CPEN 400Q / EECE 571Q Lecture 11
The quantum Fourier transform and quantum phase estimation

Tuesday 15 February 2022

Announcements

- Project group / topic selection due today
- Please upgrade to PennyLane v0.21; new `requirements.txt` file will be included later with Quiz 5 and with Assignment 3.

Quiz 5 after class today.

Last time

We introduced the quantum Fourier transform, and saw how it is the analog of the classical inverse discrete Fourier transform.

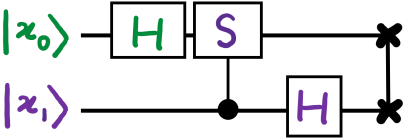
$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$$QFT = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

where for n qubits, $N = 2^n$, and $\omega = e^{2\pi i/N}$

Last time

We saw the circuits for some special cases. For 1 qubit, it is just the Hadamard. For 2 and 3 qubits:



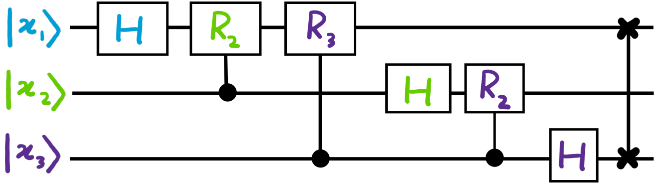


Image credit: Xanadu Quantum Codebook node F.2, F.3

Quantum Fourier transform

I showed you what the general form of the circuit looked like:

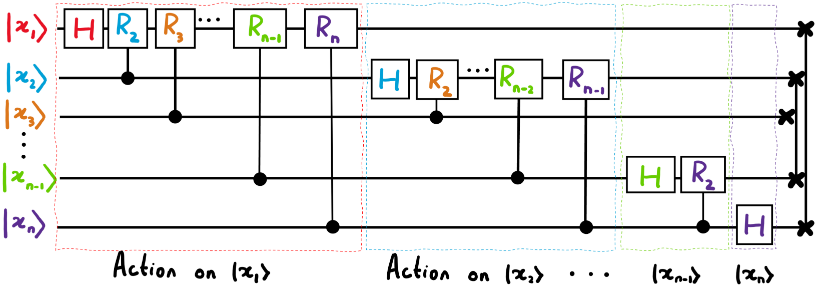


Image credit: Xanadu Quantum Codebook node F.3

Learning outcomes

- Derive the QFT circuit and implement it in PennyLane
- Describe the steps of the quantum phase estimation (QPE) subroutine
- Use the QFT to implement QPE

Example

Let $k \overset{0.}{=} k_1 k_2 k_3 k_4 = 0.1001$. The numerical value of this is:

$$
\begin{aligned}
0.1001 &= \frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} \\
&= \frac{1}{2} + \frac{1}{16} \\
&= 0.5625
\end{aligned}
$$

$$
\begin{aligned}
0.1001 &= \frac{1}{2^1} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} \\
&= \frac{1}{2} + \frac{1}{16} \\
&= 0.5625
\end{aligned}
$$

We need this for the QFT because in the exponent, we have

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle$$

and k/N is a fractional value.

What we are going to show is that

$$\text{QFT}|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

can be rewritten in the following factorized form:

$$\frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

Then, we will see how this form reveals to us the circuit that creates this state!

A circuit for the QFT

Start by rewriting k/N using fractional binary.

$k, x \in \mathbb{Z}$

$$\mathrm{QFT}|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$\omega = e^{2\pi i/N}$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x \frac{k}{N}} |k\rangle$$

$N = 2^n$

$$\frac{k}{N} = \sum_{l=1}^{n} \frac{k_l}{2^{-l}} \sim 0.k_1 k_2 \ldots$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x \left(\sum_{l=1}^{n} \frac{k_l}{2^l}\right)} |k\rangle$$

$|k\rangle = |k_1 k_2 \cdots k_n\rangle$

$$= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x \left(\sum_{l=1}^{n} \frac{k_l}{2^l}\right)} |k_1 k_2 \cdots k_n\rangle$$

$\sum_{i=1}^{n} \quad \prod_{i=1}^{n} \quad \bigotimes_{i=1}^{n}$

(keeping the last equation from the previous slide)

$$\frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x \left(\sum_{\ell=1}^{n} \frac{k_\ell}{2^\ell} \right)} |k_1 k_2 \cdots k_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x \frac{k_1}{2^1}} \cdots e^{2\pi i x \frac{k_n}{2^n}} |k_1 \cdots k_n\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \left(e^{2\pi i x \frac{k_1}{2^1}} |k_1\rangle \right) \otimes \cdots \otimes \left(e^{2\pi i x \frac{k_n}{2^n}} |k_n\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{2\pi i x \frac{k_\ell}{2^\ell}} |k_\ell\rangle$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(\sum_{k_\ell=0}^{1} e^{2\pi i x \frac{k_\ell}{2^\ell}} |k_\ell\rangle \right)$$

A circuit for the QFT

(keeping the last equation from the previous slide)

$$= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(\underbrace{\sum_{k_\ell=0}^{1} e^{2\pi i x \frac{k_\ell}{2^\ell}} |k_\ell\rangle}_{} \right)$$

$$= \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(|0\rangle + e^{2\pi i \frac{x}{2^\ell}} |1\rangle \right)$$

$$\frac{x}{2} = \frac{2^{n-1}}{2} x_1 + \frac{2^{n-2}}{2} x_2 + \dots + \frac{2^0}{2} x_n$$

$$= 2^{n-2} x_1 + 2^{n-3} x_2 + \dots + \frac{x_n}{2}$$

$$e^{2\pi i \left(2^{n-2} x_1 + \dots + \frac{x_n}{2}\right)} \sim e^{2\pi i \, 0.x_n}$$

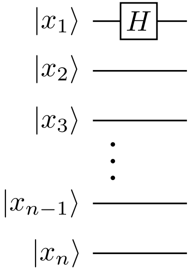
$$= \frac{1}{\sqrt{N}} \left(|0\rangle + e^{2\pi i 0.x_n} |1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n} |1\rangle\right) \cdots \left(|0\rangle + e^{2\pi i 0.x_1 x_2 \dots x_n} |1\rangle\right)$$

Starting with the state

$$|x\rangle = |x_1 \cdots x_n\rangle,$$

apply a Hadamard to qubit 1:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_1}|1\rangle\right)|x_2 \cdots x_n\rangle$$



A circuit for the QFT

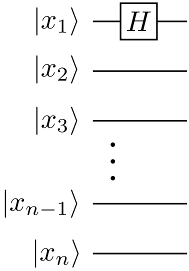
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1}|1\rangle\right)|x_2 \cdots x_n\rangle$$

If $x_1 = 0$, $e^0 = 1$ and we get the $|+\rangle$ state.

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

If $x_1 = 1$, $e^{2\pi i(1/2)} = e^{\pi i} = -1$ and we get the $|-\rangle$ state.

$$\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$



A circuit for the QFT

We are trying to make a state that looks like this:

$$|x\rangle \rightarrow \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

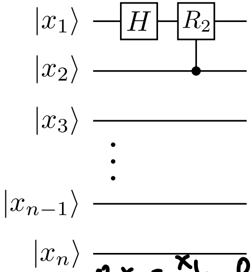
Every qubit has a different *phase* on the $|1\rangle$ state. We are going to need some way of creating this.

We define the gate:

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

Now let's apply a controlled R_2 gate from qubit 2 to qubit 1

$$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^2} \end{pmatrix}$$



$0.x_1 = \frac{x_1}{2}$ $\quad 0.x_1x_2 = \frac{x_1}{2} + \frac{x_2}{2^2}$

The first qubit picks up a phase:

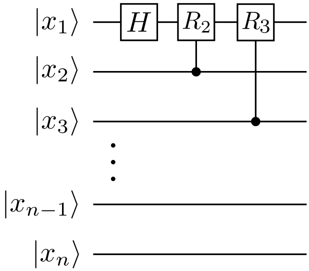
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1}|1\rangle\right)|x_2 \cdots x_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1} e^{\frac{2\pi i}{2^2}x_2}|1\rangle\right)|x_2 \cdots x_n\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_2}|1\rangle\right)|x_2 \cdots x_n\rangle$$

A circuit for the QFT

Now let's apply a controlled R_3 gate from qubit 3 to qubit 1

$$R_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^3} \end{pmatrix}$$

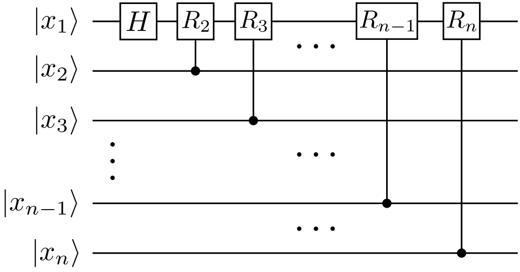


The first qubit picks up another phase:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1 x_2}|1\rangle\right)|x_2 \cdots x_n\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1 x_2 x_3}|1\rangle\right)|x_2 \cdots x_n\rangle$$

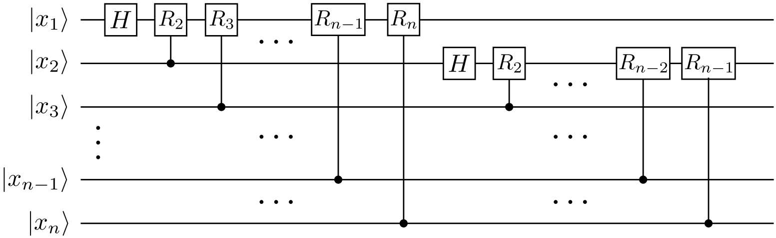
We can apply a controlled R_4 from the fourth qubit, etc. up to the n-th qubit to get

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_2\cdots x_n}|1\rangle\right)|x_2\cdots x_n\rangle$$



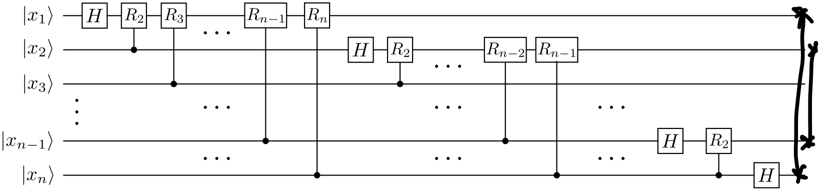
Next, ignore the first qubit and do the same thing with the second qubit: apply H, and then controlled rotations from every qubit from 3 to n to get

$$\frac{1}{\sqrt{2}^2}\left(|0\rangle + e^{2\pi i 0.x_1 x_2 \cdots x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_2 \cdots x_n}|1\rangle\right)|x_3 \cdots x_n\rangle$$



If we do this for all qubits, we get something similar to that big ugly state from earlier:

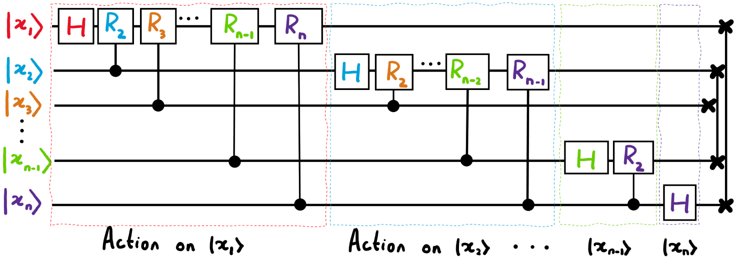
$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)}{\sqrt{N}}$$



This is almost what we want: the order of the qubits is backwards. This is easily fixed with some SWAP gates.

So the QFT can be implemented using:

- n Hadamard gates
- $n(n-1)/2$ controlled rotations
- $\lfloor n/2 \rfloor$ SWAP gates if you care about the order



The number of gates is *polynomial in n*, so this can be implemented efficiently on a quantum computer! Let's try it...

Image credit: Xanadu Quantum Codebook node F.3

Quantum phase estimation

Fun fact: eigenvalues of unitary matrices are complex numbers with magnitude 1.

Proof: Let λ_k be the eigenvalue associated with eigenvector $|k\rangle$ of a unitary U:

$$U|k\rangle = \lambda_k|k\rangle$$

$U \quad |k\rangle \rightarrow U|k\rangle = \lambda_k|k\rangle$

$|\lambda_k| = 1$

We can take the conjugate transpose of this equation:

$$\langle k|U^\dagger = \langle k|\lambda_k^*$$

$\langle k|U^\dagger = \langle k|\lambda_k^*$

Multiply the two sides together:

$$\langle k|U^\dagger U|k\rangle = \langle k|\lambda_k^*\lambda_k|k\rangle$$

$\langle k|\underbrace{U^\dagger U}_{=I}|k\rangle = \langle k|\underbrace{\lambda_k^*\lambda_k}_{=|\lambda_k|^2}|k\rangle$

$$\langle k|k\rangle = |\lambda_k|^2\langle k|k\rangle$$

$\langle k|k\rangle = |\lambda_k|^2\langle k|k\rangle$

$$1 = |\lambda_k|^2$$

$1 = |\lambda_k|^2$

Eigenvalues of unitary matrices

So we can write

$$\lambda_k = e^{2\pi i \theta_k}$$

where θ_k is some phase angle such that $|\theta_k| \leq 1$.

What if we want to *learn* an unknown θ_k?

Idea: apply U to the relevant eigenvector, because that's "what makes the phase come out".

$$U|k\rangle = e^{2\pi i\theta_k}|k\rangle$$

...but this is an unobservable *global* phase!

We have to do something different: eigenvalue estimation, or quantum phase estimation (QPE).

Quantum phase estimation

Given a unitary U and one of its eigenvectors $|k\rangle$, estimate the value of θ_k such that

$$\underbrace{U|k\rangle = e^{2\pi i\theta_k}|k\rangle}$$

Must determine:

- How to design a circuit that extracts the θ_k
- To what precision can we estimate it
- What to do if we don't know a $\underbrace{|k\rangle}$ in advance

(*You will explore the last two in your homework!*)

Let U be an n-qubit unitary; therefore $|k\rangle$ is an n-qubit state.

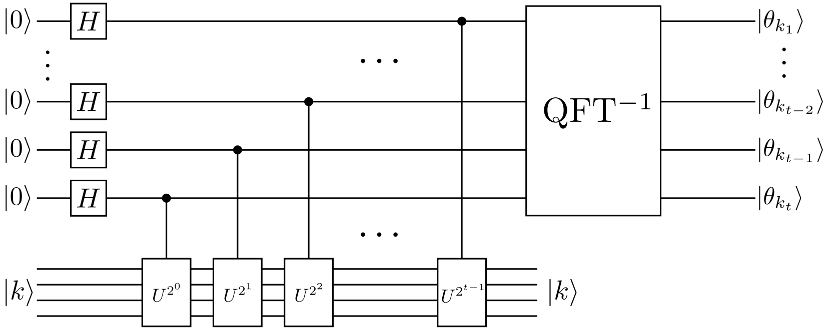
Assume for now that θ_k can be represented *exactly* using t bits in *fractional binary*:

$$\theta_k = 0.\theta_{k_1} \cdots \theta_{k_t}$$

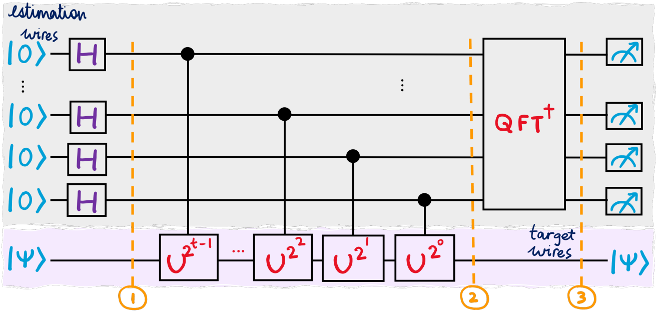
Fact: We can construct a circuit with $n + t$ qubits that recover the value of θ_k exactly by:

1. Preparing n qubits in state $|k\rangle$
2. Applying controlled applications of U to those qubits in a special way
3. Applying the inverse QFT

This is one version of the circuit:



The order of the controlled operations is irrelevant though, so you may see this too:

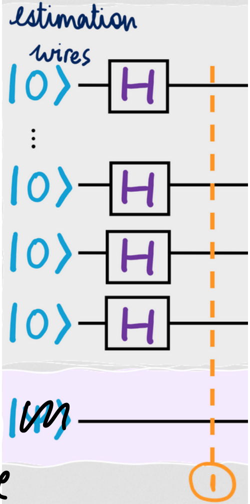


Why does this work? Let's analyze the state at points 1, 2, and 3 above.

Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 1

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$



$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|k\rangle$

eigenstate

Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 2

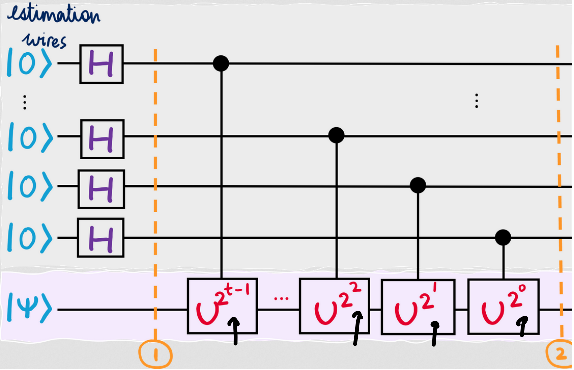
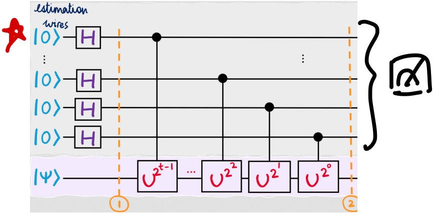


Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 2

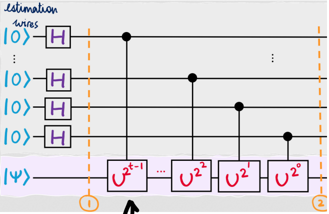


$$U|k\rangle = e^{2\pi i \theta_k}|k\rangle$$

Consider the top-most qubit:

$$(CU)^{2^{t-1}}\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|+\rangle^{\otimes t-1}|k\rangle\right] = \frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle + \frac{1}{\sqrt{2}}|1\rangle|+\rangle^{\otimes t-1}(U)^{2^{t-1}}|k\rangle$$

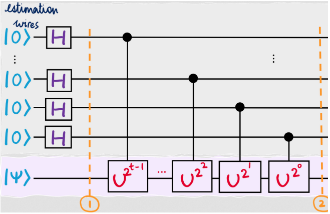
$$= \frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle + \frac{1}{\sqrt{2}}|1\rangle|+\rangle^{\otimes t-1}\left(e^{2\pi i\theta_k}\right)^{2^{t-1}}|k\rangle$$



Use phase kickback

$$\frac{1}{\sqrt{2}}|0\rangle\underbrace{|+\rangle^{\otimes t-1}|k\rangle} + \frac{1}{\sqrt{2}}\left(e^{2\pi i\theta_k}\right)^{2^{t-1}}|1\rangle\underbrace{|+\rangle^{\otimes t-1}|k\rangle}$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \left(e^{2\pi i\theta_k}\right)^{2^{t-1}}|1\rangle\right)|+\rangle^{\otimes t-1}|k\rangle$$



$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle\right)$

$|+\rangle$

$\vdots$

$|+\rangle$

$|k\rangle$

What is happening in the exponent?

$$\left(e^{2\pi i \theta_k}\right)^{2^{t-1}} = e^{2\pi i \theta_k \cdot 2^{t-1}}$$

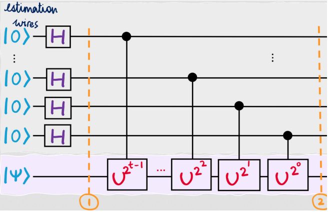
$$= e^{2\pi i \left(\frac{\theta_{k_1}}{2} + \frac{\theta_{k_2}}{2^2} + \dots + \frac{\theta_{k_t}}{2^t}\right)\cdot 2^{t-1}}$$

$$= e^{2\pi i \left(2^{t-2}\cdot \theta_{k_1} + 2^{t-2}\theta_{k} + \dots + \underbrace{\frac{\theta_{k_t}}{2}}\right)}$$

$$= e^{2\pi i \frac{\theta_{k_t}}{2}}$$

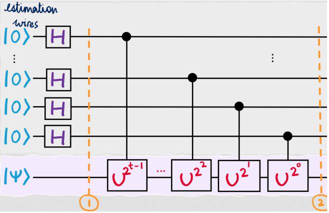
$$= e^{2\pi i\, \underbrace{0.\theta_{k_t}}}$$

Quantum phase estimation: step 2



So we have the combined state:

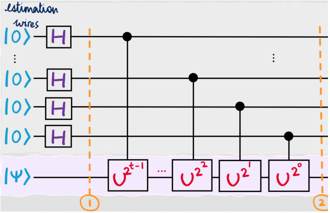
$$\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})^{2^{t-1}}|1\rangle)|+\rangle^{\otimes t-1}|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle)|+\rangle^{\otimes t-1}|k\rangle$$



Let's do the second-last qubit (ignore what happens to others for now):

$$(CU)^2\left(|+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|+\rangle|k\rangle\right)=|+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i\theta_k\cdot 2}|1\rangle)|+\rangle|k\rangle$$

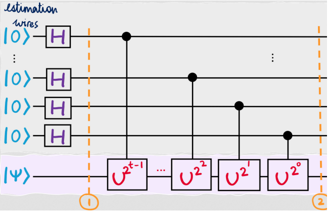
Quantum phase estimation: step 2



Again check the exponent...

$$
\begin{aligned}
(e^{2\pi i\theta_k})^2 &= e^{2\pi i\theta_k \cdot 2} \\
&= e^{2\pi i(\frac{\theta_{k_1}}{2^1} + \frac{\theta_{k_2}}{2^2} + \cdots \frac{\theta_{k_t}}{2^t})\cdot 2} \\
&= e^{2\pi i(\theta_{k_1} + \frac{\theta_{k_2}}{2} + \cdots \frac{\theta_{k_t}}{2^{t-1}})} \\
&= e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}
\end{aligned}
$$

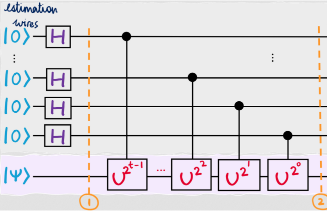
Quantum phase estimation: step 2



So we have the combined state:

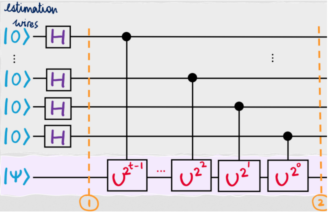
$$|+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})^2|1\rangle)|+\rangle|k\rangle = |+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)|+\rangle|k$$

Quantum phase estimation: step 2



Can show in the same way that for the last qubit

$$|+\rangle^{\otimes t-1}\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})|1\rangle)|k\rangle = |+\rangle^{\otimes t-1}\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$



$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_{t-1}}\theta_{k_t}}|1\rangle)$

$\vdots$

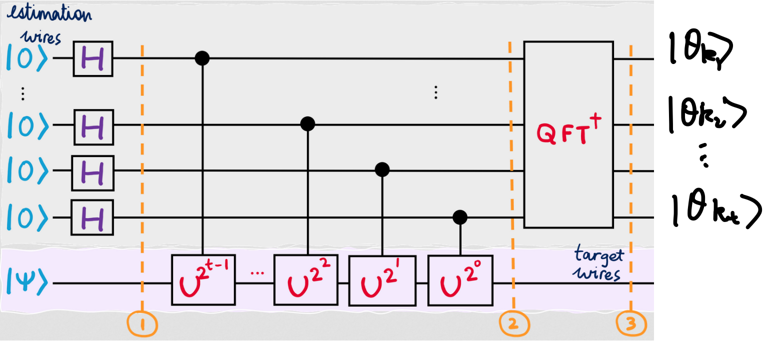
After step 2, we have the state

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle)\cdots\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Should look familiar!

$$\simeq \mathrm{QFT}|\underbrace{\theta_{k_1}\cdots\theta_{k_t}}\rangle$$

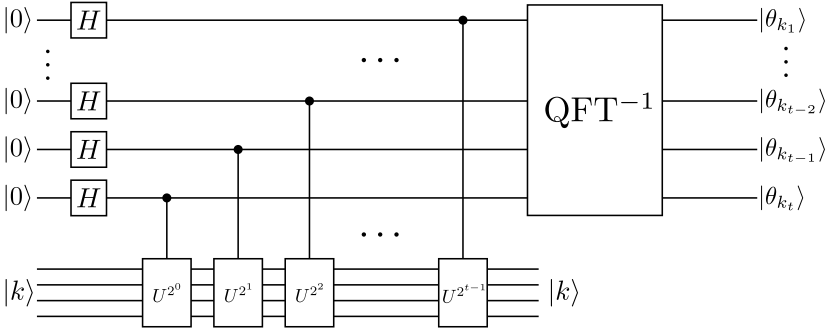
Quantum phase estimation: step 3



Last step is to apply the *inverse* QFT to recover the state...

Image credit: Xanadu Quantum Codebook node P.2

We can then measure to learn the numerical value of θ_k.



Let's implement it.

Next time

Content:

- Starting with Shor's algorithm

Action items:

1. E-mail me your project team and paper selection by end of day

Recommended reading:

- Codebook nodes F.1-F.3, P.1-P.4
- Nielsen & Chuang 5.1, 5.2