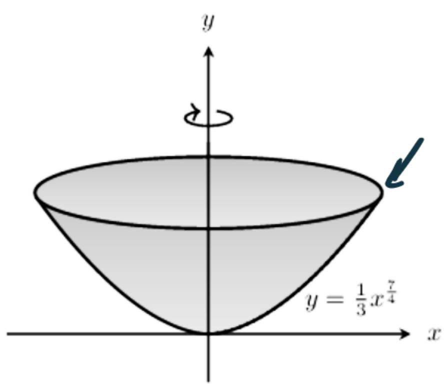
Volumes of Revolution

5.12	Area of region enclosed by a curve and x or y-axes	$A = \int_a^b \lvert y \rvert \, \mathrm{d}x$ or $A = \int_a^b \lvert x \rvert \, \mathrm{d}y$
	Volume of revolution about x or y-axes	$V = \int_a^b \pi y^2 \, \mathrm{d}x$ or $V = \int_a^b \pi x^2 \, \mathrm{d}y$

[Maximum mark: 5]

The shape of a bowl is given by revolving the curve $y = \frac{1}{3}x^{\frac{7}{4}}$, $0 \leq x \leq 6$ through 360° about the y-axis. The units are in centimetres.



1. Find the height of the bowl. 2

The bowl is completely filled with water.

2. Find the volume of water in the bowl. 3

1. $y = \frac{1}{3}(6)^{\frac{7}{4}}$ M1

$y = 7.68$ cm A1

2. $y = \frac{1}{3}x^{\frac{7}{4}}$

$3y = x^{\frac{7}{4}}$

$x = (3y)^{\frac{4}{7}}$ A1

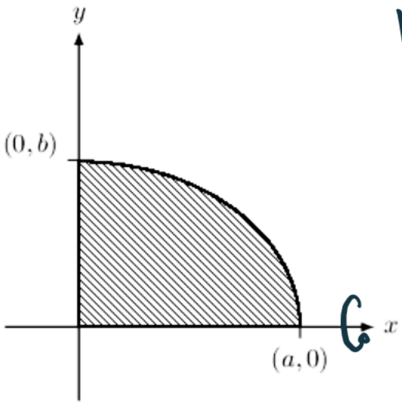
$V = \pi \int x^2 \, dy$

$= \pi \int_0^{7.68} \left[(3y)^{\frac{4}{7}}\right]^2 dy$ A1

$= 406 \text{ cm}^3$ A1

[Maximum mark: 4]

The following diagram shows the graph of the function $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$



$$V = \int_a^b \pi y^2 \, dx$$

Find an expression, in terms of a and b, for the volume of the solid formed by rotating the curve 360° about the x-axis.

$$V = \pi \int_0^a \left[\frac{b}{a}\left(\sqrt{a^2-x^2}\right)\right]^2 dx \quad \text{A1}$$

$$= \pi \int_0^a \frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$\frac{\pi b^2}{a^2}\left(\frac{3a^2x - x^3}{3}\right)\Big|_0^a$$

$$= \frac{\pi b^2}{a^2}\int_0^a \left(a^2 - x^2\right) dx$$

$$\frac{\pi b^2}{a^2}\left[\left(\frac{3a^2(a) - a^3}{3}\right) - \left(\frac{3a^2(0) - (0)^3}{3}\right)\right]$$

$$= \frac{\pi b^2}{a^2}\left(a^2\int_0^a dx - \int_0^a x^2 \, dx\right)$$

$$= \frac{\pi b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right)\Big|_0^a \quad \text{A1}$$

$$= \frac{\pi b^2}{a^2}\left[\left(a^2(a) - \frac{(a)^3}{3}\right) - \cancel{\left(a^2(0) - \frac{0^3}{3}\right)}\right] \quad \text{M1}$$

$$= \frac{\pi b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{a^3\pi b^2}{a^2} - \frac{\pi b^2 a^3}{3a^2} \quad \text{A1}$$

$$= ab^2\pi - \frac{ab^2\pi}{3} = \frac{3ab^2\pi - ab^2\pi}{3}$$

$$= \frac{2ab^2\pi}{3}$$

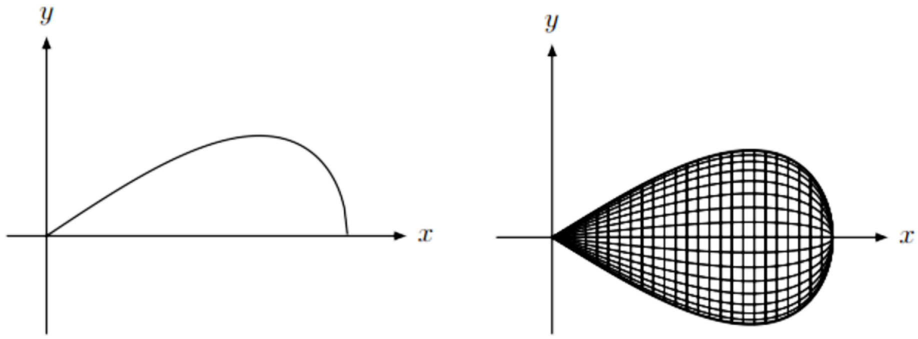
[Maximum mark: 6]

A piece of jewellery is being designed for an auction. Its shape is based on the graph of

$$y = \frac{x}{50}\sqrt{625 - x^2} \quad \text{for } 0 \leq x \leq 25$$

which is revolved 360° about the x-axis, where x and y are measured in mm.

=



1. Find the area bounded by the graph and the x-axis. 104 **2**
2. Hence or otherwise, find the volume of the jewel. 1640 mm^3 **2**
3. If 1cm^3 of silver weighs 10.5 grams, and 1 gram of silver costs \$1.50, find the cost of the silver used for the jewel to the nearest dollar. \$26 **2**

1. $$A = \int_0^{25} \frac{x}{50}\left(\sqrt{625 - x^2}\right) dx = 104 \text{ mm}^2$$

2 $$V = \pi \int_0^{25} \left(\frac{x}{50}\left(\sqrt{625 - x^2}\right)\right)^2 dx = 1640 \text{ mm}$$

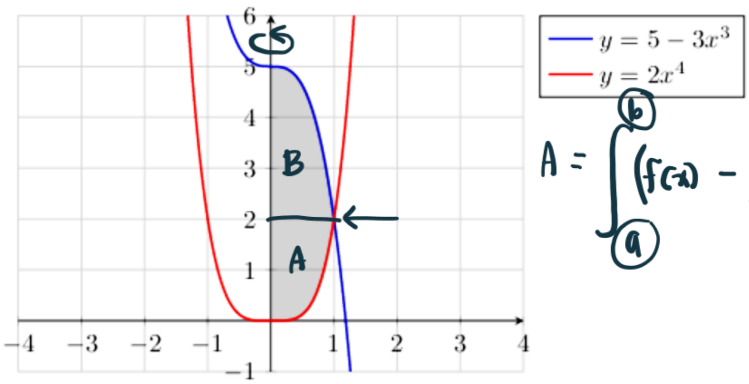
3. $$\text{Cost} = 1640\ \cancel{\text{mm}^3}\left(\frac{1\ \cancel{\text{cm}^3}}{1000\ \cancel{\text{mm}^3}}\right)\left(\frac{10.5\ \cancel{g}}{1\ \cancel{\text{cm}^3}}\right)\left(\frac{\$1.50}{1\ \cancel{g}}\right)$$

$$= \$26$$

[Maximum mark: 6]

$f(x)$ $g(x)$

The diagram below shows the graphs of functions $y = 5 - 3x^3$ and $y = 2x^4$.



$$A = \int_a^b (f(x) - g(x))\, dx$$

1. Find the volume resulting from a rotation of the region shown in the diagram through 2π about the x-axis. **2**
2. Find the volume resulting from a rotation of the region shown in the diagram through 2π about the y-axis. **4**

$y = y$

$5 - 3x^3 = 2x^4$

$2x^4 + 3x^3 - 5 = 0$

$(1, 2)$

1. $V = \pi \int_0^1 (5 - 3x^3)^2\, dx - \pi \int_0^1 (2x^4)^2\, dx$ A1

$= 57.6$ unit^3 A1

2. $y = 5 - 3x^3$

$y + 5 = -3x^3$

$-\frac{y-5}{3} = x^3$

$\sqrt[3]{\frac{5-y}{3}} = x$

$y = 2x^4$

$\sqrt[4]{\frac{y}{2}} = x$

} A1

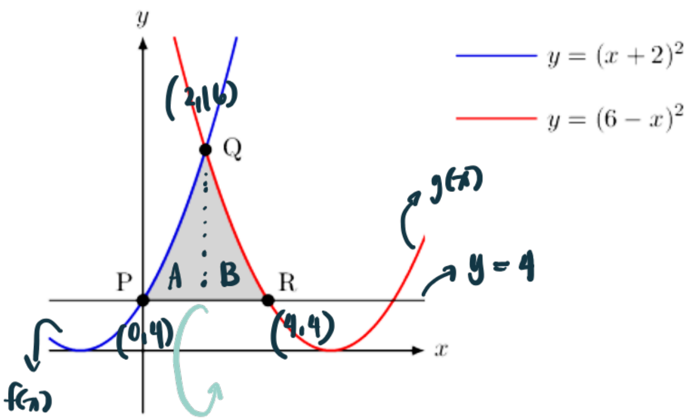
$V = \pi \int_0^2 \left(\sqrt[4]{\frac{y}{2}}\right)^2 dy + \pi \int_2^5 \left(\sqrt[3]{\frac{5-y}{3}}\right)^2 dy$ M1 A1

$= 4.19 + 5.65$

$= 9.84$ units^3 A1

[Maximum mark: 7]

The diagram below shows part of the graphs of $y = (x+2)^2$ and $y = (6-x)^2$.



1. P: $y = (0+2)^2$
$y = 4$
$(0,4)$ A1

R: $4 = (6-x)^2$
$2 = 6-x$
$-4 = -x$
$x = 4$
$(4,4)$ A1

Q: $(x+2)^2 = (6-x)^2$
$x+2 = 6-x$
$2x = 4$
$x = 2$
$y = (2+2)^2$
$y = 16$
$(2,16)$ A1

3

1. Find the coordinates of the points P, Q and R.

The shaded region is rotated through 360° about the x-axis. **4**

2. Calculate the volume of the solid obtained. $1045.522035 \approx 1050$

2. $V_A = \pi\int_a^b (f(x)-4)^2\,dx \quad V_B = \pi\int_b^c (g(x)-4)^2\,dx$

$V_T = V_A + V_B$

$= \pi\int_0^2 ((x+2)^2)^2 - 4^2\,dx + \pi\int_2^4 ((6-x)^2)^2 - 4^2\,dx$ A1 M1 A1

$= 1045.522 \approx 1050 \text{ units}^3$ A1

$V_T = \left[\pi\int_0^2 ((x+2)^2)^2\,dx + \int_2^4 ((6-x)^2)^2\,dx\right] - \int_0^4 4^2\pi\,dx$

Additional Questions

1.

Determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt[3]{x}$, $x = 8$ and the x-axis about the x-axis.

2.

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$, and the x-axis about the x-axis.

3.

Determine the volume of the solid obtained by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant about the y-axis.

4.

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

5.

Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x - 1$ about the line $x = -1$.