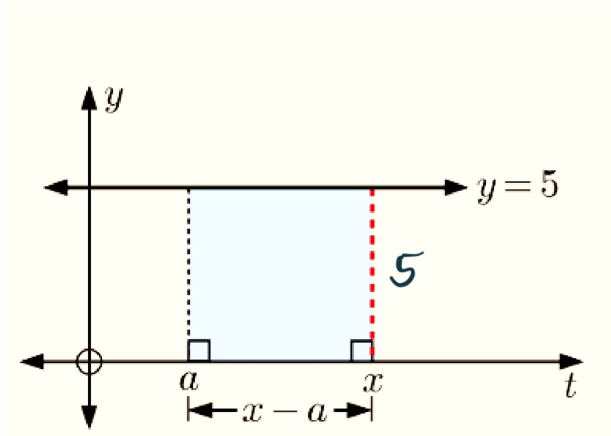
Integral Calculus

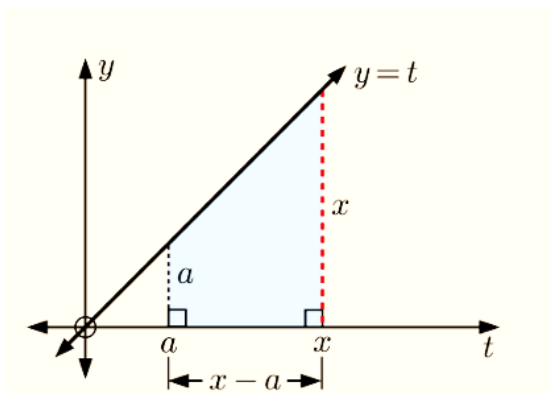
Integration is the continuous analog of a sum, which is used to calculate areas, volumes, and their generalizations.

Consider the function f(t) = 5. Find the area of the region bounded by a to x.

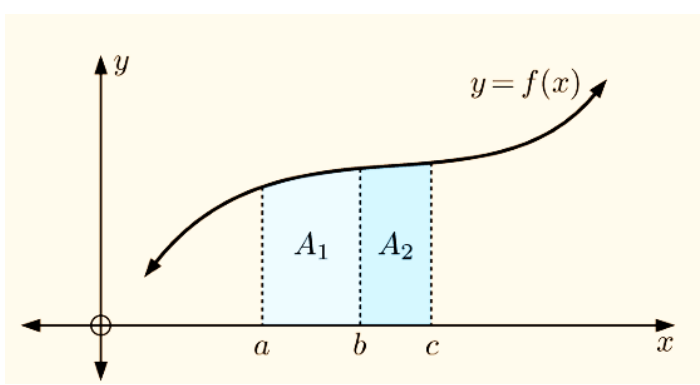
$A = 5(x-a)$ units^2



Consider the function f(t) = t. Find the area bounded by the region from a to x.



Consider as well the function y=f(x)



Approximating the area under a curve

Example 1. Approximate the area under the curve $f(x) = x^2$ from 0 to 1. $n = 4$

$$\boxed{\int_a^b f(x)\,dx}$$

$h = \frac{b-a}{n}$

$h = \frac{1-0}{4}$

$h = \frac{1}{4}$

UNDER

$= \frac{1}{4}(f(0)) + \frac{1}{4}(f(\frac{1}{4})) + \frac{1}{4}(f(\frac{1}{2})) + \frac{1}{4}(f(\frac{3}{4}))$

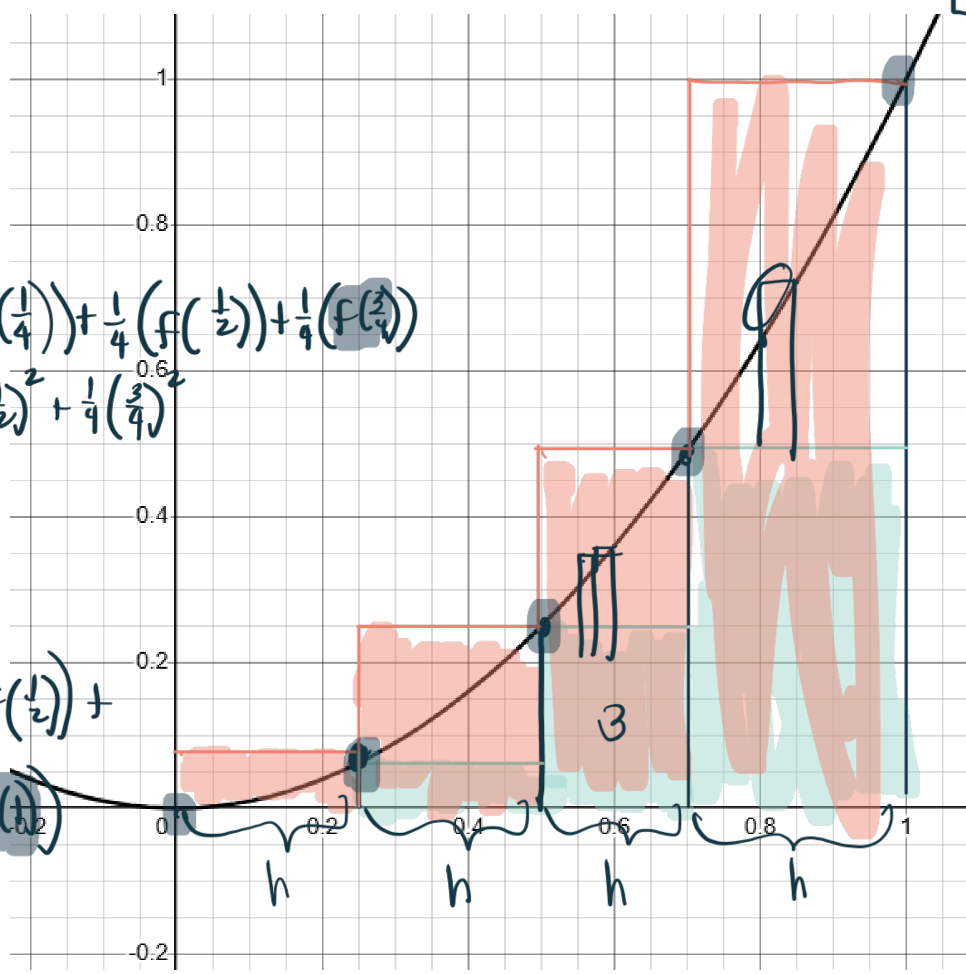
$= \frac{1}{4}(0) + \frac{1}{4}(\frac{1}{4})^2 + \frac{1}{4}(\frac{1}{2})^2 + \frac{1}{4}(\frac{3}{4})^2$

$= \frac{7}{32} \approx 0.219$

OVER

$= \frac{1}{4}(f(\frac{1}{4})) + \frac{1}{4}(f(\frac{1}{2})) + \frac{1}{4}(f(\frac{3}{4})) + \frac{1}{4}(f(1))$

$= 0.469$



Average: $\frac{0.219 + 0.469}{2} = 0.344$

upper

function, integrand

$\int_0^1 x^2\,dx = \frac{1}{3}$

variable of integration

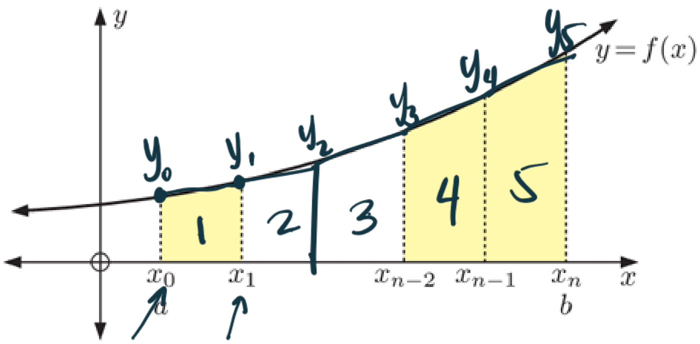
lower bound

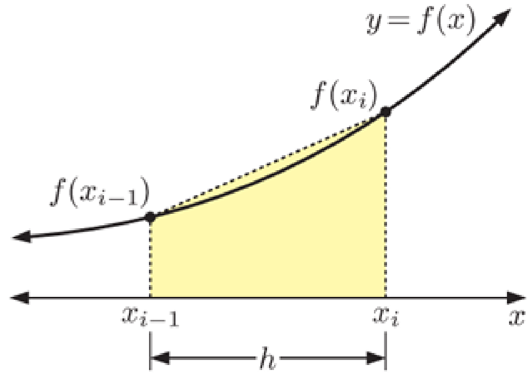
$\int_{\square}^{\square} \square\,dx$

Consider exploring if the values of n (subintervals) increase. Additional reading on The Riemann Integral

TRAPEZOIDAL RULE

If the area under the curve is evaluated, then the total area is divided into small trapezoids instead of rectangles





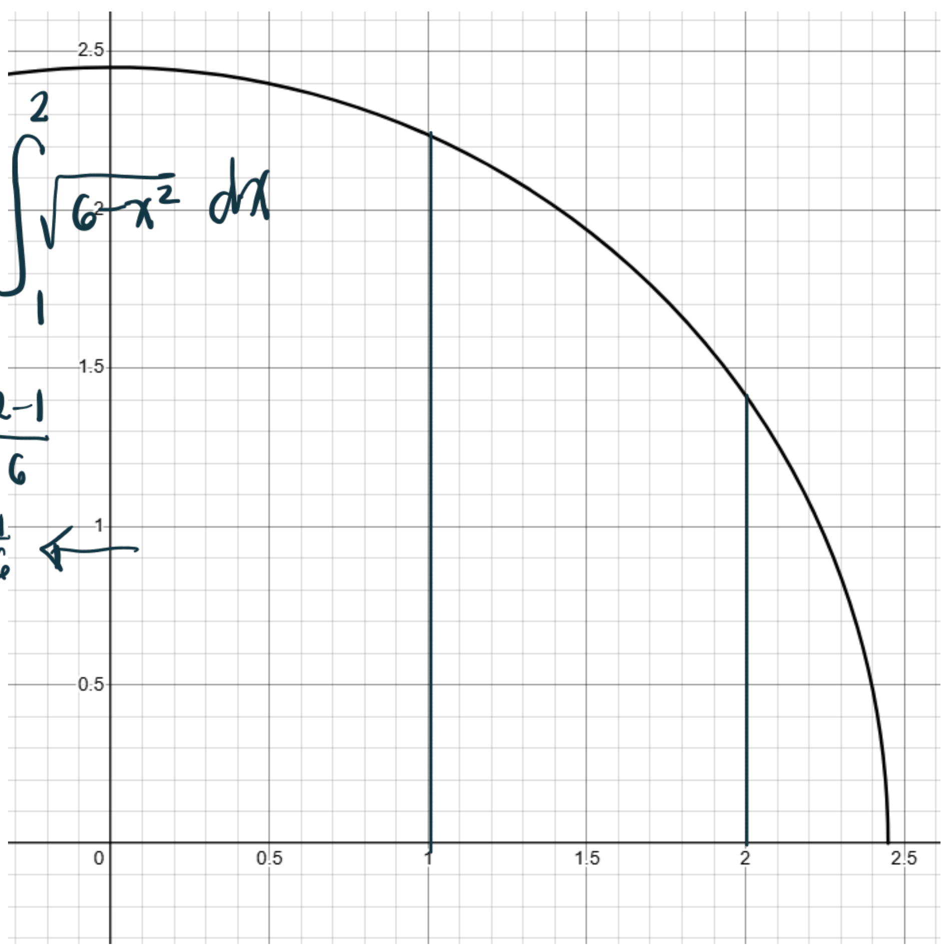
5.8	The trapezoidal rule	$\int_a^b y\,dx \approx \frac{1}{2}h\left((y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$, where $h = \frac{b-a}{n}$ | $\int_a^b y\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$

Example 2. Use the trapezoidal rule with 6 subintervals to estimate the area between $f(x) = \sqrt{6 - x^2}$ and the x-axis from x =1 to x =2.

$$A = \int_1^2 \sqrt{6 - x^2}\, dx$$

$$h = \frac{2-1}{6}$$

$$h = \frac{1}{6}$$



$y_0 = f(1)$

$y_1 = f(\frac{7}{6})$

$y_2 = f(\frac{8}{6})$

$y_3 = f(\frac{9}{6})$

$\vdots$

$y_6 = f(2)$

$f(1) =$ $f(\frac{8}{6}) = \checkmark$ $f(\frac{10}{6}) =$ $f(\frac{12}{6}) =$

$f(\frac{7}{6}) = \leftarrow$ $f(\frac{9}{6}) =$ $f(\frac{11}{6}) = \leftarrow$

$$A = \int_1^2 \sqrt{6 - x^2}\, dx = \frac{1}{2}\left(\frac{1}{6}\right)\left(\left(f(1) + f(2)\right.\right.$$

$$\int_a^b y\,dx = \frac{h}{2}\left[f(x_0) + 2\underbrace{\sum_{i=1}^{n-1} f(x_i)} + f(x_n)\right]$$

$x_1 = 1 + \frac{1}{6}(1) =$

$x_2 = 1 + \frac{2}{6} = \frac{8}{6}$

$x_5 = 1 + \frac{5}{6} = \frac{11}{6}$

$f(x_0) = f(1)$, $f(x_6) = f(2)$

$f(x_1) = 1 + \frac{1}{6}$

$f(x_2) = f(x_1) + \frac{1}{6}$

$\boxed{x_i = 1 + \frac{1}{6}i} \leftarrow$

$$\boxed{\int_1^2 \sqrt{6 - x^2}\, dx} = \frac{1}{12}\left[\sqrt{5} + 2\sum_{i=1}^{5}\sqrt{6 - (1 + \tfrac{1}{6}i)^2} + \sqrt{2}\right] \approx 1.90$$

Antidifferentiation and Indefinite Integral

5.5	Integral of x^n	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

5.11	Standard integrals	$\int \frac{1}{x} \, dx = \ln\lvert x \rvert + C$ $\int \sin x \, dx = -\cos x + C$ $\int \cos x \, dx = \sin x + C$ $\int \frac{1}{\cos^2 x} \, dx = \tan x + C$ $\int e^x \, dx = e^x + C$

Example 3. If $f(x) = 3x^2$ is the derivative of F(x), find F(x).

$F(x) = \int f(x) \, dx$

$= \int 3x^2 \, dx$

$= \frac{3x^{2+1}}{2+1} + C$

$F(x) = x^3 + C$ general solution

Indefinite Integral

$F(x) = F'(x) = f(x)$

$\frac{d}{dx}(x^3 + C) = 3x^2$

Example 4. Find the integrals of the following

a.

$$\int x^6\,dx = \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C$$

b.

$$\int (3x^2+2x)\,dx = \int 3x^2\,dx + \int 2x\,dx = x^3 + C + x^2 + C$$

$$= x^3 + x^2 + C$$

c. U - substitution

$$\int e^{2x+1}\,dx$$

$u = 2x+1$

$du = 2dx$

$\frac{du}{2} = dx$

$$\int e^u\,du \cdot \frac{1}{2} = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{2x+1} + C$$

d.

$$\int (2x+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + C = \frac{(2x+1)^4}{8} + C$$

$u = 2x+1$

$du = 2dx$

$dx = \frac{du}{2}$

e.

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2x^{\frac{3}{2}}}{3}$$

f.

$$\int \frac{1}{x\sqrt{x}}\,dx = \int \frac{1}{x \cdot x^{\frac{1}{2}}}\,dx = \int x^{-\frac{3}{2}}\,dx = \frac{x^{-\frac{3}{2}+\frac{2}{2}}}{-\frac{1}{2}} + C$$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + C = -\frac{2}{\sqrt{x}} + C$$

g.

$$\int \sin 2x \, dx$$

h.

$$\int \cos(1-5x)\,dx.$$

i.

$$\int \frac{1}{\sqrt{1-4x}}\,dx$$

j.

$$\int \frac{3\sin x}{\cos^2 x}\,dx$$

k.

$$\int e^{3x}dx$$

l.

$$\int \frac{1}{3x-2}dx$$

Particular Solutions

Example 5. If $G'(x) = 3x^2$ is the derivative of G and (1,6) lies on G, find G.

Example 6. Find f(x) given that $f'(x) = 3x^2 - 4x + 1$ and $f(0) = 12$.