PROPERTIES OF CURVES: INCREASING AND DECREASING FUNCTIONS, SIGN DIAGRAMS, STATIONARY POINTS, SECOND DERIVATIVE TEST

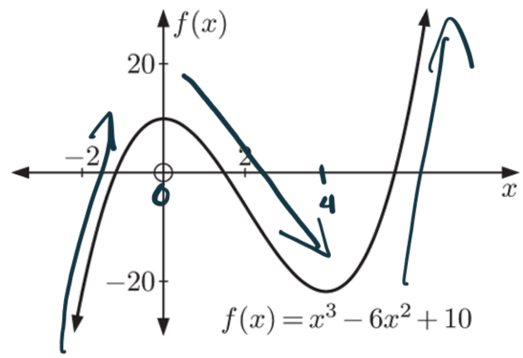
b. $x < 0$ ↑ $0 < x < 4$ ↓ $x > 4$ ↑

The graph of $f(x) = x^3 - 6x^2 + 10$ is shown alongside.

a Find $f'(x)$, and draw its sign diagram.

b Find the intervals where $f(x)$ is increasing or decreasing.

c. max / min



a. $f'(x) = 3x^2 - 12x$

$f'(x) = 0$

$3x^2 - 12x = 0$

$3x(x - 4) = 0$

$3x = 0 \quad x - 4 = 0$

$x = 0 \quad x = 4$

↳ max ↳ min

Sign diagram: + | 0 | − | 4 | +

face up $f''(a) > 0$ min

$f''(a) < 0$ max face down

$f'(-1) = \quad f'(1) = \quad f'(5) =$

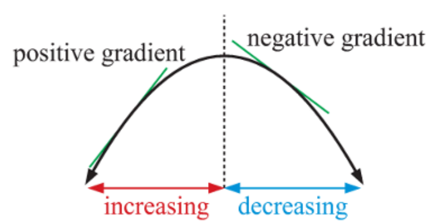
SECOND DERIVATIVE TEST → max / min

$f''(x) = 6x - 12$

max → ∩ $f''(0) = -12$ $f''(4) = 12$ ∪

↓ min

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f'(x) \leqslant 0$ for all x in S.



A sign diagram shows the intervals where a function has positive or negative outputs.

A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a **local maximum** or **local minimum**, or else a **stationary inflection**.

Example 2:

a Using technology to help, sketch a graph of $f(x) = x + \frac{1}{x}$.

b Find $f'(x)$.

c Draw a sign diagram for $f'(x)$.

d Determine the position and nature of any stationary points.

b. $f(x) = x + x^{-1}$

$f'(x) = 1 - \frac{1}{x^2}$

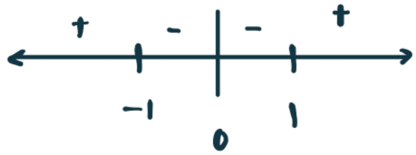
$0 = 1 - \frac{1}{x^2}$

$\frac{1}{x^2} = 1$

$1 = x^2$

$1 = \pm x$

c.



d. 2nd derivative test

$f'(x) = 1 - x^{-2}$

$f''(x) = \frac{2}{x^3}$

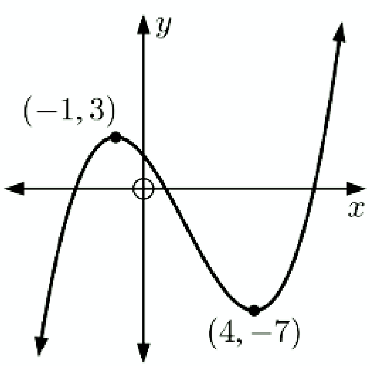
$f''(-1) = \frac{2}{-1} = -2 < 0$ $\quad f''(1) = 2 > 0$

max at $(-1,-2)$ $\quad$ min at $(1,2)$

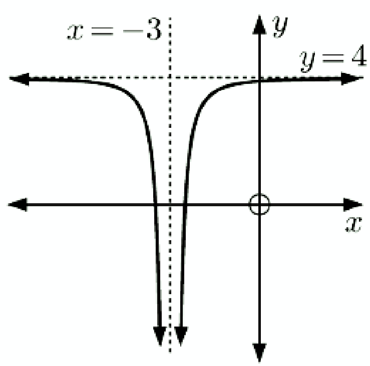
Example 3:

Find intervals where the given function is increasing or decreasing.

a



b



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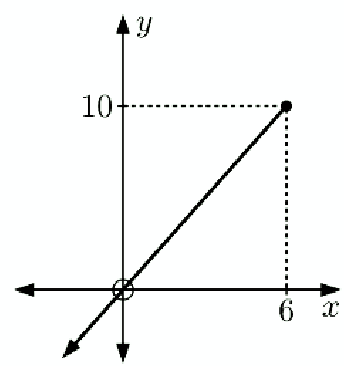


APPLICATIONS OF DIFFERENTIATION:

OPTIMISATION

Optimisation is the process of finding the maximum or minimum value of a function. The solution is often referred to as the optimal solution.

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

Step 5: Identify the optimal solution, also considering end points where appropriate.

Step 6: Write your answer in a sentence, making sure you specifically answer the question.

Example 4:

When a manufacturer makes x items per day, the profit function is $P(x) = -0.022x^2 + 11x - 720$ pounds. Find the production level that will maximise profits.

$P'(x) = -0.044x + 11$

$P'(x) = 0$

$0 = -0.044x + 11$

$-0.044x = -11$

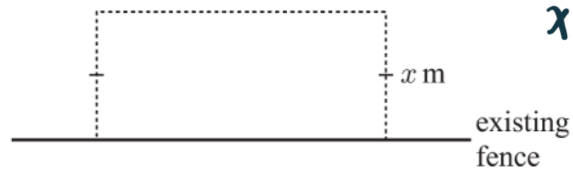
$x = 250$

$\therefore$ Production is maximised at 250 items per day

Example 5: Perimeter Area $P = L + 2x$

$A = xy$

60 metres of fencing is used to build a rectangular enclosure along an existing fence. Suppose the sides adjacent to the existing fence are x m long.



a Show that the area A of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b Find the dimensions which maximise the area of the enclosure.

a. $P = 60$ m

$60 = L + 2x$

$\rightarrow L = 60 - 2x$

$A = LW$

$= (60 - 2x)x$

$A(x) = x(60 - 2x)$ m^2

b. $A(x) = 60x - 2x^2$

$A'(x) = 60 - 4x$

$0 = 60 - 4x$

$4x = 60$

$x = 15$

1st deri

$+$ $-$

$A'(0)$ 15 $A'(16)$

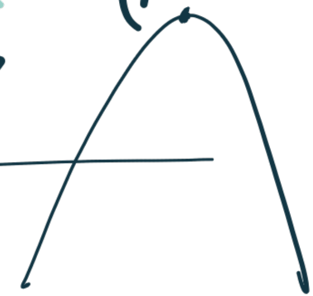
2nd deri

$A'(x) = 60 - 4x$

$A''(x) = -4$

x $A(x)$

$(15, 450)$

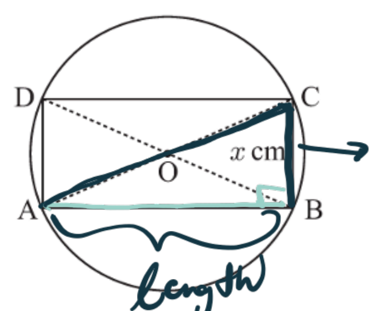


(b) $L = 60 - 2(15) = 30$

$\therefore$ The dimensions that would max the area 30×15 m

Example 6:

Consider a rectangle inscribed in a circle of diameter 10 cm. In the diagram alongside, suppose $BC = x$ cm.



a Show that if the area of the rectangle is A, then $A^2 = 100x^2 - x^4$.

b Find $\frac{d}{dx}(A^2)$ and hence find the value of x which maximises A^2.

c Hence find the dimensions of the largest rectangle which can be inscribed in the circle.

a, $AC^2 = AB^2 + BC^2$

$10^2 = AB^2 + x^2$

$AB = \sqrt{10^2 - x^2}$

Area = length $\times$ width

$A = AB\ x$

$A = x\sqrt{10^2 - x^2}$

$A^2 = x^2(100 - x^2)$

$A^2 = 100x^2 - x^4$

b, $\frac{d}{dx}(A^2) = 100x^2 - x^4$

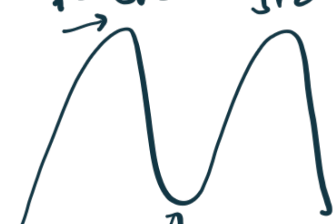
$\frac{dA^2}{dx} = 200x - 4x^3$

$0 = 200x - 4x^3$

$0 = 4x(50 - x^2)$

$x = 0$, $x = \pm 5\sqrt{2}$

$x = -5\sqrt{2}$ $5\sqrt{2}$



0

c, $5\sqrt{2} \times 5\sqrt{2}$

RATES OF CHANGE

$\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x**.

- If y increases as x increases, then $\dfrac{dy}{dx}$ will be positive.
- If y decreases as x increases, then $\dfrac{dy}{dx}$ will be negative.

RunMat → OPTN → F4 → $\dfrac{d}{dx}$ (underived function)

$2(50-t)^2$

Example 7:

In a hot, dry summer, water is evaporating from a desert oasis. The volume of water remaining after t days is $V = 2(50-t)^2$ m^3. Find:

a the average rate at which the water evaporates in the first 5 days

b the instantaneous rate at which the water is evaporating at $t = 5$ days.

a. $\dfrac{V(5)-V(0)}{5} = \dfrac{2(50-5)^2 - 2(50-0)^2}{5} = -190 \text{ m}^3/\text{day}$

b. $V = 2(50-t)^2$

$2500 -$

$V = 5000 - 200t + 2t^2$

$\dfrac{dV}{dt} = -200 + 4t \Big|_{t=5} = -180 \text{ m}^3/\text{day}$

Example 8:

Joseph starts to pump air in his bicycle tyre. From the time 1 second after he starts pumping, the volume of air in the tyre is given by $V = -\dfrac{1200}{t} + 1380$ cm^3.

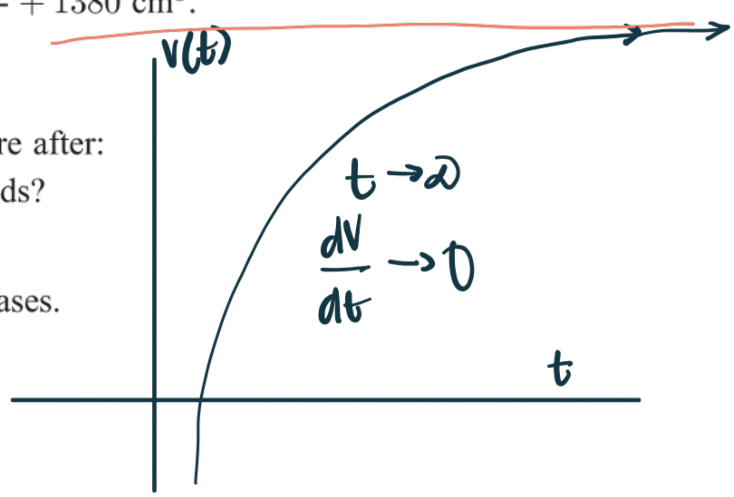
a Find $\dfrac{dV}{dt}$ and state its units.

b At what rate is air being pumped into the tyre after:

  i 2 seconds **ii** 6 seconds?

c Graph $V(t)$.

d Discuss what happens to $\dfrac{dV}{dt}$ as time increases.



a. $\dfrac{dV}{dt} = \dfrac{1200}{t^2}$ cm^3/s

b. i $\dfrac{dV}{dt}\Big|_{t=2} = 300 \text{ cm}^3/s$ ii $\dfrac{dV}{dt}\Big|_{t=6} = 33.3 \text{ cm}^3/s$

Example 9:

The cost function for producing x items each day is
$C(x) = -0.000\,007\,2x^3 + 0.0061x^2 + 18x + 14\,230$ dollars, where $0 \leqslant x \leqslant 1200$.

a Find $C(0)$ and explain what it represents.

b Find $C'(x)$ and explain what it represents.

c Find $C'(300)$ and explain what it estimates. 19.72

d Find the actual cost of producing the 301st item.

$C(301) - C(300)$?