INVERSE FUNCTION

Week 6

Learning Objective

To find the inverse of a function.

Success Criteria

- To understand that the reverse process is the inverse function.
- To find and apply the inverse function.
- To solve problems involving inverse functions.

Starter

The tables show the inputs and outputs of function.

Can you work out what $f(x)$ is and fill in the missing gaps?

$f(x) = 2x - 2$

x	$f(x)$
0	−2
1	0
2	2
3	4
4	6
10	18
15	28

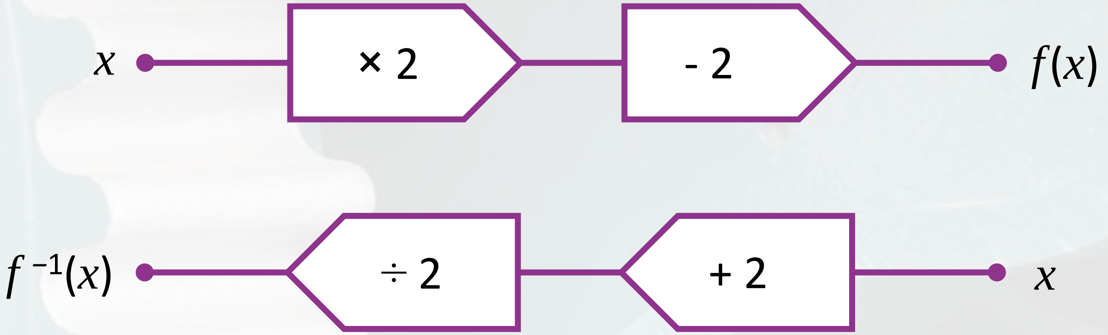
Inverse Functions

The **inverse function** is the rule that gets you from the output **back to** the input.

It is written as $f^{-1}(x)$.

Inverse Functions

If we look at the function $f(x) = 2x - 2$, we can start by thinking of the function as a function machine, then reversing the operations.



So the inverse function is $f^{-1}(x) = \frac{x+2}{2}$

x	$f(x)$
0	−2
1	0
2	2
3	4
4	6
10	18
15	28
21	40

Inverse Functions

There is a quicker way to do this than drawing out a function machine every time.
$f(x) = 2x - 2$

Take Notes!

Step one:
Start by writing your function in terms of y and x.

$y = 2x - 2$

Step two:
Switch the y and the x.

$x = 2y - 2$

Step three:
Make y the subject.

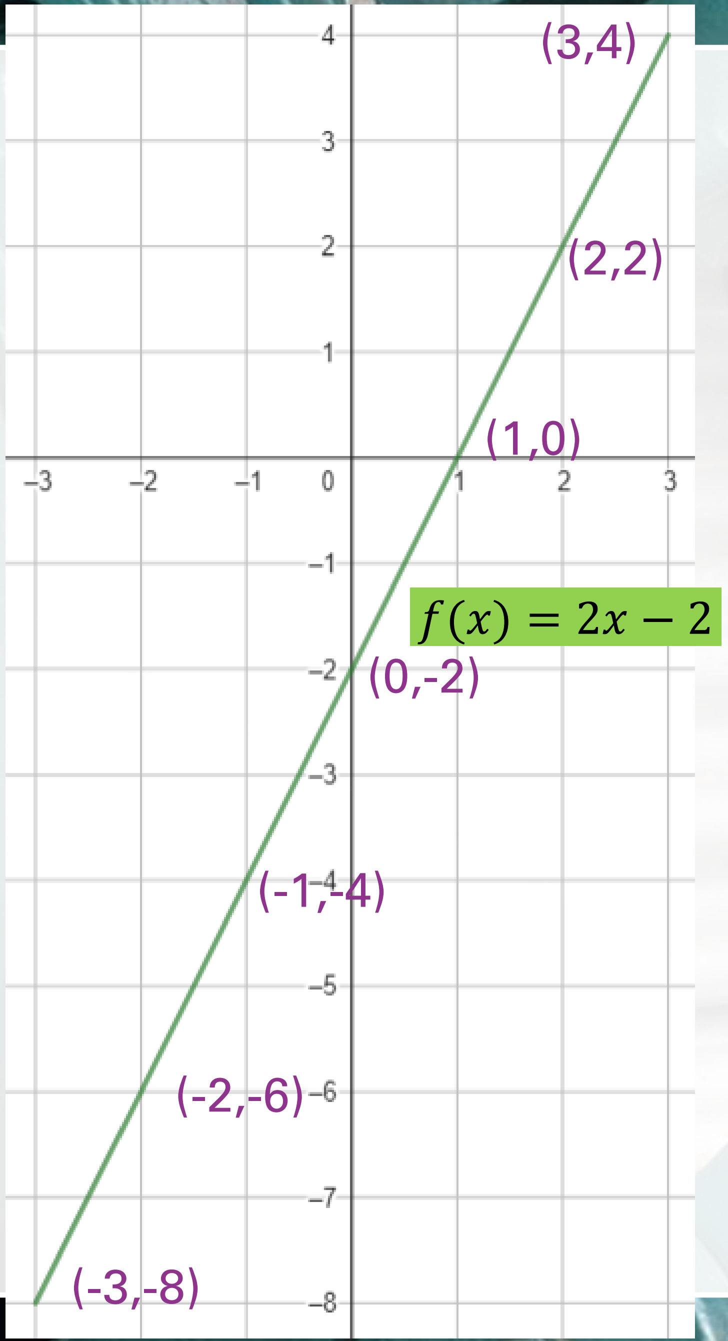
$y = \frac{x+2}{2}$

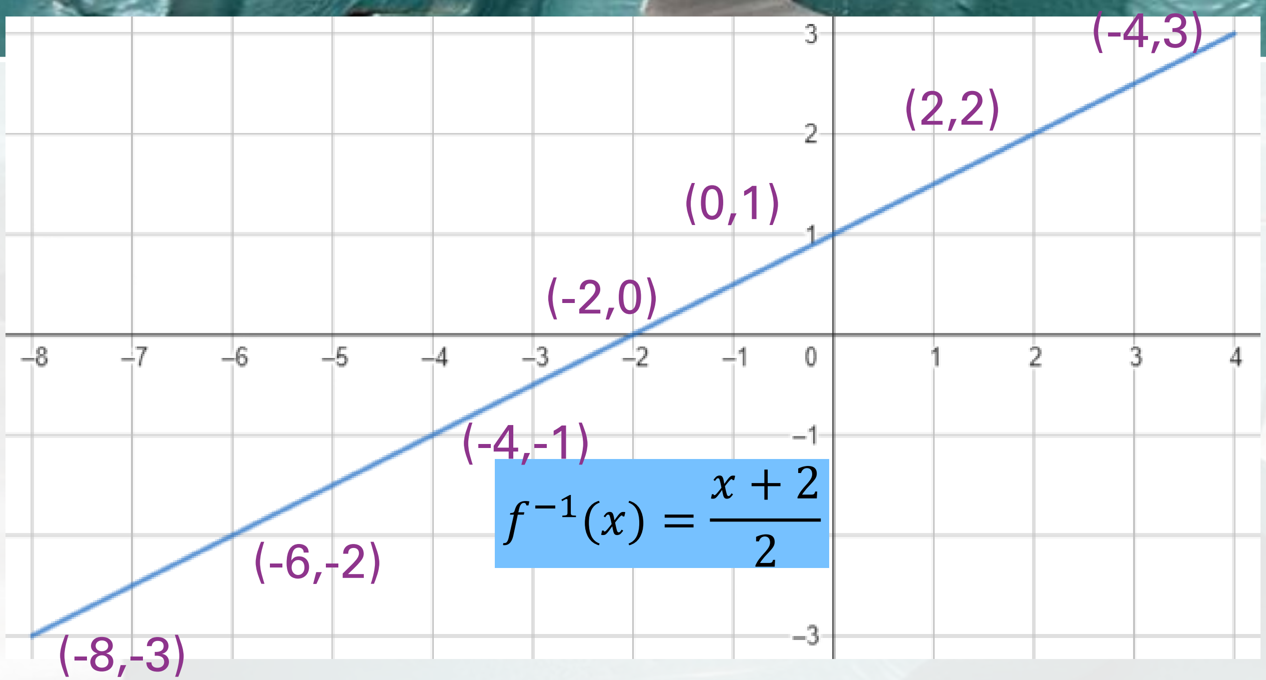
This is your inverse function: $f^{-1}(x) = \frac{x+2}{2}$

Step four:
Check by choosing an input.

$f(1) = 2(1) - 2 = 0$

$f^{-1}(0) = \frac{x+2}{2} = 1$





State the domain and range for each function on the graphs.

Discuss

Find the inverse function $f^{-1}(x)$, for the function $f(x) = 4x - 7$

Step one:
Start by writing your function in terms of y and x.

$y = 4x - 7$

Step two:
Switch the y and the x.

$x = 4y - 7$

Step three:
Make y the subject.

$y = \frac{x + 7}{4}$

So $f^{-1}(x) = \frac{x + 7}{4}$

Step four:
Check by choosing an input.

$f(1) = 4 \times 1 - 7 = -3$

$f^{-1}(-3) = \frac{-3 + 7}{2} = 1$

Example 9

The function $f(x) = 7 - \frac{1}{2}x$ is defined on the domain $-5 \leq x \leq 5$.

a Find $f^{-1}(6)$.

b Determine whether $f^{-1}(11)$ exists. If it does, find it.

c Find the range of $f(x)$.

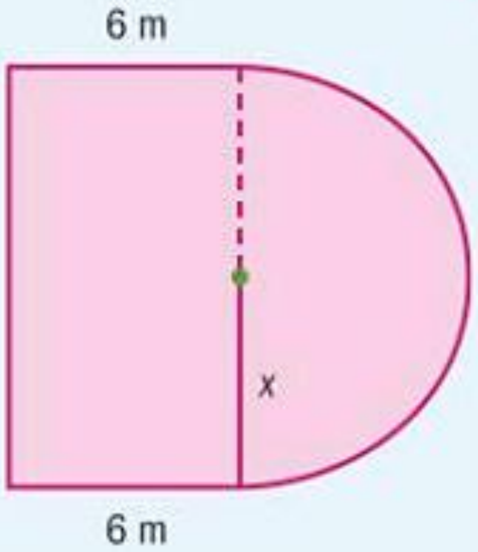
d State the domain and range of $f^{-1}(x)$.

e Solve $f^{-1}(x) = 0$.

a $f^{-1}(6) = 2$	Since $f(a) = b$ means $f^{-1}(b) = a$, $f^{-1}(6) = a$ means $f(a) = 6$. Solve $6 = 7 - \frac{1}{2}x$ algebraically or with technology.
b $f^{-1}(11)$ is not defined.	$11 = 7 - \frac{1}{2}x$ means that $x = -8$, which is outside the domain of f.
c Range: $4.5 \le y \le 9.5$	You know that $f(x)$ is linear, so you can find its values at the endpoints of the domain to find the range. $f(-5) = 9.5$, $f(5) = 4.5$
d Domain: $4.5 \le x \le 9.5$ Range: $-5 \le y \le 5$	The domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.
e $x = 7$	$f^{-1}(x) = 0$ means $f(0) = x$.

Example

Laura is planning to construct a garden consisting of a rectangle joined to a semicircle of radius x, as shown in the diagram. The sides of the rectangle perpendicular to this side will be of length 6 m. The garden must fit within an 18 m by 18 m square.



a Find a model for the perimeter P (in meters) of this garden as a function of its radius x.

b Find the reasonable domain and associated range of this model.

c Find an equation for the inverse function $P^{-1}(x)$ in gradient–intercept form.

d State the independent and dependent variables of the function $P^{-1}(x)$.

e Edging for the garden comes in three lengths: 15, 30 or 45 metres. Determine which length will give Laura a rectangle that is closest to a square.

a	$P(x) = \pi x + 2x + 12$	The rectangle has side lengths 6 and $2x$. The semicircle has circumference $\frac{2\pi x}{2} = \pi x$.
b	Domain: $0 < x \leq 9$ Range: $12 < P \leq 58.3$	x represents the radius, which must be positive. As the pool must fit within an 18×18 square, $x + 6 < 18$ and $2x < 18$, so $x < 9$. $P(0) = 12$ and $P(9) = 58.3$.
c	$P^{-1}(x) = 0.194x - 2.33$	$x = \pi y + 2y + 12$ Swap x and y. $x - 12 = (\pi + 2)y$ Factorise. $\frac{x-12}{\pi+2} = y$ Divide by the y coefficient to isolate y and convert numbers to decimal approximations.
d	Independent variable x is the perimeter. Dependent variable y is the radius associated with that perimeter.	
e	Laura should choose the 30 m edging because it results in the rectangle whose dimensions are most similar.	Find the radius for each perimeter by evaluating P^{-1}, then double the result to find one side of the rectangle. The length of the other side is 6 m: $P^{-1}(15) = 0.583$ m: 6 m × 1.166 m $P^{-1}(30) = 3.50$ m: 6 m × 7 m $P^{-1}(45) = 6.62$ m: 6 m × 12.8 m

YOUR TURN

Write your answers in your math notebook. Show all necessary workings.

Exercise 4I

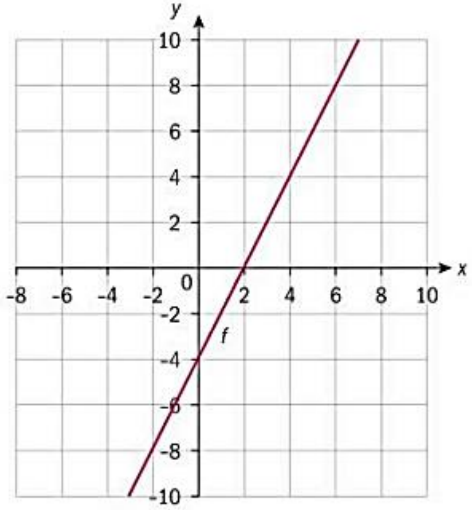
1 For each of the following functions:

i Sketch a graph of the inverse function.

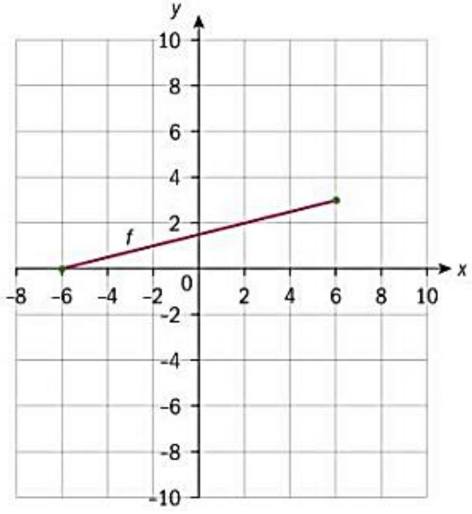
ii State the domain and range of $f^{-1}(x)$.

iii Estimate the solution of $f(x) = f^{-1}(x)$.

a



b



2 Consider $f(x) = -2.5x + 5$ for $0 \leq x \leq 3$.

a Draw the graph of f on a pair of axes. Use the same scale on both axes.

b Point A lies on the graph of f^{-1} and has coordinates $(b, 3)$. Find the value of b.

c Determine the missing entries in the table.

Function	Domain	Range
f	$0 \leq x \leq 3$	
f^{-1}		

d Draw the graph of f^{-1} on the same set of axes used in part **a**.

e Find the coordinates of the point that lies on the graph of f, the graph of f^{-1} and the graph of $y = x$.

3 Dieneke travels from Amsterdam to Budapest, a distance of 1400 km. On the first day she covers 630 km. She notes that her average speed is 90 km hr^{-1} on day 1. Assume that she continues at the same average speed on the second day without any breaks.

a Supposing that she begins driving at 8 am on day 2. Find an equation that expresses her total distance travelled from Amsterdam, d (in kilometres), as a function of time, x (in hours since 8 am of day 2).

b Find an equation for the inverse function $d^{-1}(x)$.

c Use the inverse function to predict the time (to the nearest minute) at which Dieneke will arrive at the following points:

i halfway between Amsterdam and Budapest

ii in Prague, 880 km from Amsterdam

iii in Budapest.

d Comment on which of the three places Dieneke should plan to stop for a midday meal.