Motivation: 1) $\sqrt{2}$ is irrational.

2) $(0,1)$ and $\mathbb{N}$ don't have the same card.

Upper bound.

Least upper bound.

$\mathbb{R}$ is complete, every set S has that has an upper bound a least upper bound.

Note the least upper bound does not have to be in S

Ex: a) $S = (0,1)$. $\alpha = 1$.

b) $(-\infty, 0)$ $\alpha = 0$.

c) $(1,2)$ $\alpha = 2$

d) $(1, \infty)$ ~~is not bounded~~ has no upper bound.

(Arch. prop.) $\forall\, a,b \in \mathbb{R},\ \exists\, n \in \mathbb{Z}$ s.t. $na > b$.

i.e. $\forall c \in \mathbb{R},\ \exists\, n \in \mathbb{Z}$ s.t. $n > c$.

By contradiction, assume $\exists c \in \mathbb{R}$ s.t. $\forall m \in \mathbb{N}$ $m \leq c$

$\Rightarrow \mathbb{N}$ has an upper bound $\Rightarrow$ has a least upper bound α $\Rightarrow \alpha - 1$ is not a least upper bound $\Rightarrow$

<u>Recall:</u> $S = \{x \in \mathbb{R} \mid x^2 < 2\}$.

<u>Claim:</u> $\alpha = \sup(S)$ satisfies $\alpha^2 = 2$, i.e. $\alpha = \sqrt{2}$.

<u>Pf:</u> Suppose $\alpha^2 < 2$, then $\left(\frac{2}{\alpha}\right)^2 > 2$.

Consider $y = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right)$.

By (i) $\forall a, b \in \mathbb{R}$ $\frac{a+b}{2} \geq \sqrt{ab}$, if $a \neq b$ th

$\frac{a+b}{2} > \sqrt{a \cdot b}$

Apply this to $a = \alpha, b = \frac{2}{\alpha}$ $\Rightarrow y^2 > 2$.

$\Rightarrow \left(\frac{2}{y}\right)^2 < 2$.

b/c (iii) (ii) $\Rightarrow$ $\alpha < \left(\frac{2}{y}\right)$.

$\boxed{\text{(ii)} \quad \alpha^2 < (2/y)^2.}$

<u>Contradiction.</u>

(iii) Finish to argue that $\alpha^2 \neq 2$.

A sequence a_n <u>has a limit</u> $L \in \mathbb{R}$ if

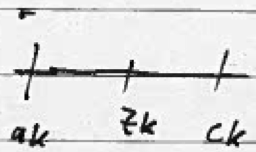
$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow |a_n - L| < \varepsilon$.

A seq. <u>converges</u> if it has a limit.

$a_1 = L$ $\qquad$ $z_k = \dfrac{a_k + c_k}{2}$

$c_1 = M.$



if $\exists$ infinitely many $x_k \le z_k$ then

$a_{k+1} = a_k$, $c_{k+1} = z_k$.

else

$a_{k+1} = z_k$, $c_{k+1} = c_k$.

$$|[a_k, c_k]| \le \frac{M-L}{2^{k-1}}$$

2\. & infinitely many x's $\in [a_k, c_k]$.

$\lim a_k = \lim c_k$ $\Leftarrow$ WS18 Now, a_k increasing, c_k decreasing and $\lim a_k - c_k = 0$.

Find (b_n).

Let $b_1 = x_1 \in [a_1, c_1]$.

Suppose $b_n \in [a_n, c_n]$.

if $[z_n, c_n]$ is the next interval.

let m be the smallest integer s.t. $x_m \in [z_n, c_n]$. define

$$b_{n+1} = x_m.$$

one such exist b/c of the $[z_n, c_n]$ was constructed.

Then $\lim a_k \le \lim b_n \le \lim c_k$.

$= N$

ER

2

Exercise: 14.28. a) (a_n) bounded and (b_n) a subsequence.
if b_n is monotone. then $\lim b_n$ converges.
Obvious.

b) A peak of (a_n) is a_n for some $n \in \mathbb{N}$ s.t.

$$a_m < a_n, \quad \forall\ m > n.$$

Review for Exam 2:

Recall: questions building up difficulty. Write anything you think will be relevant for the pf. Etc!

1) Cardinality: defn, countable, how to prove something is countable. Examples.

2) Limits: defn, how to check a seq. converges by defn. Consequences. (e.g. (a_n) is Cauchy & (a_n) is bounded).

3) Monotone Convergence Theorem: statement, how to apply it. Sufficient but not necessary.

4) Cauchy sequences: defn., how to check it, Consequences, Cauchy Convergence Thm, (i.e. (a_n) converges).

---

1) $|S| = |T|$ iff $\exists\ f: S \to T$ bijection.

$|\mathbb{N}| = |\mathbb{Z}|$ $\quad f: \mathbb{N} \to \mathbb{Z}$.

$$f(n) = \begin{cases} \dfrac{n}{2}, & n \text{ even.} \\[2ex] \dfrac{-(n+1)}{2}, & n \text{ odd.} \end{cases}$$

$|\mathbb{R}| = |(0,1)|$ by $f : (-\pi/2, \pi/2) \to \mathbb{R}$,

$f(x) = \tan(x)$.

+ massage f.

2) Consider $a_n = \dfrac{2n}{n+1}$. "$\lim a_n = 2$"

Check,

$$\frac{2n}{n+1} - 2 = \frac{2n - 2n - 2}{n+1} = \frac{-2}{n+1}.$$

$$\Rightarrow \left| \frac{2n}{n+1} - 2 \right| \leq \frac{2}{n+1} \leq \frac{2}{n}.$$

For $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\dfrac{2}{N} < \varepsilon$.

Then $\forall n \geq N$, $\dfrac{2}{n} \leq \dfrac{2}{N} < \varepsilon$.

$$\Rightarrow |a_n - 2| < \varepsilon.$$

$$\Rightarrow \lim a_n = 2.$$

(a_n) is Cauchy

$$\frac{2n}{n+1} - \frac{2m}{m+1} = \frac{2nm + 2n - 2nm - 2m}{(n+1)(m+1)} = \frac{2(n-m)}{(n+1)(m+1)}$$

Notice: $|2(n-m)| \leq 2\max\{n, m\}$.

$$\frac{1}{(n+1)(m+1)} \leq \frac{1}{n \cdot m}. \quad \Rightarrow$$

$$\left| \frac{2n}{n+1} - \frac{2m}{m+1} \right| \leq \frac{2}{\min\{n,m\}}.$$

For $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\frac{2}{N} < \varepsilon$

Then for any $n, m \geq N$, one has

$$\left(\frac{2}{\min\{n,m\}} \leq \frac{2}{N} < \varepsilon. \right).$$

$\Rightarrow$ $|a_n - a_m| < \varepsilon$. $\Rightarrow$ (a_n) is Cauchy

---

(a_n) is bounded let $\lim a_n = L$.

Consider $1 \in \mathbb{R}$, $\Rightarrow$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$.

$$|a_n - L| < 1$$

$$\Rightarrow L - 1 < a_n < L + 1.$$

Let $S = \{a_1, \dots, a_N, L-1, L+1\}$.

Consider $b = \inf(S)$, $c = \sup(S)$
$= \min(S)$ $= \max(S)$.

$\Rightarrow$ $\forall n \in \mathbb{N}$ $b \leq a_n \leq c$.

---

3) $a_n = \frac{1}{n} \cdot \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2$, $a_1 = 1$

$$a_2 = \frac{1}{2} \cdot \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{1}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + 1\right]$$

$$= 5/8.$$

$$a_{n+1} = \frac{1}{(n+1)^3}\left[\left(\sum_{i=1}^{n} i^2\right) + (n+1)^2\right].$$

$$a_n = \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right). \qquad \frac{1}{n+1} \leq \frac{3}{n}.$$

$$= \frac{1}{(n+1)^3}\left[\frac{(n+1)^3}{n^3}\sum_{i=1}^{n} i^2\right].$$

$$= \frac{1}{(n+1)^3}\left[\frac{(n^3+3n^2+3n+1)}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \frac{1}{(n+1)^3}\left[\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} i^2 + (\quad)\right].$$

$$\Rightarrow \quad a_{n+1} \leq a_n.$$

$\Rightarrow$ monotone, Hence (a_n) converges.

$a_n = \frac{(-1)^n}{n}$ converges but is not monotone.

---

4) Let $a_n = \sum_{i=1}^{n} \frac{1}{2^i}$.

let $\varepsilon > 0$, consider N s.t. $\frac{1}{2^N} < \varepsilon$.

$$a_n - a_m = \sum_{i=\min(n,m)+1}^{\max(n,m)} \frac{1}{2^i}$$

$$= \frac{1}{2^{\min}}\left(\frac{1-(1/2)^{\max-\min}}{1-1/2}\right)$$

for $n,m \geq N$. One has

$$= \frac{1-(1/2)^{|n-m|}}{2^{\min}} \qquad \frac{1}{2^{\min(n,m)}} \leq \frac{1}{2^N} < \varepsilon$$

$$|a_n - a_m| < \frac{1}{2^{\min}} < \varepsilon. \quad \Rightarrow \boxed{(a_n) \text{ is Cauchy.}}$$

Lecture 18:

2. a_n non decreasing, b_n non-increasing.

Claim: $\forall n \geq 1$, $b_n \geq a_n$.

Pf: Suppose $\exists m \in \mathbb{N}$ s.t. $a_m > b_m$.
since $\forall k \geq m$, $a_k \geq a_m$ and $b_k \leq b_m$
$\Rightarrow$ $\forall k \geq m$ $a_k > b_k$. $\Rightarrow$ $\times$ w/
$\lim (a_n - b_n) = 0$.

Claim $\Rightarrow$ a_n & b_n are bounded, thus $\lim b_n$ & $\lim a_n$ exist.

$\lim b_n \geq \lim a_n$.

3. Suppose $a_n \leq b_n \leq c_n$ & $\lim a_n = L$
$\lim c_n = L$.
Let $\varepsilon > 0$, $\exists N_1$, s.t. $\forall n \geq N_1$ $|a_n - L| < \varepsilon$
& $\exists N_2$, s.t. $\forall n \geq N_2$ $|c_n - L| < \varepsilon$.

Consider. $N = \max(N_1, N_2)$, $\forall n \geq N$ one has

$$L - \varepsilon \underbrace{\leq}_{b/c\ n \geq N_1} a_n \leq b_n \leq c_n \underbrace{\leq}_{b/c\ n \geq N_2} L + \varepsilon$$

$\Rightarrow L - \varepsilon \leq b_n \leq L + \varepsilon$, i.e. $\lim b_n = L$.

5. Consider $x_n - x_{n-1} = \dfrac{1}{x_{n-1}} - \dfrac{x_{n-1}}{2}$

$$= \frac{2-(x_{n-1})^2}{2x_{n-1}}.$$

if $x_1 > 0$.

(i) Claim: $\forall n \quad x_n - x_{n-1} \leq 0$

By induction it is enough to check

$$\frac{2-(x_n)^2}{2x_n} \leq 0 \quad \forall n.$$

$<$

$n=1$ one has $2-x_1^2 < 0$ if $x_1 > 0$ then

$$\frac{2-x_1^2}{2x_1} < 0.$$

Inductive step:

$$\left(2-\left[\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right]^2\right) \leq 0.$$

$$= 2 - \frac{1}{4}\left(x_n^2 + \frac{4}{x_n^2} + 4\right) =$$

$$= 2 - \frac{x_n^2}{4} - \frac{1}{x_n^2} \geq 0$$

$\Updownarrow \quad \cdot (4x_n^2)$

$$8x_n^2 - x_n^4 - 4 \geq 0$$

$x_n^2 > 2$. $\quad -(x_n^2-4)^2 + 8 \geq 0$

$$\left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)^2 \geq (\sqrt{2})^2 = 2.$$

$$2 - 2 = 0.$$

(ii) ~~x_1 is an upper~~ 1 is a lower bound,

since $x_n^2 > 1$, $\forall n \geq 1$.

Monotone Convergence

& $x_n \leq x_{n-1}$ $\forall n.$ $\Rightarrow$ x_n converges.

(iii)

~~Lecture 20: // Lit $\forall L \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ then $x_n > L$~~

~~Suppose $\lim x_n = L$. Then~~

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$.
$|x_n - L| < \varepsilon$.

($\exists \varepsilon > 0$, $\forall N \in \mathbb{N}$ s.t. $\exists n \geq N$
$|x_n - L| > \varepsilon$.)

~~$n \geq N \Rightarrow x_n > L$.~~

~~($x_n \leq L \Rightarrow n < N$)~~

~~(i) x_n unbounded $\Rightarrow$ $\lim x_n$ does not exist.~~

$\Updownarrow$

$\lim x_n$ exist $\Rightarrow$ x_n is bounded

Last time: Archimedean Property: $\forall a, b \in \mathbb{R}$ $a, b > 0$

$\exists N \in \mathbb{N}.$ s.t. $Na > b.$

Claim: $\underset{\subseteq \mathbb{R}}{\mathbb{N}}$ has no upper bound.

If $\mathbb{N}$ has an upper bound, then completeness $\Rightarrow \exists \alpha = \sup(\mathbb{N})$,

i.e. a least upper bound. $\Rightarrow$ $\alpha - 1$ is not an upper bound.

i.e. $\exists n \in \mathbb{N}$ s.t. $n > \alpha - 1 \Rightarrow \underset{\mathbb{N}}{\underset{\cap}{n+1}} > \alpha \Rightarrow$ Contradiction

(Claim: $\Rightarrow$ A.P.) Check this.

A sequence (x_n) is monotone if

(i) $x_n \leq x_{n+1}$

or $\qquad \forall n \in \mathbb{N}.$

(ii) $x_{n+1} \geq x_n$

It is bounded if $\exists M \in \mathbb{R}$ s.t. ~~x_n~~

(a) $x_n \leq M \quad \forall n \in \mathbb{N}$

or

(b) $+M \leq x_n \quad \forall n \in \mathbb{N}.$

If a sequence satisfies (i) and (a) $\Rightarrow$ it converges.

<u>Motivation:</u> S bounded, $\alpha = \sup(S)$.

$\Updownarrow$

α is an upper bound of S and (x_n) is a sequence in S and $\lim x_n = \alpha$.

<u>Q:</u> How to express the completeness of $\mathbb{R}$ using sequences?

Before $S \subseteq \mathbb{R}$ bounded above $\Rightarrow$ S has a least upper bou[nd] i.e. a supremum.

Find a property all convergent sequences have.

<u>Ex:</u> $a_n = \frac{n}{n+1}$, fix $\varepsilon > 0$, consider $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$. let $n,m \geq N$, then $a_n - a_m =$

$$= \frac{n}{n+1} - \frac{m}{m+1} = \frac{n-m}{(n+1)(m+1)} < \frac{n-m}{n \cdot m}$$

$$= \frac{1}{m} - \frac{1}{n} < \varepsilon. \Rightarrow \left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon.$$

<u>Defn:</u> A sequence $(x_n)_{n\geq 1}$ is a <u>Cauchy sequence</u> if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m,n \geq N$ one has $|a_n - a_m| < \varepsilon$.

<u>Proposition:</u> Every convergent sequence $(x_n)_{n\geq 1}$ is a Cauchy sequence.

too easy

not interesting

[**Example:** Consider the sequence iteratively defined:

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Prove x_n is Cauchy.]

Natural question: is the converse true?

Theorem: Every Cauchy sequence converges.

Actually, this is equivalent to the completeness axiom.

A set C is said to be complete if every Cauchy sequence in C, converges to an element of C.

~~**Pf:** let $\varepsilon > 0$, fix some N, s.t. for $m \geq N$ and $n = m+1$, one has:~~

$$x_n - x_m = x_{m+1} - x_m = \frac{x_m}{2} + \frac{1}{x_m} - x_m$$

$$= \frac{x_m^2 + 2 - 2x_m^2}{2x_m} = \frac{-x_m^2 + 2}{2x_m}.$$

$$\left|\frac{-x_m^2 + 2}{2x_m}\right| \leq \frac{|x_m|}{2} + \frac{1}{|x_m|}$$

Consider: $x_n = \sum_{i=1}^{n} \frac{1}{2^i}$. Does x_n converge?

Pf: Let N s.t. $\frac{1}{2^N} < \frac{\varepsilon}{2}$, $\forall\, n, m \geq N$ one has

$$x_n - x_m = \sum_{i=m}^{n} \frac{1}{2^i} = \frac{1}{2^m} + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^n} =$$

say $n \geq m$

$$\cancel{\frac{1}{2^n}\left(2^{n-m} + \cdots + 1\right) \leq \frac{(n-m)\cdot 2^{n-m}}{2^n} \leq \frac{2^{n-2m}}{2^n} \leq \frac{2^n}{2^{2m}}}$$

$$= \frac{1}{2^m}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m}}\right) = \frac{1}{2^m} \cdot \frac{1 - 1/2^{n-m+1}}{1 - 1/2} =$$

$$= \frac{1}{2^m}\left(2 - 1/2^{n-m}\right) \leq \frac{1}{2^{m-1}} \leq 2 \cdot \frac{1}{2^m} < 2 \cdot \frac{1}{\varepsilon} = \varepsilon$$

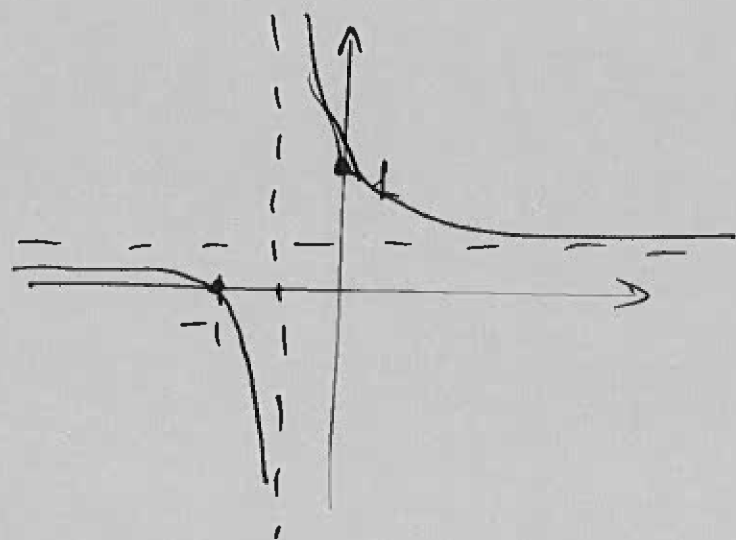
b/c $\boxed{m \geq N}$

Student A

(i) Monotone convergence theorem says a sequence converges if it's bounded and monotone.

- It's monotone

Graph $\frac{1+n}{1+2n}$



Decreasing

$\Downarrow$

Monotone

- It's bounded

$$\frac{1+n}{1+2n} \leq \frac{1+2n}{1+2n} \leq 1$$

Limit exists

(ii) $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1+n}{1+2n} = \lim_{n\to\infty} \frac{1+n+n-n}{1+2n} =$$

$$= \lim_{n\to\infty} \frac{1+2n-n}{1+2n} = \lim_{n\to\infty} 1 - \frac{n}{1+2n}$$

$$= 1 - \frac{n}{1+2n} = 1 - \frac{n \to \infty}{1+2n \to \infty} \quad \text{undefined}$$

Student B

(i) $$X_n = \frac{1+n}{1+2n} \leq \frac{1+(n+1)}{1+2n} \leq \frac{1+(n+1)}{1+(n+1)2} = X_{n+1}$$

Since this holds for all n, x_n is monotone, non-decreasing.

Also $x_n \leq 1$ for all n.

Limit exists

(ii) $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$, s.t. $\frac{1}{N} < \varepsilon$

(By previous lecture)

Let $n \geq N$, then

$$x_n = \frac{1}{1+2n} + \frac{n}{1+2n} \leq \frac{1}{2} + \frac{1}{2n}$$

$$x_n - \frac{1}{2} \leq \frac{1}{2n} < \varepsilon$$

$\cancel{= \frac{n}{1+2n} \leq x_n}$ $\leq$ $\cancel{\frac{1}{2(1+2n)}}$

$$\frac{\frac{1}{2} + (\frac{1}{2} + n)}{2(\frac{1}{2} + n)} = x_n$$

$$\frac{-\frac{1}{2} + (\frac{1}{2} + n)}{2(\frac{1}{2} + n)} \leq$$

$$\|$$

$$\frac{1}{2} - \varepsilon \leq \frac{1}{2} - \frac{1}{2(1+2n)} \leq x_n \implies \underbrace{-\varepsilon \leq x_n - \frac{1}{2}}_{\{|x_n - \frac{1}{2}| < \varepsilon\}}$$

Ye

Student C

(i) $n=1 \qquad x_n = \frac{2}{3}$

$\geq$

$n=2 \qquad x_n = \frac{1+2}{1+2\cdot 2} = \frac{3}{5}$

$\geq$

$n=3 \qquad x_n = \frac{1+3}{1+2\cdot 3} = \frac{4}{7}$

$\geq$

$n=4 \qquad x_n = \frac{1+4}{1+2 4} = \frac{5}{9}$

$\geq$

$n=5 \qquad x_n = \frac{1+5}{1+2\cdot 5} = \frac{6}{11}$

Decreasing! Limit exists

(ii) By definition $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$, so $\forall n \geq N$

$$|x_n - \tfrac{1}{2}| < \varepsilon$$

Relation between least upper bound property and sequences.

(A) ✱ Extrinsic: S has a least upper bound.

(B) ✱ Intrinsic: α is an upper bound of S and $\exists$ $(x_n)_n$ a sequence in S

s.t. $\lim x_n = \alpha$.

Suppose $\alpha = \sup S$.

Consider $\alpha - 1/n$, not an upper bound $\Rightarrow \exists$ $x_n \in S$,

s.t. $x_n > \alpha - 1/n$. Choose (x_n) as indicated.

Check: $\lim a_n = \alpha$.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be s.t. $\varepsilon > \frac{1}{N}$., then $\forall$ $n \geq N$

$$\alpha - \varepsilon < \alpha - \frac{1}{N} < x_n < \alpha < \alpha + \varepsilon.$$

$$\Rightarrow \quad |x_n - \alpha| < \varepsilon.$$

check this.

Conversely, suppose (B).

Claim: If $\beta \in \mathbb{R}$, $\beta < \alpha$, then β is not an upper bound of S. $\Rightarrow$ (A)

(Claim $\Rightarrow$ (A)) Check this.

Consider $\varepsilon = \alpha - \beta > 0$, since $\lim x_n = \alpha$, $\exists N$,

s.t. $\forall n \geq N$ $|x_n - \alpha| < \varepsilon$. $\Rightarrow$ $-(x_n - \alpha) < \varepsilon \Rightarrow x_n > \alpha - \varepsilon$

or

i.e. $x_n \in S$ and $x_n > \beta$. $(x_n - \alpha < \varepsilon.)$ can't be true.

$\Rightarrow \beta$ is not an upper bound.

Pf: Consider the set $S = \{a_1, a_2, \dots\}$.

It has $\alpha = \sup(S)$.

Claim: $\lim a_n = \alpha$ (Notice this is not automatic from previous result.)

~~$\forall \varepsilon > 0$, $\exists N$ s.t. $\varepsilon > \frac{1}{N}$~~

$\forall \varepsilon > 0$

Consider $\alpha - \varepsilon$, this is not an upper bound of S, i.e.

$\exists N \in \mathbb{N}$ s.t. $a_N > \alpha - \varepsilon$. $\Rightarrow \underset{\alpha+\varepsilon>}{\alpha \geq} \underset{\forall n \geq N}{a_n \geq a_N} > \alpha - \varepsilon$.

$\Rightarrow |a_n - \alpha| < \varepsilon \Rightarrow \underline{\lim a_n = \alpha}$.

Example: For $k \geq 2$, $x_n = \frac{1}{k^n}$ converges.

$$\frac{1}{k^n} \leq \frac{1}{2^n} < \frac{1}{n}.$$ since $n < 2^n$, $\forall n \geq 1$.

Thus, for $n > \frac{1}{\varepsilon}$, any $m \geq n$ one has

$$-\frac{1}{\varepsilon} < 0 \leq x_m < \frac{1}{\varepsilon} \Rightarrow \boxed{\lim x_n = 0.}$$

Or, $\frac{1}{k^n}$ is bounded by 1 and $x_{n+1} \leq x_n$ $\forall n$.

$\underset{\text{Conv. thm.}}{\text{Monotone}} \Rightarrow (x_n)$ has a limit.