Math 347: Exam 3
Dec. 10, 2018

Name:

This exam has five questions and lasts 50 minutes. Make sure your exam is complete before the start and don't forget to enter your name.

All of your answers need a justification, and when asked to prove a result you need to write a formal argument.

You can use the back page for scratch work, but indicate if you want something there to be graded.

You will be given partial credit, so if you don't have a complete proof explain, in words, what you know how to do.

No textbook, electronics, or notes are allowed during the exam.

Question	Points	Total
1		24
2		20
3		20
4		16
5		20
Total		100

1. Consider the set $S = \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ and define a relation $R \subset S \times S$ as follows

$$(a, b, c, d) \in R \quad \text{if} \quad ad = bc.$$

a) Prove that R is an equivalence relation, namely that it is: (i) reflexive, (ii) symmetric and (iii) transitive.

b) For $(1, 2) \in S$, what is the equivalence class $[(1, 2)]$? Find other pairs (a, b), such that

$$[(1, 2)] = [(a, b)].$$

c) Describe the set of equivalence classes of R.

2. Consider $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $a|c$ and $b|c$, then $ab|c$[1].

[1] Recall that $a|b$ means a divides b.

3. Recall that the prime factorization theorem states that for any natural number $n > 1$, n can be written as

$$p_1 \cdot \dots \cdot p_k$$

where $p_1, \dots, p_k$ are all prime numbers, uniquely up to reordering of p_i's in the above formula.

a) Using the conclusion of the above theorem, argue why 1 can't be considered a prime number[2].

b) Let a and b be integers and $m = \gcd(a, b)$. Prove that $\gcd(\frac{a}{m}, \frac{b}{m}) = 1$.

[2] It is not enough to say that 1 is the only number that divides 1.

4. Recall that the binomial coefficient theorem states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

a) Prove by $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

b) Compute in how many ways one can write 24 as a sum of 2's and 5's. (Hint: apply the theorem to $(z^2 + z^5)^{12}$.)

5. Determine if the following are true or false, only a brief explanation is necessary.

a) For R the relation on $\mathbb{Z}$ defined by $(a, b) \in R$ if $a - b$ is divisible by 6. The set $\mathbb{Z}/6\mathbb{Z}$, defined as the set of equivalence classes of R, has 5 elements.

b) The function $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ defined by sending an equivalence class $[a]$ to $[3a]$ is not injective.

c) The equation

$$39x + 104y = 3$$

has integer solutions.

d) The numbers $1,457,863$ and $1,457,865$ are relatively prime.

e) The limit

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^2}{n^3}$$

is non-zero.