CPEN 400Q Lecture 17
Mixed states, noise, and quantum channels

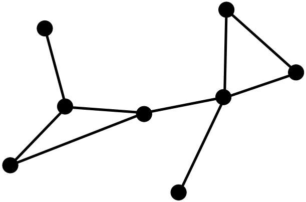
Monday 13 March 2023

Announcements

- Quiz 7 beginning of class today
- Assignment 2 due tonight at 23:59
- Friday class: video lectures on Canvas. Choice of three:
  - compilation and quantum transforms
  - error mitigation
  - Hamiltonian simulation

Last time

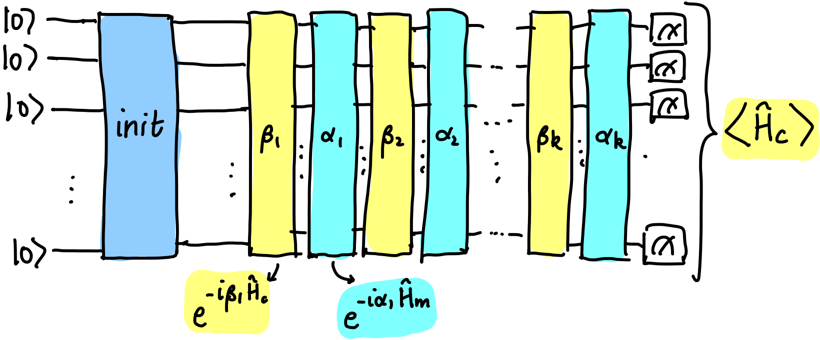
We mapped the vertex cover problem from binary variables to qubits and designed appropriate cost and mixer Hamiltonians.



$$\hat{H}_c = \sum_{ij \in E} (Z_i + Z_j + Z_i Z_j) - 2 \sum_{i \in V} Z_i$$

$$\hat{H}_m = \sum_i X_i$$

We solved the problem using the quantum approximate optimization algorithm (QAOA).

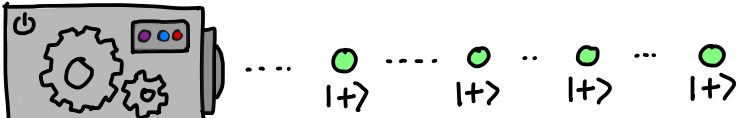


Learning outcomes

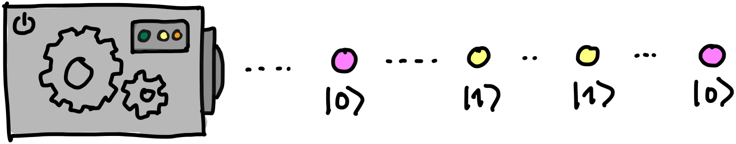
- Define a *mixed state*, and express quantum states using density matrices
- Describe the effects of common noise channels on qubit states
- Add noise to quantum circuits in PennyLane

These topics will be helpful for many of your project papers!

Suppose we have two different “boxes” that shoot particles:



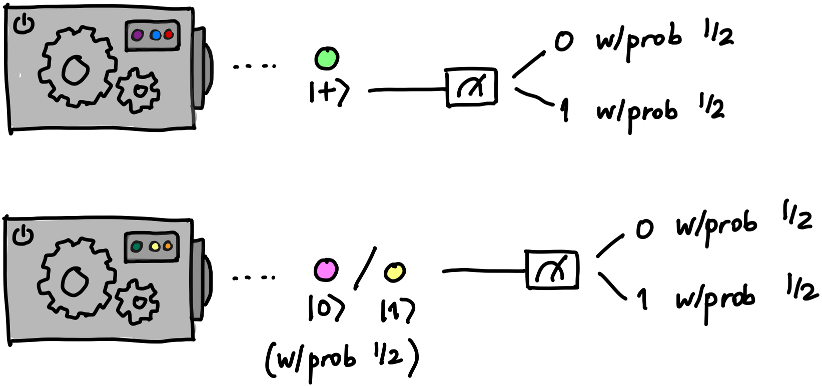
$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$



Are these the same?

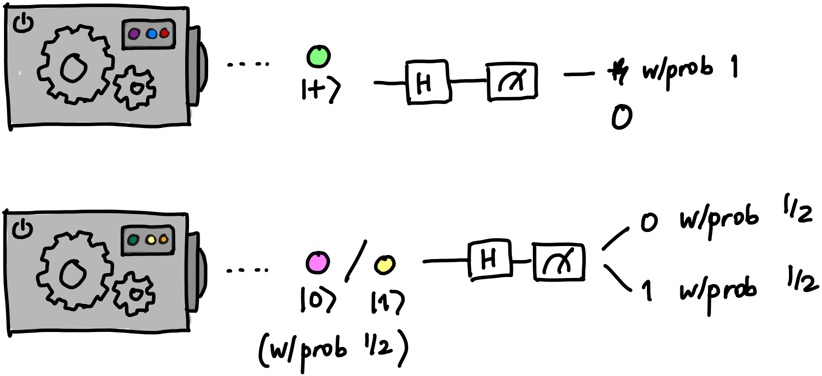
Mixed states

If we measure in the computational basis, it looks like they are.



Mixed states

But if we measure in the Hadamard basis, they are not!



What is the second box doing?

Mixed states

The second box is outputting something called a **mixed state**.

A **pure state** can be expressed as a single ket vector, e.g.,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

A **mixed state** must be expressed as a *probabilistic mixture of pure states* (it describes an ensemble of states).

$$? = ???$$

... what does that look like?

Mixed states cannot be represented as ket vectors. Instead, we use a matrix representation called a **density matrix**.

The density matrix of a pure state $|\psi\rangle$ is

rho $\leftarrow$ $$\rho = |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$$

Exercise: what are the density matrices for $|0\rangle$ and $|1\rangle$?

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

Density matrices of mixed states are linear combinations of density matrices of pure states:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| , \quad \sum_i p_i = 1$$

$(p_i \text{ real})$

Exercise: A system prepares $|+\rangle$ with probability $1/3$, and $|0\rangle$ with probability $2/3$. What is its state?

$$\rho = \begin{pmatrix} 5/6 & 1/6 \\ 1/6 & 1/6 \end{pmatrix} = \frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|0\rangle\langle 0|$$

$$= \frac{1}{3} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Density matrices

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \quad \Sigma : \text{trace}$$

Density matrices have some nice properties.

- they are Hermitian (real EVs, $B = B^\dagger$)
- they have trace 1
- they are positive semi-definite (all eigenvalues are ≥ 0)
- (for pure states only) they are projectors, i.e., $\rho^2 = \rho$

Check with our example:

$$\rho = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

- clearly Hermitian
- $\mathrm{Tr}\rho = 5/6 + 1/6 = 1$
- eigenvalues are 0.872678 and 0.127322, both ≥ 0
- not pure, so $\rho^2 \neq \rho$

Fun activity: show properties hold for general $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$

Working with density matrices and mixed states

We can do all the normal things we do to pure states (i.e., operations, measurements) with mixed states as well.

For a pure state $|\psi\rangle$ and operation U,

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$$

As mixed states,

$$\rho = |\psi\rangle\langle\psi| \longrightarrow \rho' = |\psi'\rangle\langle\psi'|$$

$$= \left(U|\psi\rangle\right)\left(\langle\psi|U^\dagger\right)$$

$$= U\rho U^\dagger$$

More generally,

$$\rho \to \rho' = U \rho U^\dagger$$

$$= U\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) U^\dagger$$

$$= \sum_i p_i \, U|\psi_i\rangle\langle\psi_i| U^\dagger$$

Recall that for a pure state $|\psi\rangle$, the probability of measuring and observing it in state $|\varphi\rangle$ is

$$\Pr(\varphi) = |\langle\varphi|\psi\rangle|^2$$

We can rewrite this...

$$\begin{aligned}\Pr(\varphi) &= |\langle\varphi|\psi\rangle|^2 \\ &= \langle\varphi|\psi\rangle\langle\psi|\varphi\rangle \\ &= \langle\psi|\underbrace{\varphi\rangle\langle\varphi}|\psi\rangle \\ &= \langle\psi|\rho_\varphi|\psi\rangle\end{aligned}$$

$|\varphi\rangle\langle\varphi|$ is the density matrix of $|\varphi\rangle$, which is a *projector*. We are projecting $|\psi\rangle$ onto $|\varphi\rangle$, and then measuring the overlap with $|\psi\rangle$.

Measurement is performed w.r.t. a basis $\{|\varphi_i\rangle\}$; there are multiple possible outcomes:

$$\Pr(\text{outcome } i) = |\langle\varphi_i|\psi\rangle|^2$$

$$\Pi_i = |\varphi_i\rangle\langle\varphi_i|$$

For mixed states, measurement outcome probabilities follow the **Born rule**:

$$\Pr(\text{outcome } i) = \mathrm{Tr}(\Pi_i \rho)$$

where the set $\{\Pi_i\}$ is called a **positive operator-valued measure (POVM)**. The elements of the POVM satisfy

$$\sum_i \Pi_i = I$$

Can see that this reduces to our original projective measurement in the case where ρ is a pure state...

$$\Pr(\text{outcome } i) = \mathrm{Tr}(\Pi_i \rho)$$
$$= \mathrm{Tr}\left(\Pi_i |\psi\rangle\langle\psi|\right)$$

For an $m \times m$ matrix A,

$$\mathrm{Tr}(A) = \sum_{k=0}^{m-1} \langle k|A|k\rangle$$

* I didn't show this part in class; but try it!

Mixed states and measurements

Can see that this reduces to our original projective measurement in the case where ρ is a pure state...

$$\Pr(\text{outcome } i) = \mathrm{Tr}(\Pi_i \rho)$$

$$= \mathrm{Tr}(\Pi_i |\psi\rangle\langle\psi|)$$

$$= \sum_{k=0}^{m-1} \langle k|\Pi_i|\psi\rangle\langle\psi|k\rangle$$

just numbers

$$= \sum_{k=0}^{m-1} \langle\psi|k\rangle\langle k|\Pi_i|\psi\rangle$$

$$= \langle\psi|\underbrace{\left(\sum_{k=0}^{m-1}|k\rangle\langle k|\right)}_{=\text{Identity}}\Pi_i|\psi\rangle$$

$$= \langle\psi|\Pi_i|\psi\rangle$$

Example: $\{|+\rangle\langle+|, |-\rangle\langle-|\}$.

$$\Pr(i) = \mathrm{Tr}(\Pi_i \rho)$$

The Born rule tells us that, given a state ρ,

$$\Pr(+) = \mathrm{Tr}(|+\rangle\langle+|\rho)$$

$$\Pr(-) = \mathrm{Tr}(|-\rangle\langle-|\rho)$$

$$\sum_i \Pi_i = I$$

Exercise: Show that $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ form a legit POVM.

$$|+\rangle\langle+| + |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mixed states and measurements

Exercise: Suppose we prepare our system in the state from earlier, ($|+\rangle$ with probability $1/3$, $|0\rangle$ with probability $2/3$). What is the probability of obtaining the POVM outcome $\Pi_+ = |+\rangle\langle+|$?

$$Pr(+) = 2/3$$

$$= \mathrm{Tr}\left(|+\rangle\langle+| \cdot \left(\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|0\rangle\langle 0|\right)\right)$$

Try:
- matrices
- algebra w/ kets

$$= \mathrm{Tr}\left(\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}\cdot|+\rangle\underbrace{\langle+|0\rangle}_{=\frac{1}{\sqrt{2}}}\langle 0|\right)$$

$$= \mathrm{Tr}\left(\frac{1}{3}|+\rangle\langle+| + \frac{2}{3\sqrt{2}}|+\rangle\langle 0|\right)$$

trace is linear

$$= \frac{1}{3}\mathrm{Tr}\left(|+\rangle\langle+|\right) + \frac{2}{3\sqrt{2}}\mathrm{Tr}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\right)$$

$$|+\rangle\langle 0| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$$

$$= \frac{1}{3} + \frac{2}{3\sqrt{2}}\cdot\frac{1}{\sqrt{2}}$$

$$= 2/3$$

Now, remember how we computed expectation values from samples back in one of the early classes:

$$\langle X \rangle = \frac{1 \cdot (\#\ {+1}\ \text{eigvals}) + (-1) \cdot (\#\ {-1}\ \text{eigvals})}{\text{num samples}}$$

$$= 1 \cdot \Pr(+) + (-1) \cdot \Pr(-)$$

$$\langle X \rangle = 1 \cdot Pr(+) + (-1) \cdot Pr(-)$$

We can compute these probabilities in terms of the trace and ρ...

$$\langle X \rangle = \mathrm{Tr}(|+\rangle\langle+|\rho) - \mathrm{Tr}(|-\rangle\langle-|\rho)$$

$$= \mathrm{Tr}((|+\rangle\langle+| - |-\rangle\langle-|)\rho)$$

$$= \mathrm{Tr}(X\rho)$$

$$\langle X \rangle = \mathrm{Tr}(|+\rangle\langle+|\rho) - \mathrm{Tr}(|-\rangle\langle-|\rho)$$

$$= \mathrm{Tr}\left(\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\rho\right)$$

$$= \mathrm{Tr}(X\rho)$$

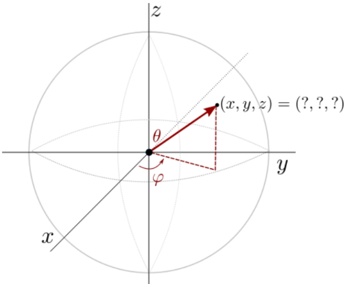
Mixed states and measurements

We can do the same for Y and Z: We can compute these probabilities in terms of the trace and ρ...

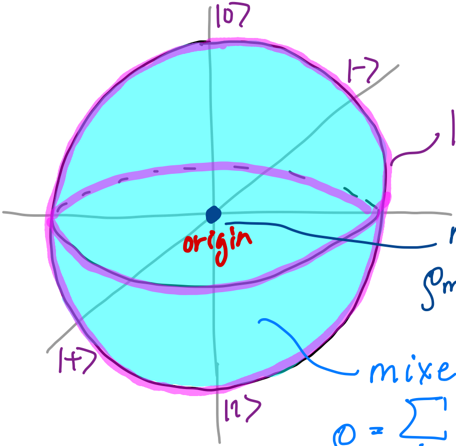
$$\langle X \rangle = \mathrm{Tr}(X\rho)$$
$$\langle Y \rangle = \mathrm{Tr}(Y\rho)$$
$$\langle Z \rangle = \mathrm{Tr}(Z\rho)$$

Remember from assignment 1:

Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:



Mixed states live *in* the Bloch sphere!



pure states

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

maximally mixed state

$\rho_{max} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1}$

mixed states

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

Mixed states

☆ We didn't discuss this in class; will show next time

More formally, we can write any ρ as

$$\rho = \frac{1}{2} I + \frac{a_x}{2} X + \frac{a_y}{2} Y + \frac{a_z}{2} Z$$

where $a_P = \mathrm{Tr}(P\rho) = \langle P \rangle$.

The case where $a_x = a_y = a_z = 0$ is the **maximally mixed state**.

(Note that all of this generalizes to multiple qubits as well)

Noise occurring in quantum systems is represented by **quantum channels**.

A quantum channel Φ *maps* states to other states.

$$\rho \to \rho' = \Phi(\rho)$$

More formally, quantum channels are linear CPTP (Completely Positive, Trace-Preserving) maps.

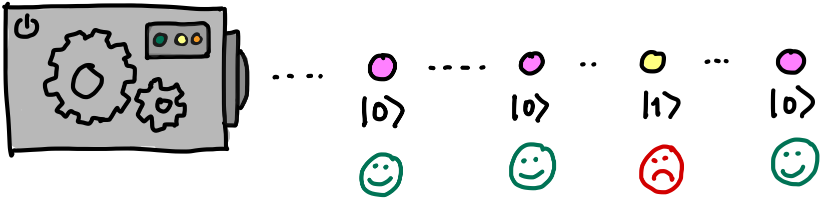
Example: applying a unitary U is a channel, $\mathcal{U}$.

$$\rho \to \Phi(\rho) = U\rho U^\dagger$$

The bit flip channel

Suppose a "bit flip" (Pauli X) error occurs with probability p.

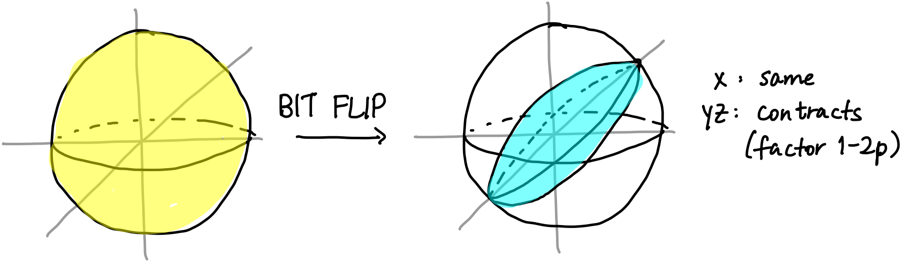
$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot X\rho X$$



We stopped here-ish on Monday.

The bit flip channel

We can visualize the effects of such a channel by observing how it deforms the Bloch sphere.



Exercise: Suppose we prepare a system in the state

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

However, a bit flip error occurs with probability $p = 0.02$. What is the probability of measuring (in the computational basis) and obtaining the $|0\rangle$ state as output?

Solution 1: solve by hand.

... too tedious, but you can evaluate

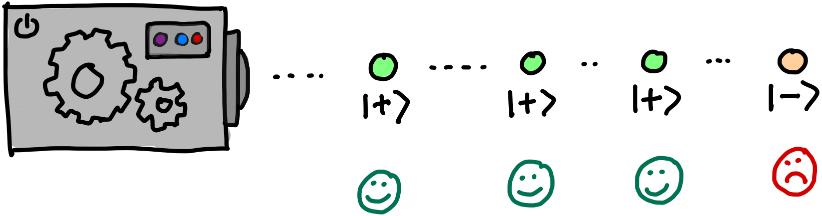
$$
\begin{aligned}
\rho_\psi &= |\psi\rangle\langle\psi| \\
\mathcal{E}(\rho_\psi) &= 0.98\rho_\psi + 0.02 X \rho_\psi X \\
\Pr(0) &= \mathrm{Tr}(|0\rangle\langle 0| \mathcal{E}(\rho_\psi))
\end{aligned}
$$

Solution 2: solve with PennyLane's ``‘default.mixed’'' device!

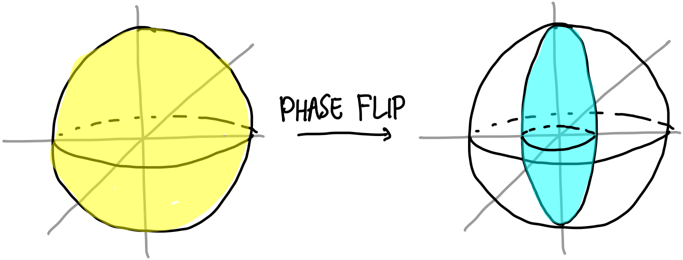
The phase flip channel

Suppose a "phase flip" (Pauli Z) error occurs with probability p.

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot Z\rho Z$$



The phase flip channel

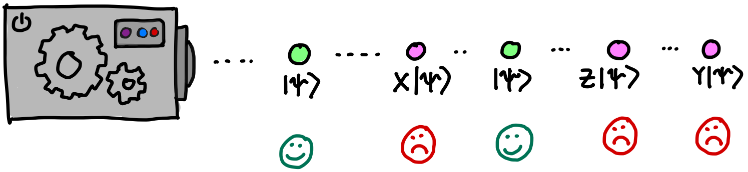


z: same
xy: contracts
(factor 1-2p)

The depolarizing channel

Suppose each Pauli error occurs with probability $p/3$. This is called the *depolarizing channel*.

$$\mathcal{E}(\rho) = (1-p)\cdot\rho + \frac{p}{3}\cdot X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z$$



The depolarizing channel

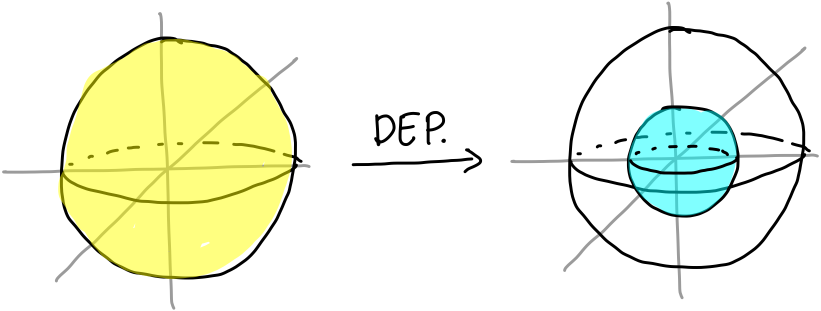
$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

can also be written as

$$\mathcal{E}(\rho) = (1-p)\rho + p \cdot \frac{I}{2}$$

Think of this as outputting ρ w/probability $1-p$, and maximally mixed state with probability p.

The depolarizing channel



Take another look at the depolarizing channel:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$$

More generally, a CPTP quantum channel $\Phi(\rho)$ is characterized by a set of **Kraus operators** $\{K_i\}$ where

$$\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$$

where

$$\sum_i K_i^\dagger K_i = I$$

Comparing density matrices

How can we quantify "how much" error occurs? How close is $\sigma = \mathcal{E}(\rho)$ to ρ?

One common metric is the **trace distance**:

$$T(\rho, \sigma) = \frac{1}{2}||\rho - \sigma||_1 = \frac{1}{2}\mathrm{Tr}\left(\sqrt{(\rho - \sigma)^\dagger(\rho - \sigma)}\right)$$

Value of trace distance is bounded by $0 \leq T(\rho, \sigma) \leq 1$, and *lower* trace distance is better.

Comparing density matrices

Another is the **fidelity**:

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$$

Value of fidelity is bounded by $0 \leq F(\rho,\sigma) \leq 1$, and *higher* fidelity is better.

Next time

Last few classes:

- Speedups, complexity, and oracle-based algorithms

Action items:

1. Prototype implementation for project
2. Assignment 2

Recommended reading:

- Codebook modules A and G