Math 2102: Homework 1

Due on: Feb. 5, 2024 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $T \in \mathcal{L}(V)$ and consider bases $B_1 = \{v_1, \dots, v_n\}$ and $B_2 = \{u_1, \dots, u_n\}$ of V. Prove that the following conditions are equivalent:

   (1) T is injective.

   (2) The columns of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in $\mathbb{F}^n$.

   (3) The columns of $\mathcal{M}(T, B_1, B_2)$ span $\mathbb{F}^n$.

   (4) The rows of $\mathcal{M}(T, B_1, B_2)$ are linearly independent in $\mathbb{F}^n$.

   (5) The rows of $\mathcal{M}(T, B_1, B_2)$ span $\mathbb{F}^n$.

2. Let V be a finite-dimensional vector space and $\mathcal{L}(V)$ the space of linear maps from V to itself. Given a linear operator $T \in \mathcal{L}(V)$.

   (i) Assume that $TS = ST$, for every $S \in \mathcal{L}(V)$. Prove that $T = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$, where Id_V is the identity operator on V.

   (ii) Assume that $T = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$. Prove that T is represented by a diagonal matrix with entries $\lambda \in \mathbb{F}$ for *any basis* of V.

   (iii) Assume that T is invertible. Let B be a basis of V and $\mathcal{M}(T) := \mathcal{M}(T, B, B)$ the matrix representing T in the basis B. Prove that $\mathcal{M}(T)$ is invertible and that $\mathcal{M}(T)^{-1}$ represents the inverse of T in the basis B.

3. Recall the following construction from Example 3 in the lecture notes. Given V a vector space over $\mathbb{R}$. We define its complexification $V_{\mathbb{C}}$ as follows:

   - as a set we let $V_{\mathbb{C}} := V \times V$;
   - $+ : V_{\mathbb{C}} \times V_{\mathbb{C}} \to V_{\mathbb{C}}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
   - scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

   (i) Check that $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.

   (ii) Given W a vector space over $\mathbb{C}$ we denote by $\bar{W}$ the same set W seen as a vector space over $\mathbb{R}$. Explain why $\bar{W}$ is a vector space over $\mathbb{R}$.

   (iii) Prove that $V \to V_{\mathbb{C}}$ is a subspace when both are seem as vector spaces over $\mathbb{R}$. Argue why any linear map $\varphi : V \to W$ into a real vector space W can be extended as an $\mathbb{R}$-linear map to $V_{\mathbb{C}}$.

   (iv) Let V be a vector space over $\mathbb{R}$ and W a vector space over $\mathbb{C}$. We will denote by $\mathcal{L}_{\mathbb{R}}(V, \bar{W})$ the set of linear maps between V and $\bar{W}$ as vector spaces over $\mathbb{R}$ and by $\mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ the set of linear maps between $V_{\mathbb{C}}$ and W as vector spaces over $\mathbb{C}$. Prove that there exists a bijective function:
   $$\varphi : \mathcal{L}_{\mathbb{R}}(V, \bar{W}) \to \mathcal{L}_{\mathbb{C}}(V_{\mathbb{C}}, W).$$
   (Hint: In how many ways can the map from (iii) be extended?)

(v) (Extra) Is the function φ above linear? Notice that you first need to answer think about why the sets are vector spaces and over which field they are vector spaces.

(vi) (Extra) Does φ preserve properties of the linear operators? For instance, if $T \in \mathcal{L}_{\mathbb{R}}(V, \bar{W})$ is injective is $\varphi(T)$ injective? Same question for surjective.

4. Let V and W be finite-dimensional vector spaces.

   (i) Given a subspace $U \subset V$, prove that there exists $T \in \mathcal{L}(V, W)$ with $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

   (ii) Prove that $T \in \mathcal{L}(V, W)$ is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $ST = \mathrm{Id}_V$, the identity operator on V.

   (iii) Prove that $T \in \mathcal{L}(V, W)$ is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that $TS = \mathrm{Id}_W$, the identity operator on W.

5. Let V_1 and V_2 be two vector spaces. Here is a definition U is a *coproduct* of V_1 and V_2 if

   (a) there are linear maps $\imath_1 : V_1 \to U$ and $\imath_2 : V_2 \to U$;

   (b) for any vector space W and linear maps $f_1 : V_1 \to W$ and $f_2 : V_2 \to W$, there exists an *unique* morphism $g : U \to W$ such that

$$g \circ \imath_1 = f_1 \text{ and } g \circ \imath_2 = f_2.$$

   (i) Give examples of U satisfying condition (a) above.

   (ii) Prove that there exists U satisfying (a) and (b) above.

   (iii) Let U and U' be two vector spaces satisfying (a) and (b) above. Prove that U and U' are isomorphic.

6. Let V_1 and V_2 be two vector spaces. A vector space Z is a *product* of V_1 and V_2 if

   (a) there are linear maps $\pi_1 : Z \to V_1$ and $\pi_2 : Z \to V_2$;

   (b) for any vector space W and linear maps $f_1 : W \to V_1$ and $f_2 : W \to V_2$, there exists an *unique* morphism $h : W \to Z$ such that

$$\pi_1 \circ h = f_1 \text{ and } \pi_2 \circ h = f_2.$$

   (i) Prove that there exists Z satisfying (a) and (b) above.

   (ii) Let U be a coproduct of V_1 and V_2 as defined in Problem 5 and Z be a product of V_1 and V_2. Prove that U and Z are isomorphic.

7. (Extra) Redo Problem 5 and 6, but considering only sets and functions. Is it the case that the coproduct and product are isomorphic (i.e. bijective as sets) in that case?