Math 347: Lecture 2 - Worksheet

August 29, 2018

1) What are the domain and image of the absolute value function? $|-|:\mathbb{R}\to\mathbb{R}$ *The domain is* $\mathbb{R}$ *and the image is the set* $\{x\in\mathbb{R} \mid X\geq 0\}$*, which can also be denoted by* $\mathbb{R}_{\geq 0}$.

2) Let $A=\{\text{January,February},\dots,\text{December}\}$. Given $x\in A$, let $f(x)$ be the number of days in x. Does f define a function from A to $\mathbb{N}$? What is its domain and range? *There are two ways of addressing this question: 1) Notice that f(February) can either take the value 28 or 29 if one considers February from a leap year or not. 2) If one assumes that February means a month that is (resp. is not) from a leap year, than the image is $\{29,30,31\}$. (resp. $\{28,30,31\}$).*

3) Define the image of the functions $f:\mathbb{R}\to\mathbb{R}$ defined below:

   a. $f(x)=\frac{x^2}{1+x^2}$; *Notice that*
   $$0\leq x^2<x^2+1.$$
   Since X^2+1 is always positive, one has
   $$0\leq\frac{x^2}{x^2+1}<1.$$
   Thus, the image is $[0,1)=\{x\in\mathbb{R}\mid 0\leq x<1\}$.

   b. $f(x)=\frac{x}{|1+x|}$. *First we notice that if $x=-1$ the expression is not defined, so the domain of the function is $\mathbb{R}\backslash\{1\}$ [1]. The image is $\mathbb{R}$. To see that notice that for $x<-1$ the expression is arbitrarily close to ∞ and for $x>-1$ it is arbitrarily close to $-\infty$.*

4) Let $f:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ be defined by $f(a,b)=\frac{(a+1)(a+2b)}{2}$.

   a. Show that the image of f is contained in $\mathbb{N}$.

   b. Determine exactly which natural numbers are in the image of f.

   a. We need to check that for any $(a,b)\in\mathbb{N}\times\mathbb{N}$ the expression
   $$\frac{(a+1)(a+2b)}{2}$$
   is actually a natural number. It is enough to show that $(a+1)(a+2b)$ is a natural number. This follows from
   $$(a+1)(a+2b)=a(a+1)+2b,$$

[1] Recall the definition of difference of sets, this means
$$\mathbb{R}\backslash\{1\}=\{x\in\mathbb{R}\mid x\neq 1\}.$$

by observing that the product of an even number by an odd number is even and that the sum of two even numbers is even[2]*. b. We notice that* $f(0,n) = n$*. So given any natural number* $n \in \mathbb{N}$ *the element* $(0,n)$ *is mapped to* n[3]*. So all natural numbers are in the image of* f*. If we denote by* S *the image of* f*. Notice that since we proved that* $S \subset \mathbb{N}$ *in item a. and that* $\mathbb{N} \subset S$ *in b. one has that* $S = \mathbb{N}$*.*

5) For S in the domain of a function f, let $f(S) = \{f(x) \mid x \in S\}$. Let C and D be subsets of the domain of f.

   a. Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

   b. Give an example where the equality doesn't hold in part a.

   a. Suppose that $x \in f(C \cap D)$*, this implies that there exists* $y \in C \cap D$ *such that* $f(y) = x$*. Since* $y \in C \cap D$*, we have that* $y \in C$ *and* $y \in D$*, thus* $x \in f(C)$ *and* $x \in f(D)$*. Hence,* $x \in f(C) \cap f(D)$*. b. Consider* $C = \{1, 2\}$*,* $D = \{1, -2\}$ *and* $f(x) = x^2$*.*

6) When $f : A \to B$ and $S \subseteq B$, we define $I_f(S) = \{x \in A \mid f(x) \in S\}$. Let X and Y be subsets of B:

   a. Determine whether $I_f(X \cup Y)$ is equal to $I_f(X) \cup I_f(Y)$.

   b. Determine whether $I_f(X \cap Y)$ is equal to $I_f(X) \cap I_f(Y)$.

   a. That is true. Let $x \in I_f(X \cup Y)$*, this gives that* $f(x) \in X \cup Y$*. This means that* $f(x) \in X$ *or that* $f(x) \in Y$*, that is* $x \in I_f(X)$ *or* $x \in I_f(Y)$*, thus* $x \in I_f(X) \cup I_f(Y)$*. b. That is also true. The proof is exactly as the previous paragraph changing "or" to "and" and* $\cup$ *to* $\cap$*.*

7) Let $S = \{x \in \mathbb{R} \mid x(x-1)(x-2)(x-3) < 0\}$. Let T be the interval $(0, 1)$ and U be the interval $(2, 3)$. Determine the relations between the sets S, T and U. *The relations are the following:*

$$T \subseteq S, \quad U \subseteq S.$$

   And there is not relation between T *and* U*, or rather* $T \cap U = \emptyset$*.*

8) Let $S = [3] \times [3]$ (the Cartesian product of $\{1, 2, 3\}$ with itself). Let T be the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq 3x + y - 4 \leq 8$. Prove that $S \subseteq T$. Does equality hold? *Consider* $(x, y) \in S$*, then we notice that*

$$3x + y \leq 12.$$

   That gives $3x + y - 4 \leq 8$*. And also that*

$$4 \leq 3x + y$$

   so also $0 \leq 3x + y - 4$*. This gives that* $(x, y) \in T$*. The equality does not hold, since* $(0, 4) \in T$*, but* $(0, 4) \notin S$*.*

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[2]Someone asked me in class how to justify these last two statements. Recall that we say that a natural number n is *even* is there exists an natural number k such that

$$n = 2k.$$

I encourage you to proof the two claims that I made:

$$(odd) * (even) = (even), \quad \text{and } (even) + (even) = (even),$$

using the above definition.

[3]This simply means that $f(0, n) = n$.