CPEN 400Q Lecture 10
The quantum Fourier transform and quantum phase estimation

Friday 10 February 2023

Announcements

- Literacy assignment 2 available (due after reading week)
- Project details posted (group and paper selection due next Friday)

We introduced the quantum Fourier transform, and saw how it is the analog of the classical inverse discrete Fourier transform.

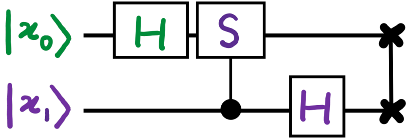
$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$$QFT = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

where for n qubits, $N = 2^n$, and $\omega = e^{2\pi i/N}$

Last time

We saw the circuits for some special cases.



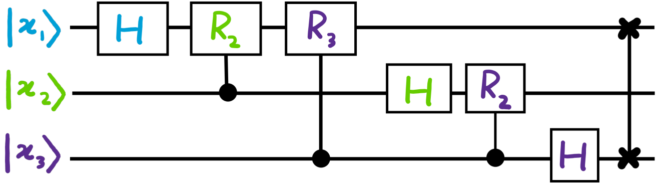


Image credit: Xanadu Quantum Codebook node F.2, F.3

Quantum Fourier transform

I showed you what the general form of the circuit looked like:

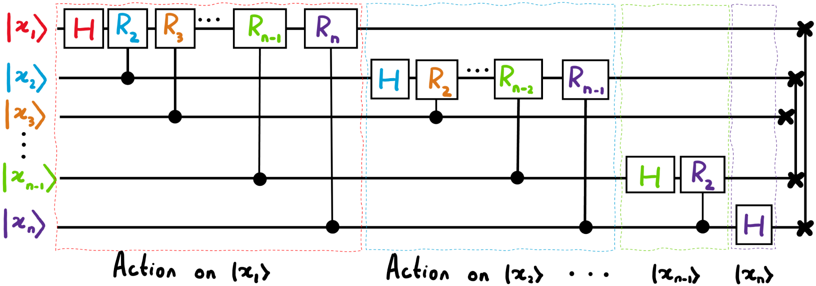


Image credit: Xanadu Quantum Codebook node F.3

Learning outcomes

- Derive the QFT circuit and implement it in PennyLane
- Describe the phase kickback trick
- Outline the steps of the quantum phase estimation (QPE) subroutine
- Use the QFT to implement QPE

Review: fractional binary notation

Example: Let $k = k_1 k_2 k_3 k_4 = 0.1001$. The numerical value of k is

We need this for the QFT because in the exponent, we have

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle$$

and k/N is a fractional value.

A circuit for the QFT

We will show that

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega^{xk}|k\rangle$$

can be factorized as:

$$\frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

This form reveals to us the circuit that creates this state!

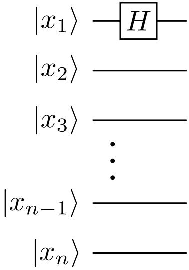
We did this last time:

$$
\begin{aligned}
|x\rangle \quad \rightarrow \quad & \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle \\
= \quad & \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x \left(\sum_{\ell=1}^{n} k_\ell 2^{-\ell}\right)} |k_1 \cdots k_n\rangle \\
= \quad & \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \\
= \quad & \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(\sum_{k_\ell=0}^{1} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \right) \\
= \quad & \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(|0\rangle + e^{2\pi i x 2^{-\ell}} |1\rangle \right) \\
= \quad & \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right) \cdots \left(|0\rangle + e^{2\pi i 0.x_1 \cdots x_n}|1\rangle\right)}{\sqrt{N}}
\end{aligned}
$$

Starting with the state

$$|x\rangle = |x_1 \cdots x_n\rangle,$$

apply a Hadamard to qubit 1:



A circuit for the QFT

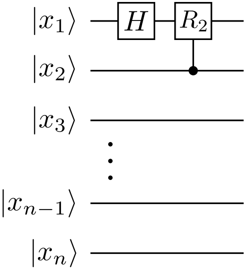
We are trying to make

$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

Every qubit has a different *phase* on the $|1\rangle$ state.

We need a gate that adds this:

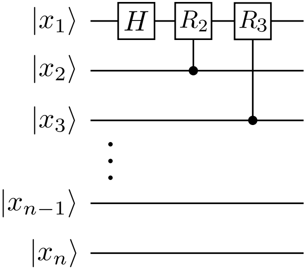
Apply controlled R_2 from qubit $2 \to 1$



First qubit picks up a phase:

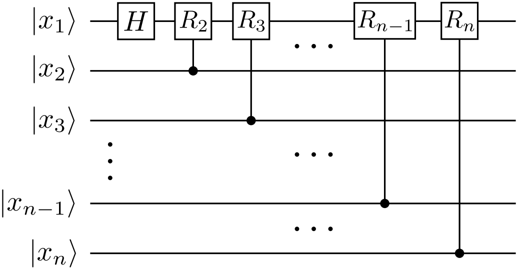
A circuit for the QFT

Apply controlled R_3 from qubit $3 \to 1$



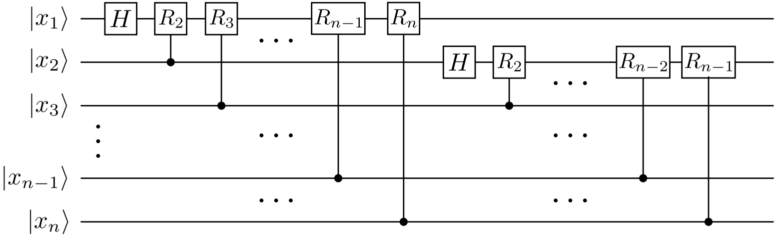
First qubit picks up another phase:

Apply a controlled R_4 from $4 \to 1$, etc. up to the n-th qubit to get



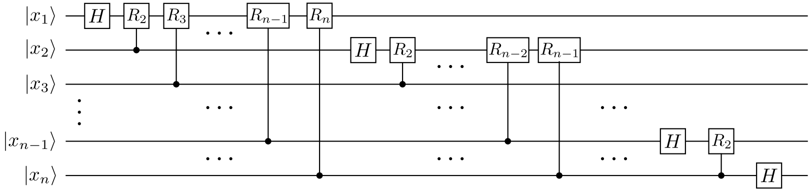
A circuit for the QFT

Next, do the same thing with the second qubit: apply H, and then controlled rotations from every qubit from 3 to n to get



Do this for all qubits to get that big ugly state from earlier:

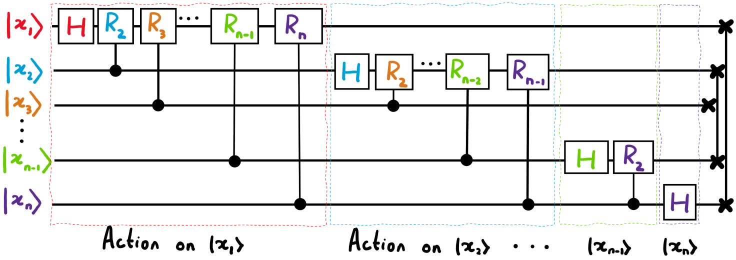
$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$



(though note that the order of the qubits is backwards - this is easily fixed with some SWAP gates)

Gate counts:

- n Hadamard gates
- $n(n-1)/2$ controlled rotations
- $\lfloor n/2 \rfloor$ SWAP gates if you care about the order



The number of gates is *polynomial in n*!

Image credit: Xanadu Quantum Codebook node F.3

Reminder: where are we going?

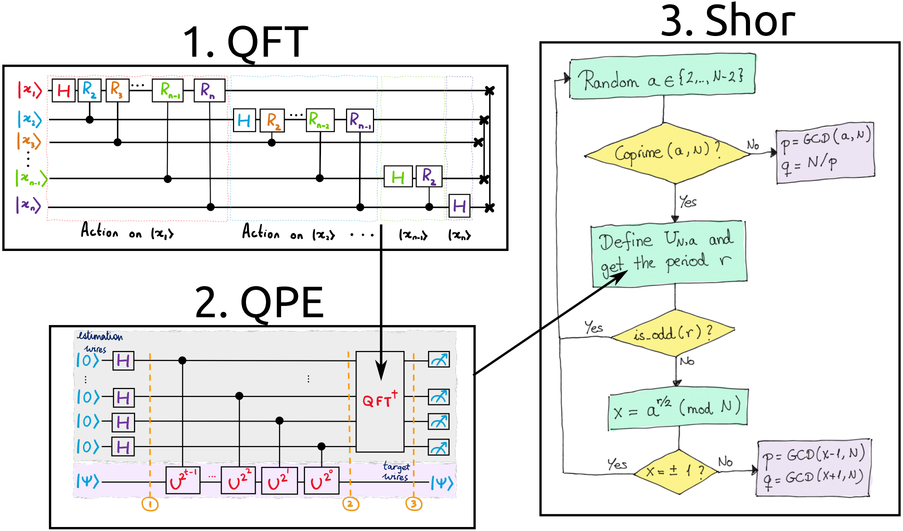


Image credit: Xanadu Quantum Codebook nodes F.3, P.3, S.4

Eigenvalues of unitary matrices

Fun fact: eigenvalues of unitary matrices are complex numbers with magnitude 1.

Proof:

So we can write

where θ_k is some phase angle such that $|\theta_k| \leq 1$.

What if we want to *learn* an unknown θ_k?

Eigenvalues of unitary matrices

Idea: apply U to the relevant eigenvector, because that's "what makes the phase come out".

...but this is an unobservable *global* phase!

We have to do something different: eigenvalue estimation, or quantum phase estimation (QPE).

Quantum phase estimation

Given a unitary U and one of its eigenvectors $|k\rangle$, estimate the value of θ_k such that

$$U|k\rangle = e^{2\pi i \theta_k}|k\rangle$$

Must determine:

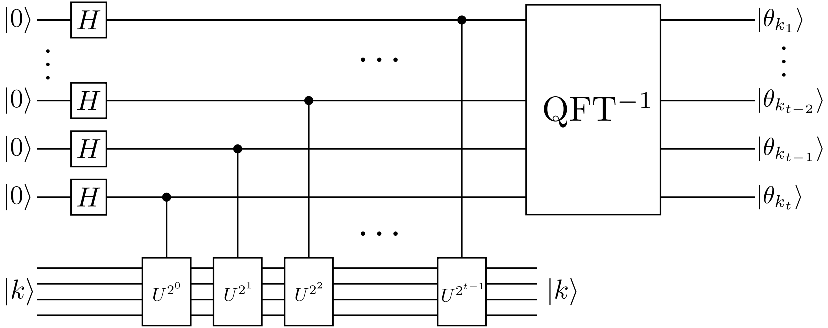
- How to design a circuit that extracts the θ_k
- To what precision can we estimate it
- What to do if we don't know a $|k\rangle$ in advance

(*You will explore the last two in your homework!*)

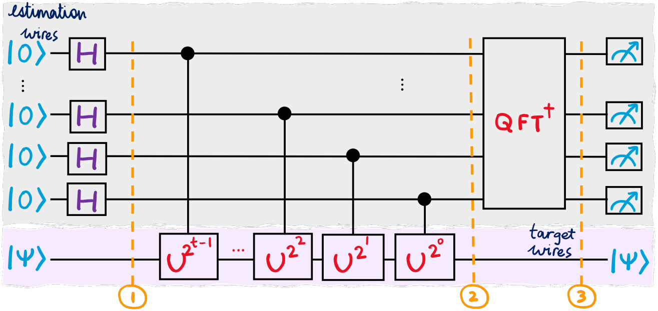
Let U be an n-qubit unitary; $|k\rangle$ is an n-qubit eigenstate.

Assume θ_k can be represented *exactly* using t bits:

$$\theta_k = 0.\theta_{k_1} \cdots \theta_{k_t}$$



You may see this version too:



Let's analyze the state at points 1, 2, and 3 above.

Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 1

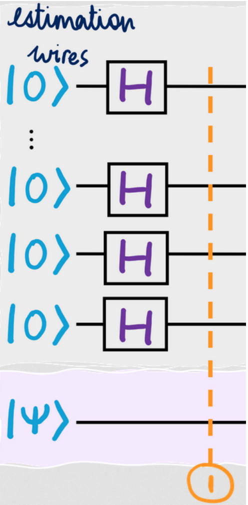
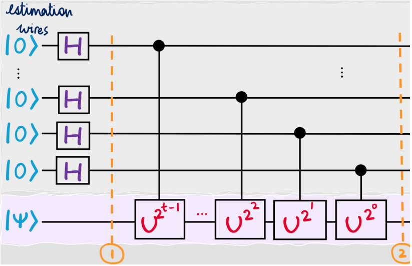


Image credit: Xanadu Quantum Codebook node P.2



We apply U to $|k\rangle$; how does the phase get to the top register?

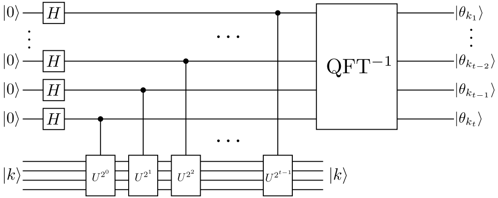


Image credit: Xanadu Quantum Codebook node P.2

The secret lies in something called *phase kickback*.

What happens when we apply a CNOT to the following state?

$$|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Phase kickback

What happens when we apply a CNOT to this state?

$$|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

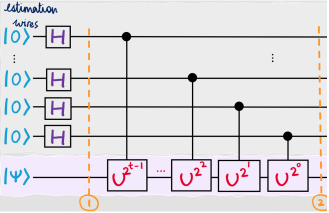
It looks like we've changed the phase of the second qubit.

The math doesn't care which qubit a global phase is attached to.

Seems the *target* qubit has done something to the *control* qubit!

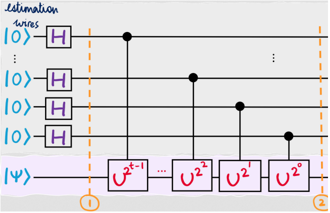
We say that the phase has been "kicked back" from the second qubit to the first.

Quantum phase estimation: step 2



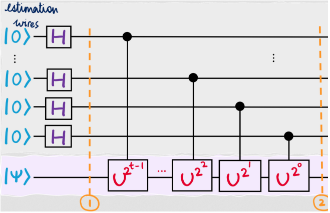
Consider the top-most qubit:

Quantum phase estimation: step 2



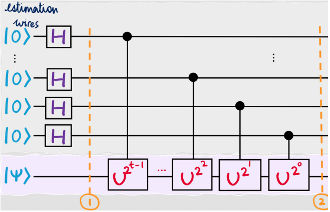
Use phase kickback

Quantum phase estimation: step 2

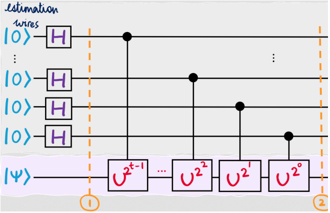


What is happening in the exponent?

Quantum phase estimation: step 2

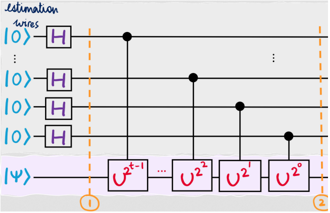


So we have the combined state:



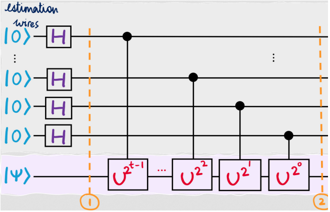
Let's do the second-last qubit (ignore what happens to others for now):

Quantum phase estimation: step 2



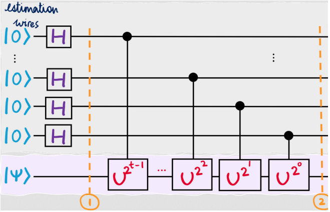
Again check the exponent...

Quantum phase estimation: step 2

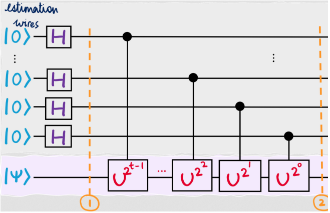


So we have the combined state:

Quantum phase estimation: step 2



Can show in the same way that for the last qubit

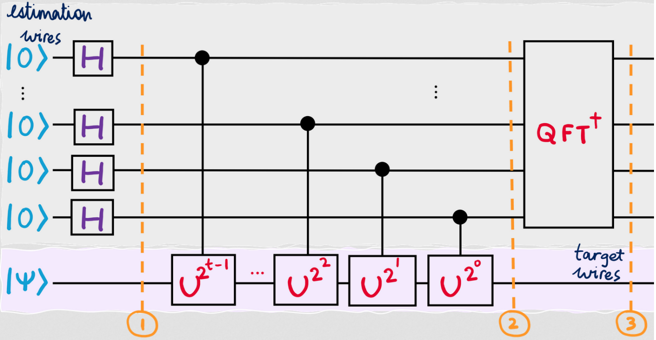


After step 2, we have the state

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle)\cdots\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Should look familiar!

Quantum phase estimation: step 3

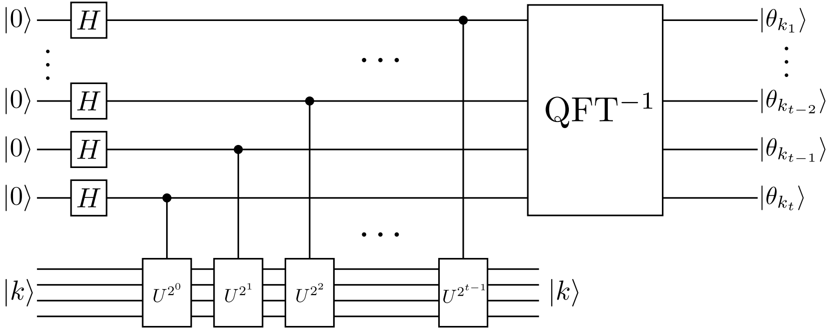


Last step is to apply the *inverse* QFT to recover the state...

Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 3

We can then measure to learn the numerical value of θ_k.



Let's implement it.

Next time

Content:

- Quiz 5 on Monday
- Continuing with QPE
- Moving towards Shor's algorithm

Action items:

1. Choose project group and paper
2. Literacy assignment 2

Recommended reading:

- Codebook nodes F.1-F.3, P.1-P.4
- Nielsen & Chuang 5.1, 5.2