Homework

Ex 2

Let $X_1 \dots X_n \sim Uniform(a, b)$, $a < b$

a) $$\begin{cases} \frac{X_1 + \dots + X_n}{n} = \frac{a+b}{2} \\ \frac{X_1^2 + \dots + X_n^2}{n} = \frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2 \end{cases}$$

$$\frac{a+b}{2} = M \Rightarrow b = 2M - a$$

$$\sigma^2 = Var(X) = \frac{(b-a)^2}{12} \Rightarrow \frac{(2M-a-a)^2}{12} \Rightarrow$$

$$(2M - 2a)^2 = 12\sigma^2$$

$$(M-a)^2 = 3\sigma^2$$

$$a = M - \sqrt{3\sigma^2}$$

$$b = 2M - (M - \sqrt{3\sigma^2}) = M + \sqrt{3\sigma^2}$$

b) $f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$

$L(a, b) = f(x_1) f(x_2) \dots = \begin{cases} \frac{1}{(b-a)^n}, & \text{if } x_i \in [a, b] \\ 0, & \text{otherwise} \end{cases}$

$a = \min X_i$

$b = \max X_i$

№5

$X_1 \dots X_n \sim \text{Poisson}(\lambda)$

$\hat{\lambda} = E[X] = \frac{X_1 + \dots + X_n}{n}$ - sample mean.

$f(k, \lambda) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

Likelihood function:

$L(\lambda) = f(X_1) \dots f(X_n) =$

$= e^{-\lambda} \frac{\lambda^{x_1}}{x_1!} \dots e^{-\lambda} \frac{\lambda^{x_n}}{x_n!}$

log-likelihood function

$$l = \ln \cdot L(\lambda) = \log\left(e^{-\lambda} \cdot \frac{\lambda^{x_i}}{x_i!}\right) =$$

$$= -\lambda + x_i \log \lambda - \log(x_i!)$$

$$\frac{dl}{d\lambda} = 0 \Rightarrow -n + \frac{x_i}{\lambda} - 0 = 0$$

$$\sum \frac{x_i}{\lambda} = n$$

$$\frac{\sum x_i}{\lambda \cdot n} = n$$

$$\frac{x_1 + \dots + x_n}{n} = \lambda \cdot n$$

score func: $S(X;\theta) = \dfrac{d \ln f(\lambda)}{d\lambda}$

$$S(X;\theta) = \frac{d(-\lambda + k\log\lambda - \log(k!))}{d\lambda} = \frac{k}{\lambda} - 1$$

$$I_n(\lambda) = \sum_{i=1}^{n} Var(S(X_i;\theta)) = \sum Var\left(\frac{X}{\lambda} - 1\right) =$$

$$= \frac{1}{\lambda^2} \cdot n \cdot \lambda = \frac{n}{\lambda}$$

$$I(\lambda) = \frac{1}{n} I_n(\lambda) = \frac{1}{\lambda}$$

N4

Let $X_1 ... X_n \sim Uniform(0, \theta)$

$$F(x) = \begin{cases} \frac{1}{\theta}, & x \in [0, \theta] \\ 0, & otherwise \end{cases}$$

$$\mathcal{L}(\theta) = f(x_1) ... f(x_n) = \begin{cases} \frac{1}{\theta^n}, & x_i \in [0, \theta] \\ 0, & otherwis \end{cases}$$

$0 \leq x_1 ... x_n \leq \theta$, therefore $\hat{\theta} = max(x_1 ... x_n)$

Let $c = \theta - \varepsilon$

$$P(\varepsilon < \theta - \hat{\theta}) = P(\varepsilon < \theta - X_1) P(\varepsilon < \theta - x_2) ...$$

$$P(X_i < \theta - \varepsilon) = \frac{\theta - \varepsilon - 0}{\theta - 0} = 1 - \frac{\varepsilon}{\theta}$$

$P(\varepsilon < \theta - \hat{\theta}) = \left(1 - \frac{\varepsilon}{\theta}\right)^n$ as $n \to \infty$,

$P(\varepsilon < \theta - \hat{\theta}) \to 0$

Consistent