physical Characteristics: In free-space propagation, the received power decays with the square of the distance according to the *Friis* transmission equation:

$$P_r = P_t \cdot G_t G_r \left(\frac{\lambda}{4\pi d}\right)^2 \tag{1.1}$$

If we consider the complex baseband envelope and normalize the transmit power to 1, the amplitude of the received signal becomes inversely proportional to the distance:

$$A(\lambda, d) \propto \frac{\lambda}{d} \tag{1.2}$$

When analyzing capacity, the key factor is the rank of the channel matrix, which determines how many independent data streams can be transmitted.
The rank is determined by whether the rows or columns of the channel matrix are linearly independent; this is primarily influenced by phase differences.

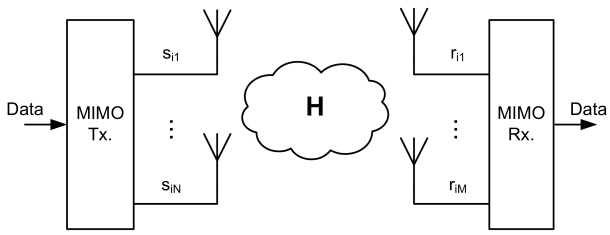


Figure 1.1: MIMO Model

When P_T, G_T and P_R are approximately equal across all links, their impact on the rank is negligible.
In free-space propagation, the complex channel coefficient is given by:

$$h_{ij} = \underbrace{\frac{\lambda}{4\pi d_{ij}}}_{\text{Amplitude attenuation}} \cdot \underbrace{e^{-j\frac{2\pi d_{ij}}{\lambda}}}_{\text{Phase rotation}} \tag{1.3}$$

Therefore, in theoretical studies, academic literature, and simulations, it is common to write:

$$h_{ij} = \frac{\lambda}{d_{ij}} e^{j\phi_{ij}} \tag{1.4}$$

where the phase term is defined as:

$$\phi_{ij} = \frac{2\pi d_{ij}}{\lambda} \tag{1.5}$$

1.1.1. LoS MIMO Channel Matrix Representation

For a 2×2 line-of-sight MIMO system, we model the received signal as:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1.6}$$

The channel matrix is given by:

$$\mathbf{H} = \begin{bmatrix} \frac{\lambda}{d_{11}} e^{-j\phi_{11}} & \frac{\lambda}{d_{12}} e^{-j\phi_{12}} \\ \frac{\lambda}{d_{21}} e^{-j\phi_{21}} & \frac{\lambda}{d_{22}} e^{-j\phi_{22}} \end{bmatrix} \tag{1.7}$$

ϕ_{ij} represents the phase shift corresponding to the path between the $i-th$ receive antenna and the $j-th$ transmit antenna.
Assuming the transmit and receive antenna spacings a and b, and the total link distance D, the path lengths are:

$$\begin{aligned} d_{11} &= \sqrt{D^2 + 0} \\ d_{12} &= \sqrt{D^2 + a^2} \\ d_{21} &= \sqrt{D^2 + b^2} \\ d_{22} &= \sqrt{D^2 + (a-b)^2} \end{aligned} \tag{1.8}$$

1.1.2. Derivation of the MIMO Channel Capacity Formula

We aim to compute the maximum mutual information:

$$C = \max \boldsymbol{I}(\mathbf{x}; \mathbf{y}) \tag{1.9}$$

The mutual information for a MIMO channel is defined as:

$$\mathbf{I}(\mathbf{x}; \mathbf{y}) = H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x}) \tag{1.10}$$

Since the channel noise is Gaussian and $\mathbf{y}|\mathbf{x}$ is conditionally Gaussian:

$$\mathbf{y}|\mathbf{x} \sim \mathcal{CN}(\mathbf{Hx}, N_0\mathbf{I}) \tag{1.11}$$

The conditional entropy is:

$$H(\mathbf{y}|\mathbf{x}) = \log\det(\pi e N_0 I) = N_R \log(\pi e N_0) \tag{1.12}$$

In Shannon theory, the input distribution $\mathbf{x}$ can be freely designed, subject to a power constraint.

$$\mathrm{Tr}(E[\mathbf{xx}^\dagger]) \leq P \tag{1.13}$$

In MIMO systems, the input signal covariance matrix is defined as

$$\mathbf{Q} = E[\mathbf{xx}^\dagger] \tag{1.14}$$

It captures:

- The power allocated to each transmit antenna (diagonal terms);
- The correlation between antenna signals (off-diagonal terms).

Shannon' s theory proves that in a Gaussian channel, the optimal input distribution that maximizes mutual information is complex Gaussian:

$$\mathbf{x} \sim \mathcal{CN}(0, \mathbf{Q}) \tag{1.15}$$

Since both $\mathbf{x}$ and $\mathbf{n}$ are Gaussian, the output is also Gaussian:

$$\mathbf{y} \sim \mathcal{CN}(0, \Sigma_y), \quad \text{where} \quad \Sigma_y = \mathbf{HQH}^\dagger + N_0\mathbf{I} \tag{1.16}$$

Then the output entropy is:

$$H(\mathbf{y}) = \log\det(\pi e(\mathbf{HQH}^\dagger + N_0\mathbf{I})) \tag{1.17}$$

Substituting into the mutual information formula:

$$\begin{aligned} I(\mathbf{x};\mathbf{y}) &= H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x}) && (1.18)\\ &= \log\det\left(I + \frac{1}{N_0}HQH^\dagger\right) && (1.19)\\ &= \log\det\left(I + \frac{P}{N_0N_T}HH^\dagger\right) && (1.20) \end{aligned}$$

1.1.3. Why Is Channel Capacity Geometry-Dependent?

Under uniform power allocation, the Shannon capacity is:

$$C = B\cdot\log_2\det\left(I + \frac{P}{N_0N_T}\mathbf{HH}^\dagger\right) = B\cdot\log_2\det\left(I + \gamma\mathbf{HH}^\dagger\right) \tag{1.21}$$

where:

- P: total transmit power
- B: Channel bandwidth (Hz)
- N_T: number of transmit antennas
- $\gamma = \frac{P}{N_0N_T}$: normalized SNR per transmit antenna
- $\mathbf{H}$: MIMO channel matrix

This capacity can also be expressed in terms of eigenvalues:

$$\boldsymbol{C} = \sum_{\boldsymbol{i=1}}^{\text{rank}(\boldsymbol{H})} \boldsymbol{B}\cdot\mathbf{log_2}\left(\mathbf{1}+\boldsymbol{\gamma\lambda_i}\right) \tag{1.22}$$

where:

- λ_i are the non-zero eigenvalues of $\mathbf{HH}^\dagger$
- $\text{rank}(\mathbf{H})$ denotes the rank of the channel matrix

1.1.4. Physical Meaning of the Rank of H

rank($\mathbf{H}$) = number of independent spatial streams the MIMO system can support.

- If rank($\mathbf{H}$)=1, the system behaves like SISO — only one data stream can be transmitted.
- If rank($\mathbf{H}$)=N_T, the system supports full spatial multiplexing of N_T independent streams.

In line-of-sight (LoS) MIMO systems:

- If antenna spacing is well designed → phase shifts are well separated → rows/-columns of $\mathbf{H}$ become orthogonal → full rank → maximum capacity.

1.2. Project analysis and simulation process analysis

This project focuses on line-of-sight (LoS) MIMO systems used in microwave and millimeter-wave point-to-point links. The main tasks are:
To analyze how channel capacity depends on:

- Transmit and receive antenna spacings a, b
- Link distance D
- Carrier frequency f

SNR / Bandwidth / Noise Density Settings:
In wireless communications, noise is typically approximated as thermal noise (thermal equilibrium noise), whose power spectral density is given by (ignoring the receiver noise figure):

$$N_0 = kT \tag{1.23}$$

where:

- $k = 1.38{\times}10^{-23} J/K$ is the Boltzmann constant;
- T is the absolute temperature in Kelvin, commonly taken as room temperature T 290K;

Thus, at room temperature:

$$N_0 = 1.38 \times 10^{-23} \times 290 \approx 4.0 \times 10^{-21}\text{W/Hz} \tag{1.24}$$

To evaluate how antenna tower displacement (by wind, modeled as a deviation x) affects capacity; To explore the capacity improvement potential under practical constraints(e.g., antenna spacing $< 10m$)

carrier freq (f)	hop length (D)
18 GHz	5km $\leq D \leq$20km
38 GHz	2km $\leq D \leq$10km
80 GHz	1km $\leq D \leq$5km
115 GHz	500m $\leq D \leq$2km
170 GHz	100m $\leq D \leq$1km

Table 1.1: Carrier Frequencies and Hop Length

1.2.1. Capacity Derivation and Analysis

Column Orthogonality Condition To maximize capacity, the channel matrix $\mathbf{H}$ must be full rank, which occurs when its columns are orthogonal:
Let:

$$\mathbf{h}_1 = \begin{bmatrix} \frac{\lambda}{d_{11}} e^{-j\phi_{11}} \\ \frac{\lambda}{d_{21}} e^{-j\phi_{21}} \end{bmatrix}, \quad \mathbf{h}_2 = \begin{bmatrix} \frac{\lambda}{d_{12}} e^{-j\phi_{12}} \\ \frac{\lambda}{d_{22}} e^{-j\phi_{22}} \end{bmatrix} \tag{1.25}$$

Orthogonality condition:

$$\mathbf{h}_1^\dagger \cdot \mathbf{h}_2 = 0 \Rightarrow \frac{\lambda^2}{d_{11}d_{12}} e^{j(\phi_{11}-\phi_{12})} + \frac{\lambda^2}{d_{21}d_{22}} e^{j(\phi_{21}-\phi_{22})} = 0 \tag{1.26}$$

Define:

$$\Delta\phi_1 = \phi_{11} - \phi_{12}, \quad \Delta\phi_2 = \phi_{21} - \phi_{22} \tag{1.27}$$

Then, we get the orthogonality condition:

$$\frac{1}{d_{11}d_{12}} e^{j\Delta\phi_1} + \frac{1}{d_{21}d_{22}} e^{j\Delta\phi_2} = 0 \tag{1.28}$$

Antenna Spacing Optimization Criterion:
Optimal spacing formula:

$$a \cdot b = \frac{\lambda D}{\min(N_{TX}, N_{RX})} \cdot K \quad (K = 1, 3, 5, \dots) \tag{1.29}$$

Objective: minimize the channel condition number

$$\kappa = \frac{\sigma_{\max}}{\sigma_{\min}} \tag{1.30}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the largest and smallest singular values of the channel matrix, respectively.

1.2.2. Water Filling

Singular Value Decomposition (SVD) of the Channel Matrix:

$$\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^{\dagger} \tag{1.31}$$

Σ: a diagonal matrix whose elements are the singular values. (where R=rank($\mathbf{H}$))
Physical meaning:
Through precoding ($\mathbf{V}$) and postcoding (U^H), the MIMO channel is decomposed into R independent SISO sub-channels. Each sub-channel gain is σ_i, corresponding to parallel transmissions!
The capacity expression becomes:

$$C = \sum_{i=1}^{r} B \cdot \log_2 \left(1 + \frac{p_i \cdot \sigma_i^2}{B \cdot N_0}\right) \tag{1.32}$$

where:

- σ_i^2: gain of the i-th subchannel (the higher, the better);
- p_i: power allocated to the i-th subchannel;
- Subject to $\sum p_i = P$ and $p_i \geq 0$

Objective: maximize the channel capacity:

$$\max_{\{p_i\}} \quad \sum_{i=1}^{r} B \cdot \log_2 \left(1 + \frac{p_i \cdot \sigma_i^2}{B \cdot N_0}\right) \tag{1.33}$$

Solving via Lagrange multipliers:
Define the Lagrangian function (with base-2 logarithm):

$$\mathcal{L}(p_1, \ldots, p_r, \mu) = \sum_{i=1}^{r} B \cdot \log_2 \left(1 + \frac{p_i \sigma_i^2}{B \cdot N_0}\right) - \mu \left(\sum_{i=1}^{r} p_i - P\right) \tag{1.34}$$

Here:

- μ : is the Lagrange multiplier (also known as the "water level");
- Each p_i is a power variable to be optimized.

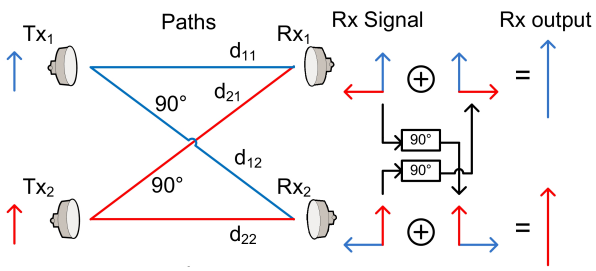


Figure 1.2: Symmetric 2x2 LoS MIMO System (orthogonal)

Take derivative with respect to each p_i and set to zero:

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{B}{\ln(2)} \cdot \frac{\sigma_i^2}{B \cdot N_0 + p_i \sigma_i^2} - \mu = 0 \tag{1.35}$$

Solve for p_i:

$$p_i = \frac{B}{\mu \ln(2)} - \frac{B \cdot N_0}{\sigma_i^2} \tag{1.36}$$

Since $\frac{B}{ln(2)}$ is a constant, we define:

$$\mu' = \frac{B}{\mu \ln(2)} \quad \Rightarrow \quad p_i = \mu' - \frac{B \cdot N_0}{\sigma_i^2} \tag{1.37}$$

Finally, by applying the non-negativity constraint $p_i \geq 0$, we arrive at the classical formula:

$$p_i = \left(\mu - \frac{B \cdot N_0}{\sigma_i^2}\right)^+ \tag{1.38}$$

This equation generally does not have a closed-form analytical solution, but an appropriate value of mu can be found through iterative methods such as bisection search or a simple for-loop approach.

1.2.3. Symmetric Spacing Case (a=b)

If the antenna spacings are symmetric, i.e., a=b, then:

$$e^{j\Delta\phi_1} = -e^{j\Delta\phi_2} \Rightarrow \Delta\phi_1 = \pi + \Delta\phi_2 + 2k\pi, \quad k \in \mathbb{Z} \tag{1.39}$$

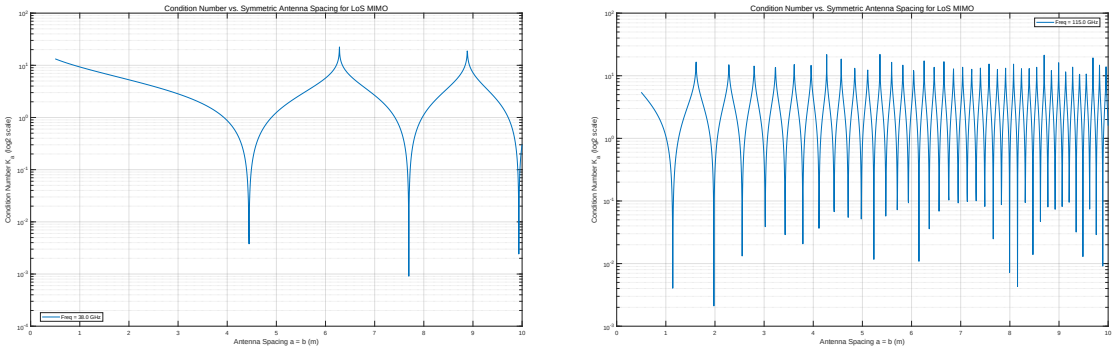
$$|\Delta\phi_1| = |\Delta\phi_2| = \frac{\pi}{2} \tag{1.40}$$

Figure 1.2 illustrates the 2x2 LoS MIMO system with a symmetric structure that satisfies the orthogonality condition, as well as the demodulation process at the receiver. Based

on Sayeed' s classical 2×2 LoS MIMO model. To achieve full rank (maximum capacity), the following orthogonality condition must be satisfied:

$$a \cdot b = \frac{\lambda D}{2} \cdot K \quad (K = 1, 3, 5, \dots) \tag{1.41}$$

According to Equation 1.21, under the condition a=b, the system's channel capacity is shown in Figure 1.4 for different carrier frequencies. In this case, the channel capacity is calculated assuming equal power allocation across all sub-channels.



(a) $f = 38$ GHz (b) $f = 115$ GHz

Figure 1.3: Condition number of channel matrices at different frequencies

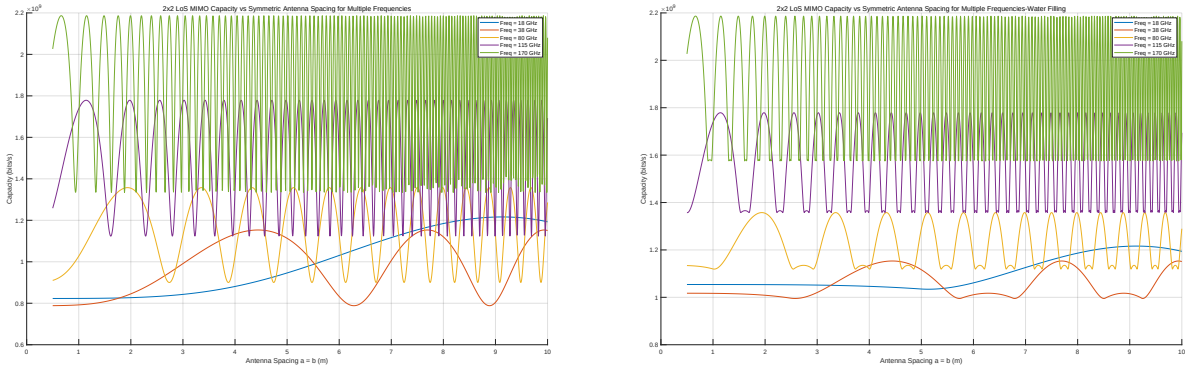
Figure 1.3 shows the variation of $log_2(condition\ \ number)$ with increasing antenna spacing a, under the condition $a = b$, for different carrier frequencies. It can be observed that the condition number periodically reaches or approaches 1. According to the orthogonality criterion, the closer the condition number is to 1, the better the channel performance. When the condition number equals 1, the channel is in its optimal state.

Channel Capacity Achieved by the Water-Filling Method:

As shown in Figure 1.4, allocating power based on channel quality leads to an increase in overall channel capacity compared to the traditional method of uniformly distributing power across all subchannels. In the case of orthogonal channels, the capacity achieved by the water-filling method is equal to that of the uniform power allocation method.

1.2.4. Impact of Asymmetric Spacing (a≠b)

According to Equation 1.21, under the condition $a \neq b$, the heatmap of the system's channel capacity using the conventional power allocation method is shown in Figure 1.5. The yellow regions follow the inverse proportional curve described by Equation 1.41, and these patterns appear periodically. As the carrier frequency increases, the channel



(a) Traditional (b) Water Filling

Figure 1.4: Capacity Comparison under Different Power Allocation Schemes (with $a = b$)

capacity becomes more sensitive to variations in a and b.

Channel capacity is optimized using the water-filling method. Figure 1.6 shows capacity heatmaps at 10 GHz, 80GHz and 170 GHz. Water-filling consistently outperforms uniform power allocation across antenna spacings and is less sensitive to spacing variations. At higher frequencies, capacity becomes more sensitive, requiring careful frequency selection based on antenna shifts.

1.2.5. Power Allocation under Different Frequencies (WF)

(Water Filling) According to Equation 1.38, the power allocation ratios for different subchannels are obtained. Figure 1.7 illustrates how the water-filling algorithm distributes power across non-orthogonal MIMO channels under various frequencies.
In the case of traditional equal power allocation, the maximum channel capacity typically occurs at orthogonal configurations, where all singular values of the channel matrix $\mathbf{H}$ are equal, and thus the subchannels are balanced.
However, under the water-filling method, the capacity is influenced not only by the channel matrix structure but also by the SNR and the distribution of singular values. Therefore, the maximum capacity does not necessarily occur at orthogonal positions. Depending on the frequency, antenna spacing, and noise level, a non-orthogonal configuration may allow the water-filling algorithm to allocate power more effectively to stronger subchannels, leading to higher capacity.
When the signal-to-noise ratio (SNR) is very high, even weak subchannels can carry information effectively, making the difference between water-filling and equal allocation negligible. But as the SNR decreases, the advantage of water-filling becomes significant, since it adapts the power allocation to channel conditions to maximize capacity.

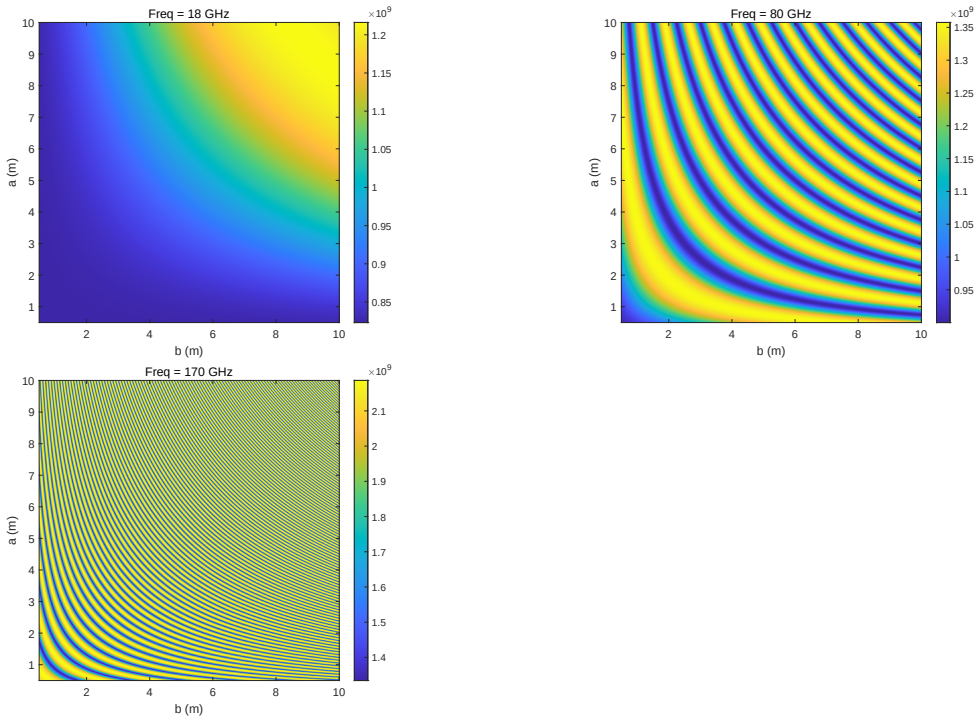


Figure 1.5: Capacity (3D) - Traditional Method

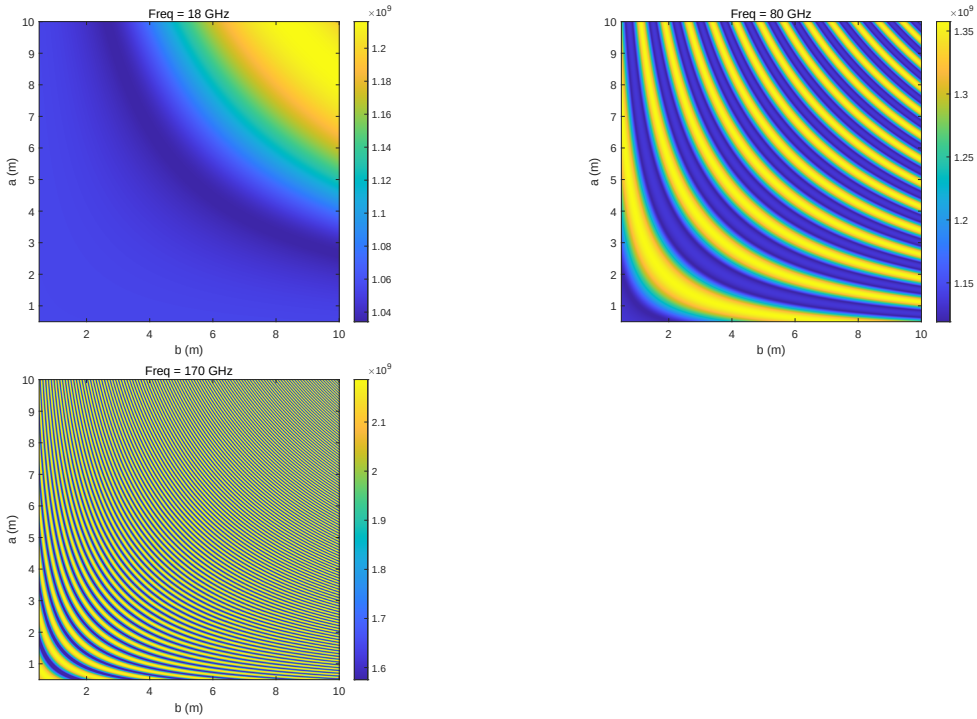
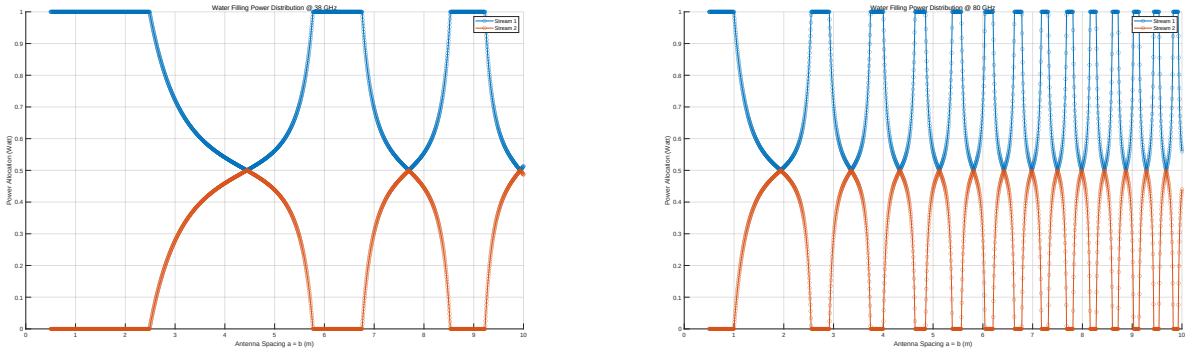


Figure 1.6: Capacity (3D) - WF



(a) $f = 38GHz$ (b) $f = 80GHz$

Figure 1.7: Power Allocation under Different Frequencies

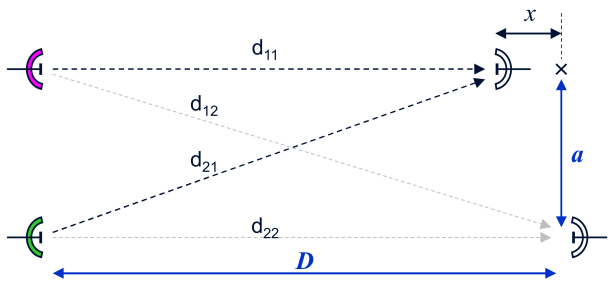


Figure 1.8: Displacement of RX

1.2.6. Impact of Receiver Lateral Displacement on Channel Capacity

In real-world deployments, receiver antennas may experience lateral displacement (i.e., moving closer to or farther from the transmitter) due to wind or tower vibrations. Although the antenna spacing remains constant, slight variations in the link distance D result in changes in the subchannel path lengths d_{ij}, which in turn cause phase shifts. These shifts affect the rank and singular value structure of the channel matrix H, leading to fluctuations in channel capacity.

1.3. Improvement Proposal

Replace omnidirectional antennas with directional antennas to reduce beamwidth, enhance directivity, and suppress inter-antenna interference, thereby improving spatial multiplexing efficiency. This is particularly beneficial in high-frequency (e.g., mmWave) LoS MIMO links, where narrower beams lead to more stable channel responses and reduced sensitivity to antenna displacement, ultimately enhancing system robustness.
Simulation results show that lateral displacement of the receiver antenna (i.e., small for-

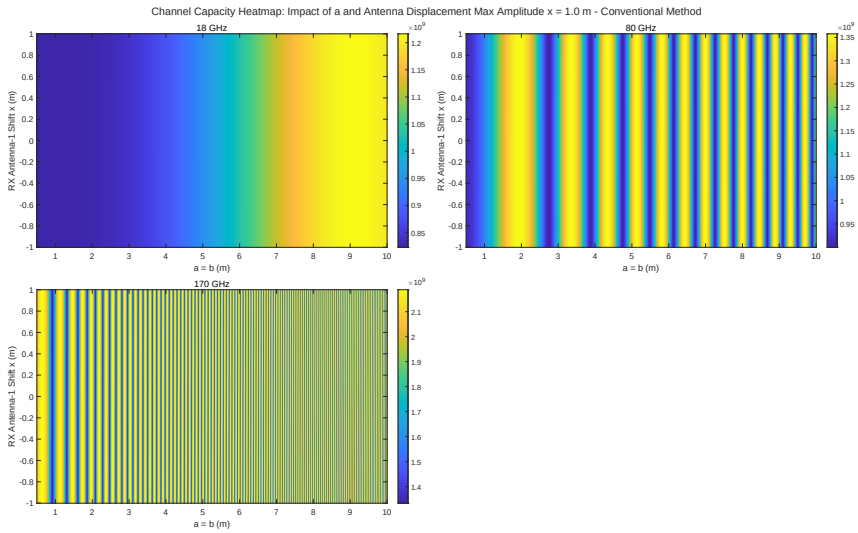


Figure 1.9: Impact of Receiver Lateral Displacement on Channel Capacity

ward/backward movement) causes minimal impact on the channel capacity, even at different carrier frequencies.

This indicates that LoS MIMO systems are robust to such displacement.

Minor shifts in receiver position do not significantly affect phase relationships or channel rank.

Therefore, receiver lateral displacement is tolerable in practical deployments.