8/27. - Lecture 1.

10 min - Introductions
Ask majors and take note:

Major	Students.

10min/5min - talk through syllabus.
* Stress Homework.
* Reading of book. / pair w/ exercises.
* Mention pdf of Chapter 1.
* Flipped classroom.

30 min: Sets. - Notation.

$A, B, C, \ldots$ $A = \{a_1, a_2, \ldots, a_n\}$.
$a_2 \in A$ belongs.
element.

$B \subseteq C$ B is a subset of C.
C contains B.

ϕ empty set, defining property?

Usual sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

proper subset $A \subset B$ if there exist an element $b \in B$ s.t. $b \notin A$.

↳ not an elem

Equality of sets:

$$S = \{x \in \mathbb{R} \mid x^2 < x\}.$$
$$T = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

Claim: $S = T$.

let $y \in S$, . $y^2 < y$

$\Rightarrow y^2 - y < 0 \Rightarrow y(y-1) < 0$.

a) $y < 0$ & $y - 1 > 0$

b) $y > 0$ & $y - 1 < 1$ only possible $\Rightarrow y \in$

Conversely, $y \in T \Rightarrow 0 < y < 1$.

$\Rightarrow 0 < y^2 < y \Rightarrow y \in S$.

Learned that $\forall a \in S$ we need to prove $a \in T$ & vice-versa.

Exercise: $S =$ piles that don't change.

$T = \{t_1, t_2, t_3, \dots\}$.

$t_1 = \cdot$

$t_2 = \vdots\ \cdot$ $\qquad \cdot =$ penny. check $S = T$.

$t_3 = \vdots\ \vdots\ \cdot$

$\vdots$

Power sets: given a set A.

$\mathcal{P}(A)$ power set of A

$= \{ B \mid B \subseteq A \}$.

Operations: 1) union $A \cup B = \{ a \mid a \in A$ or $a \in B \}$.

2) intersection. $A \cap B = \{ \_\_\_\_ \mid \& \}$.

3) difference. $A \setminus B = \{ a \in A \,\&\, a \notin B \}$.

4) Cartesian product $A \times B = \{ (a, b) \mid a \in A, b \in B$

Not discussed. 5) Power set.

Exercise: $|A|$ = # of elements in A.

Not done. if $|A| = n$, $|B| = m$.

Check: 1) $|A \cup B| \leq n + m$, 2) $|A \cap B| \geq \min(n, m)$

3) $|A \setminus B| \leq n$ 4) $|A \times B| = n \cdot m$.

$n - \min(n, m) \leq$ 5) $|\mathcal{P}(A)| = 2^n$.

Not done. Idea of universe: U = set to which all sets belong.

Sometimes it is taken to be something more concrete.

* Given a universe U. $A \subseteq U$

$A^c = U \setminus A$.

Example: S^c, for $U = \mathbb{R}$. is $\{ x \in \mathbb{R} \mid x < 0$ or $x > 1 \}$

Lecture 3 - Real numbers. ~~Exam~~

Definition 1) Field. F $(S, +, \cdot, 0, 1)$ $0 \neq 1$.

A (0) $x+y \in S$.

(i) $(x+y)+z = x+(y+z)$.

(ii) $x+y = y+x$

(iii) $x+0 = x$

(iv) given x, $\exists y \in S$ s.t. $x+y = 0$

M (0) $x \cdot y \in S$.

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(ii) $x \cdot y = y \cdot x$

(iii) $x \cdot 1 = x$.

(iv) $\forall x \neq 0$, $\exists y \in S$ s.t $x \cdot y = 1$.

(d) $x \cdot (y+z) = x \cdot y + xz$.

2) Positive set in a field F is $P \subseteq F$ s.t.

(p1) $x, y \in P \Rightarrow x+y \in P$;

(p2) $x, y \in P \Rightarrow x \cdot y \in P$;

(p3) $x \in F \Rightarrow \begin{cases} x = 0 \text{ or} \\ x \in F \text{ or} \\ -x \in F. \end{cases}$

An ordered field is F w/ $P \subseteq F$.

Define: $x < y$ to mean $y - x \in P$.

$x \leq y$ $y - x \in P$ or $y - x = 0$.

An upper bound of $S \subseteq F$, is $\beta \in S$ s.t.

$\forall x \in S$ $\beta \geq x$.

Def (Completeness). An ordered field F is complete if $\forall S \subseteq F$, $S \neq \emptyset$, if S has an upper bound it has a least upper bound.

Note: $\mathbb{R}$ is a complete ordered field.
$\mathbb{Q}$ is an ordered field, but not complete. $\{x \in \mathbb{Q} \mid x^2 < 2\}$.
$\mathbb{Z}$ is not a field why?
What is it? (Ans: ring, group).
Is $\mathbb{N}$ any of these ?

Question: ?

Exercises: 1.39 $\{x \in \mathbb{R} \mid (x-a_1)\cdot \ldots \cdot (x-a_n) < 0\}$.
$\qquad\qquad\qquad\qquad$ $=$ intervals?

1.44 $S = \{(x,y) \mid x^2+y^2 \leq 100\}$, $T = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 14\}$.

a) $S \cap T = ?$
b) $|S \cap T(\mathbb{Z})| = ?$

Remark: In general to count the number of points $\mathbb{Z}$ of says $y^3 = x^2 + 2$ is a rather hard problem.

Trivia: $x^2 - 2y^2 = -1$ has ∞-many solutions given in $\mathbb{Z}$ by $(1+\sqrt{2})^N = a + b\sqrt{2}$. (17th century)

1.56 $\exists$? a field w/ 4 elmts? what about 6?

1.55 $\{0, 1, x\}$ can be made into a field F. Check that $x + x = 1$, $x \cdot x = 1$ add. & mult. tables of F.

Sep. 7 - Lecture 5 - Compound Statements.

1) Get Homework;

2) Assign groups; (Bring cards).

3) ~~Hand~~ Post 2nd HW;

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4) <u>Logical connectives</u>

P a statement. , Q.

$\neg P$ not P, $P \wedge Q$ P and Q,

$P \vee Q$ P or Q, $P \Rightarrow Q$ and $P \Leftrightarrow Q$.

Logical equivalences: $P \Rightarrow Q$ $(\neg P) \vee Q$.

Notice we are assuming: $P \vee (\neg P)$. (excluded middle).

$$x \in A^c \Leftrightarrow \neg(x \in A).$$

$$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B).$$

<u>Ex:</u> $x \in (A \cap B)^c \Leftrightarrow \neg(x \in (A \cap B))$.

$\Leftrightarrow \neg[(x \in A) \wedge (x \in B)] \Leftrightarrow$

$\neg(x \in A) \vee \neg(x \in B)$. $\Leftrightarrow (x \in A^c) \vee (x \in B^c)$

$\Leftrightarrow$ $\underline{x \in A^c \cup B^c}$.

2.38:

Examples: $x, y \in \mathbb{Z}$.

a) xy is odd $\nLeftrightarrow$ x and y are odd;

b) xy is even $\nLeftrightarrow$ x and y are even.

2.39. start at $(0,0)$ in $\mathbb{R}^2$. each day move right or up.

$(a,b) \in P_k$ = reachable on the k^{th} day.

$\Updownarrow$

(i) $|a| + |b| \leq k$;

(ii) a and b have the same parity;

2.18: p a polynomial A = sum of even power coeffs.

B = sum of odd power coeffs.

Prove that: $A^2 - B^2 = p(1) \cdot p(-1)$.

Talk about problems in the previous Worksheet.

Among $y_1, \dots, y_n$ some number is larger as as the average.

$ax^2 + bx + c = 0$ has no rational solution if a, b and c are odd.

Suppose: a, b, c are not divisible by 3.

$$ax^2 + bx + c = 0 \qquad x = p/q.$$

$$ap^2 + b \cdot p \cdot q + c \cdot q^2 = 0. \qquad p \equiv 0,1,2 \quad q \equiv 0,1,2.$$

$$\begin{cases} c \equiv 0 \mod 3. \\ a + 2b + c \equiv 0 \mod 3. \end{cases} \quad (p,q) \in \{(0,1), (0,2), (1,2), (1,0), (2,0), (2,1)\}.$$

1, 2, 1

2, 1, 2

$(p^2, pq, q^2) \in \{(0,0,1), (0,0,1), (1,2,1), (1,0,0), (1,0,0), (0,2,\ \}$

$x \in \mathbb{Z}$

$\{0,1,2\}. \qquad x^2 \in \{0,1\}.$

$$c \equiv 0 \mod 3.$$

$\times$

or

$$a + b + c \equiv 0 \mod 3.$$

or $\quad a + 2b + c \equiv 0 \mod 3.$

$$ax^2 + b = 0. \qquad (0,1) \quad \times$$

$$ap^2 + bq^2 = 0. \qquad (1,0) \quad \times.$$

$$(1,1).$$

$$a + b \equiv 0. \ (\text{mod } 3)$$

$\Rightarrow$

Sep. 17 - Lecture 7 - Induction.

$\forall n \in \mathbb{N} = \{0, 1, 2, \dots\}$. $P(n)$ is a statement.

(i) $P(0)$ is true;
(ii) $P(n) \Rightarrow P(n+1)$ is true;

<u>Aside:</u> $P \Rightarrow Q$ T is different than
P q T.

<u>Example:</u> P = all real numbers are positive
Q = all cats are black.

Consider $x_1, \dots, x_n \in [0,1]$ then

$$\prod_{i=1}^{n} (1 - x_i) \geq 1 - \sum_{i=1}^{n} x_i .$$

<u>Pf: Case $n=1$</u> $(1 - x_1) \geq 1 - x_1$ T.

$n \Rightarrow n+1$

Consider $$\prod_{i=1}^{n+1} (1 - x_i) = \left[\prod_{i=1}^{n} (1 - x_i)\right](1 - x_{n+1}).$$

Notice $(1 - x_{n+1}) \geq 0$. , so since

$$\left[\prod_{i=1}^{n} (1 - x_i)\right] \geq 1 - \sum_{i=1}^{n} x_i \quad \text{by assumption.}$$

$$\left[\prod_{i=1}^{n} (1 - x_i)\right](1 - x_{n+1}) \geq \left(1 - \sum_{i=1}^{n} x_i\right)(1 - x_{n+1})$$

$$= 1 - \sum_{i=1}^{n} x_i - x_{n+1} + \left(\sum_{i=1}^{n} x_i\right) x_{n+1}$$

Finally, since $\left(\sum_{i=1}^{n} x_i\right) x_{n+1} \geq 0$,

$$(*) \geq 1 - \sum_{i=1}^{n} x_i - x_{n+1} = 1 - \sum_{i=1}^{n+1} x_i .$$

That is what we wanted to prove.

<u>Cor:</u> $\forall\, a \in [0,1]$, $(1-a)^n \geq 1 - n \cdot a$.

<u>Rk:</u> The above imply ~~that ...~~ explain a certain difference between compound and simple interest, or in this case deduction.

<u>Goals:</u>

Get an A grade: ⧄ ⊓

Be familiar: ⧄ ▮

Write proofs: ⧄

Further studies: |

Apply: |

<u>Reasons:</u>

Required: ⧄ ⧄ ⧄

Selective: Γ

Blah: ⊓

$n^2 \leq 2^n$ $\quad n+1 \leq 2^{n/2}$

$n \leq 2^{n/2}$

Zhengsan Chang

Also, $n+1 \leq 2^{n-1}$. $n=1$ $n=2$ $n=3$

$\leq 2^n + 2(n+1)$

$\leq 2(2^{n-1} + n + 1)$

$n^2 \leq 2^n$ by induction.

let $n=1 \Rightarrow 1 \leq 2$. Suppose it holds for n.

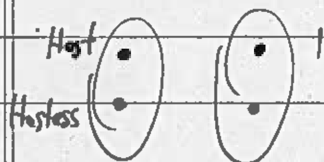
Consider $(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2(n+1)$

Lecture 9. $P(0)$ & $(\forall\, m \le k \quad P(m)$

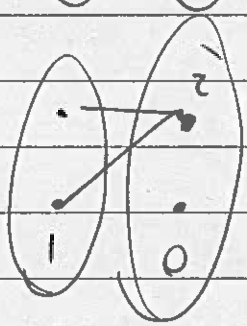
2-piles of coins example. $\Rightarrow P(k+1).)$

$f(xy) = x f(y) + y f(x) \quad \forall_{x,y} \in \mathbb{R}.$

$f(1) = 0$ and $f(u^n) = n u^{n-1} f(u). \quad \forall_{n \in \mathbb{N}}, \forall_{x \in \mathbb{R}}$

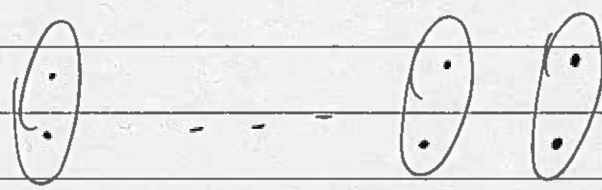


$n = 1$

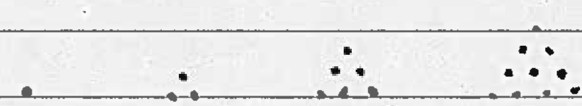


1, 2, ~~3~~
0, 1, 2

$$T_{n+1} = \frac{(n-1)\cdot n}{2} + 3n = \frac{(n-1)\cdot n + 6n}{2} = \frac{n^2 + 5n}{2} = \frac{(n+1)(n+2)}{2}$$



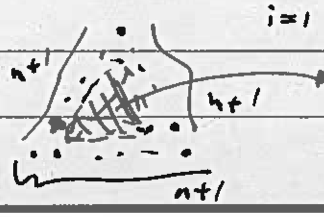
T_n triangular number. n points on each side of a equilateral Δ.



T_1 T_2 T_3 T_4 ...

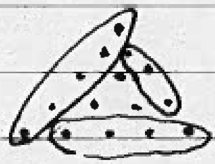
Conj: $T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Suppose: T_{n+1}:



$\to T_{n-1}$ $\Rightarrow |T_{n+1}| = |T_{n-1}|$

$n+1 + n + \cancel{n-1}$



$|T_2| + 5 + 4 + 3 = |T_5|$.

~~Pr~~ For $n = 1$, 2^0.

Suppose $\forall\, k \le n$ (and $n \geq 2$) we can represent k as described

Let $n+1$, consider. m the largest $\mathbb{Z}$ s.t.

$$2^m \le n+1 \quad (\bigstar)$$

Case 1) $2^m = n+1$ done.

2) $2^m < n+1$, then consider $n+1-2^m$.

Since $n+1-2^m \le n$ (i.e. if $m=0$ means $2 > n$.)

We can apply (strong induction to) the inductive hypothesis to $k = n+1-2^m$

to conclude

$$n+1-2^m = 2^{m_2} + \dots + 2^{m_j} \quad w/$$

$$0 \le m_j < m_{j-1} < \dots < m_2.$$

Finally, we claim that $m > m_2$.

Indeed, if $m_2 \geq m$ then

$$(n+1 - 2^m + 2^{m_2}) \geq 0$$

$$\Rightarrow \quad 2^m + 2^{m_2} \le n+1 \Rightarrow \text{contradict w/ } (\bigstar).$$

9/24.

JSmail $1 \to 4$ in Q. S1.

Lecture 11: Properties of functions

Refn: $f: A \to B$ is injective if $\forall\, x,y \in A$
$f(x) = f(y) \Rightarrow x = y$.

Zihe Wu. 14.
HW2.
13 HW

Example: $f(x) = x^3$.

$$f: \mathbb{Z} \to \mathbb{N}$$
$$n \mapsto \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$$

$$f: \mathbb{N} \to \mathbb{N}$$
$$n \mapsto 2n.$$

?.

Refn: $f: A \to B$ is surjective if $\forall\, z \in B, \exists\, x \in A$
s.t. $f(x) = z$.

Examples: $f(x) = x^3 - x$.

$$f: \mathbb{N} \to \mathbb{Z}$$
$$n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even.} \\ (-n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

$$f: \mathbb{N} \to \mathbb{N}$$
$$n \mapsto n.$$

?.

Defn: $f: A \to B$ is <u>bijective</u> if f is injective and surjective.

Prop: If $f: A \to B$ is bijective, then $\exists g: B \to A$ s.t.

$$g \circ f(x) = x \quad \forall x \in A$$

and

$$f \circ g(y) = y \quad \forall y \in B.$$

Pf: For $b \in B$, let $g(b) = a$ where $a \in f^{-1}(b)$.

Notice that g is well-defined, i.e. if $\exists a, a' \in f^{-1}(b)$, then $f(a) = f(a') \Rightarrow a = a'$ b/c f is injective. $\Rightarrow g^{-1}(b) = a = a'$.

Let $x \in A$, then $x \in f^{-1}(f(x))$ by HW 1.

$\Rightarrow g(f(x)) = x$.

Similarly the equality $f \circ g(y) = y$ holds. (cardinality)

Defn: For a set A, the <u>size</u> of A is the unique $n \in \mathbb{N}$ s.t. $\exists$

$$f: A \simeq \{1, \dots, n\}, \text{ a bijection}.$$

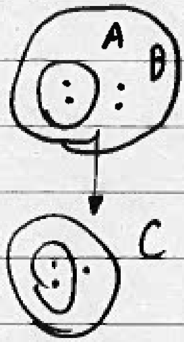
Q: $f(x) = \frac{1}{1+x^2}$ is it injective / surjective?

Defn: $f: A \to B$, $g: B \to C$ then

$h = g \circ f: A \to C$. is the composite.
$x \mapsto g(f(x))$.

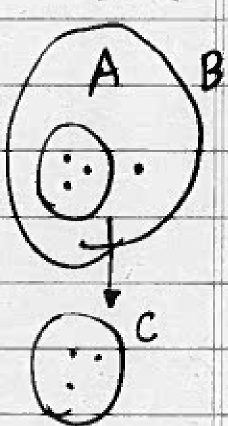
Q: If h is injective, is g or f injective?

~~let x, y s.t. $h(x) = h(y) \Rightarrow x = y$ b/c. h is injective.~~
~~$g(f(x)) = g(f(y)) \Rightarrow f(x) = f(y)$~~



let $x, y \in A$ s.t. $f(x) = f(y)$, then $h(x) = h(y) \Rightarrow x = y$. so f inj.
let $z, w \in B$ s.t. $g(z) = g(w)$, then (?).

Q: If h is surjective is g or f surjective?



Let $z \in C$, then $\exists a \in A$ s.t. $h(a) = z$, but
$f(a) \in B$ w/ $g(f(a)) = c \Rightarrow g$ surjective.
let $w \in B$, consider $g(w) \in C$, then $\exists a \in A$ s.t.
$g(f(a)) = g(w) \not\Rightarrow f(a) = w \Rightarrow f$ surjective.

Q: If f or g (or both) is injective is h injective?
let $x, y \in A$, if $h(x) = h(y)$, $g(f(x)) = g(f(y))$.
g injective $\Rightarrow f(x) = f(y)$. f inj. $\Rightarrow x = y$.
let $z \in C$, if g is surj. $\exists y \in B$ s.t. $g(y) = z$.
If f is surj., $\exists x \in A$ s.t. $f(x) = y$. $\Rightarrow h(x) = z$.
i.e. h is surj.

Q: If f or g (or both) is surjective is h surjective?

9/26. **Lecture 12.**

1) and 3). (10 min. / 15 min.) 25 min.

7) and 9). (10 min. / 15 min.)

1) $f: \mathbb{R} \to \mathbb{R} \longrightarrow T_a f: \mathbb{R} \to \mathbb{R}.$

$C = \{ g: \mathbb{R} \to \mathbb{R}$, g is a function$\}$.

then $T_a: C \to C$, similarly

$S_b: C \to C$.

Notice for each a, b one has a function.

3). If $\forall\ f, g \in C$ if $T_a f = T_a g$ then $f = g$.

$T_a f = T_a g$ if $\forall x \in \mathbb{R}.$

$$T_a f(x) = T_a g(x)$$

$$\left(f(x+a) = g(x+a) \quad \forall x \in \mathbb{R} \right.$$

$$\Updownarrow$$

$$f = g \Leftarrow \left(f(y) = g(y) \quad \forall y \in \mathbb{R} \right).$$

31. 7) Recall A, B disjoint if $A \cap B = \emptyset$.

$|A| = n$, $|B| = m$. then $\exists f: A \xrightarrow{\cong} [n]$,

$g: B \xrightarrow{\cong} [m]$. Consider $h: \{1, \dots, n, n+1, \dots, n+m\} \to A \cup B$

$$h(i) = \begin{cases} f^{-1}(i) & \text{if } i \leq n \\ g^{-1}(i) & \text{if } i \geq n+1. \end{cases}$$

<u>Claim:</u> h is bijective.

<u>h injective:</u> $h|_{[n]}$ is injective and

since $h|_{\{n+1,\dots,m+n\}}$ is injective.

and $h([n]) \cap h(\{n+1,\dots,m+n\}) = \emptyset \Rightarrow h$ injective.

Let $j \in A \cup B \Rightarrow j \in A$ or $j \in B$.

let $i = f(j)$ then $h(i) = j$	let $i = g(j)$ then $h(i) = j$.

(2) previous W.S.

9) Check that $f(x)$ is injective and surjective.

$$\left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x)$$

$$f'(x) = \pi \cdot \sec^2(\pi(x - 1/2)).$$

$\Rightarrow f'(x) > 0 \quad \forall x \in (0,1)$, i.e. $f(x)$ is monotone.

$\Rightarrow f(x)$ is injective previous W.S.

Analogously, consider $g(x) = \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2} \in (-\pi/2, \pi/2)$

$g'(x) = \frac{1}{1+x^2}$ is always positive.

$\Rightarrow g(x)$ is monotone.

Lecture 14:

1) Basic operations w/ sets.

2) What is a statement, how to negate it?

No cardinality. How to check 2 statements are equivalent?

Proof by contradiction?

3) Induction.

4) Properties of functions.

Group 2: Great! Vieta's formula

Group 5: every one.

$f: A \to B.$

$\forall x, y \in A \quad f(x) = f(y) \Rightarrow x = y.$

$g: \mathbb{Q} \to \mathbb{N}.$

$\frac{p}{q} \mapsto 2^p \cdot 3^q$ if $p \geq 0.$

$2^{-p} \cdot 3^q \cdot 5$ if $p < 0.$

$\frac{p}{q}, \frac{a}{b} \quad g(p/q) = g(a/b)$ 1

$2^p \cdot 3^q = 2^a 3^b \overset{?}{\Rightarrow} p = a$ and $q = b$