Unit 3:Geometry and Trigonometry
AIHL 1.14

MATRIX

A **matrix** is a **rectangular array** of elements, usually numbers. They consist of rows (horizontal) and columns (vertical) of elements. An m by n, or m × n, matrix is one which has m rows and n columns. Matrices can describe transformations, and all vectors are a type of matrix.

Below is a 2×3 matrix.

$$\underset{2\times 3}{R} = \begin{bmatrix} \boxed{3} & 1 & -3 \\ 6 & 3 & \boxed{7} \end{bmatrix} \quad \begin{matrix} \text{row} \\ \text{element} \end{matrix}$$

column

Order of Matrices

Lisa goes shopping at store A to buy 2 loaves of bread at $2.65 each, 3 litres of milk at $1.55 per litre, a 500 g tub of butter at $2.35. Represent the quantities purchased in a row matrix and the costs in a column matrix.

$$Q = \begin{bmatrix} 2 & 3 & 500 \end{bmatrix}_{1\times 3} \qquad C = \begin{bmatrix} 2.65 \\ 1.55 \\ 2.35 \end{bmatrix}_{3\times 1}$$

Matrix Addition and Subtraction. Matrices can be added together or subtracted from one another as long as they have the same dimensions (same number of rows and columns).

$$\begin{bmatrix} 3 & 6 & -5 \\ 4 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 7 & 4 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -9 \\ 7 & 3 & 5 \end{bmatrix}$$

If $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$$

b $\mathbf{A} + \mathbf{B} + \mathbf{C}$

$$= \begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$$

c $\mathbf{B} + \mathbf{C}$

d $\mathbf{C} + \mathbf{B} - \mathbf{A}$

Scalar Multiplication. if a scalar *t* is multiplied by a matrix A the result is matrix *t*A obtained by multiplying every element of A by t.

If $\mathbf{B} = \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix}$ find: **a** $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

$$2B = \begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix}$$

Zero Matrix. A zero matrix is a matrix in which all elements are zero.

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

Negative Matrix. The negative matrix A, denoted -A is actually -1A.

$$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}.$$

$$\text{if} \quad \mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -1 \times 3 & -1 \times -1 \\ -1 \times 2 & -1 \times 4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & -4 \end{bmatrix}$$

Matrix Algebra for Addition

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A} + \mathbf{B}$ is also a matrix.
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$
- $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$
- a half of $\mathbf{A}$ is $\frac{1}{2}\mathbf{A}$ $\left(\text{not } \frac{\mathbf{A}}{2}\right)$

Simplify

$\mathbf{A} + 2\mathbf{A}$ $= 3A$

$-\mathbf{B} + \mathbf{B}$ $= 0$

$-(2\mathbf{A} - \mathbf{C}) = -2A + C$

Find $\mathbf{X}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ if: $\mathbf{X} + \mathbf{B} = \mathbf{A}$

$X = A - B$

$A_{\underline{2}\times 3}$

If $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, find $\mathbf{X}$ if $\frac{1}{3}\mathbf{X} = \mathbf{M}$.

$B_{3\times \underline{1}}$

$X = 3M$ $\quad X = \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix}$

$A \times B_{2\times 1}$

Matrix Multiplication. the number of columns of the first matrix must be equal to the number of rows of the second matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} ae+bh & af+bi & ag+bj \\ ce+dh & cf+di & cg+dj \end{bmatrix}$$

The **product** of an $m \times n$ matrix $\mathbf{A}$ with an $n \times p$ matrix $\mathbf{B}$, is the $m \times p$ matrix (called $\mathbf{AB}$) in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of $\mathbf{A}$ with the corresponding elements in the cth column of $\mathbf{B}$.

If $\mathbf{A}$ is $2 \times n$ and $\mathbf{B}$ is $m \times 3$:

a When can we find $\mathbf{AB}$? **b** If $\mathbf{AB}$ can be found, what is its order?

c Why can $\mathbf{BA}$ never be found?

a. $n = m$

b. 2×3

c. $\underbrace{B_{m\times 3} \cdot A_{2\times n}}$ $\quad 3 \neq 2$, BA is undefined

You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver and 2 cans of white paint and your friend wants 1 hammer, 2 screwdrivers and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$7	\$3	\$19
Store B	\$6	\$2	\$22

a. Write the requirements matrix R as a 3×2 matrix.

$$R = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}_{3\times 2} \\ \text{me} \quad \text{friend} \end{array}$$

$$RP_{3\times 3}$$

b. Write the prices matrix P as a 2×3 matrix.

$$P = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix}_{2\times 3}$$

c. Find PR.

$$PR_{2\times 2} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7(1)+3(1)+19(2) & 7(1)+3(2)+19(3) \\ 6(1)+2(1)+22(2) & 6(1)+2(2)+22(3) \end{bmatrix} = \begin{array}{cc} \text{Me} & \text{friends} \\ \downarrow & \downarrow \\ \end{array} \begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix} \begin{array}{l} \leftarrow A \\ \leftarrow B \end{array}$$

d. What are your costs at store A and your friend's costs at store B?

e. Should you buy from store A or store B?

Matrix Algebra for Multiplication

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
- In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
- If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
- $\mathbf{A(B + C) = AB + AC}$ {distributive law}
- If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{AI = IA = A}$ for all 2×2 matrices $\mathbf{A}$. {identity law}
- $\mathbf{A}^n$ for $n \geqslant 2$ can be determined provided that $\mathbf{A}$ is a square and n is an integer.

Inverse of a Matrix [2x2]. When a matrix is multiplied by its inverse it gives the identity matrix.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a Matrix. Describes the transformations which the matrix represents.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det \mathbf{A} = |\mathbf{A}| = ad - bc$$

Find $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 15 + (-12) & -30 + 30 \\ 6 - 6 & -12 + 15 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 3I$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

Find $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix}$ and hence find the inverse of $\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}$

$\frac{1}{-2}\begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ $\boxed{AA^{-1} = I}$

$A^{-1} = \begin{bmatrix} -1 & 0 & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{5}{2} & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

det.

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 2\begin{vmatrix} 5 & 2 \\ -3 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 1 & -3 \end{vmatrix}$$

$$= 2(11) + 0(1) + 3(-8)$$

$$= 22 + (-24)$$

$$= -2$$

Solving Systems of Equations. Solve equations of the form **Ax = b,** where A is an m × n matrix, x is an n ×1 matrix, or vector, and b is an m × 1 matrix, or vector.

Solve the systems of equations given by

$$7x + 11y = 18$$
$$11x - 7y = -11$$

$$\Rightarrow A \cdot x = b \Rightarrow A^{-1}A \; x = A^{-1}b$$

$$\begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

$$x = \underset{2\times 2}{A^{-1}}\, \underset{2\times 1}{b}$$

$$A^{-1} = \begin{vmatrix} \frac{7}{170} & \frac{11}{170} \\ \frac{11}{170} & -\frac{7}{170} \end{vmatrix}$$