Unit 3:Geometry and Trigonometry

VECTORS

AIHL 3.11 - 3.13

Vector Equations of a line

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

Example 1.

(a) Find a vector equation, in the form $\vec{r} = \vec{a} + \lambda \vec{b}$, of the line passing through the points $A(2,-3)$ and $B(-5,2)$.

(b) Does the point $C(-12,7)$ lie on the line AB? Explain.

a. * one possible

$$r = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

b. $\begin{pmatrix} -12 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 5 \end{pmatrix}$

$-12 = 2 - 7\lambda$ $\quad$ $7 = -3 + 5\lambda$

$-14 = -7\lambda$ $\quad$ $10 = 5\lambda$

$2 = \lambda$ $\quad$ $2 = \lambda$

Since $\lambda = 2$ satisfies both equation, C lies on the line AB

Example 2.

A line passes through the point $(-1, 4, 0)$ and is parallel to the vector $2\mathbf{i} - 6\mathbf{j} + \mathbf{k}$. The point P with coordinates $(2, a, b)$ lies on the line. Find the value of a and the value of b.

$$\vec{a} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix}$$

$2 = -1 + 2\lambda$

$3 = 2\lambda$

$\frac{3}{2} = \lambda$

$a = 4 - 6\lambda$

$a = 4 - 6\left(\frac{3}{2}\right)$

$a = 4 - 9$

$a = -5$

$b = 0 + \lambda$

$b = \frac{3}{2}$

Example 3.

A line passes through the point $(-3, 5)$ and its direction is perpendicular to the vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Find the equation of the line in the form $ax + by = c$ where a, b and c are integers to be determined.

$\vec{a} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ $\quad V \cdot W = 0$ so that $V \perp W$ $\quad$ so $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$2x - y = 0$$

$$y = 2x$$

Use parametric equation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} x = -3 + \lambda \\ y = 5 + 2\lambda \end{matrix}$$

Eliminate λ

$x + 3 = \lambda$ $\quad$ so $y = 5 + 2(x+3)$

$$y = 5 + 2x + 6$$

$$y = 11 + 2x$$

$$2x - y = -11$$

Example 4.

Consider the two lines L_1 and L_2 given as follows:

$$L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ -7 \end{pmatrix} \qquad L_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$$

(a) P is the point on L_1 when $\lambda = 1$. Find the position vector of P.

(b) Show that P is also on L_2.

(c) A third line, L_3, has a direction vector of $\begin{pmatrix} a \\ 3 \\ c \end{pmatrix}$. If L_1 and L_3 are parallel, find the value of a and the value of c.

a. When $\lambda = 1$

L_1: $r = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 5 \\ -1 \\ -7 \end{pmatrix}$ $\quad r = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ $\quad \therefore P = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

b. L_2: $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$

$2 = -1 - 6\mu$ $\qquad 3 = 4 + 2\mu$ $\qquad -4 = -6 - 4\left(-\frac{1}{2}\right)$

$3 = -6\mu$ $\qquad 3 = 4 + 2\left(-\frac{1}{2}\right)$ $\qquad -4 = -6 + 2$

$-\frac{1}{2} = \mu$ $\qquad 3 = 3$ $\qquad -4 = -4$

$\mu = -\frac{1}{2}$ satisfies all equations

c. L_1 and L_3 must be scalar multiples

Direction L_1: $\begin{pmatrix} 5 \\ -1 \\ -7 \end{pmatrix}$ $\qquad \begin{pmatrix} a \\ 3 \\ c \end{pmatrix} = k \begin{pmatrix} 5 \\ -1 \\ -7 \end{pmatrix}$

Direction L_3: $\begin{pmatrix} a \\ 3 \\ c \end{pmatrix}$

$a = 5k$

$3 = -k \rightarrow -3 = k$

$c = -7k$

so $a = 5(-3)$ $\qquad c = -7(-3)$

$a = -15$ $\qquad c = 21$

Example 5.

The two lines $\vec{r}_1 = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ and $\vec{r}_2 = \begin{pmatrix} 3 \\ -10 \\ -6 \end{pmatrix} + \overset{\mu}{\lambda} \begin{pmatrix} 1 \\ -5 \\ -4 \end{pmatrix}$ intersect at point A. Find the coordinates of A.

let $\vec{r} = \vec{a} + \mu\vec{b}$

$\vec{r}_1 = \vec{r}_2$

$\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -5 \\ -4 \end{pmatrix}$

① $2 - \lambda = 3 + \mu$ ② $-3 + 4\lambda = -10 - 5\mu$ ③ $6 = -6 - 4\mu$

$12 = -4\mu$

$-3 = \mu$

$2 - \lambda = 3 - 3$

$2 - \lambda = 0$

$2 = \lambda$

so $\lambda = 2$ and $\mu = -3$

$\vec{r}_1 = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ $\vec{r}_2 = \begin{pmatrix} 3 \\ -10 \\ -6 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ -5 \\ -4 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$ $= \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

$P(0, 5, 6)$