Poisson Distribution

Distribution of Rare Events

Conditions for Poisson Distribution

Know this

1. Events occur uniformly. For a small interval the probability of the event occurring is proportional to the size of the interval.
2. The probability of more than one occurrence in the small interval is negligible. Events must not occur simultaneously.
3. Each occurrence must be **independent** of others.
4. The event must occur at random.

The parameter for the Poisson random variable is λ **(lambda)**, it is the rate and the average or mean number of occurrences over the whole interval.

Setting $\mu = \lambda t$, we define the Poisson probability distribution as:

$$P(X = x) = \frac{e^{-\mu} \times \mu^{x}}{x!}, x = 0, 1, 2, \ldots$$

The Poisson random variable is the number of such occurrences in the fixed interval.

Examples:

- X earthquakes in Japan **per** year over 6.0 on the Richter scale.
- X defects **per** 5km of internet cable
- X car accidents on a corner **per** month

An example 1:

Cars have been observed to pass a given point on a back road at a rate of 0.5 cars per hour. Find the probability that no cars pass this point in a two-hour period.

From the information we have $\lambda = 0.5$.

We define the random variable X as the number of cars that pass the given point in a two-hour period.

This means our parameter $\mu = \lambda \times 2 = 0.5 \times 2 = 1$

So our Poisson distribution is $P(X = x) = \frac{e^{-1} \times 1^x}{x!}, x = 0, 1, 2, \ldots$

So $P(X = 0) = \frac{e^{-1} \times 1^0}{0!} = 0.3679.$ Check using GDC, Distribution

D: poissonpdf(

4 steps:

i. Justify that the scenario fits the conditions for a Poisson distribution.
ii. Determine the ‘base’ rate λ.
iii. Define the random variable X.
iv. Determine the parameter, μ, that corresponds to the random variable in step iii.

Example 2:

Faults occur on a piece of string at an average of one very three metres. Bobbins, each containing 5 metres of this string, are to be used. What is the probability that a randomly selected bobbin will contain:

i. two faults

ii. At least 2 faults.

$$\lambda = \frac{1}{3} \text{ (one in 3 metres)}$$

$$\mu = \lambda \times 5 = \frac{1}{3} \times 5 = \frac{5}{3}$$

i. $P(X = 2) = \dfrac{e^{-\frac{5}{3}} \times \left(\frac{5}{3}\right)^2}{2!} \approx 0.2623$

Example 2:

Faults occur on a piece of string at an average of one very three metres. Bobbins, each containing 5 metres of this string, are to be used.
What is the probability that a randomly selected bobbin will contain:

i. two faults
ii. At least 2 faults.

ii. $P(X \geq 2)$ has unlimited calculations so we use the fact that the sum of the probabilities = 1

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \frac{e^{-\frac{5}{3}} \times \left(\frac{5}{3}\right)^0}{0!} - \frac{e^{-\frac{5}{3}} \times \left(\frac{5}{3}\right)^1}{1!} = 0.4963 \end{aligned}$$

Example 3:

A radioactive sourse emits particles at an average rate of one very 12 seconds. Find the probability that at most 5 particles are emitted in one minute.

i. Check conditions for Poisson
ii. We are given that $\lambda = \frac{1}{12}$ (one in 12 seconds)
iii. Random Variable X is the number of particles emitted after 1 minute (or 60 seconds)
iv. Parameter $\mu = \lambda \times 60 = \frac{1}{12} \times 60 = 5$

Example 3:

A radioactive sourse emits particles at an average rate of one very 12 seconds. Find the probability that at most 5 particles are emitted in one minute.

$$P(X \leq 5) = \frac{e^{-5} \times 5^0}{0!} + \frac{e^{-5} \times 5^1}{1!} + \frac{e^{-5} \times 5^2}{2!} + \cdots \frac{e^{-5} \times 5^5}{5!} \approx 0.6160.$$

Check: poissCdf(5, 0, 5)

Mean and Variance of the POISSON DISTRIBUTION.

For a random variable X having a Poisson distribution with parameter μ, then:
$E(X) = \mu$ and $Var(X) = \mu$.

Example 4:

A data entry operative finds that, on average, they make two mistakes every three screens. Assuming that the number of errors per screen follows a Poisson distribution, what are the chances that there will be two mistakes on the next screen they enter?

$E(X) = \frac{2}{3}$ and $\mu = \frac{2}{3}$

$$P(X = 2) = \frac{e^{-\frac{2}{3}} \times \left(\frac{2}{3}\right)^2}{2!} \approx 0.1141$$

Example 5

The number of flaws in metal sheets 100 cm by 150 cm is to follow a Poisson distribution. On inspecting a large number of these metal sheets, it is found that 20% of these sheets contain at least one flaw.

i. Find the average number of flaws
ii. Find the probability of observing one flaw in a metal sheet selected at random.

i. Let X be the random variable of the number of flaws per 100 cm by 150 cm metal sheet.

Example 5

The number of flaws in metal sheets 100 cm by 150 cm is to follow a Poisson distribution. On inspecting a large number of these metal sheets, it is found that 20% of these sheets contain at least one flaw.

i. Let X be the random variable of the number of flaws per 100 cm by 150 cm metal sheet.

We have $X \sim Pn(\mu)$ where μ is to be determined.

Knowing $P(X \geq 1) = 0.2$, we know the opposite $P(X = 0) = 0.8$

$P(X = 0) = \frac{e^{-\mu} \times \mu^0}{0!}$ or $e^{-\mu} = 0.8$

$\mu = 0.2231$ the average number of flaws per sheet is 0.2232

Example 5

The number of flaws in metal sheets 100 cm by 150 cm is to follow a Poisson distribution. On inspecting a large number of these metal sheets, it is found that 20% of these sheets contain at least one flaw.

ii. $$P(X = 1) = \frac{e^{-0.2231} \times 0.2231^1}{1!} \approx 0.1785$$