CPEN 400Q Lecture 04
More on measurement; multi-qubit states and gates

Friday 20 January 2023

Announcements

- Literacy assignment 1 due Wednesday 25 Jan at 23:59
- Technical assignment 1 released over next few days (will be due in 2 weeks)
- Short class on Monday
  - Need to leave at 16:00
  - Quiz 2 will be at the *end* of class

We introduced the “bra” part of the “bra-ket notation”

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \langle v| = (|v\rangle)^\dagger = (v_1^* \;\; v_2^*)$$

The inner product between two states is defined as

$$\langle v|w\rangle = v_1^* w_1 + v_2^* w_2$$

Inner product tells about the *overlap* (similarity) between states.

We introduced the concept of *orthonormal bases* for qubit states:

$$\{|\psi_1\rangle, |\psi_2\rangle\} \to \langle\psi_i|\psi_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

Examples: "Hadamard"

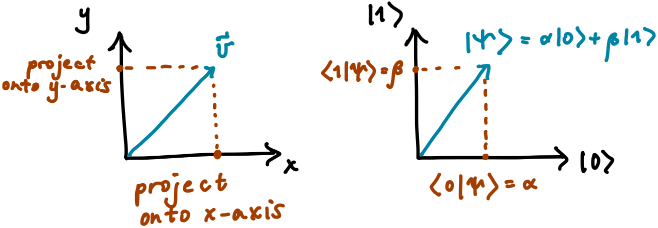
$$|0\rangle \quad \text{and} \quad |1\rangle$$

computational basis

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \quad \text{and} \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

We discussed *projective measurement* with respect to a basis.



When we measure state $|\varphi\rangle$ with respect to basis $\{|\psi_i\rangle\}$, the probability of obtaining outcome i is

$$\Pr(\text{outcome } i) = |\langle \psi_i | \varphi \rangle|^2$$

Image credit: Xanadu Quantum Codebook I.9

Learning outcomes

- Measure a single qubit in different bases
- Mathematically describe a system of multiple qubits
- Describe the action of common multi-qubit gates
- Make any gate a controlled gate
- Perform measurements on multiple qubits

Note: moving expectation values (originally L3) to a future lecture.

Exercise: consider the quantum state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle$$

If we measure in the $\{|p\rangle, |m\rangle\}$ basis, what are the measurement probabilities of the possible outcomes?

$$|p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

Basis rotations

Solution:

$\langle p|\psi\rangle =$

$$
\begin{aligned}
\langle p|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\langle 0| - i\langle 1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}e^{i\frac{5}{4}}|1\rangle\right)\\
&= \frac{\sqrt{3}}{2\sqrt{2}}\langle 0|0\rangle + \frac{i}{2\sqrt{2}}e^{i\frac{5}{4}}\langle 1|1\rangle\\
&= \frac{1}{2\sqrt{2}}\left(\sqrt{3} + ie^{i\frac{5}{4}}\right)\\
\Pr(p) &= |\langle p|\psi\rangle|^2 = \frac{1}{8}(\sqrt{3} + ie^{i\frac{5}{4}})(\sqrt{3} - ie^{-i\frac{5}{4}})\\
&= \frac{1}{8}\left(3 + \sqrt{3}i\left(e^{i\frac{5}{4}} - e^{-i\frac{5}{4}}\right) + 1\right)\\
&= \frac{1}{8}(4 + \sqrt{3}i(2i\sin(5/4)))\\
&= \frac{1}{8}(4 - 2\sqrt{3}\sin(5/4))\\
&= \approx 0.089
\end{aligned}
$$

Tedious... let's use software. But how?

Use a basis rotation to "trick" the quantum computer.

Unitary operations preserve length *and* angles between normalized quantum state vectors; there exists some unitary transformation that will convert between any basis and the computational basis.

Exercise: Suppose we want to measure in the "Y" basis. Determine a quantum circuit that sends

$$RZ(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$|0\rangle \quad \rightarrow \quad |p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$
$$|1\rangle \quad \rightarrow \quad |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$\sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$H: \quad |0\rangle \longrightarrow |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
$$|1\rangle \longrightarrow |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$S: \quad |+\rangle \longrightarrow |p\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad |-\rangle \longrightarrow |m\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

At the end of our circuit, apply the reverse (adjoint) of this transformation to rotate *back* to the computational basis.

$$|p\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow |0\rangle$$

$$|m\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \rightarrow |1\rangle$$

$UU^\dagger = \mathbb{1}$

$\downarrow$

$U = SH$

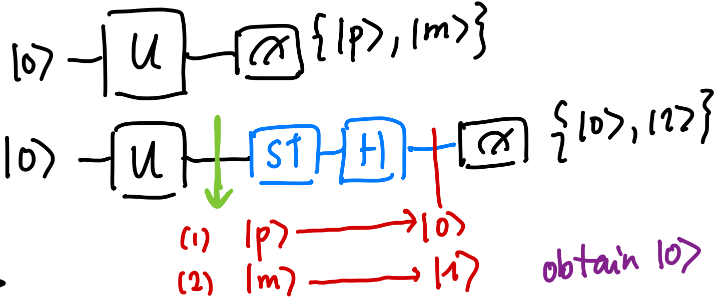
$U^\dagger = (SH)^\dagger$

$= H^\dagger S^\dagger$

$= HS^\dagger$

$|p\rangle \xrightarrow{HS^\dagger} |0\rangle$

$|m\rangle \xrightarrow{HS^\dagger} |1\rangle$



$(AB)^T = B^T A^T$

$(AB)^\dagger = B^\dagger A^\dagger$

Hands-on: adjoints

U $U^\dagger$ - adjoint

In PennyLane, we can compute adjoints of operations *and* entire quantum functions using `qml.adjoint`:

```
def some_function(x):
    qml.RZ(Z, wires=0)

def apply_adjoint(x):
    qml.adjoint(qml.S)(wires=0)
    qml.adjoint(some_function)(x)
```

`qml.adjoint` is a special type of function called a **transform**. We will cover transforms in more detail later in the course.

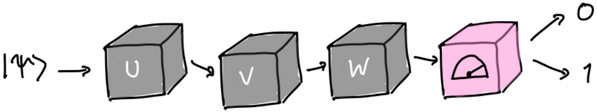
Multi-qubit systems

Recall this slide from lecture 1...

Quantum computing

Quantum computing is the act of manipulating the state of qubits in a way that represents solution of a computational problem:

1. **Prepare** qubits in a **superposition**
2. Apply **operations** that **entangle** the qubits and manipulate the amplitudes
3. **Measure** qubits to extract an answer
4. Profit



Let's simulate this using NumPy.

How do we express the mathematical space of multiple qubits?

Hilbert spaces compose under the *tensor product*.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

The tensor product of A and B, $A \otimes B$ is

$$A \otimes B = \begin{pmatrix} a\begin{pmatrix} e & f \\ g & h \end{pmatrix} & b\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ c\begin{pmatrix} e & f \\ g & h \end{pmatrix} & d\begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

np.kron

Qubit state vectors are also combined using the *tensor product*:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\[1em] 0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

An n-qubit state is therefore a vector of length 2^n.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are the computational basis vectors for 2 qubits: $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We can create arbitrary linear combinations of them as long as the normalization on the coefficients holds.

Same pattern for 3 qubits: $|000\rangle, |001\rangle, \ldots, |111\rangle$.

The tensor product is linear and distributive, so if we have

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle,$$

then they tensor together to form

$$\begin{aligned} |\psi\rangle \otimes |\varphi\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle) \\ &= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \\ &= \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} \end{aligned}$$

Single-qubit unitary operations also compose under tensor product.

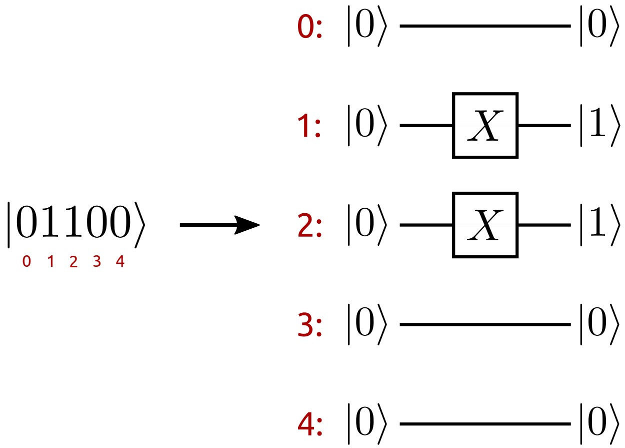
For example, apply U_1 to qubit $|\psi\rangle$ and U_2 to qubit $|\varphi\rangle$:

$$(U_1 \otimes U_2)(|\psi\rangle \otimes |\varphi\rangle) = (U_1|\psi\rangle) \otimes (U_2|\varphi\rangle)$$

If an n-qubit ket is a vector with length 2^n, then a unitary acting on n qubits has dimension $2^n \times 2^n$.

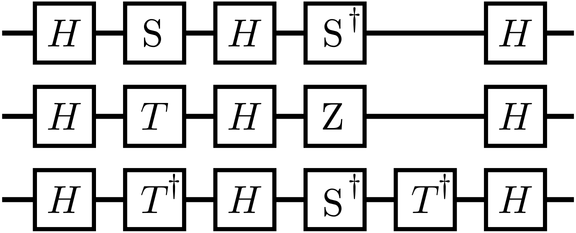
Qubit ordering (very important!)

In PennyLane:



(This is different in other frameworks!)

The few small circuits we've seen so far only involve gates on single qubits:



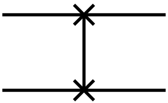
Surely this isn't all we can do...

Image credit: Xanadu Quantum Codebook I.11

We can swap the state of two qubits using the SWAP operation. First define what it does to the basis states...

$$\begin{aligned} SWAP|00\rangle &= |00\rangle \\ SWAP|01\rangle &= |10\rangle \\ SWAP|10\rangle &= |01\rangle \\ SWAP|11\rangle &= |11\rangle \end{aligned} \qquad SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Circuit element:



PennyLane: `qml.SWAP`

More generally,

$$\mathit{SWAP}\left(|\psi\rangle \otimes |\phi\rangle\right) = |\phi\rangle \otimes |\psi\rangle$$

Let's show this. Start by writing

$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \left(\gamma|0\rangle + \delta|1\rangle\right) \\ &= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle \end{aligned}$$

Now apply the SWAP:

$$\begin{aligned} \mathit{SWAP}\left(|\psi\rangle \otimes |\phi\rangle\right) &= \alpha\gamma|00\rangle + \alpha\delta|10\rangle + \beta\gamma|01\rangle + \beta\delta|11\rangle \\ &= \gamma\alpha|00\rangle + \gamma\beta|01\rangle + \delta\alpha|10\rangle + \delta\beta|11\rangle \\ &= \gamma|0\rangle(\alpha|0\rangle + \beta|1\rangle) + \delta|1\rangle(\alpha|0\rangle + \beta|1\rangle) \\ &= \left(\gamma|0\rangle + \delta|1\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right) \\ &= |\phi\rangle \otimes |\psi\rangle \end{aligned}$$

Consider a two-qubit operation U with the following action on the basis states:

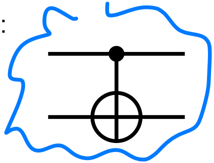
$$\begin{aligned} U|00\rangle &= |00\rangle \\ U|01\rangle &= |01\rangle \\ U|10\rangle &= |11\rangle \\ U|11\rangle &= |10\rangle \end{aligned}$$

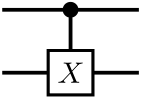
CNOT = "controlled-NOT". A NOT (X) is applied to second qubit only if first qubit is in state $|1\rangle$.

$$CNOT|00\rangle = |00\rangle$$
$$CNOT|01\rangle = |01\rangle$$
$$CNOT|10\rangle = |11\rangle$$
$$CNOT|11\rangle = |10\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Circuit elements:

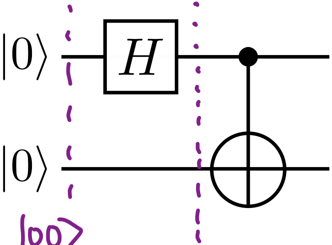




PennyLane: `qml.CNOT`

Image credit: Xanadu Quantum Codebook I.12

What does CNOT do with qubits in a superposition?



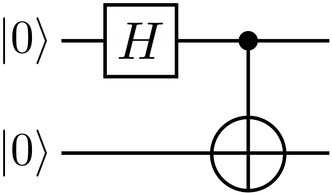
$|00\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle$

$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$

$|\psi\rangle = ??$

$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



The output state of this circuit is:

$$CNOT \cdot (H \otimes I)\,|00\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle)$$

This state is **entangled**!

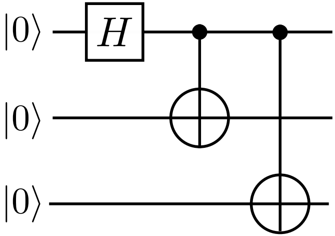
$\neq |\psi\rangle \otimes |\varphi\rangle$

We *cannot express*

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

as a tensor product of two single-qubit states.

Entanglement generalizes to more than two qubits:



GHZ state

Exercise: Express the output state of this circuit in the computational basis.

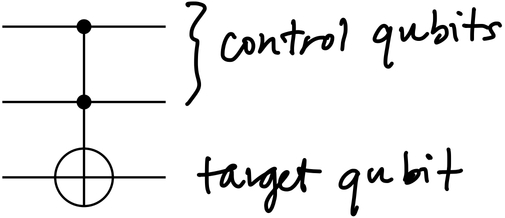
Consider the AND of two bits a and b:

a	b	ab
0	0	0
0	1	0
1	0	0
1	1	1

This gate is *not* reversible: we cannot recover inputs from outputs.

But, we can make it reversible by adding one extra bit...

The **Toffoli** implements a reversible AND gate. (It is universal for classical reversible computing).



Controlled-CNOT, or controlled-controlled-NOT.

PennyLane: `qml.Toffoli`

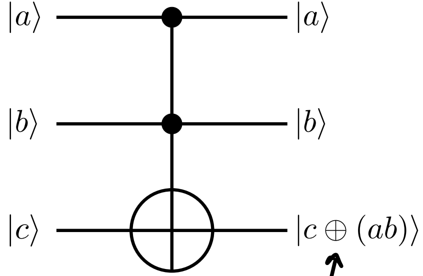
What does it do to the basis states?

$$TOF|000\rangle = |000\rangle$$
$$TOF|001\rangle = |001\rangle$$
$$TOF|010\rangle = |010\rangle$$
$$TOF|011\rangle = |011\rangle$$
$$TOF|100\rangle = |100\rangle$$
$$TOF|101\rangle = |101\rangle$$
$$TOF|110\rangle = |111\rangle$$
$$TOF|111\rangle = |110\rangle$$

Toffoli

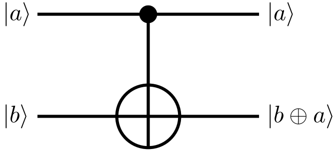
What is actually going on here?

$$
\begin{aligned}
TOF|000\rangle &= |000\rangle \\
TOF|001\rangle &= |001\rangle \\
TOF|010\rangle &= |010\rangle \\
TOF|011\rangle &= |011\rangle \\
TOF|100\rangle &= |100\rangle \\
TOF|101\rangle &= |101\rangle \\
TOF|110\rangle &= |111\rangle \\
TOF|111\rangle &= |110\rangle
\end{aligned}
$$

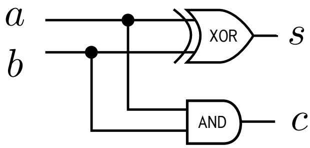


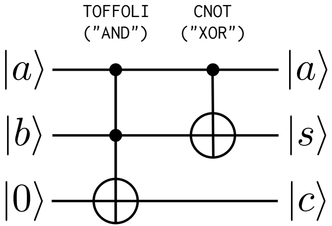
+ mod 2

We can interpret CNOT in a similar way.

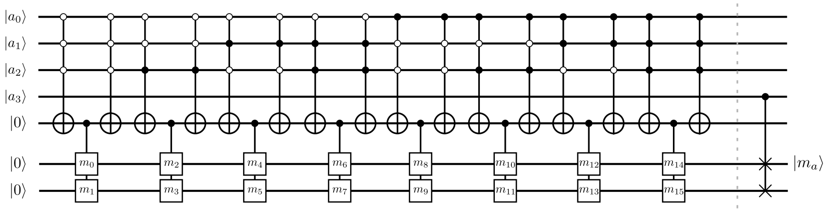


X, CNOT, TOF can be used to create Boolean arithmetic circuits.





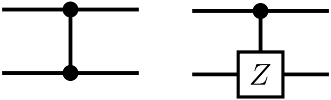
Any unitary operation can be turned into a controlled operation, controlled on any state.



Most common controls are controlled-on-$|1\rangle$ (filled circle), and controlled-on-$|0\rangle$ (empty circle).

Example: controlled-Z (CZ)

What does this operation do?



We stopped roughly here Friday; will start here next time!

PennyLane: `qml.CZ`

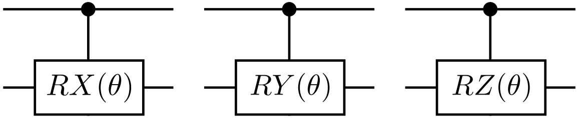
Image credit: Codebook node I.13

Example: controlled rotations (RX, RY, RZ)

Or this one?

$$CRY(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

Circuit elements:



PennyLane: `qml.CRX`, `qml.CRY`, `qml.CRZ`

Image credit: Codebook node I.14

Controlled-U

There is a pattern here:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad CRY(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

More generally,

$$CU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix} = \begin{pmatrix} I_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & U \end{pmatrix}$$

... we don't want to be writing these matrices all the time.

Hands-on: `qml.ctrl`

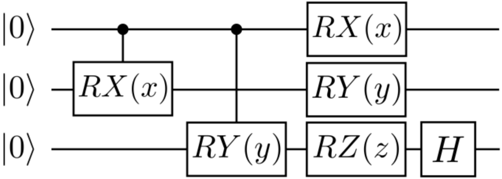
Remember from earlier, `qml.adjoint`:

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.adjoint(qml.S)(wires=0)
    return qml.sample()
```

There is a similar *transform* that allows us to perform arbitrary controlled operations (or entire quantum functions)!

```
@qml.qnode(dev)
def my_circuit():
    qml.S(wires=0)
    qml.ctrl(qml.S, control=1)(wires=0)
    return qml.sample()
```

Let's go implement this circuit:



Universal gate sets

Last class, we learned that with just

- H and T
- any two of RX, RY, and RZ,

we can implement *any* single-qubit unitary operation up to arbitrary precision.

What about for two qubits?

Universal gate sets

What about for two qubits?

- H, T, and $CNOT$
- any two of RX, RY, RZ, and $CNOT$
- H and TOF

With just 2-3 gates, we can implement *any* two-qubit unitary operation up to arbitrary precision.

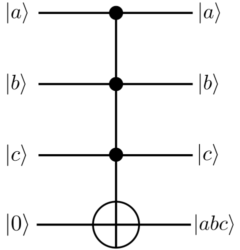
What about three or more qubits? (Same thing!)

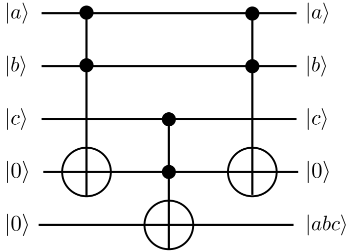
Universal gate sets

In general, finding such an implementation (*quantum circuit synthesis*, part of the quantum compilation pipeline) is computationally hard.

- sometimes we can do so for small cases (PennyLane has many decompositions pre-programmed)
- sometimes having **auxiliary** qubits around can simplify the decomposition

Auxiliary qubits are like "scratch", or "work" qubits. They start in state $|0\rangle$, and must be returned to state $|0\rangle$, but can be used to store intermediate results in a computation.





Review: single-qubit measurements

Given a state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- the probability of measuring and observing the qubit in state $|0\rangle$ is $|\alpha|^2 = \alpha\alpha^* = |\langle 0|\psi\rangle|^2$
- the probability of measuring and observing the qubit in state $|1\rangle$ is $|\beta|^2 = |\langle 1|\psi\rangle|^2$
- we can measure in different bases by "remapping" those basis states to the computational basis

We can do all this in the multi-qubit case as well.

Let

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

If we measure in the computational basis, the outcome probabilities are:

- $|\alpha|^2 = |\langle 00|\psi\rangle|^2$ for $|00\rangle$
- $|\beta|^2 = |\langle 01|\psi\rangle|^2$ for $|01\rangle$
- ...

Multi-qubit measurement outcome probabilities

Let

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

We can also measure *just one qubit*:

- The probability of the first qubit being in state $|0\rangle$ is $|\alpha|^2 + |\beta|^2$
- The probability of the second qubit being in state $|1\rangle$ is $|\beta|^2 + |\delta|^2$

Multi-qubit measurement outcome probabilities

Let

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

We can also measure *just one qubit*:

- The probability of the first qubit being in state $|0\rangle$ is $|\alpha|^2 + |\beta|^2$
- The probability of the second qubit being in state $|1\rangle$ is $|\beta|^2 + |\delta|^2$

Recap

- Measure a single qubit in different bases
- Mathematically describe a system of multiple qubits
- Describe the action of common multi-qubit gates
- Make any gate a controlled gate
- Perform measurements on multiple qubits

Next time

Content:

- Superdense coding
- The no-cloning theorem
- Quantum teleportation

$$|+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$|+0\rangle$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |0\rangle$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \neq |\psi\rangle \otimes |\phi\rangle$$

Action items:

1. Literacy assignment 1 due Wednesday
2. Technical assignment 1 posted soon
3. Quiz 2 Monday end of class on contents from this week

Recommended reading:

- From this class: Codebook nodes I.9, I.11-I.14
- For next class: Codebook nodes I.15