Random Variables and Probability Distributions

Discrete Probability Distributions

Discrete Probability Distributions

Know this

Example: Consider the probability distribution that describes the number of girls in a two-child family. Use a table, graph and function to represent this probability distribution. Let X be the random variable, number of girls in the family.

x			
P(X=x)			

Definitions

Know this

A **random variable** X represents a number, from the possible outcomes, that could occur for some random variable.

A **discrete random variable, X**, has the possible values x = 0, 1, 2, 3, 4.....
For example: the number of girls in a two-child family.

A **continuous random variable, X,** has all possible values in some interval.
For example: the volume of water in a rainwater tank during a given month.

Any random variable has a **probability distribution** associated with it.
$P(X = x_i)$ represents the probability of the event associated with x_i occurring.

Properties of a Probability Distribution

Know this

1. Each of the probabilities must lie between 0 and 1, $0 < P(X = x_i) < 1$ for all values of x_i.

2. The sum of probabilities must add to 1, that is

$$\sum_{i=1}^{n} P(X = x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_n) = 1$$

Example: Find k for the probability distribution $P(X = x) = k(x + 3)$ for $x = 0, 1, 2$.

$X = x$	0	1	2
$P(X = x)$			

[Maximum mark: 8]

A discrete random variable X has the following probability distribution.

X	0	1	2	3
$\mathrm{P}(X=x)$	0.475	$2k^2$	$\frac{k}{10}$	$6k^2$

(a) Find the value of k. [4]

(b) Write down $\mathrm{P}(X=2)$. [1]

(c) Find $\mathrm{P}(X=2 \mid X>0)$. [3]

(a) valid approach *(M1)*

eg total probability $= 1$

correct equation *(A1)*

eg $0.475 + 2k^2 + \frac{k}{10} + 6k^2 = 1,\ 8k^2 + 0.1k - 0.525 = 0$

$k = 0.25$ ***A2*** ***N3***

[4 marks]

(b) $\mathrm{P}(X = 2) = 0.025$ ***A1*** ***N1***

[1 mark]

(c) valid approach for finding $\mathrm{P}(X > 0)$ *(M1)*

eg $1 - 0.475,\ 2(0.25^2) + 0.025 + 6(0.25^2),\ 1 - \mathrm{P}(X = 0),\ 2k^2 + \frac{k}{10} + 6k^2$

correct substitution into formula for conditional probability *(A1)*

eg $\frac{0.025}{1 - 0.475},\ \frac{0.025}{0.525}$

0.0476190

$\mathrm{P}(X = 2 \mid X > 0) = \frac{1}{21}$ (exact), 0.0476 ***A1*** ***N2***

[3 marks]

Review meaning of
mean and variance

Expected Value

Know this

For a probability distribution the expectation of random variable X
or expected value of X is:

$$E(X) = \sum x_i P(X = x_i) \quad \text{or} \quad E(X) = \sum x_i p_i$$

where x_i is a particular outcome and p_i is the probability of that outcome.

E(X) is also called the mean value of X. It is a measure of central tendency
and can be thought of as a probability-weighted average or long-run average.

Note: A fair game will have expected return of zero.

1) Choose an American suburban household at random and let the random variable X be the number of cars that people in the house own. Find the number of cars owned by the average suburban American household.

X	0	1	2	3	4	5	6 or more
Probability	.08	.29	.32	.17	.08	.04	.02

2) A huge cookie jar has 60% chocolate cookies and 40% vanilla cookies. Sam chooses 3 cookies blindfolded. Let X be the number of chocolate cookies he chooses. Construct the probability distribution of X and the average number of chocolate cookies he chooses.

6) A coin is tossed three times. If heads appears on all 3 tosses, Mary will win \$16. If heads appears on 2 of the tosses, she will win \$2. The game costs \$5 to play. What is her mean expectation?

Know this

Example: Given $E(X) = 7$ and $E(X^2) = 18$ find:

a) $E(5)$

b) $E(3X)$

c) $E(X^2+2X+3)$

Variance of a Probability Distribution

Know this

If X is a discrete random variable, then its variance, σ^2, is defined as:

$$Var(X) = E\left((X-\mu)^2\right)$$

$$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 P(X = x_i)$$

To calculate Var(X) use:

$$Var(X) = E\left(X^2\right) - \left[E(X)\right]^2 \quad or \quad Var(X) = E\left(X^2\right) - \mu^2$$

Note: $Sd(X) = \sigma = \sqrt{Var(X)}$

A bag contains four gold balls and six silver balls.

(a) Two balls are drawn at random from the bag, with replacement. Let X be the number of gold balls drawn from the bag.

(i) Find $\mathrm{P}(X = 0)$.

(ii) Find $\mathrm{P}(X = 1)$.

(iii) Hence, find $\mathrm{E}(X)$. *[8 marks]*

Fourteen balls are drawn from the bag, with replacement.

(b) Find the probability that exactly five of the balls are gold. *[2 marks]*

(c) Find the probability that at most five of the balls are gold. *[2 marks]*

(d) Given that at most five of the balls are gold, find the probability that exactly five of the balls are gold. Give the answer correct to two decimal places. *[3 marks]*

(a) **METHOD 1**

(i) appropriate approach

eg $\frac{6}{10}\times\frac{6}{10},\ \frac{6}{10}\times\frac{5}{9},\ \frac{6}{10}\times\frac{5}{10}$

$P(X=0)=\frac{9}{25}=0.36$

(ii) multiplying one pair of gold and silver probabilities

eg $\frac{6}{10}\times\frac{4}{10},\ \frac{6}{10}\times\frac{4}{9},\ 0.24$

adding the product of both pairs of gold and silver probabilities

eg $\frac{6}{10}\times\frac{4}{10}\times 2,\ \frac{6}{10}\times\frac{4}{9}+\frac{4}{10}\times\frac{6}{9}$

$P(X=1)=\frac{12}{25}=0.48$

(iii) $P(X=2)=0.16$ (seen anywhere)

correct substitution into formula for $E(X)$

eg $0\times 0.36+1\times 0.48+2\times 0.16,\ 0.48+0.32$

$E(X)=\frac{4}{5}=0.8$

METHOD 2

(i) evidence of recognizing binomial (may be seen in part (ii))

eg $X\sim B(2,0.6),\ \binom{2}{0}(0.4)^2(0.6)^0$

correct probability for use in binomial

eg $p=0.4,\ X\sim B(2,0.4),\ {}^2C_0(0.4)^0(0.6)^2$

$P(X=0)=\frac{9}{25}=0.36$

(ii) correct set up

eg ${}_2C_1(0.4)^1(0.6)^1$

$P(X=1)=\frac{12}{25}=0.48$

(iii) attempt to substitute into np

eg 2×0.6

correct substitution into np

eg 2×0.4

$E(X)=\frac{4}{5}=0.8$

Let Y be the number of gold balls drawn from the bag in parts (b), (c), and (d).

(b) evidence of recognizing binomial (seen anywhere)

eg ${}_{14}C_5(0.4)^5(0.6)^9,\ B(14,0.4)$

$P(Y=5)=0.207$

(c) recognize need to find $P(Y\le 5)$

$P(Y\le 5)=0.486$

(d) recognizing conditional probability

eg $P(A\mid B),\ P(Y=5\mid Y\le 5),\ \frac{P(Y=5)}{P(Y\le 5)},\ \frac{0.207}{0.486}$

$P(Y=5\mid Y\le 5)=0.42522518$

$P(Y=5\mid Y\le 5)=0.43$ (to 2 dp)

Know this

Example: Find E(X) and Var(X) and standard deviation for the probability distribution X as defined in the table.

X	1	2	3	4
P(X=X)	0.1	0.3	0.4	0.2

Properties of Variance

Know this

1. $Var(k) = 0$
2. $Var(aX) = a^2 Var(X)$
3. $Var(aX + b) = a^2 Var(X)$

Example: X is distributed with mean 8.1 and standard deviation 2.37. If $Y = 4X - 7$, find the mean and standard deviation of the Y-distribution.

Solution

$E(X) = 8.1$, $E(4X - 7) = 4(8.1) - 7 = 25.4$

Std dev X = 2.37 so Var(X) = 5.6169

$\mathrm{Var}(4X - 7) = 4^2 \times 5.6169 = 89.8704$

Std dev of Y = $\sqrt{89.8704} = 9.48$

Example

Know this

The discrete random variable X has the following probability distribution, where $0 < \theta < \frac{1}{3}$.

x	1	2	3
$\mathrm{P}(X = x)$	θ	2θ	$1 - 3\theta$

1. Determine $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 6\theta - 16\theta^2$. *[4 marks]*

Solution

Know this

Markscheme

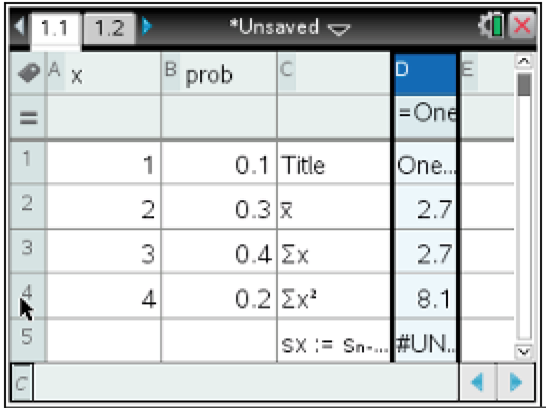
$\mathrm{E}(X) = 1 \times \theta + 2 \times 2\theta + 3(1 - 3\theta) = 3 - 4\theta$ ***M1A1***

$\mathrm{Var}(X) = 1 \times \theta + 4 \times 2\theta + 9(1 - 3\theta) - (3 - 4\theta)^2$ ***M1A1***

$= 6\theta - 16\theta^2$ ***AG***

[4 marks]

TI-Nspire



1.1
1.2
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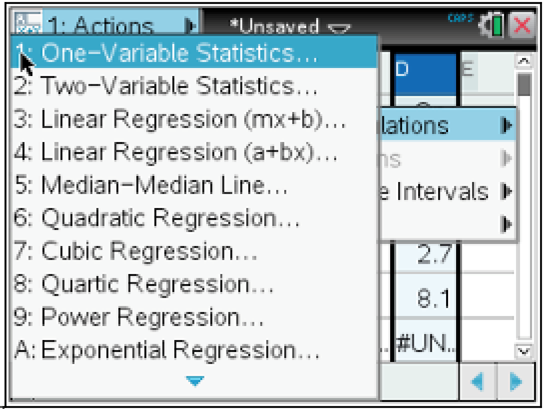
x
prob
=One
Title
One...
$\bar{x}$
Σx
Σx²
sx := sn-...
#UN..



1: Actions
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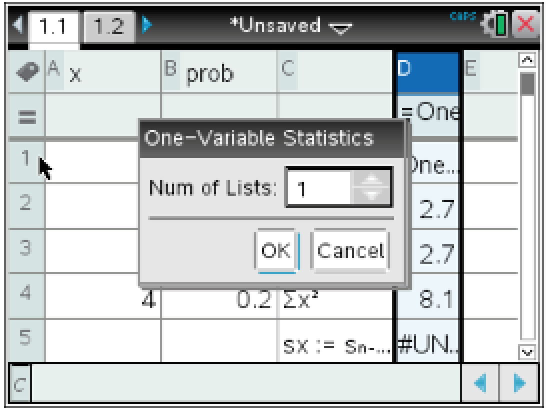
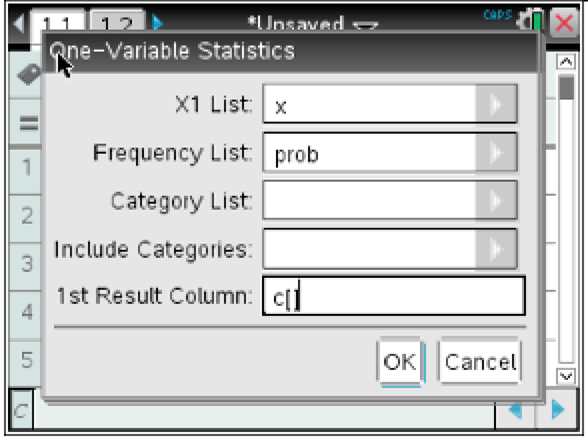
1: One-Variable Statistics...
2: Two-Variable Statistics...
3: Linear Regression (mx+b)...
4: Linear Regression (a+bx)...
5: Median-Median Line...
6: Quadratic Regression...
7: Cubic Regression...
8: Quartic Regression...
9: Power Regression...
A: Exponential Regression...



1.1
1.2
*Unsaved
One-Variable Statistics
Num of Lists: 1
OK
Cancel



One-Variable Statistics
X1 List: x
Frequency List: prob
Category List:
Include Categories:
1st Result Column: c[]
OK
Cancel

Thanks

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