Recall: Find the solutions to these functions

1. $y = 3x + 6$

if $x = 2$

$y = 12$

2. $g(x) = x^2 - 5x + 6$

$0 = (x-3)(x-2)$

$x = 3 \quad x = 2$

$\frac{d}{dx} e^{x} = e^{x}$

$\int e^{x} dx = e^{x} + C$

Try this.

3. Given that $\frac{ds}{dt} = e^{-t}$, find an expression for s in terms of t.

$ds = e^{-t} dt$

$s = -e^{-t} + C \rightarrow$ general soln

$s = -e^{-t} + 5 \quad s = -e^{-t} - 7 \rightarrow$ particular sol'n (point)

Differential Equations

A differential equation is an equation that relates one or more unknown functions and their derivatives. A differential equation tells you something about the rate of change of a quantity; in other words,it's an equation relating $\frac{dy}{dx}$ to x and y.

Example 1. The rate at which water flows out of a bucket is proportional to the square root of the volume remaining. Write this as a differential equation.

Let V = volume of water remaining (in Liters)

t = time (in seconds)

$$\frac{dV}{dt} = -K\sqrt{V}$$

Separation of Variables

If $\frac{dy}{dx} = f(x)g(y)$, then

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Example 2. [6 marks]
Solve the differential equation

$$(1 + x^2)\frac{dy}{dx} = 2xy^2$$

for y, which satisfies the initial condition $y(0) = -\frac{1}{2}$.

$$(1+x^2)\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{dx} = \frac{2xy^2}{1+x^2}$$

$$dy = \frac{2xy^2}{1+x^2} dx$$

$$\frac{dy}{y^2} = \frac{2x}{1+x^2} dx$$

$$\int \frac{dy}{y^2} = \int \frac{2x}{1+x^2} dx$$

$$\int y^{-2} dy = \int \frac{2x}{1+x^2} dx$$

$$\frac{y^{-1}}{-1} = \int \frac{du}{u} = \ln u + C$$

$$-\frac{1}{y} = \ln(1+x^2) + C$$

(General solution)

$$y = -\frac{1}{\ln(1+x^2)+C}$$

$$y(0) = -\frac{1}{2}$$

$$-\frac{1}{2} = -\frac{1}{\ln(1+0^2)+C}$$

$$C = 2$$

$$y = -\frac{1}{\ln(1+x^2)+2}$$

Example 3. [6 marks]

An experiment is conducted on a gas confined in a container. The pressure P measured in Pascals, and the volume V measured in cm^3 satisfy the differential equation

$$\frac{dP}{dV} = -\frac{P}{V}$$

Initially, the gas has a pressure of 20 000 Pascals and it is confined in a container with a volume of $100 cm^3$.

a. Solve the differential equation to show that [4]

$$P = \frac{2\,000\,000}{V}$$

$$\frac{dP}{dV} = -\frac{P}{V}$$

$$\int \frac{dP}{P} = \int \frac{dV}{-V}$$

$$\ln P = -\ln V + C$$

$$\ln P + \ln V = C$$

$$\cancel{e^{\ln}} PV = e^{C}$$

$$PV = e^{c} \quad (\text{let } A = e^{c})$$

$$PV = A$$

$$(20000)(100) = A$$

$$2{,}000{,}000 = A$$

$$PV = A$$

$$P = \frac{A}{V}$$

$$P = \frac{2\,000\,000}{V}$$

b. Calculate the pressure required to compress the gas to a different container that has a volume of $50\ cm^3$ [2]

$$P = \frac{2\,000\,00\cancel{0}}{5\cancel{0}}$$

$$P = 40000\ Pa$$

Example 4. [8 marks]

The decay rate of radium-226, a naturally occurring radioactive metal, is directly proportional to the amount observed at that instant. The half-life of radium-226 is 1600 years. At an initial measurement, there was 100 grams of radium-226 in a sample.

a. Find an expression for the amount of radium-226, R, in the sample in terms of t, where t is the time in years after the initial measurement. [6]

R = radium-226   t = time

$$\frac{dR}{dt} = -kR$$

$$dR = -kR\,dt$$

$$\int \frac{dR}{R} = \int -k\,dt$$

$$-k\int dt$$

$$\ln R = -kt + C$$

$$e^{\ln R} = e^{-kt + C}$$

$$R = e^{-kt + C}$$

$$R = e^{-kt} \cdot e^{C} \quad \text{let } A = e^{C}$$

$$R = Ae^{-kt}$$

$$R = 100 \quad t = 0$$

$$100 = Ae^{-k(0)}$$

$$50 = 100e^{-k(1600)}$$

$$\frac{1}{2} = e^{-1600k}$$

$$k = 0.00043$$

$$\boxed{R(t) = 100e^{-0.00043t}}$$

b. Find to the nearest gram the amount of radium-226 in the sample after 3000 years. [2]

$$R(3000) = 100e^{-0.00043(3000)}$$

$$= 28 \text{ grams}$$

Example 5. [8 marks]

Lisa continuously invests money into an account at a rate of 1500 euros per year. The account has an annual interest rate of 8%. The amount of money, A, in the account after t years satisfies the following differential equation:

$$\frac{dA}{dt} = 0.08A + 1500$$

The initial amount of money Lisa deposited into the account when it was set up was 730 euros.

a. Find an expression for A, in terms of t. [5]

$$\frac{dA}{dt} = 0.08A + 1500$$

$$\int \frac{dA}{0.08A + 1500} = \int dt$$

$$12.5 \ln(0.08A + 1500) = t + C$$

$$0.08A + 1500 = e^{0.08t + 0.08C}$$

$$A = Ke^{0.08t} - 18750$$

$$730 = Ke^{0.08(0)} - 18750$$

b.

i. Find $\int_0^6 (0.08A + 1500)dt$

ii. State what that value represents [3]

m = 0 —　　m = ∞ |

m = 1 /　　m = −1 \

m = 2 /　　m = −2 \

Slope Fields

A slope field is a graphical representation of the solutions to a first-order differential equation of a scalar function. This helps visualize the tangents to all of the solution curves.

Example 1.

Given the differential equation

$$\frac{dy}{dx} = 2x + y.$$

a. Construct a table showing the gradient of the slope field at the points with integer coordinates $-2 \leq x, y \leq 2$.

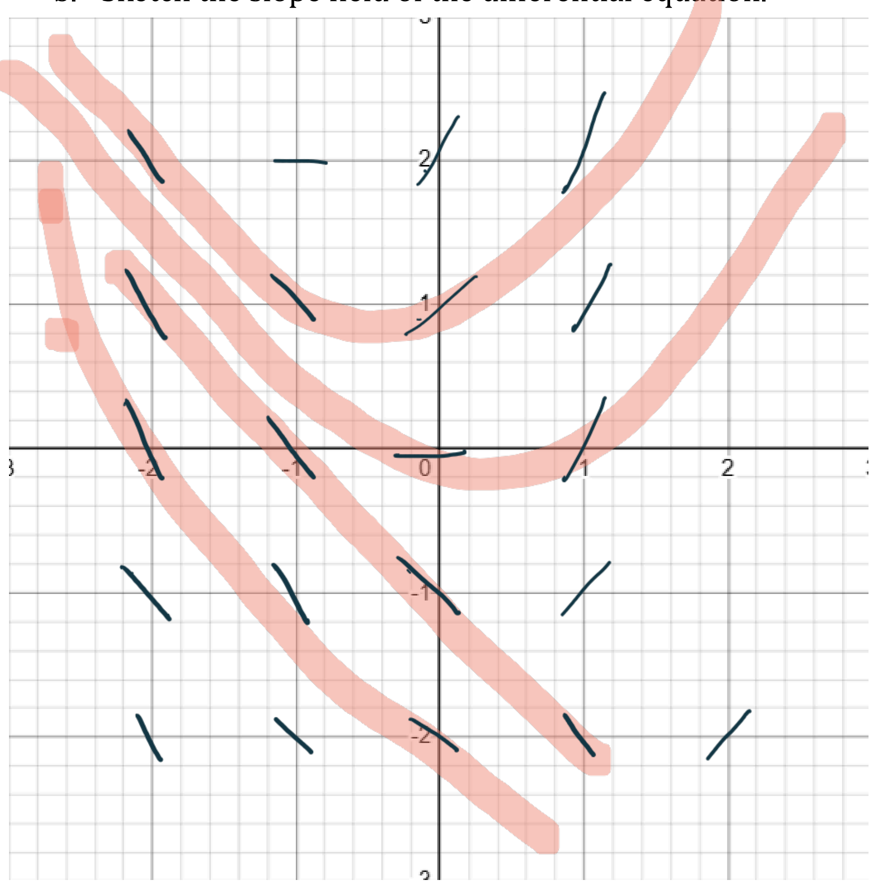
x

$\frac{dy}{dx} = 2(-1) + (-2)$

y

	-2	-1	0	1	2
-2	-6	-4	-2	-1	2
-1	-3	-3	-1	1	3
0	-4	-2	0	2	4
1	-3	-1	1	3	5
2	-2	0	2	4	7

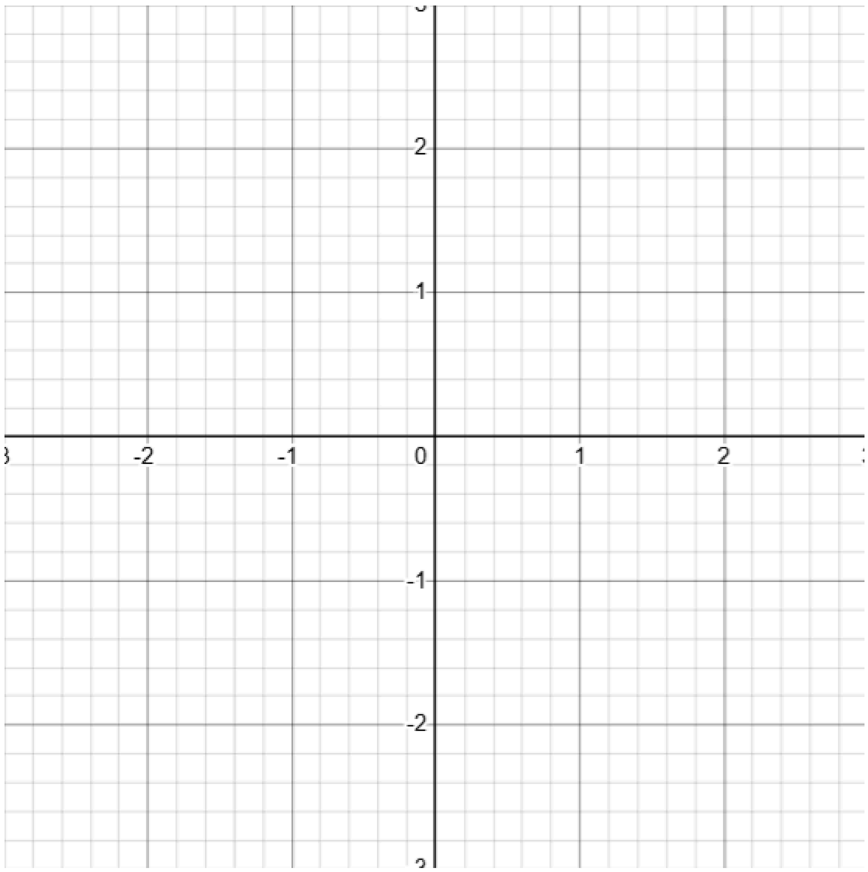
b. Sketch the slope field of the differential equation.



Example 2.

Consider the differential equation $\frac{dy}{dx} = 1 - x$, which has a general solution $y = x - \frac{1}{2}x^2 + c$.

a. Find the particular solution which passes through $(-1, 0)$.
b. Construct the slope field for the differential equation using the integer grid points for $x, y \in [-2, 2]$. Include the particular solution curve from a.

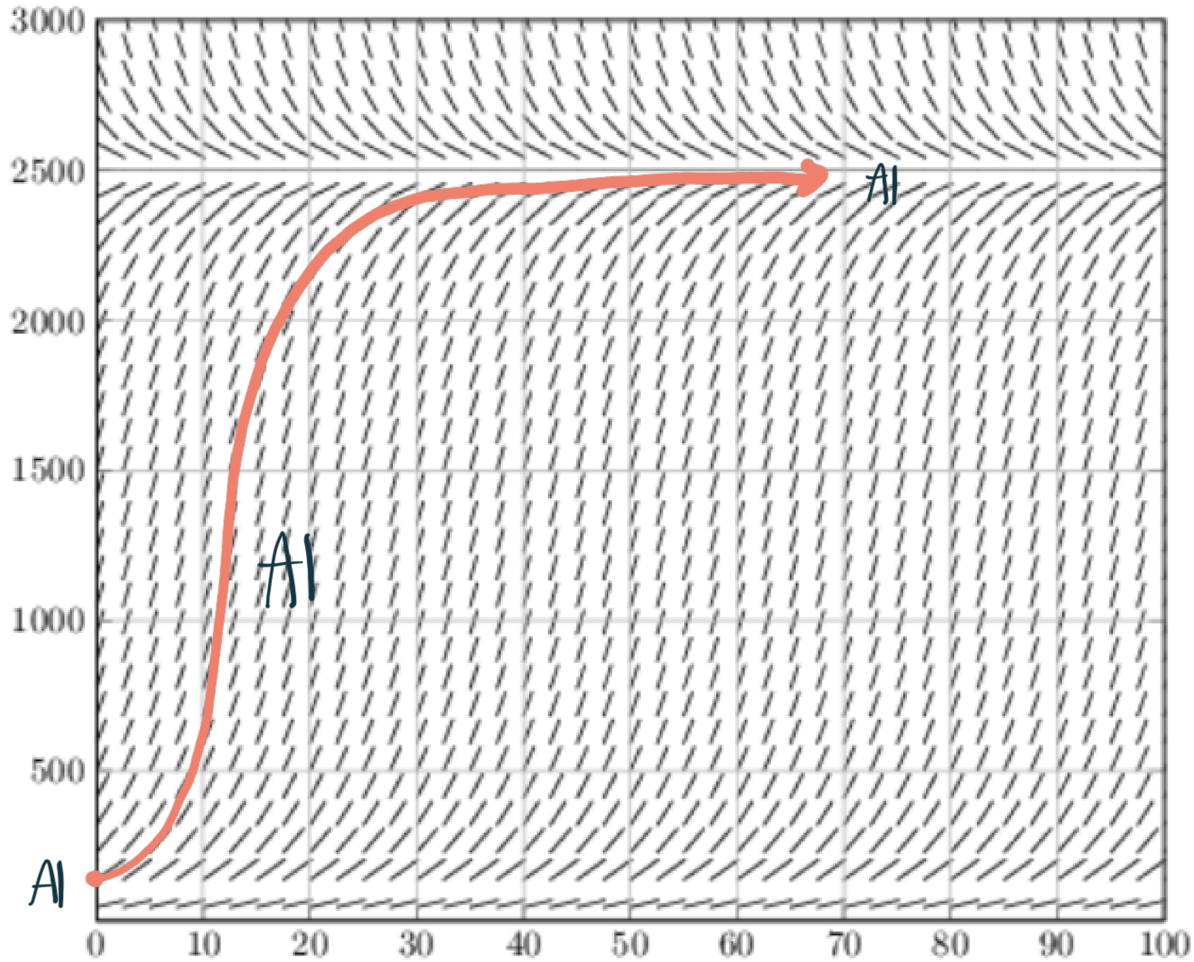


Example 3. [5 marks]

The population P of bears in a forest is expected to grow according to the model

$$\frac{dP}{dt} = 0.2P(1 - \frac{P}{2500})$$

where t is the time given in years from the beginning of 2020. The slope field for this differential equation is as follows:



a. Given that the population at the beginning of 2020 is 100, sketch the particular solution curve. [3]

b. Find the value at which the population approaches as time increases. [2]

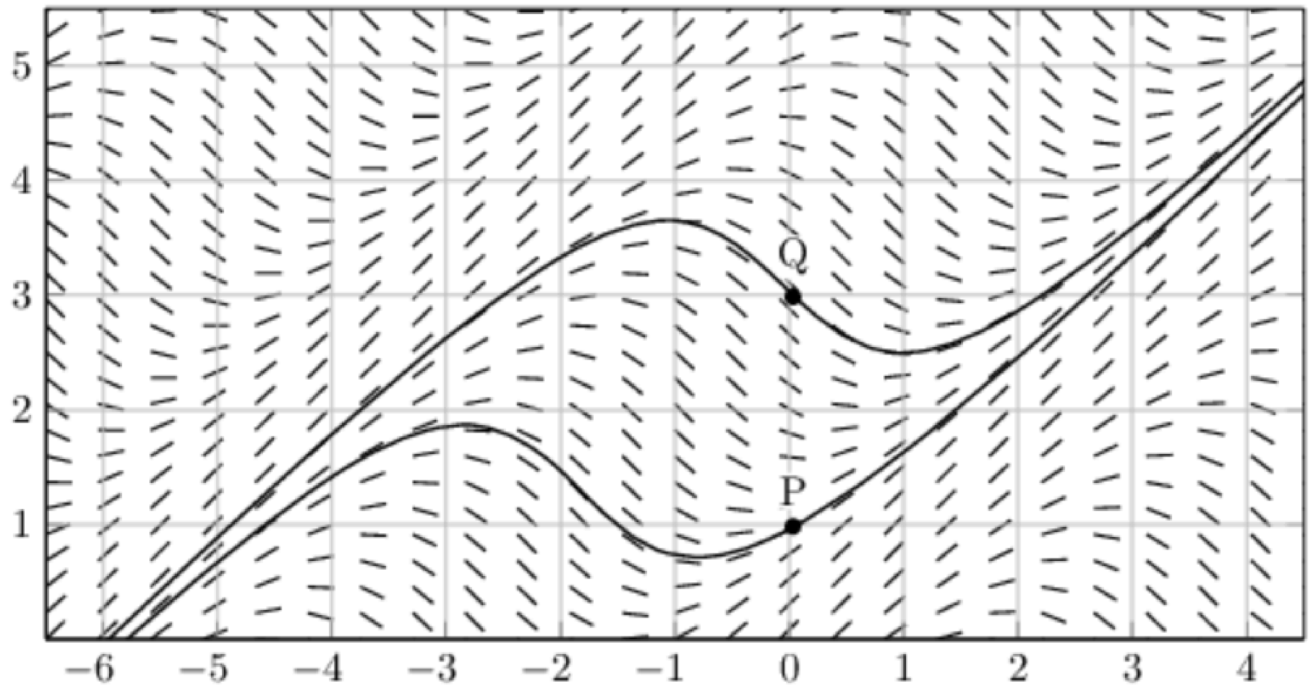
$t \to \infty \quad P \to 2500$

Example 4. [5 marks]

The diagram below shows the slope field for the differential equation

$$\frac{dy}{dx} = cos(x - y), \quad -6.5 \leq x \leq 4.5, \; 0 \leq y \leq 5.5$$

The graphs of the two solutions to the differential equation passing through points P(0,1) and Q(0,3) are drawn over the slope field.



For the two graphs given, the local maximum points lie on the straight line L_1.

a. Find the equation of L_1, giving your answer in the form y=mx + c. [3]

For the two graphs given, the local minimum points lie on the straight line L_2.

b. Find the equation of L_2, giving your answer in the form y=mx + c. [2]