Math 2102: Homework 3

Due on: Mar. 18, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $T : V \to V$ be an operator on a finite-dimensional complex vector space. Prove that the following are equivalent:

   (1) T is diagonalizable (i.e. it satisfies Proposition 6 from the Lecture Notes);

   (2) $V = \operatorname{null}(T - \lambda \operatorname{Id}_V) \oplus \operatorname{range}(T - \lambda \operatorname{Id}_V)$ for every $\lambda \in \mathbb{C}$;

   (3) the minimal polynomial $p_T = \prod_{i=1}^{m}(z - \lambda_i)$, where $\{\lambda_1, \ldots, \lambda_m\}$ are distinct;

   (4) there does not exist $\lambda \in \mathbb{C}$ such that p_T is a multiple of $(z - \lambda)^2$;

   (5) p_T and p_T' have no zeros in common;

   (6) the greatest common divisor of p_T and p_T' is the constant polynomial 1.

2. Let V be a vector space over $\mathbb{C}$. Let $\mathcal{E} \subseteq \mathcal{L}(V)$ be a subset of linear operators which commute, i.e. for any $S, T \in \mathcal{E}$ we have $ST = TS$.

   (i) Prove that for any $S, T \in \mathcal{E}$ the subspaces $\operatorname{null} p(S)$ and $\operatorname{range} p(S)$ are invariant under T.

   (ii) Prove that there is a vector in V that is an eigenvector for every element of $\mathcal{E}$.

   (iii) Prove that there is a basis of V with respect to which every element of $\mathcal{E}$ has an upper-triangular form.

3. Let V be a finite-dimensional vector space.

   (i) Let $T \in \mathcal{L}(V)$ be an invertible operator and $B_V = \{v_1, \ldots, v_n\}$ is a basis such that $\mathcal{M}(B_V, T)$ is upper triangular with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Show that $\mathcal{M}(B_V, T^{-1})$ is also upper triangular with $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ on the diagonal.

   (ii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only 0's in the diagonal but T is invertible.

   (iii) Give an example of $T \in \mathcal{L}(V)$ and B_V such that $\mathcal{M}(T, B_V)$ contains only non-zero elements in the diagonal but T is not invertible.