Binomial and Fibonacci Heaps in Racket (rkt-heaps)

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Abstract. Library[1] providing data structures viz. Binomial heap [4] and Fibonacci heap[2] are described along with primitive operations, performance and usage.

1 Introduction

$rkt - heaps$ is made for demonstration in a contest, *Lisp In Summer Projects*[2]. Any participant was expected to develop one or many applications, or libraries, using any LISP-based technology. $rkt - heaps$ uses Racket language which belongs to the Lisp/Scheme family.

Section 1 & 2 describe Binomial and Fibonacci heaps respectively. *Section 3* has performance results to ascertain $rkt - heaps$ correspond to the theoretical run-time bounds. Comparison with an existing library for Binary heaps[3] in Racket is also added for evaluation purposes. *Appendix* includes syntactics of using this library in Racket. *Section 5* has conclusions and author's learnings.

2 Binomial Heaps

Binomial heaps are a collection of heap-ordered Binomial trees with a pointer min to the root of a tree having the minimum value amongst all elements in the heap. They allow the following operations:

1. *makeheap(i)* Makes a new heap with only one element i
2. *findmin(h)* Returns the minimum value amongst all elements in the heap, h
3. *insert(h, i)* Adds element i to heap h
4. *deletemin(h)* Deletes the element with minimum value from heap h
5. *meld(h, h')* Combines two heaps h and h' into one

Before studying costs and implementation of the mentioned operations, some terms need brief explanations for reader's convenience.

[1] A software package and will be referred as $rkt - heaps$ throughout the paper
[2] http://lispinsummerprojects.org/
[3] http://docs.racket-lang.org/data/Binary_Heaps.html

Operation	Amortized Cost
makeheap	$O(1)$
findmin	$O(1)$
insert	$O(1)$
deletemin	$O(\log n)$
meld (eager)	$O(\log n)$
meld (lazy)	$O(1)$

Table 1: Amortized costs for Binomial Heaps

1. *Amortized cost* - Amortization analysis includes cost of a sequence of operations spread over the entire sequence[3]. Amortized costs may not always give the exact cost of an operation. An operation may be expensive or cheaper, but the average cost over a sequence of operations is small and defined as amortized cost.
2. *rank* - States the number of children of an element. For Binomial heap, any tree has 2^k elements rooted at some element e_k having rank k.

Making a new heap with one element - *makeheap*, and using the *min* pointer to get the minimum value - $findmin$, both require constant number of steps. Although, it may not be intuitive to see why *insert* takes just constant amortized cost. Inserting a value is equivalent to making a heap, h_1, with a single element and then merging h_1 with an existing heap, h_2. In melding, combining two trees(one from h_1 and one from h_2) with same rank k results in a tree of rank $k+1$. If a tree with rank $(k+1)$ already exists in either of the heaps - h_1 or h_2, then a tree of rank $(k+2)$ is made. This happens continuously till there is single tree of some rank $(> k)$ in the resulting heap, $h_{1,2}$. At the end of meld, $h_{1,2}$ has only one or no tree of size 2^k. The heap property[4] is imposed each time roots of two trees are combined.

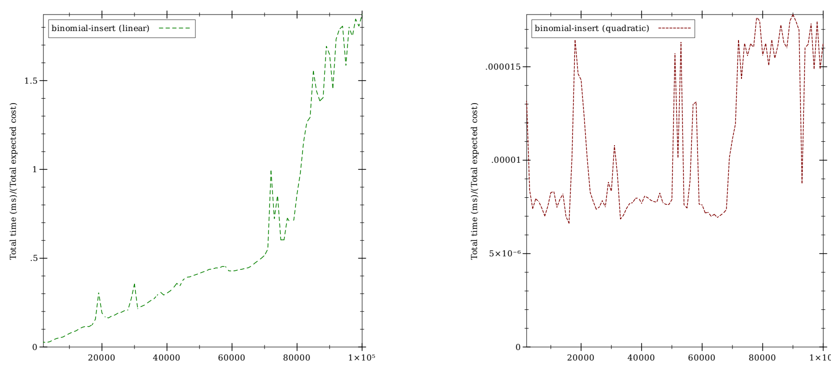
For instance, consider a Binomial heap having only one tree with rank, let's say 2. There will be no requirement to combine another tree of rank 2 until 3 elements have been inserted in trees with sizes 2^0 and 2^1. Hence, using amortized analysis the average cost over n operations will be $O(1)$ for *insert*. The technique of meld can be extended for *deletemin* operation also; take all children of the min element and meld them with the remaining trees in the heap, and update the min pointer, costing $O(\log n)$. For a more detailed analysis, refer [3].

meld eager version works in the manner as described above but for the lazy version, all roots of one heap are added to the set of roots of the other heap. This may result in having more than one binomial tree of rank k in the resulting

[4] Heap property ensures that all children of a tree rotted at r have their values ordered with its parent's value and that the same ordering applies across the heap

heap. Any subsequent call to a *deletemin* operation will correct the heap such that only one binomial tree of rank k exists. This costs $\mathrm{O}(\log n)$.

Binomial heap in $rkt-heaps$ is an array-based purely functional implementation leveraging Racket's *vectors* library. Being purely functional ensures there is no mutation of previously created data structures. Although, this leads to anomalies in runtime costs for certain operations like *insert*. With reference to Table 1, the total cost for n inserts is linear, $\mathrm{O}(n)$; but in a purely functional implementation, the insert operation cost for n inserts aggregates to $\mathrm{O}(n^2)$. The reason for such an anomaly is due to cloning of heap vector at every insert operation. This is validated in Fig:1. The curve *binomial-insert (quadratic)* signifies the divisor(total expected cost) for the dividend(total time) is quadratic, and therefore the curve has slope almost zero[5]. Alternatively, taking the divisor as linear, the total run-time cost depicts a curve with slope due to increase cost of cloning the vector, which is to say that nth insertion step clones the $(n-1)th$ heap vector[6]. Modifying the structuring of the heap to a doubly linked list(DDL) wherein all roots of trees have pointers to its right and left root elements is imperative to ensure constant number of steps for each insert (used for analyzing run-time costs in [3]). The DDL implementation and cost shall be discussed in later sections of this paper.



(a) Total expected cost $= \mathrm{O}(n)$ (b) Total expected cost $= \mathrm{O}(n^2)$

Fig. 1: Anomaly with pure functional impelementation of Binomial heaps

3 Fibonacci Heaps

Fibonacci heaps are a generalization of Binomial heaps allowing additional operations other than provided by Binomial heaps. Specifically, they allow deletion

[5] The ratio of total cost and n^2 will be constant

[6] Racket's vector API provides *(vector-append* $v_1 v_2$ *...)*, used for cloning, makes fresh copy of all elements of given vectors

of any element from the heap, and modification of the value of any element. The prototypes for additional operations are as follows:

1. *decrement(h, i, δ)* Decrements the value of i by δ in h
2. *delete(h, i)* Deletes element i from the heap h

Operation	Amortized Cost
makeheap	$O(1)$
findmin	$O(1)$
insert	$O(1)$
deletemin	$O(\log n)$
meld (lazy)	$O(1)$
decrement	$O(1)$
delete	$O(\log n)$

Table 2: Amortized costs for Fibonacci Heaps

Fibonacci heaps are implemented in $rkt - heaps$ using a DDL for its roots. Findings from array and functional based implementation of Binomial heaps, highlight the downsides of not having mutations which leads to much higher costs for performing primitive operations in contrast to what has been proved in literature. For this reason, Fibonacci heaps do not have a purely functional implementation in $rkt - heaps$.

A Fibonacci heap if operated with only *deletemin* and *meld* operations, then every tree becomes a binomial tree, and the heap Binomial. Thus the corollary that size of tree rooted at r is exponential in $rank(r)$. The exponential size of a tree is also valid otherwise; for Fibonacci heaps with operations *decrement* and *delete* [3] which requires cutting of sub-trees. This fact is essential to analyze amortized costs to be $O(1)$ and $O(\log n)$ for *decrement* and *delete* operations respectively.

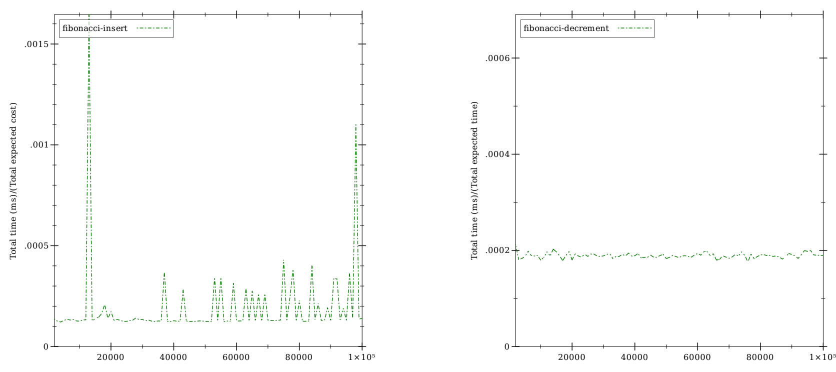
decrement operation mutates the value of the element, e, followed by check with its parent, e_p, on whether the heap property is violated or not. If it is, then the sub-tree rooted at element e, is added to the DDL of the heap. Every parent, starting at e_p is also added to DLL if a boolean flag is marked. This happens till either a root element or an element which does not have the flag marked is reached. This flag is maintained for every element e_i, except root elements, and marked as true if one of its children have been removed or false otherwise. The sub-tree rooted at e_i shall also be cut from its parent if a second child of e_i is removed i.e. the parent element is marked and a decrement operation on a child violates the heap property or a child is deleted.

delete makes the value of the element to be deleted less than the min value using *decrement* and then calls *deletemin*. This costs $O(\log n)$.

4 Evaluation

For this section, comparisons are omitted between Binomial and Fibonacci heaps since one is purely functional and other is not. The functional technique was inefficient due to cloning of vectors as shown in Section 2. It is trivial to build on the Fibonacci heap implementation by excluding the *decrement* and *deletemin* operations to get a Binomial heap with lazy melds[3] at the same cost as shown for Fibonacci heaps in this section.

In Figure 2, the curves have mountainous terrain as it the amortized total expected cost.



(a) Total expected cost = $O(n)$ (b) Total expected cost = $O(n)$

Fig. 2: Fibonacci Heaps, *insert* and *meld* operations

As stated before, *delete* involves a call to *decrement* to mutate the value of the node less than the *min* value in the heap, and subsequently to *deletemin*. In Figure 3a, there is a gradual increase in the slope of the curve. This is probably due to the increase in the constant, $O(1)$ number of steps by *decrement* operation exceeding the $O(\log n)$ number of steps by *deletemin*. It should be noted that for benchmarking a heap h of size n, the *delete* operation is called n times, and state of h is maintained in-between all *delete* operations. To find an element to be deleted, a simple random walk is performed on h. Algorithm 1 states clearly the random walk procedure. Therefore, for decrement operation which recursively removes parents while traversing up the graph, there is a high chance that the length of path traversed is positively correlated to number of calls to *delete* operation h. As a consequence, constant cost of *decrement* increases and thereby not keeping the ratio of total time over total expected cost constant for *delete* operation.

Pre-packaged library for Binary heap in *Racket* is compared with Fibonacci implementation of $rkt - heaps$ on operations which have equal expected time bounds viz. $findmin$(Figure 4a) and $deletemin$(Figure 4b). Fibonacci heaps out-

Algorithm 1: Random Walk for Fibonacci heap H_f

Require: A valid fibonacci heap H_f of size S_{H_f}

Using the fact that the rank of $H_f \leq \log_\phi S_{H_f}$, where ϕ is Golden Ratio[1], uniformly select a root element, $e_{current}$

$r := rank(e_{current})$

$d := F_{r+2}$ where F_i is the ith Fibonacci number

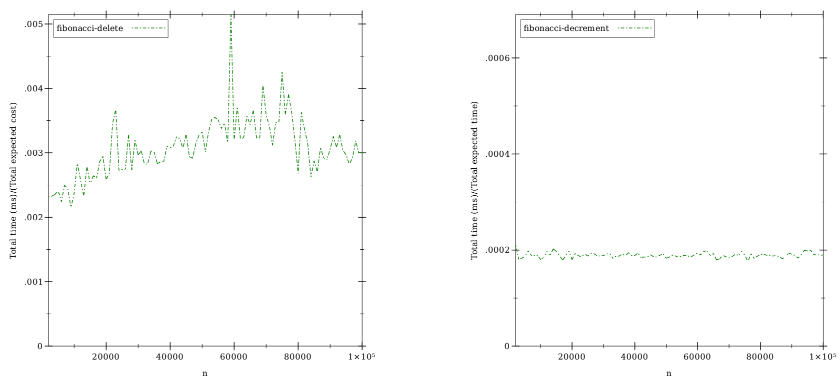
while $not\_leaf\_node(e_{current})$ and $depth(e_{current}) \leq d$ **do**

  Select uniformly a child c, from the set of children of $e_{current}$

  $e_{current} := c$

end while

return $e_{current}$



(a) Total expected cost $= \mathrm{O}\big(n \log n\big)$ (b) Total expected cost $= \mathrm{O}\big(n\big)$

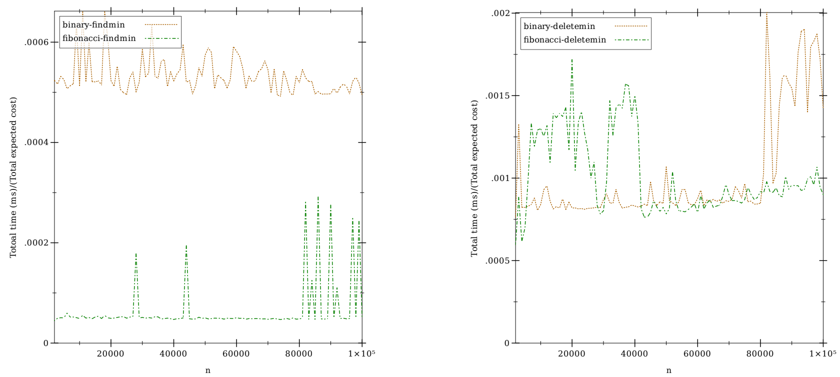
Fig. 3: Fibonacci Heaps, *delete* and *decrement* operations.

performs Binary heaps in both operations. The equivalent operations in Binary heap library are called as $heap - min$ and $heap - remove - min$.

5 Conclusions

In this paper:

- Binomial heaps included in $rkt - heaps$ do not run in expected time since they have been implemented in a functional style
- $rkt - heaps$ provide an efficient implementation of Fibonacci heaps using doubly linked lists
- Fibonacci heaps perform better than the existing Racket library implementation of Binary heaps
- Fibonacci heaps can be easily extended to work as Binomial heaps although it will not be purely functional



(a) Total expected cost $= \mathrm{O}(n)$ (b) Total expected cost $= \mathrm{O}(n \log n)$

Fig. 4: Binary Heaps vs Fibonacci Heaps, *findmin* and *deletemin* operations.

5.1 Author's learnings

- Initially, this work was started using mit-scheme but later migrated to Racket since there was better support for generation of graphs, testing and debugging
- Took a while to understand functional style of implementing heaps
- Run-time graphs[7] were essential to ascertain the algorithms were performing correctly as stated in literature
- Pure functional implementations can be slow

References

1. Richard A Dunlap. *The golden ratio and Fibonacci numbers*. World Scientific, 1997.
2. Michael L Fredman and Robert Endre Tarjan. Fibonacci heaps and their uses in improved network optimization algorithms. *Journal of the ACM (JACM)*, 34(3):596–615, 1987.
3. Dexter Kozen. *The design and analysis of algorithms*. Springer, 1992.
4. Jean Vuillemin. A data structure for manipulating priority queues. *Communications of the ACM*, 21(4):309–315, 1978.

[7] A special thanks to Dhruv Matani (http://dhruvbird.com) for his inputs in making graphs more meaningful

Binomial heaps

Version 5.3.4

October 15, 2013

```
(require binomial)
```

Binomial Heaps

A *Binomial heap* is a data structure for maintaining a collection of elements, such that new elements can be added and the element of minimum value extracted efficiently. This implmentation is purely functional hence *immutable*. *Binomial heap* allow only numbers to be stored in them.

heap-lazy? is just a check if the given argument complies with the binomial heap structure adopted in this implementation. *Binomial heaps* have a array-based implementation. All values of the heap are stored in a vector which is pointed by `car` of a pair. The `cdr` is the count/size of the heap. This pair is embedded within another pair's `car`. The `cdr` of the outer pair stores the min value of the heap.

```
(bino-makeheap val) → heap-lazy?
  val : number?
```

Returns a newly allocated heap with only one element `val`.

Examples:

```
> (define h (bino-makeheap 1))

> h
'((#(1) . 1) . 1)
```

```
(bino-findmin h) → number?
  h : heap-lazy?
```

Returns a minimum value in the heap `h`.

Examples:

```
> (define h (bino-meld (bino-makeheap 1) (bino-makeheap 2)))

> (bino-findmin h)
1
```

```
(bino-insert h val) → heap-lazy?
  h : heap-lazy?
  val : number?
```

Returns a newly allocated heap which is a copy of `h` along with `val`.

Examples:

```
> (define h (bino-makeheap 1))

> (bino-insert h 2)
'((#(#f 1 2) . 2) . 1)
```

```
(bino-deletemin h) → heap?
  h : heap?
```

Returns a newly allocated heap with the min value of the given heap `h` removed.

Examples:

```
> (define h (bino-makeheap 1))

> (bino-deletemin h)
'((#() . 0) . #f)
```

```
(bino-meld h1 h2) → heap?
  h1 : heap-lazy?
  h2 : heap-lazy?
```

Returns a newly allocated heap by coupling `h1` and `h2`.

Examples:

```
> (define h (bino-meld (bino-makeheap 1) (bino-makeheap 2)))

> h
'((#(#f 1 2) . 2) . 1)
```

```
(bino-count h) → exact-nonnegative-integer?
  h : heap-lazy?
```

Returns the count of the elements in the heap *h*

Examples:

```
> (define h (bino-meld (bino-makeheap 1) (bino-makeheap 2)))

> (bino-count h)
2
```

Fibonacci heaps

Version 5.3.4

October 15, 2013

```
(require fibonacci)
```

Fibonacci Heaps

A *Fibonacci heap* is a data structure for maintaining a collection of elements. In addition to the binomial heap operations, Fibonacci heaps provide two additional operations viz. *decrement* and *delete* exist. Although, it should be noted that that trees in *Fibonacci heaps* are not binomial trees as the implementation cuts subtrees out of them in a controlled way. The rank of a tree is the number of children of the root, and as with binomial heaps we only link two trees if they have the same rank.

Roots of binomial tress in the heap are stored in the form of a doubly linked list; each node has a reference to a left and right node.

```
(fi-makeheap val) → fi-heap?
  val : number?
```

Returns a newly allocated heap with only one element `val`.

Examples:

```
> (define h (fi-makeheap 3))

> h
#<fi-heap>
```

```
(fi-findmin h) → number?
  h : fi-heap?
```

Returns a minimum value in the heap `h`.

Examples:

```
> (define h (fi-meld! (fi-makeheap 1) (fi-makeheap 2)))
```

```
> (fi-findmin h)
1
```

```
(fi-insert! h val) → void?
  h : fi-heap?
  val : number?
```

Updates the left right pointers of the min node to accommodate the new node with `val`, `h`, with a new node having `val`.

Examples:

```
> (define h (fi-makeheap 1))
```

```
> (set! h (fi-insert! h 2))
```

```
> (fi-heap-size h)
2
```

```
(fi-deletemin! h) → fi-heap?
  h : fi-heap?
```

Updates the given heap `h` by removing the node with the minimum value and changing the reference to the new min node.

Examples:

```
> (define h (fi-meld! (fi-makeheap 1) (fi-makeheap 2)))
```

```
> (fi-deletemin! h)
```

```
> (fi-findmin h)
2
```

```
(fi-meld! h1 h2) → fi-heap?
  h1 : fi-heap?
  h2 : fi-heap?
```

Returns a newly allocated heap by coupling `h1` and `h2`.

Examples:

```
> (define h (fi-meld! (fi-makeheap 1) (fi-makeheap 2)))
```

```
> h
#<fi-heap>
```

```
(fi-decrement! h noderef delta) → void?
  h : fi-heap?
  noderef : node?
  delta : number?
```

Updates the value of *noderef* and if the heap condition is violated, then parent of the *noderef* is checked and if already marked, it is removed. This happens recursively until the root of the tree is reached or a parent which is not marked.

Examples:

```
> (define h (fi-meld! (fi-makeheap 1) (fi-makeheap 2)))

> (fi-decrement! h (fi-heap-minref h) 2)

> (fi-findmin h)
-1
```

```
(fi-delete! h noderef) → void?
  h : fi-heap?
  noderef : node?
```

Updates the heap *h* by deleting the *noderef* and updating its parent and children. Here also if the parent is marked, then it is also removed from the tree and added as a root. Happens until the root of the tree is reached or a parent is not marked.

Examples:

```
> (define h (fi-meld! (fi-makeheap 1) (fi-makeheap 2)))

> (fi-delete! h (fi-heap-minref h))

> (fi-findmin h)
2
> (fi-heap-size h)
1
```

```
(fi-heap-minref h) → node?
  h : fi-heap?
```

Returns the reference of the node which has the minimum value in *h*.

```
(fi-heap-size h) → exect-nonnegative-integer?
  h : fi-heap?
```

Returns the size of the heap *h*.

```
(fi-node-val n) → number?
  n : node?
```

Returns the value stored in the node *n*.

```
(fi-node-children n) → vector?
  n : node?
```

Returns a vector of all children of node *n*. If *n* does not have any children then a empty vector will be returned.

```
(fi-node-parent n) → node?
  n : node?
```

Returns the parent node of *n* if there exists one or #f.

```
(fi-node-left n) → node?
  n : node?
```

Returns *n*'s left node in the circular linked list. If *n* is the only node in list then a reference of *n* will be returned.

```
(fi-node-right n) → node?
  n : node?
```

Returns *n*'s right node in the circular linked list.