# 1 Homework #1, AA2024-2024: Frequency estimation and CRB

Frequency estimation and filtering are common problems in many applications. The Homework's objective is to gain experience on how to estimate the frequencies in case of single sinusoid.

Let consider a sinusoid with $N$ samples

$$x[n] = a_o cos(\omega_o n + \phi_o),$$

that is filtered by $h[n] \longleftrightarrow H(z) = \frac{2}{1+0.9z^{-1}}$ to get a resulting signal with $N$ samples affected by noise

$$y[n] = a_o cos(\omega_o n + \phi_o) * h[n] + w[n],$$

The noise is zero mean Gaussian:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\text{ww}}),$$

the $N \times N$ covariance matrix $C_{\text{ww}}$ has elements $[\mathbf{C}_{\text{ww}}]_{i,j} = \sigma_w^2 \rho^{|i-j|}$ where $|\rho| \leq 1$ is the correlation coefficient. This random process must be modelled with Matlab by the student as well as all questions below.

**Notice that no built-in codes or external macro for computation is allowed, except FFT.**

## 1.1 Noise generation

Generate using Matlab code a correlated noise for any arbitrary choice of $\rho$ as detailed abhove and compute the sample covariance $\hat{\mathbf{C}}$ from a set of $L$ samples $\{\mathbf{w}[k]\}_{k=1}^{L}$ to be chosen by the student, compare $\hat{\mathbf{C}}$ with the true covariance $\mathbf{C}_{\text{ww}}$ for the choice of $\texttt{N} = \texttt{linespace}(20, 200, 10)$ and the choice of $\rho \in \{0, .5, .99\}$,evaluate the MSE for the difference between the sample covariance $\hat{\mathbf{C}}$ and $\mathbf{C}_{\text{ww}}$. Represent graphically MSE vs N as indicated above for the choice of $\rho$. In case the student believe to be necessary, use Montecarlo analysis by designing properly $L$.

## 1.2 Frequency estimation

Assume that the goal is to evaluate the MSE on frequency estimation vs SNR compared with CRB for some use cases as speficified below. The SNR is defined for every exercise as:

$$SNR = \frac{a_o^2/2}{\sigma_w^2}$$

### 1.2.1 Filter $h[n] = \delta[n]$

Let us assume that the filter is removed, or equivalently estimate the frequency of the sinusoid and evaluate the MSE vs SNR assuming that $\phi_o \sim \mathcal{U}(0, 2\pi)$ varying realization by realization, for the following settings:

- assume that $C_{\text{ww}} = \sigma_w^2 \mathbf{I}$, (or $\rho = 0$) and $N = 100, 500, 1024$ and $\omega = \pi \times \{1/4, 1/2, 1/\sqrt{2}, 0.81\}$

Justify the results, report all figures that are necessary and compare with CRB in all cases.

### 1.2.2 Filter $h[n] \neq \delta[n]$

Let us assume that the filter is the one specified abhove with z-transform $H(z) = \frac{2}{1+0.9z^{-1}}$, estimate the frequency of the sinusoid and evaluate the MSE vs SNR assuming that $\phi_o \sim \mathcal{U}(0, 2\pi)$ varying realization by realization, for the following settings:

- assume that $C_{\text{ww}} = \sigma_w^2 \mathbf{I}$, (or $\rho = 0$) and $N = 100, 500, 1024$ and $\omega = \pi \times \{1/4, 1/2, 1/\sqrt{2}, 0.81\}$

Justify the results, report all figures that are necessary and compare with CRB in all cases.

### 1.2.3 Covariance is correlated

Repeat the exercises above for $h[n] = \delta[n]$ and $h[n] \neq \delta[n]$ considering that $[\mathbf{C}_{\mathrm{ww}}]_{i,j} = \sigma_w^2 \rho^{|i-j|}$, for $\rho = 9/10$. Pay attention to the fact that estimators and CRB needs to be redefined.

## 1.3 Frequency modulation

Generate a frequency modulated sinusoid:

$$x[n] = a \cdot cos(\gamma n^2 + \phi) + w[n]$$

where the instantaneous frequency is $\omega(n) = 2\gamma n$, design the experiment such that the instantaneous frequency variation ranges from $\pi/8$ when $n = 0$ and $3\pi/4$ when $n = N = 22 \times 10^3$ with $C_{\mathrm{ww}} = \sigma_w^2 \mathbf{I}$, phase $\phi$ is a possible degree of freedom to set the proper quadratic phase behaviour. Assume that sampling frequency is 22KHz, play the sound to figure out the signal.

a) Estimate the instantaneous frequency of the sinusoid within a proper choice of the interval of the frequency modulated sinusoid where the signal x[n] can be locally approximated as sinusoid.

b) Derive the MLE for the frequency modulated signal the model for estimating a frequency modulated signal (i.e., $\gamma$) from the set of data detailed above.