Akhil Mathew

Deformation thy & Partition Lie Algebras

$X / \mathbb{C}$ smooth variety.

Recall: 1) Deformations of X over $\mathbb{C}[\varepsilon]/(\varepsilon^2) \to H^1(T_X)$.

2) Aut (1st order def.) $\to H^0(T_X)$.

3) Given a first order deformation $y \in H^1(T_X)$, then $[y, y] \in H^2(T_X)$ vanishes $\Leftrightarrow$ y extends to a def'n. over $\mathbb{C}[\varepsilon]/(\varepsilon^3)$.

Principle: $R\Gamma(X, T_X)$ together w/ natural structure determine the def. thy of X.

Ex: Consider

$$\left\{ \begin{array}{c} \text{local Artinian} \\ \mathbb{C}\text{-algs.} \end{array} \right\} \longrightarrow \text{Sets.}$$

$$A \longmapsto \text{iso. classes of def'ns of } X \,/\, A.$$

Ex: One can regard $R\Gamma(X, T_X)$ as a DGLA. (explicitly in the Dolbeault complex).

Generally, given a DGLA $\mathfrak{g}$ one has

$$F_{\mathfrak{g}} : \text{Art. } \mathbb{C}\text{-algs.} \longrightarrow \text{Sets.}$$

$A \rightsquigarrow m_A \subseteq A$, consider $m_A \otimes \mathfrak{g}$. $\rightsquigarrow$ solutions of the Maurer-Cartan eq. $dx + \frac{1}{2}[x, x] = 0$ w/ $x \in (\mathfrak{g} \otimes m_A)^1$
$\rightsquigarrow$ quotient by gauge equivalences.

Principle: (Deligne, Drinfeld, Feigin)
Any def'n problem comes from a DGLA.

Ex: Deforming a sheaf $\mathcal{E}$ on X $\rightsquigarrow$ $R\Gamma(\underset{X}{End}(\mathcal{E}))$.

Ex: (Bogomolov-Tian-Todorov) X smooth proj., $\omega_X = \mathcal{O}_X$.

Then X has unobstructed deformations.
In fact, $R\Gamma(T_X)$ is abelian, i.e. DGLA w/ zero diff. & trivial bracket.

Ex: (Goldman-Millson)
X a compact Kähler manifold, G an alg. group.
Consider the variety of grp homo's:
$R(\pi_1 X, G) = \{f: \pi_1 X \to G\}$.

If $f \in R(\pi_1 X, G)$ corresponds to a bdd image, then has quadratic singularities at f.

Strategy: associated DGLA is formal (q.-iso. to a DGLA w/ $d=0$)
$\Rightarrow$ quadratic singularities.

This principle becomes a theorem in DAG.

Def: k a field, SCR_k
$Art_k \subset SCR_k$
$\|$
$\left\{ R \;\middle|\; \begin{array}{l} \text{(i) } H_0(R) \text{ is local Artinian;} \\ \text{(ii) } \bigoplus_{n \geq 0} H_n(R) \text{ is f.d. over } k \end{array} \right\}$

Defn: A formal moduli problem is a functor
$$F : Art_k \to Spc \quad \text{s.t.}$$

(i) $F(k) \simeq *$.

(ii) if
$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A_2 \arrow[d] \\ A_3 \arrow[r] & A_4 \end{tikzcd}$$
is a pullback in Art_k s.t. $H_0(A_2) \twoheadrightarrow H_0(A_4)$ & $H_0(A_3) \twoheadrightarrow H_0(A_4)$

$$\Rightarrow F(A_1) \simeq F(A_2) \times_{F(A_4)} F(A_3).$$

Rk: Any formal mod. prob. gives a deformation problem.

In practice, most geometric situations (e.g., formal completion of schemes, stacks, def's of alg. objects) produce formal mod. problems.

Thm (Lurie-Pridham): Suppose char $k = 0$, there is an eq. of ∞-cats:
$$\underset{\substack{\parallel \\ \left(\text{formal moduli pbs.}\right)}}{Moduli_k} \cong DGLA_k.$$
$$F \mapsto T_* F[-1].$$

Extensions of this result by: Hennion, Nuiten, Gaitsgory-Rozenblyum.

Ex: G an alg. group
$$BG^\wedge \longrightarrow \text{Lie alg. of } G.$$

In char. 0, formal groups $\simeq$ Lie Algs.

Ex: (pro-representable formal moduli pbs.)
Gives a correspondence:

$$\left\{\begin{array}{l} A \in SCR_k \text{ s.t.} \\ H_0(A) \text{ is a complete local ring} \\ + H_i(A) \text{ f.g. module} \end{array}\right\} \xrightarrow[\text{anti}]{\sim} \left\{\begin{array}{l} \mathfrak{g} \in DGLAs \text{ s.t.} \\ H^i(\mathfrak{g}) = 0 \\ \quad i \leq 0 \\ \& \text{ f.d. in each degree} \end{array}\right\}.$$

Rk: The above goes back to Quillen, where the connectivity constraints are a bit different.

Main theorem (Brantner-M.):
There is a monad Lie_π on $D(k)$ s.t.
(i) $Moduli_k \simeq Lie_\pi$-algebras $(D(k))$.
(via tangent space functor)
(ii) In char. 0, Lie_π is a free Lie algebra (up to a shift)
(iii) $Lie_\pi(V) \simeq \bigoplus_{n \geq 0} \left(C^*(\Sigma^2 \Pi_n) \otimes V^{\otimes n}\right)^{\Sigma_n}$

Defn: (Partition complex).
Π_n = nerve of poset of nontrivial partitions of $\{1, \ldots, n\}$.
Naturally a finite simplicial complex w/ $\Sigma_n \curvearrowright$.
Π_n has the homotopy type of a wedge of $(n-3)$-spheres.
+ $H_{n-3}(\Pi_n, \mathbb{Z}) \simeq n^{th}$ Lie rep. of Σ_n (Barcelo).

The cohomology of a free Lie_π-alg. can be worked out.
On $H^*(\mathfrak{g})$, $\mathfrak{g}$ a Lie_π-alg: - there is a Lie bracket.
- analogs of Steenrod ops. in top.
(satisfy Adem relations)

$\Rightarrow$ $p=2$, Goerss.

$\Rightarrow$ $p>2$, one writes down a basis of the coh. of a free algebras (Arone-Brantner).