Composite Functions

Week 7 – September 6, 2023

Objectives

- To simplify composite functions
- To evaluate composite functions
- To solve problems involving composite functions

Recall

- Let $h(x) = x + 4$. Find the range of the function when the domain is given as D = $\{1, 2, 3, 4\}$.

R = {5, 6, 7, 8}

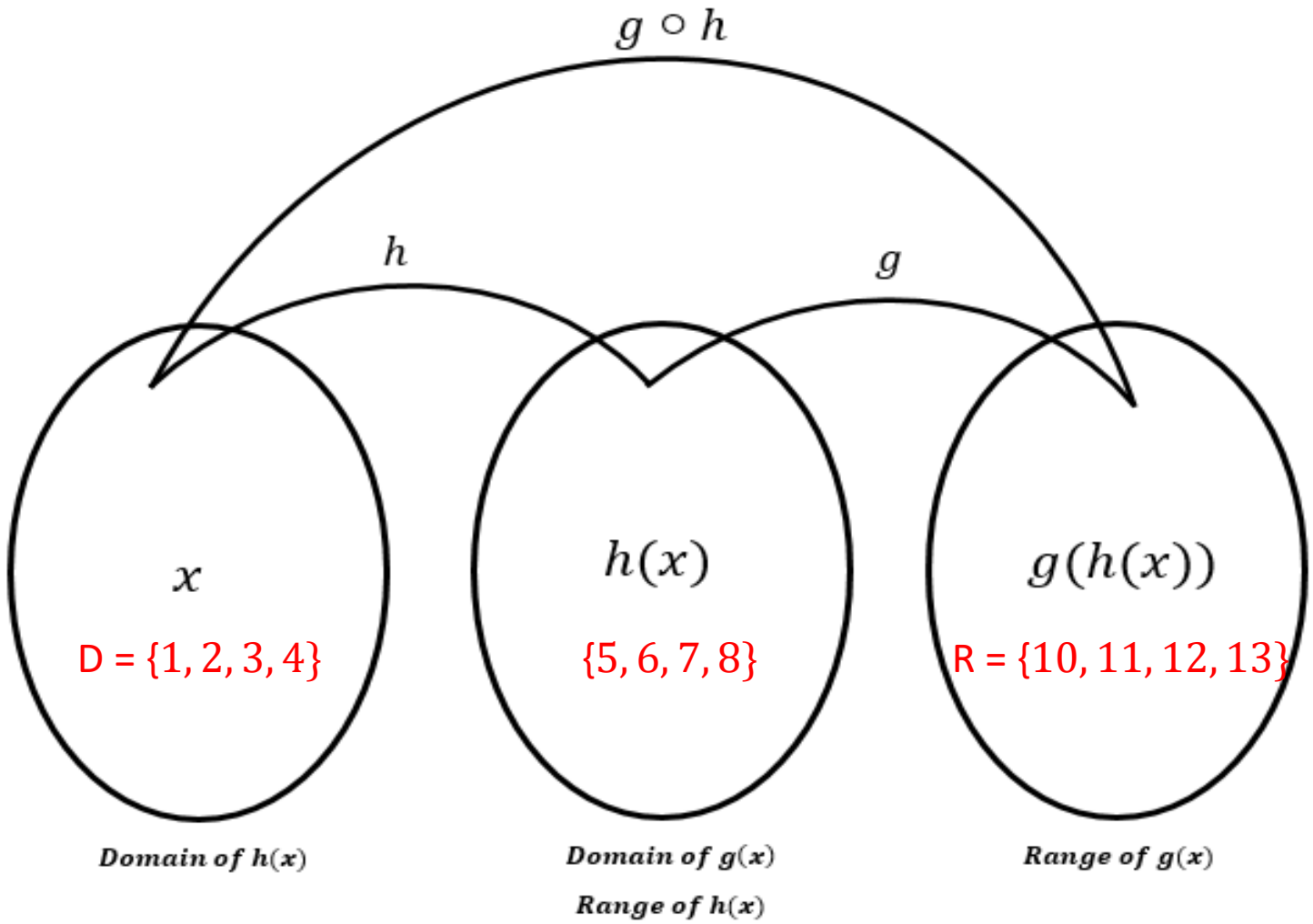
- Use each value of the range of $h(x)$ as inputs to the function $g(x) = x + 5$. Find the range of $g(x)$.

R = {10, 11, 12, 13}

- The range of $g(x)$ is the result of the composition of two functions $h(x)$ and $g(x)$.

In symbols, $g(h(x))$, where $h(x)$ is the input function and $g(x)$ is the output function.

Composite Function



- The function $h(x)$ is known as an inner function and the function $g(x)$ is referred to as an outer function.

- $\boldsymbol{g(h(x))}$ or $\boldsymbol{(g \circ h)x}$ is read as **"$\boldsymbol{g}$ of $\boldsymbol{h}$ of $\boldsymbol{x}$"** or **"$\boldsymbol{h(x)}$ is the inner function of $\boldsymbol{g(x)}$"**.

How to Solve Composition of Functions

The composition of two functions can be solved using the following steps:

- **Rewrite:** The composition written in the form $(f \circ g)x$ needs to be written as $f(g(x))$.
- **Replace:** *Substitute* the input function into the output function.
- **Review:** Make sure to *simplify* the answer obtained.

Example 1: Given that the two functions $f(x) = 4x^2 - 7x$ and $g(x) = x + 1$.

a. **Find** $(f \circ g)x$.

Rewrite: Find $f(g(x))$.

Given: output function: $f(x) = 4x^2 - 7x$

input function: $g(x) = x + 1$

Replace: *Substitute* the input function into the output function.

$$f\big(g(x)\big) = 4(x+1)^2 - 7(x+1)$$

Review: Make sure to *simplify* the answer obtained.

$$f\big(g(x)\big) = 4(x^2 + 2x + 1) - 7x - 7$$

$$f\big(g(x)\big) = 4x^2 + 8x + 4 - 7x - 7$$

$$\boldsymbol{f\big(g(x)\big) = 4x^2 + x - 3}$$

Example 1: Given that the two functions $f(x) = 4x^2 - 7x$ and $g(x) = x + 1$.

b. Hence **show** that $(f \circ g)x \neq (g \circ f)x$.

$\boldsymbol{f(g(x)) = 4x^2 + x - 3}$

Rewrite: Find $g(f(x))$.

Given: output function: $g(x) = x + 1$

input function: $f(x) = 4x^2 - 7x$

Replace: *Substitute* the input function into the output function.

$$g(f(x)) = 4x^2 - 7x + 1$$

Review: Make sure to *simplify* the answer obtained.

$$\boldsymbol{g(f(x)) = 4x^2 - 7x + 1 \text{ (Simplified form)}}$$

a. Find $(f \circ g)x$.

Rewrite: Find $f(g(x))$.

Given: output function: $f(x) = 4x^2 - 7x$

input function: $g(x) = x + 1$

Replace: *Substitute* the input function into the output function.

$$f\big(g(x)\big) = 4(x+1)^2 - 7(x+1)$$

Review: Make sure to *simplify* the answer obtained.

$$f\big(g(x)\big) = 4(x^2 + 2x + 1) - 7x - 7$$

$$f\big(g(x)\big) = 4x^2 + 8x + 4 - 7x - 7$$

$$\boldsymbol{f\big(g(x)\big) = 4x^2 + x - 3}$$

b. Hence show that $(f \circ g) \neq (g \circ f)$.

Rewrite: Find $g(f(x))$.

Given: input function: $f(x) = 4x^2 - 7x$

output function: $g(x) = x + 1$

Replace: *Substitute* the input function into the output function.

$$g\big(f(x)\big) = 4x^2 - 7x + 1$$

Review: Make sure to *simplify* the answer obtained.

$$\boldsymbol{g\big(f(x)\big) = 4x^2 - 7x + 1}$$ **(Simplified form)**

$$\boldsymbol{Since\ f\big(g(x)\big) = 4x^2 + x - 3}$$

and

$$\boldsymbol{g\big(f(x)\big) = 4x^2 - 7x + 1,}$$

$$\boldsymbol{(f \circ g)x \neq (g \circ f)x.}$$

Example 2: Given $f(x) = 6x + 4$,
a. **Find** $(f \circ f)x$.

Rewrite: Find $f(f(x))$.

Given: output function: $f(x) = 6x + 4$

input function: $f(x) = 6x + 4$

Replace: *Substitute* the input function into the output function.

$$f\big(f(x)\big) = 6(6x + 4) + 4$$

Review: Make sure to *simplify* the answer obtained.

$$f\big(f(x)\big) = 36x + 24 + 4$$

$$\boldsymbol{f\big(f(x)\big) = 36x + 28}$$

Example 2: Given $f(x) = 6x + 4$,
b. Evaluate $(f \circ f)(1)$.

We already know that:

Rewrite: Find $f(f(x))$.

Given: output function: $f(x) = 6x + 4$

input function: $f(x) = 6x + 4$

Replace: *Substitute* the input function into the output function.

$$f\big(f(x)\big) = 6(6x + 4) + 4$$

Review: Make sure to *simplify* the answer obtained.

$$f\big(f(x)\big) = 36x + 24 + 4$$

$$\boldsymbol{f\big(f(x)\big) = 36x + 28}$$

Substitute 1 and solve:

$$\boldsymbol{f\big(f(1)\big) = 36(1) + 28}$$
$$\boldsymbol{f\big(f(1)\big) = 36 + 28}$$
$$\boldsymbol{f\big(f(1)\big) = 64}$$

Example 3: Given $h(x) = x^2$ and $f(x) = x - 5$.

a. **Find** $(h \circ f)(2)$.

Rewrite: Find $h(f(2))$.

Given: output function: $h(x) = x^2$

input function: $f(x) = x - 5$

Replace: *Substitute* the input function into the output function.

$$h(f(x)) = (x-5)^2 \longrightarrow h(f(x)) = x^2 - 10x + 25$$

$$h(f(2)) = (2-5)^2 \qquad h(f(2)) = 2^2 - 10(2) + 25$$

$$\boldsymbol{h(f(2)) = 9}$$

Review: Make sure to *simplify* the answer obtained.

$$h(f(2)) = (-3)^2$$

$$\boldsymbol{h(f(2)) = 9}$$

Example 3: Given $h(x) = x^2$ and $f(x) = x - 5$.
b. **Find** $(f \circ h)(2)$.

Rewrite: Find $f(h(2))$.

Given: input function: $h(x) = x^2$

output function: $f(x) = x - 5$

Replace: *Substitute* the input function into the output function.

$$f\big(h(x)\big) = x^2 - 5$$

$$f\big(h(2)\big) = 2^2 - 5$$

Review: Make sure to *simplify* the answer obtained.

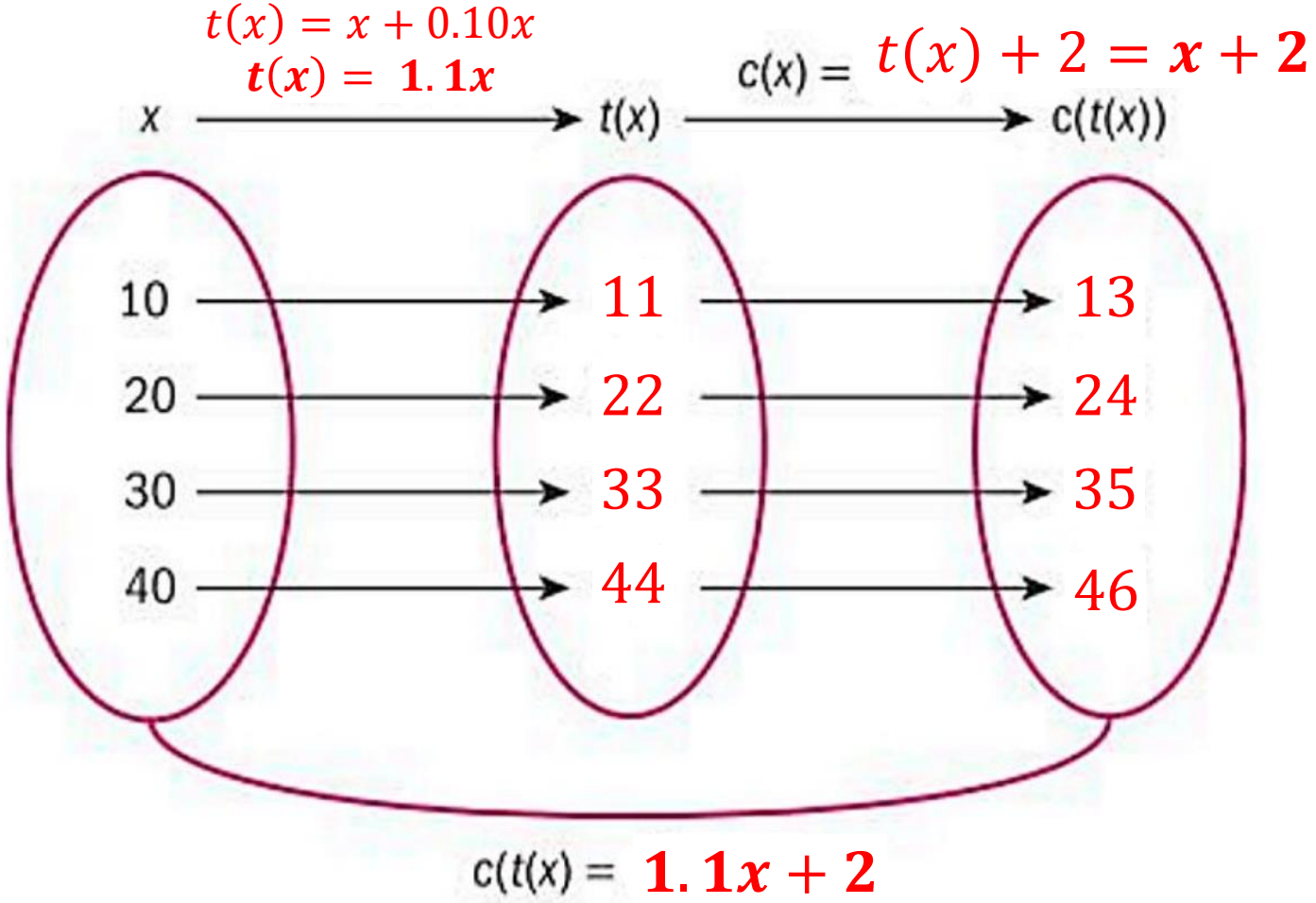
$$f\big(h(2)\big) = 4 - 5$$

$$\boldsymbol{f\big(h(2)\big) = -1}$$

Example 4: At a restaurant in Rome, you notice that €2 per table charge has been added to your bill for x euros. You also plan to add a 10% tip.

a. Write an equation that will represent the tip function, $t(x)$.

b. Complete the mapping diagram on the right that shows the final charge that would result for several different bills.



1 If $f(x) = 8x - 25$, $g(x) = 4 - x$ and $h(x) = \frac{3}{2}x$, find:

a $f \circ g(1)$

b $h \circ f(4)$

c $g \circ f(x)$

d $h \circ h \circ h(x)$

2 Returning to the example of the bill at the restaurant, recall that $c(x) = x + 2$ represents the table charge function and $t(x) = 1.1x$ represents the tip function.

a Show that $c(t(x)) \neq t(c(x))$ by finding a simplified function for each.

b Determine which order of composition will provide the waiter with the larger tip, and state, in terms of x, how much larger.

3 The number of window panes a company can produce is a function of time t, measured in hours of operation per day, and is given by $w(t) = 7t - 3$. The company's profit $\$P$ is a function of the number of windows sold, given by $P(w) = 50w - 2000$.

a Find the company's profit (or loss) if it operates for 4 hours a day.

b Find the company's profit as a function of time, expressed in the form $P(t) = mt + c$ where $m, c \in \mathbb{R}$.

c Determine the number of hours (to the nearest hour) that the factory must operate in order to earn a (positive) profit.

References

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