Unit 3:Geometry and Trigonometry

VECTOR KINEMATICS

AIHL 3.12

Vectors are used to represent forces, acceleration, velocity and momentum and enable the motion of an object to be predicted and described.

CONSTANT VELOCITY

$$\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$$

$\boldsymbol{a}$ represents the initial position of the object and vector
$\boldsymbol{b}$ represents the velocity, which takes into account the direction of motion.
parameter λ represents time

Therefore, the equation can be written as

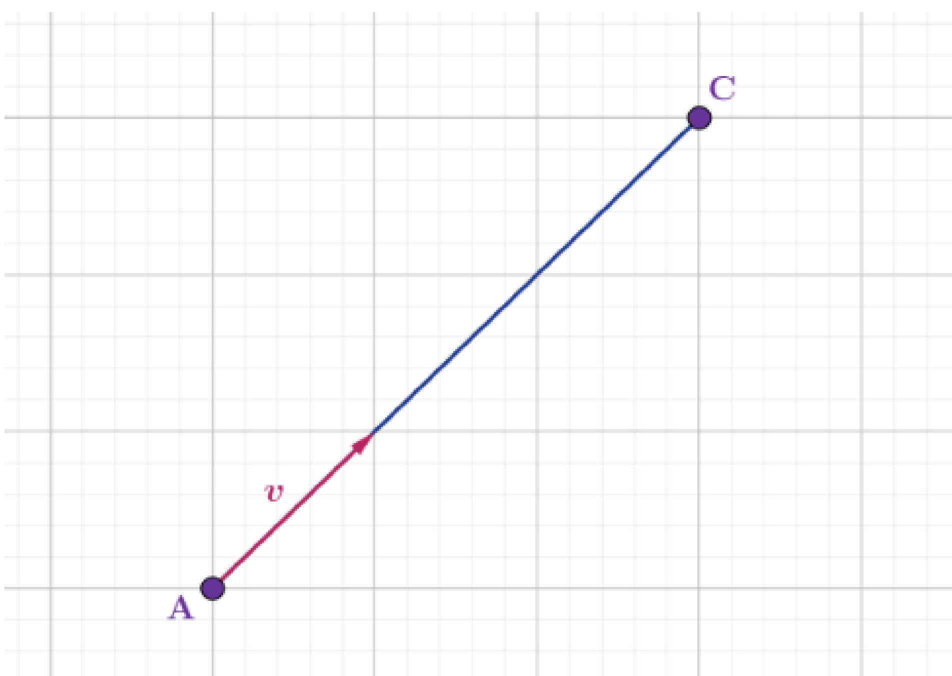
$$\boldsymbol{r}=\boldsymbol{r}_0+\boldsymbol{v}t$$

Where

$\boldsymbol{r}$ is the position vector at time t,

$\boldsymbol{r}_0$ is the initial position vector and
$\boldsymbol{v}$ is the velocity, which is constant.

Example 1. Consider a ship moving north-east with a speed of 30 km h^{-1} from port A, as shown below. How can you represent its movement using vectors?



$$V = \begin{pmatrix} 30 \cos 45^\circ \\ 30 \sin 45^\circ \end{pmatrix} \qquad \text{so} \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} 30 \times \frac{1}{\sqrt{2}} \\ 30 \times \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V = \begin{pmatrix} 15\sqrt{2} \\ 15\sqrt{2} \end{pmatrix}$$

Example 2. A ship traveling from port A, the origin, travels with $\begin{pmatrix} 30 \\ 40 \end{pmatrix}$ velocity kilometers per hour.

a) How far will the ship be from the port after 10 hours?

$$|V| = \sqrt{30^2 + 40^2}$$

$$|V| = 50 \frac{\text{km}}{\text{h}} \times 10 \text{ h} \qquad |V| = 500 \text{ km}.$$

b) Write the coordinates of the ship relative to the port.

$$p = 10 \times \begin{pmatrix} 30 \\ 40 \end{pmatrix} \qquad p = \begin{pmatrix} 300 \\ 400 \end{pmatrix} \qquad p\,(300, 400)$$

EAST NORTH

+ +

Example 3. A drone is flying in a straight line starting from point (2, 1, 2) with constant speed in ms^{-1}. After 10s its position is $\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$. Find its velocity vector and its speed. (All positions are in meters relative to a fixed origin.)

let $v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ direction vector

$r = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

at $t = 10$ $\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$

So, $\begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + 10\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

$4 = 2 + 10v_x$ $\quad 3 = 1 + 10v_y$ $\quad 3 = 2 + 10v_z$

$v_x = \frac{1}{5}$ $\quad v_y = \frac{1}{5}$ $\quad v_z = \frac{1}{10}$

$v = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$

speed $= |v| = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{10}\right)^2} = \frac{3}{10}$ m/s

Example 4. The position vector at time t seconds of a moving object, P , relative to a fixed point, O , is given by

$$\overrightarrow{OP} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} - \boldsymbol{k}) \rightarrow \overrightarrow{OP} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

a) Find the coordinates of the initial position of the object.

when $t = 0$

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The initial position is at (1,1,1)

b) Find the coordinates of the object when t = 2 seconds.

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$\overrightarrow{OP} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

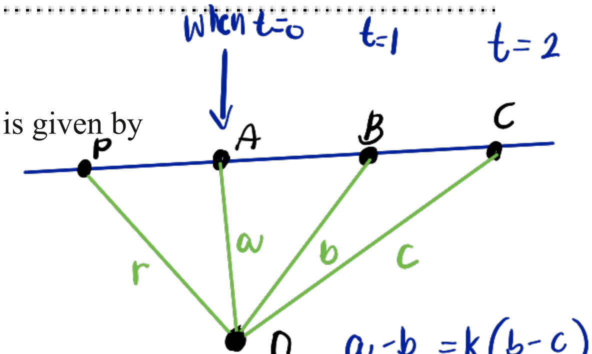
→ magnitude of the displacement vector

c) Hence, find the distance the object travelled in 2 seconds if the velocity is given in m/s. Give your answer to 3 significant figures.

$$d=\sqrt{(5-1)^2+(1-1)^2+(-1-1)^2} \qquad d=4.47$$

$$=2\sqrt{5}$$

Example 5. The position vector at time *t* seconds of a moving object is given by



Prove it as a straight line

$$\overrightarrow{OP}=\boldsymbol{r}=\begin{pmatrix}-t^2\\1\\2t\end{pmatrix}$$

Show that the path of the object is not a straight line.

$a-b=k(b-c)$

>assume it follows a line, with three more collinear points

$t=0$ then $a=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$t=1$ then $b=\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$t=2$ then $c=\begin{pmatrix}-4\\1\\4\end{pmatrix}$

$$a-b=k(b-c)$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}-\begin{pmatrix}-1\\1\\2\end{pmatrix}=k\begin{pmatrix}-1\\1\\2\end{pmatrix}-\begin{pmatrix}-4\\1\\4\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\-2\end{pmatrix}=k\begin{pmatrix}3\\0\\-2\end{pmatrix}$$

solve for k:

$1=3k$ $\qquad 0=0k \qquad -2=-2k$

$\frac{1}{3}=k$ $\qquad\qquad\qquad 1=k$

k are inconsistent

so the points are not on the same line.

Example 6. Aircraft A has an initial position vector at 0900 given by $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ km relative to an airport and is moving with velocity $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ kmh^{-1}. A second aircraft, B, has an initial position vector at 0900 given by $\begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix}$ km and is moving with velocity $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ kmh^{-1}. Show that the two aircraft will collide at a point P and write down the coordinates of P.

$\vec{A} = \vec{B}$

$$\vec{A} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} 4 \\ 5 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$0 + 3t = 4 - t \Rightarrow 4t = 4 \Rightarrow t = 1$

$2 + 4t = 5 + t \Rightarrow 3t = 3 \Rightarrow t = 1$

$3 + 5t = 5 + 3t \Rightarrow t = 1$

When $t = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + (1) \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 8 \end{pmatrix}$$

The two aircraft will collide at $(3,6,8)$ at 1000.

Example 7. Two objects are moving with a constant velocity in a straight line.

Initially, object A is at a point with coordinates (1, 3, 0) m, relative to a fixed origin, and is

moving with velocity $\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ ms^{-1}.

Object B is initially at a point with coordinates (-1, 2, -2)m, relative to the fixed origin, and is moving with

velocity $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ ms^{-1}.

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a) Show that these objects do not collide.

b) Find an equation in terms of t for the distance, d, between the objects.

c) Find their closest approach. → minimum

d) Find the velocity vector that the object B must have so that the two objects will collide after 3s.

a. $\vec{A} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ $\quad \vec{B} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$1 - 3t = -1 + 2t$ $\quad 3 + 4t = 2 - t$ $\quad 2t = -2 + 3t$

$5t = -2$ $\quad 5t = -1$ $\quad 2 = t$

$t = -\frac{2}{5}$ $\quad t = -\frac{1}{5}$ $\quad \therefore$ inconsistent, so they don't collide

b. $r_A = \begin{pmatrix} 1 - 3t \\ 3 + 4t \\ 2t \end{pmatrix}$ $\quad r_B = \begin{pmatrix} -1 + 2t \\ 2 - t \\ -2 + 3t \end{pmatrix}$

$$d = \sqrt{[(-1+2t) - (1-3t)]^2 + [(2-t) - (3+4t)]^2 + [(-2+3t) - (2t)]^2}$$

$$d = \sqrt{(-2+5t)^2 + (-1-5t)^2 + (-2+t)^2}$$

$$d = \sqrt{4 - 20t + 25t^2 + 1 + 10t + 25t^2 + 4 - 4t + t^2}$$

$$d = \sqrt{9 - 14t + 51t^2}$$

c. $d = \sqrt{9 - 14t + 51t^2}$

$(0.137, 2.835)$

Their closest approach is at 2.835 km at 0.137 s

d. $t = 3$

$$r_A = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \qquad r_B = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$1 - 3(3) = -1 + 3a \Rightarrow -7 = 3a \Rightarrow a = -\frac{7}{3}$$

$$3 + 4(3) = 2 + 3b \Rightarrow 13 = 3b \Rightarrow b = \frac{13}{3}$$

$$2(3) = -2 + 3c \Rightarrow 8 = 3c \Rightarrow c = \frac{8}{3}$$

The new velocity vector for B to collide with after 3 sec is

$$\begin{pmatrix} -\frac{7}{3} \\ \frac{13}{3} \\ \frac{8}{3} \end{pmatrix} m/s.$$