CPEN 400Q Lecture 06
Quantum teleportation

Friday 27 January 2023

Announcements

- Assignment 1 available, due Monday 6 Feb
- Quiz 3 at beginning of class on Monday

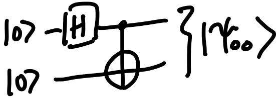
Last time

We measured multi-qubit states

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

If we measure in the computational basis, the outcome probabilities are:

- $|\alpha|^2 = |\langle 00|\psi\rangle|^2$ for $|00\rangle$
- $|\beta|^2 = |\langle 01|\psi\rangle|^2$ for $|01\rangle$
- ...



We introduced the *Bell basis*.

$$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$X \otimes \mathbb{1}$

$$|\Psi_{01}\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$|\Psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

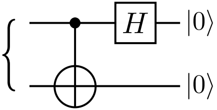
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbb{1}$ $\mathbb{I}$ I

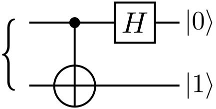
"identity"

Using an appropriate basis rotation, we can take measurements with respect to the Bell basis.

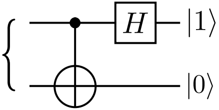
$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$



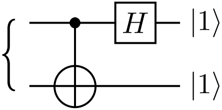
$\frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$



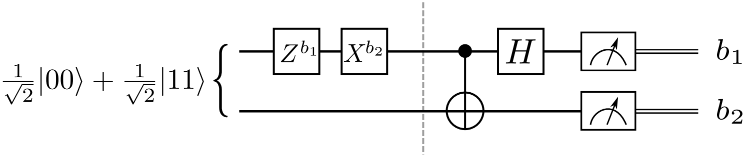
$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$



$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



We explored **superdense coding** and saw that given a pair of entangled qubits, we can send one of them to transmit two bits of information.



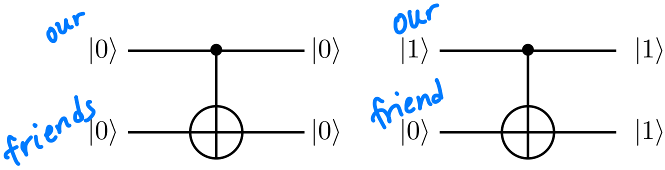
Learning outcomes

- Prove that arbitrary quantum states cannot be cloned
- Teleport a quantum state
- Compute expectation values of observables

Copying quantum states

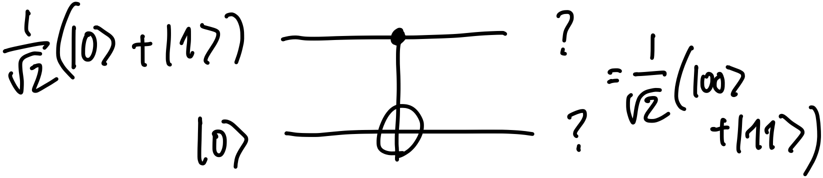
Suppose you found a really cool quantum state, and you want to send a copy to a friend. Can you?

Idea: CNOT sends $|00\rangle$ to $|00\rangle$, and $|10\rangle$ to $|11\rangle$.



Everything is linear, so will this work in general?

Can find a state for which this fails:



(Not) copying quantum states

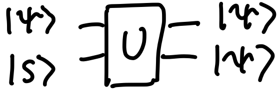
The no-cloning theorem

It is impossible to create a copying circuit that works for arbitrary quantum states.

In other words, there is no circuit that sends

$$|\psi\rangle \otimes |s\rangle \to |\psi\rangle \otimes |\psi\rangle$$

for any arbitrary $|\psi\rangle$.



Suppose we want to clone a state $|\psi\rangle$. We want to find U s.t.

$$U\left(|\psi\rangle \otimes |s\rangle\right) \to |\psi\rangle \otimes |\psi\rangle$$

where $|s\rangle$ is some arbitrary state.

If our cloning circuit is universal, then U should clone some other state, $|\varphi\rangle$.

$$U\left(|\varphi\rangle \otimes |s\rangle\right) \to |\varphi\rangle \otimes |\varphi\rangle$$

Proof of the no-cloning theorem

We purportedly have: $\left\langle |a\rangle \otimes |b\rangle , |c\rangle \otimes |d\rangle \right\rangle = \langle c|a\rangle \cdot \langle d|b\rangle$

$$
\begin{aligned}
U(|\psi\rangle \otimes |s\rangle) &= |\psi\rangle \otimes |\psi\rangle \\
U(|\varphi\rangle \otimes |s\rangle) &= |\varphi\rangle \otimes |\varphi\rangle
\end{aligned}
$$

Take the inner product of the LHS of both equations:

$$(\langle \varphi| \otimes \langle s|)\underbrace{U^\dagger \cdot U}_{=\mathbb{1}}(|\psi\rangle \otimes |s\rangle) = \langle \varphi|\psi\rangle \cdot \langle s|s\rangle = \langle \varphi|\psi\rangle$$

Now take the inner product of the RHS of both equations:

$$(\langle \varphi| \otimes \langle \varphi|)(|\psi\rangle \otimes |\psi\rangle) = \langle \varphi|\psi\rangle \cdot \langle \varphi|\psi\rangle$$

$$= (\langle \varphi|\psi\rangle)^2$$

?

1 or 0

For what states does

$$(\langle\psi|\varphi\rangle)^2 = \langle\psi|\varphi\rangle$$

Need a complex number that squares to itself... but the only numbers that square to themselves are 0 and 1!

So either the two states are orthogonal, or are just the same state. They can't be arbitrary!

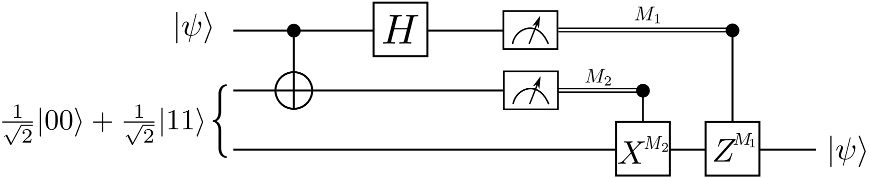
Teleportation

$|\psi\rangle$ — — $|\psi\rangle$

$|s\rangle$ — — $|\varphi\rangle \approx |\psi\rangle$?

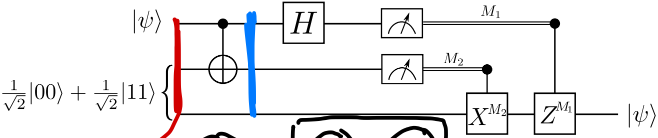
look up!

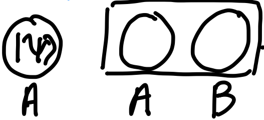
We cannot clone arbitrary qubit states, but we *can* teleport them!



Let's go one gate at a time.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



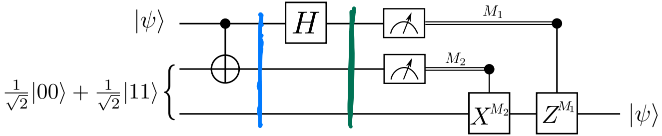


$\rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\alpha|000\rangle + \frac{1}{\sqrt{2}}\alpha|011\rangle + \frac{1}{\sqrt{2}}\beta|100\rangle + \frac{1}{\sqrt{2}}\beta|111\rangle$$

apply CNOT: $\frac{1}{\sqrt{2}}\alpha|000\rangle + \frac{1}{\sqrt{2}}\alpha|011\rangle + \frac{1}{\sqrt{2}}\beta|110\rangle + \frac{1}{\sqrt{2}}\beta|101\rangle$

Let's go one gate at a time.



$$\frac{1}{\sqrt{2}}\alpha|000\rangle + \frac{1}{\sqrt{2}}\alpha|011\rangle + \frac{1}{\sqrt{2}}\beta|110\rangle + \frac{1}{\sqrt{2}}\beta|101\rangle$$

$$\frac{1}{2}\alpha|000\rangle + \frac{1}{2}\alpha|100\rangle + \frac{1}{2}\alpha|011\rangle + \frac{1}{2}\alpha|111\rangle + \frac{1}{2}\beta|010\rangle - \frac{1}{2}\beta|110\rangle + \frac{1}{2}\beta|001\rangle - \frac{1}{2}\beta|101\rangle$$

$$\frac{1}{2}|00\rangle\otimes(\alpha|0\rangle+\beta|1\rangle) + \frac{1}{2}|10\rangle\otimes(\alpha|0\rangle-\beta|1\rangle) + \frac{1}{2}|01\rangle\otimes(\alpha|1\rangle+\beta|0\rangle) + \frac{1}{2}|11\rangle\otimes(\alpha|1\rangle-\beta|0\rangle)$$

Before measurements, the combined state of the system is (removing the $\frac{1}{2}$ for readability):

$$
\begin{array}{ccc}
|00\rangle & \otimes & (\alpha|0\rangle + \beta|1\rangle) + \\
|01\rangle & \otimes & (\alpha|1\rangle + \beta|0\rangle) + \\
|10\rangle & \otimes & (\alpha|0\rangle - \beta|1\rangle) + \\
|11\rangle & \otimes & (\alpha|1\rangle - \beta|0\rangle)
\end{array}
$$

$U(|\psi_0\rangle|s\rangle) = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

$\left(\underbrace{U(|\psi_2\rangle|s\rangle)}\right)^{\dagger} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

$(\cdots\cdots)\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

This is a *uniform* superposition of 4 distinct terms. If we measure the first two qubits in the computational basis, we are equally likely to obtain each of the four outcomes.

Quantum teleportation: the details

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

You can see that Bob's state is always some variation on the original state of Alice:

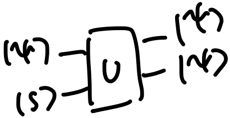
M1 M2

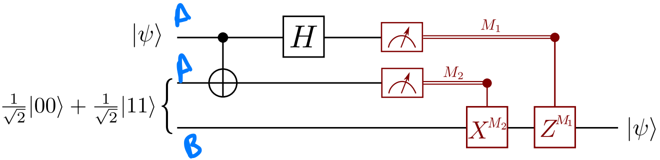
$$
\begin{aligned}
|00\rangle &\otimes (\alpha|0\rangle + \beta|1\rangle) + \\
|01\rangle &\otimes (\alpha|1\rangle + \beta|0\rangle) + \\
|10\rangle &\otimes (\alpha|0\rangle - \beta|1\rangle) + \\
|11\rangle &\otimes (\alpha|1\rangle - \beta|0\rangle)
\end{aligned}
$$

Quantum teleportation: the details

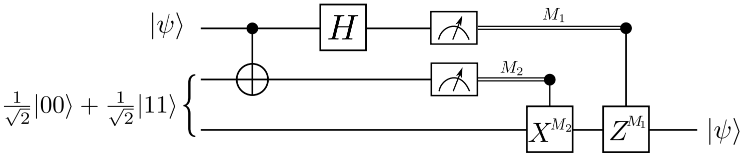
Alice measures in the computational basis and sends her results to Bob. Once Bob knows the results, he knows exactly what term of the superposition they had, and can adjust his state accordingly.

$$00 : I(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle$$
$$01 : X(\alpha|1\rangle + \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle$$
$$10 : Z(\alpha|0\rangle - \beta|1\rangle) = \alpha|0\rangle + \beta|1\rangle$$
$$11 : ZX(\alpha|1\rangle - \beta|0\rangle) = \alpha|0\rangle + \beta|1\rangle$$





Hands on: let's teleport a state



Try on Quirk circuit simulator!

What's next

Have now seen two small quantum algorithms:

- Superdense coding
- Teleportation

Next class, we will see our first *variational algorithm*: the variational quantum classifer (VQC).

Need to learn about a different type of measurement.

$$UU^\dagger = \mathbb{1}$$

Generally, we are interested in measuring real, physical quantities. In physics, these are called observables.

Observables are represented mathematically by Hermitian matrices. An operator (matrix) H is Hermitian if

$$H = H^\dagger$$

Why Hermitian? The possible measurement outcomes are given by the eigenvalues of the operator, and eigenvalues of Hermitian operators are **real**.

$$U = e^{-iHt}$$

"generator"

Example:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Z is Hermitian:

$$Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

Its eigensystem is

$$\lambda_1 = +1 \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X is Hermitian and its (normalized) eigensystem is

$$\lambda_1 = +1 \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Y is Hermitian and its (normalized) eigensystem is

$$\lambda_1 = +1, \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda_2 = -1, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

When we measure X, Y, or Z on a state, for each shot we will get one of the eigenstates (/eigenvalues).

If we take multiple shots, what do we expect to see *on average*?

Analytically, the **expectation value** of measuring the observable M given the state $|\psi\rangle$ is

$$\langle M \rangle = \langle \psi | M | \psi \rangle$$

We stopped here!

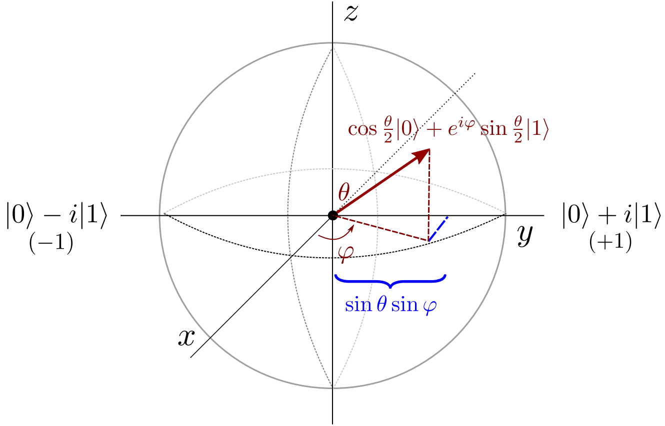
Exercise: consider the quantum state

$$|\psi\rangle = \frac{1}{2}|0\rangle - i\frac{\sqrt{3}}{2}|1\rangle.$$

Compute the expectation value of Y:

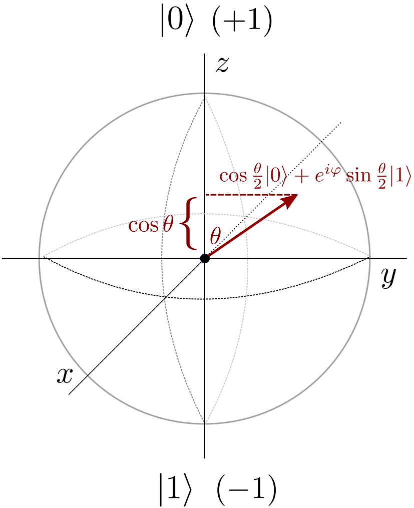
$$\begin{aligned}
|\psi\rangle &= \left(\frac{1}{2}\langle 0| + i\frac{\sqrt{3}}{2}\langle 1|\right) Y \left(\frac{1}{2}|0\rangle - i\frac{\sqrt{3}}{2}|1\rangle\right) \\
&= \left(\frac{1}{2}\langle 0| + i\frac{\sqrt{3}}{2}\langle 1|\right)\left(\frac{i}{2}|1\rangle - \frac{\sqrt{3}}{2}|0\rangle\right) \\
&= \frac{i}{4}\langle 0|1\rangle - \frac{\sqrt{3}}{4}\langle 1|1\rangle - \frac{\sqrt{3}}{4}\langle 0|0\rangle - i\frac{3}{4}\langle 1|0\rangle \\
&= -\frac{\sqrt{3}}{2}
\end{aligned}$$

The Bloch sphere offers us some more insight into what a projective measurement is.



Exercise: derive the expression in blue by computing $\langle\psi| Y |\psi\rangle$.

Expectation values and the Bloch sphere



$$
\begin{aligned}
|\psi\rangle &= \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \\
Z|\psi\rangle &= \cos\frac{\theta}{2}|0\rangle - e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \\
\langle\psi|\,Z|\psi\rangle &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\
&= \cos\theta
\end{aligned}
$$

Expectation values: from measurement data

Let's compute the expectation value of Z for the following circuit using 10 samples:

```
dev = qml.device('default.qubit', wires=1, shots=10)

@qml.qnode(dev)
def circuit():
    qml.RX(2*np.pi/3, wires=0)
    return qml.sample()
```

Results might look something like this:

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

Expectation values: from measurement data

The expectation value pertains to the measured eigenvalue; recall Z eigenstates are

$$\lambda_1 = +1, \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So when we observe $|0\rangle$, this is eigenvalue $+1$ (and if $|1\rangle$, -1). Our samples shift from

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

to

```
[-1, -1, -1, 1, -1, -1, -1, 1, -1, -1]
```

Expectation values: from measurement data

The expectation value is the weighted average of this, where the weights are the eigenvalues:

$$\langle Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

where

- n_1 is the number of $+1$ eigenvalues
- n_{-1} is the number of -1 eigenvalues
- N is the total number of shots

For our example, $\langle Z \rangle = -0.6$.

Expectation values

Let's do this in PennyLane instead:

```
dev = qml.device('default.qubit', wires=1)

@qml.qnode(dev)
def measure_z():
    qml.RX(2*np.pi/3, wires=0)
    return qml.expval(qml.PauliZ(0))
```

Multi-qubit expectation values

Example: operator $Z \otimes Z$.

Eigenvalues are computational basis states:

$$
\begin{aligned}
(Z \otimes Z)|00\rangle &= |00\rangle \\
(Z \otimes Z)|01\rangle &= -|01\rangle \\
(Z \otimes Z)|10\rangle &= -|10\rangle \\
(Z \otimes Z)|11\rangle &= |11\rangle
\end{aligned}
$$

To compute an expectation value from data:

$$\langle Z \otimes Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

Multi-qubit expectation values

Example: operator $X \otimes I$.

Eigenvalues of X are the $|+\rangle$ and $|-\rangle$ states:

$$
\begin{aligned}
(X \otimes I)|+0\rangle &= |+0\rangle \\
(X \otimes I)|+1\rangle &= |+1\rangle \\
(X \otimes I)|-0\rangle &= -|-0\rangle \\
(X \otimes I)|-1\rangle &= -|-1\rangle
\end{aligned}
$$

Fun fact: All Pauli operators have an equal number of $+1$ *and* -1 *eigenvalues!*

How to compute expectation value of X from data, when we can only measure in the computational basis?

Basis rotation: apply H to first qubit

$$
\begin{aligned}
(H \otimes I)(X \otimes I)|+0\rangle &= |00\rangle \\
(H \otimes I)(X \otimes I)|+1\rangle &= |01\rangle \\
(H \otimes I)(X \otimes I)|-0\rangle &= -|10\rangle \\
(H \otimes I)(X \otimes I)|-1\rangle &= -|11\rangle
\end{aligned}
$$

When we measure and obtain $|10\rangle$ or $|11\rangle$, we know those correspond to the -1 eigenstates of $X \otimes I$.

Hands-on: multi-qubit expectation values

Multi-qubit expectation values can be created using the `@` symbol:

```python
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0) @ qml.PauliZ(1))
```

Hands-on: multi-qubit expectation values

Can also return *multiple* expectation values, if there are no shared qubits.

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0)), qml.expval(qml.PauliZ(
                                                    1))
```

Recap

- Prove that arbitrary quantum states cannot be cloned
- Teleport a quantum state
- Compute expectation values of observables

Next time

Content:

- Variational quantum classifier

Action items:

1. Assignment 1
2. Quiz 3 on Monday

Recommended reading:

- Codebook node I.10, I.15