LINEAR ALGEBRA II
SPRING 2024 - HKU

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These notes will be updated as the semester progresses. Their goal is to present the material from the textbook and the class in a more concise form. I will often try to give slightly different phrasing (and/or proofs) than the one provided in the textbook. The intent is to make you think the concepts through and to work the concepts by yourself.

You are strongly encouraged to do the exercises as you read. They will help you parse the definitions, examples, and concepts used in the proofs of the theory. Some of these exercises will be assigned as Homework or will be discussed in class.

Points in red and blue are still being edited.

I would appreciate any comments. If you find mistakes, which are probably present, please let me know too. I normally revise part of the notes after the class in which we discussed the material, so please refer frequently to the website for the most up-to-date version.

1. Jan. 15, 2024

Plan:

- Introduce the word field, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- Define vector spaces over $\mathbb{F}$.
- Examples: $\mathbb{F}^S$ for S a set.
- Proposition with basic properties: uniqueness of additive identity and inverse, multiplication by 0 gives the zero vector, the zero vector is stable and -1 multiplication gives the additive inverse.

1.1. Fields.

In your previous linear algebra class (Math 2101) you defined a vector space over the real numbers. The very same definition works in a slightly more general context, we start by introducing some terminology for that.

Definition 1. A *field* is a triple $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set, and we have operations (i.e. functions):

- addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$,
- multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$;

satisfying the following list of axioms:

(a) addition and multiplication are associative;

(b) addition and multiplication are commutative;

(c) there exists $0 \in \mathbb{F}$, such that $a + 0 = 0 + a = a$, for all $a \in \mathbb{F}$;

(d) there exists $1 \in \mathbb{F}$, such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in \mathbb{F}$;

(e) $0 \neq 1$;

(f) every $a \in \mathbb{F}$ has an *additive inverse*, i.e. an element $b \in \mathbb{F}$ such that $a + b = b + a = 0$;

(g) every $a \in \mathbb{F} \backslash \{0\}$ has a multiplicative inverse;

(h) distributivity, i.e. for every $a, b, c \in \mathbb{F}$ one has: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Date: Last updated January 25, 2024.

Notation 1. We will omit the $\cdot$ when writing the multiplication operation, i.e. for any $a, b \in \mathbb{F}$ we will write ab for $a \cdot b$.

Example 1. (i) The real numbers $\mathbb{R}$ form a field with usual addition and multiplication.

(ii) The complex numbers $\mathbb{C}$ form a field with usual addition and multiplication.

(iii) The *rational numbers* $\mathbb{Q} := \{\frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{Z}\backslash\{0\}\}$ are a field.

Exercise 1. Write out explicitly what conditions (a-b) and (g) above are and check them in one of the examples in Example 1.

Exercise 2. Let p be a prime number and consider $\mathbb{F}_p = \{0, 1 \ldots, p-1\}$, then for $a, b, c \in \mathbb{F}_p$ we define:

$$a + b := c \text{ if } (a + b - c) \text{ is a multiple of } p, \qquad a \cdot b := c \text{ if } (a \cdot b - c) \text{ is a multiple of } p.$$

(i) Check that $+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ and $\cdot : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ are well-defined.

(ii) Prove that $\mathbb{F}_p$ is a field.

Exercise 3. Can you come up with another example of a field?

1.2. **Vector spaces.** In a previous Linear Algebra class you probably approached vector spaces by concrete examples. The main point of this class is to develop the theory from an abstract point of view focused on proofs, mostly basis-free, and applicable to general fields of characteristic zero, until later results that might require $\mathbb{F}$ to be the real or complex numbers.

Let $\mathbb{F}$ be a field.

Definition 2. A *vector space* over $\mathbb{F}$ is the data of

(i) a set V;

(ii) an operation $+ : V \times V \to V$;

(iii) a *scalar multiplication* operation $\cdot : \mathbb{F} \times V \to V$.

These are subject to the following axioms:

(a) the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and inverse;

(b) the operation $\cdot$ is associative;

(c) for every $v \in V$ one has $1 \cdot v = v$;

(d) scalar multiplication distributes over vector addition (i.e. the operation $+$ on V).

Example 2. (i) The set $\{0\}$ is a vector space over any field $\mathbb{F}$.

(ii) Given a set S consider $\mathbb{F}^S$ the set of functions $f : S \to \mathbb{F}$. The operations are defined by pointwise addition and multiplication, i.e. given $f, g \in \mathbb{F}^S$ and $a \in \mathbb{F}$ we let:

$$(f + g)(s) := f(s) + g(s), \qquad (a \cdot f)(s) := a \cdot f(s),$$

and $0_{\mathbb{F}^S}$ is the zero function.

(iii) For any $n \geq 1$, the set $\mathbb{F}^n$ is a vector space, where the operations are defined as follows. Let $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$, $w = (w_1, \ldots, w_n) \in \mathbb{F}$, and $a \in \mathbb{F}$, then:

$$v + w := (v_1 + w_1, \ldots, v_n + w_n), \qquad a \cdot v := (av_1, \ldots, av_n),$$

and $0_{\mathbb{F}^n} := (0, \ldots, 0)$.

(iv) For any $n, m \geq 1$ the set $M_{m\times n}(\mathbb{F})$ of $m \times n$ matrices with coefficients in $\mathbb{F}$ equipped with matrix addition and scalar multiplication is a vector space over $\mathbb{F}$.

(v) The set $\mathbb{F}^{\mathbb{N}}$ of sequences with value in $\mathbb{F}$ is a vector space with termwise addition and scalar multiplication.

Remark 1. A set G equipped with an operation $+ : G \times G \to G$ satisfying condition (a) above is an *Abelian group*. These objects are very important in algebra and are studied in more detail in an abstract algebra course, e.g. Math3301 (Algebra I).

Lemma 1. *Let V be a vector space over $\mathbb{F}$.*

1. *Given $v \in V$ such that $v + w = w$ for all $w \in V$, then $v = 0_V$.*
2. *The additive inverse is unique.*
3. *For every $v \in V$, we have $0 \cdot v = 0_V$.*
4. *For every $a \in \mathbb{F}$, we have $a \cdot 0_V = 0_V$.*
5. *For every $v \in V$ we have $v + (-1) \cdot v = 0$, i.e. the additive inverse of v is given by $-v := (-1) \cdot v$.*

Proof.
(1) We have $v = v + 0_V = 0_V$, where the first equality follows from Definition 2 (a) and the second from the assumption.
(2) Assume there exists $u_1, u_2 \in V$ such that $u_1 + v = 0_V = u_2 + v$. Then we have: $u_1 = u_1 + 0_V = u_1 + u_2 + v = u_2 + u_1 + v = u_2 + 0_V = u_2$.
(3) Notice $v + 0 \cdot v = (1 + 0) \cdot v = 1 \cdot v = v$. Thus by (1), we have $0 \cdot v = 0_V$.
(4) For any $a \in \mathbb{F}$, we have: $a \cdot 0_V = a \cdot (0_V + 0_V) = a \cdot 0_V + a \cdot 0_V$. By (1), we have $a \cdot 0_V = 0_V$.
(5) Notice $v + (-1) \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0_V$, where in the last step we used (3).

$\square$

Notation 2. (1) Notice that for $a, b \in \mathbb{F}$ and $v \in V$ we have:

$$(ab) \cdot v = a \cdot (b \cdot v)$$

by Defintition 2 (b). Thus, we can omit the $\cdot$ for the operation of scalar multiplication as we omitted it for multiplication in a field (see) without causing ambiguity.

(2) We will denote the additive inverse of v by $-v$.

(3) We will denote 0_V simply by 0. This should not be confused with $0 \in \mathbb{F}$ the identity of the operation $+$ in $\mathbb{F}$, as these live in different sets, except when $V = \mathbb{F}$, in which case the notation is consistent.

Remark 2. (i) The empty set $\emptyset$ is not a vector space. Namely, it fails condition (a) from Definition 2.

(ii) Condition (a) from Definition 2 can be substituted by

(a)' the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and (3) from Lemma 1 holds.

Indeed, assume (a)', then we have $0_V = 0 \cdot w = (1 + (-1)) \cdot w = w + (-1)w$ for every $w \in V$. Thus (a) holds.

Example 3. Let V be a vector space over $\mathbb{R}$. We can define a vector space over the complex numbers $V_\mathbb{C}$, called the *complexification* of V as follows:

- as a set we let $V_\mathbb{C} := V \times V$;
- $+ : V_\mathbb{C} \times V_\mathbb{C} \to V_\mathbb{C}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
- scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

The reader should check that $V_\mathbb{C}$ is a vector space over $\mathbb{C}$.

Exercise 4. Universal property of complexification. Let V be a vector space over $\mathbb{R}$ and W a vector space over $\mathbb{C}$. Notice that W can be seen as a vector space over $\mathbb{R}$, where $a \cdot w := (a + i0) \cdot w$, i.e. using the natural inclusion of $\mathbb{R}$ into $\mathbb{C}$. Let $\mathrm{Hom}_\mathbb{R}(V, W)$ denote the set of linear operators between V and W, where W is seen as a vector space over $\mathbb{R}$ and let $\mathrm{Hom}_\mathbb{C}(V_\mathbb{C}, W)$ denote the set of linear operator between $V_\mathbb{C}$ and W as vector spaces over $\mathbb{C}$. Prove that there exists a bijection:

$$\mathrm{Hom}_\mathbb{R}(V, W) \xrightarrow{\sim} \mathrm{Hom}_\mathbb{C}(V_\mathbb{C}, W).$$

2. Jan. 18, 2024

2.1. Subspaces.

Definition 3. Let V be a vector space, a subset $U \subseteq V$ is said to be a *subspace* if:

(a) $0 \in U$;

(b) the restrictions $+_U : U \times U \to V$ and $\cdot_U : \mathbb{F} \times U \to V$ factors as:

$$\begin{tikzcd} U \times U \arrow[r, "+'_U"] \arrow[dr, "+_U"'] & U \arrow[d] \\ & V \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \times U \arrow[r, "\cdot'_U"] \arrow[dr, "\cdot_U"'] & U \arrow[d] \\ & V \end{tikzcd}.$$

Given a subspace $U \subseteq V$ we will simply write $+ : U \times U \to U$ and $\cdot : \mathbb{F} \times U \to U$ for $+'_U$ and $\cdot'_U$, respectively.

Exercise 5. (i) Check that Definition 3 agrees with Definition (1.33) from the textbook.

(ii) Show that only requiring condition (b) in Definition 3 would not agree with the notion as defined in the textbook.

Example 4. (i) let $U \subset \mathbb{F}^n$ defined as $U := \{(v_1, \dots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}$;

(ii) let $p \in \mathbb{F}[x, y, z]$ be a polynomial of the form $p(x, y, z) = ax + by + cz$, for some constants $a, b, c \in \mathbb{F}$, then $U := \{(v_1, v_2, v_3) \in \mathbb{F}^3 \mid p(v_1, v_2, v_3) = 0\}$ is a subspace;

(iii) the subset of functions $f : [0, 1] \to \mathbb{R}$ which are continuous is a subspace of all the functions from $[0, 1]$ to $\mathbb{R}$;

(iv) let $U \subset \mathbb{F}[x]$ denote the subset of polynomials p such that $p(0) = p'(0) = \cdots = p^{(k)}(0) = 0$;

(v) the set of all sequences of complex numbers whose limit is 0 is a subspace of $\mathbb{C}^{\mathbb{N}}$.

Exercise 6. (i) Let $p \in \mathbb{R}[x, y, z]$ be a polynomial of degree 1 and define the subset:

$$U_p := \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid p(v_1, v_2, v_3) = 0\}.$$

Show that U_p is a subspace if and only if p is of the form taken in (ii) of Example 4.

(ii) With the notation as in (i), assume that $p_1, p_2 \in \mathbb{R}[x, y, z]$ are polynomials of degree 1 with no constant term, prove that

$$U_{p_1} \cap U_{p_2} = U_{p_1 p_2}$$

is a subspace of $\mathbb{R}^3$.

(iii) Can you guess which types of polynomials $p \in \mathbb{R}[x, y, z]$ have the property that U_p is a subspace of $\mathbb{R}^3$.

Exercise 7. Let $\mathbb{F}^{\mathbb{N}}$ be the vector space of sequences over $\mathbb{F}$. For an integer $p \geq 1$, we define the subset $S_p \subset \mathbb{F}^{\mathbb{N}}$ of sequences $(a_n)_{n \geq 1}$ satisfying:

$$\sum_{n=1}^{\infty} |a_n|^p < \infty.$$

Proof or disproof S_p is a subspace for every integer $p \geq 1$.

Definition 4. Given $U_1, U_2 \subseteq V$ two subspaces of V we define the *sum* $U_1 + U_2 \subseteq V$ as the subset of elements $v \in V$ such that there exist $u_1 \in U_1$ and U_2 such that $u_1 + u_2 = v$. For $U_1, \dots, U_k$ a collection of k subspaces of V, we inductively define:[1]

$$U_1 + \cdots + U_k := U_1 + (U_2(\cdots + U_k)).$$

Example 5. (i) let $U_i = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_i \neq 0\}$ for $i = 1, 2, 3, 4$. Then

$$U_2 + U_3 + U_4 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}, \qquad U_1 + U_2 + U_3 + U_4 = \mathbb{F}^4.$$

[1] In fact, this definition is independent of the choice of parenthesization, hence justifying the notation.

(ii) let $U_1 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_3 + v_4 = 0 \text{ and } v_1 + v_2 = 0\}$, let $U_2 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}$, then $U_1 + U_2 = \mathbb{F}^4$.

Exercise 8. In the condition in Definition 4 are the vectors u_1 and u_2 uniquely determined? Compare (i) and (ii) in Example 4.

Definition 5. Given $U_1, U_2 \subseteq V$ two subspaces of V we say that $U_1 + U_2$ is a *direct sum* if u_1 and u_2 are uniquely determined. In this case, we use the notation $U_1 \oplus U_2$[2]. Similarly, given subspaces $U_1, \ldots, U_k \subset V$ we say that $U_1 + \cdots + U_k$ is a direct sum if any vector $v \in U_1 + \cdots + U_k$ can be written in an unique way as $v = u_1 + \cdots u_k$, where $u_j \in U_j$ for $1 \leq j \leq k$. We denote the direct sum by $U_1 \oplus \cdots \oplus U_k$.

Exercise 9. Let $V = \mathbb{F}^4$. Provide three distinct subspaces $U_1, U_2, U_3 \subseteq V$ such that:

$$V_1 + V_2 = V_1 \oplus V_2, V_2 + V_3 = V_2 \oplus V_3, \text{ but } V_1 + V_2 + V_3 \neq V_1 \oplus V_2 \oplus V_3.$$

2.2. Span and linear dependence.

Definition 6. Given a subset $S \subseteq V$ we define $\operatorname{Span} S$ the *span of* S to be the subset of V consisting of vectors $v \in V$ such that

$$v = a_1 v_1 + \ldots + a_k v_k$$

for some $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{F}$, and $v_1, \ldots, v_k \in S$. It is convenient to define $\operatorname{Span} \emptyset = \{0\}$. If $\operatorname{Span} S = V$ we say that S *spans* V.

Remark 3. It is clear that $\operatorname{Span} S$ is a vector space and that it contains S. We claim that $\operatorname{Span} S$ is the smallest subspace of V containing S. Consider a subspace $U \subseteq V$ such that $S \subseteq U$, we claim that $\operatorname{Span} S \subseteq U$. Indeed, given $v \in \operatorname{Span} S$ we have $v = v = a_1 u_1 + \ldots + a_k u_k$ for some $a_1, \ldots, a_k \in \mathbb{F}$, and $v_1, \ldots, v_k \in S$. Since $u_1, \ldots, u_k \in U$ we have $v \in U$. Thus, it follows that $\operatorname{Span} S$ belongs to the intersection of all subspaces of V containing $\operatorname{Span} S$.

Example 6. (i) Consider $\{e_1, \ldots, e_n\} \subseteq \mathbb{F}^n$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 is in the ith position. Then $\operatorname{Span}\{e_1, \ldots, e_n\} = \mathbb{F}^n$.

(ii) Let $a, b, c \in \mathbb{F}$ and consider $S = \{(b, -a, 0), (0, c, -b)\}$, then we have $\operatorname{Span} S = U_p$, where U_p is defined as in Example 4 (ii).

Exercise 10. Let e_i be as in Example 6 (i) and consider the set $S = \{je_i - ie_j\}_{1=i<j=n}$, then

$$\operatorname{Span} S = \{(v_1, \ldots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}.$$

Definition 7. A vector space U is *finite-dimensional* if there exists a finite subset $S \subseteq U$ such that $\operatorname{Span} S = U$.

Example 7. (i) $\mathbb{F}^n$ is finite-dimensional;

(ii) the set $\mathcal{P}_n(\mathbb{F})$ of polynomials of degree at most n;

(iii) for any S a finite set $\mathbb{F}^S$ is a finite-dimensional vector space.

Exercise 11. Check which of the examples of vector spaces defined so far are finite-dimensional.

Definition 8. (1) A *polynomial* with coefficients in $\mathbb{F}$ is a function $p : \mathbb{F} \to \mathbb{F}$ such that

$$p(x) = a_0 + a_1 x + \cdots a_n x^n \tag{1}$$

for some $n \in \mathbb{N}$ and $a_i \in \mathbb{F}$.

(2) We let $\mathbb{F}[x]$ denote the set of polynomials in $\mathbb{F}$, notice that the textbook uses the notation $\mathcal{P}(\mathbb{F})$.

(3) Given a polynomial $p \in \mathbb{F}[x]$ the *degree* of p is the smallest natural number $n \in \mathbb{N}$ such that p can be written as (1). By convention, we set the degree of the zero polynomial to be $-\infty$.

(4) Let $\mathcal{P}_n(\mathbb{F})$ denote the set of polynomials of degree at most n.

[2] At the moment this notation might seem unmotivated, but it will be clearer when we consider this operation on vector spaces.

Exercise 12. Check that $\mathcal{P}_n(\mathbb{F})$ forms a vector space.

Exercise 13. Assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $p : \mathbb{F} \to \mathbb{F}$ be a function. Check that $p \in \mathcal{P}_n(\mathbb{F})$ if and only if $p^{(n+1)} = 0$.

Exercise 14. The set $\mathbb{F}[x] = \mathcal{P}(\mathbb{F})$ is a vector space. Think about how we can formally define this.

Definition 9. Let V be a vector space over $\mathbb{F}$. Given a finite subset $S = \{v_1, \ldots, v_n\} \subset V$ we say that S is *linearly independent* if

$$a_1v_1 + \cdots + a_nv_n = 0 \implies a_1 = \cdots = a_n = 0,$$

where $a_1, \cdots, a_n \in \mathbb{F}$. By convention, we declare that $S = \emptyset$ is linearly independent. We say that a subset $S \subset V$ is linearly dependent if it is not linearly independent.

Example 8. (i) For every $k \in \{1, \ldots, n\}$, the set $S = \{e_1, \ldots, e_k\} \subset \mathbb{F}^n$, where e_i's are defined as in Example 6 (i), is linearly independent.

(ii) For any $k \geq 0$ the set $S_k := \{1, x, \ldots, x^k\} \subset \mathbb{F}[x]$ is linearly independent.

(iii) Given $\{v, w\} \subset V$, then $\{v, w\}$ is linearly independent if and only if $v \neq aw$ for every $a \in \mathbb{F}$ and $bv \neq w$ for every $b \in \mathbb{F}$.

Exercise 15. Given a finite subset $S \subset V$. Prove that S if $0 \in S$ then S is linearly dependent.

Example 9. (1) the subset $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_1\} \subset \mathbb{F}^3$ is linearly dependent.

(2) the subset $S = \{x^2, x^2 - 2x, 3x\} \subset \mathbb{F}[x]$ is linearly dependent.

Exercise 16. Given $S = \{(2, 3, 1), (1, -1, 2), (7, 3, c)\} \subset \mathbb{F}^3$. Check that S is linearly independent if and only if $c = 8$.

Exercise 17. Given $\{v_1, v_2, v_3, v_4\} \subset V$ a linearly independent set. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.

The next result is extremely useful in many future proofs since it allows one to make a linearly dependent set smaller.

Lemma 2. *Let $\{v_1, \ldots, v_n\} \subset V$ be a linearly dependent subset of a vector space V. Then there exists $k \in \{1, \ldots, n\}$ such that*

$$v_k \in \operatorname{Span}\{v_1, \ldots, v_{k-1}\},$$

when $k = 1$ the right-hand side above should be interpreted as $\operatorname{Span}\{v_1\}$. Moreover, one has:

$$\operatorname{Span}\{v_1, \ldots, v_n\} = \operatorname{Span}\{v_1, \ldots, v_n\} \backslash \{v_k\}.$$

Proof. Since $\{v_1, \ldots, v_n\}$ is linearly dependent there exists $a_1, \ldots, a_n \in \mathbb{F}$ not all zero such that

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Thus, let $k \in \{1, \ldots, n\}$ such that $a_k \neq 0$, then we have:

$$v_k = -a_k^{-1}(a_1v_1 + \cdots + a_{k-1}v_{k-1} + a_{k+1}v_{k+1} + \cdots + a_nv_n),$$

where the expression on the right above works for $k \in \{2, \ldots, n-1\}$, we leave it to the reader to write the correct expression for the edge cases. To prove the last assertion we notice that clearly $\operatorname{Span}\{v_1, \ldots, v_n\} \backslash \{v_k\} \subseteq \operatorname{Span}\{v_1, \ldots, v_n\}$. Now suppose that $w \in \operatorname{Span}\{v_1, \ldots, v_n\}$ and let $w = a_1v_1 + \cdots + a_nv_n$. Since $v_k \in \operatorname{Span}\{v_1, \ldots, v_{k-1}\}$, there exists $b_1, \ldots, b_{k-1} \in \mathbb{F}$ such that $v_k = b_1v_1 + \cdots + b_{k-1}v_{k-1}$. Then

$$w = (a_1 + b_1)v_1 + \cdots + (a_{k-1} + b_{k-1})v_{k-1} + \sum_{i=k+1}^{n} a_iv_i,$$

so $w \in \operatorname{Span}\{v_1, \ldots, v_n\} \backslash \{v_k\}$. This finishes the proof. $\square$

Definition 10. Let $T \subseteq V$ be a subset of a vector space. We say that T is a *spanning set of* V if $\operatorname{Span} T = V$.

Lemma 3. *Let V be a finite-dimensional vector space. Consider $S, T \subseteq V$ subsets of a vector space V. Suppose that $\operatorname{Span} S = V$ and that T is a linearly independent subset. Then $|T| \leq |S|$.*

Proof. Let $v_1 \in T$ and consider $S \cup \{v_1\}$. Since $\operatorname{Span} S = V$ we have that $v_1 \in \operatorname{Span} S$, so $S \cup \{v_1\}$ is linearly dependent. By Lemma 2 there exists $u_1 \in S$ such that $\operatorname{Span} S \cup \{v_1\} = \operatorname{Span} S \cup \{v_1\} \backslash \{u_1\}$. Now let $T_1 := T \backslash \{v_1\}$ and $S_1 := T \backslash \{u_1\}$.

Notice that we can repeat this process k times, where $k = |T|$ to obtain two sequences:

$$\emptyset \subset T_k \subset \cdots \subset T_1 \subset T, \qquad \text{and} \qquad S_k \subset \cdots \subset S_1 \subset S$$

where $|S_i| = |S| - i$ and $|T_i| = |T| - i$ for every $i \in \{1, \ldots, k\}$. This implies that $|S| \geq k = |T|$.

This is the same argument as in (2.22) in the textbook. □

Corollary 1. *Let $U \subseteq V$ be a subset of a finite-dimensional vector space V, then U is finite-dimensional.*

Proof. We do an induction on the number of vectors necessary to span U. The base case is $U = \operatorname{Span} \emptyset = \{0\}$, in which case U is finite-dimensional. Assume that $U \neq \{0\}$ and let $v_1 \in U$ be a non-zero vector. Then if $U = \operatorname{Span} v_1$ we are done, otherwise there exists $v_2 \in U$ such that $v_2 \notin \operatorname{Span} v_1$ and we can consider $\operatorname{Span}\{v_1, v_2\}$. We claim that repeating this step k times gives $\operatorname{Span}\{v_1, \ldots, v_k\} = U$ for some $k \in \mathbb{N}$. Indeed, let $S \subset V$ be a finite set such that $\operatorname{Span} S = V$. Such a set exists since V is finite-dimensional. Then consider $\{v_1, \ldots, v_k\} \subseteq \{v_1, \ldots, v_k\} \cup S$. Since $S \subseteq \{v_1, \ldots, v_k\} \cup S$ spans V, we have that $k \leq |S|$, thus k is finite. □

3. Jan. 22, 2024

3.1. **Basis.** The following concept is extremely important in linear algebra. One could say that the main difference between this course and Math2101 is that in Math2101 one is choosing a basis for every vector space that is considered by default, whereas in Math2102 we are not.

Definition 11. A subset $S \subset V$ is a *basis* if it satisfies:

(a) $\operatorname{Span} S = V$;

(b) S is linearly independent.

Example 10. (i) The set $\{e_1, \ldots, e_n\}$ as defined in Example 6 (i) is a basis of $\mathbb{F}^n$.

(ii) The set $\{1, \ldots, x^4\}$ is a basis of $\mathcal{P}_4(\mathbb{F})$ the vector space of polynomials of degree at most 4.

(iii) The sets $\{(7, 5), (-4, 9)\}$ and $\{(1, 2), (3, 5)\}$ are both basis of $\mathbb{F}^2$.

Remark 4. A subset $S = \{v_1, \ldots, v_n\} \subset V$ is a basis of V if and only if every element $u \in V$ can be written as:

$$u = a_1 v_1 + \cdots + a_n v_n$$

for an unique choice of $a_1, \ldots, a_n \in \mathbb{F}$. Indeed, suppose that there are two n-uples $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{F}^n$ such that

$$u = a_1 v_1 + \cdots + a_n v_n, \qquad u = b_1 v_1 + \cdots + b_n v_n,$$

and $a_i \neq b_i$ for some $i \in \{1, \ldots, n\}$. Then we have:

$$\begin{aligned} 0 = u - u &= (a_1 v_1 + \cdots + a_n v_n) - (b_1 v_1 + \cdots + b_n v_n) \\ &= (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n. \end{aligned}$$

Since S is linearly independent, we have that $a_i = b_i$ for all $i \in \{1, \ldots, n\}$.

One of the consequences of Lemma 2 is that any finite spanning set contains a subset which is a basis.

Lemma 4. *Let $T \subset V$ be a finite spanning subset of V. Then there exists $S \subseteq T$ such that S is a basis.*

Proof. We proceed by downward induction. If T is linearly independent we are done. If T is linearly dependent, by Lemma 2 there exists $v \in T$ such that $\operatorname{Span} T \backslash \{v\} = \operatorname{Span} T = V$ and $|T \backslash \{v\}| < |T|$. Since T is finite this process stops and we obtain a basis. □

We get two immediate consequences:

Corollary 2. *(1) Every finite-dimensional vector space V admits a basis.*

(2) Any linearly independent subset $S = \{v_1, \ldots, v_k\} \subset V$ extends to a basis.

Proof. For (1) let T be a finite set such that $\operatorname{Span} T = V$. By Lemma 4 there exists $S \subseteq T$ such that S is a basis of V.

For (2) let $T = \{w_1, \ldots, w_n\}$ be a finite set such that $\operatorname{Span} T = V$. Then $\operatorname{Span} S \cup T = V$. Order the set $T \cup S$ as follows $\{v_1 < v_2 < \cdots < v_k < w_1 < \cdots < w_n\}$, then running the argument in the proof of Lemma 4 we notice that we obtain a subset $R \subset V$ such that:

$$S \subseteq R \subset S \cup T \qquad \text{and} \qquad \operatorname{Span} R = V.$$

□

The following result is interesting because it uses that $\mathbb{F}$ is a field in a serious way. In other words, certain concepts so far would make sense for more general objects as (commutative) rings, i.e. the integers $\mathbb{Z}$, however, the following result is on of the first to fail.

Lemma 5. *Let V be a finite vector space and consider a subspace $U \subseteq V$. Then there exists a subspace $W \subseteq V$ such that $U \oplus W = V$.*

Proof. Notice that U is also finite-dimensional. Let T be a basis for U (it exists by Corollary 2 (1)). By Corollary 2 (2) we can find $T \subset R$ such that R is a basis of V. We claim that $W := \operatorname{Span} R \backslash T$ satisfies $U \oplus W = V$. Indeed, it is clear that $U + W = V$, by Lemma **??** we need to check that $U \cap W = \{0\}$. We give names to the elements of $U = \{v_1, \ldots, v_k\}$ and $W = \{v_{k+1}, \ldots, v_n\}$. Assume by contradiction that there exists a non-zero vector $v \in U \cap W$, then we have

$$v = a_1 v_1 + \cdots a_k v_k = a_{k+1} v_{k+1} + \cdots + a_n v_n.$$

Thus, $a_1 v_1 + \cdots a_k v_k - (a_{k+1} v_{k+1} + \cdots + a_n v_n) = 0$, and since $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is linearly independent, we have that $a_i = 0$ for all $i \in \{1, \ldots, n\}$. So we get a contradiction with $U \cap W \neq \{0\}$. This finishes the proof. □

3.2. **Dimension.** The notion of dimension is rather intuitive. The next result justifies that one can define it in a naïve way.

Lemma 6. *Given T, S two basis of a vector space V, we have $|S| = |T|$.*

Proof. Notice that S and T are both linearly independent sets and spanning sets for V. Thus, Lemma 3 implies that $|S| \leq |T|$ and $|T| \leq |S|$. □

Definition 12. The *dimension* of a vector space V, denoted by $\dim V$, is the size of any basis of V.

Exercise 18. Go through all the examples of finite-dimensional vector spaces we had so far and find out their dimension.

Here are a couple of easy consequences of the defintion.

Lemma 7. *Assume that V is finite-dimensional.*

(1) For any subspace $U \subseteq V$, we have $\dim U \leq \dim V$.

(2) Let $S \subseteq V$ be a linearly independent set, if $|S| = \dim V$, then S is a basis.

(3) Given a subspace $U \subseteq V$ such that $\dim U = \dim V$, then $U = V$.

(4) Let $S \subseteq V$ such that $\operatorname{Span} S = V$ and $|S| = \dim V$, then S is a basis.

Proof. (1) Let $S \subset U$ be a basis of U. Notice that $S \subset V$ is also linearly independent. Now Lemma 3 implies that $|S| \leq |T|$ for any basis T of V, i.e. $|S| \leq \dim V$.

(2) Assume that S is not a basis, then by Corollary 2 (2) there exists $S \subset S'$ such that S' is a basis. However, this would imply that $\dim V = |S| < |S'|$ where S' is a basis of V, which is a contradiction.

(3) Let $S \subset U$ be a basis of U. Since $S \subset V$ is linearly independent in V and $|S| = \dim U = \dim V$, by (2) we have S is a basis of V. Thus, $U = \operatorname{Span} S = V$.

(4) Assume that S is not a basis, i.e. S is linearly dependent, then by Lemma 2 there exist $v \in S$ such that $\operatorname{Span} S \backslash \{v\} = V$. This gives $\dim V \leq |S| - 1$, which is a contradiction. □

Now we investigate how the notion of dimension interacts with sums of subspaces.

Lemma 8. *Let $U_1, U_2 \subset V$ be subspaces of V. Then we have:*

$$\dim U_1 + U_2 = \dim U_1 + \dim U_2 - \dim U_1 \cap U_2.$$

Proof. Let S_{12} be a basis of $U_1 \cap U_2$. By Corollary 2 (2) there exists $S_{12} \subset S_1$ and $S_{12} \subset S_2$ such that S_i is a basis of U_i, for $i = 1, 2$. We claim that $S_1 \cup S_2$ is a basis of $U_1 + U_2$. Clearly, we have $S_1 \cup S_2 \subset U_1 + U_2$, this gives that $\operatorname{Span} S_1 \cup S_2 \subset U_1 + U_2$. Since $U_1 \subseteq \operatorname{Span} S_1 \cup S_2$ and $U_2 \subseteq \operatorname{Span} S_1 \cup S_2$, we obtain $\operatorname{Span} S_1 \cup S_2 = U_1 + U_2$.

Now, we need to check that $S_1 \cup S_2$ is linearly independent. For this we actually need to give names to the elements of S_{12}, S_1 and S_2. Let $S_{12} = \{u_1, \dots, u_i\}$, $S_1 \backslash S_{12} = \{v_1, \dots, v_j\}$ and $S_2 \backslash S_{12} = \{w_1, \dots, w_k\}$. Suppose we have an equation:

$$a_1 u_1 + \cdots + a_i u_i + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0,$$

for some $a_1, \dots, a_i, b_1, \dots, b_j, c_1, \dots, c_k \in \mathbb{F}$. Then solving for $w := c_1 w_1 + \cdots + c_k w_k$ we have that $w \in U_1$. But this also gives that $w \in U_2$. Thus, there exist some scalars $d_1, \dots, d_i$ such that

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots d_i u_i.$$

Now, since $\{w_1, \dots, w_k\} \cup \{u_1, \dots, u_i\} = S_2$ is a linearly independent set, we get that all c_ℓ's vanish. Thus we have that $a_1 u_1 + \cdots + a_i u_i + b_1 v_1 + \cdots + b_j v_j = 0$. Since $S_1 = \{v_1, \dots, v_j\} \cup \{u_1, \dots, u_i\}$ is linearly independent, we get that all a_ℓ's and b_ℓ's also vanish. This finishes the proof. $\square$

The previous result is an example of how questions about vector subspaces can be reduced to set-theoretic questions by using bases. We will return to this in later sections.

Exercise 19. Let V be a ten-dimensional vector space.

(1) Suppose that $U_1, U_2 \subset V$ are subspaces of dimension 6. Prove that there exists two vectors $u_1, u_2 \in U_1 \cap U_2$ such that neither is a scalar multiple of the other.

(2) Suppose that $U_1, U_2, U_3 \subset V$ are subspaces such that $\dim U_1 = \dim U_2 = \dim U_3 = 7$, prove that $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

4. Jan. 25, 2024

4.1. **Linear Maps.** Most objects we encounter in mathematics only have "real" meaning when compared in an appropriate way to other objects of the same type. For instance, when study sets we are naturally lead to studying functions and comparing sets using them.

Vector spaces, a more structure kind of set, need to be compared with each other using a more structured kind of function. We introduce that now:

Definition 13. Let V and W be two vector spaces. A *linear map*, sometimes also called a *linear transformation*, is a function $T : V \to W$ satisfying:

(a) (additivity) $T(u + v) = T(u) + T(v)$ for every $u, v \in U$;

(b) (homogeneity) $T(au) = aT(u)$ for every $a \in \mathbb{F}$ and $u \in U$.

We will let $\mathcal{L}(V, W)$ denote the set of linear maps between V and W, and simply write $\mathcal{L}(V) := \mathcal{L}(V, V)$ for the set of linear maps from V to itself. Also notice that for any linear map T, one has $T(0) = 0$.

Example 11. Let V be a vector space.

(i) The zero map $0 : V \to V$, where $0(V) := 0_V = 0$ is a linear map.

(ii) The identity map $\mathrm{Id}_V : V \to V$ given by $\mathrm{Id}_V(v) = v$.

(iii) Given any $a \in \mathbb{F}$ then $T_a(v) := a \cdot v$ is a linear map.

(iv) Differentiation $D : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear map, i.e. $D(p) := p'$.

(v) Integration $T_{[0,1]} : \mathbb{R}[x] \to \mathbb{R}$ given by $T_{[0,1]}(p) := \int_0^1 p(x)dx$.

(vi) Let $q \in \mathbb{R}[x]$ then $T_q(p)(x) := p(q(x))$ is a linear map.

(vii) Let consider a collection of scalars $a_{i,j} \in \mathbb{F}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. $a_{1,1}, \ldots, a_{n,1}, a_{1,2}, a_{2,2}, \ldots, a_{n,2}, \ldots, a_{1,m}, \ldots, a_{n,m} \in \mathbb{F}$, then

$$T(v_1, \ldots, v_n) := (a_{1,1}v_1 + \cdots + a_{1,n}v_n, \ldots, a_{m,1}v_1 + \cdots + a_{m,n}v_n)$$

is a linear map.

Exercise 20. Prove that any linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ is of the form given in Example 11.(vii).

Lemma 9. *Let V and W be finite-dimensional vector spaces and $\{v_1, \ldots, v_n\} \subset V$ be a basis of V. Given any subset $\{w_1, \ldots, w_n\} \subset W$ there exists an unique linear map $T : V \to W$ such that*

$$T(v_i) = w_i \qquad \text{for } i \in \{1, \ldots, n\}. \tag{2}$$

Proof. We first define T. Given any $v \in V$ can be written as $v = a_1v_1 + \cdots + a_nv_n$ we let $Tv := c_1w_1 + \cdots + c_nw_n$. Notice that this is well-defined, since there is only one single way of written v as above and that it satisfies the conditions required. It is clear that it is a linear operator, we leave the details to be checked to the reader. Finally, assume that there exists $T' \in \mathcal{L}(V, W)$ satisfying equations (2). Then for any $a_i \in \mathbb{F}$ we have $T'(a_iv_i) = a_iw_i$, thus for any $v \in V$, which can be written uniquely as $v = a_1v_1 + \cdots + a_nv_n$ we obtain:

$$T'(v = a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n = T(v).$$

This finishes the proof of uniqueness.

We leave the details to check that T is a linear map to the reader. □

We now observe that the set $\mathcal{L}(V, W)$ can be naturally endowed with the structure of a vector space.

Lemma 10. *For any two vector spaces V, W the set $\mathcal{L}(V, W)$ is a vector space with addition and scalar multiplication defined as:*

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \qquad \textit{and} \qquad (a \cdot T)(v) := a \cdot T(v).$$

Proof. The details are left to the reader. □

Exercise 21. Given U, V, and W vector spaces we can consider the composition operation:

$$\begin{aligned}(-) \circ (-) : \mathcal{L}(V, W) \times \mathcal{L}(U, V) &\to \mathcal{L}(U, W) \\ (S, T) &\mapsto S \circ T(v) := S(T(v)).\end{aligned}$$

(i) Check that $\circ$ defined above is a linear map.

(ii) Check that the operation $\circ$ is associative and that it has identity elements. Part of the exercise is making sense of what that means.

Remark 5. This is an abstract remark and can be skipped as we will not use this concept in this course.. A mathematical concept that is really helpful in organizing certain mathematical objects is that of a category. You can look up its definition here. We essentially just showed that vector spaces together with linear maps form a category Vect. In fact, the category Vect has many nice properties.

Exercise 22. Let V be a vector space, such that $\dim V > 1$. Prove that there exists $T, S \in \mathcal{L}(V)$ such that $ST \neq TS$.

4.2. **Null spaces and ranges.** In this subsection we define subspaces that are naturally associated to a linear operator.

Definition 14. Let $T : V \to W$ be a linear map.

(1) the *null space* of T:

$$\operatorname{null} T := \{v \in V \mid Tv = 0\};$$

(2) the *range* of T:

$$\operatorname{range} T := \{w \in W \mid Tv = w, \text{ for some } v \in V\}.$$

We check that $\operatorname{null} T$ and $\operatorname{range} T$ are in fact subspaces of V and W, respectively. Let $u, v \in \operatorname{null} T$ we have

$$T(u+v) = T(u) + T(v) = 0, \qquad \text{and} \qquad T(a \cdot u) = a \cdot T(u) = a \cdot 0 = 0.$$

Assume that $w_1, w_2 \in \operatorname{range} T$, then there exist $u_1, u_2 \in U$ such that $T(u_i) = w_i$, for $i = 1, 2$. Then we have

$$T(u_1 + u_2) = w_1 + w_2, \qquad \text{and} \qquad T(a \cdot u_1) = a \cdot w_1,$$

thus $\operatorname{range} T$ is a vector space.

Remark 6. The null space is sometimes also called *kernel* of T. The range is sometimes called *image* of T.

The next results shows how the kernel and range related to the notions of injective and surjective.

Lemma 11. *Let $T : V \to W$ be a linear operator.*

(1) T is injective if and only if $\operatorname{null} T = \{0\}$*;*

(2) T is surjective if and only if $\operatorname{range} T = W$*.*

Proof. For (1), first assume that T is injective. Assume that there exists a non-zero vector $v \in \operatorname{null} T$. However, this is a contradiction with $Tv \neq 0$, since $v \neq 0$. Now assume that T is injective, and suppose that $Tv = 0$. Since $Tv = T(0) = 0$ and T is injective, we get $v = 0$. For (2) there is nothing to check. □

Theorem 1 (Fundamental theorem of linear maps)**.** *Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof. Let $\{u_1, \ldots, u_n\}$ be a basis of $\operatorname{null} T$. By Lemma 2 (2) we can extend it to $\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+m}\}$ a basis of V. We claim that $\{Tu_{n+1}, \ldots, Tu_{n+m}\}$ is a basis of $\operatorname{range} T$. Indeed, let $w \in W$, then there exist $v \in V$ such that $Tv = w$. Since $\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+m}\}$ is a basis of V, there exists scalars a_i's such that $v = a_1 u_1 + \cdots + a_{n+m} u_{n+m}$ and we have:

$$w = T(v) = T(a_1 u_1 + \cdots + a_{n+m} u_{n+m}) = a_1 T(u_1) + \cdots a_n T(u_n) + a_{n+1} T(u_{n+1}) + \cdots + a_{n+m} T(u_{n+m}).$$

Since the first n terms vanish, we get $w = a_{n+1} T(u_{n+1}) + \cdots + a_{n+m} T(u_{n+m})$. This shows that $\{Tu_{n+1}, \ldots, Tu_{n+m}\}$ is a spanning set. We now check that it is also linearly independent. Assume that there exists scalars $b_1, \ldots, b_m \in \mathbb{F}$, not all zero, such that we have

$$b_1 T(u_{n+1}) + \cdots + b_m T(u_{n+m}) = 0$$

this implies that $b_1 u_{n+1} + \cdots + b_m u_{n+m} \in \operatorname{null} T$. However, since $\{u_1, \ldots, u_n\}$ is a basis of $\operatorname{null} T$, it means there are scalars $c_1, \cdots, c_n \in \mathbb{F}$, not all zero, such that:

$$c_1 u_1 + \cdots + c_n u_n = b_1 u_{n+1} + \cdots + b_m u_{n+m}.$$

However, this is a contradiction with $\{u_1, \ldots, u_{n+m}\}$ being a basis of V.

This finishes the proof. □

Here are a couple of easy consequences of the previous result.

Corollary 3. *Let V and W be finite-dimensional vector spaces.*

(1) Assume that $\dim V > \dim W$*, then any linear map $T : V \to W$ is not injective.*

(2) Assume that $\dim V < \dim W$*, then any linear map $T : V \to W$ is not surjective.*

Proof. (1) Assume by contradiction that there exist $T : V \to W$ an injective linear map. By Lemma 11.(1) we have $\operatorname{null} T = 0$ and Theorem 1 implies $\dim V = \dim \mathit{range} T \leq \dim W$, which is a contradiction. (2) Assume by contradiction that there exist $T : V \to W$ a surjective linear map. Then Lemma 11.(2) and Theorem 1 implies that $\dim W = \dim \operatorname{range} T \leq \dim T$, since $\dim \operatorname{null} T \geq 0$, which is also a contradiction. □

In fact, we can deduce some statements you will be familiar with from Math 2101 from Corollary 3.

Remark 7. (1) Any homogeneous system of linear equations with more variables than equations has a nonzero solution. Consider $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$, a system of linear equations is given by an $m \times n$ matrix A, which gives a linear operator $T_A : \mathbb{F}^n \to \mathbb{F}^m$, by Exercise 20. By Corollary 3(1) we get that there exists $v \in \mathbb{F}^n$ such that $Tv = 0$.

(2) Any inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms. Again consider $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ and a linear transformation $T : V \to W$ encoding the system of linear equations. If $m > n$, then by Corollary 3.(2) there exists $w \in W$ such that $Tv = w$ has no solution.

4.3. **Matrices.** Given two positive natural numbers $n, m \geq 1$ and a field $\mathbb{F}$ an *m-by-n matrix* with coefficients in $\mathbb{F}$ is a list $(a_{i,j})_{1\leq i\leq m,\, 1\leq j\leq n}$ sometimes denoted by:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

We let $\mathbb{F}^{m,n}$ denote the vector space of m by n matrices. You should make sure you understand why this is a vector space.

Definition 15. Given $T \in \mathcal{L}(V, W)$ a linear map between two vector spaces V, W. Let $B_V = \{v_1, \dots, v_n\}$ be a basis of V and $B_W = \{w_1, \dots, w_m\}$ be a basis of W. The *matrix A associated to T and these basis* is defined by:

$$Tv_j = a_{1,j}w_1 + \cdots + a_{m,j}w_m \qquad \text{for } 1 \leq j \leq n.$$

Sometimes we emphasize the dependence on the basis by denoting $A := \mathcal{M}(T, B_V, B_W)$.

Example 12. (i) Let $T : \mathcal{P}_3(\mathbb{F}) \to \mathcal{P}_2(\mathbb{F})$ be the linear map associated to the differentiation. Consider the bases $B_3 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$ of $\mathcal{P}_3(\mathbb{F})$ and $\mathcal{P}_2(\mathbb{F})$, respectively. Then we have:

$$\mathcal{M}(T, B_3, B_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(ii) Let

(iii) Consider $T : \mathbb{R}^4 \to \mathbb{R}^4$ the linear map given by $T(v_1, v_2, v_3, v_4) = (v_1 + v_2, v_3, 0, v_2)$. On the basis $B_V = \{e_1, e_2, e_3, e_4\}$ for the source $V = \mathbb{R}^4$ and $B_w = \{e_1, e_2, e_3, e_4\}$ for the target, we have:

$$\mathcal{M}(T, B_V, B_W) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, if we take the basis $B_W^{(2)} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4\}$ then we have:

$$\mathcal{M}(T, B_V, B_W^{(2)}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

Exercise 23. (i) Make sure that you understand how to obtain the matrices in the Example above.

(ii) Can you come up with a basis of $\mathbb{R}^4$ in Example 12.(iii) where all of the columns are non-zero?

(iii) Can you come you with a basis B_W such that the set of column vectors of the matrix in Example 12.(iii) are linearly independent?

Remark 8. The construction from Definition 15 sends the operations of addition and scalar multiplication of linear maps to the corresponding operations between matrices as defined in Math 2101. Indeed, given two linear maps $T, S \in \mathcal{L}(V, W)$, a scalar $a \in \mathbb{F}$, and two bases B_V of V and B_W of W. Then we have:

$$\mathcal{M}(T + S, B_V, B_W) = \mathcal{M}(T, B_V, B_W) + \mathcal{M}(T, B_V, B_W) \qquad \text{and} \qquad \mathcal{M}(aT, B_V, B_W) = a\mathcal{M}(T, B_V, B_W).$$

Question: When did you learn matrix multiplication? Have you every thought what was the *meaning* behind the rule of how to multiply matrices?

Lemma 12. *Consider U, V, and W three vector spaces and suppose that we picked bases B_U, B_V and B_W, respectively. Consider $T : U \to V$ and $S : V \to W$ two linear maps. Then*

$$\mathcal{M}(S \circ T, B_U, B_W) = \mathcal{M}(S, B_V, B_W)\mathcal{M}(T, B_U, B_V), \tag{3}$$

where on the righthand side above we consider the multiplication of matrices.

Proof. This is done on page 73 of the textbook. We leave the details to the reader. □

Problem 1. Does the formula fail if we take different bases for V, i.e. do we have

$$\mathcal{M}(S \circ T, B_U, B_W) = \mathcal{M}(S, B'_V, B_W)\mathcal{M}(T, B_U, B_V)$$

for B'_V different than B_V?

Let's recall a couple of concepts from Math 2101.

Definition 16. Given a matrix A represented as follow:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}. \tag{4}$$

(1) The *column space of* A, denoted $\mathrm{col}(A)$, is defined as the span of $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^m$, where $v_i = (a_{1,i}, \ldots, a_{m,i})$ for $1 \leq i \leq n$.

(2) The *row space of* A is the span of $\{w_1, \ldots, w_m\}$ in $\mathbb{R}^n$ spanned by $w_j = (a_{j,1}, \ldots, a_{j,n})$ for $1 \leq j \leq m$.

Lemma 13. *For any $m \times n$ matrix A we have*

$$\dim \mathrm{col}(A) = \dim \mathrm{row}(A).$$

Proof. **TODO:** Write this. □

4.4. **Isomorphisms.** The following notion is going to allow us to compare vector spaces and identify when there are "essentially" the same for all purposes of linear algebra.

Definition 17. A linear map $T : V \to W$ is *invertible* if there exists a map $S : W \to V$ such that

$$S \circ T = \mathrm{Id}_V \qquad \text{and} \qquad T \circ S = \mathrm{Id}_W.$$

Remark 9. Notice that if an inverse exists it is unique. Indeed, assume that S_1 and S_2 are inverses of T. Then we have $S_1 = S_1 \circ T \circ S_2 = S_2$. Thus, we will denote by T^{-1} the uniquely determied inverse, if it exists.

Example 13. Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v_1, v_2, v_3) = (v_1 + v_2 + v_3, v_2, v_3)$. Then $T^{-1}(u_1, u_2, u_3) = (u_1 - u_2 - u_3, u_2, u_3)$.

Exercise 24. Determine which ones of the linear maps that we consider so far are invertible and determine their inverses.

Lemma 14. *Let $T : V \to W$ be a linear map. The following are equivalent:*

(1) T is invertible;

(2) T is injective and surjective;

(3) $\mathrm{null}\, T = \{0\}$ and $\mathrm{range}\, T = W$.

Proof. The equivalence (2) ⇔ (3) is Lemma 11. Assume (1). Let $v, u \in V$ such that $Tv = Tu$ then we have $v = T^{-1}Tv = T^{-1}Tu = u$, thus T is injective. Now let $w \in W$ be any vector, then $T^{-1}w \in V$ satisfies $TT^{-1}w = w$, so T is surjective.

Assume (2). For any $w \in W$ we let $S(w) = v$ for any $v \in V$ such that $Tv = w$. This is well-defined, since by the injectivity of T, there is only one v satisfying $Tv = w$. We claim that S is an inverse of T. Indeed, we have $TSw = w$ and $STv = v$ for every $w \in W$ and $v \in V$. □

In the case where our vector space is finite-dimensional, then we have a stronger version of Lemma 14.

Corollary 4. *Assume that V and W are finite-dimensional vector space and that $\dim V = \dim W$ and consider $T \in \mathcal{L}(V, W)$. The following are equivalent:*

(1) *T is invertible;*

(2) *T is injective;*

(2)' $\operatorname{null} T = \{0\}$*;*

(3) *T is surjective;*

(3)' $\operatorname{range} T = W$*.*

Proof. The implications $(1) \Rightarrow (2)/(2)'$ and $(1) \Rightarrow (3)/(3)'$ are clear.

The equivalences $(2) \Leftrightarrow (2)'$ and $(3) \Leftrightarrow (3)'$ were establishes in Lemma 11.

Assume (2)', then by Theorem 1 we have $\dim V = \dim \operatorname{range} T = \dim W$, so we have (3)'. Now (2)' and (3)' imply (1) by Lemma 14.

Assume (3)', then by Theorem 1 we have $\dim V = \dim \operatorname{range} T - \dim \operatorname{null} T = \dim W - \dim \operatorname{null} T$. Thus, $\dim \operatorname{null} T = 0$, so we have (2)'. Now (2)' and (3)' imply (1) by Lemma 14. □

Remark 10. It is important to notice that the assumption that V and W are finite-dimensional in Corollary 4 is crucial. Indeed, consider the linear map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ given by

$$T(f)(n) = \begin{cases} 0 \text{ if } n = 0, \\ f(n+1) \text{ else.} \end{cases}$$

This is injective but not an isomorphism.

Exercise 25. Write an example of a linear map which is surjective but not an isomorphism.

Exercise 26. Prove that there exists a polynomials $p \in \mathbb{R}[x]$ such that $((x^2 + 5x + 7)p)'' = q$ for any $q \in \mathbb{R}[x]$.

Exercise 27. Consider $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, V)$ two linear maps. Assume that $\dim V = \dim W < \infty$. Prove that $ST = \mathrm{Id}_W$ if and only if $TS = \mathrm{Id}_V$.

The next concept plays the role for vector spaces of what bijections are for sets.

Definition 18. Two vector spaces V and W are said to be *isomorphic* if there exists an invertible linear map $T : V \to W$, equivalently if there exists an invertible linear map $S : W \to T$. In this case either morphism $T : V \to W$ or $S : W \to V$ are called *isomorphisms*.

Notation 3. We will sometimes simply write $V \simeq W$ to say that V and W are isomorphic.

It turns out that it is rather easy to determine if two finite-dimensional vector spaces are isomorphic or not as the next result shows.

Lemma 15. *Let V and W be two finite-dimensional vector spaces. The following are equivalent:*

(1) $V \simeq W$*.*

(2) $\dim V = \dim W$*.*

Proof. Assume (1) and let $T : V \to W$ be an invertible linear map. By Theorem 1 we have:

$$\dim T = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{range} T = \dim W,$$

where the second and third equalities above follow from Corollary 4.

Assume (2) and let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_n\}$ be a basis of W. Let $T(c_1v_1 + \cdots + c_nv_n) := c_1w_1 + \cdots + c_nw_n$, which is well-defined as argued in the proof of Lemma 9. We claim T is injective. Indeed, assume that $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n = 0$ for some non-zero combination of c_i's, then linear independence of w_i's imply that all c_i's are zero which gives that $c_1v_1 + \cdots + c_nv_n = 0$. Thus, by Corollary 4 we are done. □

The next result justify the idea that on can always think of a linear map between finite-dimensional vector spaces as a matrix. Notice however that this depends on the choice of bases.

Lemma 16. *Let V and W be finite-dimensional vector spaces of dimensions $n = \dim V$ and $m = \dim W$. Then $\mathcal{L}(V, W) \simeq \mathbb{F}^{m,n}$.*

Proof. Let $B_V = \{v_1, \ldots, v_n\}$ be a basis of V and $B_W = \{w_1, \ldots, w_m\}$ be a basis of W. We claim that

$$\mathcal{M} := \mathcal{M}(-, B_V, B_W) : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$$

as defined in Definition 15 is an isomorphism. First notice that $\mathcal{M}$ is a linear map by Remark 8. First we prove that $\mathcal{M}$ is injective. Indeed, assume that $\mathcal{M}(T) = 0$, then $Tv = 0$ for every $v \in B_V$ since B_V is a basis we get $Tu = 0$ for every $u \in V$, thus $T = 0$. Now we prove that $\mathcal{M}$ is surjective. Let $A \in \mathbb{F}^{m,n}$, represented as equation (4). Let $Tv_i := a_{1,i}w_1 + \cdots + a_{m,i}w_m$ for $1 \le i \le n$. This is a well-defined linear operator and it is clear that $\mathcal{M}(T) = A$. □

Here is a nice consequence of the discussion so far:

Corollary 5. *For any finite-dimensional vector spaces V and W, we have $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$.*

Proof. By Lemma 16 and Lemma 15 we have that

$$\dim \mathcal{L}(V, W) = \mathbb{F}^{\dim W, \dim V}.$$

The result now follows from calculating the dimension of the space of $\dim V$ by $\dim W$ matrices. □

The following is variation on Definition 15.

Definition 19. Let V be a vector space of dimension n, given a basis $B_V = \{v_1, \ldots, v_n\}$ we let $\mathcal{M}(V, B_V) : V \to \mathbb{F}^n$ denote the linear map determined as follows

$$\mathcal{M}(V, B_V)(v) = (a_1, \ldots, a_n) \text{ if } a_1v_1 + \cdots + a_nv_n = v.$$

You should check this is well-defined and indeed a linear map.

Remark 11. Notice that $\mathcal{M}(V, B_V)$ is always an isomorphism. Indeed, the same argument as in the proof of Lemma 16 works, we simply take W to be $\mathbb{F}$.

The next result relates a notion from Math 2101 which was defined using a basis, namely the column space of a matrix, with the range of a linear map which didn't depend on a basis to be defined.

Proposition 1. *Let $T : V \to W$ be a linear map between finite-dimensional vector spaces and let $B_V = \{v_1, \ldots, v_n\}$ and $B_W = \{w_1, \ldots, w_n\}$ be basis of V and W, respectively. The restriction of $\mathcal{M}(W, B_W)$ to $\operatorname{range} T$ has the following factorization:*

$$\begin{tikzcd}
\operatorname{range} T \arrow[r, "\varphi"] \arrow[d, "\subset"'] & \operatorname{col}(\mathcal{M}(T, B_V, B_W)) \arrow[d] \\
W \arrow[r, "\mathcal{M}(W, B_W)"'] & \mathbb{F}^{\dim W}
\end{tikzcd}$$

and φ is an isomorphism.

Proof. Let $w \in \operatorname{range} T$ we need to check that $\mathcal{M}(W, B_W)(Tv)$ belons to $\operatorname{col}(\mathcal{M}(T, B_V, B_W))$. Let $(a_{i,j})_{1 \le i \le m; 1 \le j \le n}$ denote the entries of the matrix $\mathcal{M}(T, B_V, B_W)$. Given $w \in \operatorname{range} T$ there exists $b_1, \ldots, b_n \in \mathbb{F}$ such that $w = T(b_1v_1 + \cdots + b_nv_n)$, we have

$$w = b_1T(v_1) + \cdots + b_nT(v_n).$$

Now we notice that $\mathcal{M}(W, B_V)(T(v_1)) = (a_{j,1}, \ldots, a_{j,m}) \in \mathbb{F}^m$ (Check this!). By definition we have

$$\operatorname{col}(\mathcal{M}(T, B_V, B_W)) = \operatorname{Span}\left(\{(a_{1,1}, \ldots, a_{1,m}), \ldots, (a_{n,1}, \ldots, a_{n,m})\}\right).$$

Thus, linearity of $\mathcal{M}(W, B_W)$ implies

$$\mathcal{M}(W, B_W)(w) \in \operatorname{Span}\{\mathcal{M}(W, B_W)(Tv_1), \ldots, \mathcal{M}(W, B_W)(Tv_n)\} = \operatorname{col}(\mathcal{M}(T, B_V, B_W))$$

as required. This shows that φ factors as claimed.

We now check that φ is an isomorphism. We first notice that it is injective since it is the restriction of an injective linear map. Finally, let $w \in \operatorname{col}(\mathcal{M}(T, B_V, B_W))$, then it can be written as $w = d_1\mathcal{M}(W, B_W)(Tv_1) + \cdots + d_n\mathcal{M}(W, B_W)(Tv_n)$ for some $d_1, \ldots, d_n \in \mathbb{F}$. By the linearity of $\mathcal{M}(W, B_W)$ we have

$$w = \mathcal{M}(W, B_W)(d_1Tv_1 + \cdots d_nTv_n)$$

and it is clear that $d_1Tv_1 + \cdots d_nTv_n \in \operatorname{range} T$. This finishes the proof. $\square$

The proposition immediately imply:

Corollary 6. *Let V, W be finite-dimensional vector spaces, $T : V \to W$ a linear map and B_V and B_W bases, then*

$$\dim \operatorname{range} T = \dim \operatorname{col}(\mathcal{M}(T, B_V, B_W)) = \dim \operatorname{row}(\mathcal{M}(T, B_V, B_W)).$$

We emphasize that in Corollary 6 the quantity $\dim \operatorname{range} T$ is defined without any appeal to a basis either of V or W.

We end this subsection stating a useful formula for change of bases. Before it we introduce some notation to simplify the formula. Let V be a finite-dimensional vector space and consider $T : V \to V$ a linear map and B_V a basis of V, then we simply write:

$$\mathcal{M}(T, B_V) := \mathcal{M}(T, B_V, B_V).$$

Lemma 17. *Let V be a finite-dimensional vector space and consider $T : V \to V$ a linear map and B_V^1 and B_V^2 two bases of V. Then we have:*

$$\mathcal{M}(T, B_V^1) = \mathcal{M}(\operatorname{Id}_V, B_V^1, B_V^2)^{-1}\mathcal{M}(T, B_V^2)\mathcal{M}(\operatorname{Id}_V, B_V^1, B_V^2).$$

Proof. This is (3.84) from the textbook. Write details. $\square$

5. Jan. 29, 2024

5.1. Products and Quotients of Vector spaces.