POWER FUNCTIONS AND MODELS

October 23, 2023

OBJECTIVES

- To define power functions
- To identify the special properties that a power function may exhibit.
- To apply the properties in graphing and identifying power functions.

INVESTIGATE

1 Sketch the graphs of the following power functions.

a $f(x) = x^2$ b $f(x) = x^3$ c $f(x) = x^4$

d $f(x) = x^5$ e $f(x) = x^6$ f $f(x) = x^7$

Complete the table below and answer the questions that follow.

Function	minimum	maximum	y-intercept

INVESTIGATE

2. Can you **group** power functions into smaller categories?

Yes. Even and Odd Positive Power Functions

3. How are they the same and how are they different from a **parabola**?

Even Positive Power Functions form a parabolic graph.

4. How are they the same and how are they different from a **cubic** graph?

Odd Positive Power Functions have graphs that are similar to a cubic graph.

INVESTIGATE

5. How many **maximum and minimum points** can a power function have?

Even Positive Power Functions: One minimum, No maximum

Odd Positive Power Functions: None

6. Are power functions **symmetric**? If yes, what type of symmetry do they possess?

*Yes. Even Positive Power Functions have an **axis** of symmetry at the y-axis.*

*Odd Positive Power Functions have a **point** of symmetry at the origin.*

7. What happens to the graph of power functions as the value of the **power increases**?

The graph tends to become flatter around the origin.

POWER FUNCTIONS

A **power function** is a function of the form $f(x) = ax^n$, where a and n are the parameters of the function and n is a member of $\mathbb{R}$.

Parameter $a \neq 0$, since $a = 0$ would make the function zero.

- If $n = 0$, the function is constant.
- If $n = 1$, the function is linear.
- If $n = 2$, the function is quadratic.
- If $n = 3$, the function is cubic.

The fundamental (simplest) power function is $f(x) = x^n$.

POWER FUNCTIONS

The graphs of all **even positive** power functions have a shape similar to a parabola and are symmetric about the y-axis.

The graphs of all **odd positive** power functions have a shape similar to a cubic function and are rotationally symmetric about the origin.

As the index of positive power functions increases, the graphs tend to become flatter around the origin.

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

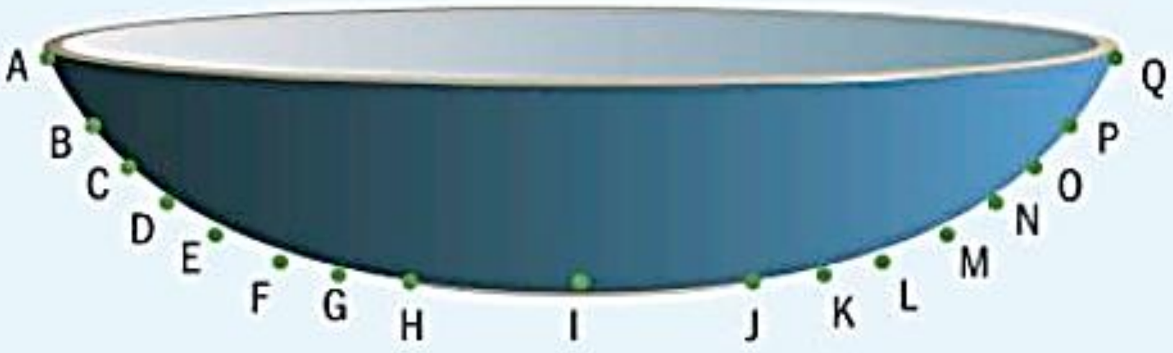
The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Which functions are power functions?

$$f(x) = 2x^2 \cdot 4x^3$$
$$g(x) = -x^5 + 5x^3 - 4x$$
$$h(x) = \frac{2x^5-1}{3x^2+4}$$

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

A designer wants to create a model function for a bowl that she sketched by hand in order to be able to process it digitally. To do so, she put the sketch over a grid and marked some data points, with the centre of the bottom of the bowl being the origin.



The data points are the following:

Point	A	B	C	D	E	F	G	H	I
x	−16.5	−15.1	−14	−12.8	−11.3	−9.3	−7.5	−5.3	0
y	7	4.9	3.62	2.53	1.54	0.7	0.3	0.07	0

Point	J	K	L	M	N	O	P	Q
x	5.3	7.5	9.3	11.3	12.8	14	15.1	16.5
y	0.07	0.3	0.7	1.54	2.53	3.62	4.9	7

a Plot the given data on your GDC or other technology.

She first thought of using a quadratic function to model the shape.

b Explain why a quadratic function could be suitable to model this shape.

c Use your GDC to determine the quadratic model function for this set of data.

d Find the coefficient of determination, and comment on your answer.

e Sketch the model function over the scatter plot and comment on the closeness of fit to the original data.

Not being satisfied with the model function she created, she decided to determine a new quartic model function.

f Explain why the designer might have not been satisfied with the model function she created and why a quartic function could be a suitable alternative model.

g Use your GDC to determine the quartic model function for this set of data.

h Find the coefficient of determination, and comment on its significance in relation to the previous model.

i Sketch the model function over the scatter plot and comment on the closeness of fit to the original data and compare it to the previous model.

Using the quartic model, the designer want to create a model for another bowl whose dimensions are double in size to the original.

j Determine the equation of the model function for the larger bowl.

Viola is conducting an experiment to demonstrate the relation between the radius and the volume of a sphere. She gathered the following data.

r (cm)	1	2	3	4	5	6	7	8
V (cm^3)	4.19	33.5	113	268	524	905	1440	2140

a Plot the given data on your GDC.

b Use your GDC to determine the power function model for this set of data.

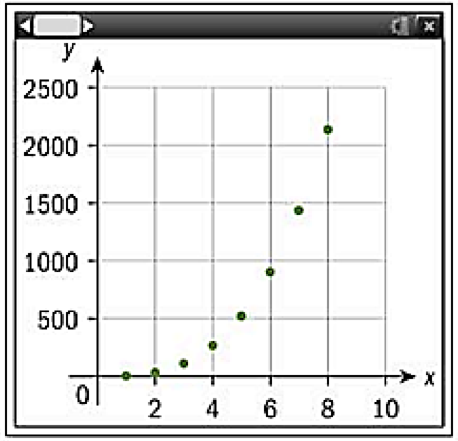
c Comment on the choice of model by determining the coefficient of determination.

d Sketch the model function over the scatter plot and comment on the closeness of fit to the original data.

e Use the model function to determine the volume of a sphere with radius 10 cm.

f Comment on how the value found in part **e** compares with the actual value of the volume of a sphere of radius 10 cm found using the appropriate formula.

a

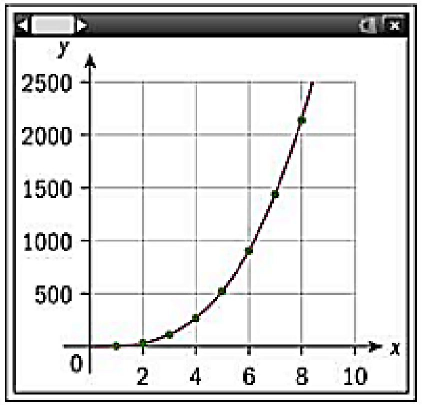


x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

b $V = 4.19\,r^{3.00}$

c $R^2 = 1$ so the power model fits this data perfectly

d



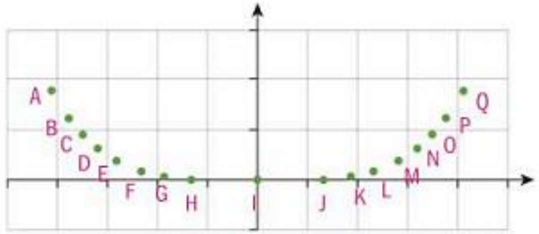
x_1	1	2	3	4	5	6	7	8
y_1	4.19	33.5	113	268	524	905	1440	2140

Power function is a perfect fit.

e 4190

f 4188.79

a



b A quadratic formula would be suitable to model this shape since the data points seem to have vertical symmetry and also because of their constant concavity.

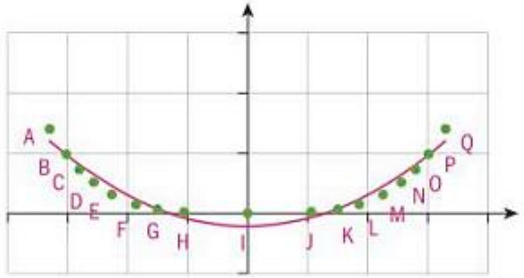
c The best fit parabola is: $f(x) = 0.0263x^2 - 1.15$

Using quadratic regression in the GDC.

d $R^2 = 0.933$

The coefficient of determination shows that the model function describes the data set very closely.

e



Although the model function goes through (or close to) most points, it still misses the bottom point by a significant amount.

f The quadratic model had some major problems in describing the shape of the bowl, the most important being at the bottom of the bowl.

A quartic function can be a better fit, since in addition to depicting symmetry and constant concavity, it also is flatter close to the origin.

g The best fit quartic function is:

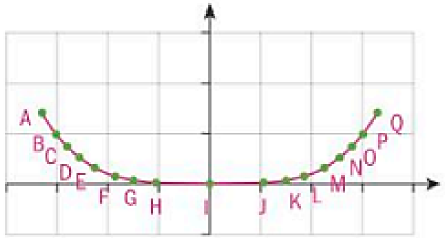
$f(x) = 0.0000946x^4 - 0.0000649x^2 + 0.000214$

Using quartic regression in the GDC.

h $R^2 = 0.999$

The coefficient of determination shows almost perfect fit and is of course significantly better than the quadratic.

i



The model function goes perfectly through all the data points and describes the shape of the bowl in a much better way than the quadratic.

j $g(x) = 2f\left(\frac{x}{2}\right) = 0.0000942x^4 = 0.0000118x^4$

The enlarged bowl is double the size both in the x and in the y directions.

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