Period-doubling and Chaos in Nonlinear Difference Equations: Applications to Models of Biological Populations

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Introduction

While continuous ordinary differential equations are more apt to model biological populations with overlapping generations, a more appropriate tool to model populations which display non-overlapping generations are difference equations. The nonlinear logistic difference equation, (1), is the most ubiquitous equation in this field

$$N_{n+1} = f(N_n) = rN_n\left(1 - \frac{N_n}{K}\right) \tag{1}$$

where $r > 0$ is the intrinsic growth rate of the population and $K > 0$ is the carrying capacity. This equation appears simple upon first inspection but investigation reveals it can exhibit rich dynamical behaviour. In certain parameter regions, the behaviour of the equation becomes unpredictable and chaotic. In this essay, we shall investigate the behaviour of the map.

Dynamical Behaviour

We define $N_n \in [0, 1]$ and $0 < r < 4$, in order to ensure all iterations of the map confined to the interval $[0, 1]$. The carrying capacity determines where the population will saturate and does not influence dynamical behaviour. We set $K = 1$ for simplicity. In order to analyse the dynamics of the system, we first identify the steady states of the model. These are levels of the population which remain constant as time progresses. A steady state N_* satisfies

$$f(N_*) = N_* \tag{2}$$

Thus, to find the steady states, we solve

$$rN_*(1 - N_*) = N_*$$

An obvious solution is the trivial case $N_* = 0$. Considering $N_* \neq 0$, we find

$$r(1 - N_*) = 1$$

Solving this equation, we find the only non-trivial steady state of (1) is

$$N_* = 1 - \frac{1}{r}$$

Biological relevance dictates $N_n \geq 0$ always, as we can't have negative populations. Thus, this steady state only exists for $r > 1$.

Linear Stability Analysis

We investigate how populations behave around the steady states. To do this, we conduct a linear stability analysis. The derivative of (1) will be necessary to determine stability. We find

$$f'(N_n) = r(1 - 2N_n) \tag{3}$$

$N_* = 0$:

$$|f'(0)| = |r| \tag{4}$$

For stability, we require $|r| < 1 \implies -1 < r < 1$. If we enforce the bounds placed on r, we find $0 < r < 1$. This means the trivial steady state is monotonically asymptotically stable for these values of r, as $r > 0$ is always true.

Considering instability, we find $|r| > 1 \implies r > 1$, discarding the case $r < -1$. Thus, we find $N_* = 0$ is a monotonically unstable steady state for $r > 1$.

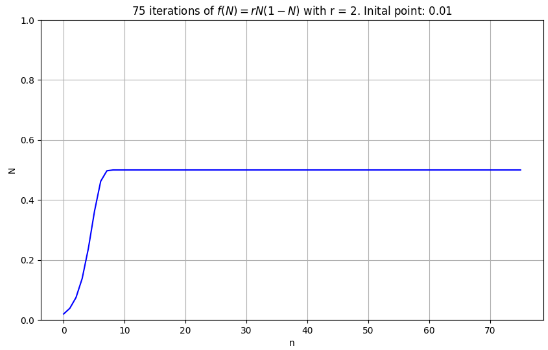
$N_* = 1 - \frac{1}{r}$:

$$f'\left(1 - \frac{1}{r}\right) = 2 - r \tag{5}$$

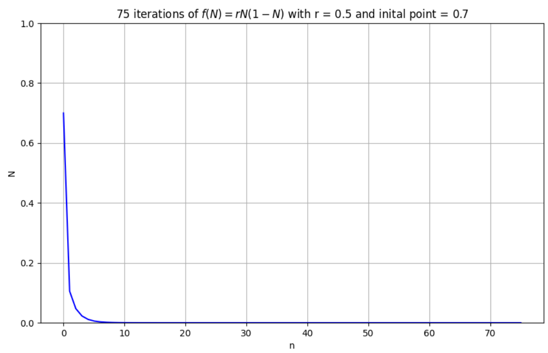
Considering stable cases first, we see for $1 < r < 2 \implies 0 < f'(N_*) < 1$, so the steady state is monotonically asymptotically stable for these values of r. For $2 < r < 3 \implies -1 < f'(N_*) < 0$, meaning the steady state is oscillatory asymptotically stable. For $r = 3$, $f'(N_*) = -1$, so the steady state is stable but not asymptotically stable here.

Now considering instability. We first consider if the steady state is monotonically unstable, that is $2 - r > 1 \implies 1 > r$. This is a contradiction, as $r > 1$ necessarily for the steady state to exist. We conclude that this steady state is never monotonically unstable. Now considering oscillatory instability, which implies $2 - r < -1 \implies r < 3$. This is compatible with the values of r and also consistent with the parameter ranges we have already determined.

We can verify this numerically. Setting $r = 2$, with an initial point 0.01, Figure 1a shows the iterations converge on to a steady value at $N = \frac{1}{2}$, which is consistent with the work done thus far.



(a) For $r = 2$, iterations converge to a non-zero steady state



(b) For $r = \frac{1}{2}$, we see iterations converge to zero.

Figure 1: Iterations converging monotonically to steady states

Period-2 Dynamics

We now investigate behaviour for $r > 3$. When a non-trivial stable steady state loses its stability, a stable period-2 orbit emerges. We find this orbit by solving

$$f^2(N_*) = f(f(N_*)) = r^2 N_*(1-N_*)(1 - rN_*(1-N_*)) = N_* \tag{6}$$

We know $N_* = 0$ and $N_* = 1 - \frac{1}{r}$ will satisfy this equation $(f(f(N_*)) = f(N_*) = N_*)$. However, they are not counted as period-2 points of $f(N_n)$, as 2 is not the lowest integer for which they are fixed points of the map. Dividing by $N_*(N_* - \frac{r-1}{r})$ yields

$$r^2 N_*^2 - r(r+1)N_* + r + 1 = 0$$

which has solutions

$$N_{\pm} = \frac{(1+r) \pm \sqrt{(r-3)(r+1)}}{2r}$$

Clearly these solutions only exist for $r > 3$, which is consistent with our earlier linear stability analysis. To verify this is a period-2 orbit, $\{N_+, N_-\}$ must satisfy

$$f(N_+) = N_- \quad f(N_-) = N_+$$

It can be verified that this does indeed hold, confirming $\{N_+, N_-\}$ is a period-2 orbit which exists for $r > 3$. We can apply further linear stability analysis to determine the stability of this periodic orbit. The orbit $\{N_+, N_-\}$ is stable for $|f'(N_+)f'(N_-)| < 1$ and unstable for $|f'(N_+)f'(N_-)| > 1$. We find

$$f'(N_{\pm}) = -1 \mp \sqrt{(r-3)(r+1)}$$
$$\implies |f'(N_+)f'(N_-)| = |-r^2 + 2r + 4|$$

Thus $|-r^2+2r+4| < 1$ implies the period-2 orbit $\{N_+, N_-\}$ is stable for $3 < r < 1 + \sqrt{6}$, which is consistent with the theory developed so far. Figure 2 verifies this numerically.

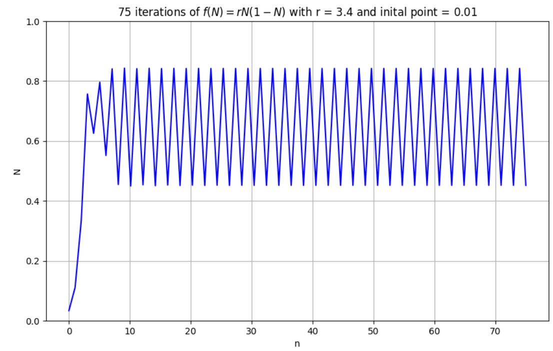


Figure 2: Stable period-2 orbit of $N_n = f(N_n)$ after 75 iterations for $r = 3.4$

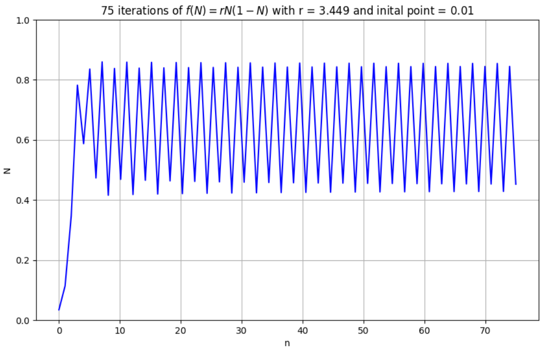
Chaotic Behaviour

The phenomenon of a stable period-2N orbit emerging when a period-N orbit becomes unstable is called a period doubling *bifurcation*. Analysing further period doubling bifurcations analytically is difficult and numerical methods are often more appropriate. If we continue our approach, we can see the period-2 orbit becomes unstable around $r = 3.449$, at which point a stable period-4 orbit emerges (Figure 3a). This bifurcates into a stable period-8 orbit around $r = 3.54409$ (Figure 3b), at which point the period-4 orbit ceases to be stable. This period-8 orbit then gives way to a period-16 orbit. This pattern of period doubling continues infinitely as r increases. The distance between doubling points decreases at each iteration, making it impossible to accurately define where higher orbits begin and end. The results of the first five bifurcations are seen in Table 1.

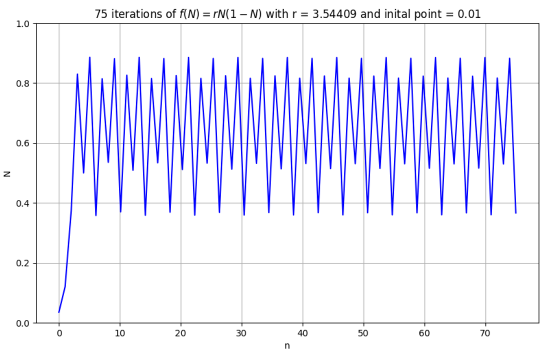
Table 1: Table of Period Doubling Bifurcation Points

n	r_n	**Period**
1	3	2
2	3.449...	4
3	3.54409...	8
4	3.5644...	16
5	3.568759...	32

We can see that r is converging to a value $r_\infty = 3569946...$. This is where the system becomes *chaotic*. Here behaviour is highly unpredictable and volatile. The system will never settle down into a long term periodic orbit, opting to oscillate between values seemingly at random. This behaviour is influenced by the initial point of N_n, a defining characteristic of chaos. We can visualise this behaviour with a

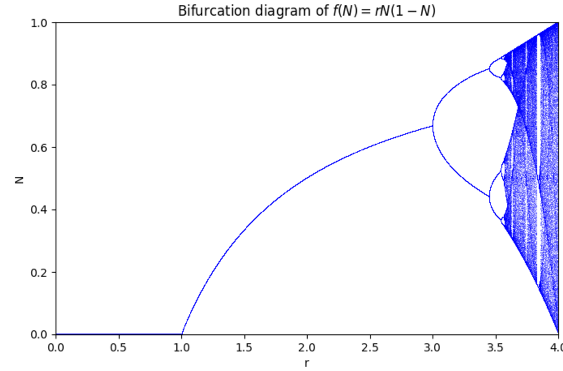


(a) Stable period-4 orbit; $r = 3.449$

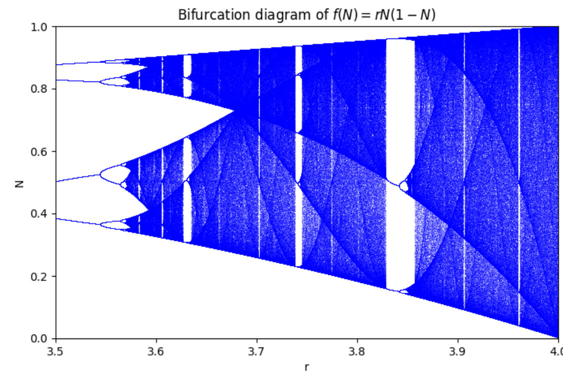


(b) Stable period-8 orbit; $r = 3.54409$

Figure 3: Stable Periodic Orbits



(a) $r \in [0, 4]$



(b) $r \in [3.5, 4]$

Figure 4: Bifurcation Diagram of $f(N_n)$

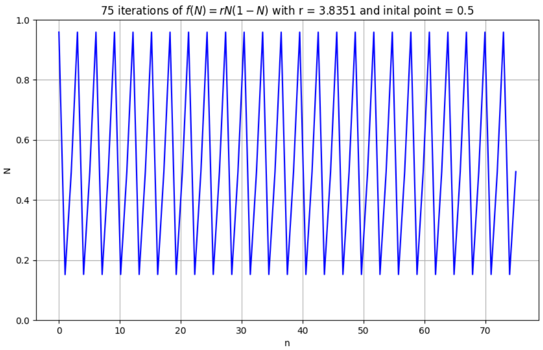
bifurcation diagram (Figure 4). This plot provides a graphical representation of the transition from single steady states into chaos.

Periodic Windows

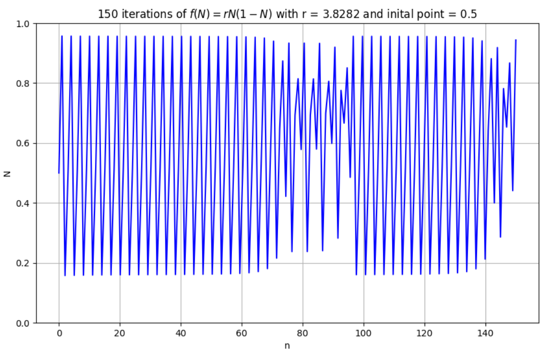
Upon close inspection of Figure 4b, we notice there are gaps between the chaos. It would seem that stable periodic cycles emerge within the chaotic regime. We call these periodic windows. Let's first verify that they are indeed periodic cycles. The most prominent window is a period-3 cycle at roughly $r = 3.8351$. A plot of this region can be seen in Figure 5a. This is indeed a stable period-3 cycle. It only appears for a brief moment as r increases, before collapsing back into chaos. Why does this happen? Explaining the period-3 window explains all periodic windows, as they all operate according to the same mechanism. Consider $f^3(N_n)$. Figure 6a shows this map against N_n at $r = 3.835$. We can see that there are a number of steady states as indicated by the intersections with N_n. Consider if we decrease r to $r = 3.8$ (Figure 6b). The peaks and troughs of $f^3(N_n)$ shrink and we are left with just two steady states. However, these steady states are not counted as steady states of $f^3(N_n)$, as they are steady states of $f(N_n)$. This implies that somewhere between $r = 3.835$ and $r = 3.8$, there is a point which N is tangent to $f^3(N)$. At this point, the stable and unstable cycles become one and then disappear in what is referred to as a *tangent bifurcation*. This phenomena is what marks the birth of a periodic window. It can be shown that the tangent value is $r = 1 + \sqrt{8} = 3.8284....$ Interestingly, it has been proven by Li and Yorke (Li and Yorke 1975) that if a difference equation of the form $N_{n+1} = N_n f(N_n)$ has a 3-cycle, it also has an k cycle, $k > 0$, $k \in \mathbb{Z}$, which will never settle into a periodic cycle and thus will be chaotic. This is consistent with what we have seen thus far.

Intermittency

We conclude the analysis by studying an interesting phenomenon observed about periodic windows. Consider Figure 5b, which shows iterations at $r = 3.8282$. We see that a period-3 cycle seems to appears here. This is peculiar, as the period-3 cycle doesn't exist in this region. This seems to be the ghost of that cycle. This is a phenomena known as intermittency, first explored in great detail in a paper by Pomeau and Manville (Manneville and Pomeau 1980). We see that the value of r at which we observe the ghost cycles, or intermittency, is very close to the tangent bifurcation value of r. Thus, the curves of $f^3(N_n)$ are very close but not quite tangent to N_n. This causes iterations to behave uniformly for a brief window, before becoming chaotic again. I have illustrated this phenomenon in Figure 7. Figure 7b shows a zoomed view of the phenomenon, where we can see that iterations become temporarily 'trapped'. This is what accounts for the period-3 behaviour we see in the iteration plot.

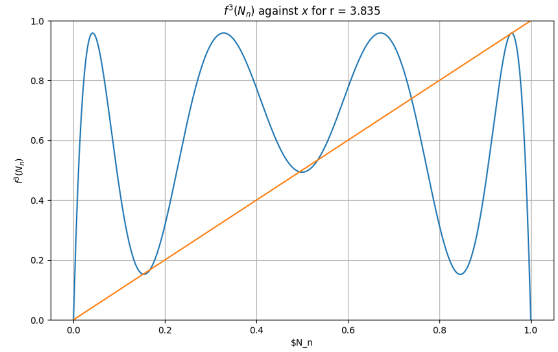


(a) Stable period-3 orbit. Initial value $N = 0.5$

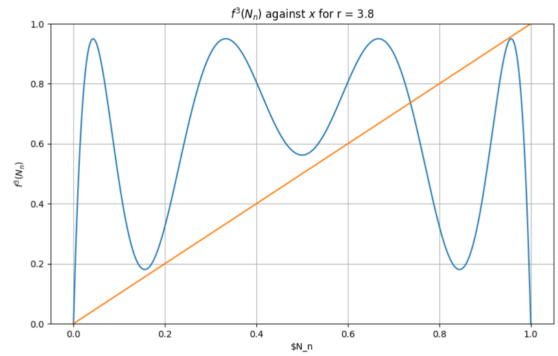


(b) 'Ghost' period-3 orbit at $r = 3.8282$

Figure 5



(a) $r = 3.835$



(b) $r = 3.8$

Figure 6: $f^3(N_n)$ against N_n

Conclusion

We began by studying the behaviour of the logistic difference equation using analytic techniques. We were able to identify fixed points of the map and determine their stability, using linear stability analysis. We also found a period-2 orbit and performed a similar linear stability analysis. When analytic techniques become unwieldy, we resorted to numerical ones. We saw the period doubling phenomena continues infinitely until the system is *chaotic*. We then investigated many chaotic phenomena, such as periodic windows and intermittency.

References

Li, Tien-Yien and James A. Yorke (1975). "Period Three Implies Chaos". In: *The American Mathematical Monthly* 82.10, pp. 985–992. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2318254` (visited on 04/07/2024).

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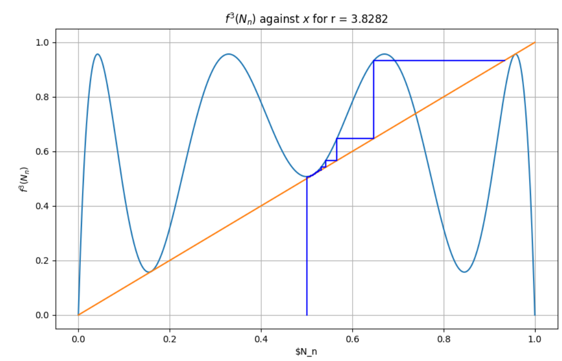
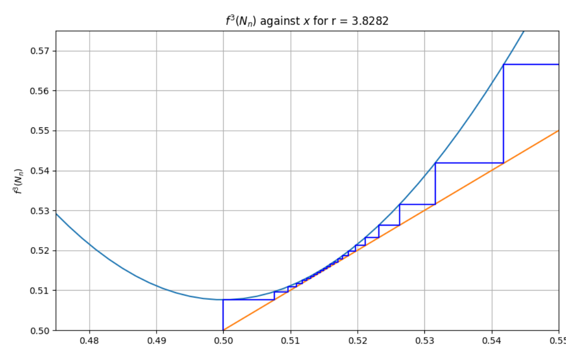


(a) $r = 3.8282$



(b) Briefly, $f^3(N_n) \approx N_n$

Figure 7: Intermittency visualised