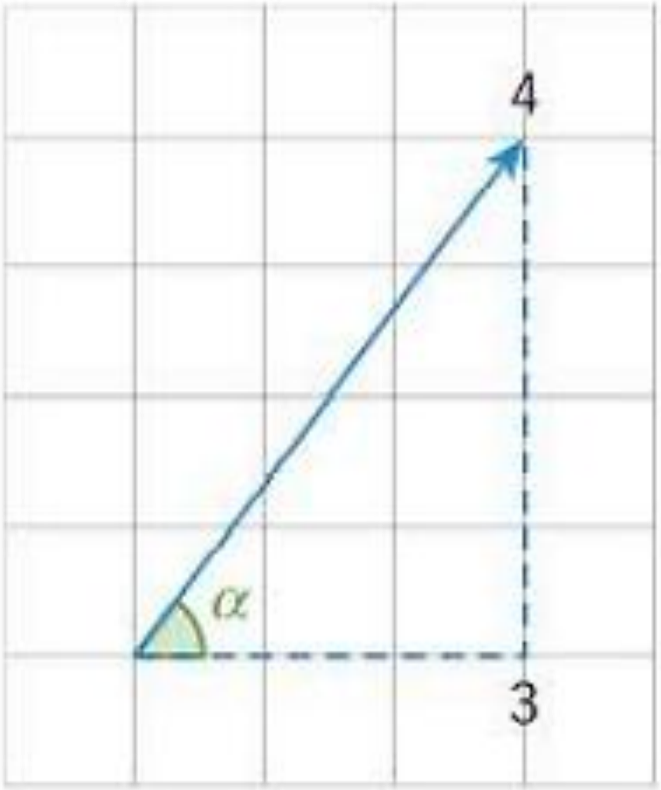
THE MAGNITUDE AND DIRECTION OF A VECTOR

Week 35



The magnitude of a vector $\boldsymbol{v}$ is its length. It is written as $|\boldsymbol{v}|$ and can be found using Pythagoras' theorem.

The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{vmatrix} 3 \\ 4 \end{vmatrix} = \sqrt{3^2 + 4^2} = 5$

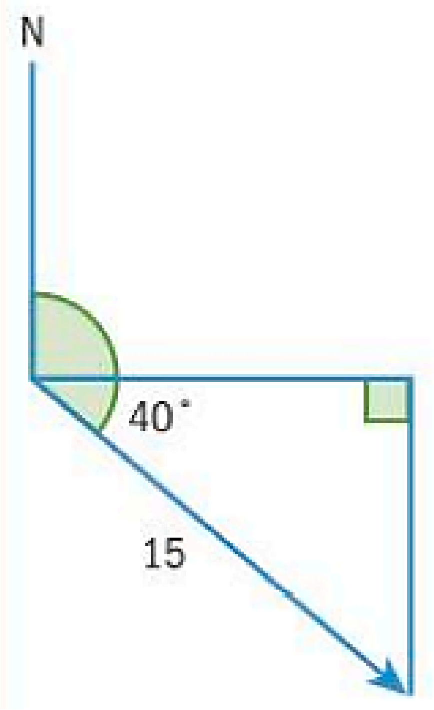
The direction of a vector is normally given as an angle. Within a Cartesian coordinate system the angle is normally measured anti-clockwise from the positive x-axis.

The direction of the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is the angle α where $\tan \alpha = \frac{4}{3}$ hence $\alpha = 53.1°$.

DIRECTION OF A VECTOR

Quadrant in which (x, y) lies	θ (in degrees)
1	α
2	$180° - \alpha$
3	$180° + \alpha$
4	$360° - \alpha$

Write the following displacement as a column vector and in i, j form: 15m on a bearing of $130°$.



Vector is $\begin{pmatrix} 15 \cos 40° \\ -15 \sin 40° \end{pmatrix}$

$= \begin{pmatrix} 11.5 \\ -9.64 \end{pmatrix}$ m or $11.5\boldsymbol{i} - 9.64\boldsymbol{j}$

In order to find the entries for the column vector first create a right-angled triangle and then use trigonometry.

From the direction of the vector you will be able to see which entries should be positive and which negative.

1 For the resultant of each of the vector sums below, find the

- **i** magnitude
- **ii** direction

a $\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}$

b $(5\boldsymbol{i}+2\boldsymbol{j})+(-6\boldsymbol{i}-4\boldsymbol{j})$

c $2\begin{pmatrix}3\\2\end{pmatrix}-3\begin{pmatrix}-4\\-1\end{pmatrix}$

d $5(\boldsymbol{i}+2\boldsymbol{j})+3(\boldsymbol{i}-3\boldsymbol{j})$

2 The magnitude of a vector $\begin{pmatrix}a\\b\end{pmatrix}$ can be written as $\begin{vmatrix}a\\b\end{vmatrix}$.

a Verify that $\begin{vmatrix}48\\20\end{vmatrix}$ is equal to $4\begin{vmatrix}12\\5\end{vmatrix}$.

b By first taking out a factor and without using a GDC, find the magnitude of

i $\begin{pmatrix}18\\24\end{pmatrix}$ **ii** $\begin{pmatrix}-30\\40\end{pmatrix}$ **iii** $\begin{pmatrix}28\\-21\end{pmatrix}$

3 A designer needs to construct a line segment of a given length in a given direction. His software requires him to enter the line segment as a single column vector.

Find the column vector he needs to input in the following situations, using the fact that a vector which is in the same direction as a vector $\boldsymbol{u}$ can be written as $k\boldsymbol{u}$. $k > 0$

a A vector that is in the same direction as $\begin{pmatrix}3\\4\end{pmatrix}$ but with a magnitude of 8.

b A single vector which is equivalent to the resultant of a vector in the same direction as $\begin{pmatrix}0\\1\end{pmatrix}$ followed by the vector $\begin{pmatrix}5\\0\end{pmatrix}$ and has magnitude $\sqrt{74}$.

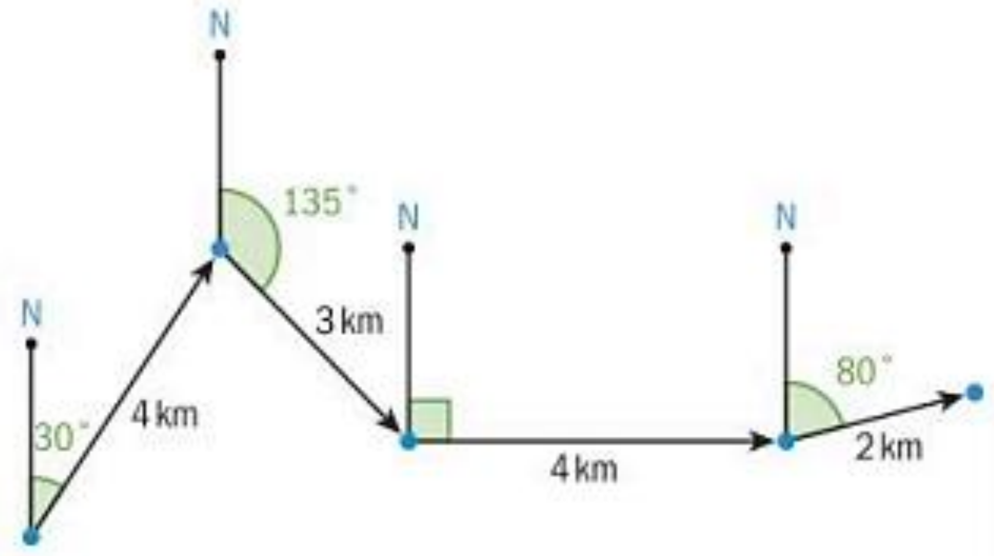
c A vector which is equivalent to the resultant of a vector in the same direction as $\begin{pmatrix}1\\1\end{pmatrix}$, followed by the vector $\begin{pmatrix}1\\-5\end{pmatrix}$ and has magnitude $\sqrt{50}$.

4 A man walking in a large field walks 200 m north-east and 175 m west.

   a Write each of the displacements as a column vector.

   b Hence find his final distance from his starting point.

5 A boat sails 4 km on a bearing of 030°, followed 3 km south-east, then 4 km due east and 2 km on a bearing of 080° as shown on the right. Determine its final distance from the starting point. Find also the bearing it would have to travel on to return directly to the starting point.



1. Given | Resultant | Magnitude | Direction

a.) $\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}=\boxed{\begin{pmatrix}4\\4\end{pmatrix}}$

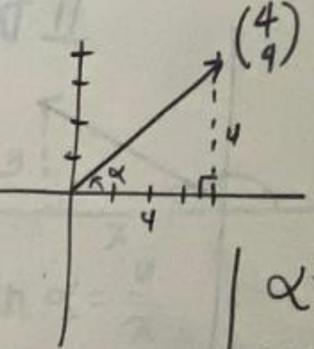
$\left|\begin{matrix}4\\4\end{matrix}\right|=\sqrt{4^2+4^2}=\sqrt{16+16}=\boxed{4\sqrt{2}}$

Direction

$\tan\alpha=\frac{4}{4}$

$\alpha=\tan^{-1}\left(\frac{4}{4}\right)$

$\boxed{\alpha=45°}$

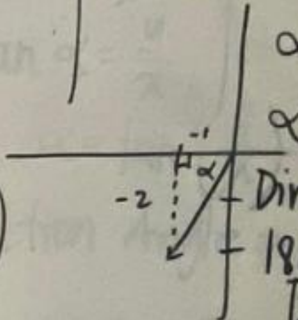


b.) $(5i+2j)+(-6i-4j)=\boxed{-i-2j}$

$\left|\begin{matrix}-1\\-2\end{matrix}\right|=\sqrt{(-1)^2+(-2)^2}=\boxed{\sqrt{5}}$

$\alpha=\tan^{-1}\left(\frac{-2}{-1}\right)$

$\alpha\approx63.4°$



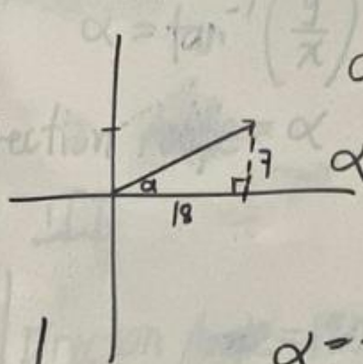
Direction Angle $=180°+63.4°\approx\boxed{243.4°}$

c.) $2\begin{pmatrix}3\\2\end{pmatrix}-3\begin{pmatrix}-4\\-1\end{pmatrix}=\begin{pmatrix}6\\4\end{pmatrix}+\begin{pmatrix}12\\3\end{pmatrix}=\boxed{\begin{pmatrix}18\\7\end{pmatrix}}$

$\left|\begin{matrix}18\\7\end{matrix}\right|=\sqrt{18^2+7^2}=\boxed{\sqrt{373}}$

$\alpha=\tan^{-1}\left(\frac{7}{18}\right)$

$\alpha\approx21.3°$



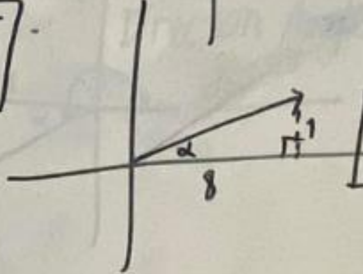
d.) $5(i+2j)+3(i-3j)$

$=(5i+10j)+(3i-9j)=\boxed{8i+j}$

$\left|\begin{matrix}8\\1\end{matrix}\right|=\sqrt{8^2+1^2}=\boxed{\sqrt{65}}$

$\alpha=\tan^{-1}\left(\frac{1}{8}\right)$

$\boxed{\alpha\approx7.13°}$



2.) a.)

$$\left|\begin{matrix}48\\20\end{matrix}\right| = \sqrt{48^2+20^2}$$

$$= 52$$

$$4\left|\begin{matrix}12\\5\end{matrix}\right| = 4\sqrt{12^2+5^2}$$

$$= 4(13)$$

$$= 52$$

$$\left|\begin{matrix}48\\20\end{matrix}\right| = \sqrt{48^2+20^2}$$

$$= \sqrt{2704}$$

$$= \sqrt{(16)(169)}$$

$$= 4\sqrt{169}$$

$$= 4\sqrt{12^2+5^2}$$

$$= 4\left|\begin{matrix}12\\5\end{matrix}\right|$$

3.) a.) $k\begin{pmatrix}3\\4\end{pmatrix} = k\left|\begin{matrix}3\\4\end{matrix}\right| = 8$

$= k\sqrt{3^2+4^2} = 8$

$= k\sqrt{25} = 8$

$\sqrt{25} = \pm 5$

$k(5) = 8$ and $k(-5) = 8$

$\frac{5k}{5} = \frac{8}{5}$ $\qquad$ $\frac{-5k}{-5} = \frac{8}{-5}$

$\boxed{k = \frac{8}{5}}$ and $k = -\frac{8}{5}$

So, a vector that is in the same direction as $\begin{pmatrix}3\\4\end{pmatrix}$ is: $k\begin{pmatrix}3\\4\end{pmatrix} = \frac{8}{5}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}\frac{24}{5}\\ \frac{32}{5}\end{pmatrix} = \boxed{\frac{1}{5}\begin{pmatrix}24\\32\end{pmatrix}}$

b) $k\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\k\end{pmatrix} + \begin{pmatrix}5\\0\end{pmatrix} = \begin{pmatrix}5\\k\end{pmatrix} = \sqrt{74}$

$= \left|\begin{matrix}5\\k\end{matrix}\right| = \sqrt{74}$

$= \left(\sqrt{5^2+k^2}\right)^2 = \left(\sqrt{74}\right)^2$

$= 5^2 + k^2 = 74$

$= 25 + k^2 = 74$

$k^2 = 74 - 25$

$k^2 = 49$

$\boxed{k = 7}$

So, $\begin{pmatrix}5\\k\end{pmatrix} = \boxed{\begin{pmatrix}5\\7\end{pmatrix}}$

c) $k\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}k\\k\end{pmatrix}+\begin{pmatrix}1\\-5\end{pmatrix}=\begin{pmatrix}k+1\\k-5\end{pmatrix}\rightarrow\left|\begin{matrix}k+1\\k-5\end{matrix}\right|=\sqrt{50}$

$$\left(\sqrt{(k+1)^2+(k-5)^2}\right)^2=\left(\sqrt{50}\right)^2$$

$$(k+1)^2+(k-5)^2=50$$

$$k^2+2k+1+k^2-10k+25=50$$

$$2k^2-8k+26=50$$

$$\frac{2k^2-8k-24}{2}=\frac{0}{2}$$

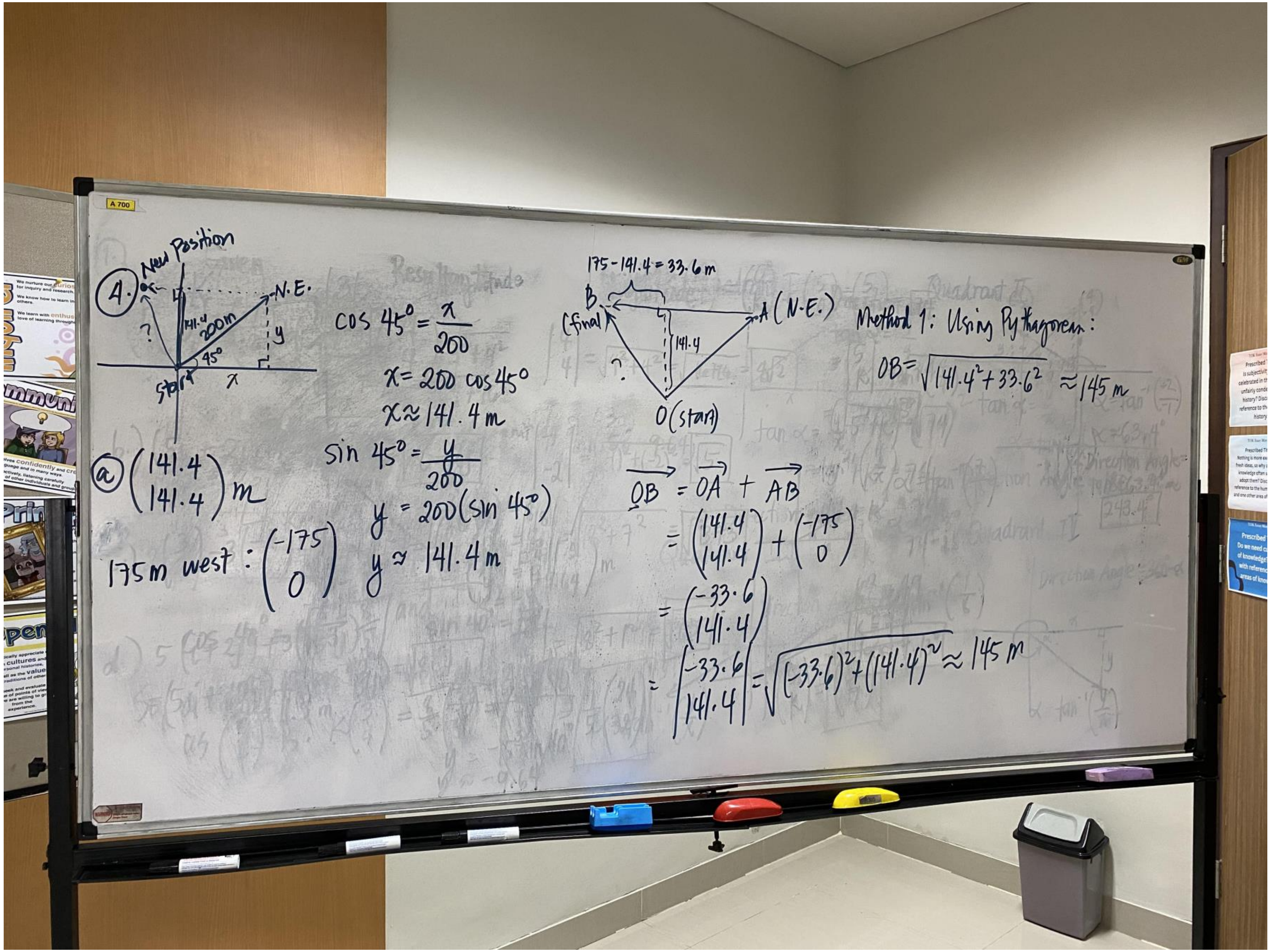
$$k^2-4k-12=0$$

$$(k-6)(k+2)=0$$

$$k-6=0 \quad k+2=0$$

$$\boxed{k=6} \quad k=-2$$

So, $\begin{pmatrix}k+1\\k-5\end{pmatrix}=\begin{pmatrix}6+1\\6-5\end{pmatrix}=\boxed{\begin{pmatrix}7\\1\end{pmatrix}}$


A.
New Position
N.E.
141.4
200m
45°
y
x
start
?
cos 45° = x/200
x = 200 cos 45°
x ≈ 141.4 m
sin 45° = y/200
y = 200(sin 45°)
y ≈ 141.4 m
a (141.4, 141.4) m
175m west : (-175, 0)
175 − 141.4 = 33.6 m
B (final)
A (N.E.)
141.4
O (start)
OB = OA + AB
= (141.4, 141.4) + (-175, 0)
= (-33.6, 141.4)
= |(-33.6, 141.4)| = √((-33.6)² + (141.4)²) ≈ 145 m
Method 1: Using Pythagorean:
OB = √(141.4² + 33.6²) ≈ 145 m