Zili Zhang

$P = W$, a strange duality for Hitchin systems.

§ Non abelian Hodge theory

Fix C a curve over $\mathbb{C}$ and $n \geq 1$.

First moduli: $M_B = \{\rho : \pi_1(C) \to GL_n(\mathbb{C})\} / \sim$ conjugation.

Second moduli: $M_{dR} = \{(V, \nabla) :$ V a complex v.b. rk. n. &
$\nabla : V \to V \otimes A^1$ a connection C^∞.
$\nabla^2 = 0$ + stability cond.s.$\} / \sim$.

Third moduli: $M_{Dol} = \{(E, \theta) \mid E$ deg. 0 rk n. hol. v.b. on C
$\theta : E \to E \otimes \Omega^1_C$ map of $\mathcal{O}_C$-sheaves
+ stability $\}. / \sim$.

Thm (Simpson): 1. M_B, M_{dR} & M_{Dol}. are algebraic varieties.
2. $\exists$ a canonical diffeomorphism:

$$M_B \underset{\text{diff.}}{\simeq} M_{dR} \underset{\text{diff.}}{\simeq} M_{Dol}.$$

Rk: M_{dR} is algebraic by a theorem that says $\nabla = \partial + \bar{\partial}$ w/ ∂ giving a hol. str. to V and $\bar{\partial}$ an alg. conn. on it.

Example: For C an elliptic curve, $n = 1$.

$$\begin{tikzcd} M_B = \mathbb{C}^* \times \mathbb{C}^* \arrow[r, "\text{diff.}"] & (\mathbb{R}^1 \times S^1) \times (\mathbb{R}^1 \times S^1) \\ M_{Dol} = C^\vee \times \mathbb{C} \arrow[r, "\text{diff.}"] & (S^1 \times S^1) \times (\mathbb{R}^1 \times \mathbb{R}^1) \arrow[u, "\text{diff.}"'] \end{tikzcd}$$

Rk: It is not possible to find an alg. map or even rational map between M_B & M_{Dol}.

§2 Some geometry

~~Ex~~ ~~ℂ×~~

M_B is affine, i.e. $\pi_1(C) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle / (\prod_{i=1}^{g} [\alpha_i, \beta_i] = id)$

& $M_B = \{ (A_i, B_i)_{i=1}^{g} \in GL(n, \mathbb{C})^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = id \} // PGL(n, \mathbb{C})$.

$M_{Dol.} = \{ (E, \theta) \mid E$ deg. 0 rk n. hol. v.b. $/ C$, $\theta : E \to E \otimes \Omega^1_C$ map of $\mathcal{O}_C$-sheaves $\} / \sim$.

$(E, \theta) \rightsquigarrow \det(\lambda I - \theta) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$.

w/ $a_i \in H^0(C, Sym^i(\Omega^1_C))$.

$$\begin{array}{ccc} M_{Dol.} & & (E, \theta) \\ \downarrow h & & \downarrow \\ \bigoplus_{i=1}^{n} H^0(C, Sym^i \Omega^1_C) & \ni & (a_1, \ldots, a_n) \end{array}$$

Thm (Hitchin, Nitsure): h is proper. w/ fibers abelian varieties.

§3. Hodge theoretic interpretation of M_B & M_{Dol}

$$0 \subset W_0 H^k(X) \subset W_1 H^k(X) \subset \cdots \subset W_{2k} H^k(X) = H^k(X).$$

s.t. $W_i H^k(X) / W_{i-1} H^k(X) = \bigoplus_{p+q=i} H^{p,q}$ &

$$H^{p,q} \simeq \overline{H^{q,p}}.$$

Eg: $H^1(\mathbb{C}^\times) \simeq W_2 H^1(\mathbb{C}^\times) / \left(\underset{\substack{\shortparallel \\ 0}}{W_1 H^1(\mathbb{C}^\times)} \right)$

<u>Natural question:</u> $W_\bullet H^*(M_B) \simeq W_\bullet H^*(M_{Dol})$?

<u>Ans:</u> No.

<u>Rk:</u> M_{Dol} is smooth $\Rightarrow$ weights $\in [k, 2k]$.
Contracting it to the fiber at 0, which is projective gives weights $\in [0, k]$ b/c projective & sing.

§4 <u>Perverse filtration & $P=W$ conjectures</u>

$f: X^d \to Y$ f is proper.

$$\mathbb{H}({}^p\tau_{\le k} Rf_* \underline{\mathbb{Q}}_X[d]) \xrightarrow{2k} \mathbb{H}(Rf_* \underline{\mathbb{Q}}_X[d]).$$

$$\overset{\text{smooth}}{(r = \dim X \times_Y X - \dim X)} \qquad (*) \qquad \overset{\text{is}}{H^*(X)[d]}$$

$$P_k H^*(X) := \mathrm{Im}\, 2k.$$

Q: Is there any result about vanishing outside of certain degrees for smooth/proj. fiber. ?

(i) $P_0 H^*(X) \subset P_1 H^*(X) \subset \ldots \subset H^*(X)$.

(ii) $P_\cdot H^*(X)$ depends on f & Y.

In the case $h: \overset{2d}{M_{Dol}} \to \mathbb{A}^d$ proper.

general flag: $\emptyset = \Lambda_{-1} \subset \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_d = \mathbb{A}^d$.

$$X_i = h^{-1}\Lambda_i \subset M_{Dol}.$$

Q2: A gen. description of perverse filtration in more general cases?

$$P_k H^i(X) \overset{\text{defn}}{:=} \ker\{H^i(M_{Dol}) \to H^i(X_{i-k-1})\}. \quad (**)$$

<u>Rk:</u> The equivalence between $(*)$ & $(**)$ is proved by de Cataldo & Migliorini.

$\underline{P = W \text{ conjecture}}$: (dCHM08)

$$H^*(M_B) = H^*(M_{Dol.})$$

$$P_k H^*(M_{Dol.}) = W_{2k} H^{*}(M_B).$$

$$W_{2k} H^*(M_B) = W_{2k+1} H^*(M_B).$$

Rk:

$$\begin{tikzcd} & C \arrow[dl, "top"'] \arrow[dr, "alg"] & \\ M_B \arrow[d, "alg."'] & & M_{Dol.} \arrow[d, "top."] \\ \text{Hodge str.} & & \text{Perverse str.} \end{tikzcd}$$

Thm: (de Cataldo, Hausel, Migliorini) $g \geq 2$, $n = 2$. $P = W$ (✓)

Thm: (de Cataldo, Maulik, Shen '19) $g = 2$, any n. $P = W$ (✓)

$\tilde{D}_4$ $\quad \mathbb{P}^1$ $\quad p_1, \dots, p_4$.

$$M_B = \left\{ \rho : \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_4\}) \to GL(2, \mathbb{C}) \;\middle|\; \rho(\gamma_i) = C_i \right\} / \sim .$$

$C_1, \dots, C_4$ c.c. in $GL(2, \mathbb{C})$ w/ $\prod \det C_i = 1$.

$\underline{P = W}$ (✓)

$$M_{Dol} = \{ (E, \Theta, l_1, \dots, l_4) \mid \Theta : E \to E \otimes \Omega^1 C(p_1 + \dots + p_4) \}.$$

~~Res~~ $\text{res}\, \Theta_i : E|_{p_i} \to l_i \to 0$.

+ stab.

$\boxed{\text{(1)} \ddagger \text{(1)}}$