Math 347: Lecture 6 - Worksheet

September 10, 2018

1) Consider $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers, $S = P(\mathbb{N})$ the power set of $\mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}$ a function given by $f(n) = n^2 + 1$. Are the following true or false? Justify.

   (i) $\{1\} \subset P(\mathbb{N})$?
   (ii) $\{1\} \in \mathbb{N}$?
   (iii) $\{1, 2\} \in P(\mathbb{N})$?
   (iv) $\{\{2\}\} \subset P(\mathbb{N})$?
   (v) $f^{-1}(0) \in \mathbb{N}$?
   (vi) $f^{-1}(1) \in \mathbb{N}$?
   (vii) $f^{-1}(f(\{1, 2\})) \subset \mathbb{N}$?
   (viii) $\{1, 2\} \subset \mathbb{N}$?
   (ix) $\mathbb{N} \subset P(\mathbb{N})$?
   (x) $\emptyset \in P(\mathbb{N})$?

   (i) False, the correct would be $\{\{1\}\} \subset P(\mathbb{N})$*; (ii) False,* $1 \in \mathbb{N}$*; (iii) True; (iv) True; (v) False,* $f^{-1}(0) = \emptyset \subset \mathbb{N}$*; (vi) False,* $f^{-1}(1) = \{0\} \subset \mathbb{N}$*; (vii) True,* $f^{-1}(f(\{1, 2\})) = \{0, 1\} \subset \mathbb{N}$*; (viii) True; (ix) False,* $\{\mathbb{N}\} \subset P(\mathbb{N})$ *or* $\mathbb{N} \in P(\mathbb{N})$*; (x) True.*

2) Consider

$$P(x) = \text{"}x^2 \text{ is positive."}, \quad Q(x) = x^2, \quad R = \text{"all real numbers are positive."}$$

   Are the following true or false? Justify.

   (i) $P(x)$ is a statement.
   (ii) $(\forall x \in \mathbb{R}) P(x)$ is a statement.
   (iii) $(\exists x \in \mathbb{R}) P(x)$ is a statement.
   (iv) $Q(2)$ is a statement.
   (v) $Q(2) = 5$ is a statement.
   (vi) R is a statement.
   (vii) $R \Rightarrow (Q(2) = 5)$ is true.
   (viii) $(x \in \mathbb{R}) \Rightarrow P(x)$.

   (i) False, P(x) is not a statement until we define x; (ii) True, this is a statement [1]*; (iii) True; (iv) False; (v) True; (vi) True; (vii) True, "false implies anything"; (viii) True.*

3) What is the contrapositive of the statement: "For $f(x) = x^2 + b$ if $x \neq y$ and $x \neq -y$, then $f(x) \neq f(y)$."? Prove that the contrapositive is true. What can you deduce? *The contrapositive is for* $f(x) = x^2 + b$ *if* $f(x) = f(y)$ *then either* $x = y$ *or* $x = -y$*. Indeed, if*

$$x^2 + b = y^2 + b \;\Rightarrow\; x^2 = y^2 \;\Rightarrow\; x = y \;\text{ or }\; x = -y.$$

   Since the contrapositive is true, the initial statement is also true.

[1] We are not saying whether it is true or false as a statement.

4) Prove the statement "If a is bigger than or equal to any real number smaller than b, then $a \geq b$." Consider proof by contradiction. *Suppose by contradiction that a is bigger than or equal to any $c \in \mathbb{R}$, such that $c < b$ and that $a < b$. Consider the number $d = a + \frac{b-a}{2}$, this satisfies $d < b$, since $a < \frac{a+b}{2} < b$[2] However our hypothesis applies to d, thus $a \geq d$ which gives*

$$a < a + \frac{b-a}{2} \leq a$$

that is a contradiction. So we proved by contradiction that $a \geq b$. .

5) Consider the numbers $x_1, \ldots, x_4$. Prove that at least one of them is smaller than or equal to their average. *Let $S = \frac{x_1+x_2+x_3+x_4}{4}$ be the average of those numbers. Suppose by contradiction that $x_i > S$ for $i = 1, \ldots, 4$. Hence, one has*

$$x_1 + x_2 + x_3 + x_4 > 4S.$$

Dividing bothe terms by 4, one obtains

$$S < \frac{x_1 + x_2 + x_3 + x_4}{4}.$$

This is a contradiction, so we proved that at least one of x_i is smaller than or equal to S.

6) (Challenge) Suppose that a and b are integer numbers not divisible by 3. Prove that $ax^2 + b = 0$ has a rational solution if and only if a and b don't have the same reminder when divided by 3, i.e. $a \not\equiv b \pmod 3$. *Let $x = p/q$ be a rational solution written in minimal form, i.e. p and q are integers with no common factor. One obtains that*

$$ap^2 + bq^2 = 0.$$

There are some cases to analyze. Either p is divisible by 3, or it is not. Case 1: if p is divisible by 3, this implies that ap^2 is divisible by 3, since 0 is divisible by 0, so is bq^2. However, b is not divisible by 3 by assumption, so q^2 has to be divisible by 3. By the Claim after the proof we know that q is divisible by 3, which is a contradiction with p/q being written in minimal form. Case 2: p has reminder 1 or 2, this implies that p^2 has reminder 1 when divided by 3. Thus the reminder of ap^2 is the same as that of a. If this reminder is 1 this implies that bq^2 needs to have reminder 2 when divided by 3, otherwise they can't add to 0 which has reminder 0. Since q^2 can only have reminder 1, this gives b has reminder 2. Similarly, if a had reminder 2 this would imply that b needs to have reminder 1 for the equation to have a solution.

Claim: *Suppose that p is a prime and that p divides n^2 for $n \in \mathbb{N}$, then p divides n. We will prove this result in the third part of class, as of now it is important to realize that this is not automatic. For instance if p were not a prime the above result is not true.*

[2] I.e. the average of two numbers is always between them, and not equal to either if they are different.