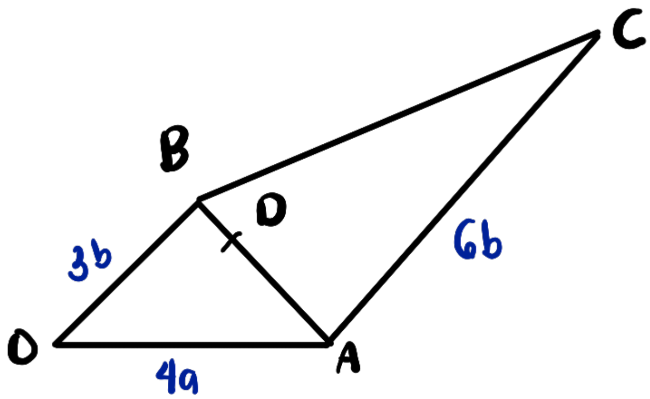
GEOMETRIC PROOFS

AIHL 3.13

Example 1. The figure below shows a trapezium OBCA where OB is parallel to AC .



The point D lies on BA so that BD DA : 1: 2 = . Let OA = 4a , OB = 3b and AC = 6b .

a) Find simplified expressions, in terms of a and b , for each of the vectors OC , AB , AD and OD .

$\vec{OC} = \vec{OA} + \vec{AC}$

$= 4a + 6b$

$\vec{AB} = \vec{AO} + \vec{OB}$

$= -4a + 3b$

$\vec{AD} = \frac{2}{3}\vec{AB}$

$= \frac{2}{3}(-4a + 3b)$

$= -\frac{8}{3}a + 2b$

$\vec{OD} = \vec{OA} + \vec{AD}$

$= 4a + \left(-\frac{8}{3}a + 2b\right)$

$= \frac{4}{3}a + 2b$

b) Deduce, showing your reasoning, that O , D and C are collinear and state the ratio of : OD:DC .

$\vec{DC} = \vec{DA} + \vec{AC}$

$= \left(\frac{8}{3}a - 2b\right) + 6b$

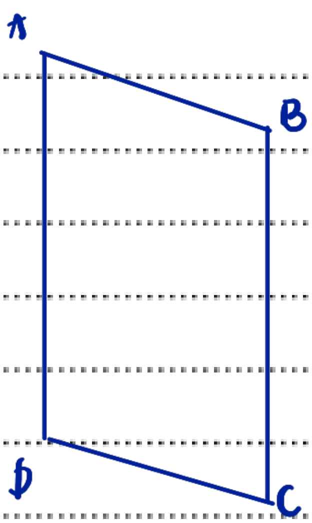
$= \frac{8}{3}a + 4b$

$= 2\left(\frac{4}{3}a + 2b\right)$

$\vec{OD} = \frac{4}{3}a + 2b$

$OD : DC = 1 : 2$

Example 2. Use vectors to prove that the points A, B, C and D with position vectors a = (3i - 5j - 4k), b = (8i - 7j - 5k), c=(3i -2j +4k) and d =(5k – 2i) are the vertices of a parallelogram.



$$\vec{AB} = b - a = \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix}$$

$$\vec{BC} = c - b = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 9 \end{pmatrix}$$

$$\vec{CD} = d - c = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$\vec{DA} = a - d = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 9 \end{pmatrix}$$

$\therefore\ AB = -CD$

$BC = -DA$

Example 3. The points A(5,1,3) , B(3,1,5) , C(5,3,5) and D(4,0,3) are given.
a) Show that the triangle ABC is equilateral and find its area.
b) Show further that
AD=λ AB +μAC, stating the exact values of the scalar constants λ and μ .
c) Find the size of the angle BAD .

a. $|\vec{AB}| = |b-a| = \sqrt{(-2)^2+0^2+2^2} = \sqrt{8}$ $\qquad A = \frac{1}{2}|AB||AC| \sin 60^\circ$

$|\vec{BC}| = |c-b| = \sqrt{2^2+2^2+0^2} = \sqrt{8}$ $\qquad = 2\sqrt{3}$

$|\vec{CA}| = |a-c| = \sqrt{0^2+(-2)^2+(-2)^2} = \sqrt{8}$

b. $AD = d - a = (-1,-1,0)$

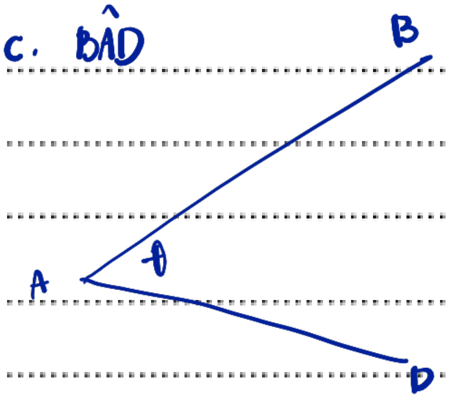
$(-1,-1,0) = \lambda(-2,0,2) + \mu(0,2,2)$

$= -2\lambda, 2\lambda, 2\mu, 2\mu)$

$-1 = -2\lambda \qquad -1 = 2\mu$

$\frac{1}{2} = \lambda \qquad -\frac{1}{2} = \mu$

c. $B\hat{A}D$



$\overline{AB} \cdot \overline{AD} = |AB||AD| \cos\theta$

$\frac{AB \cdot AD}{|AB||AD|} = \cos\theta$

$\theta = 60^\circ$