Talk 4: ∞-categories, functors, coherent nerve and Spc & Cat$_\infty$.

One can use $h\mathcal{C}$ to identify isomorphisms in $\mathcal{C}$.

Def'n: A morphism $f: X \to Y$ in an ∞-category $\mathcal{C}$ is said to be an isomorphism if $[f]: X \to Y$ in $h\mathcal{C}$ is an isomorphism.

Equivalently, $\exists\, g, h: Y \to X$ s.t. $g \circ f \sim id_X$, $f \circ h \sim id_Y$.

Examples: · $\mathcal{C}$ an ∞-cat. then $\mathcal{C}$ is a Kan complex $\iff$ $h\mathcal{C}$ is a groupoid $\iff$ all morphisms in $\mathcal{C}$ are invertible

· X top. space then $h\,\mathrm{Sing}(X) \simeq$ fundamental groupoid of X.

Discussion of mapping spaces from last notes. 3] $\to$ 5].)

Examples (of functors): (i) $\{f: C \to D\} \simeq \{f: N C \to N D\}$.

(ii) $\{f: \mathcal{C} \to N D\} \simeq \{f: h\mathcal{C} \to D\}$.

(iii) $\mathcal{C}$ an ∞-cat. X a top. space.

$$\{f: \mathcal{C} \to \mathrm{Sing}_\bullet(X)\} \simeq \{\text{cont. fcts. } |\mathcal{C}| \to X\}.$$

$$|\mathcal{C}| := \operatorname*{colim}_{[n]\in\Delta^{op}} \mathcal{C}_n .$$

Prop $\Rightarrow$ $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an ∞-category.

$$\begin{tikzcd} \text{Cats.} \arrow[r, hook, "N" '] & \infty\text{-Cats.} \arrow[l, bend right, "h(-)" ', "\perp"] & \text{Kan.} \arrow[l, hook'] \arrow[d, shift right, "|-|" '] \\ & & \text{Top.} \arrow[u, shift right, "\mathrm{Sing}" '] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

We will later write adjoints to Kan $\to$ ∞-Cats, even at the level of ∞-categories.

<u>A word on models.</u> The intuition of ∞-categories is that for any pair of objects X, Y in $\mathcal{C}$ we have a space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Maybe a more intuitive approach would have started w/ the theory of categories enriched over topological spaces. This works, but it can be technically difficult.

(Model options): - quasi-categories.
- topological categories.
- simplicial categories.
- Segal categories. $X_\bullet : \Delta^{op} \to \mathrm{Top}$. + cond.
$X_0 \in \mathrm{Sets}$, $X_n \simeq X_1 \times_{X_0} \cdots \times X_1$.

- Complete Segal spaces: $X_\bullet : \Delta^{op} \to \mathrm{Top}$. • $X_n \to X_{n_1} \times_{X_0} X_{n_2}$ for $[n_1] \amalg [n_2] = [n]$.

$+ \; X_1^{inv} \overset{w.e.}{\simeq} X_0$.

- Relative categories: $(\mathcal{C}, W)$.

<u>Rk:</u> • Some of these models are "better" in the sense they form an "$(\infty,2)$-category" enriched in $(\infty,1)$-cats. (∞-cosmoi.)

• Models are important even when developing the thy in a single model, e.g. Lurie uses simpl. categories in discussion of Straightening / Unstraightening.

<u>More examples of ∞-categories:</u> // Homotopy coherent nerve.
for $n \geq 0$, let $\mathrm{Path}([n])$ be the simplicial cat:
- objects are $\mathrm{Ob}([n])$; $\mathrm{Hom}_{\mathrm{Path}([n])}(x, y) := N_\bullet(\text{poset } x = x_0 < x_1 \cdots < x_m = y)$.
reverse inclusion order.

(analogously, T_n top. cat. w/ $Ob(T_n) = Ob([n])$.
$$\mathrm{Hom}_{T_n}(i,j) = \begin{cases} \{ f \in [0,1]^{\{i,\dots,j-1,j\}} \;\; f(i)=f(j)=1 \} & j \geq i \\ \emptyset & \text{else} \end{cases}$$

$N^{hc}_\bullet(\underline{C}) := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}([n]), \underline{C})$ (=: simplicial functors.)

($N^{top}_\bullet(\underline{C}) := \mathrm{Hom}_{\mathrm{Cat}_{\mathrm{Top}}}(T_n, \underline{C})$ = topological functors.)

<u>Prop.</u> Let $\underline{C}$ be a simplicial category s.t. $\forall X, Y \in \underline{C}$ $\mathrm{Hom}_C(X,Y)$ is a Kan complex.

Then $N^{hc}_\bullet(\underline{C})$ is an ∞-category.

2. DG nerve. Let $\underline{C}$ be a category enriched in chain complexes of abelian groups. ($\mathrm{Hom}(Y,Z)_n \times \mathrm{Hom}(X,Y)_m \to \mathrm{Hom}(X,Z)_{n+m}$).

$N^{dg}_\bullet(\underline{C})_n := \{ \{X_i\}_{0 \leq i \leq n}, f_I \}$. $-X_i \in Ob(\underline{C})$.

$- I = \{ i_0 > \dots > i_k \} \subseteq [n]$, $|I| \geq 2$.
$f_I \in \mathrm{Hom}(X_{i_k}, X_{i_0})_{k-1}$ satisfying:
$$df_I = \sum_{j=1}^{k-1} (-1)^j \left(f_{\{i_0 > \dots > i_j\}} \circ f_{\{i_j > \dots > i_k\}} - f_{I \setminus \{i_j\}} \right)$$

e.g.: $N^{dg}(\underline{C})_1 = \{ f \in \mathrm{Hom}(X,Y)_0 \mid df = 0 \}$.

$N^{dg}(\underline{C})_2 = \{ f \in \mathrm{Hom}(X,Y)_0, g \in \mathrm{Hom}(Y,Z)_0, h \in \mathrm{Hom}(X,Z)_0,$
$\sigma \in \mathrm{Hom}(X,Z)_1$
$(df = dg = dh = 0)$ $h - g \circ f = d\sigma \}$.

Prop: $N^{dg}(\mathcal{C})$ is an ∞-category.

Rk: One can do the above for certain 2-categories, see Duskin nerves.

The ∞-category $\mathcal{S}pc$:

Let $\mathrm{Kan} \subseteq \mathrm{Sets}_\Delta$, the category Sets_Δ is enriched over simplicial sets, i.e. $\mathrm{Hom}_{\mathrm{Sets}_\Delta}(X,Y)_n := \mathrm{Hom}_{\mathrm{Sets}_\Delta}(\Delta^n \times X, Y)$.

Fact: (Easy). for any pair K, K' of Kan complexes. $\mathrm{Hom}_{\mathrm{Sets}_\Delta}(K,K')_\bullet$ is a Kan complex.

Prop: $\Rightarrow$ $N^{hc}_\bullet(\mathrm{Kan})$ is an ∞-category.

$$N^{hc}_\bullet(\mathrm{Kan}) = \mathcal{S}pc$$

The ∞-category $\mathcal{C}at_\infty$:

Consider the simplicial category $\underline{\mathcal{C}at_\infty}$ where:

- objects are quasi-categories. (∞-cats);
- $\mathrm{Hom}_{\underline{\mathcal{C}at_\infty}}(\mathcal{C}, \mathcal{D}) := (\mathrm{Fun}(\mathcal{C}, \mathcal{D}))^{\simeq}$

Here: given $\mathcal{C}$ an ∞-category $\mathcal{C}^{\simeq}$ is the simplicial subset of $\mathcal{C}$ ~~[illegible]~~ consisting of $\sigma: \Delta^n \to \mathcal{C}$ s.t. each edge of σ is an isomorphism.

[Kerodon §4.4.3] Properties: (i) $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is a subcategory.

(ii) $\mathcal{C}^{\simeq}$ is an ∞-groupoid, i.e. Kan complex.

$$\mathcal{C}at_\infty := N^{hc}\,\underline{Cat_\infty}$$

<u>Rk</u>: There are many other models for Cat_∞, some have better technical properties. Namely, taking the coherent nerve of the cat. of fibrant-cofibrant objects in marked simplicial sets. This proves $\mathcal{C}at_\infty$ has (small) limits & colimits.