EIGENVALUES and EIGENVECTORS

Recall:

1. Identify the type of transformation of an identity matrix in

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

2. Consider the transformation $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix}$ applied to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. Consider, as well, $\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \text{x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

In general,
if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{x} \begin{bmatrix} p \\ q \end{bmatrix} = \lambda \times \begin{bmatrix} p \\ q \end{bmatrix}$ we say that $\begin{bmatrix} p \\ q \end{bmatrix}$ is an **eigenvector** and λ an **eigenvalue** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Example

Find the eigenvalues of, and an eigenvector for $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

The eigenvalues of $\mathbf{A}$ are the solutions to $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

The **characteristic polynomial** of an $n \times n$ matrix $\mathbf{A}$ is $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$.
The eigenvalues of $\mathbf{A}$ are the solutions to $p(\lambda) = 0$.

Example

Find the eigenvalues of, and an eigenvector for $\begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}$.

TRY:

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$.

MATRIX DIAGONALIZATION

A non-zero square matrix is said to be **diagonal** if the elements *not* on its leading diagonal are zero.

A square matrix **A** is **diagonalisable** if there exists a matrix **P** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. We say that **P** diagonalises **A**.

If **A** is a 2×2 matrix with *distinct* eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, then $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ diagonalises **A**, and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Example

The matrix $A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$,

with corresponding eigenvectors $x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively.

Show that and both diagonalize A.

....

....

....

....

....

....

....

....

....

....

....

....

Example.

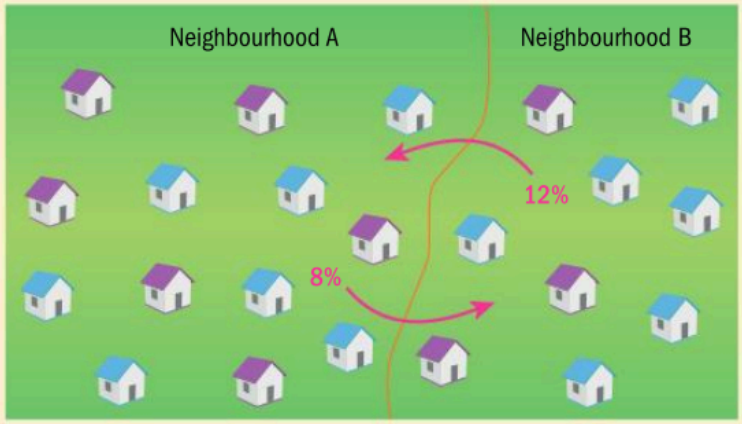
a. Find a diagonalization of $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

b. Hence, find an expression for A^4 in the form PD^4P^{-1}.

c. Find an expression for A^4 as a product of 3 matrices with no exponents.

Example

Consider the situation where people move between two neighbourhoods in a particular city. Each year since 2015, 8% of people currently living in neighbourhood A move to neighbourhood B and 12% of people currently living in neighbourhood B move to neighbourhood A. You may assume any movement other than between the two neighborhoods exactly balances those arriving with those leaving.



1 Let T be the transition matrix where T_{ji} represents the probability of a person moving from neighbourhood i to neighbourhood j. Explain why the transition matrix is

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0.92 & 0.12 \\ 0.08 & 0.88 \end{pmatrix} \end{array}$$

2 The populations of neighbourhoods A and B are 24,500 and 45,200 respectively at the beginning of 2015. After 1 year the population can be expressed as the matrix $P_1 = TP_0 = \begin{pmatrix} 0.92 & 0.12 \\ 0.08 & 0.88 \end{pmatrix}\begin{pmatrix} 24500 \\ 45200 \end{pmatrix}$

What is the population of neighbourhood A after 1 year? Neighbourhood B?

Let $S_0 = \begin{pmatrix} 24500 \\ 45200 \end{pmatrix}$. The population after n years is given by the expression $T^n S_0$.

3 Find the eigenvalues and eigenvectors for $\boldsymbol{T}$.

4 Diagonalize $\boldsymbol{T}$ and use the relationship $\boldsymbol{T}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$ to show that

$$\boldsymbol{T}^n = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & 0.8^n \end{pmatrix}\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & -0.6 \end{pmatrix};$$

5 As $n \to \infty$, what will $\boldsymbol{D}^n = \begin{pmatrix} 1^n & 0 \\ 0 & 0.8^n \end{pmatrix}$ approach? What will $\boldsymbol{T}^n$ approach?

6 Hence find the long-term population of the neighbourhoods.

The administration of both neighborhoods would like a formula to tell them how many residents are expected to be in each community n years after 2015.

7 Multiply out your expression found in question 3 to find a suitable formula for each of the school boards.

8 Use your formula to write down the populations of neighborhood A in 2020.