THE COMPLEX PLANE

Week 23

Objectives

- To represent complex numbers on the Argand plane

- To find the modulus and argument of a complex number and use these to graph complex numbers.

The Complex Plane

- We can plot any complex number on a plane as a unique ordered number pair. We refer to the plane as the **complex plane** or the **Argand plane**.

- On the complex plane, the $x - axis$ is the **real axis** and the $y - axis$ is the **imaginary axis**.

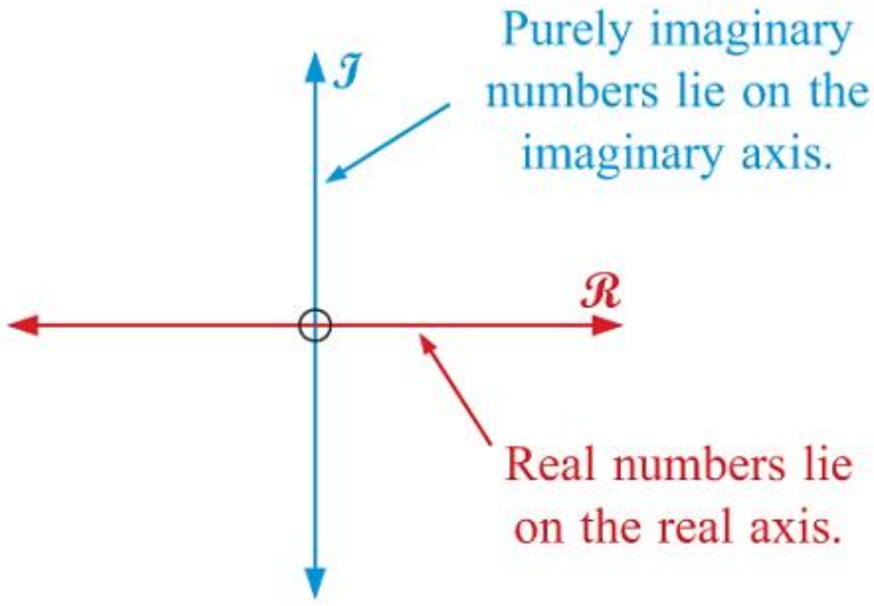
- We can illustrate complex numbers using vectors on the Argand plane. We call this an **Argand diagram**.

International Mindedness

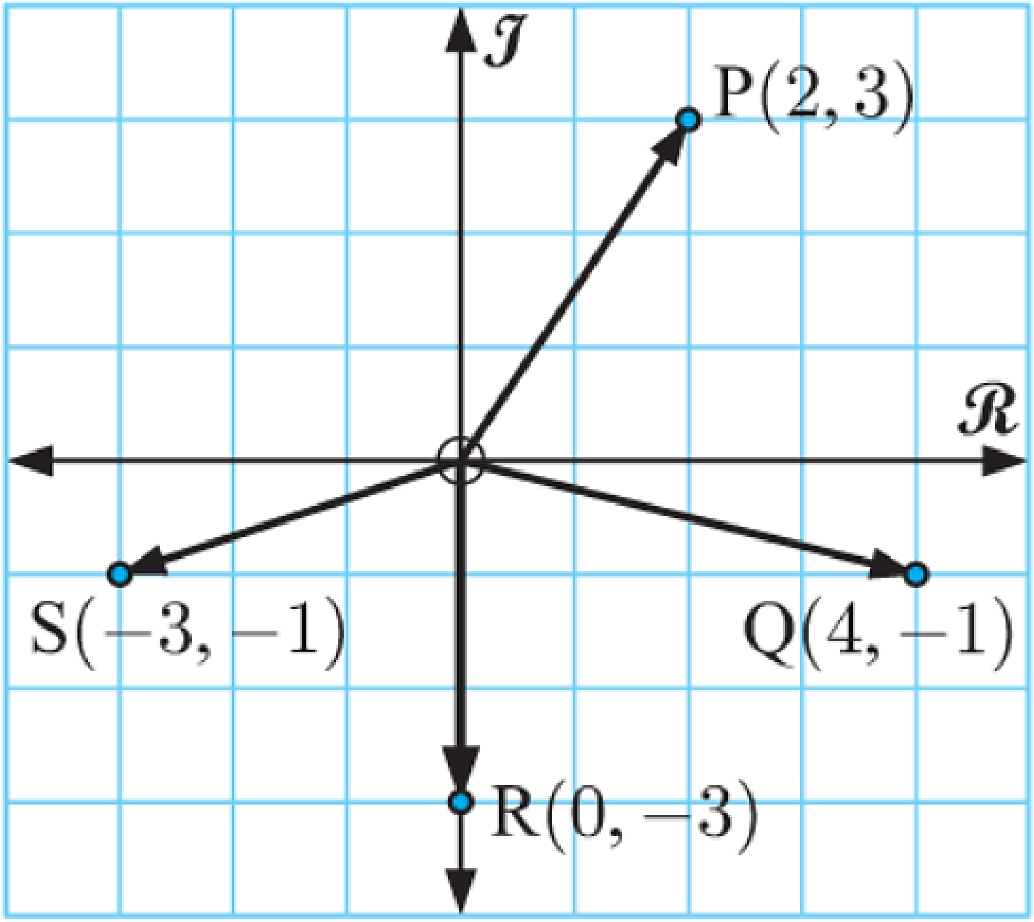
John-Robert Argand and Carl Friedrich Gauss developed the concept of the complex plane during roughly the same time period. Although one was working in France and the other was working in Germany, they came up with similar ideas independently. Hence, the Argand diagram is also referred to as the Gaussian plane.

The Complex Plane

- All real numbers with $b = 0$ lie on the real axis.
- All purely imaginary numbers with $a = 0$ lie on the imaginary axis.
- The origin (0, 0) lies on both axes. It corresponds to $z = 0$, a real number.
- Complex numbers that are neither real nor purely imaginary (a and b are both not 0) lie in one of the four quadrants.



On an Argand diagram, $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$.



$\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ represents $2 + 3i$

$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ represents $4 - i$

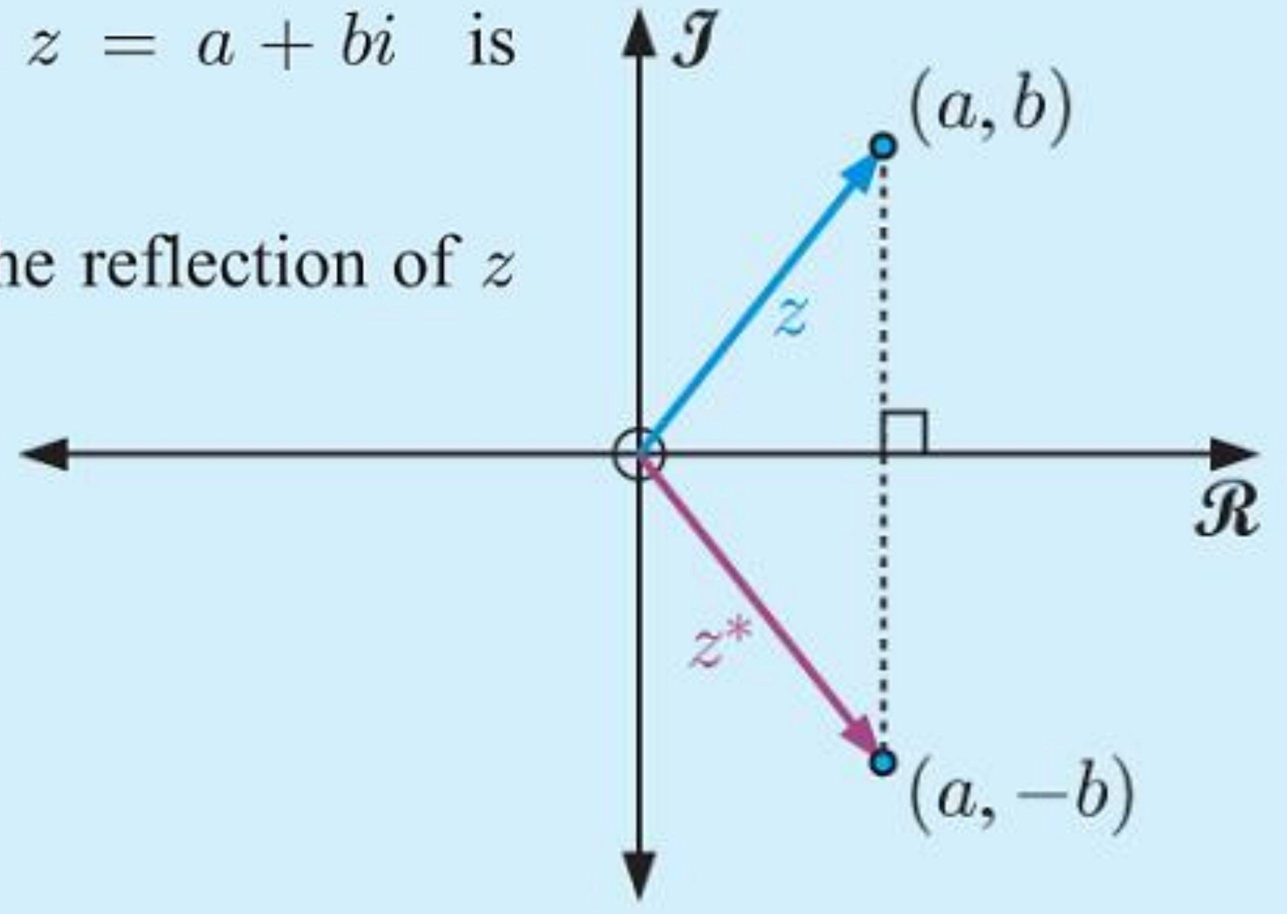
$\overrightarrow{OR} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ represents $-3i$

$\overrightarrow{OS} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ represents $-3 - i$

The Complex Plane

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

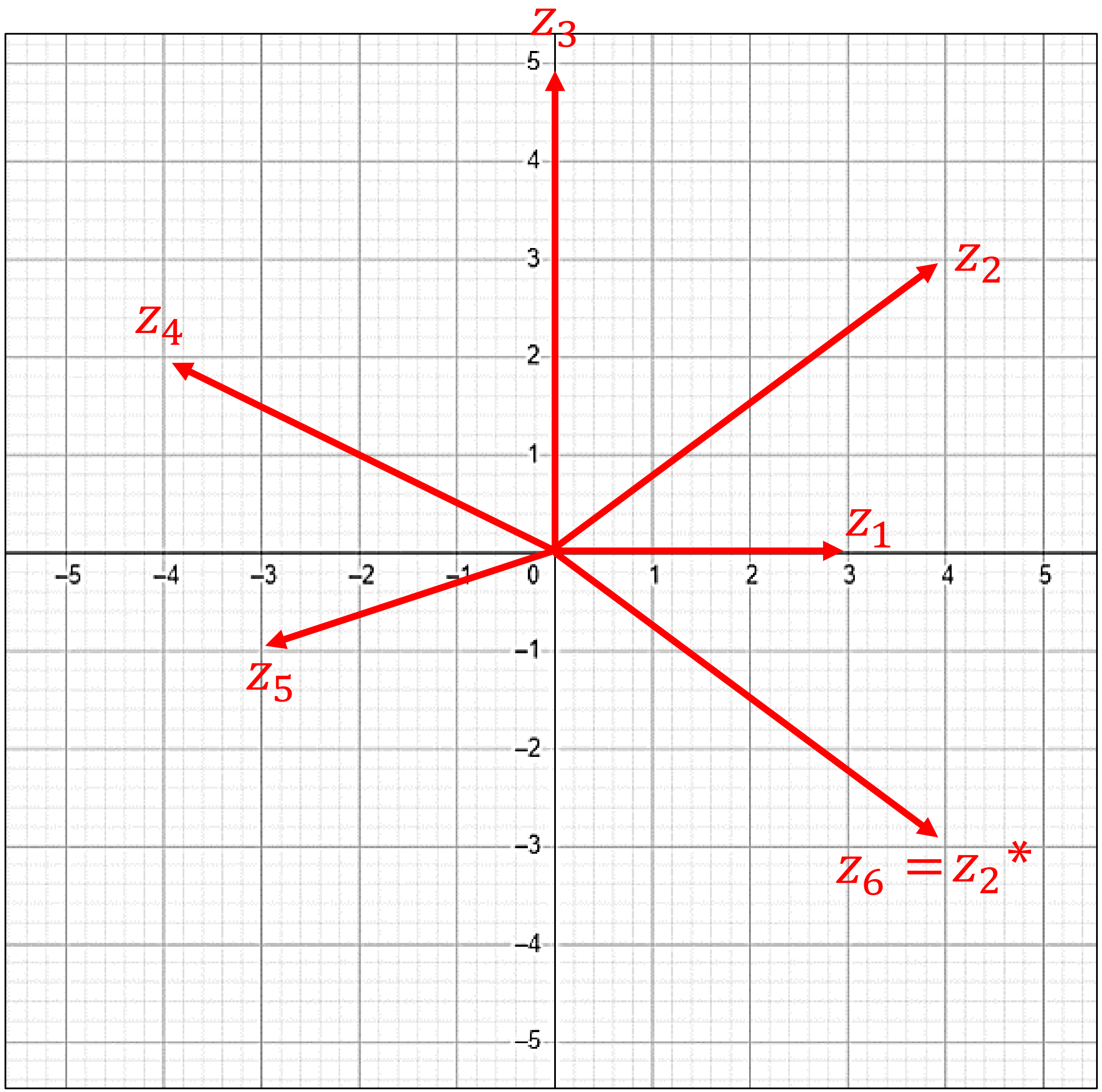
In the complex plane, z^* is the reflection of z in the real axis.



Ex. 1: Illustrating

Illustrate the positions of the following complex numbers on an Argand diagram:

$$z_1 = 3, \qquad z_2 = 4 + 3i,$$
$$z_3 = 5i, \qquad z_4 = -4 + 2i,$$
$$z_5 = -3 - i, \qquad z_6 = {z_2}^*$$

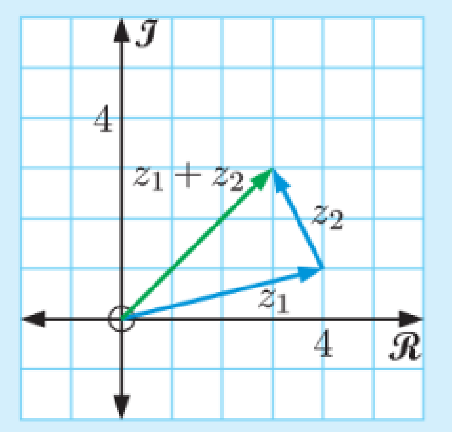


Ex.2: Adding and Subtracting Complex Numbers

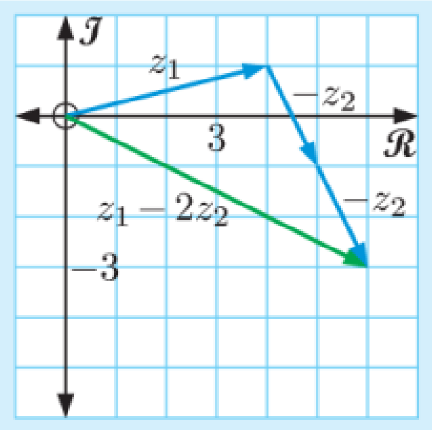
Suppose $z_1 = 4 + i$ and $z_2 = -1 + 2i$. Find using both algebra and vectors:

a $z_1 + z_2$ **b** $z_1 - 2z_2$ **c** $2z_1 + {z_2}^*$

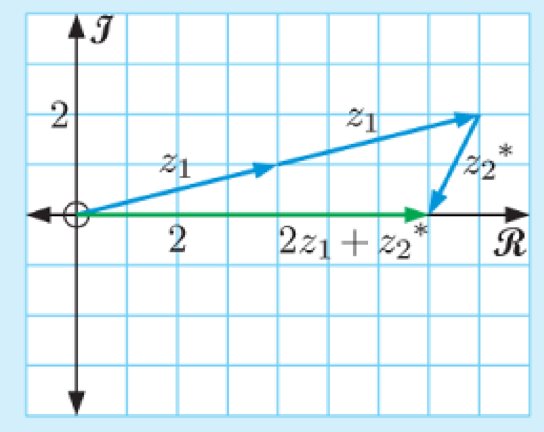
a
$$\begin{aligned} & z_1 + z_2 \\ &= (4+i) + (-1+2i) \\ &= 4 + i - 1 + 2i \\ &= 3 + 3i \end{aligned}$$



b
$$\begin{aligned} & z_1 - 2z_2 \\ &= (4+i) - 2(-1+2i) \\ &= 4 + i + 2 - 4i \\ &= 6 - 3i \end{aligned}$$



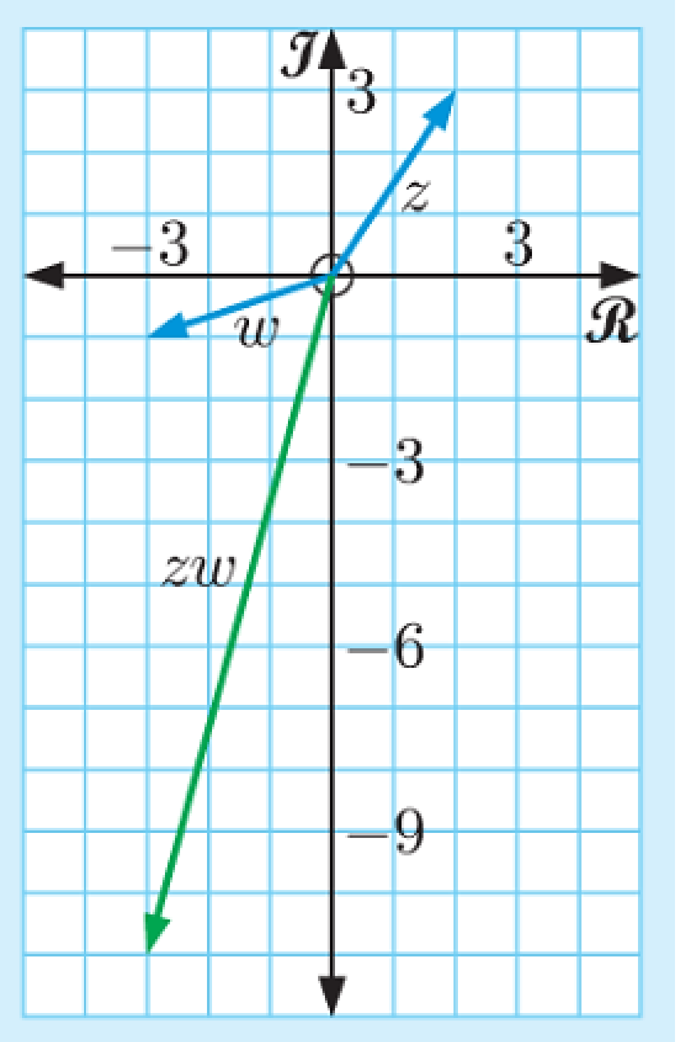
c
$$\begin{aligned} & 2z_1 + {z_2}^* \\ &= 2(4+i) + (-1-2i) \\ &= 8 + 2i - 1 - 2i \\ &= 7 \end{aligned}$$



Ex.2: Multiplication of Complex Numbers

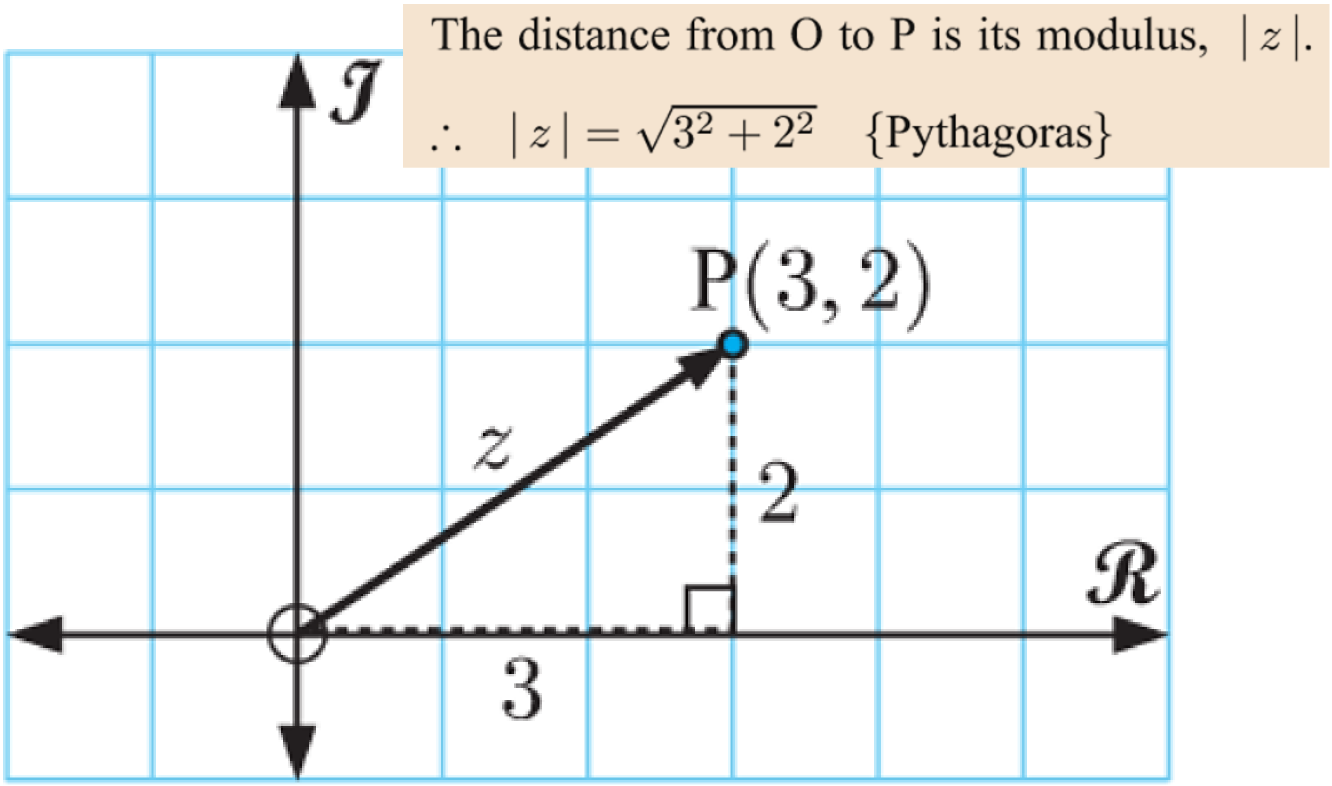
Let $z = 2 + 3i$ and $w = -3 - i$. Find zw, and illustrate z, w, and zw on the same Argand diagram.

$$\begin{aligned} zw &= (2+3i)(-3-i) \\ &= -6 - 2i - 9i + 3 \\ &= -3 - 11i \end{aligned}$$



Modulus $|z|$ and Argument ($\arg z$)

- Modulus – the <u>magnitude</u> of a complex number, which is also the <u>length</u> of the vector

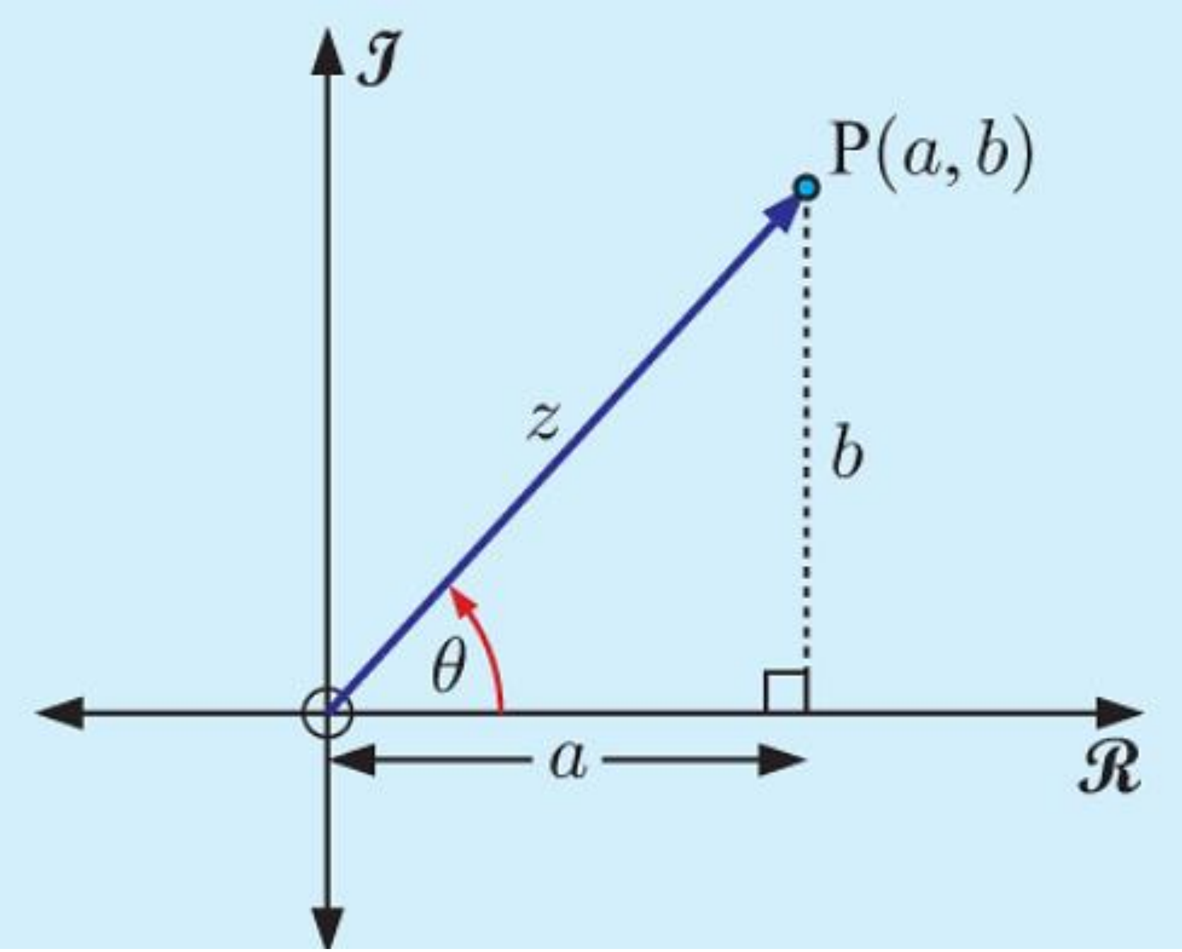


Modulus $|z|$ and Argument $(\arg z)$

- Argument – the <u>direction</u> (an angle measurement, in either degree or radian)

Suppose the complex number $z = a + bi$ is represented by the vector $\overrightarrow{OP}$.

The **argument** of z, or simply $\arg z$, is the angle θ in the interval $-\pi < \theta \leqslant \pi$ which is measured anticlockwise between the positive real axis and $\overrightarrow{OP}$.



Be aware

You should always make a sketch of the Argand diagram for the complex number if you are working with the modulus and argument.

Examples

Find $|z|$ and $\arg z$ for z equal to:

a $1+3i$ **b** $-1+3i$ **c** $-3-i$

Important

Given that $z = x + yi$, then the principal value of the argument is

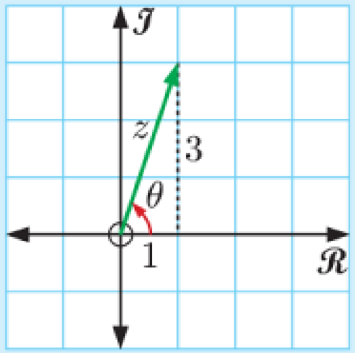
- in the first and fourth quadrants when $Re\,(z) > 0, \mathrm{arg(z)} = \tan^{-1}\left(\frac{y}{x}\right)$
- in the second quadrant when $Re\,(z) < 0$ and $Im\,(z) > 0$, $\mathrm{arg(z)} = 180 + \tan^{-1}\left(\frac{y}{x}\right)$
- in the third quadrant when $Re\,(z) < 0$ and $Im\,(z) < 0$, $\mathrm{arg(z)} = -180 + \tan^{-1}\left(\frac{y}{x}\right)$.

Examples

Find $|z|$ and $\arg z$ for z equal to:

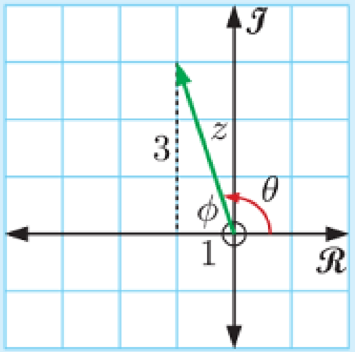
a $1+3i$ **b** $-1+3i$ **c** $-3-i$

a $|z| = \sqrt{1^2+3^2}$
$= \sqrt{10}$



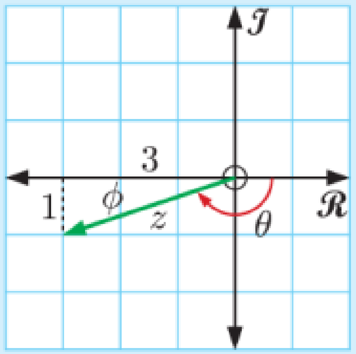
$\tan\theta = \frac{3}{1} = 3$
$\therefore \quad \arg z \approx 1.25$

b $|z| = \sqrt{(-1)^2+3^2}$
$= \sqrt{10}$



$\tan\phi = \frac{3}{1} = 3$
$\therefore \quad \phi \approx 1.25$
But $\quad \theta = \pi - \phi$
$\therefore \quad \arg z \approx 1.89$

c $|z| = \sqrt{(-3)^2+(-1)^2}$
$= \sqrt{10}$



$\tan\phi = \frac{1}{3}$
$\therefore \quad \phi \approx 0.322$
But $\quad \theta = -\pi + \phi$
$\therefore \quad \arg z \approx -2.82$

YOUR TURN

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