

# **CPEN 400Q / EECE 571Q Lecture 16**

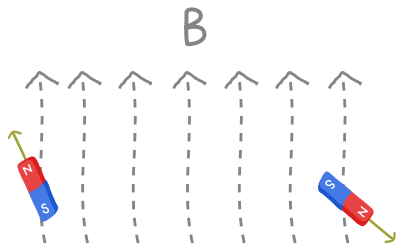
## **Mixed states, noise, and quantum channels**

Thursday 10 March 2022

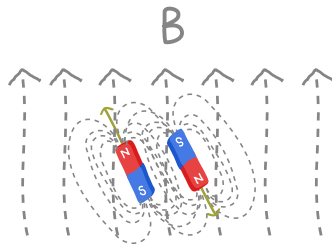
- Assignment 3 due Friday 11 March 23:59
- Project prototype meetings
  - Today: come to my office first (KAIS 3043), and we can snag KAIS 3028 if free
  - Tomorrow: KAIS 3065 booked from 15:00-17:00
  - Both days: Zoom (use my office hours link)

## Last time

We introduced the idea of Hamiltonians, Hermitian operators that describe the energy of physical systems. They can be expressed as linear combinations of Pauli operators.



$$\hat{H} = -\alpha Z_0 - \alpha Z_1$$



$$\hat{H} = -\alpha Z_0 - \alpha Z_1 + \beta(X_0 X_1 + Y_0 Y_1 + Z_0 Z_1)$$

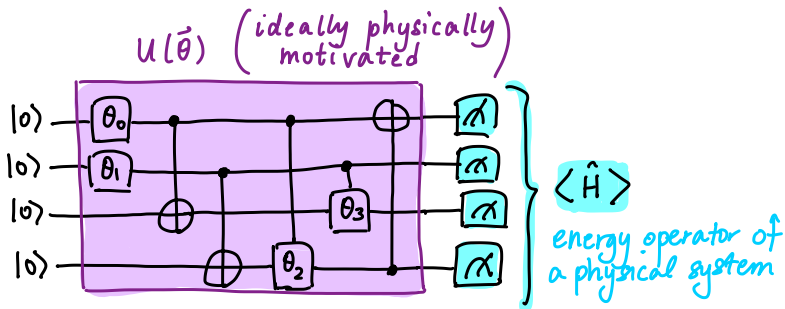
Image credits: Xanadu Quantum Codebook node H.5

The energy of a system is the *expectation value of the Hamiltonian*. It is computed as a linear combination of the expectation values of its Pauli constituents.

$$\begin{aligned}\hat{H} = \sum_i c_i P_i \quad \Rightarrow \quad \langle \hat{H} \rangle &= \langle \psi | \hat{H} | \psi \rangle \\ &= \langle \psi | \left( \sum_i c_i P_i \right) | \psi \rangle \\ &= \sum_i c_i \langle \psi | P_i | \psi \rangle \\ &= \sum_i c_i \langle P_i \rangle\end{aligned}$$

## Last time

We computed the ground state energy of a small quantum system with a variational eigensolver.



$$\min_{\vec{\theta}} \langle \hat{H} \rangle = \min_{\vec{\theta}} \langle 0 | U^\dagger(\vec{\theta}) \hat{H} U(\vec{\theta}) | 0 \rangle \rightarrow E_g$$

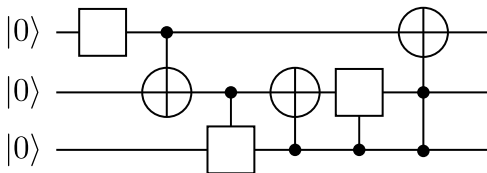
(ground state energy)

## Solution to quiz 7

Use the VQE to determine the ground state energy of

$$\hat{H} = X_0X_1 + 2X_1X_2 + 3X_0X_2 - Z_0 - 2Z_1 - 3Z_2$$

You were given the template circuit

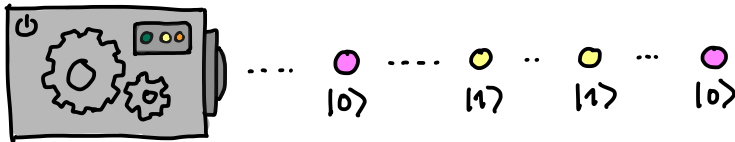
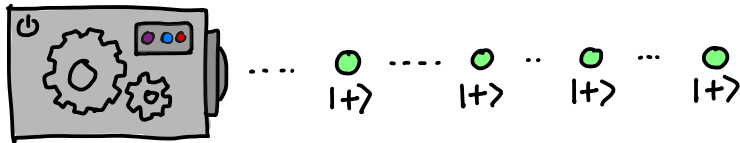


Let's code it up.

- Define a *mixed state*, and express quantum states using density matrices
- Describe the effects of common noise channels on qubit states
- Add noise to quantum circuits in PennyLane

# Mixed states

Suppose we have two different “boxes” that shoot particles:

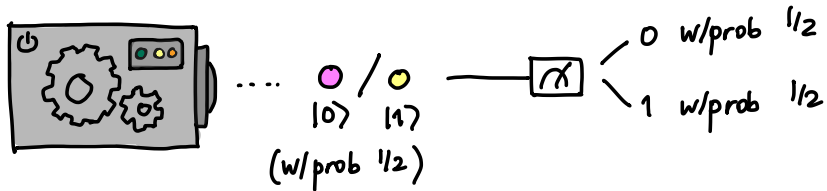
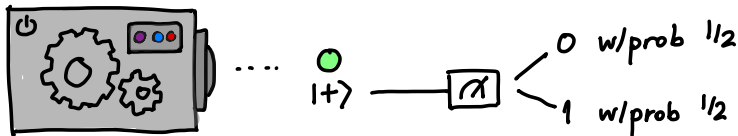


Are these the same?



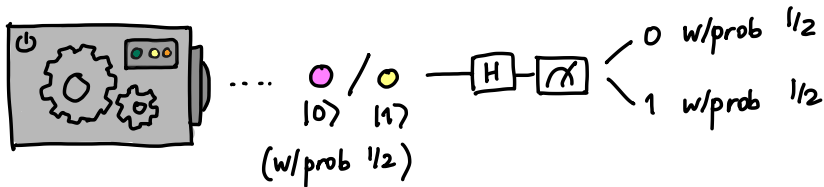
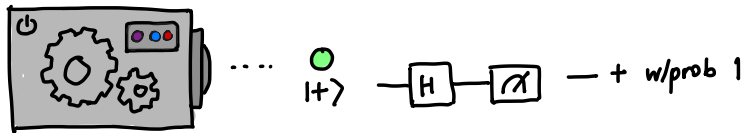
## Mixed states

If we measure in the computational basis, it looks like they are.



## Mixed states

But if we measure in the Hadamard basis, they are not!



What is the second box doing?

The second box is outputting something called a **mixed state**.

A state is a **pure state** if it can be expressed as a single ket vector, e.g.,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

A state is a **mixed state** if it can be expressed as a *probabilistic mixture of pure states* (it describes an ensemble of states).

$$? = ???$$

... what does that look like?

## Density matrices

Mixed states cannot be represented as ket vectors. Instead, we use a matrix representation called a **density matrix**.

The density matrix of a pure state  $|\psi\rangle$  is

For example,

## Density matrices

Density matrices of mixed states are linear combinations of density matrices of pure states:

For example, suppose we have a box that prepares  $|+\rangle$  with probability  $1/3$ , and  $|0\rangle$  with probability  $2/3$ :

Density matrices have some nice properties.

- they are Hermitian
- they have trace 1
- they are positive semi-definite (all eigenvalues are  $\geq 0$ )
- (for pure states only) they are projectors, i.e.,  $\rho^2 = \rho$

Check with our example:

$$\rho = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

- clearly Hermitian
- $\text{Tr}\rho = 5/6 + 1/6 = 1$
- eigenvalues are 0.872678 and 0.127322, both  $\geq 0$
- not pure, so  $\rho^2 \neq \rho$

Fun activity: show properties hold for general  $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$

## Working with density matrices and mixed states

We can do all the normal things we do to pure states (i.e., operations, measurements) with mixed states as well.

For a pure state  $|\psi\rangle$  and operation  $U$ ,

As mixed states,



More generally,

## Mixed states and measurements

What about measurements?

Recall that for a pure state  $|\psi\rangle$ , the probability of measuring and observing it in state  $|\varphi\rangle$  is computed using the inner product:

We can rewrite this...

$|\varphi\rangle\langle\varphi|$  is the density matrix of  $|\varphi\rangle$ , which is a *projector*. We are projecting  $|\psi\rangle$  onto  $|\varphi\rangle$ , and then measuring the overlap with  $|\psi\rangle$ .

## Mixed states and measurements

Measurement is performed w.r.t. a basis  $\{|\varphi_i\rangle\}$ ; there are multiple possible outcomes:

For mixed states, measurement outcome probabilities follow the **Born rule**:

where the set  $\{P_i\}$  is called a **positive operator-valued measure (POVM)**. The elements of the POVM satisfy

## Mixed states and measurements

Can see that this reduces to our original projective measurement in the case where  $\rho$  is a pure state...

For an  $m \times m$  matrix  $A$ ,

## Mixed states and measurements

Can see that this reduces to our original projective measurement in the case where  $\rho$  is a pure state...

## Mixed states and measurements

Example POVM:  $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ .

First, check the criteria:

For a particular  $\rho$ ,

## Mixed states and measurements

Now, remember how we computed expectation values from samples back in one of the early classes:

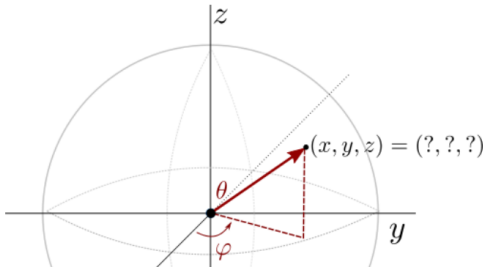
We can compute these probabilities in terms of the trace and  $\rho$ ...

# Mixed states and measurements

We can do the same for  $Y$  and  $Z$ : We can compute these probabilities in terms of the trace and  $\rho$ ...

Remember from assignment 1:

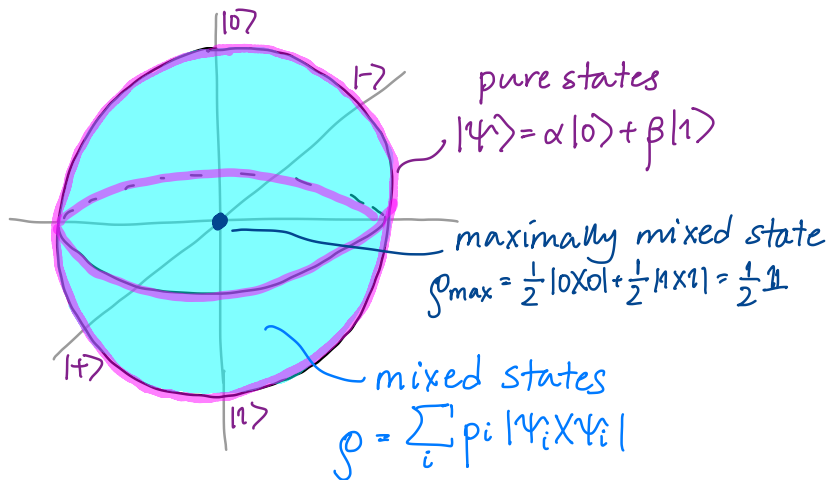
Write a function that uses quantum measurements to extract its Bloch vector, i.e., its coordinates in 3-dimensional space, as demonstrated by the following graphic:





## Mixed states on the Bloch sphere

Mixed states live *in* the Bloch sphere!



## Mixed states

More formally, we can write any  $\rho$  as

where  $a_P = \text{Tr}(P\rho) = \langle P \rangle$ .

(Should know such an expansion is possible since  $\rho$  is Hermitian, and Paulis are a basis for Hermitian matrices)

The case where  $a_x = a_y = a_z = 0$  is the **maximally mixed state**.

(Note that all of this generalizes to multiple qubits as well)

## Quantum channels and noise

## Quantum channels

Noise occurring in quantum systems is represented by **quantum channels**.

A quantum channel  $\Phi$  *maps* states to other states.

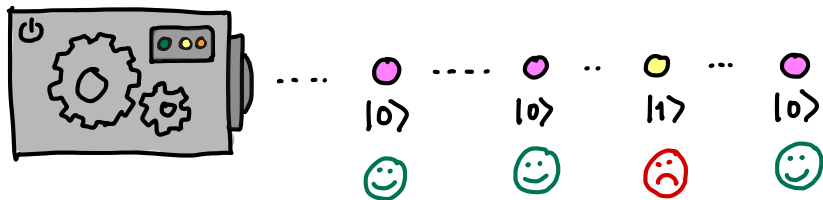
*More formally, quantum channels are linear CPTP (Completely Positive, Trace-Preserving) maps.*

Example: applying a unitary  $U$  is a channel,  $\mathcal{U}$ .

# The bit flip channel

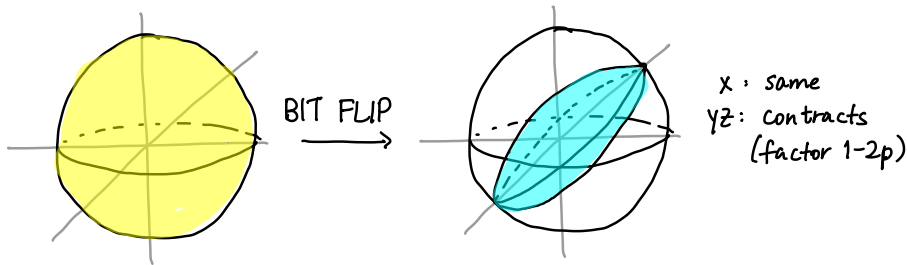
Suppose a “bit flip” (Pauli  $X$ ) error occurs with probability  $p$ .

$$\mathcal{E}(\rho) = (1-p) \cdot \rho + p \cdot X \rho X$$



# The bit flip channel

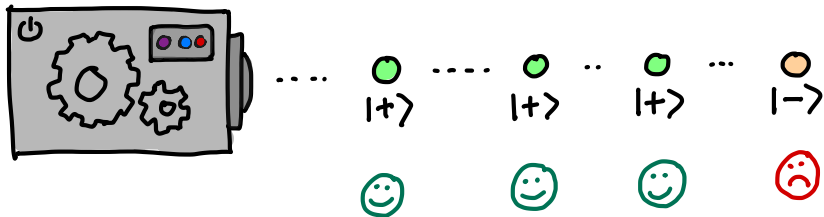
We can visualize the effects of such a channel by observing how it deforms the Bloch sphere.



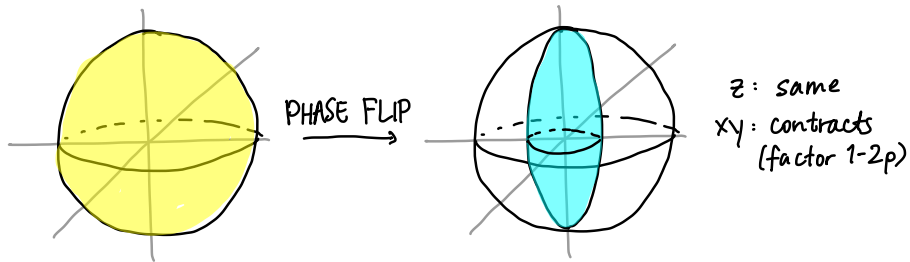
# The phase flip channel

Suppose a “phase flip” (Pauli  $Z$ ) error occurs with probability  $p$ .

$$\mathcal{E}(\rho) = (1-p) \cdot \rho + p \cdot Z \rho Z$$



# The phase flip channel

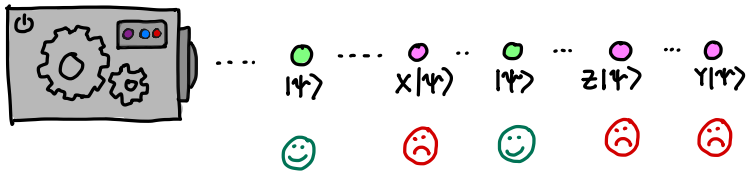




# The depolarizing channel

Suppose each Pauli error occurs with probability  $p/3$ . This is called the *depolarizing channel*.

$$\mathcal{E}(\rho) = (1-p) \cdot \rho + \frac{p}{3} \cdot X \rho X + \frac{p}{3} Y \rho Y + \frac{p}{3} Z \rho Z$$



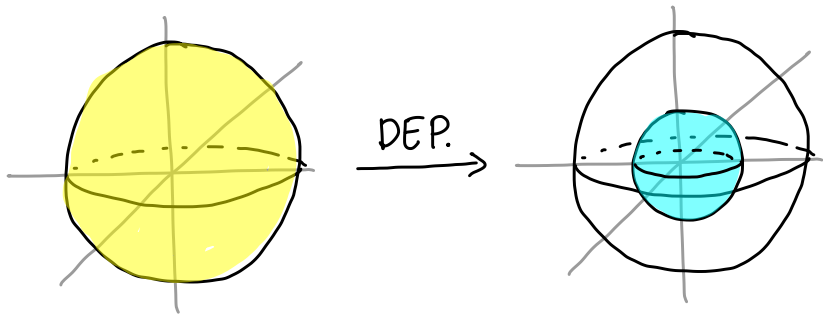
# The depolarizing channel

The depolarizing channel

can also be written as

Think of this as outputting  $\rho$  w/probability  $1 - p$ , and maximally mixed state with probability  $p$ .

# The depolarizing channel



# Comparing density matrices

How can we quantify “how much” error occurs? How close is  $\sigma = \mathcal{E}(\rho)$  to  $\rho$ ?

One common metric is the **trace distance**:

Value of trace distance is bounded by  $0 \leq T(\rho, \sigma) \leq 1$ , and *lower* trace distance is better.

Another is the **fidelity**:

Value of fidelity is bounded by  $0 \leq F(\rho, \sigma) \leq 1$ , and *higher* fidelity is better.

# Next time

## Content:

- VQE part II: VQE for real molecules
- What does *actual* hardware noise look like?
- How do we process noisy results?

## Action items:

1. Prototype implementation for project
2. Assignment 3

## Recommended reading:

- Nielsen and Chuang Ch. 8