CPEN 400Q Lecture 20
The oracle, query complexity, and Deutsch's algorithm

Friday 24 March 2023

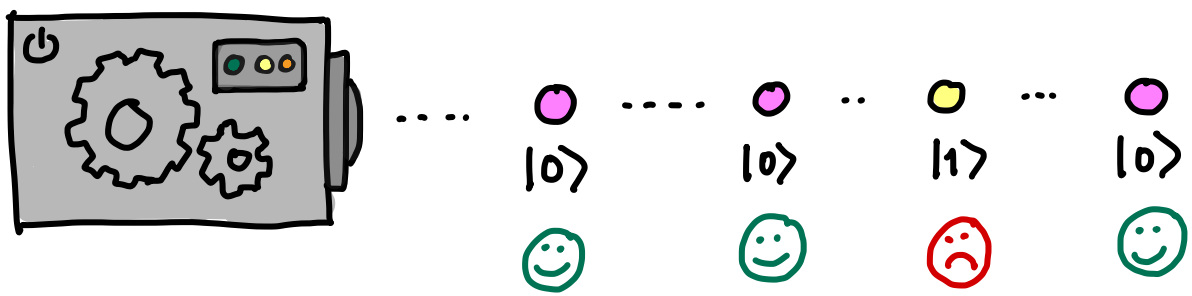
Announcements

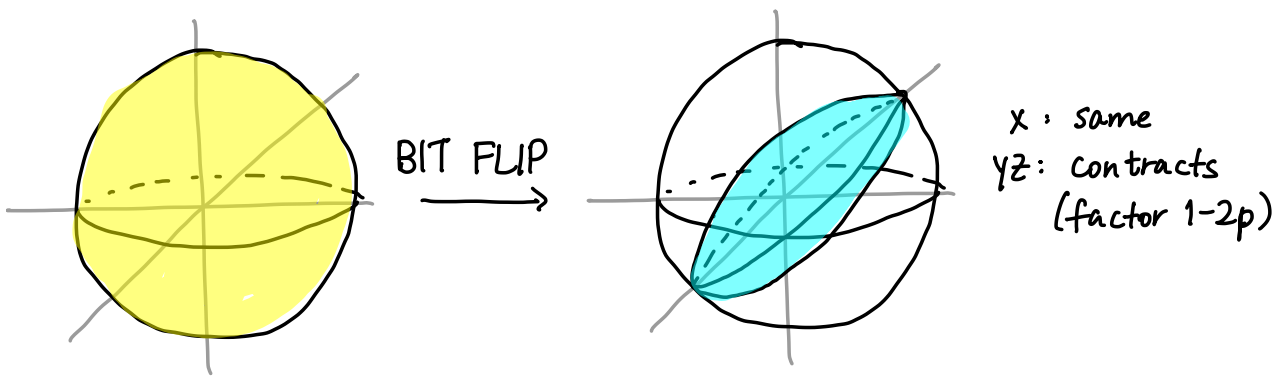
- Project schedule posted on Piazza
- Full grading rubric posted in PrairieLearn
- Quiz 9 (last quiz) on Monday about *today's* material
- Literacy assignment 3 due 29 March (Wednesday) at 23:59
- Assignment 3 available Monday (2 questions)

Last time

We formalized the idea of quantum channels and discussed some common error channels

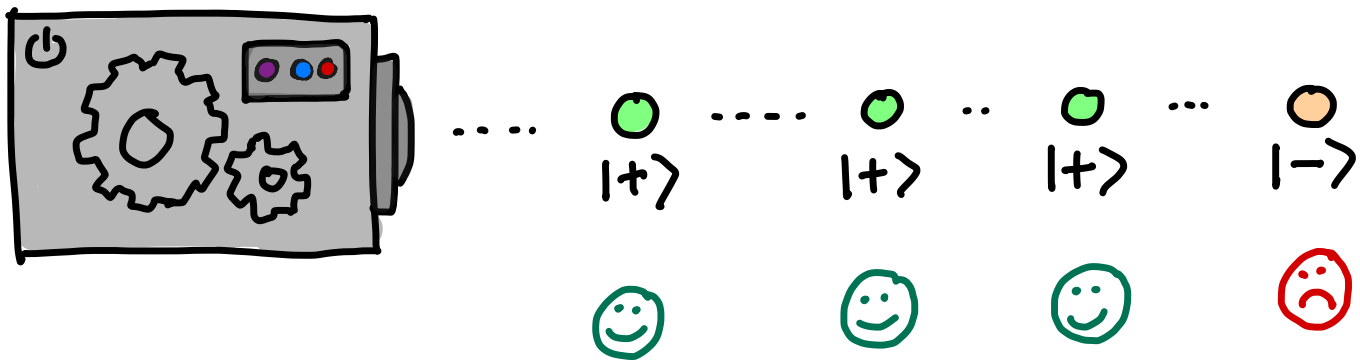
$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot X\rho X$$

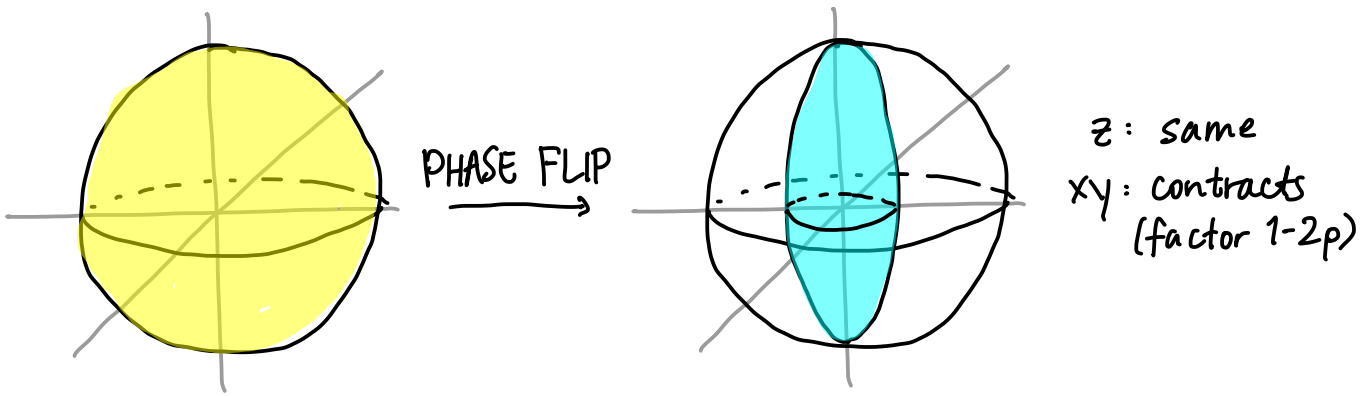




Last time

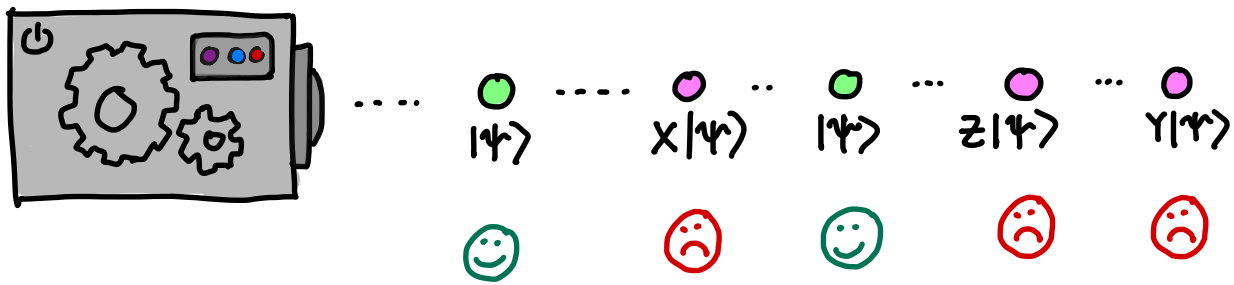
$$\mathcal{E}(\rho) = (1-p)\cdot\rho + p\cdot Z\rho Z$$

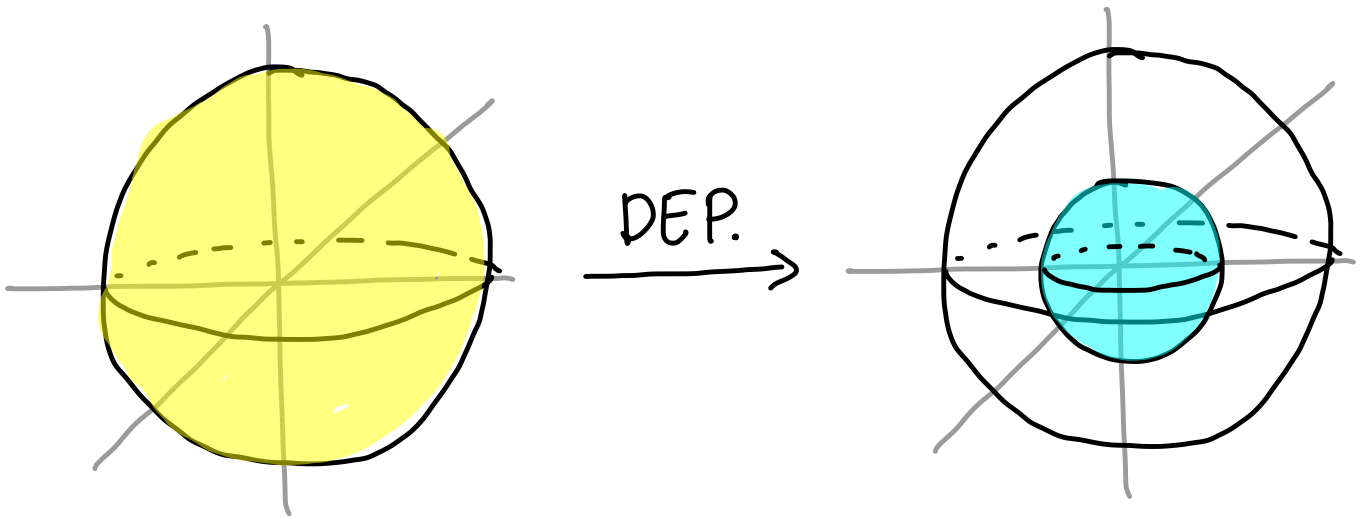




The depolarizing channel

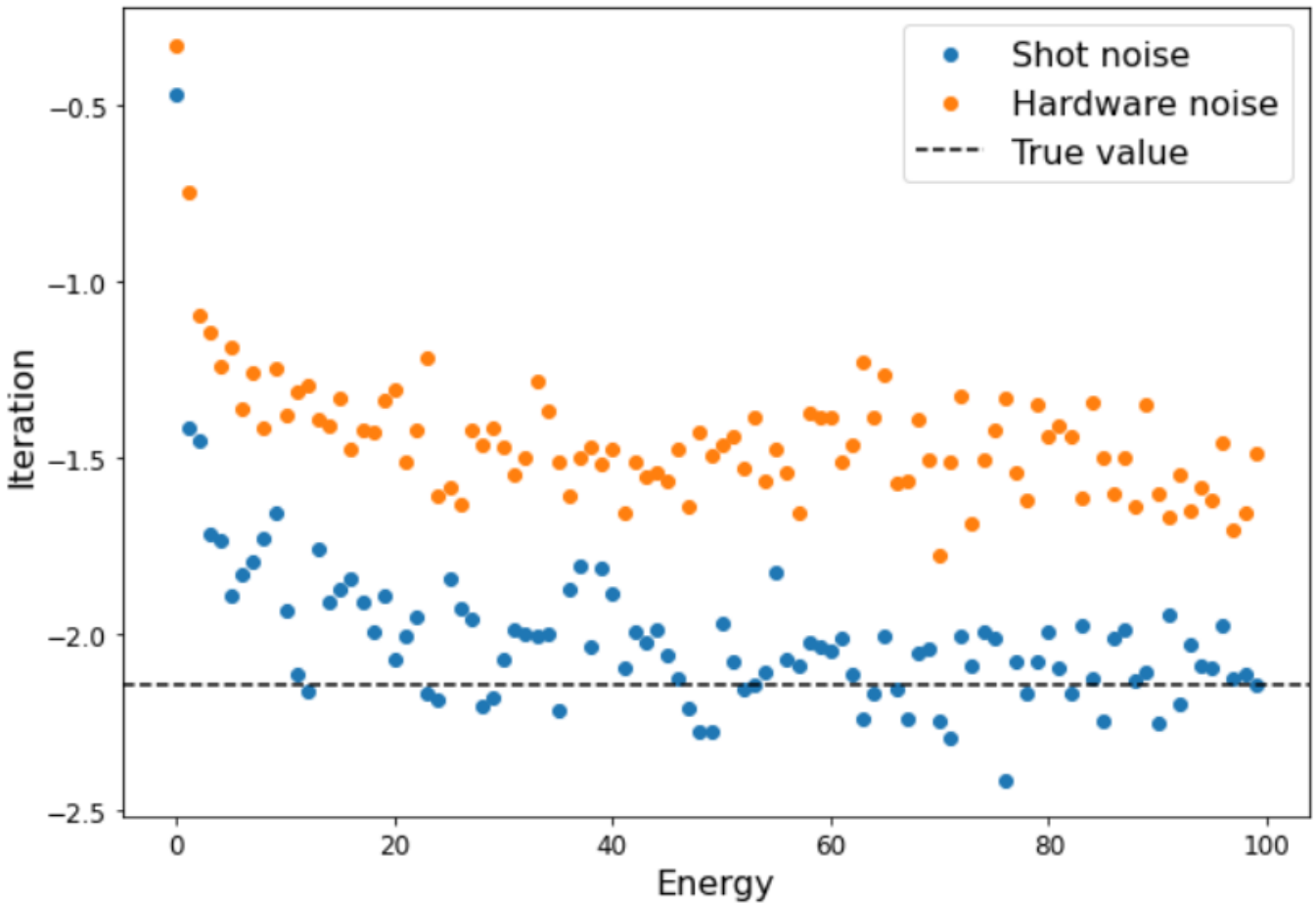
$$\mathcal{E}(\rho) = (1-p)\cdot\rho + \frac{p}{3}\cdot X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z$$





The depolarizing channel

We discussed metrics for quality of noisy hardware, and did a more realistic run of the deuteron VQE using simulated hardware noise.



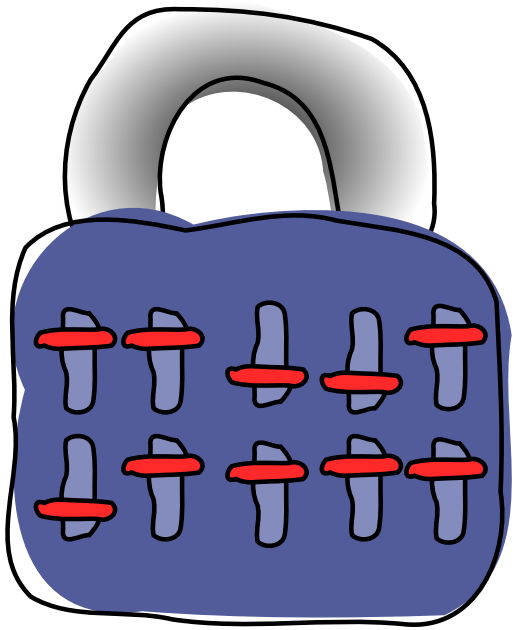
To make this better, see the video lecture on Canvas about *error mitigation*.

Learning outcomes

- Define the query complexity of an algorithm
- Describe multiple strategies for incorporating an *oracle* query into a quantum circuit
- Implement Deutsch's algorithm in PennyLane

Oracles: motivating problem

Suppose we would like to find the combination for a "binary" lock:



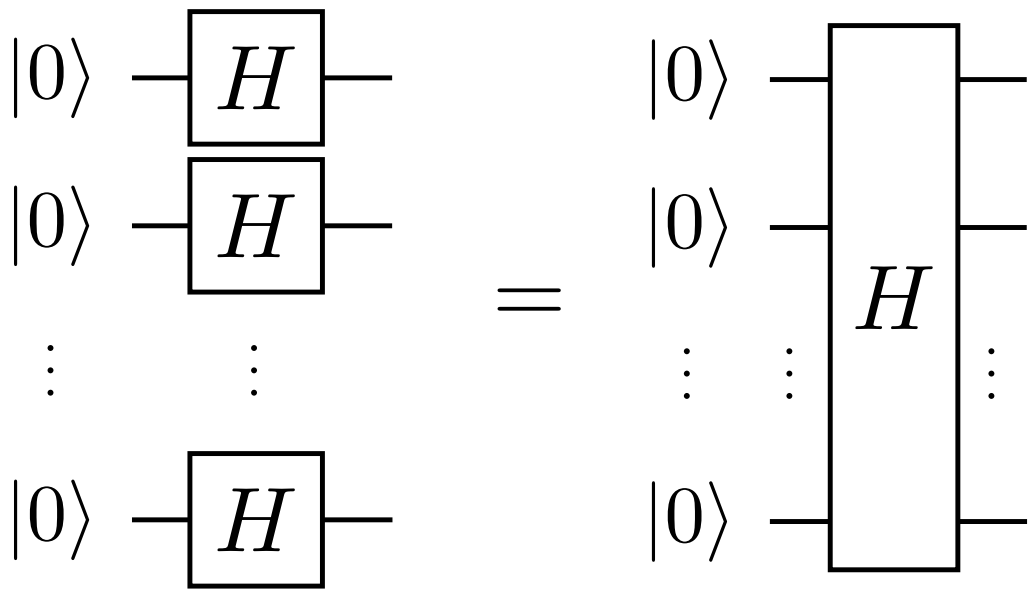
How do we solve this classically?

Image credit: Codebook node A.1

Idea: use superposition

Can we do better with a quantum computer?

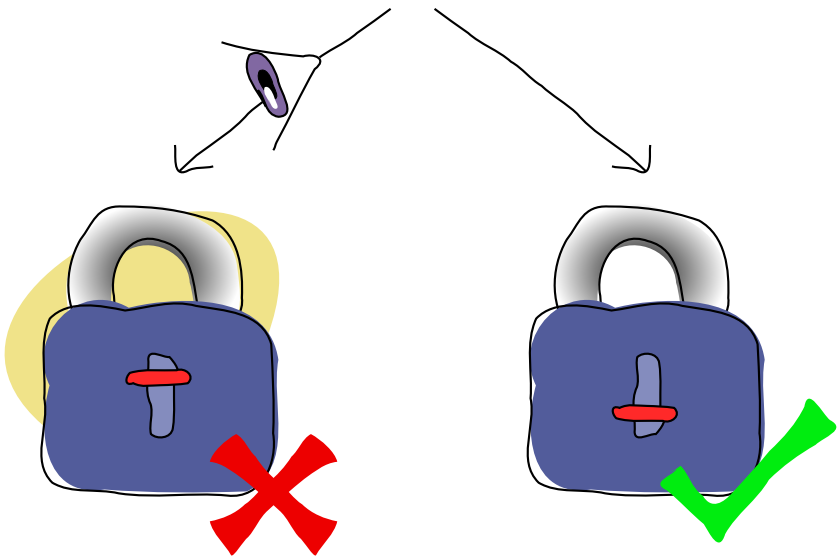
What if we take n qubits and put them in a superposition with all possible combinations?



Often called the *Hadamard transform*.

Idea: use superposition

Measurements are probabilistic - just because we put things into a uniform superposition of states, and our solution is "in" there, doesn't mean we are any closer to solving our problem.



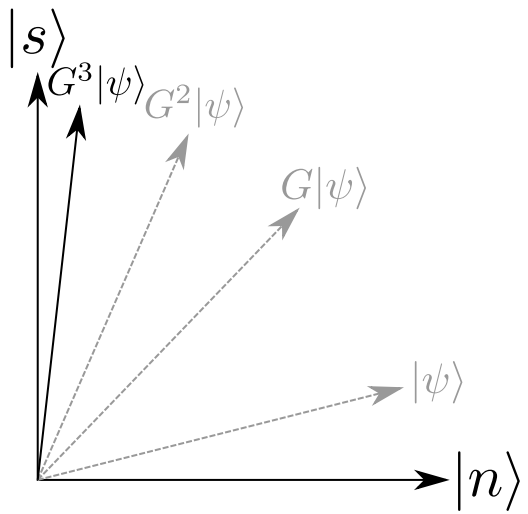
Quantum computers are **NOT** faster because they can "compute everything at the same time."

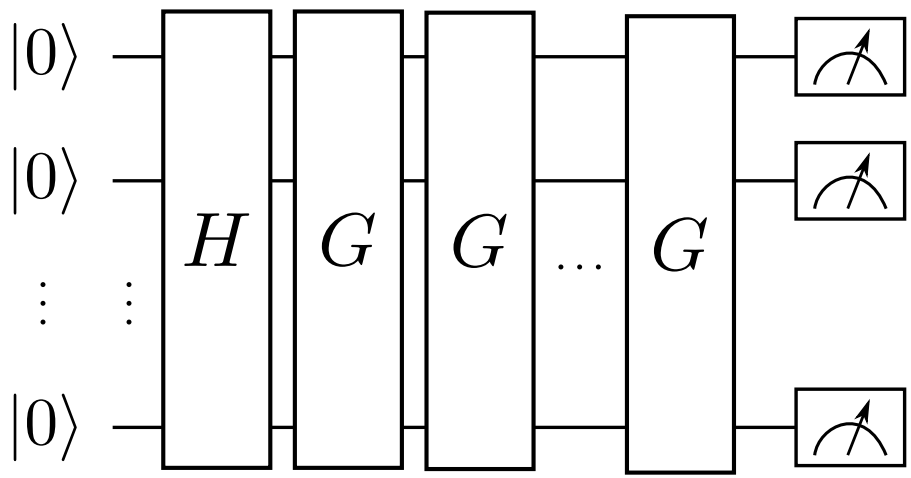
Image credit: Codebook node A.1

Solving problems with quantum computers

Can we solve this problem better with a quantum computer?

Yes: **amplitude amplification**, and **Grover's algorithm**





We will explore the algorithmic primitives that are involved, and some other cases where we can do better with quantum computing.

Oracles

Motivating problem

Suppose we would like to find combination for a "binary" lock:

Classically, we would have to try every possible combination. If there are n bits, that's 2^n possible tries. Can we do better with a quantum computer?

Image credit: Codebook node A.1

6 / 26

What is a "try"?

Oracles

We often express these tries as **evaluations** of a function that tells us whether we have found the correct answer.

Let

- **x** be an n-bit string that represents an input to the lock
- **s** be the solution to the problem (i.e., the correct combination)

We can represent trying a lock combination as a function:

Oracles

We don't necessarily care *how* this function gets evaluated, only that it gives us an answer (more specifically, a yes/no answer).

$$f(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = \mathbf{s} \\ 0 & \text{otherwise.} \end{cases}$$

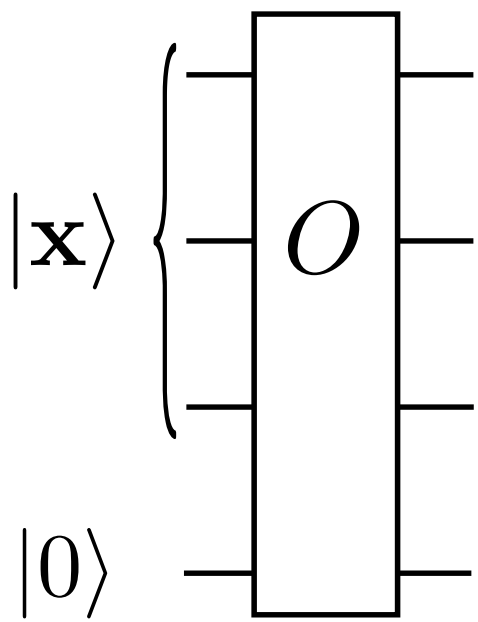
We consider this function as a black box, or an **oracle**.

Every time we try a lock combination, we are **querying the oracle**. The amount of queries we make is the **query complexity**.

Quantum oracles

To solve this problem using quantum computing, we need some circuit that plays the role of the oracle.

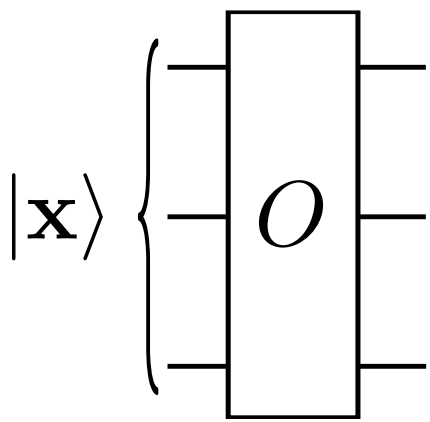
Idea 1: encode the result in the state of an additional qubit.



$$
\begin{aligned}
O|000\rangle|0\rangle &= |000\rangle|0\rangle \\
O|001\rangle|0\rangle &= |001\rangle|0\rangle \\
O|010\rangle|0\rangle &= |010\rangle|0\rangle \\
O|011\rangle|0\rangle &= |011\rangle|0\rangle \\
O|100\rangle|0\rangle &= |100\rangle|0\rangle \\
O|101\rangle|0\rangle &= |101\rangle|0\rangle \\
O|110\rangle|0\rangle &= |110\rangle|1\rangle \\
O|111\rangle|0\rangle &= |111\rangle|0\rangle
\end{aligned}
$$

Quantum oracles

Idea 2: encode the result in the phase of a qubit.



$$
\begin{aligned}
O|000\rangle &= |000\rangle \\
O|001\rangle &= |001\rangle \\
O|010\rangle &= |010\rangle \\
O|011\rangle &= |011\rangle \\
O|100\rangle &= |100\rangle \\
O|101\rangle &= |101\rangle \\
O|110\rangle &= -|110\rangle \\
O|111\rangle &= |111\rangle
\end{aligned}
$$

Deutsch's algorithm

Motivation: You are given access to an oracle and are promised that it implements one of the following 4 functions:

Name	Action	Name	Action
f_1	$f_1(0) = 0$ $f_1(1) = 0$	f_2	$f_2(0) = 1$ $f_2(1) = 1$
f_3	$f_3(0) = 0$ $f_3(1) = 1$	f_4	$f_4(0) = 1$ $f_4(1) = 0$

Functions f_1 and f_2 are *constant* (same output no matter what the input), and f_3 and f_4 are balanced.

Deutsch's algorithm

How many queries do you need to make to the oracle to determine if the function is constant or balanced? (i.e., either one of f_1/f_2, or one of f_3/f_4).

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f_3	$f_3(0) = 0$ $f_3(1) = 1$	f_4	$f_4(0) = 1$ $f_4(1) = 0$

Classical solution: 2

We always need to query both inputs 0 and 1 to find out the nature of the function.

Deutsch's algorithm

How many queries do you need to make to the oracle to determine if the function is constant or balanced? (i.e., either one of f_1/f_2, or one of f_3/f_4).

Name	Action	Name	Action
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f_3	$f_3(0) = 0$ $f_3(1) = 1$	f_4	$f_4(0) = 1$ $f_4(1) = 0$

Quantum solution: 1
How???

Phase kickback

The secret relies on *phase kickback*.

Remember: what happens when we apply a CNOT to the following two two-qubit states?

$$|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right), \quad |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

We get

$$\begin{aligned} CNOT \left(|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) &= \\ CNOT \left(|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) &= \end{aligned}$$

Phase kickback

We can write a general version of this effect:

$$CNOT\left(|b\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right) =$$

But what does this have to do with Deutsch's algorithm and figuring out if a function is constant or balanced?

Deutsch's algorithm

This is where our oracle comes in. Suppose we have a black box, U_f, that implements any of these four functions, f:

$$U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

Setting $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ will allow us to 'extract' the value of $f(0) \oplus f(1)$ with just a single query.

Let's work through the math.

Deutsch's algorithm

$$U_f|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) =$$

$$=$$

If $f(x) = 0$, we get

$$U_f|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) =$$

If $f(x) = 1$, we get

$$U_f|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) =$$

Deutsch's algorithm

So just like the case of the CNOT where we wrote the general version

$$CNOT\left(|b\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right)=(-1)^b|b\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right),$$

we can write

Essentially, before the CNOT was just playing the role of U_f for the specific function $f(0)=0, f(1)=1$.

Deutsch's algorithm

This doesn't look like much on its own - we want to get a *combination* of $f(0)$ and $f(1)$. How can we do this?

By setting $|x\rangle$ to be a superposition!

$$U_f \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$=$$

$$=$$

Let's pull out a phase factor of $(-1)^{f(0)}$, since global phase doesn't matter anyways.

Deutsch's algorithm

$$=$$

$$=$$

Now let's look at how this state is different when f is a constant vs. a balanced function.

Deutsch's algorithm

$$U_f \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \frac{|0\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle}{\sqrt{2}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

If the function is constant, $f(0) \oplus f(1) = 0$ and the state is

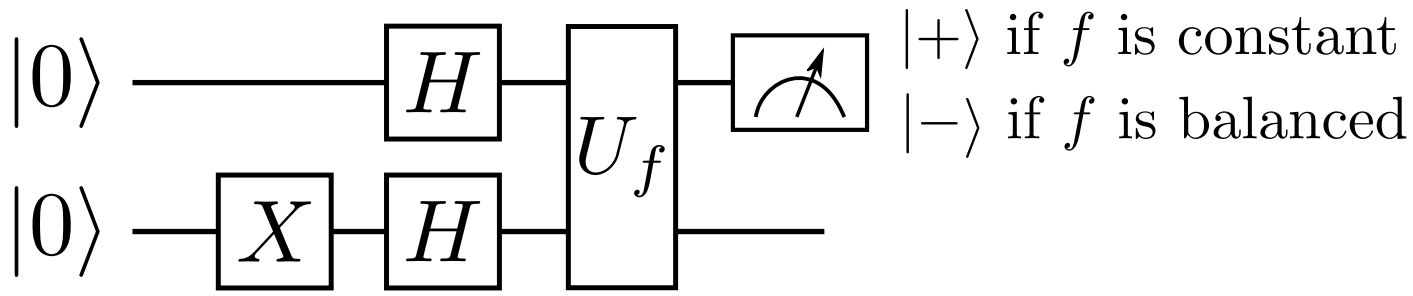
$$U_f \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) =$$

But if the function is balanced, $f(0) \oplus f(1) = 1$ and the state is

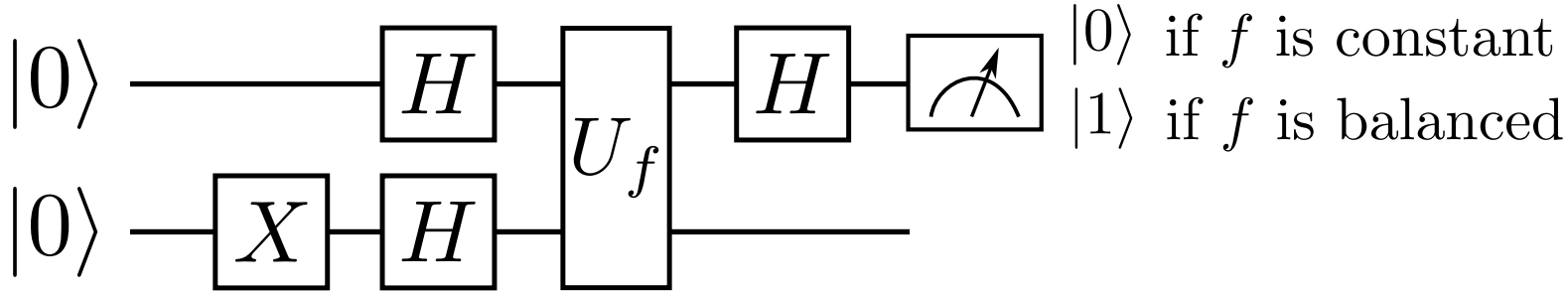
$$U_f \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) =$$

Deutsch's algorithm

As a circuit, Deutsch's algorithm looks like this:

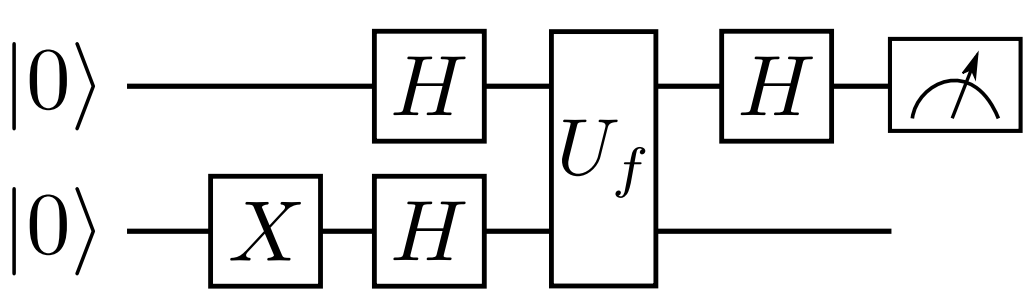


Or equivalently,

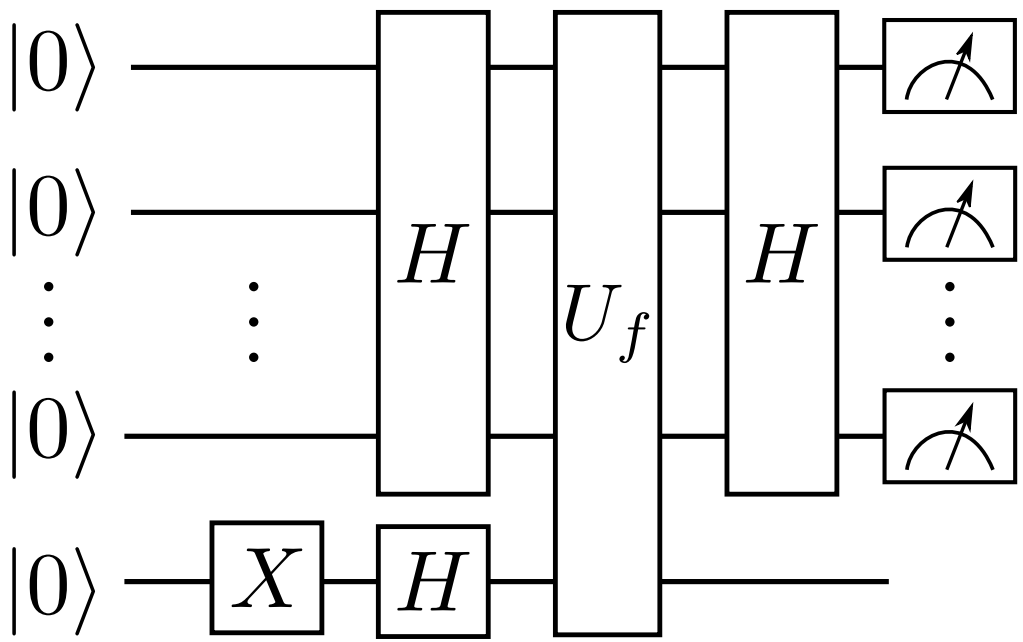


We call U_f just once, but obtain information about the relationship between $f(0)$ and $f(1)$! Let's implement it.

Generalization: Deutsch-Jozsa algorithm



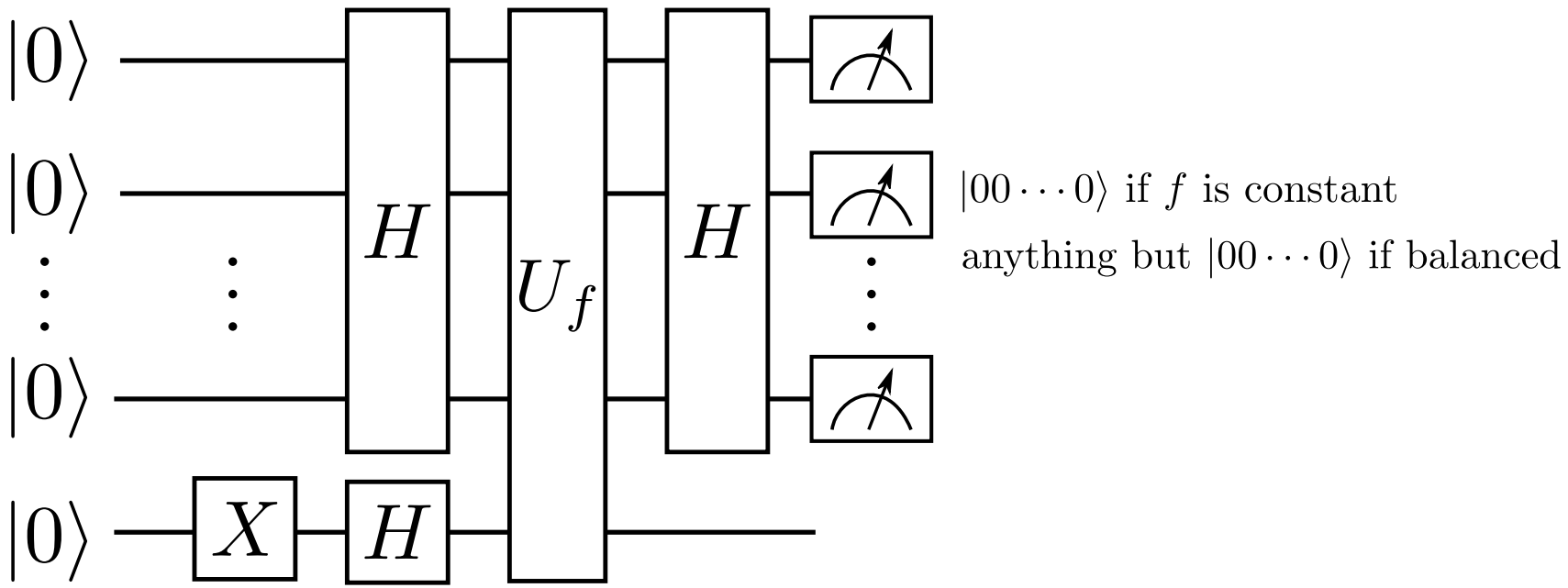
$|0\rangle$ if f is constant
$|1\rangle$ if f is balanced



$|00\cdots0\rangle$ if f is constant
anything but $|00\cdots0\rangle$ if balanced

Generalization: Deutsch-Jozsa algorithm

$2^{n-1} + 1$ classical queries in worst case; still only 1 quantum query.



(Challenge: try implementing it yourself to check if this works!)

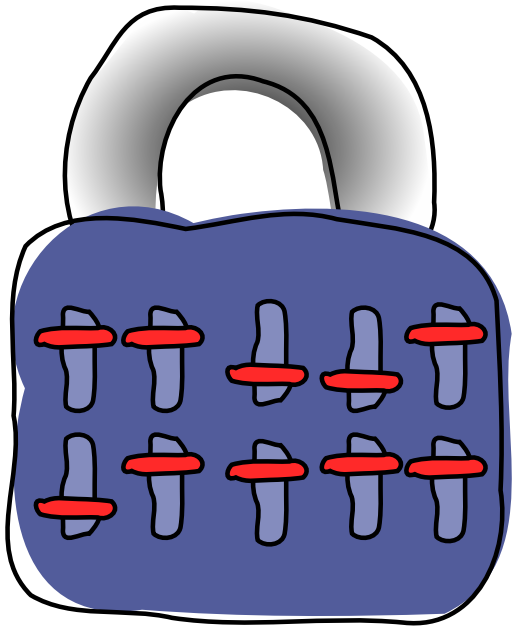
Oracle-based algorithms

A few other interesting ones to look into in addition to Deutsch-Josza:

Bernstein-Vazirani algorithm (will see on A3)
Given $f : \{0,1\}^n \to \{0,1\}$ such that $f(x) = x \cdot s$ for some secret bitstring s. Find s using the fewest number of queries to the oracle.

Simon's algorithm
Given $f : \{0,1\}^n \to \{0,1\}^n$ and promised that for some non-trivial bit string s, $f(x) = f(y)$ iff $x \oplus y = s$. Find s using the fewest queries to the oracle

Let's break that lock!

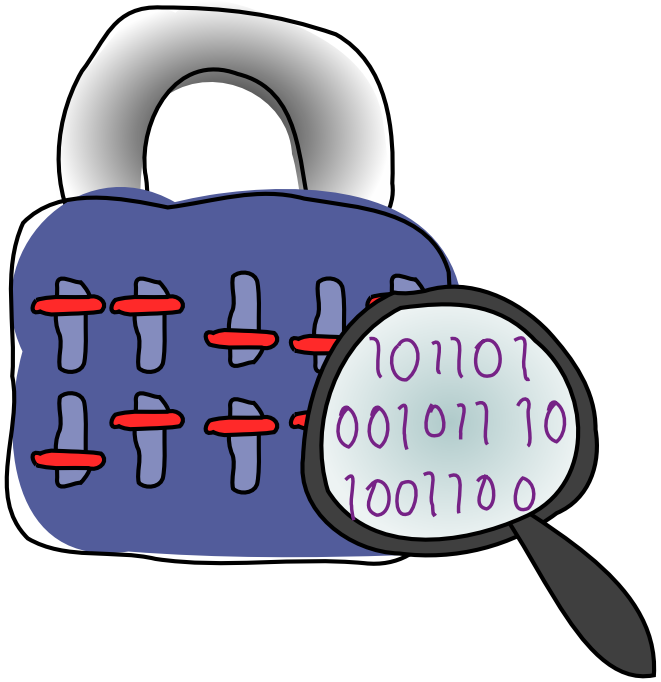


We input the combination to the lock as an n-bit (binary) string. The correct combination is labelled $\mathbf{s}$.

Image credit: Codebook node A.1

Grover's quantum search algorithm

How many times must we query the oracle to find the solution?



Classical: in the worst case, times

Quantum: times

Grover's quantum search algorithm

Idea: start with a uniform superposition and then *amplify* the amplitude of the state corresponding to the solution.

In other words, go from the uniform superposition

to something that looks more like this:

Q: Why do we want a state of this form?

$$|\psi'\rangle = (\text{big number})|\mathbf{s}\rangle + (\text{small number}) \sum_{\mathbf{x} \neq \mathbf{s}} |\mathbf{x}\rangle$$

Q: Why do we want a state of this form?

$$|\psi'\rangle = (\text{big number})|\mathbf{s}\rangle + (\text{small number}) \sum_{\mathbf{x} \neq \mathbf{s}} |\mathbf{x}\rangle$$

A: When we make a measurement, we will very likely get the solution to our problem!

We will see how to do this next time!

Next time

Last few classes:

- Amplitude amplification, Grover's algorithm

Action items:

1. Literacy assignment 3
2. Project code and presentation

Recommended reading:

- Codebook modules A and G