Math 347 Worksheet

Worksheet 11: Permutations of n elements

November 7, 2018

1) Consider S_n the set of permutations of n elements.

(i) Prove that any element $s \in S_n$ can be written as the composite

$$s = t^1 \circ \ldots t^k,$$

where each t^i is a transposition.

Solution. *For any $n \geq 2$ we denote by $[n]$ the set $\{1, \ldots, n\}$, we let $t^n_{i,j}$ denote the permutation that swaps the numbers i and j, we also denote by e^n the identity function on $[n]$.*

We proceed by induction, the base case of $n = 2$ is trivially true, i.e. the only non-trivial permutation of two elements is $t_{1,2}$ and we have

$$e^2 = t_{1,2} \circ t_{1,2}.$$

Suppose that we proved the theorem for n. Consider σ a permutation of $\{1, \ldots, n+1\}$. There are two possibilities:

a) $\sigma(n+1) = n+1$, in this case the function[1]

$$\sigma|_{[n]} : [n] \to [n]$$

is a permutation and by the induction hypothesis one has

$$\sigma|_{[n]} = \prod_{k \in K} t^n_{i_k, j_k}$$

for some finite set K. This gives that

$$\sigma = \prod_{k \in K} t^n_{i_k, j_k},$$

and prove the result for $n+1$, in this case.

b) $\sigma(n+1) = m$, for $m \neq n+1$. Then we consider the permutation

$$t^{n+1}_{m,n+1} \circ \sigma.$$

However, this reduces to the first case, and we are finished with the proof.

(ii) Fix $\ell \in [n]$. Prove that the transpositions in the above can all be taken to be

$$t^i = t_{\ell,j}$$

for some $j \in [n]$, where $t_{\ell,j}$ is the transposition that swaps ℓ and j.

[1] See Exercise 1 from Homework 4 for what this notation means.

Solution. *It is enough to prove that for any $i, j \in [n]$ one can write*

$$t^n_{i,j} = t^n_{\ell,i} \circ t_{\ell,j} \circ t_{\ell,i},$$

which is clear by inspection.

Indeed, by (i) we can write any σ as a product

$$\prod_{k\in K} t^n_{i_k,j_k} = \prod_{k\in K} t^n_{\ell,i_k} \circ t_{\ell,j_k} \circ t_{\ell,i_k}.$$

2) Prove that for any element $s \in S_n$ there exists $k \le n$ such that[2]

$$s^k = e.$$

Solution. *Suppose by contradiction that there exists $\sigma \in S_n$ such that for all $i \in [n]$*

$$\sigma^{n+1}(i) = i.$$

This implies that the set

$$\{i, \sigma(i), \sigma^2(i), \ldots, \sigma^{n+1}(i)\} \tag{1}$$

has $n+1$ distinct elements[3]*. However, the σ is a function from $[n]$ to $[n]$, which implies that the set (1) is a subset of $[n]$, which is a contradiction.*

3) Draw a functional digraph of an element $s \in S_n$. What does the condition from the previous exercise mean in terms of the graph?

Solution. *It means that every path in the digraph has length less than or equal to n.*

4) For a natural number n a *partition* of n is a way of writing n as a sum of positive numbers.

(i) list the partitions of 6;

Solution. *One has* 6, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$ *and* $1+1+1+1+1+1$.

(ii) prove that the number of partitions of n with k parts equals the number of partitions of n with largest part k.

Solution. *(Use Young diagrams.)*

5) (Extra) Consider a square in $\mathbb{R}^2$ with vertices $(1,1), (1,-1), (-1,-1)$ and $(-1,1)$. Let r be the function $r : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation of 90 degrees, let $s : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that reflects the points around the line $x = y$ (Make a drawing.). Let D_4 be the set of functions obtained by considering compositions of r or s any number of times.

(a) prove that D_4 is finite and compute its size;

(b) prove that any element $a \in D_4$ preserves the square;

(c) prove that D_4 is not equal to S_n for any n. Can you find an n such that $D_4 \subset S_n$?

[2] Here s^k means one composes s with itself k times.

[3] Justify this to yourself if it is unclear.