Math 347 Worksheet

Lecture 16: Completeness, Limits and Convergence

October 10, 2018

Consider $S = \{x \in \mathbb{R} \mid x^2 < 2\}$ and let $\alpha = \sup(S)$. We want to prove that $\sup(S) = \alpha$ satisfies $\alpha^2 = 2$.

(i) Prove that for any positive real numbers a, b one has

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Moreover, if $a \neq b$, then the above is a strict inequality.

For a solution, check Prop. 1.4 pg. 5 of the book.

(ii) Consider $y = \frac{1}{2}\left(\alpha + \frac{2}{\alpha}\right)$, prove that

$$\alpha^2 < \left(\frac{2}{y}\right)^2.$$

(iii) Prove that if a, b are positive real numbers, then $a^2 < b^2$ implies $a < b$.

(iv) Finish the argument from class to show that $\alpha^2 \neq 2$.

For a solution of (ii-iv), see 13.7 Solution pg. 257-258 of the book.

1. Prove that for any a, b positive real numbers, there exist $n \in \mathbb{N}$ such that $an > b$.

   This is argued in Theorem 13.9, pg 258.

2. For $a, b \in \mathbb{R}$ is the statement below true or false?

   The interval $(a, b]$ contains its supremum and infimum.

   See Example 13.6, pg 257.

3. Suppose that the sequence $(x_n)_{n \geq 0}$ does not converge to zero. Express this as a formal statement.

4. Consider the sequences $x_n = n$ and $y_n = 1/n$. Are they bounded or monotone? Does any of them converge? Check this using the definitions.

5. Consider $\lim a_n < \lim b_n$ for two sequence a_n and b_n that admit limits. Prove that for N sufficiently large $a_N < b_N$.

   See 13.13 Definition, pg. 260 for the meaning of sufficiently large.

I. If $S \subset \mathbb{R}$ then $\alpha = \sup(S)$ if and only if α is an upper bound of S and there exists a sequence in S converging to α.

II. If a_n and b_n are two nondecreasing sequences, and

$$\lim(a_n - b_n) = 0.$$

Then both sequences converge and have the same limit. Prove that this is not the case if they are not monotone, i.e. nondecreasing.

III. Proof of the existence of square roots. We want to check that for every $x \geq 0$ a real number, there exists $\sqrt{x}$ a positive real number, such that $(\sqrt{x})^2 = x$.

a. Prove that if a sequence a_n converges to L, then a_n^2 converges to L^2.

b. Prove that if $k \geq 2$, then $a_n = \frac{1}{k^n}$ converges to 0.

c. Consider the sequences ℓ_n defined as[1]

$$\ell_n = \text{ largest multiple of } \frac{1}{10^n}, \text{ s. t. } \ell_n^2 \leq x;$$

and

$$r_n = \text{ smallest multiple of } \frac{1}{10^n}, \text{ s. t. } r_n^2 \geq x.$$

First argue that both sequences are well-defined. Then use II. and a. and b. to show that both sequences converge to $\sqrt{x}$.

[1]Here, the $\frac{1}{10^n}$ is not a typo, an important part of the exercise is to understand what does it mean for a number y to a multiple of $\frac{1}{n}$ for some $n \geq 1$.