Math 347 Worksheet

Worksheet 10: Binomial Theorem and Binomial Coefficients

October 31, 2018

1) Use the binomial theorem to prove that $|P(S)| = 2^{|S|}$ for a finite set S.

Solution. *In the binomial coefficients formula*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},$$

we take $x = y = 1$, *this gives us:*

$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

Thus, if S *is a set with* n *elements, we only need to prove that* $|P(S)| = \sum_{i=0}^{n} \binom{n}{i}$.
Let for $0 \leq i \leq n$, *let* $P_i \subset P(S)$ *be the subset such that*

$$T \in P_i, \quad \text{if} \quad |T| = i.$$

We notice that

$$P(S) = P_0 \cup P_1 \cup \cdots \cup P_n,$$

and for each $i \neq j$ *one has*

$$P_i \cap P_j \neq \emptyset.$$

Thus, the number of elements of $P(S)$ *is equal to the sum*

$$\sum_{i=0}^{n} |P_i|.$$

But P_i *is exactly the set of subsets of* S *that contain* i *elements, and there are exactly* $\binom{n}{i}$ *of those.*

2) Prove the following identities about binomial coefficients:

(i) Basic identity

$$\binom{n}{k} = \binom{n}{n-k};$$

Solution. *With the notation introduced above, we notice that*

$$T \mapsto S \backslash T$$

gives a bijection between the sets P_i *and* P_{n-i}, *thus*

$$|P_i| = |P_{n-i}|.$$

(ii) Pascal's identity, for all $0 \leq k \leq n$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1};$$

Solution. *Let $x \in S$, for any $0 \leq i \leq n$ one can define two subsets $R_x \subset P_i$ and $R_{\mathrm{notx}} \subset P_i$ as follows*

$$T \in R_x \quad \textit{if} \quad x \in T, \quad \textit{and } T \in R_{\mathrm{notx}} \quad \textit{if} \quad x \notin T.$$

We notice that $R_X \cap R_{\mathrm{notx}} = \emptyset$ and one has the following

$$|R_x| = \binom{n-1}{k-1}, \quad \textit{and} \quad |R_{\mathrm{notx}}| = \binom{n-1}{k},$$

since if x is (resp. not) one of the elements of T there are only $k-1$ (resp. k) choices left out of the set $S \backslash \{x\}$, which has $n-1$ elements.

(iii) Chairperson identity

$$k\binom{n}{k} = n\binom{n-1}{k-1};$$

Solution. *A mathematics department has n faculty members and needs to form a committee with k members to go through the graduate students applications, and the committee needs to have a chairperson, responsible for breaking ties and contacting the accepted students.*

Professor A says we should first pick the chair of the committee and then pick $k-1$ other members from the rest of the faculty.

Professor B says we should first pick the committee and then pick a chair among the people in the committee.

Professor A is counting the right-hand side and professor B is counting the left-hand side.

(iv) Summation identity

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

Solution. *Consider the set $S = \{1, \ldots, n+1\}$. The righthand side is counting the way of choosing $k+1$ elements from this set. We can divide this set into disjoint subsets R_i, where $T \in R_i$ if the largest number in T is $i+1$. We notice that there are*

$$\binom{i}{k}$$

sets in R_i. Indeed, since $i+1 \in T$, one has that $T \backslash \{i+1\}$ has k elements that necessarily have to be chosen from the set $\{1, \ldots, i\}$, thus i choose k. Adding the cardinality of all the sets R_i is exactly the lefthand side of the formula.

3) Calculate the number of non-negative integer solutions of $x_1 + x_2 + x_3 + x_4 = m$. What about the equation $x_1 + \cdots + x_k = n$?

Solution. *We are trying to arrange m dots and 3 bars, thus we have*

$$\binom{m+3}{3}$$

options. More generally, there are

$$\binom{m+k-1}{k-1}$$

non-negative integer solutions to $x_1 + \cdots + x_k = n$.

4) Suppose that $n! + m! = k!$ for some $n, m, k \in \mathbb{N}$. Prove that $n = m = 1$ and $k = 2$.

Solution. *Suppose that n, m, k is a solution and we can suppose that $n \geq m$. Since $m > 0$ one has*

$$k! > n! \geq m!.$$

If we divide the equation by $n!$ one obtains

$$1 + \frac{m!}{n!} = k \cdot (k-1) \cdot \dots \cdot (n+1).$$

Since each term on the right-hand is an integer and 1 is an integer this gives that $\frac{m!}{n!}$ is an integer, thus $m \geq n$, so $m = n$. Now for

$$2 = k \cdot k(-1) \cdot \dots \cdot (n+1),$$

the only possibility is $n+1 = 2$. This gives the solution $m = n = 1$ and $k = 2$, but that is indeed the only one.

5) By using a counting argument, prove that

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

Solution. *Consider $[n] = \{1, \dots, n\}$, and let P be the set of pairs (A, B) such that*

$$A \subset B \subset [n],$$

with $|A| = j$ and $|B| = k$.

Now there are two ways of producing an element (A, B) as above.

1) We pick a subset B of $[n]$ with k elements and then we pick A a subset of B with j elements. This gives the lefthand side of the formula.

2) We pick A a subset of j elements of $[n]$. Now to produce B we need $k - j$ elements which are not in A, namely we pick $k - j$ elements from $[n] \backslash A$. This gives the righthand side of the formula.

6) A proof that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ without induction.

(a) Prove that

$$i^2 = 2\binom{i}{2} + i.$$

Solution. *The lefthand side is counting the ways to pick an ordered pair of not necessarily distinct elements from a set with i elements. One the righthand, the term $2\binom{i}{2}$ is counting how many pairs (x, y), with $x, y \in [i]$, have $x \neq y$ and the term i is counting how many pairs have (x, x), for $x \in [i]$.*

(b) Use the above result to find and prove the formula above.

Solution. *We can rewrite each term of the sum using the formula from (i). This gives*

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (2\binom{i}{2} + i) = \sum_{i=1}^{n} 2\binom{i}{2} + \sum_{i=1}^{n} i.$$

The first term on the right is

$$2\binom{n+1}{3},$$

from the summation formula. The second term is

$$\frac{n(n+1)}{2}$$

from the result for the summation of i.

If one puts the two terms together, this gives

$$2\frac{(n+1)n(n-1)}{6} + \frac{n(n+1)}{2} = \frac{(2n+1)(n+1)n}{6}.$$

7) What other summation formulas can you prove using the trick from Question 6)?