CPEN 400Q Lecture 10
The quantum Fourier transform and quantum phase estimation

Friday 10 February 2023

Announcements

- Literacy assignment 2 available (due after reading week)
- Project details posted (group and paper selection due next Friday)

We introduced the quantum Fourier transform, and saw how it is the analog of the classical inverse discrete Fourier transform.

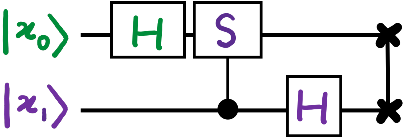
$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle$$

$$QFT = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}$$

where for n qubits, $N = 2^n$, and $\omega = e^{2\pi i/N}$

Last time

We saw the circuits for some special cases.



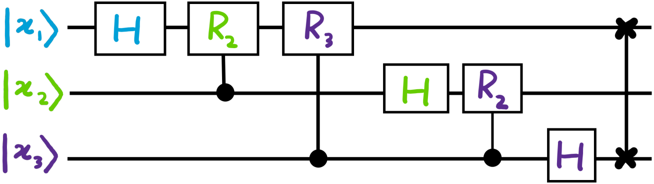


Image credit: Xanadu Quantum Codebook node F.2, F.3

I showed you what the general form of the circuit looked like:

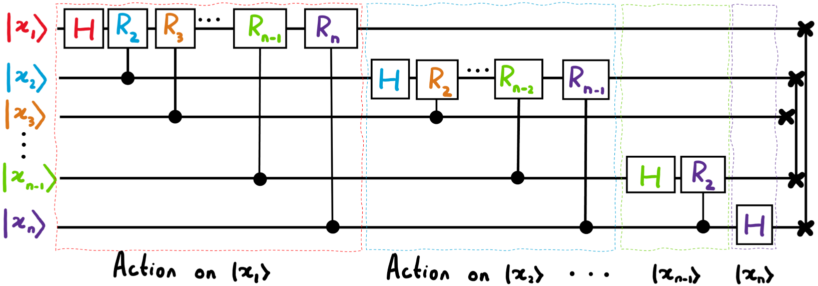


Image credit: Xanadu Quantum Codebook node F.3

Learning outcomes

- Derive the QFT circuit and implement it in PennyLane
- Describe the phase kickback trick
- Outline the steps of the quantum phase estimation (QPE) subroutine
- Use the QFT to implement QPE

Review: fractional binary notation

$k = 2^{n-1} \cdot k_1 + 2^{n-2} k_2 + \dots + 2k_{n-1} + k_n$

Example: Let $k = k_1 k_2 k_3 k_4 = \underline{0.1001}$. The numerical value of k is

$$\sum_{\ell=1}^{n} \frac{k_\ell}{2^\ell}$$

$$\begin{aligned} 0.1001 &= \frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} \\ &= \frac{1}{2} + \frac{1}{16} \\ &= 0.5625 \end{aligned}$$

We need this for the QFT because in the exponent, we have

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle$$

$\omega = e^{\frac{2\pi i}{N}}$, $N \rightarrow 2^n$

and k/N is a fractional value.

We will show that

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\omega^{xk}|k\rangle$$

can be factorized as:

$$\frac{\big(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\big) \otimes \big(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\big)\cdots \otimes \big(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\big)}{\sqrt{N}}$$

This form reveals to us the circuit that creates this state!

A circuit for the QFT

We did this last time:

$$
\begin{aligned}
|x\rangle \quad \rightarrow \quad & \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{xk} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x (k/N)} |k\rangle \\
= \quad & \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i x \left(\sum_{\ell=1}^{n} k_\ell 2^{-\ell}\right)} |k_1 \cdots k_n\rangle \\
= \quad & \frac{1}{\sqrt{N}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{\ell=1}^{n} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \\
= \quad & \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(\sum_{k_\ell=0}^{1} e^{2\pi i x k_\ell 2^{-\ell}} |k_\ell\rangle \right) \\
= \quad & \frac{1}{\sqrt{N}} \bigotimes_{\ell=1}^{n} \left(|0\rangle + e^{2\pi i x 2^{-\ell}} |1\rangle \right) \\
= \quad & \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right) \cdots \left(|0\rangle + e^{2\pi i 0.x_1 \cdots x_n}|1\rangle\right)}{\sqrt{N}}
\end{aligned}
$$

convert to bits

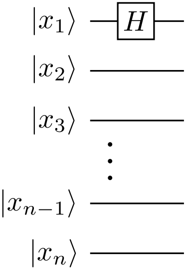
distribute phase across qubits

Starting with the state

$$|x\rangle = |x_1 \cdots x_n\rangle,$$

apply a Hadamard to qubit 1:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_1}|1\rangle\right)|x_2 \cdots x_n\rangle$$



$x_1 = 0$: $\quad 0.x_1 = 0 \quad \Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|x_2 \cdots x_n\rangle$

$x_1 = 1 \quad 0.x_1 = \frac{x_1}{2} \quad e^{2\pi i \cdot \frac{x_1}{2}} = e^{\pi i} = -1 \Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \cdot |x_2 \cdots x_n\rangle$

$0.x_1 x_2 \cdots x_n = \frac{x_1}{2} + \frac{x_2}{2^2} + \dots + \frac{x_n}{2^n}$

We are trying to make

$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$

Every qubit has a different *phase* on the $|1\rangle$ state.

We need a gate that adds this:

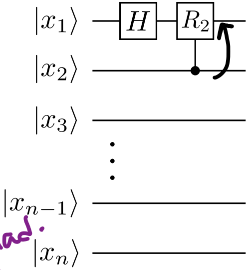
$$R_z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix}$$

A circuit for the QFT

Apply controlled R_2 from qubit $2 \rightarrow 1$



$$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^2} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^2}} \end{pmatrix}$$

$$e^{2\pi i \cdot \frac{1}{2^2}}$$

Had.

$$\frac{x_1}{2} + \frac{x_2}{2^2}$$

$-R_2$

First qubit picks up a phase:

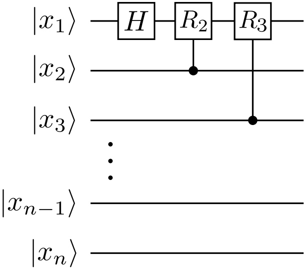
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1}|1\rangle\right)|x_2 \cdots x_n\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1 x_2}|1\rangle\right)|x_2 \cdots x_n\rangle$$

$$\frac{1}{\sqrt{2}}\left(\underbrace{|0\rangle + e^{2\pi i\, 0.x_1}|1\rangle}\right)|x_2 \cdots x_n\rangle$$

$$\overset{C\text{-}R_2}{=} \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.x_1 x_2}|1\rangle\right)|x_2 \cdots x_n\rangle$$

Apply controlled R_3 from qubit $3 \to 1$

$$R_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^3}} \end{pmatrix}$$



First qubit picks up another phase:

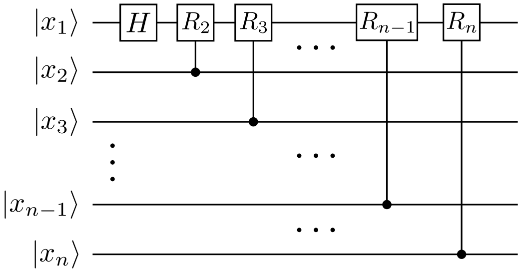
$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_2}|1\rangle\right)|x_2\cdots x_n\rangle$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_2x_3}|1\rangle\right)|x_2\cdots x_n\rangle$$

A circuit for the QFT

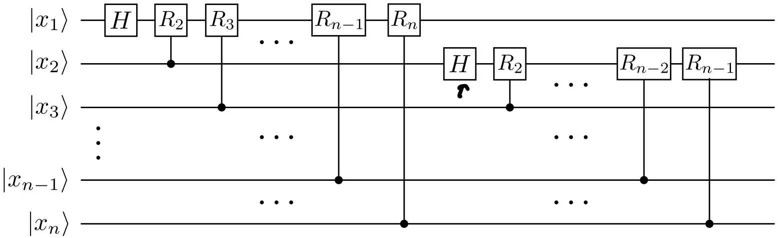
Apply a controlled R_4 from $4 \to 1$, etc. up to the n-th qubit to get

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i 0.x_1x_2\cdots x_n}|1\rangle\right)|x_2\cdots x_n\rangle$$



Next, do the same thing with the second qubit: apply H, and then controlled rotations from every qubit from 3 to n to get

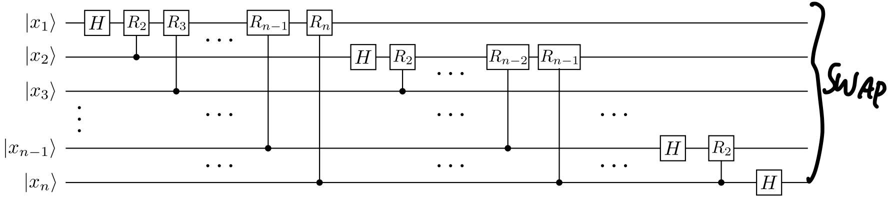
$$\frac{1}{(\sqrt{2})^2}\left(|0\rangle + e^{2\pi i 0.x_1 x_2 \cdots x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_2 x_3 \cdots x_n}|1\rangle\right)|x_3 \cdots x_n\rangle$$



A circuit for the QFT

Do this for all qubits to get that big ugly state from earlier:

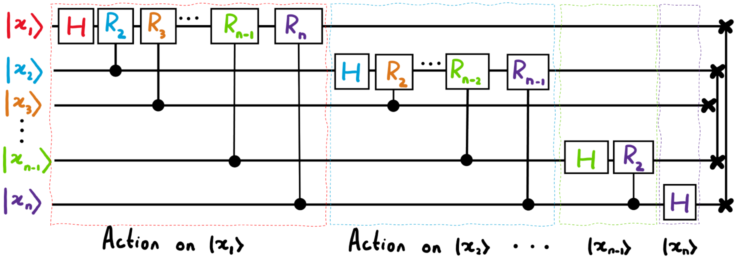
$$|x\rangle \to \frac{\left(|0\rangle + e^{2\pi i 0.x_n}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.x_{n-1}x_n}|1\rangle\right)\cdots\left(|0\rangle + e^{2\pi i 0.x_1\cdots x_n}|1\rangle\right)}{\sqrt{N}}$$



(though note that the order of the qubits is backwards - this is easily fixed with some SWAP gates)

Gate counts:

- n Hadamard gates
- $n(n-1)/2$ controlled rotations
- $\lfloor n/2 \rfloor$ SWAP gates if you care about the order



The number of gates is *polynomial in n*! Efficient!

Image credit: Xanadu Quantum Codebook node F.3

Reminder: where are we going?

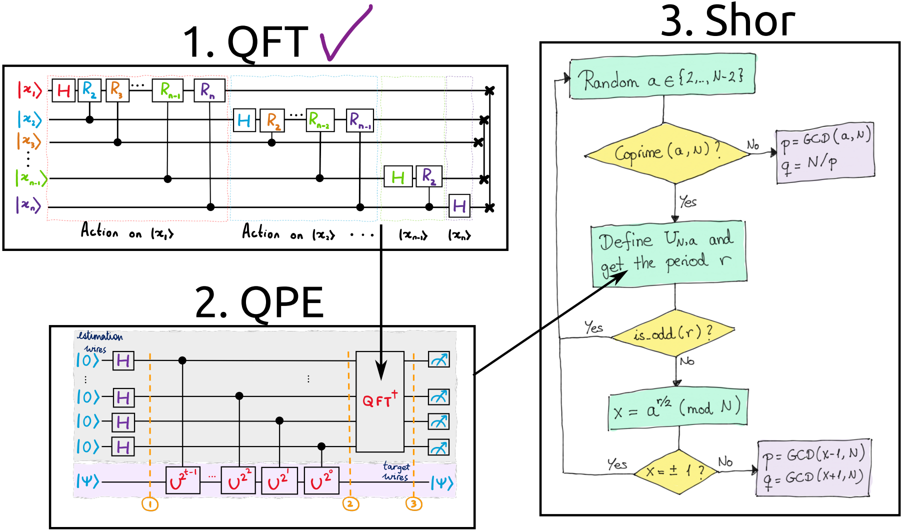


Image credit: Xanadu Quantum Codebook nodes F.3, P.3, S.4

Fun fact: eigenvalues of unitary matrices are complex numbers with magnitude 1.

$$U \leftrightarrow e^{i\boxed{\theta}}$$

Proof: Let λ_k be the eigenvalue associated with eigenvector $|k\rangle$ of a unitary U:

λ_k $\quad |k\rangle$

properly normalized state

$$U|k\rangle = \lambda_k|k\rangle$$

$$U|k\rangle = \lambda_k|k\rangle \quad ①$$

We can take the conjugate transpose of this equation:

$$\langle k|U^\dagger = \langle k|\lambda_k^*$$

$$(U|k\rangle)^\dagger = (\lambda_k|k\rangle)^\dagger \Rightarrow \langle k|U^\dagger = \lambda_k^*\langle k| \quad ②$$

Multiply the two sides together:

$$\langle k|U^\dagger U|k\rangle = \langle k|\lambda_k^*\lambda_k|k\rangle$$

$$\langle k|k\rangle = |\lambda_k|^2$$

$$1 = |\lambda_k|^2$$

② × ① :

$$\underbrace{\langle k|U^\dagger U|k\rangle}_{\langle k|I|k\rangle = \langle k|k\rangle} = \lambda_k^*\langle k|\cdot\lambda_k|k\rangle$$

$$= \lambda_k^*\lambda_k\langle k|k\rangle$$

$$1 = |\lambda_k|^2 \Rightarrow \lambda_k = e^{i\theta_k}$$

So we can write

$$\lambda_k = e^{2\pi i\theta_k}$$

$0.\theta_{k_1}\cdots\theta_{k_t}$

where θ_k is some phase angle such that $|\theta_k| \leq 1$.

What if we want to *learn* an unknown θ_k?

Eigenvalues of unitary matrices

Idea: apply U to the relevant eigenvector, because that's "what makes the phase come out".

$$U|k\rangle = e^{2\pi i\theta_k}|k\rangle$$

...but this is an unobservable *global* phase!

We have to do something different: eigenvalue estimation, or quantum phase estimation (QPE).

Given a unitary U and one of its eigenvectors $|k\rangle$, estimate the value of θ_k such that

$$U|k\rangle = e^{2\pi i\theta_k}|k\rangle$$

Must determine:

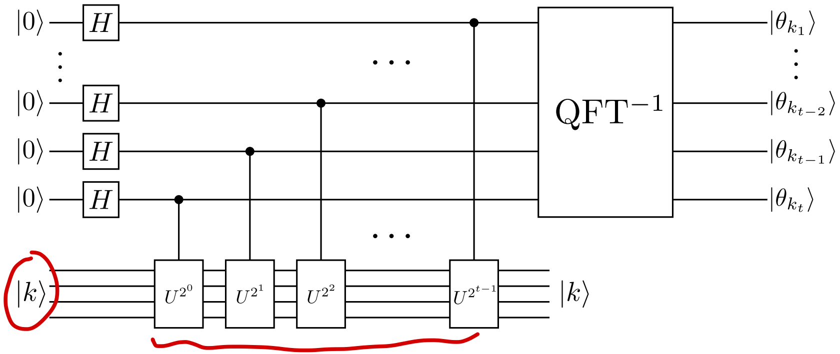
- How to design a circuit that extracts the θ_k
- To what precision can we estimate it
- What to do if we don't know a $|k\rangle$ in advance

(*You will explore the last two in your homework!*)

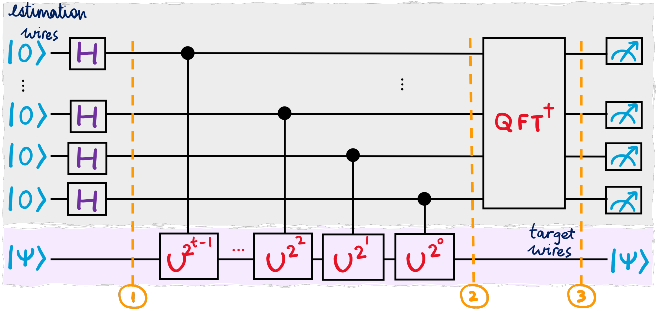
Let U be an n-qubit unitary; $|k\rangle$ is an n-qubit eigenstate.

Assume θ_k can be represented *exactly* using t bits:

$$\theta_k = 0.\theta_{k_1} \cdots \theta_{k_t}$$



You may see this version too:

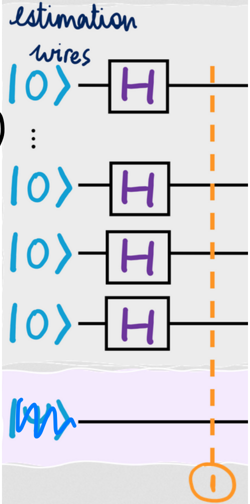


Let's analyze the state at points 1, 2, and 3 above.

Image credit: Xanadu Quantum Codebook node P.2

Quantum phase estimation: step 1

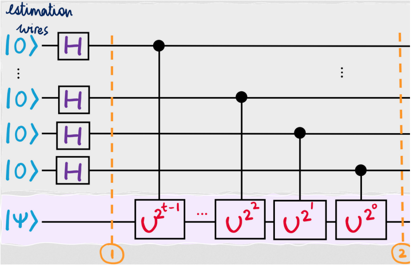
$$|0\rangle|0\rangle$$
$$\downarrow$$
$$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$
$$=\frac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)$$



$|k\rangle$

uniform superposition

Image credit: Xanadu Quantum Codebook node P.2



We apply U to $|k\rangle$; how does the phase get to the top register?

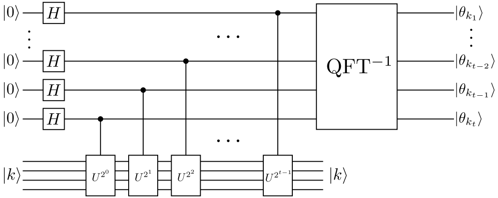
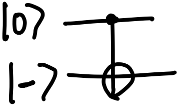


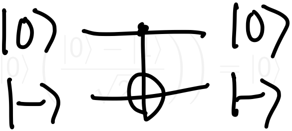
Image credit: Xanadu Quantum Codebook node P.2

The secret lies in something called *phase kickback*.

What happens when we apply a CNOT to the following state?



$$|0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \quad = |0\rangle|-\rangle$$

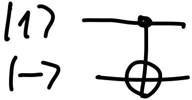


$$CNOT\left(|0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) = |0\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

We stopped here.

What happens when we apply a CNOT to this state?

$$|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$



$$\Big\downarrow \text{CNOT}$$

$$|1\rangle\left(\frac{|1\rangle-|0\rangle}{\sqrt{2}}\right) = |1\rangle\left(-|-\rangle\right) = \left(-|1\rangle\right)|-\rangle$$

$$CNOT\left(|1\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) = |1\rangle\left(\frac{|1\rangle-|0\rangle}{\sqrt{2}}\right) = |1\rangle\left(-\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right)$$

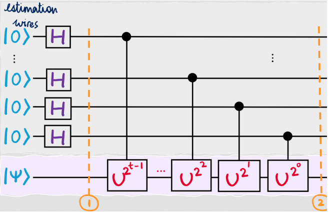
It looks like we've changed the phase of the second qubit.

The math doesn't care which qubit a global phase is attached to.

$$\mathit{CNOT}\left(|1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\right) = (-|1\rangle)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

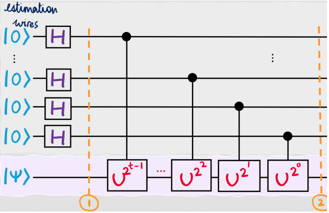
Seems the *target* qubit has done something to the *control* qubit!

We say that the phase has been "kicked back" from the second qubit to the first.



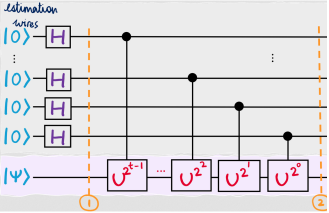
Consider the top-most qubit:

$$
\begin{aligned}
(CU)^{2^{t-1}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|+\rangle^{\otimes t-1}|k\rangle\right) &= (CU)^{2^{t-1}}\left(\frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle\right)\\
&\quad + (CU)^{2^{t-1}}\left(\frac{1}{\sqrt{2}}|1\rangle|+\rangle^{\otimes t-1}|k\rangle\right)\\
&= \left(\frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle\right)\\
&\quad + \left(\frac{1}{\sqrt{2}}|1\rangle|+\rangle^{\otimes t-1}(e^{2\pi i\theta_k})^{2^{t-1}}|k\rangle\right)
\end{aligned}
$$



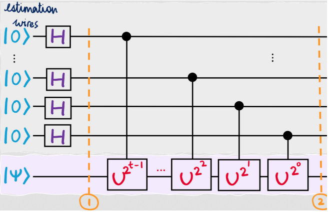
Use phase kickback

$$\left(\frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle\right)+\left(\frac{1}{\sqrt{2}}|1\rangle|+\rangle^{\otimes t-1}(e^{2\pi i\theta_k})^{2^{t-1}}|k\rangle\right)$$
$$=\left(\frac{1}{\sqrt{2}}|0\rangle|+\rangle^{\otimes t-1}|k\rangle\right)+\left(\frac{1}{\sqrt{2}}(e^{2\pi i\theta_k})^{2^{t-1}}|1\rangle|+\rangle^{\otimes t-1}|k\rangle\right)$$
$$=\frac{1}{\sqrt{2}}(|0\rangle+(e^{2\pi i\theta_k})^{2^{t-1}}|1\rangle)|+\rangle^{\otimes t-1}|k\rangle$$



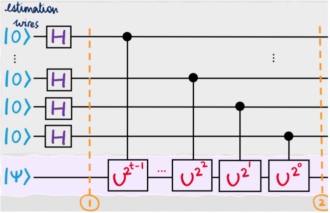
What is happening in the exponent?

$$
\begin{aligned}
(e^{2\pi i\theta_k})^{2^{t-1}} &= e^{2\pi i\theta_k \cdot 2^{t-1}} \\
&= e^{2\pi i(\frac{\theta_{k_1}}{2^1} + \frac{\theta_{k_2}}{2^2} + \cdots \frac{\theta_{k_t}}{2^t})\cdot 2^{t-1}} \\
&= e^{2\pi i(2^{t-2}\theta_{k_1} + 2^{t-3}\theta_{k_2} + \cdots \frac{\theta_{k_t}}{2})} \\
&= e^{2\pi i \frac{\theta_{k_t}}{2}} \\
&= e^{2\pi i 0.\theta_{k_t}}
\end{aligned}
$$



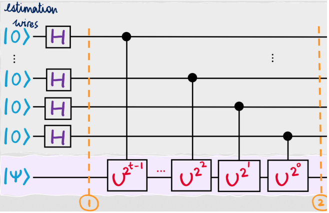
So we have the combined state:

$$\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})^{2^{t-1}}|1\rangle)|+\rangle^{\otimes t-1}|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_t}}|1\rangle)|+\rangle^{\otimes t-1}|k\rangle$$



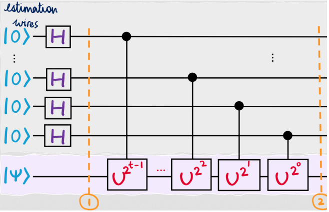
Let's do the second-last qubit (ignore what happens to others for now):

$$(CU)^2\left(|+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|+\rangle|k\rangle\right) = |+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i\theta_k\cdot 2}|1\rangle)|+\rangle|k\rangle$$



Again check the exponent...

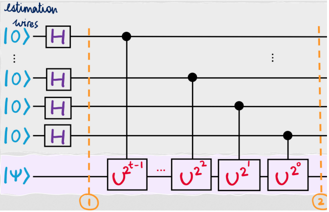
$$
\begin{aligned}
(e^{2\pi i\theta_k})^2 &= e^{2\pi i\theta_k \cdot 2} \\
&= e^{2\pi i(\frac{\theta_{k_1}}{2^1} + \frac{\theta_{k_2}}{2^2} + \cdots \frac{\theta_{k_t}}{2^t})\cdot 2} \\
&= e^{2\pi i(\theta_{k_1} + \frac{\theta_{k_2}}{2} + \cdots \frac{\theta_{k_t}}{2^{t-1}})} \\
&= e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}
\end{aligned}
$$



So we have the combined state:

$$|+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})^2|1\rangle)|+\rangle|k\rangle = |+\rangle^{\otimes t-2}\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)|+\rangle|k$$

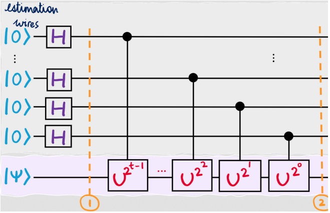
Quantum phase estimation: step 2



Can show in the same way that for the last qubit

$$|+\rangle^{\otimes t-1}\frac{1}{\sqrt{2}}(|0\rangle + (e^{2\pi i\theta_k})|1\rangle)|k\rangle = |+\rangle^{\otimes t-1}\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Quantum phase estimation: step 2

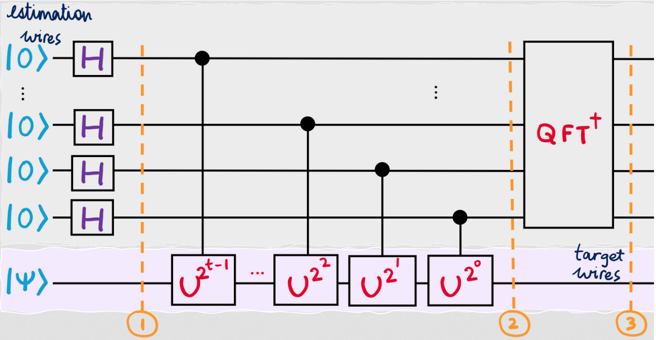


After step 2, we have the state

$$\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_t}}|1\rangle)\cdots\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_2}\cdots\theta_{k_t}}|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+e^{2\pi i 0.\theta_{k_1}\cdots\theta_{k_t}}|1\rangle)|k\rangle$$

Should look familiar!

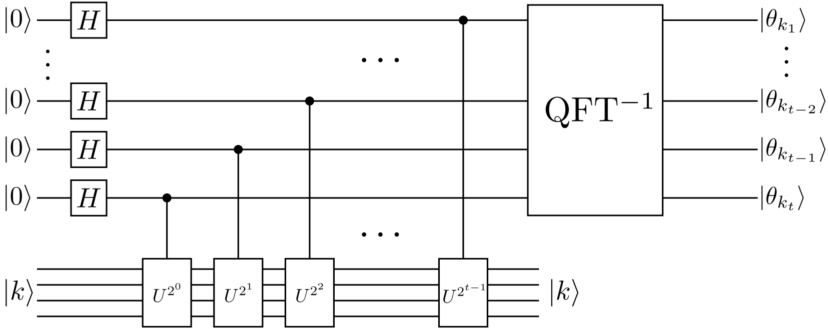
Quantum phase estimation: step 3



Last step is to apply the *inverse* QFT to recover the state...

Image credit: Xanadu Quantum Codebook node P.2

We can then measure to learn the numerical value of θ_k.



Let's implement it.

Next time

Content:

- Quiz 5 on Monday
- Continuing with QPE
- Moving towards Shor's algorithm

Action items:

1. Choose project group and paper
2. Literacy assignment 2

Recommended reading:

- Codebook nodes F.1-F.3, P.1-P.4
- Nielsen & Chuang 5.1, 5.2