REPRESENTING SYSTEMS:

TRANSITION MATRICES

MARKOV CHAINS and STEADY STATES

T_n

TRANSITION MATRIX. A matrix that represents the **probability of moving** from the current state of a system to the next state, or travelling from one vertex in a graph to any other vertex in the graph.
STATE MATRIX. A matrix S_n shows the state of a system at time n. The state matrix may represent either: • the probability or proportion of the population that are in each state, or • the actual number from the population that are in a given state. The **initial state** matrix is S_0.

A town has two schools, School A and School B. At the start of each school year, 10% of students move from School A to School B and 5% move from B to A. At the start of this school year, School A has 280 students and School B has 320 students.

a. State the initial state of the system

$$S_0 = \begin{bmatrix} 280 \\ 320 \end{bmatrix}$$

b. Let $S_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ represent the number of the students in each school at the start of the next school year, where a is the number of students in School A and b the number in School B. Show that $S_1 = \begin{pmatrix} 268 \\ 332 \end{pmatrix}$.

$$S_n = T^n S_0$$

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix} \end{array}$$

$$S_1 = T^1 S_0 = \begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix} \begin{bmatrix} 280 \\ 320 \end{bmatrix} = \begin{matrix} 0.90 \times 280 + 0.05 \times 320 \\ 0.10 \times 280 + 0.95 \times 320 \end{matrix} = \begin{bmatrix} 268 \\ 332 \end{bmatrix}$$

c. Let $T = \begin{pmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{pmatrix}$. Show that S_2, which represents the number of students in each school at the start of the school year two years from now, is . $S_2 = T^2 \begin{pmatrix} 280 \\ 320 \end{pmatrix}$

$S_2 = TS_1$ $\quad\quad S_n = T^n S_0$

$= TTS_0$

$S_2 = T^2 S_0$

$S_2 = T^2 \begin{pmatrix} 280 \\ 320 \end{pmatrix}$

$Ax - \lambda x = 0$

$Ax = \lambda x$

$\begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d. Write an equation for the S_n in terms of matrix multiplication of the diagonalised matrix for T.

$S_n = T^n S_0$

$T^n = PD^nP^{-1}$ ✓

$T = \begin{bmatrix} 0.90 & 0.05 \\ 0.10 & 0.95 \end{bmatrix}$

$\begin{vmatrix} 0.90-\lambda & 0.05 \\ 0.10 & 0.95-\lambda \end{vmatrix}$

$(0.90-\lambda)(0.95-\lambda) - (0.10)(0.05)$

$\lambda^2 - 1.85\lambda + 0.85 = 0$

$\lambda = 0.85 \quad \lambda = 1$

$D = \begin{bmatrix} 0.85 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

eigenvector of λ

$S_n = T^n S_0$

$= PD^nP^{-1}S_0$

$S_n = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0.85 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 280 \\ 320 \end{bmatrix}$

$\lambda_1 = 0.85 \qquad \lambda_2 = 1$

$$\begin{bmatrix} 0.90 - \lambda & 0.05 \\ 0.10 & 0.95 - \lambda \end{bmatrix}$$

$\lambda = 0.85$

$$\begin{bmatrix} 0.05 & 0.05 \\ 0.10 & 0.10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Mat Rref

Rref(Mat A)

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$\lambda = 0.85$ if $x_2 = 1$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} -0.10 & 0.05 \\ 0.10 & -0.05 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 - \frac{1}{2}x_2 = 0$

$2x_1 = x_2$

if $x_1 = 1$

$x_2 = 2$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

e. Hence, find the number of students in each school as n approaches infinity. Explain the significance of this result.

$n \to \infty, \quad D^{n} \to \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$S_n = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 280 \\ 320 \end{bmatrix} = \begin{bmatrix} 200 \\ 400 \end{bmatrix}$$

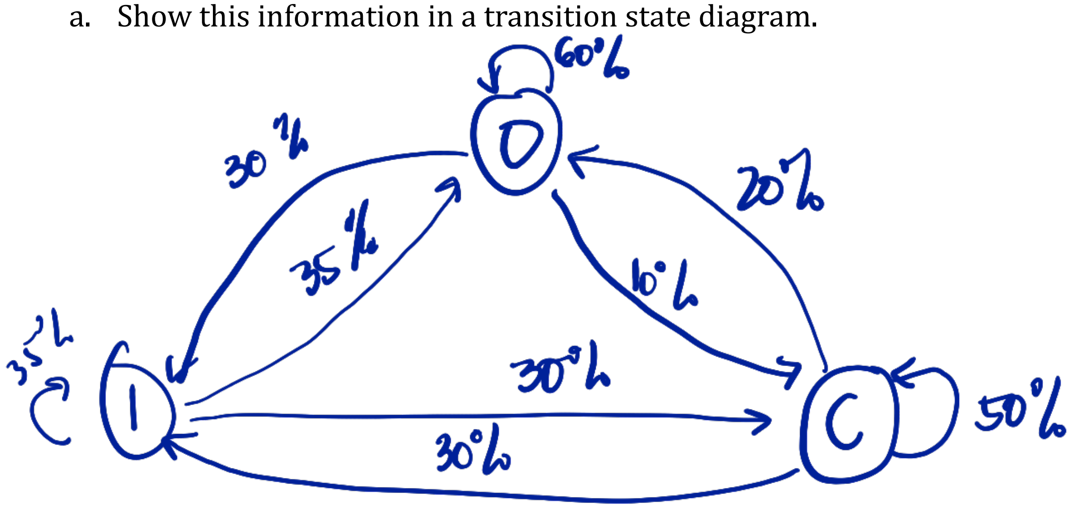
Shimano cycle company hires bicycles in a city through a mobile app. Users can unlock a bicycle with their smartphone, ride it to their destination and lock the bicycle.

Shimano divides the city into three zones: Inner (I), Outer (O) and Central Business District (C).

By tracking their bicycles with a GPS over several weeks, the company finds that at the end of each day:

- 50% of the bicycles rented in Zone C remain in Zone C, 30% were left in Zone I, and 20% were left in Zone O.
- 60% of the bicycles rented in Zone O remain in Zone O, 30% were left in Zone I, and 10% were left in Zone C
- 35% of the bikes rented in Zone I remain in Zone I, 35% were left in Zone O, and 30% were left in Zone C.

a. Show this information in a transition state diagram.



b. Show this information in a transition matrix.

$$\begin{array}{cc|ccc} & & & \text{FROM} & \\ & & C & I & O \\ \hline \text{TO} & C & 0.50 & 0.30 & 0.10 \\ & I & 0.30 & 0.35 & 0.30 \\ & O & 0.20 & 0.35 & 0.60 \end{array}$$

$$T = \begin{bmatrix} 0.50 & 0.30 & 0.10 \\ 0.30 & 0.35 & 0.30 \\ 0.20 & 0.35 & 0.60 \end{bmatrix}$$

c. Determine the probability that after three days, a bicycle that started in C is now in O.

$$T^3 = \begin{bmatrix} 0.307 & 0.280 & 0.243 \\ 0.316 & 0.316 & 0.316 \\ 0.377 & 0.405 & 0.441 \end{bmatrix}$$

MARKOV CHAIN. A stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.
STEADY STATE. The steady state vector of a square transition matrix, **T**, is the solution matrix, **q**, that satisfies the equation **Tq = q**.

A large company has 964 employees. The number of employees that eat at the cafeteria each day can be predicted using the transition matrix shown below:

$$
\begin{array}{lcc}
 & & \text{Eats at cafeteria today?} \\
 & & \begin{array}{cc}\text{yes} & \text{no}\end{array} \\
\text{Eats at cafeteria tomorrow?} & \begin{array}{c}\text{yes} \\ \text{no}\end{array} & \begin{bmatrix} 0.85 & 0.10 \\ 0.15 & 0.90 \end{bmatrix}
\end{array}
$$

Show that the steady state vector for this system is

$$q = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$q = \begin{bmatrix} 385.6 \\ 578.4 \end{bmatrix}$$

$$\begin{bmatrix} 0.85 & 0.10 \\ 0.15 & 0.90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$0.85x + 0.10y = x \longrightarrow -0.15x + 0.10y = 0 \quad (2)$$

$$0.15x + 0.90y = y \longrightarrow 0.15x - 0.10y = 0 \quad (3)$$

$$x + y = 964 \quad (1)$$

$$\boxed{\text{Rref}}$$

$$\begin{bmatrix} 1 & 1 & 964 \\ -0.15 & 0.10 & 0 \\ 0.15 & -0.10 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 0 & 385.6 \\ 0 & 1 & 578.4 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \longrightarrow \begin{array}{l} x = 385.6 \\ y = 578.4 \end{array}$$

Each year 3% of the population living in a certain city will move to the suburbs and 6% of the population living in the suburbs will move into the city. At present 975000 people live in the city itself, and 525000 live in the suburbs. Assuming that the total population does not change, find the distribution of the population:

a. One year from now
b. 10 years from now

c. 50?

$$S_n = T^n S_0$$

c. what is the steady state

$$Tq = q$$