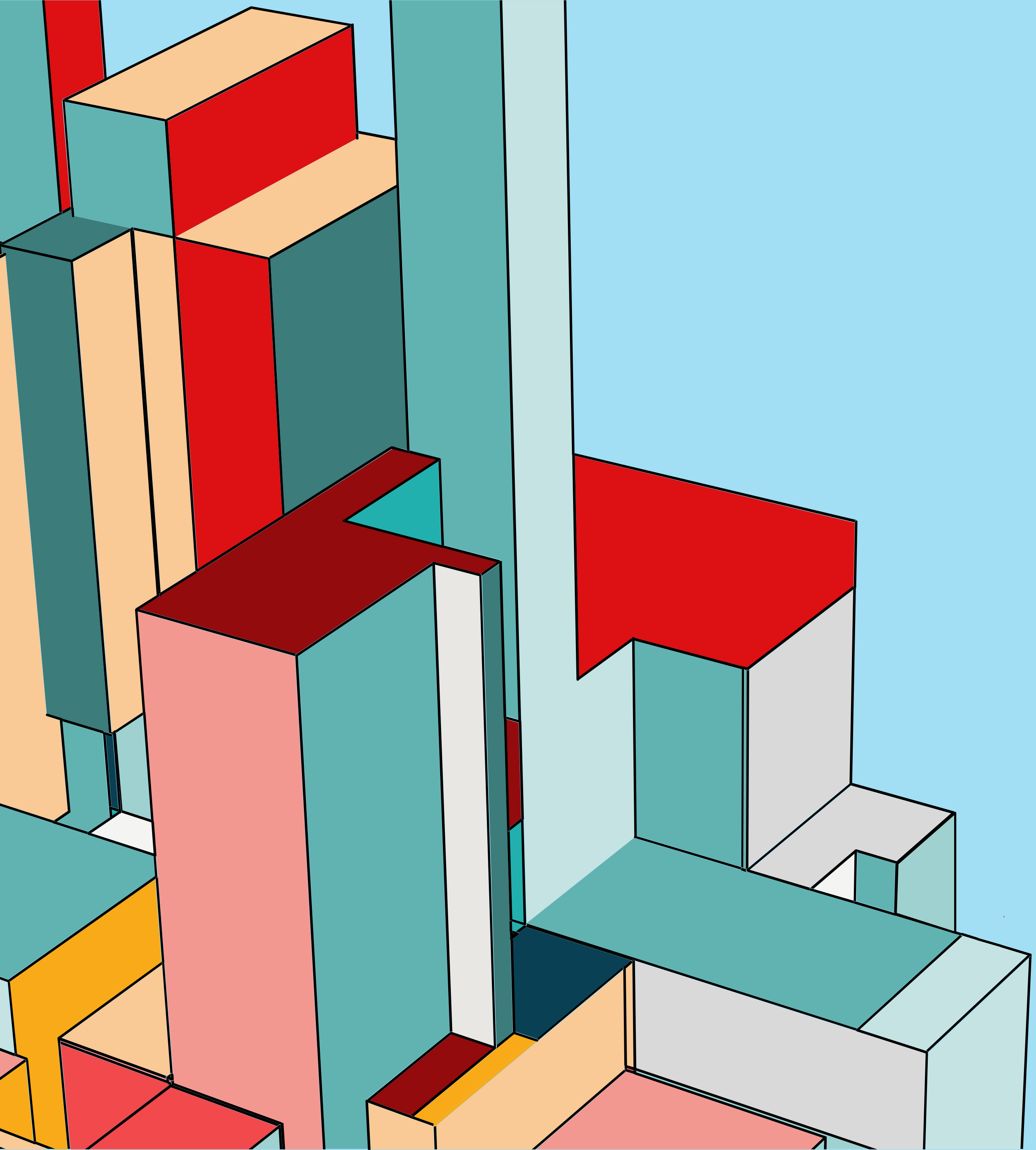


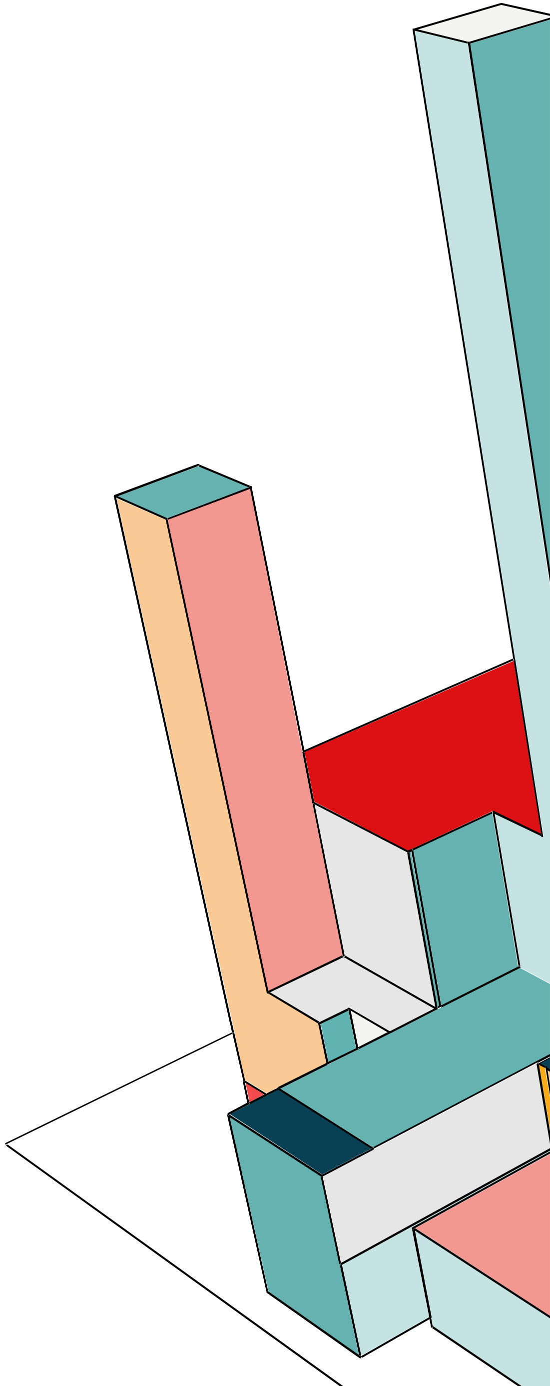
POLAR FORM

OBJECTIVE

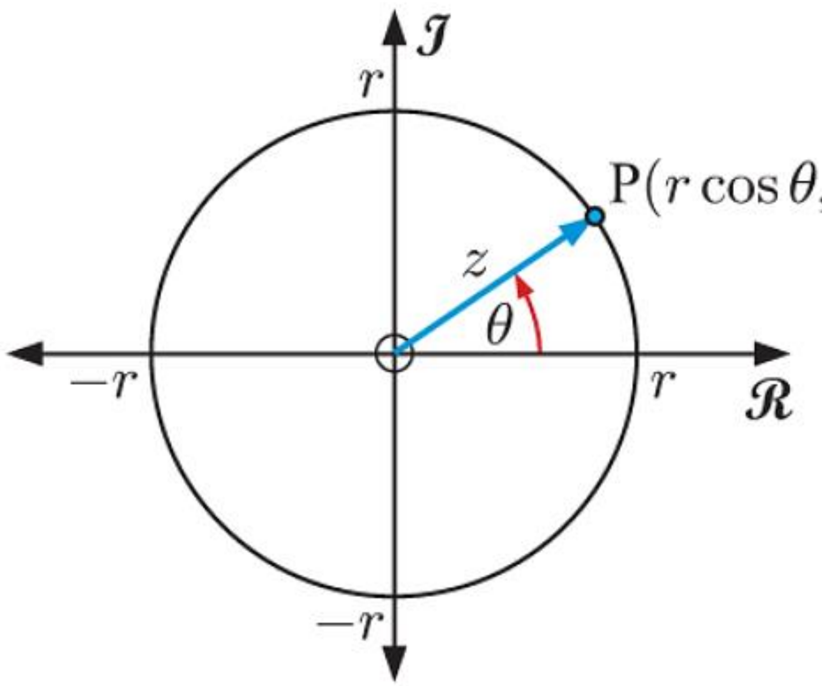
- to write complex numbers into polar and exponential forms

CARTESIAN FORM

$$z = a + bi$$



POLAR FORM



$\therefore$ on the Argand plane, the complex number represented by $\overrightarrow{\mathrm{OP}}$ is

$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg z$.

We define $\mathbf{cis}\,\boldsymbol{\theta} = \mathbf{cos}\,\boldsymbol{\theta} + \boldsymbol{i}\,\mathbf{sin}\,\boldsymbol{\theta}$ so that $z = |z|\operatorname{cis}\theta$.

Any point P which lies on a circle with centre $\mathrm{O}(0,\ 0)$ and radius r, has Cartesian coordinates $(r\cos\theta,\ r\sin\theta)$.

POLAR FORM

A complex number z has **polar form** $\boxed{z = |z| \operatorname{cis} \theta}$ where $|z|$ is the **modulus** of z, θ is the **argument** of z, and $\operatorname{cis} \theta = \cos \theta + i \sin \theta$.

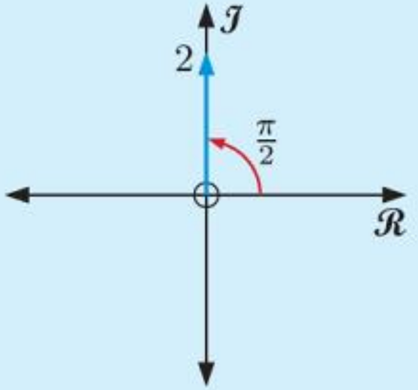
Polar form is also called the **modulus-argument form.**

EXAMPLES: CARTESIAN TO POLAR FORM

Write in polar form:

a $2i$ **b** -3 **c** $1-i$

a

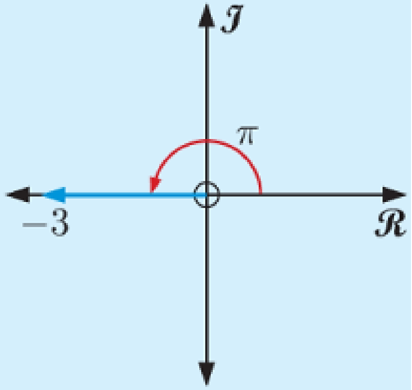


$|2i| = 2$

$\arg(2i) = \frac{\pi}{2}$

$\therefore \quad 2i = 2\,\text{cis}\,\frac{\pi}{2}$

b

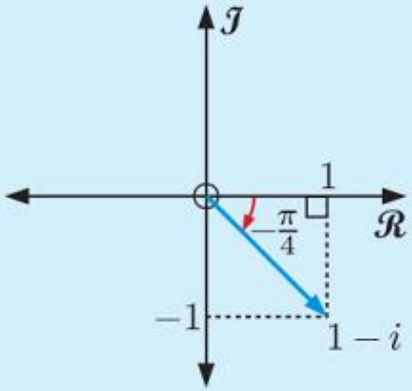


$|-3| = 3$

$\arg(-3) = \pi$

$\therefore \quad -3 = 3\,\text{cis}\,\pi$

c



$|1-i| = \sqrt{1+1} = \sqrt{2}$

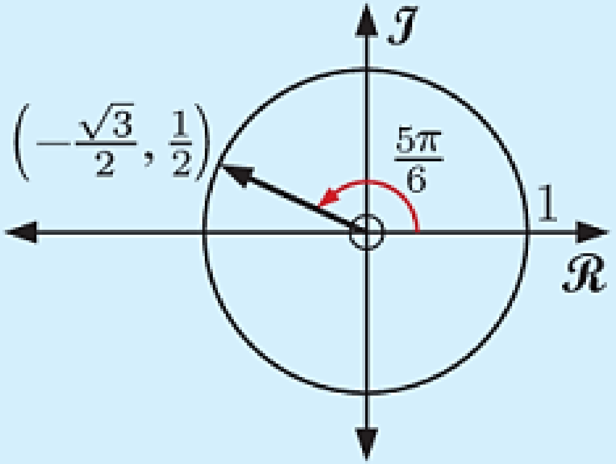
$\arg(1-i) = -\frac{\pi}{4}$

$\therefore \quad 1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

EXAMPLE: POLAR TO CARTESIAN FORM

Convert $\sqrt{3}\,\mathrm{cis}\,\frac{5\pi}{6}$ to Cartesian form.

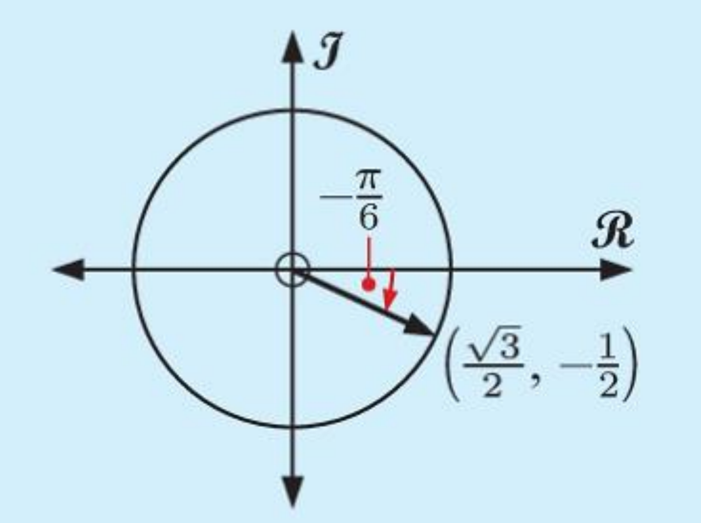
We expand $\mathrm{cis}\,\frac{5\pi}{6}$ using a unit circle diagram.



$$
\begin{aligned}
&\sqrt{3}\,\mathrm{cis}\,\frac{5\pi}{6} \\
&= \sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \\
&= \sqrt{3}\left(-\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right) \\
&= -\frac{3}{2} + \frac{\sqrt{3}}{2}i
\end{aligned}
$$

EXAMPLE: POLAR TO CARTESIAN FORM

Simplify $\operatorname{cis}\frac{107\pi}{6}$.



$$\operatorname{cis}\frac{107\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

EXAMPLES

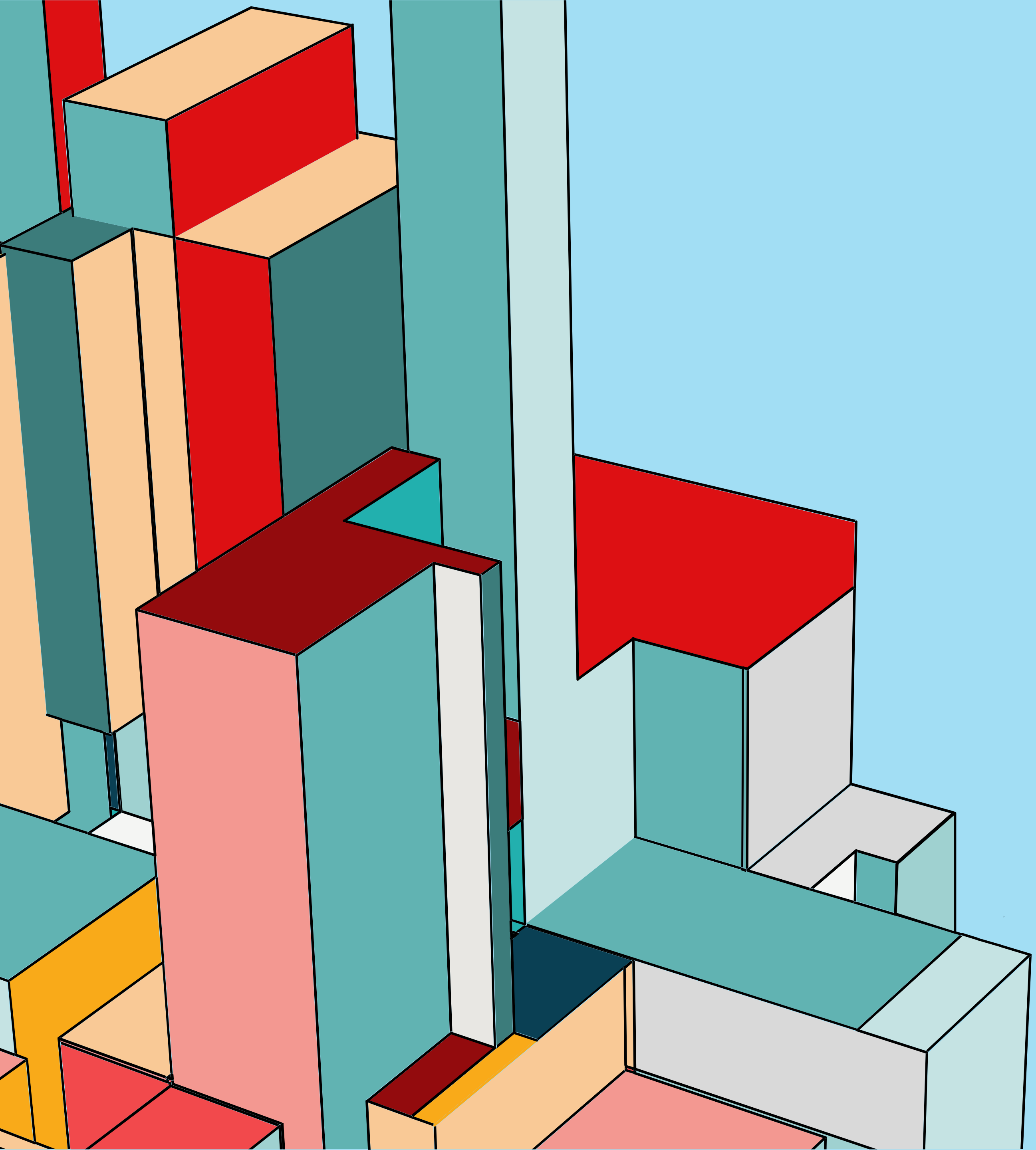
Use your calculator to convert:

a $5\operatorname{cis}(1.9)$ to Cartesian form

b $6-5i$ to modulus-argument form.

a $5\operatorname{cis}(1.9) \approx -1.62+4.73i$

b $6-5i \approx \sqrt{61}\operatorname{cis}(-0.695)$



EXPONENTIAL FORM

HISTORICAL NOTE EULER'S BEAUTIFUL EQUATION

One of the most remarkable results in mathematics is known as **Euler's beautiful equation** $e^{i\pi} = -1$ named after **Leonhard Euler**.

It is called beautiful because it links together three great constants of mathematics: Euler's constant e, the imaginary number i, and the ratio of a circle's circumference to its diameter, which is π.

Harvard lecturer **Benjamin Pierce** said of $e^{i\pi} = -1$,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

EXPONENTIAL FORM

$$\text{For any } \theta \in \mathbb{R}, \quad e^{i\theta} = \cos\theta + i\sin\theta.$$

This identity allows us to write any complex number $z = |z| \operatorname{cis} \theta$ in the **exponential form** or **Euler form** $z = |z| e^{i\theta}$.

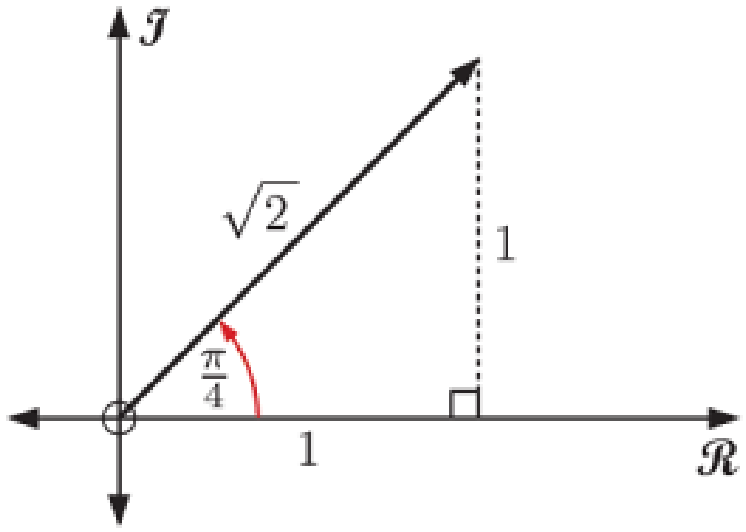
For example, consider $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$,

$\therefore \quad 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4} = \sqrt{2} e^{i\frac{\pi}{4}}$

So, $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ is the **polar form** of $1 + i$

and $\sqrt{2} e^{i\frac{\pi}{4}}$ is the **exponential form** of $1 + i$.

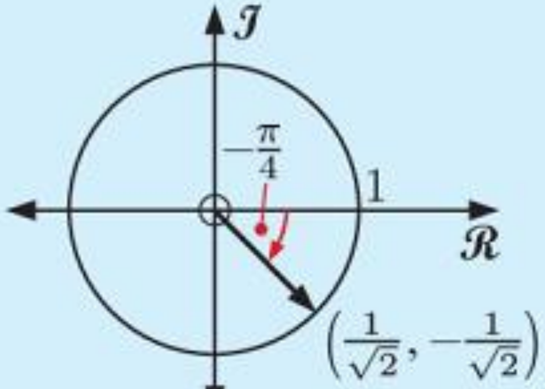


EXAMPLES

Evaluate: **a** $e^{-i\frac{\pi}{4}}$ **b** i^{-i}

a $e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$



b Now $i = 0 + 1i$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= e^{i\frac{\pi}{2}}$

$\therefore \quad i^{-i} = \left(e^{i\frac{\pi}{2}}\right)^{-i}$

$= e^{-i^2\frac{\pi}{2}}$

$= e^{\frac{\pi}{2}}$

EXAMPLES

Write:

a $-1+i$ in polar form and exponential form

b $3\operatorname{cis}\left(-\frac{\pi}{6}\right)$ in Cartesian form and exponential form

c $2e^{i\frac{2\pi}{3}}$ in Cartesian form and polar form.

a $\sqrt{2}\operatorname{cis}\frac{3\pi}{4}, \quad \sqrt{2}e^{i\frac{3\pi}{4}}$

b $\frac{3\sqrt{3}}{2}-\frac{3}{2}i, \quad 3e^{-i\frac{\pi}{6}}$

c $-1+\sqrt{3}i, \quad 2\operatorname{cis}\frac{2\pi}{3}$

EXAMPLES

Use your calculator to convert to Cartesian form:

a $e^{1.2i}$

Use your calculator to convert to exponential form:

a $5 + 2i$

a $\approx 0.362 + 0.932i$ **b** $\approx 3.06 - 2.58i$

c $\approx -7.32 + 4.11i$ **d** $\approx 0.324 - 0.528i$

a $\approx 5.39e^{0.381i}$ **b** $\approx 9.85e^{2.72i}$ **c** $\approx 3.16e^{-2.54i}$

d $\approx 4.90e^{-1.10i}$

NEXT MEETING...

Exam-style Questions: Solving Problems involving Complex Numbers