Unit 3:Geometry and Trigonometry

VECTOR MULTIPLICATIONS

Scalar Product vs Vector Product
AIHL 3.13

SCALAR PRODUCTS

The **SCALAR PRODUCT** of two vectors gives information about the angle between the two vectors

- If the scalar product is **positive** then the angle between the two vectors is **acute**.
- If the scalar product is **negative** then the angle between the two vectors is **obtuse**.
- If the scalar product is **zero** then the angle between the two vectors is **a right angle**.

Scalar product	$\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\boldsymbol{v}\cdot\boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

Example 1.

Calculate the scalar product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and

$\boldsymbol{w} = 3\mathbf{j} - 2\mathbf{k} - \mathbf{i}$ using:

i)
the formula $\boldsymbol{v}\cdot\boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$,

$v \cdot w = (2)(3) + (0)(-2) + (-5)(-1)$

$= 6 - 0 + 5$

$= 11$

ii)
the formula $\boldsymbol{v}\cdot\boldsymbol{w}=|v||w|\cos\theta$, given that the angle between the two vectors is 66.6°.

$|v| = \sqrt{2^2+0^2+(-5)^2}$ $\quad |w| = \sqrt{3^2+(-2)^2+(-1)^2}$

$= \sqrt{29}$ $\quad = \sqrt{14}$

$v \cdot w = (\sqrt{29})(\sqrt{14}) \cos 66.6°$

≈ 8.00

ANGLE BETWEEN VECTORS

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1+v_2w_2+v_3w_3}{\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert}$

Example 2.

Calculate the angle formed by the two vectors $\boldsymbol{v}=\begin{pmatrix}-1\\3\\2\end{pmatrix}$ and $\boldsymbol{w}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$.

$v \cdot w = |v||w|\cos\theta$

$\dfrac{v \cdot w}{|v||w|} = \cos\theta$

$\cos\theta = \dfrac{7}{\sqrt{14}\,\sqrt{26}}$

$\theta = 68.48$

$|v| = \sqrt{(-1)^2+3^2+2^2}$

$= \sqrt{14}$

$|w| = \sqrt{3^2+4^2+(-1)^2}$

$= \sqrt{26}$

Example 3.

Find the value of t such that the two vectors $\boldsymbol{v}=\begin{pmatrix}2\\t\\5\end{pmatrix}$ and $\boldsymbol{w}=(t-1)\mathbf{i}-\mathbf{j}+\mathbf{k}$ are perpendicular to each other.

$v \cdot w = 0$

$2(t-1) + (t)(-1) + 5 = 0$

$2t - 2 - t + 5 = 0$

$t = -3$

VECTOR PRODUCTS

The **VECTOR PRODUCT** is a vector in plane that is **perpendicular** to the two vectors from which it was calculated. The direction of the vector product is perpendicular to both v and w.

- If two vectors are parallel then the vector product is zero.

Vector product	$\boldsymbol{v}\times\boldsymbol{w}=\begin{pmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{pmatrix}$, where $\boldsymbol{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$, $\boldsymbol{w}=\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$ $\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$

The vector cross product of $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ is

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\mathbf{k}$$

Example 4.

Calculate the magnitude of the vector product between the two vectors $\boldsymbol{v}=\begin{pmatrix}2\\0\\-5\end{pmatrix}$ and $\boldsymbol{w}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ using

$$|v\times w| = \begin{bmatrix} 0(-1) - (-5)(-2) \\ -5(3) - (2)(-1) \\ 2(-2) - (0)(3) \end{bmatrix} = \begin{pmatrix} 0-10 \\ -15+2 \\ -4 \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$$

i)

the formula $\boldsymbol{v}\times\boldsymbol{w}=\begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$,

$$|v\times w| = \sqrt{(-10)^2+(-13)^2+(-4)^2}$$

$$= \sqrt{285} \approx 16.9$$

$$v\times w = \left|\begin{matrix} i & j & k \\ 2 & 0 & -5 \\ 3 & -2 & -1 \end{matrix}\right|\begin{matrix} i & j \\ 2 & 0 \\ 3 & -2 \end{matrix}$$

$$= 0i - 15j - 4k$$

$$- \left[0k + 10i + (-2j)\right]$$

$$= -10i - 13j - 4k$$

ii)

the formula, given that the angle between them is 1 radian.

$$|v\times w| = (\sqrt{29})(\sqrt{14})\sin 1$$

$$= 16.955$$

$$\approx 17.0$$

AREA using VECTOR PRODUCT

The area of the parallelogram with two adjacent sides formed by the vectors v and w is equal to the magnitude of the vector product of two vectors v and w.

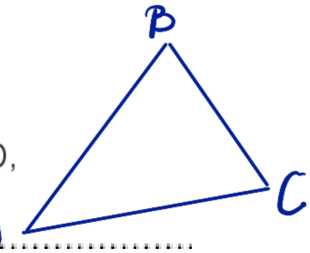
Area of a parallelogram	$A=\lvert\boldsymbol{v}\times\boldsymbol{w}\rvert$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram

So $A_\Delta = \frac{1}{2}|v\times w|$

Example 5.

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

A B C



$\vec{AB} = (2, -1, -3) \qquad \vec{AC} = (1, 0, -6)$

$$A_{\Delta} = \frac{1}{2}\left| \vec{AB} \times \vec{AC} \right| \qquad = 5.43 \text{ sq units}$$

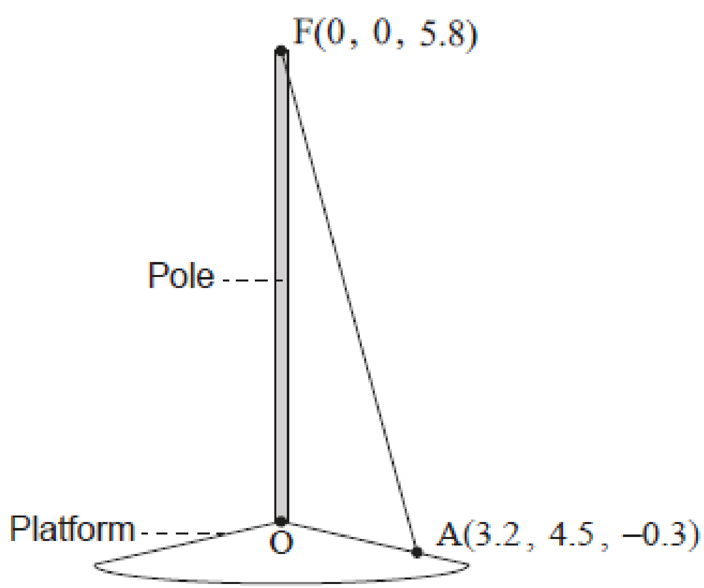
$$= \frac{1}{2} \cdot \sqrt{6^2 + 9^2 + 1^2}$$

$$= \frac{\sqrt{118}}{2}$$

Example 6.

A vertical pole stands on a sloped platform. The bottom of the pole is used as the origin, O, of a coordinate system in which the top, F, of the pole has coordinates (0, 0, 5.8). All units are in metres.

diagram not to scale



The pole is held in place by ropes attached at F.

One of these ropes is attached to the platform at point A(3.2, 4.5, −0.3). The rope forms a straight line from A to F.

a. Find $\overrightarrow{AF}$ $= \begin{pmatrix} -3.2 \\ -4.5 \\ 6.1 \end{pmatrix}$ [1]

b. Find the length of the rope. [2]

$|\overline{AF}| = \sqrt{(-3.2)^2 + (-4.5)^2 + 6.1^2} = 8.23$

c. Find FÂO, the angle the rope makes with the platform. [5]

$O\ (0,0,0)$

$\overrightarrow{AO} \begin{pmatrix} -3.2 \\ -4.5 \\ 0.3 \end{pmatrix}$

$|\overrightarrow{AO} \times AF| = |\overrightarrow{AO}||\overline{AF}| \sin\theta$

$$\frac{\sqrt{(-26.1)^2 + (18.56)^2 + 0^2}}{\left(\sqrt{(-3.2)^2 + (-4.5)^2 + (0.3)^2}\right)\left(\sqrt{(-3.2)^2 + (-4.5)^2 + (6.1)^2}\right)} = \sin\theta$$

$\theta = 44.7°$

$\overline{\text{IF}}$ using dot products ?