GRAPH THEORY

AIHL 3.14, 3.15

A **graph** is a mathematical structure that is used to **represent objects and the connections between them**. They can be used in modeling many real-life applications, e.g. electrical circuits, flight paths, maps etc.

Parts of a Graph

1. A **vertex (point)** represents an object or a place
   Adjacent vertices are connected by an edge
   The degree of a vertex can be defined by how many edges are connected to it

2. An **edge (line)** forms a connection between two vertices
   Adjacent edges share a common vertex
   An edge that starts and ends at the same vertex is called a loop
   There may be multiple edges connecting two vertices

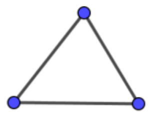
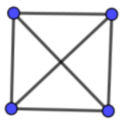
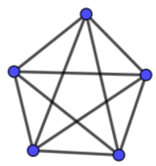
Types of Graphs

1. A **complete graph** is a graph in which each vertex is connected by an edge to each of the other vertices
   The edges in a weighted graph are assigned numerical values such as distance or money
   The edges in a directed graph can only be travelled along in the direction indicated the in-degree of a vertex is the number of edges that lead to that vertex the out-degree is the number of edges that leave from that vertex

2. A **simple graph** is undirected and unweighted and contains no loops or multiple edges
   Given a graph G, a subgraph will only contain edges and vertices that appear in G
   In a connected graph it is possible to move along the edges and vertices to find a route between any two vertices
   If the graph is strongly connected, this route can be in either direction between the two vertices

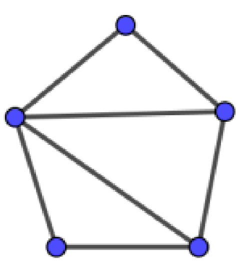
3. A **tree** is a graph in which any two vertices are connected by exactly one path
   A **spanning tree** is a subgraph, which is also a tree, of a graph G that contains all the vertices from G

K_n is a complete simple graph

K_1	
K_2	
K_3	
K_4	
K_5	

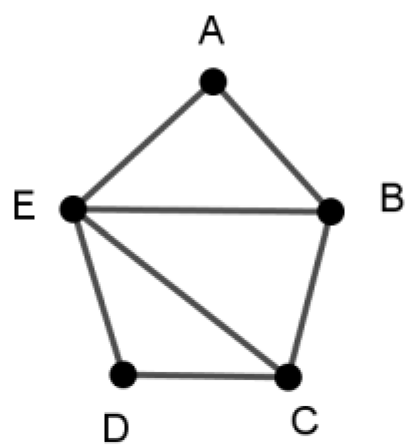
How many edges does K_n have?

Simple graphs

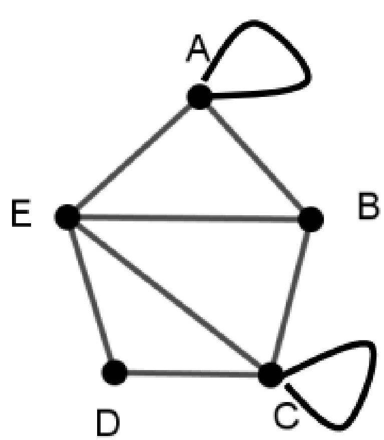


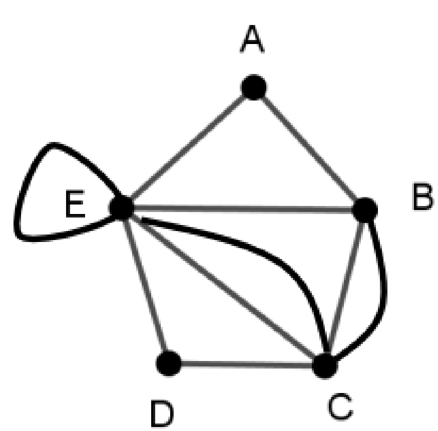
Consider a company of five people: A, B, C, D, E. If they are all friends to each other, this “warm friendship” can be represented by a complete graph K_5.

Degree of a Vertex



Non-simple Graphs





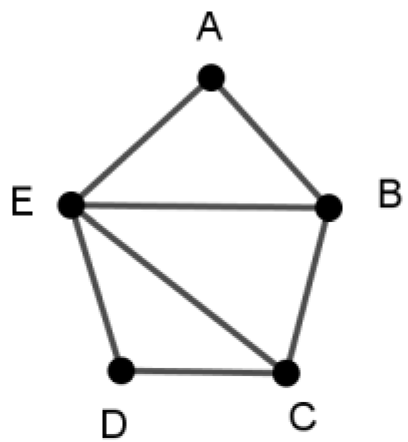
Formal Definition of Graphs

A graph G [or a graph **G(V,E)**] consists of:

a set of vertices **V**

a set of edges between vertices **E**

Example 4. Name the vertices, edges, at least two adjacent vertices, at least one adjacent edge, the degrees of each of the vertices. State as well whether the graph is simple and reason out.

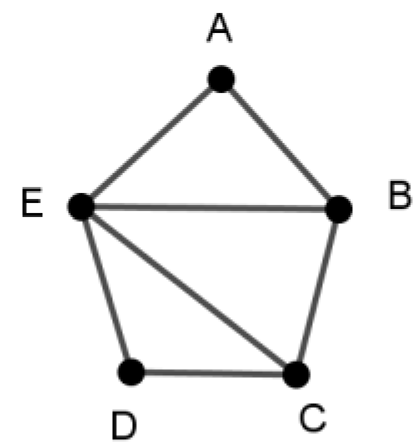


Routes of the graph

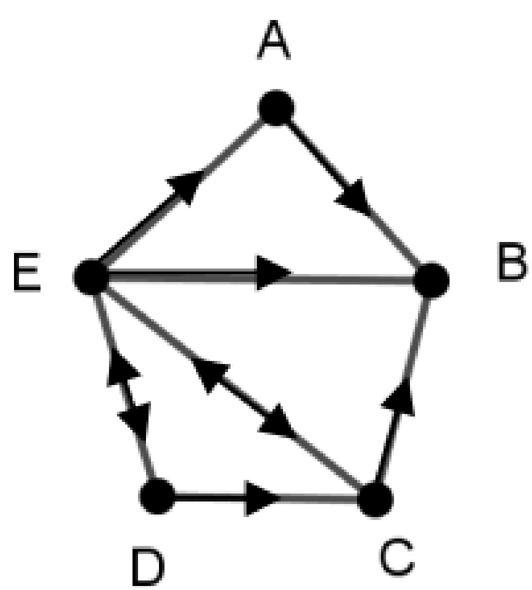
1. **Path** : no vertex is revisited
2. **Trail** : no edge is revisited
3. **Walk** : Walk as you wish
4. **Cycle** : derived from a path which closes to the initial vertex
5. **Circuit**: a trail which closes to the initial vertex

Adjacency Matrix. An adjacency matrix is a **square matrix** used to represent a finite graph. The elements of the matrix indicate **whether pairs of vertices are adjacent or not** in the graph. In the special case of a finite simple graph, the adjacency matrix is a-matrix with zeros on its diagonal

	A	**B**	**C**	**D**	**E**	
A						
B						
C						
D						
E						



Directed Graphs



	in					
		A	B	C	D	E	Out degree
from	A						
	B						
	C						
	D						
	E						
	In degree						

Weighted Graphs

Identify the length and the total weight of the path ABCD

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....

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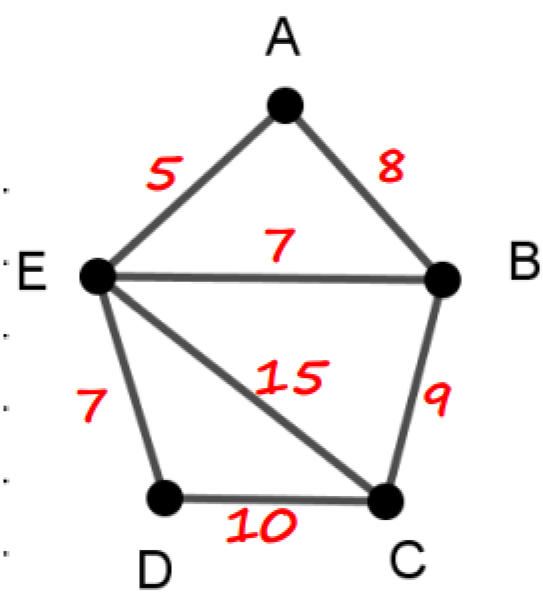
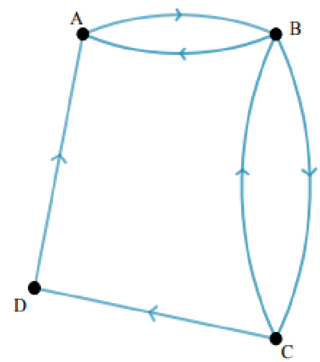


Represent the weighted graph by a weighted adjacency table

	A	B	C	D	E
A					
B					
C					
D					
E					

Practice Exercises:

1.



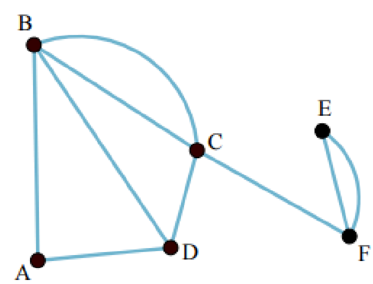
Choose **all** of the features that apply to the graph.

- ☐ Weakly-connected
- ☐ Strongly-connected
- ☐ Weighted
- ☐ Directed

[1]

2.

Consider the graph G shown opposite.

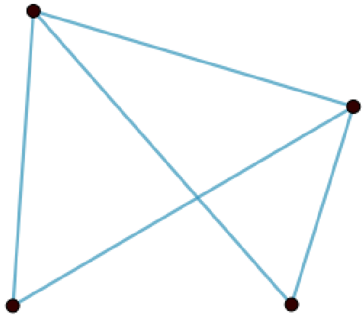


Choose whether the following statements about graph G are true or false.

	True	**False**
Vertices C and E are adjacent.	☐	☐
G is not connected.	☐	☐
Vertex C has degree 3.	☐	☐
G has 9 edges.	☐	☐
B is an even vertex.	☐	☐
G is a simple graph.	☐	☐

[2]

3. Complete the number of edges and vertices in the graph shown.



____ vertices

____ edges

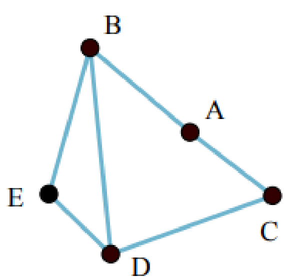
[2]

4.

	A	B	C	D	E
A		1	1		
B	1			1	1
C	1			1	
D		1	1		1
E		1		1	

Choose **all** of the graphs that represent the information shown in the table.

☐



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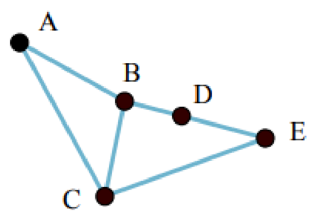

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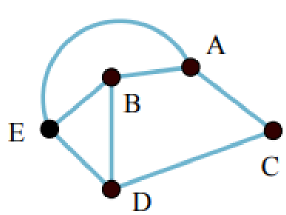

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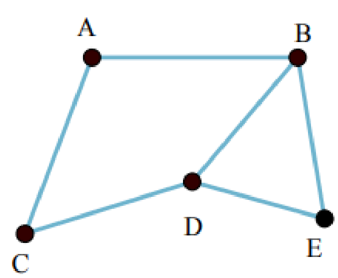
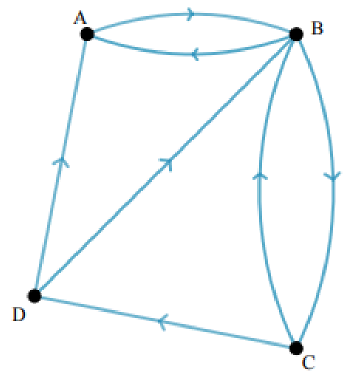


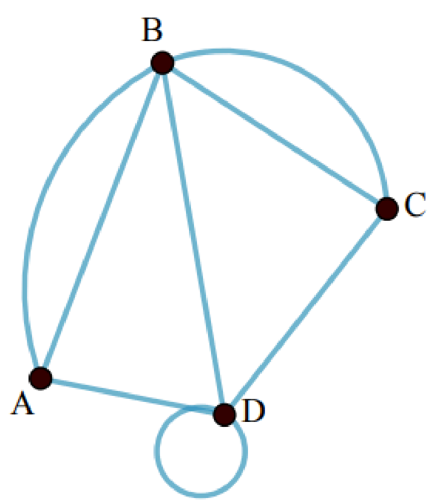
[1]

5. Fill in the adjacency matrix for the directed graph below.



Answer: ________________ [1]

6. Fill in the adjacency matrix for the graph below.



Answer: ________________ [1]

7.

The elements of the adjacency matrix represent the number of direct public transport routes between 4 towns A, B, C and D.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

a) Select the town which is not connected to any other town by a direct transport route.

☐ A ☐ D ☐ B ☐ C

b) Explain why the leading diagonal of the adjacency matrix only contains zeros.

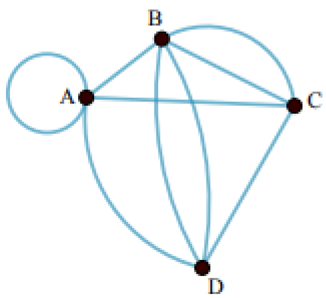
c) Explain why the matrix is symmetric. [3]

8.

Choose all the graphs that correspond to the adjacency matrix shown opposite.

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

☐



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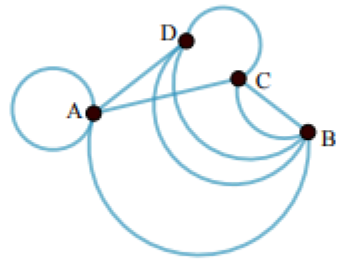

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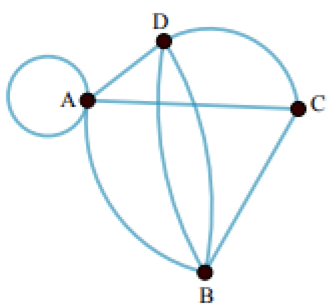

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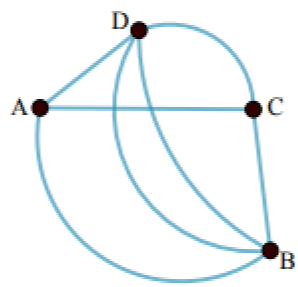


[2]

9. Draw a weighted graph represented by the table below.

	A	B	C	D	E
A			8	5	
B				9	12
C	8			4	
D	5	9	4		11
E		12		11	

[2]

10. Consider the table below where A, B , C and D are vertices in a graph.

	A	B	C	D
A		5	7	
B	5		6	4
C				
D	2		5	

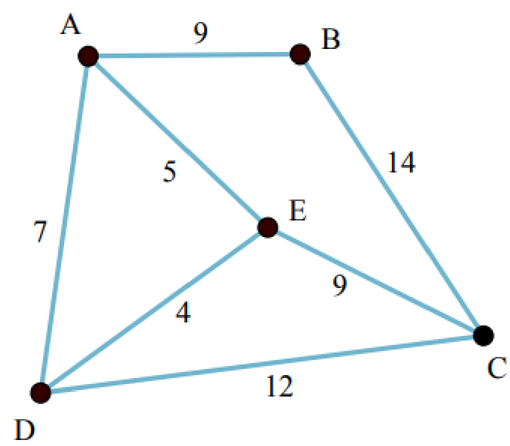
a) Draw a weighted, directed graph using the information in the table.

b) Using your graph, or otherwise, complete the table for the in-degree and out-degree of each vertex.

	in-degree	out-degree
A	____	____
B	____	3
C	____	3
D	1	____

[4]

11. Use the weighted graph to complete the table.



	A	B	C	D	E
A		9		7	5
B	9		____		
C		____		____	____
D	7		12		____
E	5		____	4	

[2]