CPEN 400Q Lecture 12
Order finding and Shor's algorithm

Friday 17 February 2023

Announcements

- First questions of assignment 2 available very soon (final one still in testing)
- Project group and paper selection due **today** (use Piazza to find teammates)

Last time

We implemented quantum phase estimation.

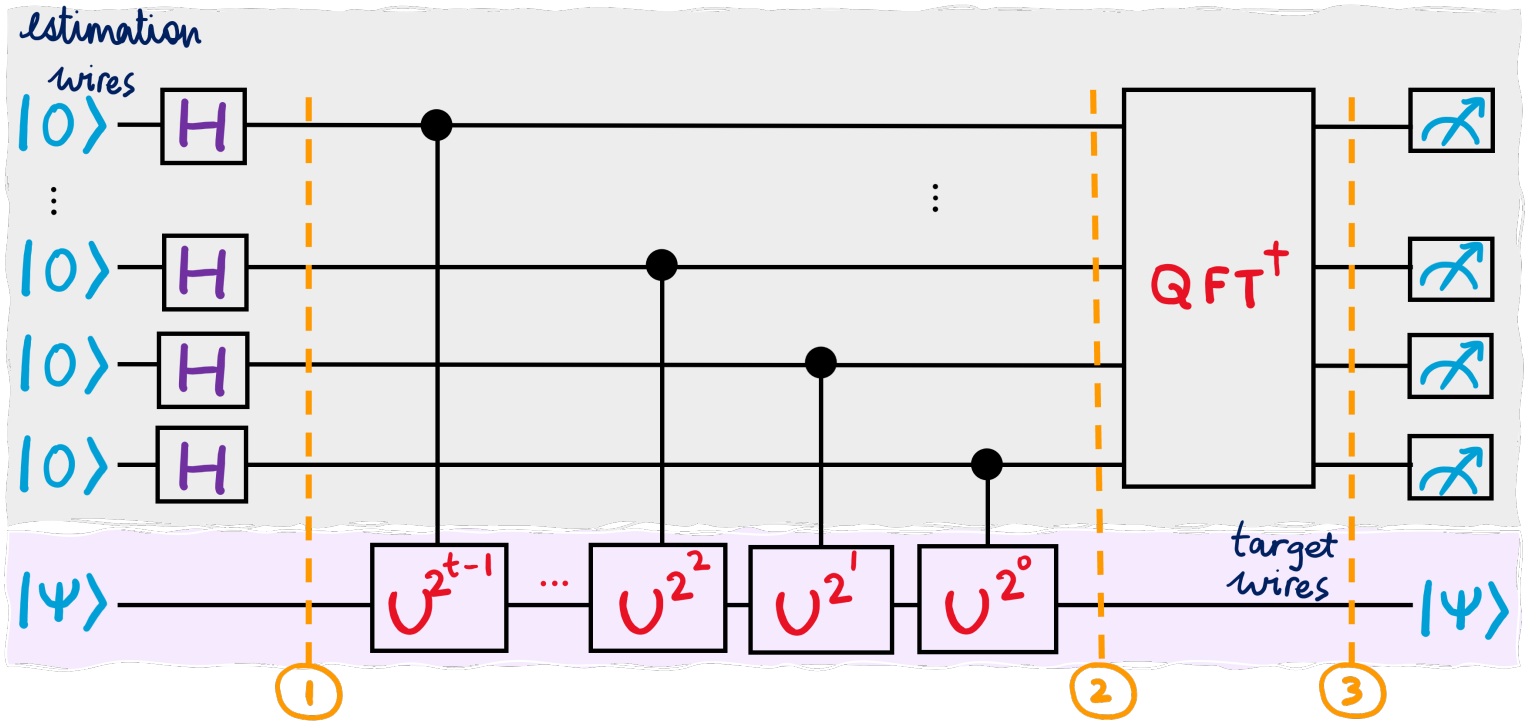
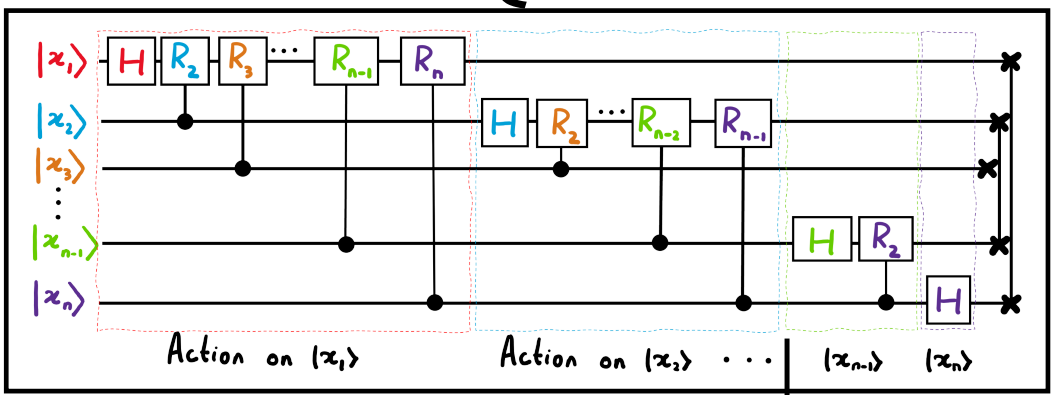


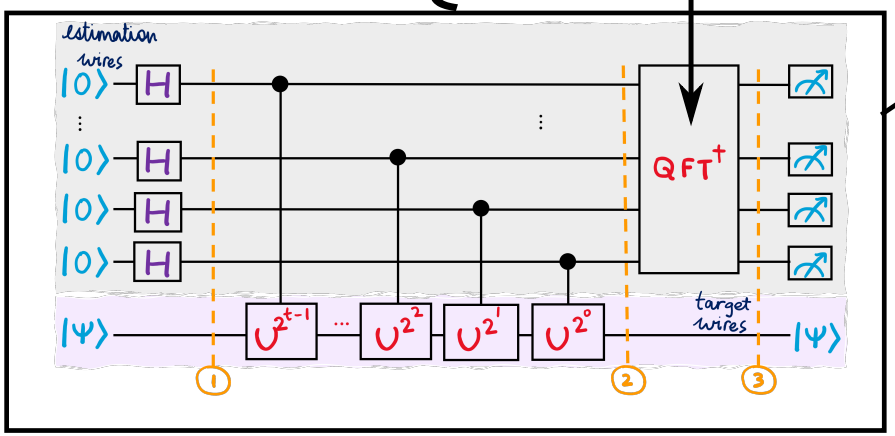
Image credit: Xanadu Quantum Codebook node P.2

Last time

1. QFT



2. QPE



3. Shor

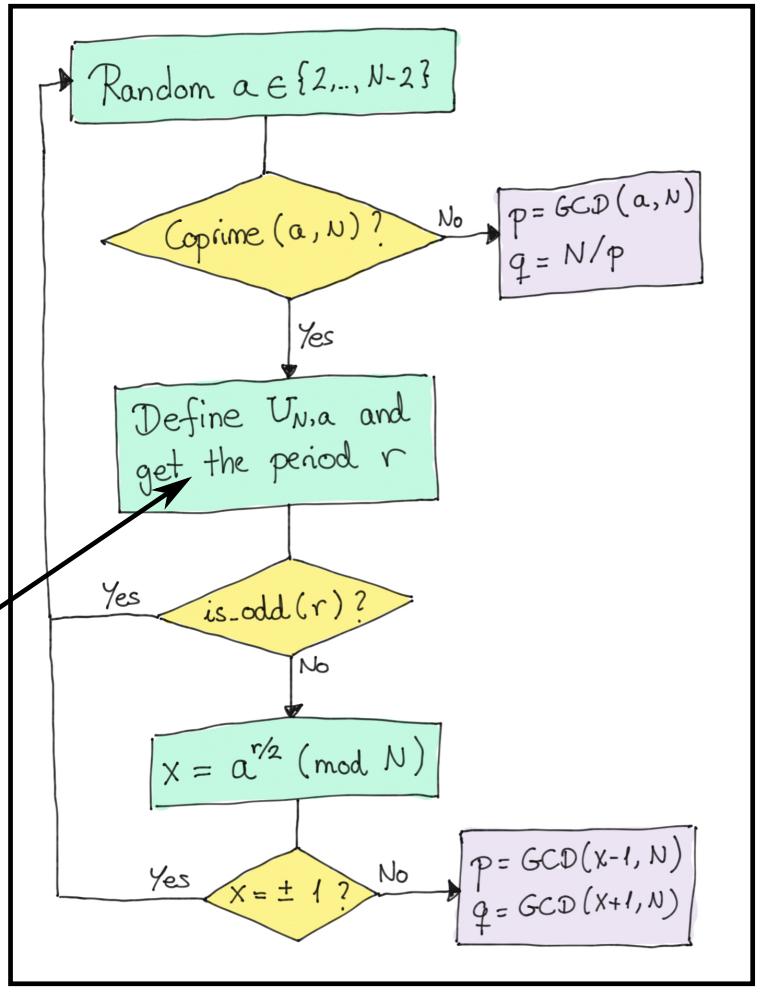


Image credit: Xanadu Quantum Codebook nodes F.3, P.3, S.4

Learning outcomes

- Use QPE to implement the order finding algorithm
- Implement both classical and quantum components of Shor's algorithm to factor numbers
- Describe the steps involved in the RSA cryptosystem, and identify the vulnerability to quantum computers

Order finding on a quantum computer

Suppose we have a function

$$f(x)$$

over the integers modulo N.

If there exists $r \in \mathbb{Z}$ s.t.

$$f(x+r) = f(x) \quad \forall\, x$$

$f(x)$ is periodic with period r.

Order finding on a quantum computer

Suppose

$a \in \mathbb{Z}$

$$f(x) = a^x \bmod N$$

The *order* of a is the smallest m such that

$$f(m) = a^m \bmod N \equiv 1 \bmod N$$

Note that this is also the period:

$$f(x+m) = a^{x+m} = a^x a^m = a^x = f(x)$$

Order finding on a quantum computer

More formally, define

$$f_{N,a}(m) = a^m \equiv 1 \bmod N$$

Define a unitary operation that performs

$$U_{N,a}|k\rangle = |ak \bmod N\rangle$$

comp. basis state

If m is the order of a, and we apply $U_{N,a}$ m times,

$$U_{N,a}^m |k\rangle = |a^m k \bmod N\rangle = |k\rangle$$

$a^m \equiv 1 \bmod N$

So m is also the order of $U_{N,a}$! We can find it efficiently using a quantum computer.

Order finding on a quantum computer

Let U be an operator and $|\phi\rangle$ any state. How do we find the minimum r such that

$$U^r|\phi\rangle = |\phi\rangle$$

→ order

QPE does the trick if we set things up in a clever way:

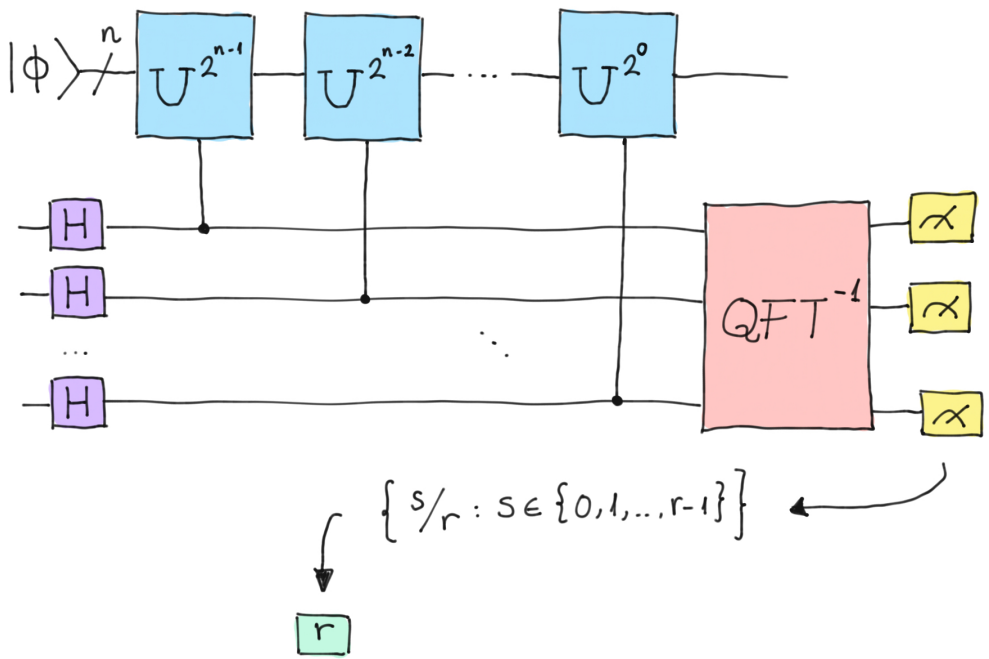


Image credit: Xanadu Quantum Codebook node S.3

Order finding on a quantum computer

$$U^r|\phi\rangle = |\phi\rangle$$

Consider the state

$$|\psi_0\rangle = \frac{1}{\sqrt{r}}\left(|\phi\rangle + U|\phi\rangle + U^2|\phi\rangle + \dots + U^{r-1}|\phi\rangle\right)$$

If we apply U to this:

$$U|\psi_0\rangle = \frac{1}{\sqrt{r}}\left(U|\phi\rangle + U^2|\phi\rangle + \dots + U^r|\phi\rangle\right)$$

$$= \frac{1}{\sqrt{r}}\left(U|\phi\rangle + U^2|\phi\rangle + \dots + |\phi\rangle\right)$$

$$= 1\cdot|\psi_0\rangle$$

eigenstate

Order finding on a quantum computer

Now consider the state

$$|\psi_1\rangle = \frac{1}{\sqrt{r}}\left(|\phi\rangle + e^{\frac{-2\pi i}{r}} U|\phi\rangle + e^{\frac{-2\cdot 2\pi i}{r}} U^2|\phi\rangle + \cdots + e^{\frac{-(r-1)\cdot 2\pi i}{r}} U^{r-1}|\phi\rangle\right)$$

If we apply U to this:

$$U|\psi_1\rangle = \frac{1}{\sqrt{r}}\left(U|\phi\rangle + e^{\frac{-2\pi i}{r}} U^2|\phi\rangle + e^{\frac{-2\cdot 2\pi i}{r}} U^3|\phi\rangle + \cdots\right) =$$

$$= \frac{e^{\frac{2\pi i}{r}}}{\sqrt{r}}\left(e^{\frac{-2\pi i}{r}} U|\phi\rangle + e^{\frac{-2\cdot 2\pi i}{r}} U^2|\phi\rangle + \cdots\right)$$

$$= e^{\frac{2\pi i}{r}}|\psi_1\rangle$$

Order finding on a quantum computer

This generalizes to $|\Psi_s\rangle$

$$|\Psi_s\rangle = \frac{1}{\sqrt{r}}\left(|\phi\rangle + e^{-s\cdot\frac{2\pi i}{r}} U|\phi\rangle + \ldots + e^{-(r-1)s\cdot\frac{2\pi i}{r}} U^{r-1}|\phi\rangle\right)$$

It has eigenvalue

$$U|\Psi_s\rangle = e^{\frac{2\pi i s}{r}}|\Psi_s\rangle$$

$$U^r|\phi\rangle = |\phi\rangle$$

Idea: if we can create *any* one of these $|\Psi_s\rangle$, we could run QPE and get an estimate for s/r, and then recover r.

Order finding on a quantum computer

Problem: to construct any $|\Psi_s\rangle$, we would need to know r in advance!

Solution: construct the uniform superposition of all of them.

$$|\psi\rangle = \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |\psi_s\rangle$$

But what does this equal?

Order finding on a quantum computer

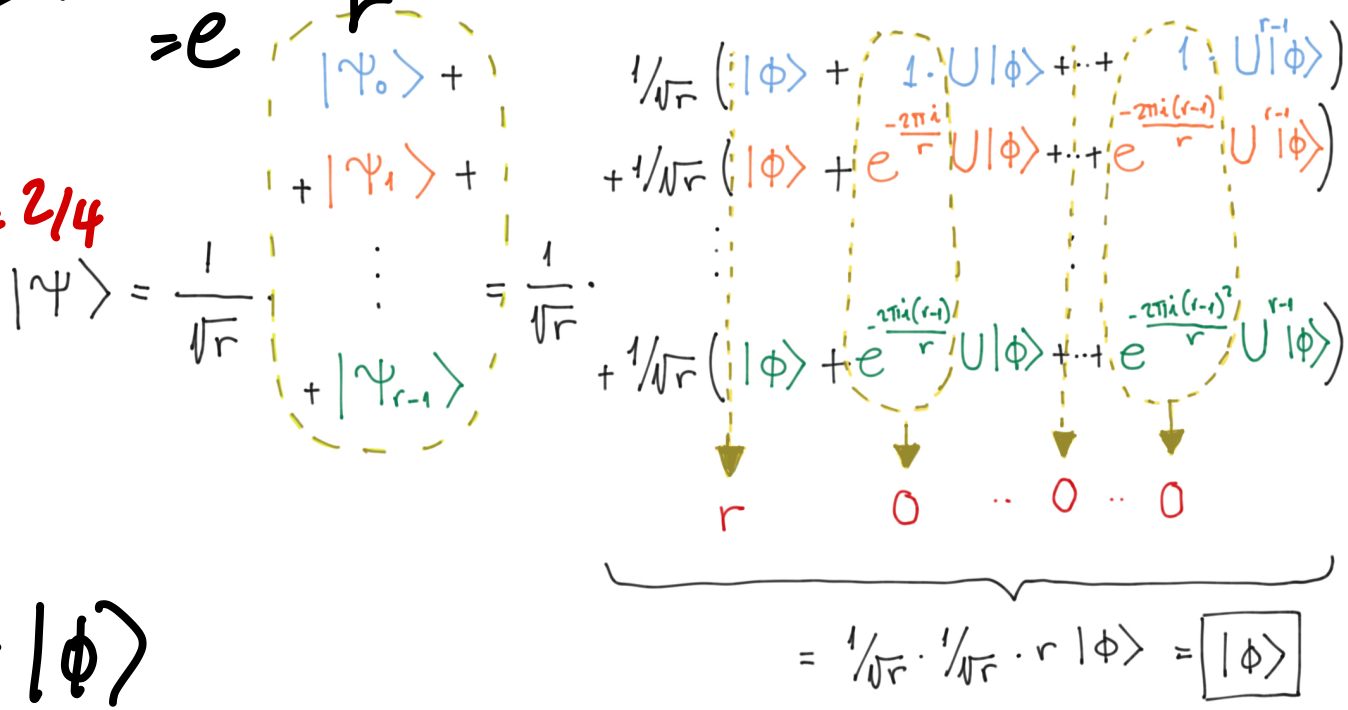
The superposition of all $|\Psi_s\rangle$ is just our original state $|\phi\rangle$!

$\frac{s}{r}$ $e^{2\pi i \theta k} = e^{2\pi i \frac{s}{r}}$

0.5 1/2 = 2/4

0.75 3/4

0.25 1/4



$U^r|\phi\rangle = |\phi\rangle$

↳ $|1\rangle$ $U|k\rangle = |ak \bmod N\rangle$ $U^r|1\rangle = |a^r \bmod N\rangle = |1\rangle$

If we run QPE, the output will be s/r for one of these states.

Image credit: Xanadu Quantum Codebook node S.3

Overview

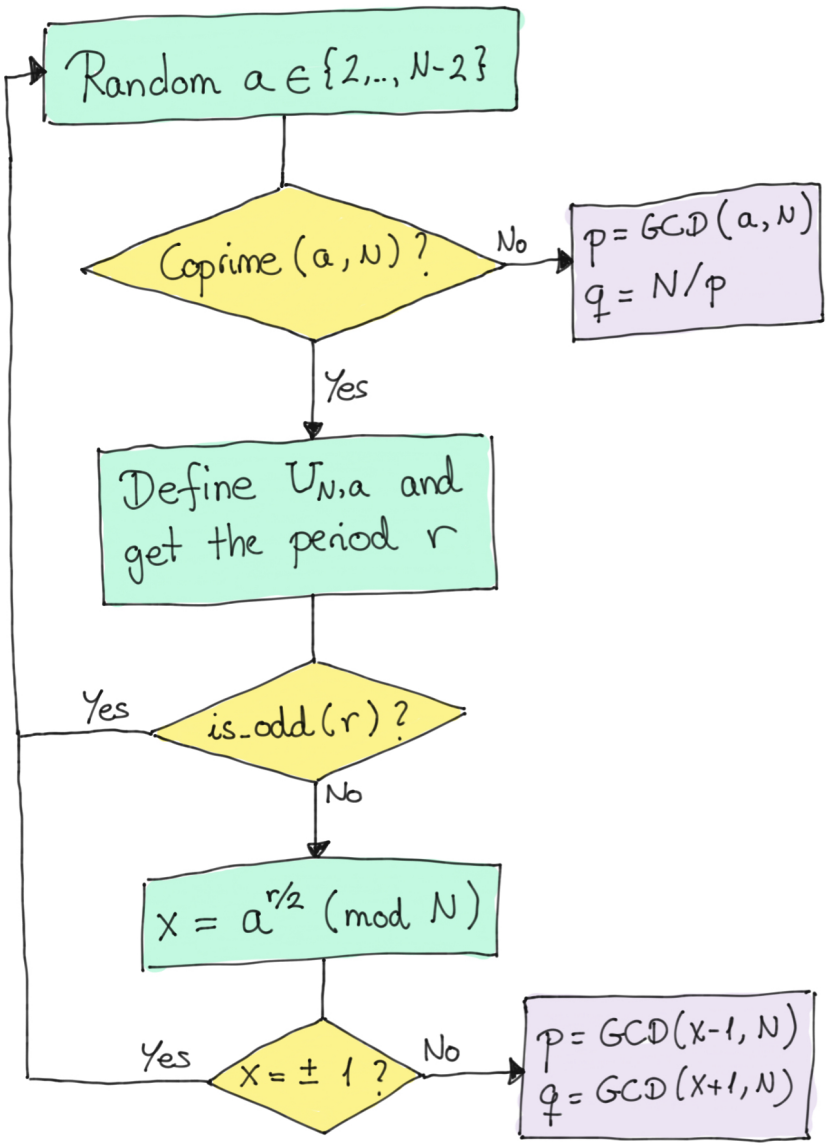
Shor's algorithm is used to factor some number N into

$$N = pq$$

where p and q are prime.

A quantum computer runs order finding, and the result is used to obtain p and q.

The rest of the algorithm is classical.



Non-trivial square roots

Idea: find a *non-trivial square root* of N, i.e., some $x \neq \pm 1$ s.t.

$$x^2 \equiv 1 \mod N$$

If we find such an x, then we know

$$x^2 \equiv 1 \mod N$$
$$x^2 - 1 \equiv 0 \mod N$$
$$(x-1)(x+1) \equiv 0 \mod N$$

This means that

$$(x-1)(x+1) = kN$$

$N = p \cdot q$

for some integer k.

Non-trivial square roots

If

$$(x-1)(x+1) = kN = k \cdot p \cdot q$$

then $x-1$ is a multiple of one of p or q, and $x+1$ is a multiple of the other. If

$$x-1 = sp$$
$$x+1 = tq$$

$$N = pq$$

we can compute the values of p and q by finding their *gcd* with N:

$$p = \gcd(x-1, N)$$
$$q = \gcd(x+1, N)$$

But... how do we find such an x?

Non-trivial square roots and factoring

$$x^2 \equiv 1 \bmod N$$

It's actually okay to find any *even* power of x for which this holds:

$$x^r = x^{2r'} = (x^{r'})^2 = 1 \bmod N$$

We can use order finding to find such an r, and it is an even number, then we can find an x and factor N.

use to find p, q

Shor's algorithm

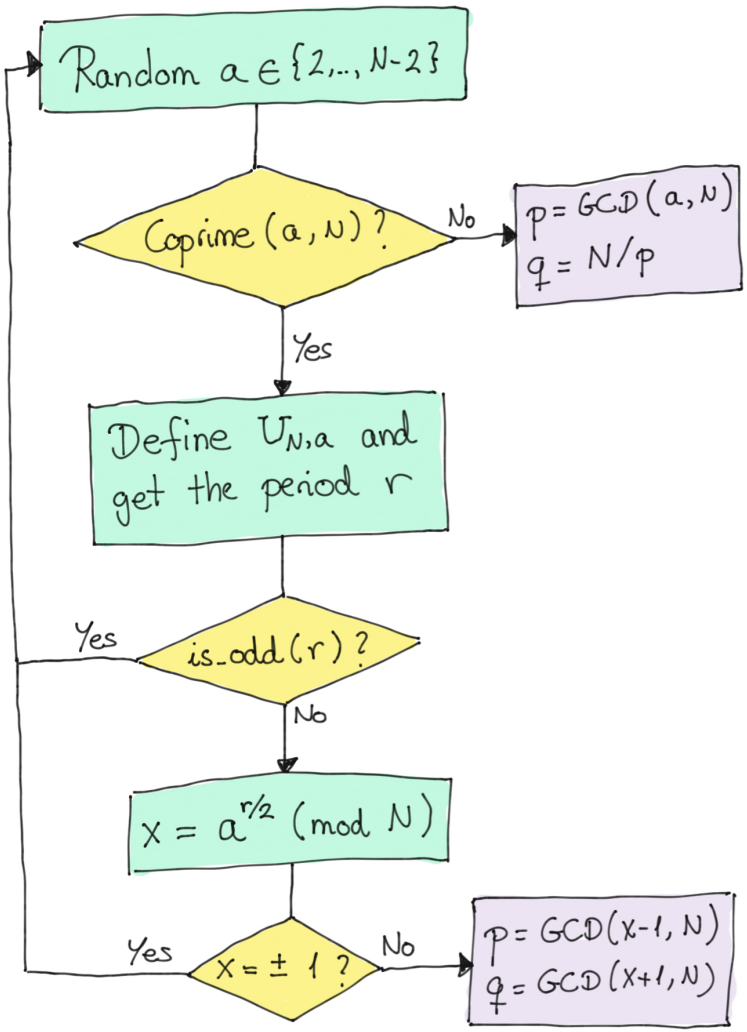
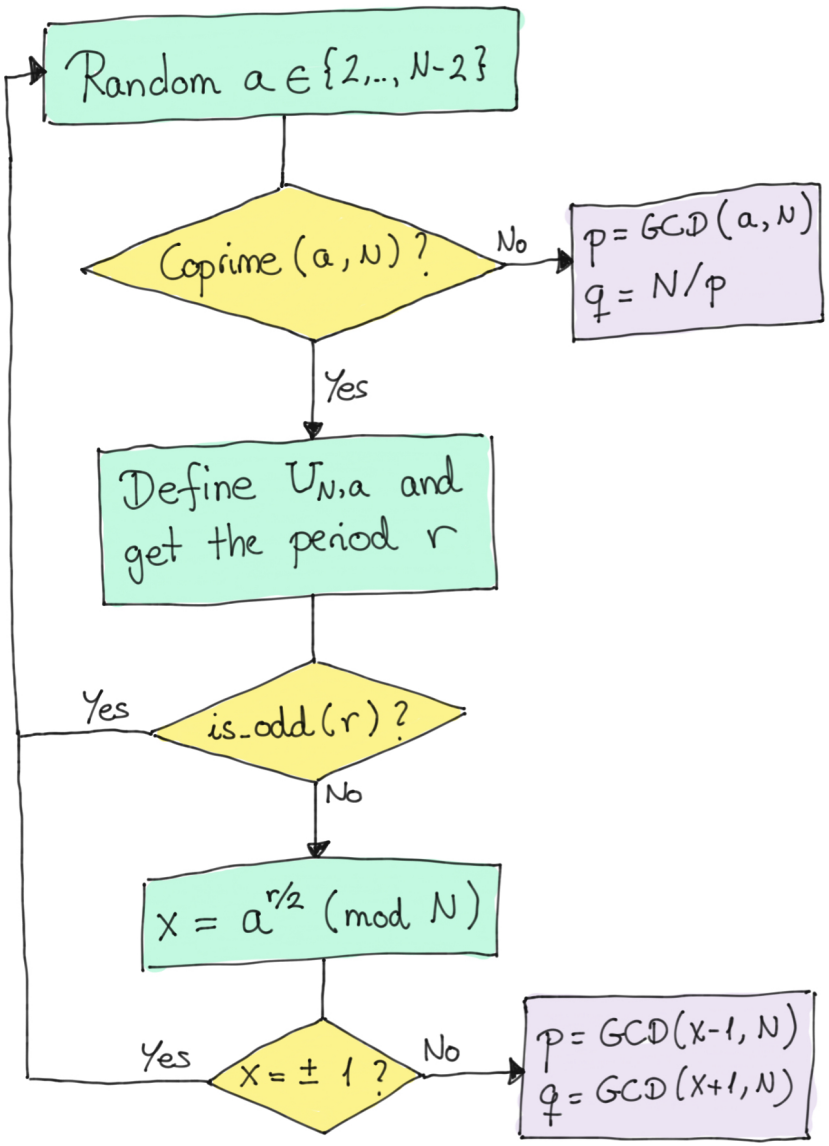


Image credit: Xanadu Quantum Codebook node S.4

Is this really efficient?

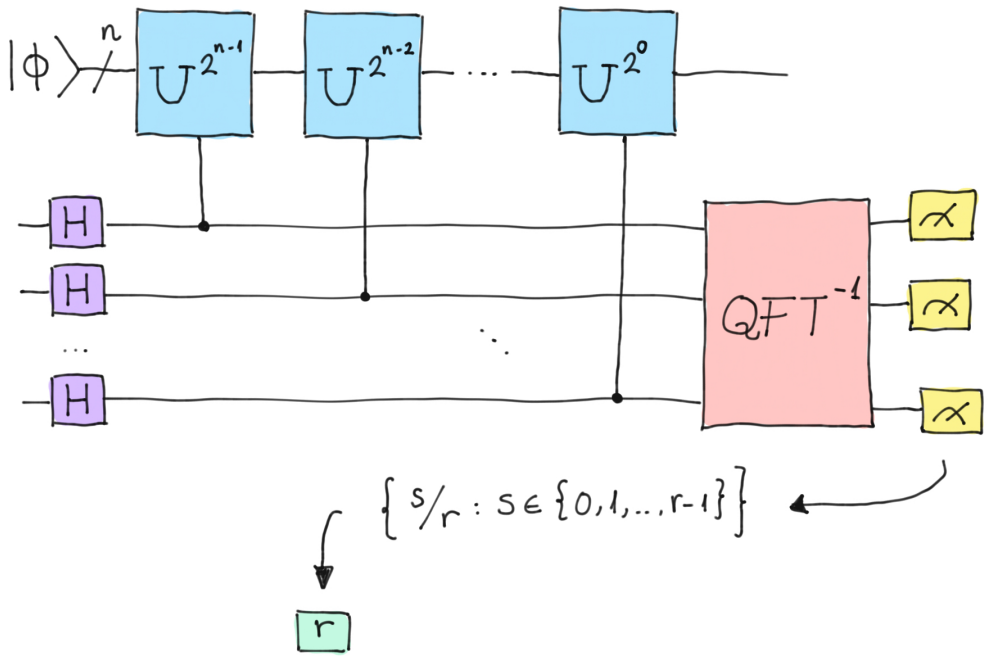
GCD: polynomial w/Euclid's algorithm

Modular exponentiation: can use exponentiation by squaring, other methods to reduce number of operations and memory required



Is this really efficient?

Quantum part: let $L = \lceil \log_2 N \rceil$.



QFT: polynomial in number of qubits $O(L^2)$

Controlled-U gates: implemented using something called *modular exponentiation* in $O(L^3)$ gates.

Public-key cryptosystems

Two different types of cryptosystems:

- Symmetric: the key used to decode is the same as (or can easily be obtained from) the one used to encode
- Asymmetric / public-key: the key used to decode is different than the one used to encode

Both are used in modern infrastructure and each has its own advantages/disadvantages, attack vectors, and vulnerabilities.

RSA (Rivest–Shamir–Adleman) is a public-key cryptosystem.

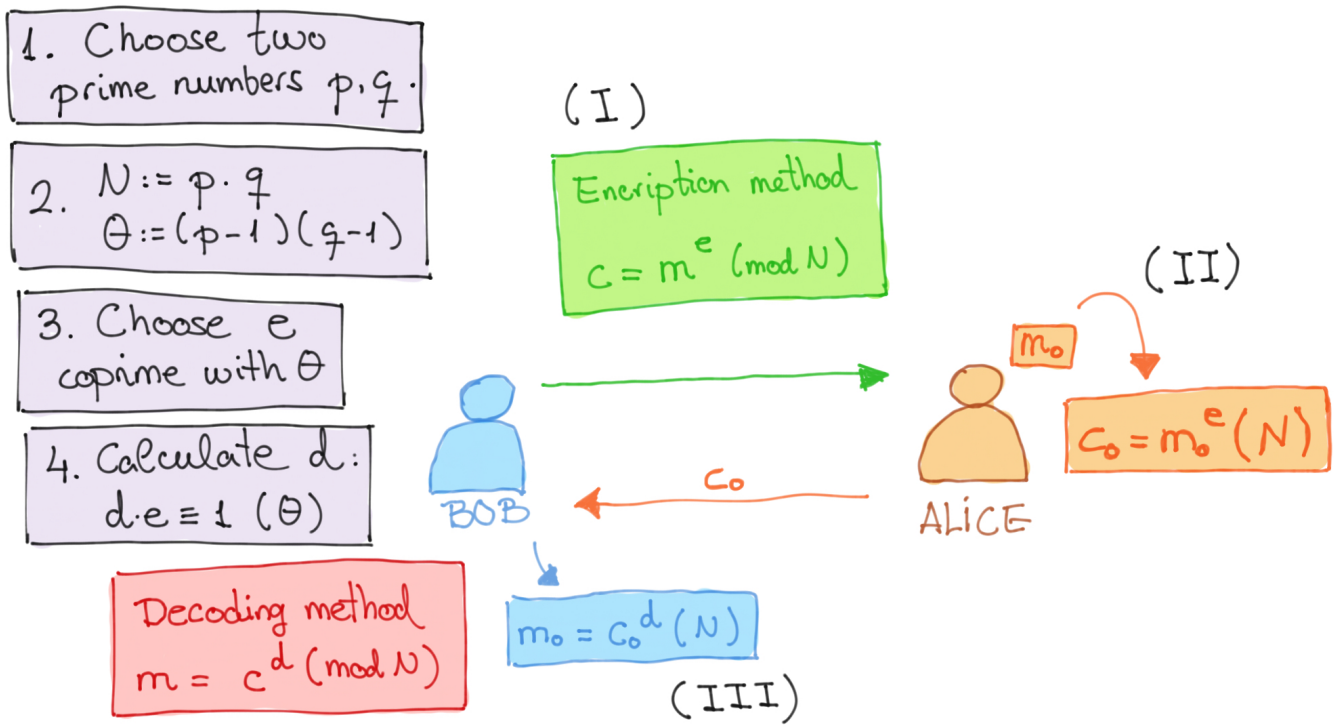


Image credit: Xanadu Quantum Codebook node S.5

RSA: brief review of number theory concepts

The math behind RSA involves modular arithmetic and a few other number-theoretic ideas:

Greatest common divisor (gcd): $gcd(a, b)$ is the largest integer factor that divides perfectly into both a and b

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Fermat's little theorem: if p prime and a is an integer, then $a^p = a \bmod p$. If a is not divisible by p, then $a^{p-1} \equiv 1 \bmod p$.

For the full notes on RSA, see the file lecture-12-supplement.pdf on GitHub

Step 1

Choose two *prime numbers*, p and q.

Step 2

Compute:

$$N = p \cdot q$$
$$\theta = (p-1)(q-1)$$

Step 3

Choose a value e that is *co-prime* with θ.

Step 4

Compute the inverse of e mod θ, i.e., find d s.t.

$$d \cdot e \equiv 1 \bmod \theta$$

The *public key* is the pair (e, N).

The *private key* is the pair (d, N).

Encoding

Acquire the public key (e, N) of the party you wish to send something to. To send the message m, encode it as

$$c = m^e \bmod N$$

Decoding

If you receive c and have the private key (d, N), decode like so:

$$c^d \bmod N = (m^e)^d \bmod N = m$$

Two cases to consider to understand why this works.

Since $ed = 1 \bmod \theta$, there exists integer k such that $ed = 1 + k\theta$.

Case 1: m co-prime with N

See lecture-12-supplement.pdf

It is a known result in number theory that if m and N are co-prime, then $m^\theta \equiv 1 \bmod N$ where $\theta = (p-1)(q-1)$. Thus,

Case 2: m not co-prime with N

Then, $gcd(m, N) > 1$. Must be p or q, because those are the only two factors of N.

Suppose $gcd(m, N) = p$. Then, p also divides m,

$$m \equiv 0 \mod p, \quad m^{ed} \equiv 0 \equiv m \mod p$$

But q does not. q is prime, so $gcd(q, m) = 1$. So by Fermat's little theorem,

$$m^{(q-1)} \equiv 1 \mod q, \quad m^{(p-1)(q-1)} = m^{\theta} \equiv 1 \mod q$$

Case 2: m not co-prime with N

Again, since $ed = 1 \bmod \theta$, there exists k such that $ed = 1 + k\theta$.

$$m^{ed} \bmod q = mm^{k\theta} \bmod q$$
$$= m \bmod q$$

So we have

$$m^{ed} \equiv m \bmod p$$
$$m^{ed} \equiv m \bmod q$$

It follows that

$$m^{ed} \equiv m \bmod N$$

RSA and factoring

- To decrypt the message, we must learn d
- We know e, and that $de \equiv 1 \bmod \theta$

We have only e, and N.

However, N and θ are based on the same two prime numbers:

$$N = p \cdot q$$
$$\theta = (p-1)(q-1)$$

If we can factor $N = pq$, we find θ and can decode the message!

The security of RSA relies on the fact that factoring for large numbers is computationally intractable.

Computational complexity of breaking RSA

Current recommended key (N) sizes are 2048- and 4096-bit.

The largest key size cracked so far is 829 bits in 2020. See: `https://en.wikipedia.org/wiki/RSA_numbers`

Best-known classical algorithm:

General number field sieve

From Wikipedia, the free encyclopedia

In number theory, the **general number field sieve** (**GNFS**) is the most efficient classical algorithm known for factoring integers larger than 10^{100}. Heuristically, its complexity for factoring an integer n (consisting of $\lfloor \log_2 n \rfloor + 1$ bits) is of the form

$$\exp\left(\left(\sqrt[3]{\frac{64}{9}} + o(1)\right)(\ln n)^{\frac{1}{3}}(\ln \ln n)^{\frac{2}{3}}\right) = L_n\left[\frac{1}{3}, \sqrt[3]{\frac{64}{9}}\right]$$

Computational complexity of breaking RSA

On a quantum computer, there exists an algorithm that can help us solve the problem in *polynomial time*.

But it is still going to be a long time before that happens.

RSA-2048	**Old estimates**				**Current estimates**			
p_g	n_ℓ	n_p	quantum resources	time	n_ℓ	n_p	quantum resources	time
10^{-3}	6190	19.20	1.17	1.46	8194	22.27	0.27	0.34
10^{-5}	6190	9.66	0.34	0.84	8194	8.70	0.06	0.15

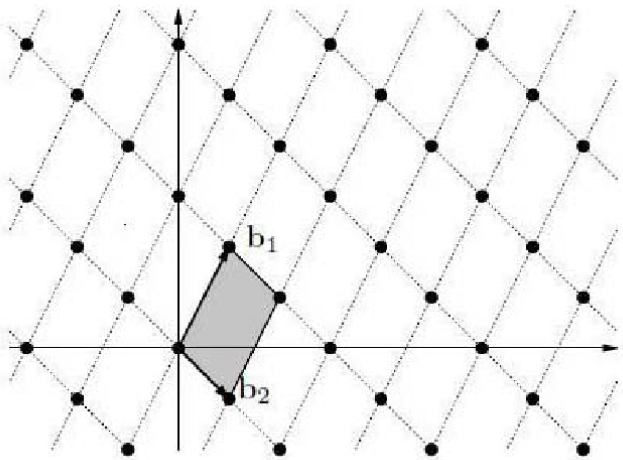
Table 2. RSA-2048 security estimates. Here n_ℓ denotes the number of logical qubits, n_p denotes the number of physical qubits (in millions), time denotes the expected time (in hours) to break the scheme, and *quantum resources* (*quantum resources*) are expressed in units of megaqubitdays. The corresponding classical security parameter is 112 bits.

Image: V. Gheorghiu and M. Mosca, *A resource estimation framework for quantum attacks against cryptographic functions - recent developments.* Feb. 15 2021.

What can / should we do about this?

RSA will not be cracked tomorrow. BUT, it's only a matter of time. We need to use this time wisely to re-tool our infrastructure. One option: post-quantum cryptography.

Figure 2 A 2-dimensional lattice generated by the basis $B = [b_1, b_2]$



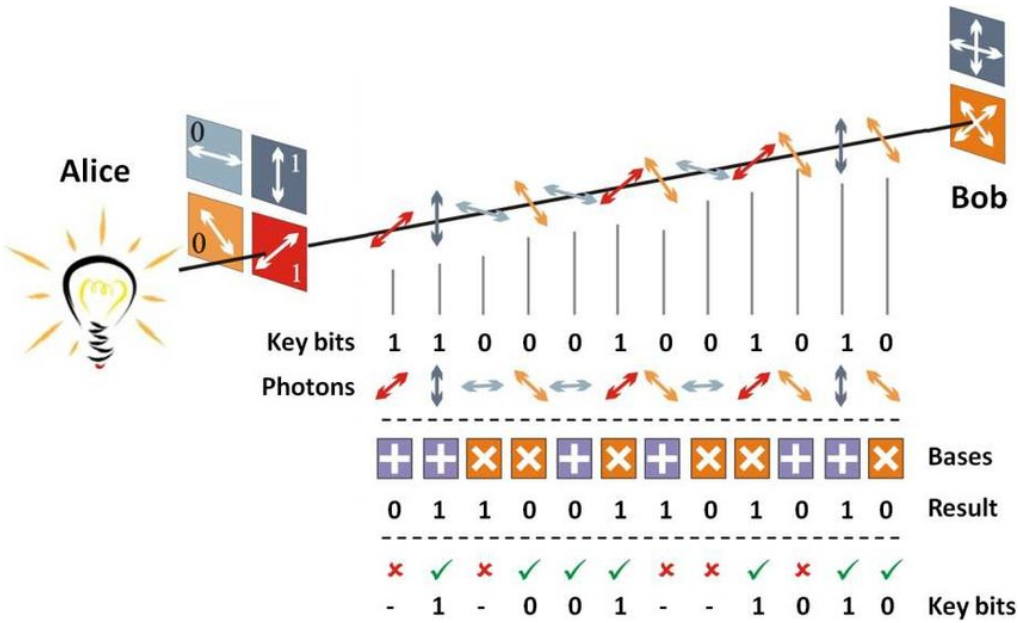
The LWE problem was firstly defined by Regev (2005). In

Use hard problems that *don't* have (known) efficient quantum solutions.

Image credit: G. Zhang, J. Qin. *Lattice-based threshold cryptography and its applications in distributed cloud computing*. Int. J. High Perform. Comput. Netw. 2015

What can / should we do about this?

Symmetric crypto remains largely secure; use quantum key distribution to perform key exchange for it rather than RSA.



Theoretically secure, but can be challenging to implement, and other potential attack vectors such as hardware.

Image credit: Carrasco-Casado, Marmol, Denisenko. (2016) *Free-Space Quantum Key Distribution*. Optical Wireless Communications - An Emerging Technology (pp.589-607)

What can / should we do about this?

Ethical dilemma: if we know quantum computers have malicious applications like breaking our cryptography infrastructure

- Should we still build them?
- Who should get to build them?
- Who should get to *use* them?

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Next time

Content:

- Moving back to variational algorithms

Action items:

1. Work on A2 once available
2. Start working on prototype implementation for project

Recommended reading:

- Codebook nodes S.1-S.5
- Nielsen & Chuang 5.3, Appendix A.5