Hangjian Su — Motivic Chern classes & Iwahori invariants of principal series.

joint w/ P. Aluffi, L. Mihalcea & J. Schürmann.
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① The problem.

$F$ a non Arch. local field, $\mathcal{O}_F \subseteq F$ its ring of ints.
$k_F$ = residue field $= \mathbb{F}_q$.

$G$ a reductive split alg. grp / $\mathcal{O}_F$. Fix $B \supseteq T$ a Borel and maximal torus.

The diagram:

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G(\mathcal{O}_F) \arrow[d] \\ B(k_F) \arrow[r, hook] & G(k_F) \end{tikzcd}$$

defines the Iwahori subgroup $I$

The Iwahori-Hecke alg. is $\mathcal{H} = (\mathbb{C}_c[I \backslash G(F)/I], *)$ convolution.

Principal series reps.

For $\tau$ an unramified character of $T(F)$.
(trivial on $T(\mathcal{O}_F)$)

$G(F)$.

$$\rightsquigarrow \mathrm{Ind}_{B(F)}^{G(F)} \tau := \left\{ \begin{array}{l} \text{locally cst. fcts} \\ f: G(F) \to \mathbb{C} \end{array} \middle| \begin{array}{l} f(bg) = \tau(b)\, \delta^{1/2}(b)\, f(g) \\ b \in B(F) \end{array} \right\}.$$

$$I(\tau) := \left(\mathrm{Ind}_{B(F)}^{G(F)} \tau\right)^I \text{"="} \mathbb{C}_c[B(F) \backslash_\tau G(F)/I] \circlearrowleft \mathcal{H}$$

by convolution.

Two bases in $I(\tau)$.

a/ Standard basis: $\{\varphi_w \mid w \in W\}$.

Relation between basis ?

$$G(F) = \coprod_{w \in W} B(F) w I$$

$$\varphi_w = 1_{B(F) w I}.$$

b/ Casselman basis: $\{f_w \mid w \in W\}$.

assume $\tau$ is regular.

For $x \in W$,

$$A_x : \mathrm{Ind}_{B(F)}^{G(F)} \tau \to \mathrm{Ind}_{B(F)}^{G(F)} x^{-1}\tau$$

$f_w$ is defined by the condition:

$$A_x(f_w)(1) = \delta_{x,w}.$$

Reformulation (Bump-Nakasuji)

Mobius transformation: define $\phi_w := \sum_{u \geq w} \varphi_u$.

$$\phi_w = \sum_{z \in W} m_{w,z} f_z$$

(?).

Example: for $w = id \in W$. $\phi_{id} = \sum_{w \in W} \varphi_w \in (\mathrm{Ind}_{B(F)}^{G(F)} \tau)^{G(O)}$

Spherical vector.

$$\phi_{id} = \sum_{z} \underbrace{\prod_{\substack{\alpha > 0 \\ s_\alpha z < z}} \frac{1 - q^{-1} e^{\check{\alpha}}(\tau)}{1 - e^{\check{\alpha}}(\tau)}}_{= m_{id, z}} f_z \qquad \check{\alpha} \text{ is a coroot}$$

Gindikin-Karpelevich formula.

<u>Conjecture:</u> (Bump-Nakasuji)

$G$ is simply-laced.

① $w \leq z \in W$.

If $P_{w_0 z, w_0 w}(q) = 1$ Kazhdan-Lusztig poly.

$w_0$ longest element in $W$.

$$\Rightarrow m_{w,z} = \prod_{\substack{\alpha > 0 \\ w \leq s_\alpha z \leq z}} \frac{1 - q^{-1} e^{\check{\alpha}}(\tau)}{1 - e^{\check{\alpha}}(\tau)} \cdot (*)$$

② $\prod_{w \leq s_\alpha z \leq z} (1 - e^{\check{\alpha}}) m_{w,z}$ is analytic on $\check{T}(\mathbb{C}) = \mathbb{C}^\times \otimes_{\mathbb{Z}} \check{X}(T)$

- $m_{w,z}$ depends on the unramified char. $\tau \in \check{T}(\mathbb{C})$.

<u>Refined conjecture:</u> (Naruse).

Observation: $G$ is A.D.E.

$P_{w_0 z, w_0 w}(q) = 1 \Leftrightarrow$ the opposite Schubert variety $Y(w) := \check{B}^- w \check{B}^\vee / \check{B}^\vee \subseteq \check{G}/\check{B}$ is smooth at the point $z\check{B} \in \check{G}/\check{B}$

① $(*)$ holds $\Leftrightarrow$ $Y(w)$ is smooth at $z\check{B} \in \check{G}/\check{B}$.

② the same.

Steinberg variety.

<u>Strategy</u>

$K_{\check{G}\times\mathbb{C}^*}(St_{\check{G}}) \simeq \mathbb{H} = \mathbb{C}_c[I \backslash G(F)/I]$

Kazhdan-Lusztig.

$K_{T^\vee}(\check{G}/\check{B})$ Ginzburg. $\xleftrightarrow{\simeq}$ $I(\tau)$.

(dual) motivic Chern classes $\longleftrightarrow$ $\varphi_w$

answer the ? on this side.

fixed point basis $\longleftrightarrow$ $f_w$ ?

② Motivic Chern classes.

(K-theoretic generalization of MacPherson classes).

$X/\mathbb{C} \rightsquigarrow K^\circ(Var/X)$

$\rightsquigarrow K^\circ(Coh(X)) =: K(X)$.

<u>Thm</u> [Brasselet-Schurmann-Yakura]:

$\exists!$ natural transformation

formal variable.

$MC_y : K^\circ(Var/X) \longrightarrow K(X)[y]$ s.t.

if $X$ is smooth, $MC_y([X \xrightarrow{id} X]) = \sum_i y^i [\Lambda^i T^* X]$.

For $X = \check{G}/\check{B}$. $X(w)^\circ = B^\vee w B^\vee/B^\vee$, $Y(w)^\circ = B^{\vee -} w B^\vee/B^\vee$

$[X(w)^\circ \hookrightarrow X] \in K^\circ(Var/X)$.

$$MC_y(X(w)^\circ) = MC_y([X(w)^\circ \hookrightarrow X]) \in K_{\check{T}}(\check{G}/\check{B}).$$

By localization

$$K_{\check{T}}(X)_{loc} := K_{\check{T}}(X) \otimes_{K_{\check{T}}(pt)} \mathrm{Frac}(K_{\check{T}}(pt)).$$

has a basis $\{[\mathcal{O}_{w\check{B}}] \mid w \in W\}$. $w\check{B} \in (\check{G}/\check{B})^{\check{T}}$.

- Hecke action (Lusztig)

$\check{\alpha}$ a simple root, $P_\alpha^\vee \supseteq B_\alpha^\vee \rightsquigarrow G^\vee/B^\vee \xrightarrow{\pi_\alpha} \check{G}/P_\alpha^\vee$.

$$T_{\alpha^\vee} := (1 + y\mathcal{L}_{\check{\alpha}}) \cdot (\pi_\alpha^* \pi_{\alpha,*} - id) \curvearrowright K_{\check{T}}(X).$$

$$\mathcal{L}_{\check{\alpha}} = \check{G} \times_{\check{B}} \mathbb{C}_{\check{\alpha}} \in \mathrm{Pic}_{\check{T}}(X).$$

<u>Thm:</u> $MC_y(X(w)^\circ) = T_{w^{-1}}([\mathcal{O}_{X(id)}]) \in K_{T^\vee}(X)[y]$.

$\mathcal{O}_{X(id)} :=$ structure sheaf of $\check{B} \in \check{G}/\check{B}$.

③ The proof.

- $\tau$ unramified of $T \to \tau \in \check{T}(\mathbb{C})$.

- $K_{\check{T}}(X)$ is a module over $K_{\check{T}}(pt) = K^0(\mathrm{Coh}(X)) = K^0(\mathrm{Var}/X)$.
$= \mathbb{Z}[\check{T}]$.

- $K_{\check{T}}(pt) \xrightarrow{ev_\tau} \mathbb{C}$.

Thm: $\exists!$ isomorphism of $H(W)$-modules

$$\varphi: K_{\check T}(\check G/\check B)[y] \underset{K_{\check T}(pt)}{\otimes} \mathbb{C}_\tau \xrightarrow{\sim} I(\tau).$$

s.t. • $y \mapsto -q = -|k_p|$.

• $[\mathcal{O}_{w\check B}] \mapsto f_w$.

• $MC_y(X(w)^\circ) \mapsto \varphi_w$.

$$\phi_w = \sum_{u \geq w} \varphi_u = \sum_z m_{w,z} f_z.$$

Cor: $m_{w,z} = \left(\dfrac{MC_{-q}(Y(w))|_z}{MC_{-q}(Y(z))|_z}\right)^\vee(\tau). \in K_{\check T}(pt).$ $\overset{\shortparallel}{\mathbb{Z}[\check T]}.$

$$(e^\lambda)^\vee = e^{-\lambda}.$$

Thm: $Y(w)$ is smooth at $z\check B \in \check G/\check B$.

$\Updownarrow$ Kumar.

$$[Y(w)]|_z = \prod_{\substack{\alpha>0 \\ w \not\leq s_\alpha z}} \alpha \in H^*_{\check T}(pt).$$

$\Updownarrow$ (AMSS)

$$MC_y(Y(w))|_z = \prod_{\substack{\alpha>0 \\ z s_\alpha z \geq w}} (1+ye^{z\alpha}) \prod_{\substack{\alpha>0 \\ w \not\leq z s_\alpha}} (1-e^{z\alpha}).$$

Refined Conj. 1 $\Leftarrow$ Thm' + Cor.

Conj. 2 is reduced to (via Cor.)

$$MC_y(Y(w)^\circ)|_z \text{ is divisible by } \prod_{\substack{\alpha>0 \\ w \not\geq z s_\alpha < z}} (1-e^{z\alpha^\vee})$$

$$\Uparrow$$

$$K_T^\vee(pt) \quad \Leftarrow \text{ follows from a GKM argument.}$$