Math 2102: Homework 4
Due on: Apr. 4 at 11:59 pm.

All assignments must be submitted via Moodle. Precise and adequate explanations should be given to each problem. Exercises marked with Extra might be more challenging or a digression, so they won't be graded.

1. Let $T, S : V \to V$ be two operators on a finite-dimensional inner product space. Assume that $TS = ST$.
   (i) Prove that there is an orthonormal basis of V with respect to which T and S are upper-triangular.
   (ii) Assume that T is normal. Use (i) to give a different proof of the complex spectral theorem.
2. Let $\mathbb{F} = \mathbb{R}$, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{C}$. Recall the definition of $T_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ from Exercise 3 in HW 2.
   (i) Show that $u + iv \in G(\lambda, T_{\mathbb{C}})$ if and only if $u - iv \in G(\lambda, T_{\mathbb{C}})$.
   (ii) Show that the (algebraic) multiplicity of λ as an eigenvalue of $T_{\mathbb{C}}$ is the same as the (algebraic) multiplicity of $\overline{\lambda}$ as an eigenvalue of $T_{\mathbb{C}}$.
   (iii) Use (ii) to show that if $\dim V$ is an odd number, then $T_{\mathbb{C}}$ has a real eigenvalue.
   (iv) Use (iii) to give an alternative proof of Proposition 6 in the Lecture Notes, namely that $\dim V$ is odd then T has an eigenvalue.
3. Assume $\mathbb{F} = \mathbb{C}$ and consider $T \in \mathcal{L}(V)$ an operator on a finite-dimensional vector space. Prove that there does not exist a decomposition of V into a direct sum of two T-invariant subspaces if and only if the minimal polynomial of T is $(z - \lambda)^{\dim V}$ for some $\lambda \in \mathbb{C}$.
4. Let V and W be two finite-dimensional inner product spaces.
   (i) Prove that $\langle S, T \rangle := \operatorname{tr}(T^* S)$ determines an inner product on $\mathcal{L}(V, W) \times \mathcal{L}(V, W)$.
   (ii) Let $B_V = \{e_1, \ldots, e_n\}$ be an orthonormal basis of V and $B_W = \{f_1, \ldots, f_m\}$ be an orthonormal basis of W. Let $\langle -, - \rangle_{\text{std}} : \mathbb{F}^{mn} \times \mathbb{F}^{mn} \to \mathbb{F}$ be the standard inner product on $\mathbb{F}^{mn}$ (i.e. Example 23 (i) and (ii) from the Lecture Notes). Let $\mathcal{M}(-, B_V, B_W) : \mathcal{L}(V, W) \xrightarrow{\cong} \mathbb{F}^{mn}$ be the isomorphism given by the matrix coefficients. Prove that

   $$\langle S, T \rangle = \langle \mathcal{M}(S, B_V, B_W), \mathcal{M}(T, B_V, B_W) \rangle_{\text{std}}$$

   for all $S, T \in \mathcal{L}(V, W)$.