# Math 347: Final Exam

**Dec. 20, 2018**

**Name:**

**Instructor name:**

This exam has eight questions and lasts 3 hours. Make sure your exam booklet has all the questions before the start and don't forget to enter your name.

All of your answers need a justification, and when asked to prove a result you need to write a formal argument.

Let me know if you need any scratch paper, and indicate if you want any of your work on scratch paper graded.

You will be given partial credit, so if you don't have a complete proof, explain in detail what you know how to do.

No textbook, electronics, or notes are allowed during the exam.

| Question | Points | Total |
|---|---|---|
| 0 | | 2 |
| 1 | | 24 |
| 2 | | 20 |
| 3 | | 30 |
| 4 | | 24 |
| 5 | | 24 |
| 6 | | 20 |
| 7 | | 32 |
| 8 | | 24 |
| Total | | 200 |

1. Let $n \geq 2$ be a natural number.

   a) Assume $n$ is a prime number. Prove that if $n$ divides $ab$, then $n$ divides $a$ or $n$ divides $b$.

   b) Does a) still hold if $n$ is not a prime number? Prove your statement or give a conterexample.

   c) At a Christmas party, a couple suddenly realizes that their wedding is exactly six months away. Moreover, they noticed that both the Christmas party and their wedding are happening on a Saturday. Is the couple getting married on a leap year? (You may assume that the Christmas party is in December.)

2. Prove that every Cauchy sequence is bounded.

3. a) Let $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. Prove that there exist $n$ and $m$ such that

$$na + mb = 1.$$

b) Does the equation
$$3x + 15y = 6.$$
have integers solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$?

c) Write all the solutions to the equation
$$3x + 15y = 6.$$

4. a) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly monotone function[1] prove that $f$ is injective.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function. Is $f$ also surjective? Give an example or proof to justify your claim.

c) Do the sets $(0, 1)$ and $\mathbb{R}$ have the same cardinality?

[1] Recall that means that for any $x, y \in \mathbb{R}$ if $x < y$, then $f(x) < f(y)$.

5. Let $S$ be the set of sequences of non-zero real numbers, i.e. functions from $\mathbb{N}$ to $\mathbb{R}\backslash\{0\}$ . Consider the relation $R$ on $S$,

$$((a_n),(b_n)) \in R, \quad \text{if} \quad \lim_n \frac{a_n}{b_n} = 1.$$ [2]

a) Prove that $R$ is an equivalence relation.

b) Give three examples of sequences in the equivalence class of $a_n = n^3$.

c) Prove that $a_n = \frac{n^2}{2}$ and $b_n = \sum_{i=1}^{n} i$ are in the same equivalence class.

[2]This equation conveys two statements: (i) that the limit $\lim_n \frac{a_n}{b_n}$ exists, and (ii) that the value of this limit is 1.

6. a) For $n \geq 1$, prove that[3]

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

b) Find the number of positive integer[4] solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

[3] If $a < b$ we define

$$\binom{a}{b} = 0.$$

[4] That is $x_i > 0$ for all $1 \leq i \leq k$.

7. Give examples of the following structures or argue why no example exists. You also need to explain why your examples satisfy the required properties.

   a) A set $S$ and a relation $R$ that is transitive and reflexive but not symmetric.

   b) A set $S$ and a realtion $R$ that is transitive and satisfies the following condition:

   $$\forall (x, y) \in S \times S \ \text{ if}(x, y) \notin R, \text{ then } (y, x) \in R.$$

   c) An equivalence relation $R$ on $\mathbb{N}$ that has finitely many equivalence classes and an equivalence relation $R'$ that has infinitely many equivalence classes.

   d) Two functions $f$ and $g$, such that $g \circ f$ is injective but $g$ is not injective.

e) A non-bounded Cauchy sequence.

f) Sets $A, B$, a function $f : A \to B$, and a subset $T \subseteq B$, such that

$$f^{-1}(f(T)) \neq T.$$

g) Sets $A, B$ and $C$ such that

$$(A \cap B) \backslash C = A \cap (B \backslash C).$$

h) Sets $S$ such that the set of functions $X \to S$ has bigger cardinality than $P(X)$, the power set of $X$.

8. Determine if the following are true or false, and give a brief explanation.

a) The statements
$$(P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$
and
$$Q \wedge \neg Q$$
are logically equivalent.

b) Fix $a, L \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$. Consider the statements
$$P = (\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in \mathbb{R})\ [(0 < |x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon),$$
and
$$Q = (\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x \in \mathbb{R})\ [(0 < |x - a| < \delta) \Rightarrow (|\frac{f(x) - f(a)}{x - a} - L| < \epsilon).$$
Then $Q \Rightarrow P$.

c) For any $n \geq 2$ there exists a unique set of primes $\{p_1, \ldots, p_k\}$, $p_i \in \mathbb{N}$ for all $1 \leq i \leq k$, such that
$$n = p_1 \cdots p_k.$$

d) Let $n, a \in \mathbb{N}$, such that $\gcd(a, n) = 1$. Consider the statements

$$P = (\exists x)(ax \equiv 1 \bmod n),$$

and

$$Q = ([a]\cdot : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \text{ is injective.})^5.$$

Then $Q \Rightarrow P$ is true.

e) For all $n \in \mathbb{N}$, the statement

$$(\gcd(n, n+2) = 1) \vee (\exists k \in \mathbb{Z}, \text{ s.t. } \; n = 2k)$$

is true.

f) Let $A \subseteq B$ be two finite sets. Suppose that there are as many subsets of $B$ containing $A$ as subsets of $B$ not containing $A$. Then $|A| = |1|$.

[5] Recall that this function is explicitly defined as $[a] \cdot [x] = [a \cdot x]$, for any $[x] \in \mathbb{Z}/n\mathbb{Z}$.