Math 2102: Homework 2
Due on: Feb. 26, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $U \subseteq V$ be a subspace and denote by $\pi : V \to V/U$ the quotient map. Let $\pi^* \in \mathcal{L}((V/U)^*, V^*)$ denote the associated dual linear map.
   (i) Show that π^* is injective.
   (ii) Show that $\operatorname{range} \pi^* = U^0$.
   (iii) Conclude that π^* is an isomorphism between $(V/U)^*$ and U^0.

2. Consider $V = \mathbb{C}((x))$ the set of Laurent series, i.e. $a \in \mathbb{C}((x))$ is given by a series $a(x) = \sum_{i\in\mathbb{Z}} a_i x^i$, where $a_i \in \mathbb{C}$, and such that there exists $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$.
   (i) Check that V is a vector space. Is the set $\{x^n\}_{n\in\mathbb{Z}}$ a basis of V?
   (ii) Given any $g \in V$ consider the map $L_g : V \to \mathbb{C}$ given by
   $$L_g(f) = \operatorname{Res}(gf) := \text{coefficient of } x^{-1} \text{ in } g(x)f(x),$$
   Prove that L_g is a well-defined linear map.
   (iii) Consider $\varphi : V \to V^*$ given by $\varphi(g) := L_g$. Prove that φ is an isomorphism.
   (iv) Let $\mathbb{C}[[x]] \subset \mathbb{C}((x))$ be the subset of Taylor series, i.e. $a \in \mathbb{C}[[x]]$ if $a = \sum_{n\geq 0} a_n x^n$ for some $a_i \in \mathbb{C}$ and let $\mathbb{C}[x^{-1}] \subset \mathbb{C}((x))$ denote the subset of $a \in \mathbb{C}((x))$ such that $a = \sum_{n\leq 0} a_n x^n$, with $a_m = 0$ for $m \ll 0$. Prove that $\mathbb{C}[[x]]$ and $\mathbb{C}[x^{-1}]$ are subspaces.
   (v) Determine the range of φ restricted to $\mathbb{C}[[x]]$ and $\mathbb{C}[x^{-1}]$.
   (vi) (Extra) Conclude that $\mathbb{C}[[x]] \simeq \mathbb{C}[x]^*$ and that $\mathbb{C}[x]^* \simeq \mathbb{C}[[x]]$.

3. Let $T : V \to W$ be a linear operator between real vector spaces. We define:
   $$T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}, \qquad T(u + iv) := T(u) + iT(v).$$
   (i) Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $T_{\mathbb{C}}$.
   (ii) Prove that λ is an eigenvalue of $T_{\mathbb{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbb{C}}$.

4. Let V be a finite-dimensional vector space and consider $T \in \mathcal{L}(V)$ and $U \subseteq V$ a subspace invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by:
   $$T/U : V/U \to V/U, \qquad T/U(v + U) := T(v) + U.$$
   (i) Check that T/U is well-defined.
   (ii) Show that each eigenvalue of T/U is an eigenvalue of T.
   (iii) Prove that the minimal polynomial of T is a multiple of the minimal polynomial of T/U.
   (iv) Prove that $p_{T/U}p_{T|U}$ is a multiple of p_T, here p_S is the minimal polynomial of the operator S.