SOLUTIONS FOR PART IV

13. THE REAL NUMBERS

13.1. *Sequences.*

a) $\langle x\rangle$ *defined by* $x_n = n$ *is monotone but not bounded.* Each succeeding term is larger than the previous term, so the sequence is increasing and hence monotone. The set of terms is the set of natural numbers, which is unbounded, so the sequence is unbounded.

b) $\langle y\rangle$ *defined by* $y_n = 1/n$ *is monotone but not bounded.* Each succeeding term is smaller than the previous term, so the sequence is decreasing and hence monotone. Since $0 < y_n \le 1$ for all n, the sequence is bounded.

13.2. *The proverb "A lot of a little makes a lot" describes Theorem 13.9*; if a is "a little", then n can be made large enough so that na is "a lot".

13.3. *Every bounded sequence of real numbers converges—FALSE.* The sequence $\langle x\rangle$ with $x_n = (-1)^n$ is a counterexample. However, it is true that every bounded *monotone* sequence convergers.

13.4. *The interval* (a, b) *contains its infimum and its supremum—FALSE.* The infimum and supremum are a and b, which are not in the open interval. The closed interval $[a, b]$ does contain its infimum and its supremum.

13.5. *If the sequence* $\langle x\rangle$ *does not converge to zero, then there exists* $\epsilon > 0$ *so that for all* n, $|x_n| > \epsilon$*—FALSE.* However, it is true that when $\langle x\rangle$ does not converge to zero, there exists $\varepsilon > 0$ so that for infinitely many n, $|x_n| > \varepsilon$.

13.6. *A countable sequence of real numbers.* Listing numbers according to the position of the last nonzero digit in their decimal expansions lists only numbers with finitely many nonzero digits in their expansions. All such numbers are rational, so the set listed is countable.

13.7. *Every infinite subset of a countable set is countable.* Let A be an infinite subset of a countable set B. Since B is countable, there is a bijection $f : \mathbb{N} \to B$; it lists the elements of B in some order. The elements of A occur as a subsequence of this, and thus we also have a sequence listing the elements of A. Thus A is countable.

Every set that contains an uncountable set is uncountable. Let A be a subset of a set B. If B is finite or countable, then bijections make A also finite or countable. The contrapositive states that if A is uncountable, then B is uncountable.

For example, to show that $\mathbb{R}$ is uncountable it suffices to show that $[0, 1]$ is uncountable.

13.8. *If* S *is a bounded set of real numbers, and* S *contains* $\sup(S)$ *and* $\inf(S)$*, then* S *is a closed interval — FALSE.* Counterexamples include the finite set $S = \{0, 1\}$ and the uncountable set $S = [0, 1] \cup [2, 3]$.

13.9. *If* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = \frac{2x-8}{x^2-8x+17}$*, then the supremum of the image of* f *is 1—TRUE.* We show that 1 is an upper bound on $f(x)$ and that 1 is in the image. The latter claim follows from $f(5) = 2/2 = 1$.

Since $x^2 - 8x + 17 \le (x-4)^2 + 1$, this quadratic polynomial is never zero. Hence the inequality $f(x) \le 1$ is equivalent to $2x - 8 \le x^2 - 8x + 17$, which is equivalent to $0 \le x^2 - 10x + 25$. Since $x^2 - 10x + 25 = (x-5)^2 \ge 0$, the desired inequality is always true.

13.10. *Every positive irrational number is the limit of a nondecreasing sequence of rational numbers—TRUE.* For each irrational number α, let α_n denote the decimal expansion of α to n places. This defines a nondecreasing sequence of rational numbers with limit α.

13.11. *a) If* $\langle a\rangle$ *and* $\langle b\rangle$ *converge and* $\lim a_n < \lim b_n$*, then there exists* $N \in \mathbb{N}$ *such that* $n \ge N$ *implies* $a_n < b_n$ *— TRUE.* Let $L = \lim a_n$ and $M = \lim b_n$. Let $\varepsilon = (M-L)/2$. The definition of convergence implies that there exist N_1 and N_2 such that $n \ge N_1$ implies $|a_n - L| < \varepsilon$ and $n \ge N_2$ implies $|b_n - M| < \varepsilon$. Let $N = \max\{N_1, N_2\}$. For $n \ge N$, we have $a_n < L + \varepsilon = M - \varepsilon < b_n$.

b) If $\langle a\rangle$ *and* $\langle b\rangle$ *converge and* $\lim a_n \le \lim b_n$*, then there exists* $N \in \mathbb{N}$ *such that* $n \ge N$ *implies* $a_n \le b_n$ *— FALSE.* If $a_n = 2/n$ and $b_n = 1/n$, then $\lim a_n = 0 = \lim b_n$, so $\lim a_n \le \lim b_n$, but $a_n > b_n$ for all n.

13.12. *If* S *is a bounded set of real numbers, and* $x_n \to \sup(S)$ *and* $y_n \to \inf(S)$*, then* $\lim x_n + y_n \in S$*—FALSE.* Consider $S = \{1, 2\}$. If $x_n = 1$ for all n, and $y_n = 2$ for all n, then $x_n + y_n$ converges to 3, which is not in S.

The counterexample still works when we consider $\lim \frac{x_n+y_n}{2}$, since $\frac{x_n+y_n}{2} = 3/2 \notin S$.

13.13. *If* $x > 0$ *and* $x^2 \ne 2$*, then* $y = \frac{1}{2}(x + 2/x)$ *satisfies* $y^2 > 2$. We show that $y^2 - 2$ is a square. We have

$$y^2 - 2 = \left[\tfrac{1}{2}\left(x + \tfrac{2}{x}\right)\right]^2 - 2 = \tfrac{1}{4}\left(x^2 + 4 + \tfrac{4}{x^2}\right) - \tfrac{8}{4}$$

$$= \tfrac{1}{4}\left(x^2 - 4 + \tfrac{4}{x^2}\right) = \tfrac{1}{4}\left(x - \tfrac{2}{x}\right)^2 > 0.$$

Note that $x^2 \ne 2$ implies that $x - 2/x \ne 0$.

13.14. *To six places, the base 3 expansion of* $1/10$ *is .002200.* We have $(73/729) > (1/10) > (72/729)$. The base 3 expansion of 72 is 2200, since $72 = 2 \cdot 27 + 2 \cdot 9 + 0 \cdot 3 + 0 \cdot 1$. Dividing by $729 = 3^6$ yields .002200. Since

1/10 exceeds 72/729 by less than 1/729, the expansion of 1/10 agrees with this through the first six places.

13.15. *Reciprocals of positive integers with one-digit expansions.* In base k, we seek positive integer solutions to $\frac{1}{n} = \frac{i}{k}$ with $1 \le i < k$. Rewriting this as $n = k/i$, we get a solution for each divisor of k less than k. For $k = 10$, the fractions are 1/2, 1/5, 1/10. For $k = 9$, they are 1/3, 1/9. For $k = 8$, they are 1/2, 1/4, 1/8.

13.16. *In base 26, the string BAD represents the decimal number 679.* $D(26)^0 + A(26)^1 + B(26)^2 = 3 + 0 + 1(676) = 679$.

In base 26, the string $.MMMMMMMMMMMMM\cdots$ represents 12/25. Let x be the desired value. Note that the value of M is 12. From $26x = M.MMMMMMMMMMMM\cdots$, we have $26x = 12 + x$, and thus $x = 12/25$.

13.17. *When q is odd, the base q expansion of* $1/2$ *consists of* $(q-1)/2$ *in each position.* See the more general result in the next solution.

13.18. *When* $q \equiv 1 \pmod 3$*, the base q expansion of* $1/3$ *consists of* $(q-1)/3$ *in each position.* In general, we prove that if $q \equiv 1 \pmod k$, then the base q expansion of $1/k$ consists of $(q-1)/k$ in each position.

The alternative expansion of 1 in base q consists of $q-1$ in every position. Since $k|(q-1)$, the distributive law for series allows us to divide the sum of the series $\sum(q-1)q^{-n}$ by dividing each coefficient to obtain the series expansion $1/k = \sum \frac{q-1}{k} q^{-n}$.

13.19. *If f is a bounded function on an interval I, then* $\sup(\{-f(x)\colon x \in I\}) = -\inf(\{f(x)\colon x \in I\})$. Let $\alpha = \sup(\{-f(x)\colon x \in I\})$, and $S = \{f(x)\colon x \in I\}$. We have $\alpha \ge -f(x)$ and hence $-\alpha \le f(x)$ for all $x \in I$, so $-\alpha$ is a lower bound for S.

On the other hand, Prop 13.15 yields a sequence $\langle x \rangle$ of numbers in I such that $-f(x_n) \to \alpha$. Thus $f(x_n) \to -\alpha$. We now apply the analogue of Prop 13.15 for infimum. Since $-\alpha$ is a lower bound for S and $-f(x_n)$ defines a sequence of elements of S converging to $-\alpha$, we conclude that $-\alpha = \inf(S)$.

13.20. *Sequence converging to infimum or to supremum.*

a) $S = \{x \in \mathbb{R}\colon 0 \le x < 1\}$. We have $x_n = 1 - 1/(n+1) \to 1 = \sup(S)$ and $y_n = 1/(n+1) \to 0 = \inf(S)$.

b) $S = \{\frac{2+(-1)^n}{n}\colon n \in \mathbb{N}\}$. The set S consists of the terms of a sequence that begins $1, 3/2, 1/3, 3/4, \ldots$. The constant sequence converges to the supremum: $x_n = 3/2 = \sup(S)$. A monotone sequence converging to the infimum is given by $y_n = 3/(2n) \to 0 = \inf(S)$.

13.21. *The Least Upper Bound Property holds for an ordered field* $\mathbf{F}$ *if and only if the Greatest Lower Bound Property holds for* $\mathbf{F}$. Given a set S, let $-S$ denote $\{x : -x \in S\}$. Upper bounds on $-S$ are the negatives of lower bounds on S, and lower bounds on $-S$ are the negatives of upper bounds on S. The LUB Property implies for nonempty S that $-S$ has a least upper bound α, which implies that S has a greatest lower bound $-\alpha$, and the GLB Property follows. Conversely, the GLB Property implies for nonempty S that $-S$ has a greatest lower bound α, which implies that S has a least upper bound $-\alpha$, and the LUB Property follows.

13.22. *Determination of* $\sup(S)$ *and* $\inf(S)$.

a) $S = \{x\colon x^2 < 5x\}$. Rewrite S as $S = \{x\colon x(x-5) < 0\}$. Thus $x \in S$ if and only if x and $x-5$ have opposite signs. This requires $x > 0$ and $x < 5$, and that suffices, so S is the open interval $(0, 5)$. This set is bounded (by 0 and 5), and $\sup(S) = 5$ and $\inf(S) = 0$.

b) $S = \{x\colon 2x^2 < x^3 + x\}$. Rewrite S as $S = \{x\colon x(x-1)^2 > 0\}$. The condition holds if and only if $x > 0$ and $x \ne 1$. This set is unbounded, but its infimum is 0.

c) $S = \{x\colon 4x^2 > x^3 + x\}$. The inequality is equivalent to $x(x^2 - 4x + 1) < 0$. The zeros of the quadratic factor are at $x = 2 \pm \sqrt{3}$. Thus $S = (-\infty, 0) \cup (2-\sqrt{3}, 2+\sqrt{3})$. The set has no lower bound, but $\sup(S) = 2 + \sqrt{3}$.

13.23. *If $A, B \subset \mathbb{R}$ have upper bounds and $C = \{x + y \in \mathbb{R}\colon x \in A, y \in B\}$, then C is bounded and* $\sup C = \sup A + \sup B$. Let $\alpha = \sup A$ and $\beta = \sup B$. We prove first that $\alpha + \beta$ is an upper bound for C. For each $z \in C$, the definition of C implies that $z = x + y$ for some $x \in A$ and $y \in B$. By the definition of upper bound, $x \le \alpha$ and $y \le \beta$. Hence $z = x + y \le \alpha + \beta$, and $\alpha + \beta$ is an upper bound for C.

To prove that $\alpha + \beta$ is the least upper bound, consider q such that $q < \alpha + \beta$. Thus $q = \alpha + \beta - \varepsilon$ for some $\varepsilon > 0$. Since $\alpha = \sup A$, the number $\alpha - \varepsilon/2$ is not an upper bound for A, and there exists $x \in A$ with $x > \alpha - \varepsilon/2$. Similarly, there exists $y \in B$ with $y > \beta - \varepsilon/2$. This constructs $z \in C$ such that $z = x + y > \alpha + \beta - \varepsilon = q$. Hence q is not an upper bound for C.

Alternative proof: Instead of showing directly that C has no smaller upper bound, it also suffices to show that C contains the elements of a sequence converging to $\alpha + \beta$. This can be obtained by taking a sequence $\langle x \rangle$ in A converging to α and a sequence $\langle y \rangle$ in B converging to β. The sum consists of elements of C: $z_n = x_n + y_n \to \alpha + \beta$.

Comment: Since $\alpha + \beta$ may not lie in C, one cannot prove that $\alpha + \beta$ is the least upper bound for C without using the properties of supremum. For example, if $A = \{x \in \mathbb{R}\colon 0 < x < 1\}$ and $B = \{x \in \mathbb{R}\colon 2 < x < 3\}$, then $C = \{x \in \mathbb{R}\colon 2 < x < 4\}$; none of these sets contains its supremum.

13.24. *When $f, g: \mathbb{R} \to \mathbb{R}$ are bounded functions such that $f(x) \le g(x)$ for all x, with images F, G respectively, the following possibilities may occur* (pictures omitted):

a) $\sup(F) < \inf(G)$. Let $f(x) = 0$ and $g(x) = 1$ for all x.

b) $\sup(F) = \inf(G)$. Let $f(x) = g(x) = 0$ for all x.

c) $\sup(F) > \inf(G)$. Let $f(x) = |x|$ for $|x| \le 1$ and $f(x) = 1$ for $|x| > 1$. Let $g(x) = |x|$ for $|x| \le 2$ and $g(x) = 2$ for $|x| > 2$. Now $\sup(f(x)) = 1$ and $\inf(g(x)) = 0$.

13.25. $\lim \sqrt{1+n^{-1}} = 1$. Given $\varepsilon > 0$, let $N = \lceil 1/\varepsilon \rceil$. Note that $\sqrt{1+n^{-1}} < 1 + n^{-1}$ when $n > 0$. For $n \ge N$, we have $\left|\sqrt{1+n^{-1}} - 1\right| < \left|1+n^{-1}-1\right| = n^{-1} \le N^{-1} \le \varepsilon$. Thus $\sqrt{1+n^{-1}} \to 1$, by the definition of limit.

Comment: Let $a_n = \sqrt{1+n^{-1}}$. A less efficient approach first uses MCT to prove that $\langle a \rangle$ converges. Letting $L = \lim a_n$, we have $a_n^2 \to L^2$. Proving $a_n^2 \to 1$ directly yields $L = \pm 1$, and positivity of a_n then yields $L = 1$.

13.26. *If* $\lim a_n = 1$, *then* $\lim[(1+a_n)^{-1}] = \frac{1}{2}$.

Consider $\varepsilon > 0$. Because $\lim a_n = 1$, the definition of limit tells us that there exists $N_1 \in \mathbb{N}$ such that $n \ge N_1$ implies $|a_n - 1| < \varepsilon$. Also $|1 + a_n| = |1 + 1 - 1 + a_n| \le 2 + |a_n - 1| < 2 + \varepsilon$. Let $N = N_1$. Now $n \ge N$ implies

$$\left|\frac{1}{1+a_n} - \frac{1}{2}\right| = \left|\frac{2-1-a_n}{2(1+a_n)}\right| = \frac{|1-a_n|}{2(1+a_n)} < \frac{\varepsilon}{2(2+\varepsilon)} < \varepsilon.$$

Thus $\lim[(1+a_n)^{-1}] = 1/2$, by the definition of limit.

13.27. *If* $a_n = \sqrt{n^2+n} - n$, *then* $\lim a_n = \frac{1}{2}$. We multiply and divide a_n by $\sqrt{n^2+n} + n$, simplify the result, and use Exercises 13.25–13.26. Thus

$$a_n = \sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{(\sqrt{n^2+n} + n)}$$
$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1} \to \frac{1}{2}.$$

13.28. *If* $x_n \to 0$ *and* $|y_n| \le 1$ *for* $n \in \mathbb{N}$, *then* $\lim(x_n y_n) = 0$. One cannot argue that $\lim(x_n y_n) = \lim(x_n)\lim(y_n) = 0 \cdot \lim(y_n) = 0$, since $\lim(y_n)$ need not exist.

A correct proof uses $|y_n| \le 1$ to argue that $|x_n y_n| = |x_n|\,|y_n| \le |x_n|$. Given $\varepsilon > 0$, the convergence of $\langle x \rangle$ yields $N \in \mathbb{N}$ such that $n \ge N$ implies $|x_n| < \varepsilon$. By our first computation, $|x_n y_n| \le |x_n| < \varepsilon$ for such n, and thus $\lim x_n y_n = 0$.

13.29. *The limit of the sequence* $\langle x \rangle$ *defined by* $x_n = (1+n)/(1+2n)$ *is* $1/2$. Since the denominator exceeds the numerator and both are positive, we have $0 < x_n < 1$ for all $n \in \mathbb{N}$. We also compute

$$x_{n+1} - x_n = \frac{n+2}{2n+3} - \frac{n+1}{2n+1} = \frac{(2n+1)(n+2) - (2n+3)(n+1)}{(2n+3)(2n+1)}$$
$$= \frac{-1}{(2n+3)(2n+1)} < 0.$$

Since $\langle x \rangle$ is a decreasing sequence bounded below, the Monotone Convergence Theorem implies that $\lim_{n\to\infty} x_n$ exists.

To prove that $\lim_{n\to\infty} x_n = 1/2$, we compute $x_n - 1/2 = \frac{n+1}{2n+1} - \frac{1}{2} = \frac{1}{4n+2}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 4/\varepsilon$. Now $n > N$ implies that $|x_n - 1/2| = \frac{1}{4n+2} < \varepsilon$. Since this holds for each $\varepsilon > 0$, we have $x_n \to 1/2$, by the definition of limit.

13.30. *The sequence* $\langle x \rangle$ *defined by* $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$ *converges.* By the Monotone Convergence Theorem, it suffices to prove that $\langle x \rangle$ is increasing and bounded above by 1. For the first statement

$$x_{n+1} - x_n = \sum_{i=1}^{n+1} \frac{1}{n+1+i} - \sum_{i=1}^{n} \frac{1}{n+i} = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0.$$

For the second statement, $x_n = \sum_{i=1}^n \frac{1}{n+i} < \sum_{i=1}^n \frac{1}{n+1} = \frac{n}{n+1} \le 1$.

13.31. $x_n = (1+(1/n))^n$ *defines a bounded monotone sequence.* Let $r_n = x_{n+1}/x_n$. We show that $r_n > 1$ to prove that $\langle x \rangle$ is increasing. Writing x_n as $(\frac{n+1}{n})^n$, we have

$$r_n = \left[\frac{n+2}{n+1}\Big/\frac{n+1}{n}\right]^n \frac{n+2}{n+1} = \left(\frac{n^2+2n}{n^2+2n+1}\right)^n \frac{n+2}{n+1} = \left(1 - \frac{1}{(n+1)^2}\right)^n \frac{n+2}{n+1}.$$

Since $(1-a)^n \ge 1 - na$ (Corollary 3.20) when $\alpha > 0$, we have

$$r_n \ge \left(1 - \frac{n}{(1+n)^2}\right)\frac{n+2}{n+1} = \frac{n^2+n+1}{n^2+2n+1}\frac{n+2}{n+1} = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1.$$

To show that $\langle x \rangle$ is bounded, we write $x_n = (1+1/n)^n = \sum_{k=0}^n \binom{n}{k} n^{-k}$. Since $\prod_{i=0}^{k-1}(n-i) < n^k$, we obtain $x_n \le \sum_{k=0}^n 1/k!$. Thus it suffices to show that this sum is bounded. We have $1/k! < 1/2^k$ for $k \ge 4$. Therefore,

$$\sum_{k=0}^n \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \sum_{k=4}^n \frac{1}{k!} < \frac{8}{3} + \sum_{k=4}^\infty 1/2^k = \frac{8}{3} + \frac{1}{8} = \frac{67}{24}.$$

13.32. *The Nested Interval Property.* A *nested sequence* of closed intervals, with I_n of length d_n, satisfies $I_{n+1} \subseteq I_n$ for all n and $d_n \to 0$. The Nested

Interval Property states that for such a sequence, there is exactly one point that belongs to each I_n. Let $I_n = [a_n, b_n]$.

a) The Completeness Axiom implies the Nested Interval Property. For all n, we have $a_n \le a_{n+1} \le b_n \le b_1$ and $b_n \ge b_{n+1} \ge a_n \ge a_1$. Thus both $\langle a \rangle$ and $\langle b \rangle$ are bounded monotone sequences. By the Montone Convergence Theorem, $\langle a \rangle$ increases to its supremum, A, and $\langle b \rangle$ decreases to its infimum, B. Since $b_n - a_n = d_n \to 0$, we obtain $\lim a_n = \lim b_n$, so $A = B$. This implies that A belongs to every I_n.

b) The Nested Interval Property implies the Completeness Axiom. Assume that the Nested Interval Property holds, and let S be a nonempty set with an upper bound b_1. Choose $x_1 \in S$. We construct a sequence of intervals $\{[x_n, b_n] \colon n \in \mathbb{N}$.

Having constructed $[x_n, b_n]$, consider the midpoint $z_n = (x_n + b_n)/2$ of the interval. If z_n is an upper bound for S, then let $b_{n+1} = z_n$ and $x_{n+1} = x_n$. Otherwise, choose x_{n+1} as an element of S larger than z_n, and let $b_{n+1} = b_n$.

The Nested Interval Property implies that $2^{-n} \to 0$. Since $0 \le b_n - x_n \le (b_1 - x_1)/2^{n-1}$, also $d_n = x_n - b_n \to 0$. By the Nested Interval Property, there is exactly one point α belonging to each I_n.

Since $x_n \to \alpha$, S contains a sequence of elements converging to α, and there is no upper bound less than α. Since $b_n \to \alpha$, no element of S is larger than α, because this would contradict that every b_n is an upper bound. Hence α is the least upper bound for S, and the supremum indeed exists.

13.33. *The k-ary expansion of* $1/2$. When k is even, with $k = 2n$, the expansion is $.n$. When k is odd, with $k = 2n+1$, the expansion $1 = .(2n)(2n)(2n)\cdots$ yields $(1/2) = .nnn\cdots$.

13.34. *There is a rational number between any two irrational real numbers and an irrational number between any two rational numbers.* The rational numbers expressible as fractions whose denominators are powers of 10 are the numbers with terminating decimal expansions.

Let a, b be the canonical decimal expansions of two irrational numbers α, β with $\alpha < \beta$. Since they are distinct real numbers, there is a first digit where they differ. Truncate the expansion of β at that digit to obtain the expansion of a rational number γ. Since β is irrational, its decimal expansion cannot terminate, and thus $\gamma < \beta$. Since α is irrational, its decimal expansion cannot end with all 9's. Since its expansion is less than that of γ in the first place where they differ, we thus have $\alpha < \gamma$.

Now suppose that a, b are distinct rational numbers with $a < b$.

Proof 1 (by irrationality of $\sqrt{2}$). Let $c = a + (\sqrt{2}/2)(b - a)$. Since $0 < \sqrt{2}/2 < 1$, the number c is between a and b. If c is rational, then closure of operations on rational numbers implies that $2(c - a)/(b - a)$ is rational, but this equals $\sqrt{2}$.

Proof 2 (by uncountability of the interval $(0, 1)$). Map the interval (a, b) to the interval $(0, 1)$ by $f(x) = (x - a)/(b - a)$. This function is a bijection, and thus the intervals have the same cardinality. Since the interval $(0, 1)$ is uncountable and the set of rationals is countable, there must be a number in the interval (a, b) that is not rational.

Proof 3 (by k-ary expansions). Express a and b as fractions via $a = p/q$ and $b = r/s$. Let k be the least common multiple of q and s. Now a and b are expressible as fractions with denominator k. Thus they have finite k-ary expansions; indeed, the expansions have only one nonzero term in the fractional part. Form the k-ary expansion of c by appending to the k-ary expansion of a the k-ary expansion of any irrational number.

13.35. *A real number has more than one k-ary expansion if and only if it is expressible as a fraction with a denominator that is a power of k.* Suppose that $\alpha = m/k^n$ for some positive integer m. We may assume that $m < k^n$ (otherwise, we subtract the integer part) and that m is not a multiple of k (otherwise, we cancel a factor of k to obtain such an expression with a smaller power of k). Now α has the terminating k-ary expansion $a_1 a_2 \cdots a_n$ with $a_n \ne 0$. Also α has the nonterminating k-ary expansion $\langle c \rangle$, where $c_i = a_i$ for $i < n$, $c_n = a_n - 1$, and $c_i = k - 1$ for $i > n$.

For the converse, suppose that α has k-ary expansions a and a'. A k-ary expansion of α yields a bounded increasing sequence with limit α. If the sequence of digits is $\langle a \rangle$, then the sequence $\langle b \rangle$ converging to α is defined by $b_n = \sum_{i=1}^{n} a_i/k^i$. Similarly, we have $b'_n \to \alpha$, where $b'_n = \sum_{i=1}^{n} a'_i/k^i$. If a and a' differ, let n be the first position where they differ. We may assume that $a_n < a'_n$. We have $b_n \le b'_n - 1/k^n$. The contribution from all remaining terms of the expansion a is at most $1/k^n$, with equality achieved only if all the remaining positions in a are $k - 1$. Hence $\alpha \le b_n + 1/k^n \le b'_n \le \alpha$. Since equality holds throughout, the contribution from all remaining positions in the expansion a' must be 0. This requires that every remaining position in b' is 0. Hence $b'_n = \alpha$ is a rational number expressible with denominator k^n, and $\langle a \rangle$ is its alternative nonterminating expansion.

13.36. *a) Long division of a by b (in base 10) yields the decimal expansion of a/b.* Let $\alpha = a/b$, with $b > a$. The decimal expansions is $.c_1 + .0c_2 + .00c_3 + \cdots$. For k-ary expansion in general, we want $a = b(c_1 k^{-1} + c_2 k^{-2} + c_3 k^{-3} + \cdots$. In the long division process, we find one digit at a time, always computing a remainder. In the first step, we append a 0 to the k-ary representation of a, multiplying it by k. We then apply the Division Algorithm to write $ak = c_1 b + r_1$. Thus $r_1 = ak - c_1 b$, where c_1 is the first digit in the k-ary expansion, and r_1 is the remainder.

Long division proceeds using r_1 instead of a; the expansion of this follows c_1. We add a zero to the end of r_1 (multiplying it by k) and use the Division Algorithm to write $r_1k = c_2b + r_2$. In general, we generate c_j and r_j by $r_{j-1}k = c_jb + r_j$. By induction on j, this yields $a/b = (\sum_{i=1}^{j} c_ik^{-i}) + r_j/(bk^{-j})$, and thus this process produces a k-ary expansion. The proof of the induction step is

$$\frac{a}{b} = (\sum_{i=1}^{j-1} \frac{c_i}{k^{-i}}) + \frac{r_{j-1}}{bk^{-(j-1)}} = (\sum_{i=1}^{j-1} \frac{c_i}{k^{-i}}) + \frac{c_jb + r_j}{bk^{-j}} = (\sum_{i=1}^{j} \frac{c_i}{k^{-i}}) + \frac{r_j}{bk^{-j}}.$$

b) Given $a, b \in \mathbb{N}$, the decimal expansion of a/b has a period of length less than b. In the argument of part (a), there are b possible remainders that can arise when using long division to produce the decimal expansion of a/b. If 0 arises, then all subsequent remainders are 0, and we have a terminating expansion, which can be viewed as period 1. Otherwise, there are $b - 1$ possible nonzero remainders, and the repetition must occur after at most $b - 1$ steps. Thus the period is less than b.

13.37. *The Cantor diagonalization argument does not prove that $\mathbb{Q}$ is uncountable.* We could write "we list the expansions of numbers in $\mathbb{Q}$ and create an expansion $\langle a \rangle$ for a number y that is not on our list. This contradicts the hypothesis that $\mathbb{Q}$ is countable." The proof does not work, because there is no contradiction; the resulting y is not in $\mathbb{Q}$.

13.38. *The set of subsets of $\mathbb{N}$ and the set of real numbers in $[0, 1)$ have the same cardinality.* Each subset of $\mathbb{N}$ is specified by a binary sequence. Each number in $[0, 1)$ is specified by a binary expansion, except that the rational numbers expressible as fractions with a power of 2 in the denominator have two such expansions. Let T' be this set of real numbers. Let N' be the subsets of $\mathbb{N}$ specified by binary sequences ending in all 1's or all 0's (i.e., the set or its complement is finite). Each of N', T' is the union of countably many finite sets and thus is countable; thus there is a bijection between them. We define a bijection from $\mathbb{N} - N'$ to $[0, 1) - T'$ by mapping the subset specified by the binary sequence α to the number with binary expansion α.

13.39. *$\mathbb{R} \times \mathbb{R}$ has the same cardinality as $\mathbb{R}$.* We define a bijection from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Given two real numbers, we interleave their canonical decimal exansions, centered at the decimal point. For example, the pair $(11.625, 3.1415926\cdots)$ becomes $1013.61245105090206\cdots$ (trailing and leading zeros are added as needed. The result is the canonical decimal expansion of a real number. To invert the process, we extract alternate digits to obtain the expansion of the original pair.

13.40. *An ordered field in which $\mathbb{N}$ is a bounded set.* Let F be the set of formal expressions $a = \sum_{i \in \mathbb{Z}} a_ix^i$ such that each a_i belongs to $\mathbb{R}$ and $\{i < 0 :\ a_i \neq 0\}$ is finite. Let $a \in F$ be *positive* if the least-indexed nonzero

coefficient a_k in the expression for a is positive. Let the *sum* $c = a + b$ be defined by $c_i = a_i + b_i$ for all $i \in \mathbb{Z}$. Let the *product* $d = ab$ be defined by $d_j = \sum_{i \in \mathbb{Z}} a_ib_{j-i}$ for all $j \in \mathbb{Z}$.

For each element of F, there is by definition a smallest index for which the coefficient in the formal expression is nonzero. Given two elements $a, b \in F$, let m, n be these indices.

a) The sum and product of two elements of F is an element of F, Both c_i and d_j, defined using arithmetic operations on real numbers, are real. The smallest indices for which c_i and d_j are nonzero are $\min\{m, n\}$ and $m + n$, respectively. The sum and product thus have finitely many nonzero coefficients for negative indices and both belong to F.

b) F is an ordered field. Part (a) verifies the closure axioms. Associativity and commutativity of addition follow immediately from associativity and commutativity of real number addition.

Commutativity of multiplication. Fix $j \in \mathbb{Z}$. As l runs over all integers, also $i = j - l$ runs over all integers. Thus

$$[b \cdot a]_j = \sum_{l \in \mathbb{Z}} b_la_{j-l} = \sum_{i \in \mathbb{Z}} a_ib_{j-i} = [a \cdot b]_j.$$

Associativity of multiplication. Since each of summation used to compute a coefficient in the product of two elements of F is finite, the summations involved below in computing the product of three are also finite.

$$[(a \cdot b) \cdot c]_j = \sum_{i \in \mathbb{Z}} (a \cdot b)_ic_{j-i} = \sum_{i \in \mathbb{Z}} \left[\sum_{k \in \mathbb{Z}} a_kb_{i-k} \right] c_{j-i} = \sum_{i,k \in \mathbb{Z}} a_kb_{i-k}c_{j-i}.$$

$$[a \cdot (b \cdot c)]_j = \sum_{r \in \mathbb{Z}} a_r \cdot (b \cdot c)_{j-r} = \sum_{r \in \mathbb{Z}} a_r \left[\sum_{s \in \mathbb{Z}} b_sc_{j-r-s} \right] = \sum_{r,t \in \mathbb{Z}} a_rb_{t-r}c_{j-t}.$$

In the last step, for fixed r we set $t = r + s$ and observe that as s runs over all $\mathbb{Z}$, also t runs over all $\mathbb{Z}$. Now writing i as t and k as r show that corresponding coefficients are the same.

Identity. The additive identity $\mathbf{0}$ is the element with all coefficients 0. The multiplicative identity is the element $1 \cdot x^0$ (all other coefficients 0). Thus $(a + \mathbf{0})_j = a_j$ and $(a \cdot \mathbf{1})_j = a_j$ for all $a \in F$ and $j \in \mathbb{Z}$.

Inverses. The additive inverse of $a \in F$ is $b \in F$ defined by $b_j = -a_j$ for all $j \in \mathbb{Z}$. For the multiplicative inverse, suppose m is the least index with nonzero coefficient in a. Then the least index with nonzero coefficient in $b = a^{-1}$ is $-m$, with $b_{-m} = 1/a_m$. This yields $(a \cdot b)_0 = \sum_{i \in \mathbb{Z}} a_ib_{-i} = a_m/a_m =$

1. Each successive coefficient b_{-m+j} is then determined by $0 = (a \cdot b)_j = \sum_{i \in \mathbb{Z}} a_i b_{j-i}$ for $j = 1, 2, \cdots$.

Distributive Law. We have $a \cdot (b+c) = a \cdot b + a \cdot c$, because corresponding coefficients are equal, since

$$[a \cdot (b+c)]_j = \sum_{i \in \mathbb{Z}} a_i (b+c)_{j-i} = \sum_{i \in \mathbb{Z}} a_i (b_{j-i} + c_{j-i})$$
$$= \sum_{i \in \mathbb{Z}} a_i b_{j-i} + \sum_{i \in \mathbb{Z}} a_i c_{j-i} = (a \cdot b)_j + (a \cdot c)_j.$$

We have now verified that F is a field. For the order axioms, define m, n as above for $a, b \in F$, and suppose $a_m, b_n > 0$, so that a and b are positive. *Closure under Addition*: The least index with nonzero coefficient in $a + b$ is $\min\{m, n\}$. The coefficient is a_m, $a_m + b_n$, or b_n in the cases $m < n$, $m = n$, or $m > n$; all of these are positive. *Closure under Multiplication*: The least-indexed coefficient in $a \cdot b$ is $a_m b_n$, which is positive. *Trichotomy*: Given $a \in F$, the element $-a$ is the element b defined by $b_j = -a_j$ for all $j \in \mathbb{Z}$. Thus trichotomy for F follows from trichotomy for $\mathbb{R}$ when applied to the coefficient a_m.

c) $\mathbb{N}$ is a bounded set in F. We interpret $n \in \mathbb{N}$ as the element $n = n \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + \cdots$. Let $a = 1 \cdot x^{-1} + 0 \cdot x^0 + 0 \cdot x^1 + \cdots$. By our definitions, $a - n$ is positive for all $n \in \mathbb{N}$. Thus a is an upper bound for the set $\mathbb{N} \subset F$.

14. SEQUENCES AND SERIES

14.1. *An unbounded sequence that has no convergent subsequence.* Let $x_n = n$. The sequence $\langle x \rangle$ is unbounded, as are all its subsequences.

An unbounded sequence that has a convergent subsequence. Let $y_{2n} = 0$ and $y_{2n+1} = n$ for all n. The sequence $\langle y \rangle$ is unbounded, but it has a constant and therefore convergent subsequence.

14.2. *Unbounded increasing sequences satisfying additional conditions.*

a) $\lim(a_{n+1} - a_n) = 0$. Let $a_n = \sqrt{n}$. We have $a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = 1/(\sqrt{n+1} + \sqrt{n}) \to 0$.

b) $\lim(a_{n+1} - a_n)$ *does not exist.* Let $a_n = n^2$. We have $a_{n+1} - a_n = (n+1)^2 - n^2 = 2n + 1$. Thus $\{a_{n+1} - a_n\}$ is unbounded, and the sequence has no limit.

c) $\lim(a_{n+1} - a_n) = L$, *where* $L > 0$. Let $a_n = nL$. Since $L > 0$, $\langle a \rangle$ is unbounded. We have $a_{n+1} - a_n = (n+1)L - nL = L \to L$.

14.3. *Examples of sequences $\langle a \rangle$ and $\langle b \rangle$ such that* $\lim a_n = 0$, $\lim b_n$ *does not exist, and the specified condition holds.*

a) $\lim(a_n b_n) = 0$. Let $a_n = 0$ and $b_n = (-1)^n$ for all n. Since $a_n b_n = 0$ for all n, $\lim(a_n b_n) = 0$.

b) $\lim(a_n b_n) = 1$. Let $a_n = 1/n$ and $b_n = n$ for all n. Since $a_n b_n = 1$ for all n, $\lim(a_n b_n) = 1$.

c) $\lim(a_n b_n)$ *does not exist.* Let $a_n = 1/n$ and $b_n = n(-1)^n$ for all n. Since $a_n b_n = (-1)^n$ for all n, $\lim(a_n b_n)$ does not exist.

14.4. *If $x_{n+1} = \sqrt{1 + x_n^2}$ for all $n \in \mathbb{N}$, then $\langle x \rangle$ does not converge.* If $\langle x \rangle$ converges, with $\lim x_n = L$, then the properties of limits yield $L = \sqrt{1 + L^2}$. This requires $L^2 = 1 + L^2$, which is impossible.

14.5. *A counterexample to the following false statement: "If $a_n < b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges."* Let $b_n = 0$ and $a_n = -1$ for all n, then $a_n < b_n$ for all n, and $\sum b_n = 0$, but $\sum a_n$ diverges.

14.6. *The expression* $.111\ldots$ *is the k-ary expansion of* $\frac{1}{k-1}$. The expansion evaluates to the geometric series $\sum_{n=1}^{\infty} (1/k)^n$. This equals $1/k$ times $\sum_{n=0}^{\infty} (1/k)^n$. Since $\sum_{n=0}^{\infty} (1/k)^n = \frac{1}{1-1/k}$, we obtain $.111\ldots = \frac{1}{k} \frac{1}{1-1/k} = \frac{1}{k-1}$.

14.7. *The binary expansions of $2/7$ and $\sqrt{2}$ to six places are $.010010$ and 1.011010, respectively.* We have $(19/64) > (2/7) = (18/63) > (18/64)$. The binary expansion of 18 is $1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$; thus $18/64 = .010010$, since $64 = 2^6$. Since $2/7$ exceeds $18/64$ by less than $1/64$, its expansion agrees with that of $18/64$ through six places.

Using the bisection algorithm produces the same result. $2/7$ is below $1/2$, above $1/4$, below $3/8$, below $5/16$, above $9/32$, below $19/64$. Again the expansion begins $.010010$.

For $\sqrt{2}$, we want the largest multiple of $1/2^6$ whose square is less than 2. The fastest route with a calculator may be to compare squares with $2 \cdot 2^{12} = 8192$. This exceeds $81 \cdot 100 = 90^2$, and $91^2 = 8281$. Thus we want the binary expansion of 90, shifted by six places. $90 = 2^6 + 2^4 + 2^3 + 2^1$, so the expansion begins 1.011010.

14.8. Let $\langle x \rangle$ be a sequence of real numbers.

a) If $\langle x \rangle$ is unbounded, then $\langle x \rangle$ has no limit—TRUE. The contrapositive of this statement is immediate from the definition of convergence.

b) If $\langle x \rangle$ is not monotone, then $\langle x \rangle$ has no limit—FALSE. The sequence defined by $x_n = (-1)^n/n$ is not monotone, but it converges.

14.9. *Properties of convergence.* Suppose that $x_n \to L$.

a) For all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < \varepsilon$—TRUE. Since $x_{n+1} \to L$ and $x_n \to L$, the difference has limit 0, so it is less than ε for sufficiently large n.

b) There exists $n \in \mathbb{N}$ such that for all $\varepsilon > 0$, $|x_{n+1} - x_n| < \varepsilon$—FALSE. The quantifier on ε requires that $x_{n+1} - x_n = 0$, but there are convergent sequences with no consecutive values equal.

c) There exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| < \varepsilon$—TRUE. As mentioned in part (a), the difference converges to 0. Hence it is a bounded sequence, with some bound M on $|x_{n+1} - x_n|$. Choose $\varepsilon = 2M$.

d) For all $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $|x_{n+1} - x_n| < \varepsilon$—TRUE. Let $y_n = |x_{n+1} - x_n|$. This is now the statement that $y_n \to 0$, which was verified in part (a).

14.10. *a) If $\langle x \rangle$ converges, then there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < 1/2^n$—FALSE.* Let $x_n = \sum_{k=1}^{n} (2/3)^{k-1}$. This is the sequence of partial sums of a geometric series, converging to $1/(1 - 2/3)$, which equals 3. However, $x_{n+1} - x_n = (2/3)^n$, which is larger than $(1/2)^n$.

b) If $|x_{n+1} - x_n| < 1/2^n$ for all $n \in \mathbb{N}$, then $\langle x \rangle$ converges—TRUE. We show that $\langle x \rangle$ is a Cauchy sequence. Given $\varepsilon > 0$, choose N so that $1/2^N < \varepsilon/2$. For $m > n \geq N$, we have

$$|x_m - x_n| = \left|\sum_{i=1}^{m-n} (x_{n+j} - x_{n+j-1})\right| \leq \sum_{i=1}^{m-n} \left|x_{n+j} - x_{n+j-1}\right|$$
$$\leq \sum_{i=1}^{m-n} \frac{1}{2^{n+j-1}} = \frac{1}{2^n} \sum_{i=1}^{m-n} \frac{1}{2^{j-1}} < \frac{2}{2^n} < \varepsilon$$

14.11. *a) If $x_1 = 1$ and $x_{n+1} = x_n + 1/n$ for $n \geq 1$, then $\langle x \rangle$ is bounded—FALSE.* For $n \geq 2$, we have $x_n = 1 + \sum_{k=1}^{n-1} (1/k)$. If $\langle x \rangle$ is bounded, then $\sum_{k=1}^{\infty} (1/k)$ converges, which is false.

b) If $y_1 = 1$ and $y_{n+1} = y_n + 1/n^2$ for $n \geq 1$, then $\langle y \rangle$ is bounded—TRUE. Since $\sum_{k=1}^{\infty} (1/k^2)$ converges to a number α, we have $1 \leq y_n = 1 + \sum_{k=1}^{n-1} (1/k^2) < 1 + \alpha$, and hence $\langle y \rangle$ is bounded.

14.12. *If $a_n \to 0$ and $b_n \to 0$, then $\sum a_n b_n$ converges—FALSE.* Let $a_n = b_n = 1/\sqrt{n}$. We have $a_n \to 0$ and $b_n \to 0$, but $\sum a_n b_n = \sum (1/n)$. This is the harmonic series, which diverges.

14.13. *If $\langle a \rangle$ converges, then every subsequence of $\langle a \rangle$ converges and has the same limit as a.* Let $\langle b \rangle$ be a subsequence of $\langle a \rangle$, with $b_k = a_{n_k}$. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$, where $L = \lim a_n$. Let K be the minimum k such that $n_k \geq N$. Now $k \geq K$ implies $|b_k - L| = \left|a_{n_k} - L\right| < \varepsilon$. Thus $\langle b \rangle$ also satisfies the definition of convergence to L.

14.14. *If $a_n \to L$ and $b_n \to M \neq 0$, then $a_n/b_n \to L/M$.* We may assume that $\{b_n\}$ has no 0's, by deleting corresponding terms from both sequences

if 0's occur in $\langle b \rangle$. In the text we have proved that the limit of the product of two sequences is the product of the limits. Hence it suffices to prove that $1/b_n \to 1/M$, because then we can apply the rule for the limit of a product of sequences.

Since $b_n \to M \neq 0$, there exists N such that $n \geq N$ implies $|b_n - M| < |M|/2$ and thus $|b_n| > |M|/2$ and $|1/b_n| < 2/|M|$. Thus the reciprocals of the terms in $\langle b \rangle$ form a bounded sequence. Let M' be a bound: always $|1/b_n| < M'$. Now $|1/b_n - 1/M| = |M - b_n|/(M|b_n|) < |M - b_n|(M'/M)$. The sequence $c_n = |M - b_n|(M'/M)$ is a constant times a sequence converging to 0 (since $b_n \to M$), so $c_n \to 0$. By Proposition 13.12, we conclude that $1/b_n \to 1/M$.

To prove that $1/b_n \to 1/M$ using the definition, we must determine N for each $\varepsilon > 0$ such that $n \geq N$ implies $|1/b_n - 1/M| < \varepsilon$. By the convergence of $\langle b_n \rangle$, we can make $|b_n - M|$ as small as desired; we choose N_1 such that $n \geq N_1$ implies $|b_n - M| < |M|/2$. This means $|b_n| > |M|/2$ and hence $n \geq N_1$ implies $|1/b_n| < 2/|M|$. We can also choose N_2 such that $n \geq N_2$ implies $|b_n - M| < \varepsilon|M|^2/2$. Choose $N = \max\{N_1, N_2\}$. For $n \geq N$, we have

$$|1/b_n - 1/M| = |M - b_n|/(|M|^{|}b_n|) < (\varepsilon|M|^2/2)(1/|M|)(2/|M|) = \varepsilon.$$

Alternatively, one can apply the definition directly to a_n/b_n, using $\frac{a_n}{b_n} - \frac{L}{M} = \frac{a_nM - b_nL}{b_nM} = \frac{(a_n - L)M - L(b_n - M)}{b_nM}$. In this approach, it is still necessary to choose n large enough to obtain an appropriate bound on $|1/b_n|$.

14.15. *If $b \leq L + \varepsilon$ for all $\varepsilon > 0$, then $b \leq L$.* We prove the contrapositive. If $b > L$, then let $\varepsilon = (b - L)/2$. Since the average of two numbers is between them, we have $b > (b + L)/2 = L + (b - L)/2 = b + \varepsilon$.

14.16. *If $a_n = p(n)/q(n)$, where p and q are polynomials and q has larger degree than p, then $a_n \to 0$.* Let k, l be the degrees of p, q, and let the leading coefficient of q be b. Let $g(n) = p(n)/n^l$ and $h(n) = q(n)/n^l$, so $a_n = g(n)/h(n)$. The sequence given by $g(n)$ is a sum of finitely many sequences whose terms have the form c/n^j, where $c \in \mathbb{R}$ and $j \in \mathbb{N}$. By the properties of limits, such sequences have limit 0; hence also their sum $g(n) \to 0$. The value of $h(n)$ is b plus another expression of this form, so $h(n) \to b$. Thus

$$\lim \frac{p(n)}{q(n)} = \lim \frac{n^l g(n)}{n^l h(n)} = \frac{\lim g(n)}{\lim h(n)} = \frac{0}{b} = 0.$$

14.17. *If $a_n = p(n)x^n$, where p is a polynomial in n and $|x| < 1$, then $a_n \to 0$.* If $x = 0$, then $a_n = 0$ and $a_n \to 0$. Thus we may assume that $x \neq 0$. We prove that $|a_{n+1}/a_n| \to x$. We have $a_{n+1}/an = xp(n+1)/p(n)$.

We have $p(n) = \sum_{i=0}^{d} c_i n^{d-i}$ for some constants d and $a_0, \ldots, a_d$. By expanding powers of $n+1$ using the binomial theorem, we can also write $p(n+1) = \sum_{i=0}^{d} b_i n^{d-i}$ as a polynomial in n. To study $p(n+1)/p(n)$, we divide numerator and denominator by n^d. Since $n^{-k} \to 0$ when $k > 0$ and the sum of finitely many sequences approaching 0 also has limit 0, we have Thus $|a_{n+1}/a_n| \to x$. Since $|x| < 1$, Proposition 14.11 yields $a_n \to 0$.

Alternatively, one can *reduce* the problem to the case $p(n) = n^d$. When $p(n) = \sum_{i=0}^{d} c_i n^{d-i}$, we have a_n as the sum of $d+1$ terms of the form $c_i n^{d-i} x^n$. Since $cb_n \to 0$ when $b_n \to 0$ and the sum of finitely many sequences converging to 0 also converges to 0, it suffices to prove that $n^d x^n \to 0$ for each d. Letting $a_n = n^d x^n$, we have $a_{n+1}/a_n = (1+1/n)^d x \to x$. Again we complete the proof using Proposition 14.11.

14.18. *If $a_1 = 1$ and $a_n = \sqrt{3a_{n-1}+4}$ for $n > 1$, then $a_n < 4$ for all $n \in \mathbb{N}$.* We use induction on n. For the basis, $a_1 = 1 < 4$. For the induction step,

$$a_n = \sqrt{3a_{n-1}+4} \le \sqrt{3 \cdot 4 + 4} = 4$$

14.19. *If $x_1 = 1$ and $2x_{n+1} = x_n + 3/x_n$ for $n \ge 1$, then $\lim_{n\to\infty} x_n = \sqrt{3}$.* If $\lim_{n\to\infty} x_n$ exists and equals L, then the two sides of the recurrence approach $2L$ and $L + 3/L$, respectively, so the solutions to $2L = L + 3/L$ are the only possible limits. Thus $L^2 = 3$, and the possible limits are $\pm\sqrt{3}$.

Given $x_1 = 1$, we have $x_2 = 2$ and $x_3 = 7/4$. We claim that thereafter the sequence is nondecreasing and bounded below by 0. It must thus have a limit, since every bounded monotone sequence has a limit. Since the sequence has no negative terms, the paragraph above implies that the limit must be $\sqrt{3}$.

The recurrence immediately yields $x_{n+1} > 0$ when $x_n > 0$, so the sequence remains positive. To study monotonicity, we study the sign of the difference $x_{n+1} - x_n$. We have

$$x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{3}{x_n} - x_n = \frac{1}{2}(\frac{3}{x_n} - x_n).$$

Thus the difference is nonpositive if and only if $(3/x_n) - x_n \le 0$, which holds if and only if $3 \le x_n^2$. Since x_n is positive, the condition for $x_{n+1} - x_n \le 0$ is $x_n \ge \sqrt{3}$.

It thus suffices to prove that all terms after the first are at least $\sqrt{3}$. Since $x_{n+1} = \frac{1}{2}(x_n + 3/x_n)$, the number x_{n+1} is the average of x_n and $3/x_n$. The AGM Inequality states that $(a+b)/2 \ge \sqrt{ab}$ when $a, b \ge 0$. Thus we have $x_{n+1} \ge \sqrt{x_n(3/x_n)} = \sqrt{3}$, as desired.

14.20. *If $x_1 > -1$ and $x_{n+1} = \sqrt{1+x_n}$ for $n \ge 1$, then $\langle x \rangle$ converges, and $\lim_{n\to\infty} x_n = (1+\sqrt{5})/2$.* No matter what x_1 is, x_2 is positive, and for $n > 2$

each x_n exceeds 1. The criterion for $x_{n+1} < x_n$ is thus the criterion for $\sqrt{1+x_n} < x_n$. If this holds, then we successively obtain $1 + \sqrt{1+x_n} < 1 + x_n$ and $1 + x_{n+1} < x_{n+1}^2$ and $\sqrt{1+x_{n+1}} < x_{n+1}$, and thus the criterion is maintained. The criterion is equivalent to $1 + x_n < x_n^2$, which for positive x_n is the condition $x_n > (1+\sqrt{5})/2$. If $x_1 > (1+\sqrt{5})/2$, we obtain a monotone decreasing sequence bounded below by $(1+\sqrt{5})/2$. If $x_1 < (1+\sqrt{5})/2$, we obtain a monotone increasing sequence bounded above by $(1+\sqrt{5})/2$. By the Monotone Convergence Theorem, $\langle x \rangle$ converges.

Let $L = \lim_{n\to\infty} x_n$. The right side of $x_{n+1}^2 = 1 + x_n$ must converge to L^2, and the right side converges to $1 + L$. Thus $L^2 = 1 + L$, and the limit must be the positive solution of the equation, $L = (1+\sqrt{5})/2$.

14.21. *If $c > 1$, and $\langle x \rangle$ is the sequence defined by $x_1 = c$ and $x_{n+1} = x_n^2$ for $n \ge 1$, then $\langle x \rangle$ is unbounded.* Each x_n is positive. If the sequence is bounded, then its set of values has a supremum α. The set contains a sequence converging to α. This implies that $x_n > \sqrt{\alpha}$ for some n. This yields $x_{n+1} > \alpha$. Thus there cannot be an upper bound.

14.22. *If $c > 0$, then $c^{1/n} \to 1$.* When a sequence of positive numbers converges to a positive limit, the sequence of reciprocals converges to the reciprocal of the limit. Hence it suffices to prove the statement when $c > 1$.

For $c > 1$, the sequence defined by $x_n = c^{1/n}$ is monotone decreasing. Also it is bounded below by 1, so the Monotone Convergence Theorem implies that it converges to a limit L, and $L \ge 1$.

Every subsequence of a convergent sequence converges to the same limit. Hence it suffices to determine the limit L for the sequence defined by $y_k = c^{1/2^k}$. Since $y_k = x_{2^k}$, $\langle y \rangle$ is a subsequence of $\langle x \rangle$.

Note that $y_{k+1} = \sqrt{y_k}$. The properties of limits now yield $L = \sqrt{L}$. Since $L \ge 1$, we conclude that $L = 1$.

14.23. *If $f_1 = x$ and $f_{n+1} = (f_n)^2/2$ for $n \ge 1$, then $\langle f \rangle$ can converge only to 0 or 2.* The properties of limits yield $L = L^2/2$, with solution set $\{0, 2\}$.

The sequence is constant if and only if $x \in \{0, 2\}$. If $|x|_2$, then the sequence is strictly increasing and unbounded. If $0 < x < 2$, then the sequence is strictly decreasing and converges to 0. If $-2 < x < 0$, then $0 < f_2 < 2$, after which the sequence decreases to 0.

14.24. *Sequences $\langle x \rangle$ satisfying the recurrence $x_{n+1} = x_n^2 - 4x_n + 6$.*

a) If $\lim_{n\to\infty} x_n$ exists and equals L, then $L \in \{2, 3\}$. If $\langle x \rangle$ converges, then both sides of the recurrence relation must have the same limit. By the properties of limits, $\lim_{n\to\infty}(x_n^2 - 4x_n + 6) = L^2 - 4L + 6$. Hence L must satisfy $L = L^2 - 4L + 6$. The quadratic formula yields $L \in \{2, 3\}$ as the roots of this equation, so these are the only possible values of the limit.

b) The behavior of x_n as $n \to \infty$. Defining $\langle y \rangle$ by $y_n = x_n - 2$ (suggested by "completing the square" to obtain $x_{n+1} = (x_n - 2)^2 + 2$) and substituting for $\langle x \rangle$ in terms of $\langle y \rangle$ in the recurrence yields $y_{n+1} = y_n^2$. If $-1 < y_0 < 1$, we have $y_n \to 0$. If $|y_n| > 1$, we have $y_n \to \infty$. The following table describes all the cases and interprets them in terms of x_0 to obtain the behavior of x_n for large n.

start	limit	start	limit
$y_0 < -1$	$y_n \to \infty$	$x_0 < 1$	$x_n \to \infty$
$y_0 = -1$	$y_n = 1$	$x_0 = 1$	$x_n = 3$
$-1 < y_0 < 1$	$y_n \to 0$	$1 < x_0 < 3$	$x_n \to 2$
$y_0 = 1$	$y_n = 1$	$x_0 = 3$	$x_n = 3$
$y_0 > 1$	$y_n \to \infty$	$x_0 > 3$	$x_n \to \infty$

14.25. *If $x_n = x_{n-1}^2 + Ax_{n-1} + B$ for $n \ge 1$, then the possible values of $\lim_{n\to\infty} x_n$ are $L = (-a \pm \sqrt{a^2 - 4B})/2$, where $a = A - 1$.* If $x_n \to L$, then L must satisfy $L = L^2 + AL + B$. By the quadratic formula, $L = (-a \pm \sqrt{a^2 - 4B})/2$, where $a = A - 1$.

Limiting behavior of x_n in terms of x_0, A, and B. "Completing the square yields $x_n = (x_{n-1} + A/2)^2 + B - A^2/4$. Letting $y_n = x_n + A/2$, this becomes $y_n = y_{n-1}^2 + B - A^2/4 + A/2$.

This reduces the problem to studying $y_n = y_{n-1}^2 + c$. If $c > 1/4$, then there is never a limit, regardless of the starting value. This is the same as the earlier condition $a^2 - 4B \ge 0$.

The full analysis can be completed as in Exercise 14.24 or with geometric analysis.

14.26. *If $a_{n+2} = (\alpha + \beta)a_{n+1} - \alpha\beta a_n$ with $\beta \ne \alpha$, and $a_0 = a_1 = 1$, then $\lim a_{n+1}/a_n$ equals whichever of α and β has larger absolute value.* By the techniques of Chapter 12, the recurrence has characteristic roots α and β, and the general solution is $a_n = A\alpha^n + B\beta^n$. The initial conditions require $A + B = 1$ and $A\alpha + B\beta = 1$. If $|\beta| > |\alpha|$, then dividing the denominator by $B\beta^n$ and the numerator by $B\beta^{n+1}/\beta$ yields

$$\frac{a_{n+1}}{a_n} = \beta \frac{(A/B)(\alpha/\beta)^{n+1} + 1}{(A/B)(\alpha/\beta)^n + 1} \to \beta.$$

14.27. *If $0 < c < 1$, then $(c^n + 1)^{1/n} \to 1$.* Let $x_n = (c^n + 1)^{1/n}$. Since $1 < x_n < 1 + c^n$, the sequence $\langle x \rangle$ is squeezed between two sequences converging to 1. By the Squeeze Theorem, $x_n \to 1$.

$(1 + 1)^{1/n} \to 1$. This sequence is decreasing and bounded below by 1; therefore it converges to its infimum. If $\alpha > 1$, then $\alpha^n > 2$ for sufficiently large n, so no number larger than 1 is a lower bound for the seqence.

$\lim_{n\to\infty}(a^n + b^n)^{1/n} = \max\{a, b\}$. Let $x_n = (a^n + b^n)^{1/n}$. By symmetry, we may assume that $a \le b$; let $c = a/b$. Since $x_n(a^n + b^n)^{1/n} = b([a/b]^n + 1)^{1/n}$, we have $\lim_{n\to\infty} x_n = b \lim_{(\to\infty} c^n + 1)^{1/n} = b$.

14.28. *Alternative proof of the Bolzano-Weierstrass Theorem* (every bounded sequence has a convergent subsequence).

a) Every bounded sequence with a monotone subsequence has a convergent subsequence. A monotone subsequence of a bounded sequence is a bounded monotone sequence. The Monotone Convergence Theorem implies that this subsequence converges.

b) Every bounded sequence has a monotone subsequence. In a sequence $\langle a \rangle$, call index n is a *peak* if $a_m < a_n$ for $m > n$. If n and n' are peaks with $n' > n$, then $a_{n'} < a_n$. If there are infinitely many peaks, then these terms form a monotone decreasing subsequence of $\langle a \rangle$.

If there are finitely many peaks, then let n_0 be an index *after* the last peak. Since n_0 is not a peak, there is an index n_1 with $n_1 > n_0$ such that $a_{n_1} > a_{n_0}$. In general, having defined an increasing list of indices $n_0, \dots, n_k$, we can choose n_{k+1} such that $n_{k+1} > n_k$ and $a_{n_{k+1}} > a_{n_k}$, since n_k is not a peak. This generates a monotone increasing subsequence of $\langle a \rangle$.

14.29. *Multiple limit points.* A limit point of a sequence is any limit of an infinite convergent subsequence. The sequence $a_n = (-1)^n$ has limit points at ± 1. The sequence 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, $\cdots$ has every natural number as a limit point.

14.30. *If $\langle x \rangle$ is defined by $x_1 = 1$ and $x_{n+1} = 1/(x_1 + \cdots + x_n)$ for $n \ge 1$, then $x_n \to 0$.* If $\{x_1, \dots, x_n\}$ are positive, then x_{n+1} is positive, because the sum of positive numbers and the reciprocal of a positive number are positive. Since also x_1 is positive, induction on n yields that x_n is positive for every n. Next, if x_n is positive, then $\sum_{i=1}^n x_i > \sum_{i=1}^{n-1} x_i > 0$, which yields $x_{n+1} < x_n$. Hence $\langle x \rangle$ is a decreasing sequence bounded below by 0. By the Monotone Convergence Theorem, $\langle x \rangle$ converges. Furthermore, since $\langle x \rangle$ is bounded below by 0, $\lim x_n \ge 0$. There are several ways to determine the limit.

Proof 1 (monotonicity) Suppose that the limit L is positive. Since $\lim x_n$ is the infimum when $\langle x \rangle$ is decreasing, $x_n > L$ for all n. Therefore, $\sum_{k=1}^M x_k > ML$, and $x_{M+1} = 1/\sum_{k=1}^M x_k < 1/(ML)$. For $M \ge 1/L^2$, we obtain $x_{M+1} < L$, which contradicts $x_n \to L$. Thus L can only be 0.

Proof 2 (arithmetic properties of limits). Because $\sum_{i=1}^{n-1} x_i = 1/x_n$, we can rewrite the recurrence as $x_{n+1} = 1/(x_n^{-1} + x_n)$. If $\langle x \rangle$ converges to L and $L! = 0$, then the recurrence yields $L = 1/(L^{-1} + L)$. This equation simplifies to $1 = L(L^{-1} + L) = 1 + L^2$, which implies that $L = 0$ and contradicts the hypothesis. (Comment: arguing that L must satisfy $L = 1/(\sum_{i=1}^n L)$ is not valid, because a fixed term in the sequence, such as x_1, does not "approach" anything.)

Proof 3 (convergence of series). If $x_{n+1} \to L$ and $L \neq 0$, then $\sum_{i=1}^{n} x_i = 1/x_{n+1} \to 1/L$. Thus $\sum_{i=1}^{\infty} x_i$ is a convergent series. The terms of a convergent series must converge to 0, so $x_n \to 0$.

14.31. *If $x_1 \geq 0$ and $x_{n+1} = \frac{x_n+2}{x_n+1}$ for $n \geq 0$, then $x_n \to \sqrt{2}$.* We show that the sequence defined by $y_n = \left|x_n - \sqrt{2}\right|$ decreases by at least a factor of 2 with each step and hence converges to 0. This yields $x_n \to \sqrt{2}$. We compute

$$\left|x_{n+1} - \sqrt{2}\right| = \left|\frac{x_n+2}{x_n+1} - \sqrt{2}\right| = \left|\frac{x_n + 2 - \sqrt{2}x_n - \sqrt{2}}{x_n+1}\right|$$
$$= \left|\frac{(x_n - \sqrt{2}) - \sqrt{2}(x_n - \sqrt{2})}{x_n+1}\right| = \left|x_n - \sqrt{2}\right| \left|\frac{1-\sqrt{2}}{x_n+1}\right| < \frac{1}{2}\left|x_n - \sqrt{2}\right|.$$

The last step uses $x_n \geq 0$, which follows from $x_1 \geq 0$ and the recurrence.

14.32. *The fly and the train.*

Proof 1. The fly travels at 200 miles per hour for the time it takes the train to travel 2 miles. Since the fly is traveling twice as fast as the train, it travels 4 miles.

Proof 2. If the fly is at the train when it is x miles from the wall, and y is the distance from the wall when the fly is next at the train, then the fly travels $x+y$ while the train travels $x-y$. If this takes time t, then $\frac{x+y}{200} = t = \frac{x-y}{100}$; hence $y = x/3$. The fly has traveled $x + y = 4x/3$. The next segment is $(4/3)(x/3)$, etc. Starting with $x = 2$, the fly travels $(4/3)2\sum_{n=0}^{\infty}(1/3)^n = 2\frac{1}{1-1/3} = 4$ miles. (Comment: Here also, one can observe that in each segment of time the fly travels twice as far $(4x/3)$ as the train $(2x/3)$, so without the geometric series the answer is still 4 miles.

14.33. *If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge to A and B, respectively, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges and equals $A + B$.* Let $c_k = a_k + b_k$. The nth partial sum of the sequence $\langle c\rangle$ is the sum of the nth partial sum of $\langle a\rangle$ and the nth partial sum of $\langle b\rangle$. Thus the sequence $\langle s\rangle$ of partial sums of $\langle c\rangle$ is the sum of the sequences of partial sums of $\langle a\rangle$ and $\langle b\rangle$. Since they converge, the properties of limits imply that $\langle s\rangle$ converges to the sum of their limits, which is $A + B$.

Alternative proof. One can also prove this directly from the definition of convergence of series by using an $\varepsilon/2$-argument.

14.34. *Ternary expansions.* The number 1/2 satisfies $1/2 = (1/3) + (1/2)(1/3)$. Hence its ternary expansion starts with 1, and the rest of it is the ternary expansion of 1/2 again (shifted by one position). Hence $1/2 = .1111111..._{(3)}$. Using the geometric series, we can verify this by computing $\sum_{k=1}^{\infty}(1/3)^k = (1/3)/(1-1/3) = 1/2$.

Similarly, we have $.121212... = 5\sum_{k=1}^{\infty}(1/9) = 5(1/9)/(1-1/9) = 5/8$.

14.35. *Expansions of rational numbers.*

a) In base 10, $.141414\cdots$ *is* $14/99$. We have $100x = x + 14$.

b) In base 5, $.141414\cdots$ *is* $3/8$. Written in base 10, we have $25y = y + 9$.

14.36. *Expansions of rational numbers.*

a) In base 10, $.247247247\cdots$ *is* $247/999$. We have $1000x = x + 247$.

b) In base 8, $.247247247\cdots$ *is* $167/511$. Written in base 10, we have $512y = y + 167$.

14.37. *Numbers in the interval $[0, 1]$ whose ternary expansions contain no 1's.* The set is obtained geometrically by iteratively deleting the middle third of each interval that remains. The set is uncountable, because we can define a bijection f from $[0, 1]$ to the desired set S by taking the binary expansion of x and replacing each 1 with a 2 to obtain the ternary expansion of $f(x)$. Prove that the set is uncountable.

14.38. *For every rational number α, the k-ary expansion of α is eventually periodic* (after some initial portion, the remainder is a repeating finite list). Since the k-ary expansion of the integer part of α is finite, we may assume that $0 \leq \alpha < 1$. Since α is rational, we may choose integers r, s such that $\alpha = r/s$. Let $\langle a\rangle$ be the sequence of integers in the k-ary expansion of α; by definition, a_n is the integer j such that $j/k^n \leq \alpha - b_n < (j+1)/k^n$, where $b_n = \sum_{i=1}^{n-1} a_i/k^i$.

The proof that $\langle a\rangle$ is eventually periodic is modeled on long division. We can obtain the decimal expansion of r/s by dividing s into r. At each step we maintain an integer remainder between 0 and $s-1$. If the remainder is ever 0, then the expansion terminates. Otherwise, we generate remainders in $\{1, \ldots, s-1\}$. By the pigeonhole principle, we get a repetition within at most s steps. Once we have a repetition, the list of remainders repeats.

An example shows the role of the remainders. When expanding $14/23$, we divide 23 into 14 to obtain $.652\cdots$ with successive remainders $12, 5, 4, \cdots$. To express $14/23$ as partial expansion plus remainder, we have $14 = .6\cdot 23 + 12\cdot 10^{-1}$, $14 = .65\cdot 23 + 5\cdot 10^{-2}$, $14 = .652\cdot 23 + 4\cdot 10^{-3}$, etc. Here $r = 14$, $s = 23$, $b_3 = .652$, $\alpha = b_3 + (r_3/s)k^{-3}$. The nth remainder r_n is a number between 0 and $s-1$. If the remainder were at least s, then by the procedure for producing a_n we would enlarge the value j that we choose for a_n.

Viewing long division more explicitly yields a procedure for the k-ary expansion. Begin with $a_1 = j$, where $j/k \leq \alpha < (j+1)/k$. The first remainder is the integer r_1 such that $r/s = (a_1/k) + (r_1/s)k^{-1}$; in long division we compute this by $r_1 = (r - b_1 s)k$. To complete the expansion of $\alpha = r/s$, we find the expansion of r_1/s and shift it by one position, tacking it on the end of the expansion a_1 found so far. The next step generates a_2 and a remainder r_2 between 0 and $s-1$, via $r_1/s = (a_2/k) + (r_2/s)k^{-1}$. Each successive remainder r_{n+1} is the first remainder when we expand the ratio r_n/s.

As remarked earlier, the sequence of remainders has fewer than s possible values and must repeat. If $r_m = r_n$ with $m > n$, then the portion of the expansion after position m (expanding r_m/s) is the same as the portion of the expansion after position n (expanding r_n/s). The latter portion repeats the portion beginning at position $n+1$, and this argument iterates to establish the successive repetitions.

b) If the k-ary expansion of x is eventually periodic, then x is rational. Since every integer is rational and the sum of two rational numbers is rational, it suffices to prove the claim for $0 < \alpha < 1$.

Suppose that the expansion eventually repeats with period s. We have $\alpha = \sum_{n=1}^{\infty} c_n k^{-n}$. Let $\beta = k^s\alpha$. By the hypothesis, the expansion of β eventually agrees with the expansionof α. Thus $\beta - \alpha = (k^s - 1)\alpha = \sum_{n=1}^{r} c_n k^{s-n}$ for some r. Since this sum is finite, $(k^s - 1)\alpha$ is a rational number, and hence also α is rational.

14.39. *The series $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ converges to an irrational number.* The decimal expansion of the sum has a 1 in position $n!$ for each n, and it has a 0 in each other position. Since the 1s are successively farther and farther apart, the expansion is never eventually periodic.

14.40. *The geometric series.* Suppose that $|x| < 1$.

a) Given y_0, the sequence defined by $y_{n+1} = 1 + xy_n$ for $n \ge 0$ converges to $1/(1-x)$. A particular solution to the inhomogeneous recurrence is given by $C = 1 + xC$, which yields $C = 1/(1-x)$. The general solution is $y_n = Ax^n + 1/(1-x)$. For every A, this converges to $1/(1-x)$, since $|x| < 1$.

b) Given y_0, the sequence defined by $y_n = 1 + xy_{n+1}$ for $n \ge 0$ does not converge. We have $y_{n+1} = \frac{1}{x}y_n - \frac{1}{x}$. The particular solution satisfies $C = C/x - 1/x$, so again $C = 1/(1-x)$. However, the general solution is now $y_n = A(1/x)^n + 1/(1-x)$, which diverges for all y_0 because $|1/x| > 1$.

14.41. *Guaranteeing a constant total error.* For each $n \ge 1$, make the nth measurement within $1/2^n$. The total error is then at most $\sum_{n=1}^{\infty} 1/2^n = 1$.

14.42. *A union of countably many sets of measure zero also has measure zero.* A countable collection has a bijection to $\mathbb{N}$, so we can indexed our sets of measure zero as $\{S_n : n \in \mathbb{N}\}$. Fix $\varepsilon > 0$. Since each S_n has measure zero, we can cover S_n using a countable collection of intervals whose lengths sum to less than $\varepsilon/2^n$. Let $\mathbf{I}$ be the union of all these collections of intervals. Since unions of countably many countable sets are countable, $\mathbf{I}$ is a countable collection of intervals. Furthermore, the lengths of all the intervals in $\mathbf{I}$ sum to less than $\sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$. Thus $\cup S_n$ has measure 0.

Let $Z(n)$ be the set of integer multiples of $1/n$. The set $\mathbb{Q}$ of rational numbers is $\bigcup_{n\in\mathbb{N}} Z(n)$. Each $Z(n)$ has mesure zero, being covered by intervals centered at its elments. Since there are countably many such sets, also $\mathbb{Q}$ has measure zero.

14.43. *$\sum_{n=1}^{\infty} (\frac{x}{x+1})^n = x$ when $x > -1/2$, and otherwise the sum diverges.* Since x is fixed, this is a geometric series (lacking its first term). It converges if and only if $\left|\frac{x}{x+1}\right| \le 1$. This condition is equivalent to $x^2 < (x+1)^2$, which is equivalent to $2x + 1 > 0$.

When $\sum_{n=1}^{\infty} z^n$ converges, its value is $z/(1-z)$, and we have

$$\sum_{n=1}^{\infty} (\frac{x}{x+1})^n = \frac{x}{x+1}/(1 - \frac{x}{x+1}) = x/(x+1-x) = x.$$

14.44. *The value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is between 1 and 2.* We compare with $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, which we can compute using the definition of convergence of a series. If $a_k = \frac{1}{k(k+1)}$, then $a_k = \frac{1}{k} - \frac{1}{k+1}$. For the partial sum, we have $s_n = \sum_{i=1}^{n} \frac{1}{k} - \sum_{i=1}^{n} \frac{1}{k+1}$. These sums telescope to yield $s_n = 1 - 1/(n+1)$, which has limit 1. Hence $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Since $1/k^2 > 1/k(k+1)$ and all the terms are positive, we have $\sum_{k=1}^{\infty} \frac{1}{k^2} > 1$. If we separate out the first term, we can compare $1/(k+1)^2 < 1/k(k+1)$ and apply the comparison test to obtain convergence and an upper bound: $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} < 1 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 2$.

14.45. *If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ equals $1/n$, for $n \ge 1$, then $a_1 = 1$ and $a_n = -1/[n(n-1)]$ for $n > 1$.* The nth partial sum is $s_n = \sum_{k=1}^{n} a_k$. We have $a_1 = s_1 = 1$. For $n > 1$, we have $a_n = s_n - s_{n-1} = 1/n - 1/(n-1) = -1/[n(n-1)]$.

14.46. *If $b_k = c_k - c_{k-1}$, where $c_0 = 1$ and $\lim_{k\to\infty} c_k = 0$, then $\sum_{k=1}^{\infty} b_k = -1$.* We use the definition of series and the telescoping property of the sum to compute $\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} c_k - c_{k-1} = \lim_{n\to\infty} c_n - c_0 = -1$.

14.47. *If $\sum a_n^2$ and $\sum b_n^2$ converge to M and N, then $\sum a_n b_n$ converges.* Using the AGM Inequality, we have $\sum |a_n b_n| \ge \frac{1}{2}\sum a_n^2 + \frac{1}{2}\sum b_n \le (M+N)/2$. Hence $\sum a_n b_n$ converges, by the comparison test.

14.48. *The tennis game to four points.* Each list with k points for the server and l for the other player has probability $p^k(1-p)^l$ of occurrence. Given k and l, there are $\binom{k+l}{l}$ such lists. With $k = 4$ and $l < 4$, the total probability that the server wins is $\sum_{l=0}^{3} \binom{4+l}{l} p^k (1-p)^l$.

14.49. *If $\sum_{k=1}^{\infty} c_k$ diverges to ∞, and $a_k \ge c_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ diverges to ∞.* Let $s_n = \sum_{k=1}^{n} c_k$ and $r_n = \sum_{k=1}^{n} a_k$. We are given that $\langle s \rangle$ has no upper bound. Also, $r_n \ge s_n$, and hence also $\langle r \rangle$ has no upper bound.

14.50. *The series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges.* If $\sum 1/(2n-1)$ converges, then $\sum 2/(2n-1)$ converges. Since $1/n < 2/(2n-1)$, we conclude by the comparison test that $\sum 1/n$ converges. Since in fact $\sum 1/n$ diverges, our original hypothesis that $\sum 1/(2n-1)$ converges is false.

14.51. *The number e is irrational.* We have defined $e = \sum_{k=0}^{\infty} 1/k!$. If e is rational, then let $e = m/n$ be an expression of e as a fraction in lowest terms. Now $n!e$ is an integer equal to $\sum_{k=0}^{\infty} n!/k!$. Since the terms up to $k = n$ are integers, we conclude that $\sum_{k=n+1}^{\infty} n!/k!$ is an integer.

We obtain a contradiction by proving that $\sum_{k=n+1}^{\infty} n!/k! < 1/n$. Canceling $n!$ leaves $\prod_{i=1}^{k-n} \frac{1}{n+i}$ as the term for k. This is bounded by $1/(n+1)^{k-n}$. Therefore,

$$\sum_{k=n+1}^{\infty} \frac{n!}{k!} < \sum_{k=n+1}^{\infty} \left(\frac{1}{n+1}\right)^{k-n} = \sum_{j=1}^{\infty} \left(\frac{1}{n+1}\right)^{j} = \frac{1/(n+1)}{1-1/(n+1)} = \frac{1}{n}.$$

14.52. *If the sequence $\langle a \rangle$ alternates in sign and the sequence of absolute values is nondecreasing and converges to 0, then $\sum_{k=1}^{\infty} a_k$ converges.* By definition, the series converges if and only if the sequence $\langle s \rangle$ of partial sums (defined by $s_n = \sum_{k=1}^{n} a_k$) converges. We prove that $\langle s \rangle$ converges.

a) Using Cauchy convergence criterion. We use the definition to show that $\langle s \rangle$ is a Cauchy sequence. We prove that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n, m > N$ implies $|s_m - s_n| < \varepsilon$.

Because $a_{n+1} \to 0$ and $a_{n+1} = s_{n+1} - s_n$, we have $s_{n+1} - s_n \to 0$. For any $\varepsilon > 0$, the definition of convergence yields $N \in \mathbb{N}$ such that $k > N$ implies $|s_{k+1} - s_k| < \varepsilon$. The crucial point is that the conditions on $\langle a \rangle$ imply that for all m, s_{m+1} lies between s_{m-1} and s_m, and hence all partial sums after s_{k+1} lie between s_{k+1} and s_k. This implies that the difference between any two of them is at most $|s_{k+1} - s_k|$, which is less than ε if $m, n > k > N$.

If $s_{m-1} < s_m$, then a_m is positive and a_{m+1} is negative, but smaller in magnitude than a_m. This implies that $s_{m-1} < s_{m+1} < s_m$. If $s_{m-1} > s_m$, then a_m is negative and a_{m+1} is positive, but smaller in magnitude than a_m. This implies that $s_{m-1} > s_{m+1} > s_m$.

b) Using properties of convergence. We define sequences $\langle c \rangle$ and $\langle d \rangle$ such that $\langle c \rangle$ is nondecreasing, $\langle d \rangle$ is nonincreasing, $d_n - c_n \to 0$, and $c_n \le s_n \le d_n$. The first three properties imply that $\langle c \rangle$ and $\langle d \rangle$ converge to the same limit (details in the text). The fourth property and the Squeeze Theorem then yield the convergence of $\langle s \rangle$.

Next we obtain the desired $\langle c \rangle$ and $\langle d \rangle$. By symmetry, we may assume $a_0 > 0$. Then the sequence $c_j = s_{2j+1}$ of partial sums ending an an odd index is an increasing sequence, since each a_{2i} is positive and $a_{2i} + a_{2i+1} > 0$. On the other hand, the sequence $d_j = s_{2j}$ is a decreasing sequence, since $a_{2j-1} + a_{2j} < 0$. Also $d_j - c_j = -a_{2j+1} \to 0$. By part (a), $\langle c \rangle$ and $\langle d \rangle$ converge and have the same limit. Now make the sequences $\langle c' \rangle$ and $\langle d' \rangle$ by taking each term twice, so $c'_u b2j + 1 = c'_u b2j = s_{2j+1}$ and $d'_u b2j + 1 = d'_u b2j = s_{2j}$. These have the same limit as $\langle c \rangle$ and $\langle d \rangle$. Also $c'_n \le s_n \le d'_n$ for all $n \ge 1$ (actually, $s_n = c'_n$ if n is odd, and $s_n = d'_n$ if n is even). By the Squeeze Theorem, the sequence $\langle s \rangle$ of partial sums of the series has a limit, meaning the series converges.

14.53. *Summation dependent on order of terms.* Consider $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. By Exercise 15.32, this series converges. The first three terms sum to 5/6, and thereafter each successive pair has negative sum, so the sum of the series is less than 5/6. On the other hand, the series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ has sum greater than 5/6, because again the first three terms sum to 5/6, but now each successive triple of terms has positive sum, since $1/(4n-3) + 1/(4n-1) - 1/(2n)$ is positive for $n \ge 1$ (note that each term appears in exactly one triple).

To find a reordering of the terms whose sum is finite but exceeds 3/2, group enough positive terms with the each negative term so that each group has positive sum and the first group has sum at least 3/2. To absorb the $-1/2$, we take reciprocals of enough odd numbers to reach at least 2; this happens at 1/15. We have taken eight positive terms and one negative. Continuing in this fashion yields groups of the form $-1/(2n) + \sum_{k=1}^{8} 1/(16n - 2k + 1)$. Since each positive quantity is bigger than $1/(16n)$, the sum in each group (eight positive terms and one negative term) is positive. Thus the sum of the series exceeds 3/2.

The series produced by this ordering is convergent because we can group successive terms to express it as an alternating series. We have argued that $\sum_{k=1}^{8} 1/(16n - 2k + 1) > 1/(2n)$. The next positive group of eight terms sums to $\sum_{k=1}^{8} 1/(16n + 16 - 2k + 1) = \sum_{k=1}^{8} 1/(16n + 2k - 1)$. Here each summand exceeds $1/(16n)$, so we conclude that $1/(2n) > \sum_{k=1}^{8} 1/(16n + 16 - 2k + 1)$. Thus taking positive terms in groups of eight and alternating these sums with single negative terms yields an alternating series of terms with decreasing magnitudes that approach zero.

14.54. *If $\sum a_k$ converges, $\sum |a_k|$ diverges, and $L \in \mathbb{R}$, then the terms of $\langle a \rangle$ can be reordered to obtain a series that converges to L.* We imitate the idea in the solution to Exercise 14.34. Because the sum of the positive terms diverges, the remainder still diverges after deleting any initial portion, and the same statement holds for the negative terms in the other direction. Thus we can choose enough of the positive terms to exceed L, then enough of the negative terms so the partial sum is less than L, then enough positive terms to exceed L again, and so on. Furthermore, we stop taking terms of the current type when we first exceed L. Since $a_n \to 0$, the amount by which we can differ from L is bounded by a sequence that approaches 0.

14.55. *If $a_k \to 0$ and the sequence of partial sums is bounded, then $\sum_{k=1}^{\infty} a_k$ converges—FALSE.* We let $a_k = \pm(1/k)$, with signs chosen as follows. Suppose we have chosen the signs up through a_n. If the partial sum $\sum_{k=1}^{n} a_k$ is now nonpositive, then we choose just enough of the forthcoming terms to be positive to bring the partial sum up as high as 1. If the partial sum is now at least 1, then we choose just enough of the forthcoming terms to be negative to bring the partial sum down as far as 0. Since $\sum(1/k)$ is unbounded, this procedure works, and the resulting series has bounded partial sums but does not converge.

14.56. *Ratio test for divergence— If $\langle a\rangle$ is a sequence such that $|a_{k+1}/a_k| \to \rho$ for some $\rho > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.* Choose β so that $1 < \beta < \rho$. By the hypothesis, there is some N such that $n \geq N$ implies that $|a_{k+1}/a_k| > \beta$. Thus $|a_n| > |a_N|\,\beta^{n-N}$ for $n \geq N$. Since $\beta > 1$, this prevents $a_n \to 0$, which is required for the series to converge.

14.57. $\lim(F_{n+1}/F_n) = (1+\sqrt{5})/2$, *where $\langle F\rangle$ is the Fibonacci sequence.* By the ratio test, the series $\sum_{n=0}^{\infty} F_n x^n$ converges if $\lim x F_{n+1}/F_n$ exists and is less than 1. Since the generating function is $1/(1-x-x^2)$, this occurs whenever x is less than the smallest magnitude R of a root of $1-x-x^2$. By the ratio test, this yields $\lim \frac{F_{n+1}}{F_n} = \frac{1}{R}$.

To find R, set $1-x-x^2 = 0$. By the quadratic formula, $x = (-1 \pm \sqrt{5})/2$. The zero with smaller magnitude is $(-1+\sqrt{5})/2$, and its reciprocal is $(1+\sqrt{5})/2$. This value is the limit, which agrees with the formula for the Fibonacci numbers in Solution 12.23.

14.58. *Limit comparison test: If $\langle a\rangle$ and $\langle b\rangle$ are sequences of positive numbers, and a_k/b_k converges to a nonzero real number L, then $\sum_{k=1}^{\infty} b_k$ converges if and only if $\sum_{k=1}^{\infty} a_k$ converges.* Since a_k/b_k converges to a nonzero real number if and only if b_k/a_k converges to a nonzero real number, it sufficent to prove one direction of the equivalence. We prove that convergence of $\sum_{k=1}^{\infty} b_k$ implies convergence of $\sum_{k=1}^{\infty} a_k$. Let $L = \lim a_k/b_k$.

Proof 1 (Cauchy sequences). Suppose that $\sum b_k$ converges, and let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. It suffices (by the Cauchy Convergence Criterion) to prove that $\langle s\rangle$ is a Cauchy sequence. When $m \geq n$, we have $s_m - s_n = \sum_{k=n+1}^{m} a_k$. We want to approximate this by the corresponding sum $\sum_{k=n+1}^{m} b_k$, which we can make small.

For $\varepsilon > 0$, we need to choose N such that $n, m \geq N$ implies $|s_m - s_n| < \varepsilon$. Since $\langle t\rangle$ converges and thus is a Cauchy sequence, there exists N_1 such that $n, m > N_1$ implies $|t_m - t_n| < \varepsilon/(L+1)$. Since $a_k/b_k \to L$, there exists N_2 such that $k > N_2$ implies $|a_k/b_k - L| < 1$ and hence that $a_k < (L+1)b_k$. Choose $N = \max\{N_1, N_2\}$. Now $n, m \geq N$ (we may assume that $m \geq n$) implies

$$|s_m - s_n| = \sum_{k=n+1}^{m} a_k < \sum_{k=n+1}^{m} (L+1)b_k = (L+1)|t_m - t_n| < (L+1)\frac{\varepsilon}{L+1} = \varepsilon$$

Proof 2 (comparison of partial sums). Define e_k by $a_k = Lb_k + e_k$. Since $a_k/b_k = L + e_k/b_k$, we have $e_k/b_k \to 0$. A convergent sequence is bounded, so there is a constant C such that $|e_k/b_k| < C$ for all k. Now

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(Lb_k + e_k) \leq L\sum_{k=1}^{n} b_k + \sum_{k=1}^{n} Cb_k = (L+c)\sum_{k=1}^{n} b_k.$$

Since partial sums of $\sum a_k$ are bounded by a constant multiple of the corresponding sums for $\sum b_k$, convergence of $\sum b_k$ implies convergence of $\sum a_k$.

14.59. *If $\langle a\rangle$ is a convergent sequence of positive numbers, then $\sum_{k=1}^{\infty} \frac{1}{ka_k}$ diverges.* Let $L = \lim a_k$. Let $b_k = 1/k$ and $c_k = 1/(ka_k)$. We have $b_k/c_k = a_k \to L$. By the limit comparison test, $\sum_{k=1}^{\infty} c_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. Since this is the harmonic series, both diverge.

14.60. *Applications of limit comparison test.*

a) $\sum_{n=1}^{\infty} \frac{2n^2+15n+2}{n^4+3n+1}$ *converges.* Let $a_n = 1/n^2$ and $b_n = \frac{2n^2+15n+2}{n^4+3n+1}$. Since $\lim(b_n/a_n) = 2$ and $\sum_{n=1}^{\infty} a_n$ converges, also $\sum_{n=1}^{\infty} b_n$ converges.

b) $\sum_{n=1}^{\infty} \frac{2n^2+15n+2}{n^3+3n+1}$ *diverges.* Let $a_n = 1/n$ and $b_n = \frac{2n^2+15n+2}{n^3+3n+1}$. Since $\lim(b_n/a_n) = 2$ and $\sum_{n=1}^{\infty} a_n$ diverges, also $\sum_{n=1}^{\infty} b_n$ diverges.

c) $\sum_{n=1}^{\infty} \frac{3+5n+n^2}{2^n}$ *converges.* Let $a_n = n^2/2^n$ and $b_n = \frac{3+5n+n^2}{2^n}$. Since $\lim(b_n/a_n) = 1$ and $\sum_{n=1}^{\infty} a_n$ converges, also $\sum_{n=1}^{\infty} b_n$ converges.

14.61. *If p is a polynomial of degree d, and q is a polynomial of degree at least $d+2$ that is nonzero on positive integers, then $\sum_{n=1}^{\infty} \frac{p(n)}{q(n)}$ converges.* Let r be the difference in the degrees of p and q. Let $a_n = n^{-r}$ and $b_n = \frac{p(n)}{q(n)}$. Since b_n/a_n converges to the ratio of the leading coefficients of p and q (a constant), and $\sum_{n=1}^{\infty} a_n$ converges, also $\sum_{n=1}^{\infty} b_n$ converges.

14.62. *Ratio test by limit comparison test.* We need a variation on the limit comparison test. That test says that if $b_k/a_k \to L$, where L is nonzero, then $\sum b_k$ converges if and only if $\sum a_k$ converges. However, if L is zero, then still the convergence of $\sum a_k$ implies the convergence of $\sum b_k$.

We want to prove that $b_{k+1}/b_k \to \rho$ with $\rho < 1$ implies that $\sum b_k$ converges. By the comparison test, we may assume that $\langle b\rangle$ is a sequence of positive numbers. Choose r with $\rho < r < 1$. By the definition of limit, there exists N such that $b_{k+1}/b_k < r$ for $k \geq N$. For $k \geq N$, we thus have

$$\frac{b_{k+1}}{r^{k+1}} = \frac{b_{k+1}}{b_k}\frac{b_k}{r^{k+1}} < r\frac{b_k}{r^k}\frac{1}{r} = \frac{b_k}{r^k}.$$

Thus $\langle c \rangle$ with $c_k = b_k/r^k$ is a decreasing sequence of positive numbers for $k \geq N$ and hence converges. In fact, it converges to 0, but the argument for the limit comparison test still applies in this direction. Thus the convergence of the geometric series $\sum r^k$ implies the convergence of $\sum b_k$.

14.63. *Cauchy condensation test.*

a) If $\langle a \rangle$ is a decreasing sequence of positive numbers, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{0=1}^{\infty} 2^k a_{2^k}$ converges. We view the term $2^k a_{2^k}$ as representing 2^k copies of the term a_{2^k}. Hence when we think of the first series as $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots$, we can think of the second as $a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + \cdots$ and compare them term by term.

Define $\langle c \rangle$ to consist of 1 copy of a_1, then 2 of a_2, etc. Since $\sum_{k=0}^{m-1} 2^k = 2^m - 1$, the copies of a_{2^m} in $\langle c \rangle$ occur at positions 2^m through $2^{m+1} - 1$, and we compare them with a_{2^m} through $a_{2^{m+1}-1}$. Since $\langle a \rangle$ is decreasing and positive, this yields $a_n \leq c_n$ for all n. If the series $\sum_{n=1}^{\infty} c_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

For the converse, think of twice the first series as the series $a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + \cdots$, defining $\langle d \rangle$ by setting $d_{2n} = d_{2n-1} = a_n$. As before, the copies of a_{2^m} in $\langle c \rangle$ occur at positions 2^m through $2^{m+1} - 1$. The terms of $\langle d \rangle$ at these positions are $a_{2^{m-1}}$ through a_{2^m}, with one copy of the first and last and two each of those between. Since $\langle a \rangle$ is decreasing and positive, this yields $c_n \leq d_n$ for all n. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} d_n$ is twice that and also converges, and then $\sum_{n=1}^{\infty} c_n$ converges by the comparison test.

b) $\sum_{n=1}^{\infty} n^{-c}$ converges if $c > 1$ is a fixed real number. Here $a_n = n^{-c}$. Applying the test above, we form the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k (2^k)^{-c} = \sum_{k=0}^{\infty} (2^{1-c})^k.$$

This is the geometric series $\sum_{k=0}^{\infty} x^k$ with $x = 2^{1-c}$. This converges if $|x| < 1$, which is true if and only if $c > 1$, and diverges if $x \geq 1$. By the Cauchy condensation test, the original series also converges if $c > 1$.

14.64. *Given that $\langle a \rangle$ and $\langle b \rangle$ are sequences of positive numbers such that $\frac{b_{k+1}}{b_k} \leq \frac{a_{k+1}}{a_k}$ for sufficiently large k, convergence of $\sum_{k=1}^{\infty} a_k$ implies convergence of $\sum_{k=1}^{\infty} b_k$.* Let N be an integer such that $b_{n+1}/b_n \leq a_{n+1}/a_n$ for $n \geq N$. For $k \geq 0$, this yields $b_{N+k}/b_N \leq a_{N+k}/a_N$ (by telescoping product). Hence $b_{N+k} \leq a_{N+k}(b_N/a_N)$ for $k \geq 0$. Since $\sum_k a_{N+k}$ converges, so does $\frac{b_n}{a_N} \sum_k a_{N+k}$, and therefore $\sum_k b_{N+k}$ also converges, by the comparison test. Since this omits only finitely many terms of $\sum_j b_j$, also that sum converges.

14.65. *Raabe's test.*

a) If $0 < x < 1$, then $(1 - px) < (1 - x)^p$, where p is a real number greater than 1. The text does not actually define $(1 - x)^p$ until p349 in Chapter 17. We also apologize for using differentiation here. Define f on the interval $[0, 1]$ by $f(x) = (1-x)^p + px - 1$. Now $f'(x) = p - p(1-x)^{p-1} = p(1 - (1 - x)^{p-1})$. Thus $f'(0) = 0$, and $f'(x) > 0$ otherwise. Therefore f is increasing, and $f(x) > f(0)$ for $x > 0$. Thus $(1 - x)^p > 1 - px$.

b) If $\langle b \rangle$ is a sequence of positive numbers such that $b_{k+1}/b_k \leq 1 - p/k$ for sufficiently large k, then $\sum_{k=1}^{\infty} b_k$ converges. Let $a_k = 1/k^p$. By part (a), $a_{k+1}/a_k = (1 - \frac{1}{k+1})^p > 1 - \frac{p}{k+1}$. Thus if $b_{k+1}/b_k < 1 - p/(k + 1)$, then $b_{k+1}/b_k < a_{k+1}/a_k$. Since $\sum a_k$ converges, so does $\sum b_k$, by Exercise 14.64.

14.66. *Every nonzero rational number is a finite sum of reciprocals of distinct integers.* Such a sum is called an *Egyptian fraction*. It suffices to show this for a positive rational number x. Since $1/k > 0$ for each $k \in \mathbb{N}$, and $\sum 1/k$ diverges, there is a unique $n \in \mathbb{N}$ such that $x - \sum_{j=1}^{n}(1/j) > 0$ but $x - \sum_{j=1}^{n+1}(1/j) \leq 0$.

To write x as an Egyptian fraction, it therefore suffices to find an Egyptian fraction for $x - \sum_{j=1}^{n}(1/j)$ using only reciprocals of integers greater than n. Since this value is less than 1, it suffices to find such a representation for each rational number p/q with $0 < p/q < 1$. Given p/q, the value of n is the unique integer k such that $1/(k + 1) \leq p/q < 1/k$.

Having found n, observe that $np < q$. Let $p'/q' = p/q - 1/(n+1)$. Note that $\frac{p'}{q'} = \frac{(n+1)p-q}{q(n+1)} = \frac{p+np-1}{q(n+1)}$. Since $np < q$, we have $p' < p$; we obtain a smaller rational number with a smaller numerator. Also $p'/q' < 1/(n + 1)$, so continuing the process uses only reciprocals of larger integers. After at most p iterations, we have written p/q as an Egyptian fraction.

Illustration: $5/11 = 1/3 + 4/33 = 1/3 + 1/9 + 3/297 = 1/3 + 1/9 + 1/99$ Since $p/q - 1/(n + 1) < 1/n - 1/(n + 1) = 1/(n^2 + n)$, the denominators grow rather quickly in this process.

14.67. $\exp(x + y) = \exp(x)\exp(y)$. By the series definition and the Binomial Theorem, we have $\exp(x + y) = \sum_n (x + y)^n/n! = \sum n \sum_j \frac{x^j y^{n-j}}{j!(n-j)!}$. Since the series for $\exp(x + y)$ is absolutely convergent, we can rearrange the terms to get $\sum x^j/j! \sum y^k/k!$, which equals $\exp(x)\exp(y)$.

14.68. *Root test.* Let $\langle a \rangle$ be a sequence such that $|a_n|^{1/n} \to \rho$.

a) If $\rho < 1$, then $\sum_{k=1}^{\infty} a_k$ converges. Choose N so that $n \geq N$ implies $|a_n|^(1/n) < \rho < \rho' < 1$. Now $|a_n| < (\rho')^n$, so the series converges by comparison with the geometric series.

b) If $\rho > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges. Since $a_n- > \rho$, and $\rho > 1$, the terms don't then to 0, and hence the series diverges.

c) If $\rho = 1$, then $\sum_{k=1}^{\infty} a_k$ may converge or diverge. Let $a_n = (1/n)^s$ for fixed s. The series converges when $s = 2$ but diverges when $s = 1$, yet in each case $(a_n)^{1/n} \to 1$, since $(1/n^s)^{1/n} = (n^{1/n})^{-s} \to 1^{-s}$.

14.69. *Root test with* lim sup. Let $\langle a \rangle$ be a sequence such that $L = \limsup |a_n|^{1/n}$.

a) If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ converges. Choose L' with $L < L' < 1$, and choose N so that $n \geq N$ implies $|a_n|^{1/n} < L'$. For $n \geq N$, we have $|a_n| < (L')^n$. Hence the series converges by comparison with the geometric series.

b) If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges. Choose L' with $L > L' > 1$, and choose N so that $n \geq N$ implies $|a_n|^{1/n} > L'$. For $n \geq N$, we have $|a_n| > (L')^n > 1$. Since a_n does not tend to 0, the series diverges.

15. CONTINUITY

15.1. *There is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(k) = (-1)^k$ for $k \in \mathbb{Z}$—TRUE.* Let $f(x) = \cos(\pi x)$ for $x \in \mathbb{R}$. Without the cosine function, it is easy to sketch the graph of such a function f.

15.2. *There is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if and only if $x \in \mathbb{Z}$—TRUE.* Let $f(x) = \sin(\pi x)$ for $x \in \mathbb{R}$. Without the sine function, it is easy to sketch the graph of such a function f.

15.3. *If f is continuous on $\mathbb{R}$, and $f(x) = 0$ for $x \in \mathbb{Q}$, then f is constant—TRUE.* If f is continuous and $f(x_0) \neq 0$, then $f(x)$ is nonzero in an interval around x_0. This interval contains rational numbers, and f is zero on $\mathbb{Q}$. Thus $f(x_0) = 0$ for all $x_0 =\in \mathbb{R}$.

15.4. *There exists $x > 1$ such that $\frac{x^2+5}{3+x^7} = 1$—TRUE.* Let $f(x) = \frac{x^2+5}{3+x^7}$. Since $f(1) = 3/2 > 1$ and $f(2) = 9/131 < 1$ and f is continuous on the interval $[1, 2]$, the Intermediate Value Theorem guarantees the existence of $x \in (1, 2)$ such that $f(x) = 1$.

15.5. *The function f defined by $f(x) = |x|^3$ is continuous at all $x \in \mathbb{R}$—TRUE.* The absolute value and cubing functions are continuous. Since it is a composition of continuous functions, f is continuous.

15.6. *If $f+g$ and fg are continuous, then f and g are continuous—FALSE.* Let $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = -1$ for $x \notin \mathbb{Q}$. Let $g(x) = -1$ for $x \in \mathbb{Q}$ and $g(x) = 1$ for $x \notin \mathbb{Q}$. Then $(f + g)(x) = 0$ and $(fg)(x) = -1$ for all $x \in \mathbb{R}$. These are continuous, but f and g are not.

15.7. *If f, g, h are continous on the interval $[a, b]$, and $f(a) < g(a) < h(a)$ and $f(b) > g(b) > h(b)$, then there exists $c \in [a, b]$ such that $f(c) = g(c) = h(c)$—FALSE.* Consider $a = 0$ and $b = 1$. On the interval $[0, 1]$, let $f(x) = x$, $g(x) = 2/3$, and $h(x) = 1 - x$.

15.8. *If $|f|$ is continuous, then f is continuous—FALSE.* Let $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = -1$ for $x \notin \mathbb{Q}$. Now $|f|$ is constant, but f is nowhere continuous.

15.9. Let f and g be continuous on $\mathbb{R}$.

a) If $f(x) > g(x)$ for all $x > 0$, then $f(0) > g(0)$—FALSE. Let $f(x) = x$ and $g(x) = 0$ for all $x \in \mathbb{R}$.

b) f/g is continuous at all $x \in \mathbb{R}$—FALSE. Let $g(x) = 0$ for all $x \in \mathbb{R}$, and let f be any continuous function that is nonzero somewhere.

c) If $0 < f(x) < g(x)$ for all x, then there is some $x \in \mathbb{R}$ such that $f(x)/g(x)$ is the maximum value of f/g—FALSE. Let $f(x) = 1 + x^2$ and $g(x) = 2 + x^2$ for $x \in \mathbb{R}$. The image of f/g is the $\{y : 1/2 \leq y < 1\}$. The supremum of this set is 1, but $f(x)/g(x) < 1$ for all x.

d) If $f(x) \leq g(x)$ for all x, and g is never 0, then f/g is bounded—FALSE. Let $f(x) = -x^2$ and $g(x) = 1$ for all x.

e) If $f(x)$ is rational for each x, then f is constant—TRUE. If a continuous function attains two values, then by the Intermediate Value Theorem it attains all values between them, which includes both rational and irrational values.

15.10. *a) If f is continuous on $\mathbb{R}$, then f is bounded—FALSE.* Let $f(x) = x$ for $x \in \mathbb{R}$.

b) If f is continuous on $[0, 1]$, then f is bounded—TRUE. By the Maximum-Minimum Theorem.

c) There is a function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous at exactly one point—TRUE. Let $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$.

d) If f is continuous on $\mathbb{R}$ and is bounded, then f attains its supremum—FALSE. Let $f(x) = x^2/(1 + x^2)$ for $x \in \mathbb{R}$. The image of f is $\{y \in \mathbb{R} : 0 \leq y < 1\}$. The supremum is 1, but $f(x) < 1$ for all x.

15.11. *The Intermediate Value Theorem remains true when the hypothesis $f(a) < y < f(b)$ is replaced with $f(a) > y > f(b)$.* Suppose that f is continuous on $[a, b]$ and $f(a) > y > f(b)$. Let $g = -f$. The function g is continuous, and $g(a) = -f(a) < -y < -f(b) = g(b)$. Hence the Intermediate Value Theorem guarantees an x in (a, b) with $g(x) = -y$. That is, $f(x) = y$.

15.12. *Construction of a function f and positive sequences $\langle a \rangle$ and $\langle b \rangle$ converging to 0 such that $f(a_n)$ converges but $f(b_n)$ is unbounded.* (**Note:** The requirement that the terms of the sequences be positive will be added in the next edition to make the problem more valuable.) As long as $\langle a\rangle$ and $\langle b\rangle$ have no common terms, we can define f arbitrarily on the two sequences, such as $f(x)=0$ for $x=a_n$ and $f(x)=1/x$ for $x=b_n$.

No such function f is continuous at 0. If f is continuous at 0, then $\lim f(b_n)=f(0)$ whenever $b_n\to 0$. If the sequence $f(b_n)$ is unbounded, then it cannot converge.

15.13. *The absolute value function is continuous.* By squaring both sides, we observe that $||x|-|a||\le |x-a|$. With $f(x)=|x|$, taking $\delta=\varepsilon$ then implies that $|f(x)-f(a)|<\varepsilon$ when $|x-a|<\delta$.

15.14. *If $f(x)=1/x$, then $f(x)$ is within .1 of $f(.5)$ when x is within $1/42$ of .5, and no wider interval around .5 has this property.* We are asked to find the supremum of the set $\{\delta\colon |x-.5|<\delta\Rightarrow |1/x-2|<.1\}$.

As x moves away from 2^{-1}, we reach a point above or below where $|1/x-2|=.1$. This occurs at $x=2.1^{-1}$ or $x=1.9^{-1}$. Since $\left|2.1^{-1}-2^{-1}\right|=1/42$ and $\left|1.9^{-1}-2^{-1}\right|=1/38$, we conclude that the implication $|x-.5|<\delta\Rightarrow\left|x^{-1}-2\right|<.1$ is true when $\delta\le 1/42$ and not when $\delta>1/42$.

15.15. *Given $f(x)=x^2+4x$, choosing $|x|<\sqrt{(4+\varepsilon)}-2$ implies $|f(x)|<\epsilon$ (given $\varepsilon<4$).* We require δ so that $|x|<\delta$ implies $|x^2+4x|<\epsilon$. Let $g(x)=x^2+4x$. Since g is increasing for $x>-2$, g is injective. We solve each of $g(x)=\epsilon$ and $g(x)=-\epsilon$. By the quadratic formula, $g(-2+\sqrt{(4+\varepsilon)})=\varepsilon$ and $g(\sqrt{(4-\varepsilon)}-2)=-\varepsilon$. The solution with smaller absolute value is $-2+\sqrt{(4+\varepsilon)}$, which we call δ. Now $|x|<\delta$ implies $|x^2+4x|<\epsilon$.

15.16. *If $\lim_{x\to 0}f(x)=0$, then for all $n\in\mathbb{N}$, there exists x_n such that $|f(x_n)|<1/n$.* By the definition of limit, for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-0|<\delta$ when $0<|x-0|<\delta$. To choose x_n, let $\varepsilon=1/n$, and choose x_n from the resulting interval $(-\delta,\delta)$.

15.17. Let $f(a,n)=(1+a)^n$, where a and n are positive.

a) For fixed a, $\lim_{n\to\infty}f(a,n)=\infty$. Powers of a number greater than 1 grow without bound. *For fixed n, $\lim_{a\to 0}f(a,n)=1$.* When n is fixed, we can apply the arithmetic properties of limits to obtain $\lim_{a\to 0}(1+a)^n=(1+\lim_{a\to 0})^n=1$.

b) If L is a real number with $L\ge 1$, then there exists $\langle a\rangle$ such that $a_n\to 0$ and $f(a_n,n)\to L$ as $n\to\infty$. We can obtain $(1+a_n)^n=L$ by setting $a_n=L^{1/n}-1$. To show that $\lim_{n\to\infty}a_n=0$, it suffices to show that $c^{1/n}\to 1$ when $c\ge 1$. This fact will be added to Chapter 14.

15.18. *Often discontinuous functions.*

a) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=0$ if $x\in\mathbb{Q}$ and $f(x)=1$ if $x\notin\mathbb{Q}$ is discontinuous at every point. For each $x\in\mathbb{R}$, there are rational numbers and irrational numbers arbitrarily close to x. Hence there are points where f is 0 and points where f is 1 arbitrarily close to x. To show that f fails the definition of continuity at x, we choose $\varepsilon=1/2$. Now for every $\delta>0$, there exists y with $|y-x|<\delta$ such that $|f(y)-f(x)|=1>\varepsilon$.

b) The function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=0$ if $x\in\mathbb{Q}$ and $g(x)=x$ if $x\notin\mathbb{Q}$ is continuous only at 0. We show first that g is continuous at $x=0$. Given $\varepsilon>0$, let $\delta=\varepsilon$. If $|y-0|<\delta$, i.e. $|y|<\delta$, then $|f(y)-f(0)|=|f(y)|\le|y|<\varepsilon$, whether y is rational or not.

We show that g is not continuous at $x\ne 0$ by showing that the definition of continuity fails. Let $\varepsilon=x$. If $x\in\mathbb{Q}$, then there are irrational points y arbitarily close to x such that $|f(y)-f(x)|\ge\varepsilon$. If $x\notin\mathbb{Q}$, then there are rational points y arbitrarily close to x such that $|f(y)-f(x)|\ge\varepsilon$.

15.19. *If $|f(x)-f(a)|\le c|x-a|$ for some positive constant c and all x, then f is continuous at a.*

Proof 1. For $\varepsilon>0$, let $\delta=\varepsilon/c$. If $|x-a|<\delta$, then $|f(x)-f(a)|=|f(x)|\le c|x|<c\delta\le\varepsilon$. Thus f satisfies the definition of continuity at a.

Proof 2. The function f is squeezed between the two continuous functions $f(a)+c(x-a)$ and $f(a)-c(x-a)$, which are equal at a. (They are continuous because each is a combination of continuous functions via binary operations and compositions, all of which preserve continuity.)

15.20. *Continuity of arithmetic combinations of functions.*

a) If f and g are continuous at a, then $f+g$, $f\cdot g$ and cf are continuous at a, and every polynomial is continuous. The hypothesis means $\lim_{x\to a}f(x)=f(a)$ and $\lim_{x\to a}g(x)=g(a)$. By the properties of limits, these imply $\lim_{x\to a}(f+g)(x)=\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=f(a)+g(a)=(f+g)(a)$. Hence $f+g$ is continuous at a. Since $\lim_{x\to a}f(x)g(x)=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$, the analogous computation implies $\lim_{x\to a}(f\cdot g)(x)=(f\cdot g)(a)$, so $f\cdot g$ is also continuous at a.

Since the function g defined by $g(x)=c$ for all $x\in\mathbb{R}$ is continuous, the result above about products implies that $c\cdot f$ is continuous when f is continuous. Finally, every polynomial is a finite sum of products of finitely many continuous functions. Using the statements above and induction on the degree, every polynomial is continuous.

b) the ratio of two polynomials is continuous at every point where the denominator is nonzero. Computations like those above show that the ratio of two continuous functions is continuous at every point where the denominator is nonzero.

15.21. *There exists $x\in[1,2]$ such that $x^5+2x+5=x^4+10$.* Let $f(x)=x^5-x^4+2x-5$ (the difference of the two sides). We have $f(1)=-3$ and $f(2)=15$. By the Intermediate Value Theorem, there exists $x\in[1,2]$ such that $f(x)=0$. For this x, the desried equality holds.

15.22. *a) If f and g are continuous on the closed interval $[a, b]$, and also $f(a) > g(a)$ and $f(b) < g(b)$, then there exists $c \in [a, b]$ such that $f(c) = g(c)$.* Let $h = f - g$; the difference of continuous functions is continuous. We have $h(a) > 0$ and $h(b) < 0$. By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $h(c) = 0$, and $h(c) = 0$ requires that $f(c) = g(c)$.

b) If $f(a) = (1/2)g(a)$ and $f(b) = 2g(b)$, then there need not exist $c \in [a, b]$ with $f(c) = g(c)$. If g is a linear function that is negative at a and positive at b, and f is the linear function with $f(a) = (1/2)g(a)$ and $f(b) = 2g(b)$, then $f(x) > g(x)$ for $a \le x \le b$. If f and g have this relation at endpoints, with $g(x) \ge 0$ for $x \in [a, b]$, then $f(a) < g(a)$ and $f(b) > g(b)$, and the argument of part (a) applies.

15.23. *For f and g continuous on $[a, b]$ with $f(a) = (1/2)g(a)$ and $f(b) = 2g(b)$, there need not exist $c \in [a, b]$ such that $f(c) = g(c)$.* Let $a = 0$ and $b = 3$. Let $f(x) = x - 1$ and $g(x) = x - 2$. Now $f(0) = -1 = (1/2)g(0)$ and $f(3) = 2 = 2g(3)$, but always f is one unit above g.

Such a c must exist if $g(x) \ge 0$ for $x \in [a, b]$. If $g(x) \ge 0$ for $x \in [a, b]$, then $f(a) < g(a)$ and $f(b) > g(b)$, and the Intermediate Value Theorem applies to $f - g$. (One can also apply it to f/g.)

15.24. *Every polynomial of odd degree has at least one real zero.* If p has odd degree, then $p(x)$ and $p(-x)$ have opposite signs when x is sufficiently large. By applying the Intermediate Value Theorem to the resulting interval $[-x, x]$, there exists y such that $p(y) = 0$.

15.25. *Given $\varepsilon > 0$, there exists $c > 0$ (depending only ε) such that $|xy| \le \varepsilon x^2 + cy^2$ for all $x, y \in \mathbb{R}$.* We seek $\varepsilon x^2 - xy + cy^2 \ge 0$, so we choose c so that the left side of this inequality can be expressed as a square. Since $(\sqrt{\varepsilon}x - y/2\sqrt{\varepsilon})^2 = \varepsilon x^2 - xy + y^2/(4\varepsilon)$, the desired inequality holds for all $x, y \in \mathbb{R}$ when we choose $c = 1/(4\varepsilon)$.

15.26. *A continuous function on $[a, b]$ has a lower bound.* Let f be a continuous function on $[a, b]$. If f has no lower bound, then for all n there exists $x_n \in [a, b]$ such that $f(x_n) < -n$. Since $a \le x_n \le b$ for all n, the sequence $\langle x \rangle$ is bounded, and hence it has a convergent subsequence. Let $\{x_{n_k} : k \ge 1\}$ be the terms of this sequence. We have $x_{n_k} \to L$ as $k \to \infty$, where $L \in [a, b]$. By the sequential definition of continuity, $f(x_{n_k}) \to f(L)$ as $k \to \infty$. Since it converges, the sequence of values $f(x_{n_k})$ is bounded. This contradicts $f(x_{n_k}) < -n_k$. Hence f must indeed have a lower bound.

15.27. *A function f is continuous if and only if $-f$ is continuous.* This follows immediately from Corollary 15.12 with g being the function that is -1 everywhere. Alternatively, note that $|f(x) - f(y)| = |-f(x) - [-f(y)]|$. Therefore, $|x - y| < \delta$ implying $|f(x) - f(y)| < \varepsilon$ is equivalent to $|x - y| < \delta$ implying $|-f(x) - [-f(y)]| < \varepsilon$.

If f is continuous on $[a, b]$ and $f(a) > y > f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$. Let $g = -f$. If $f(a) > y > f(b)$, then $g(a) < -y < g(b)$. Also g is continuous on $[a, b]$, by part (a). By the Intermediate Value Theorem, there is a point $c \in [a, b]$ such that $g(c) = -y$. Now $f(c) = -g(c) = y$, and c is the desired point.

15.28. Let P be the set of positive real numbers. Let $f : P \to P$ be continuous and injective.

a) The inverse of f, defined on the image of f, is continuous. An injective function is a bijection from its domain to its image, so there is an inverse function defined on the image. A continuous injective function is strictly monotone. (If $x_1 < x_2 < x_3$ and the function values are not in order, then we may assume by symmetry that $f(x_1)$ is between $f(x_2)$ and $f(x_3)$, and applying the Intermediate Value Theorem to the interval $[x_2, x_3]$ gives us $c \in [x_2, x_3]$ such that $f(c) = f(x_1)$, contradicting the injectivity of f.)

We now show that the inverse of a strictly monotone continuous function is continuous. To visualize this, think of the graph of a continuous monotone function. Distance from a point (x, y) on the graph to the point $(x, 0)$ is the value of $f(x)$. Distance from (x, y) on the graph to $(0, y)$ is the value of $f^{-1}(y)$. To prove that $f^{-1}(y)$ is continuous, consider a arbitrary b, ε. Let $a = f^{-1}(b)$. Since $f^{-1}(b) = a$, it suffices to choose δ small enough in terms of ε (and b) so that $|y - b| < \delta$ implies $|f^{-1}(y) - a| < \varepsilon$.

Consider the interval $(a-\varepsilon, a+\varepsilon)$. Because f is continuous and monotone, the values of f on this interval form the interval $(f(a - \varepsilon), f(a + \varepsilon))$ (if f is increasing) or $(f(a + \varepsilon), f(a - \varepsilon))$ (if f is decreasing). In particular, if $f(a - \varepsilon) < y < f(a + \varepsilon)$, then $a - \varepsilon < f^{-1}(y) < a + \varepsilon$. We choose $\delta = \min\{f(a+\varepsilon) - b, b - f(a-\varepsilon)\}$. This guarantees that $|y - b| < \delta$ implies $|f^{-1}(y) - a| < \varepsilon$. The proof when f is decreasing is similar.

b) If the sequence $\langle x \rangle$ satisfies $x_1 = c$ for some $c \in P$ and $x_{n+1} = f(\sum_{j=1}^{n} x_j)$ for $n \ge 1$, and $\langle x \rangle$ converges, then its limit is 0. If $x_{n+1} \to L$ and $L \ne 0$, we must have $\sum_{j=1}^{n} x_j \to f^{-1}(L)$, by the continuity of f^{-1}. If the series converges, the terms must converge to 0. We conclude that $L = 0$.

15.29. *Computation of $\lim_{n\to\infty} f_n(x)$, where $f_n(x) = (x^n + 1)^{1/n}$ for $x \ge 0$.* The graph of f_1 is the ray $\{(x, x + 1) : x \ge 0\}$; this is the ray with slope 1 rising from the point $(0, 1)$. The graph of f_2 starts at the same point and is always below f_1 but has slope increasing to 1 as $\lim_{x\to\infty}$.

For fixed x, the limit of $(x^n + 1)^{1/n}$ is computed in Exercise 14.27. This yields $g(x) = 1$ for $x \le 1$ and $g(x) = x$ for $x > 1$.

15.30. *To four decimal places, $\sqrt{10} = 3.16228$.* Since $x^2 - 10$ has value -1 at $x = 3$ and 6 at $x = 4$, the Intermediate Value Theorem guarantee a zero in the interval $(3, 4)$. Iteratively bisect the remaining interval and use the value at the midpoint to restrict the search to half of it. Let $f(x) = x^2 - 10$. We compute $f(3.5) = 2.25$, $f(3.25) = .5625$, $f(3.125) = -.234375$, $f(3.1875) = .160156$, $f(3.15625) = -.049449$, $f(3.171875) = .060791$, $f(3.1640625) = .011291$, $f(3.1601562) = -.013413$, $f(3.162109375) = -.001065$. One point of this exercise is that the convergence is rather slow; we have not yet reached four decimal places of accuracy. Newton's Method in Chapter 16 is faster.

To two decimal places (and more), the real solution of $x^7 - 5x^3 + 10 = 0$ *is* -1.64152. Using the same method, let $f(x) = x^7 - 5x^3 + 10$. We have $f(0) = 10$, $f(-2) = -78$, $f(-1) = 14$, $f(-1.5) \approx 9.8$, $f(-1.75) \approx -13$, $f(-1.625) \approx 10$, $f(-1.6875) \approx -5$, $f(-1.65625) \approx -1.5$, and so on.

15.31. *Even when all limits to be taken exist, it need not be true that* $\lim_{y\to 0} \lim_{x\to 0} f(x, y) = \lim_{x\to 0} \lim_{y\to 0} f(x, y)$. The limit when approaching the origin in $\mathbb{R}^2$ may depend on the direction of approach, just as when $g(x) = x/|x|$ the limit is $+1$ when approaching 0 from the positive side and -1 when approaching 0 from the negative side. There are many explicit counterexamples; we present one.

Let $f(x, y) = |x|^{|y|}$ if $x \neq 0$, and let $f(0, y) = 0$. Now $\lim_{y\to 0} f(x, y) = 1$ for every fixed $x \neq 0$, so $\lim_{x\to 0} \lim_{y\to 0} f(x, y) = 1$. On the other hand, $\lim_{x\to 0} f(x, y) = 0$ for every fixed $y \neq 0$, so $\lim_{y\to 0} \lim_{x\to 0} f(x, y) = 0$.

15.32. *Fuel along a circular track.* The car starts at some place, presumably at a gas container to have some fuel. We scale the problem so that the track is one mile long, the car travels one mile per gallon, and the total amount of fuel is one gallon. Graph a function that starts with $f(0)$ being the amount of fuel in the first container. As x increases, let $f(x)$ decline at rate 1 until the position of the next container is reached; the value of f then jumps up by the amount in that container. Continue this process.

The value of f may become negative before $x = 1$ is reached. Since the total amount of gas in containers is exactly enough, $f(1) = 0$. Although f is not continuous, if there are finitely many containers we can select a position where which f is minimal. Starting at this point merely shifts the graph. The new function is never negative, so starting here works.

15.33. *If* n *is a positive integer, and* f *is continuous on* $[0, 1]$ *with* $f(0) = f(1)$*, then the graph of* f *has a horizontal chord of length* $1/n$. Since $f(0) = f(1)$, we have $\sum_{i=1}^{n}(f(\frac{i}{n}) - f(\frac{i-1}{n})) = 0$. Since these numbers sum to 0, they cannot be all positive or all negative, and thus for some $k \le n-1$ we have $(f(\frac{k}{n}) - f(\frac{k-1}{n}))(f(\frac{k+1}{n}) - f(\frac{k}{n})) \le 0$. If the product is 0, then we have a horizontal chord of length $1/n$ starting at $x = (k-1)/n$ or $x = k/n$. Otherwise, the Intermediate Value Theorem yields a point c between $(k-1)/n$ and k/n such that $f(c) - f(c + 1/n) = 0$, and we have the desired chord starting at $x = c$.

If a real number α *between 0 and 1 is not the reciprocal of an integer, then there is a continuous function on* $[0, 1]$ *that vanishes at the endpoints and whose graph has no horizontal chord of length* α. Let n be the unique natural number such that $1/(n+1) < \alpha < 1/n$. Let $ai = i/n$ for $0 \le i \le n$. Let $bi = i/(n+1)$ for $0 \le i \le n+1$. Through $(a_0, 0), \dots, (a_n, 0)$ draw a family of parallel lines with positive slope. Through $(b_1, 0), \dots, (b_n, 0)$ draw a family of parallel lines with positive slope. Note that the horizontal coordinates of these points are in the order $a_0, b_1, a_1, \dots, b_n, a_n$. Taking the segment of each line that intersects the x-axis and lies between two lines of the other family thus yields the graph of a continuous function. Furthermore, all its chords have length at most $1/(n+1)$ (if confined to the space between two segments in the second family) or at least $1/n$.

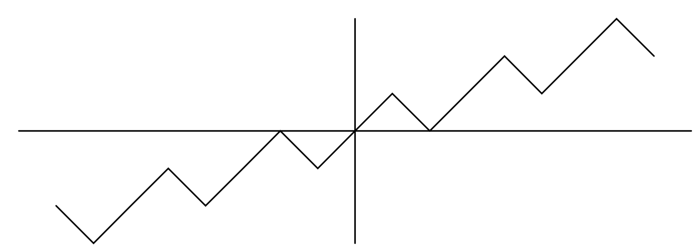
15.34. *Let* f *be continuous on an interval* I*. For each* $a \in I$ *and* $\varepsilon > 0$*, let* $m(a, \varepsilon) = \sup(\{\delta\colon |x - a| < \delta$ *implies* $|f(x) - f(a)| < \varepsilon\})$*. The function* f *is uniformly continuous on* I *if and only if* $\inf_a m(a, \varepsilon) > 0$ *whenever* $\varepsilon > 0$.

Necessity. Suppose that f is uniformly continuous, and consider $\varepsilon > 0$. Since f is uniformly continuous, there exists δ' such that $|x - a| < \delta'$ implies $|f(x) - f(a)| < \varepsilon$ for all $x, a \in I$. Thus for all $a \in I$, we have $\delta' \le m(a, \varepsilon)$. Therefore $\inf_a m(a, \varepsilon) \ge \delta' > 0$.

Sufficiency. Consider $\varepsilon > 0$. If $\inf_a m(a, \varepsilon) = \delta' > 0$, then $|x - a| < \delta'$ implies $|x - a| < m(a, \varepsilon)$, and hence $|f(x) - f(a)| < \varepsilon$. Hence choosing this δ' in terms of ε establishes that f is uniformly continuous on I.

15.35. *Continuous functions with constant multiplicity.*

a) A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ *such that every real number occurs as the image of exactly three numbers.* Let f be the functions that repeatedly takes "two steps up and one step down" as suggested by the graph below. To define f precisely, we first give its value at integers. It maps integers to integers, defined by the inverse image of $n \in \mathbb{Z}$ being $\{3n-2, 3n, 3n+2\}$. Extend f to all of $\mathbb{R}$ by letting f be linear between neighboring integers. Now f takes on the value x at three points in the interval $[3x - 2, 3x + 2]$ and nowhere else.



b) If $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous, and each* $z \in \mathbb{R}$ *is the image of exactly* k *numbers, then* k *is odd.* Let $x_1, \dots, x_k$ be the zeros of f. By multiplying

f by -1 if necessary and using the Intermediate Value Theorem and the hypothesis that the image is all of $\mathbb{R}$, we may assume that $f(x) > 0$ for $x > x_k$ and $f(x) < 0$ for $x < x_1$

We show next that there exists $\varepsilon > 0$ such that $f(x) = \varepsilon$ for only one x with $x > x_k$. To prove this, observe that $f(x) = 1$ for at most k such x; let x' be the largest of these. For $x > x'$, we have $f(x) > 1$. For $x_k < x < x'$, the function has finitely many local minima. Let c be the least of these minima. Since x_k is the last zero, $c > 0$. All values y with $0 < y < c$ occur as $f(x)$ for only one x larger than x_k. Let $\varepsilon = c/2$.

Because there are only finitely many extreme points for f in the interval $[x_1, x_k]$, we can choose ε' from an open interval about ε so that ε' does not occur as the value of f at a local extremum in $[x_1, x_k]$. Thus ε' occurs as the value of f an even number of times in each subinterval $[x_i, x_{i+1}]$. Since it also occurs once after x_k and never before x_1, ε' occurs an odd number of times. Hence k is odd.

16. DIFFERENTIATION

16.1. *For $x \neq 0$, $\lim_{h\to 0} \frac{1}{h}(\frac{1}{(x+h)^2} - \frac{1}{x^2}) = \frac{-2}{x^3}$.* The limit is $\lim_{h\to 0} \frac{1}{h}(f(x+h) - f(x))$, where $f(x) = 1/x^2$. This is the definition of the derivative, $f'(x)$. When $f(x) = x^{-2}$, we have $f'(x) = -2x^{-3}$.

16.2. *Chain rule for linear functions.* When f and g are linear functions, the chain rule says that the slope of the composition is the product of the slopes.

16.3. *Interpretation of interpolation.* When reading tables of values, interpolation uses a linear approximation to an underlying function between two given points.

16.4. *Rate of changing in temperature.* We assume that temperature is a continuously differentiable function of time. Near the time when it reaches its high for the day, temperature is changing slowly, because the derivative reaches zero at the extreme, and we have assumed that the derivative is continuous.

16.5. *A function f such that f^2 is differentiable at every point while f is differentiable at no point.* Let $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = -1$ for $x \notin \mathbb{Q}$.

16.6. *There is a function f such that $f(x+h) = f(x) + h$ for all $x, h \in \mathbb{R}$—TRUE.* The requirement $f(x+h) = f(x) + h$ yields $\frac{1}{h}(f(x+h) - f(x)) = 1$ for all x, and hence $f'(x)$ exists and always equals 1. This suggests letting $f(x) = x + c$, where c is constant, and indeed every such function satisfies the desired property.

16.7. *There is a function f such that $f(x+h) = f(x) + h^2$ for all $x, h \in \mathbb{R}$—FALSE.* If so, then $\frac{1}{h}(f(x+h) - f(x)) = h$ for all x and h. Letting $h \to 0$ yields that f is differentiable and $f'(x) = 0$ for all x. Thus such a function f is constant, but if f is constant then $f(x+h) - f(x)$ is 0, not h^2.

16.8. *There is a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$—FALSE.* The hypotheses force $f(x) = x + c$ for $x > 0$ and $f(x) = -x + d$ for $x < 0$. If f is differentiable at 0, then also f is continuous there, which requires $c = d = f(0)$. When computing $\frac{1}{h}(f(0+h) - f(0))$, we thus obtain 1 for $h > 0$ and -1 for $h < 0$. Hence the limit of the difference quotient does not exist when $x = 0$.

16.9. *If both $f + g$ and fg are differentiable, then f and g are differentiable—FALSE.* Let $f(x) = 1$ and $g(x) = -1$ for $x \in \mathbb{Q}$, and let $f(x) = -1$ and $g(x) = 1$ for $x \notin \mathbb{Q}$. Now $f + g$ is identically 0 and fg is identically 1. Both are differentiable, but f and g are not even continuous.

16.10. *Differentiation of $f(x) = \prod_{j=1}^{n}(x + a_j)$, where $a_1, \ldots, a_n \in \mathbb{R}$.* The product rule and induction on n yield $f'(x) = \sum_{k=1}^{n} \prod_{j\in[n]-\{k\}}(x + a_j)$.

16.11. *The product rule for differentiation, via difference quotients.* We assume that f and g are differentiable at x. The difference quotient for fg at x is

$$\begin{aligned}
\frac{(fg)(x+h) - (fg)(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x_h) - f(x+h)g(x) + f(x+h)g(x) - f(x)}{h} \\
&= f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}
\end{aligned}$$

Letting $h \to 0$ yields $f(x)g'(x) + g(x)f'(x)$, by the properties of limits.

16.12. *A flawed argument to compute $\frac{d}{dx}\frac{1}{g(x)}$.* "Given that g is differentiable and nonzero at x, applying the product rule to $1 = g(x) \cdot \frac{1}{g(x)}$ yields $0 = \frac{g'(x)}{g(x)} + g(x)\frac{d}{dx}\frac{1}{g(x)}$. Solving for $\frac{d}{dx}\frac{1}{g(x)}$ yields $\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{(g(x))^2}$." The argument obtains the only possible value for $\frac{d}{dx}\frac{1}{g(x)}$, but it does not show that the derivative exists.

16.13. *Quotient rule for differentiation.*

Proof 1. If f and g are differentiable at $x = a$ and $g(a) \neq 0$, then the function ϕ defined by $\phi(x) = f(x)/g(x)$ is differentiable at a, with $\phi'(a) = [g(a)f'(a) - g'(a)f(a)]/[g(a)]^2$. To prove this, we apply the product rule to

$f(x)g(x)^{-1}$ after finding $h'(x)$, where $h = g(x)^{-1}$. Differentiating the function on both sides of $1 = g(x)g(x)^{-1}$, we have $0 = g'(x)h(x)+g(x)h'(x)$, which is $h'(x) = -g'(x)/[g(x)]^2$. Now $\phi'(a) = f'(a)/g(a) + f(a)[-g'(a)/[g(a)]^2]$, which equals the formula claimed.

Proof 2. Let $M = g(a)g(a+h)$. We compute

$$\begin{aligned}\left[\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}\right]\frac{1}{h} &= [f(a+h)g(a) - f(a)g(a+h)]\frac{1}{Mh}\\ &= [f(a+h)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(a+h)]\frac{1}{Mh}\\ &= g(a)[f(a+h) - f(a)]/Mh - f(a)[g(a+h) - g(a)]\frac{1}{Mh}.\end{aligned}$$

Taking the limit as $h \to 0$, the continuity of g at a (which follows from differentiability of g at a) implies $M \to [g(a)]^2$, and we obtain $[g(a)f'(a) - f(a)g'(a)]/[g(a)]^2$.

16.14. *When $f(x) = x^{1/3}$, $f'(x) = (1/3)x^{-2/3}$.* Using the factorization of the difference of cubes, we have

$$h = (x+h) - x = ((x+h)^{1/3} - x^{1/3})((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}).$$

Thus the difference quotient $[f(x+h) - f(x)]/h$ equals $1/((x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3})$. As $h \to 0$, this approaches $1/(3x^{2/3})$.

16.15. *Correction of an inductive argument for $(d/dx)x^n = nx^{n-1}$.* The failed argument: "Basis step ($n = 0$): $\lim_{h\to 0}(1-1)/h = 0$. Inductive step ($n > 0$): Using the induction hypothesis for $n-1$ and the product rule for differentiation, we compute $(d/dx)x^n = (d/dx)xx^{n-1} = x \cdot (n-1)x^{n-2} + 1 \cdot x^{n-1} = nx^{n-1}$."

The induction step uses the result $(d/dx)(x) = 1$ and thus is not valid when $n = 1$. To correct the error, prove the case $n = 0$ separately, prove the case $n = 1$ as the basis step, and then keep the induction step as is.

16.16. (•) Let $r = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We define x^r to be $(x^p)^{1/q}$. Determine $f'(x)$, where $f(x) = x^r$. (Hint: We have determined this already for $r \in \mathbb{N}$. Derive the formula first for $p = 1$ and then for $r \in \mathbb{Q}$. Comment: When $r \in \mathbb{R}$, the same formula holds for f'. The proof uses properties of the exponential function and appears in Exercise 17.16.)

16.17. *If e_1 and e_2 are error functions and $e_1(h) \le e(h) \le e_2(h)$ for all h in a neighborhood of 0, then e is an error function.* We have $\frac{e_1(h)}{h} \le \frac{e(h)}{h} \le \frac{e_2(h)}{h}$. Since e_1 and e_2 are error functions, the upper and lower bounds have limit 0 as $h \to 0$. By the Squeeze Theorem, also $\lim_{h\to 0}\frac{e(h)}{h} = 0$, and hence e is an error function.

16.18. *If $f(x) = x + x^2$ when x is rational, and $f(x) = x$ when x is irrational, then f is differentiable at $x = 0$.* In particular, we prove that $f'(0) = 1$. Since $f(0) = 0$, the difference quotient is $f(h)/h$. To prove that $f(h)/h \to 1$, we prove that $|\frac{f(h)}{h} - 1| \le |h|$ for $h \ne 0$. When $h \in \mathbb{Q}$, we have $|\frac{f(h)}{h} - 1| = |\frac{(h+h^2)}{h} - 1| = |h|$. When $h \notin \mathbb{Q}$, we have $|\frac{f(h)}{h} - 1| = |\frac{h}{h} - 1| = 0$.

16.19. *Differentiability at a point.*

a) If $|f(x)| \le x^2 + x^4$ for all x, then f is differentiable at 0. We prove that $f'(0) = 0$. It suffices to show that the absolute value of the difference quotient is bounded by a function of h having limit 0 as $h \to 0$. The hypothesis yields $f(0) = 0$. Thus

$$\left|\frac{f(0+h) - f(0)}{h}\right| = \left|\frac{f(h)}{h}\right| \le \frac{|h|^2 + |h|^4}{|h|} = |h| + |h|^3 \to 0.$$

b) If $|f(x)| \le g(x)$, where $g(x) \ge 0$ for all x and $g'(0) = g(0) = 0$, then f is differentiable at 0. We prove that $f'(0) = 0$. Again the hypothesis yields $f(0) = 0$. Thus

$$\left|\frac{f(0+h) - f(0)}{h}\right| = \left|\frac{f(h)}{h}\right| \le \left|\frac{g(h)}{h}\right|.$$

The last quantity has limit 0, because $g(0) = g'(0) = 0$.

c) If g is a bounded function and $f(x) = (x-a)^2 g(x)$ for all x, then $f'(a)$ exists. We prove that $f'(a) = 0$. Since g is bounded, there exists M such that $|g(x)| \le M$ for all x. Also, the hypothesis yields $f(a) = 0$. Now

$$\left|\frac{f(a+h) - f(a)}{h}\right| = \left|\frac{h^2 g(a+h)}{h}\right| = |h|\,|g(a+h)| \le M\,|h|.$$

Since $M\,|h| \to 0$, the result follows.

16.20. *A condition for derivative 0.* If $|f(x) - f(y)| \le |g(x) - g(y)|$ for all $x, y \in \mathbb{R}$, and g is differentiable at a with $g'(a) = 0$, then f is differentiable at a and $f'(a) = 0$. To prove this, let $x = a+h$ and $y = a$ and apply the condition. Since $g'(a) = 0$, for any $\varepsilon > 0$ we can choose δ such that $|[g(a+h) - g(h)]/h| < \varepsilon$ for $|h| < \delta$. Hence for $|h| < \delta$ we have $[f(a+h) - f(a)]/h \le |[g(a+h) - g(a)]/h| < \varepsilon$ and $[f(a+h) - f(a)]/h \ge -|[g(a+h) - g(a)]/h| > -\varepsilon$. Since this is true for all $\varepsilon > 0$, we have $\lim_{h\to 0}[f(a+h) - f(a)]/h = 0$.

16.21. *Properties of $g(x) = f(x)/x$ (for $x \neq 0$) when $f(0) = 0$ and f is differentiable.*

a) If $g(0) = f'(0)$, then g is continuous at 0. Continuity at 0 requires $g(0) = \lim_{h\to 0} g(h) = \lim_{h\to 0} f(h)/h = f'(0)$.

b) If $g(0)$ is defined so that g is continuous at 0, then g need not be differentiable at 0. For example, let $f(x) = x^{3/2}$. Now f is differentiable at 0, with $f'(0) = 0$. However, $g(x) = f(x)/x = x^{1/2}$ is not differentiable at 0.

16.22. *The shrinking ball.* The relationship between volume v and radius r is $v = \frac{4}{3}\pi r^3$. Differentiating as a function of time yields $\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}$. If air escapes at the rate of 36 cubic inches per second, then when the radius is 6 inches we obtain $-36 = 4\pi 36 \frac{dr}{dt}$. Hence at this time the reate of change in the radius is $-1/(4\pi)$.

16.23. *The real number most exceeding its square is* $1/2$. We maximize $x - x^2$. Without calculus, this follows from $x - x^2 = \frac{1}{4} - (x - \frac{1}{2})^2 \leq \frac{1}{4}$.

16.24. *If $f(x) = ax^2 + bx + c$ with $a > 0$, then the minimum of f on $\mathbb{R}$ is $\frac{c-b^2}{4a}$, which is positive if and only if $b^2 < 4ac$.* Since f increases without bound as $|x|$ grows, the minimum occurs when $f'(x) = 0$. Since $f'(x) = 2ax + b$, which is 0 when $x = -b/2a$, we have $f(x) \geq f(-b/2a) = \frac{c-b^2}{4a}$.

16.25. *An extremal problem.* We assume that the liquid has no cost, so we are finding the extreme values of the total cash inflow. When the price is x, the inflow is $f(x) = xg(x) + 50\sqrt{g(x)}$, where $g(x) = 1000/(5 + x)$. We consider nonnegative values of x.

Since f is continuous for nonnegative x and differentiable for positive x, at a local extremum we must have $f'(x) = 0$. Using the product rule, chain rule, etc., we have

$$f'(x) = \frac{1000}{5+x} + \frac{-1000x}{(5+x)^2} + 50(-1/2)(5+x)^{-3/2}.$$

We set this to 0 and solve for x, obtaining $x = 35$. The candidates are $x = 35$ and the endpoints, at $x = 0$ and arbitrarily large x. We have $f(0) = 500\sqrt{2}$, $f(35) = 1150$, and $\lim_{x\to\infty} f(x) = 1000$. So, the maximum is at $x = 35$ and the minimum is at $x = 0$.

16.26. *If $m_1, \ldots, m_k$ are nonnegative real numbers with sum n, then $\sum_{i<j} m_i m_j \leq (1 - \frac{1}{k})\frac{n^2}{2}$, with equality only when $m_1 = \cdots = m_k$.*

a) Proof by calculus and induction. Basis step: $k = 1$. The sum is empty and both sides of the equality are 0. Always equality holds and the claimed condition for equality is vacuously satisfied.

Induction step: $k > 1$. Let $f_k(n)$ denote the maximum for k numbers summing to n. When $m_k = x$, the sum equals $(\sum_{i<j<k} m_i m_j) + x(\sum_{i<k} m_i)$. With x fixed, the maximum is $\max f_{k-1}(n - x) + x(n - x)$. By the induction hypothesis, this equals $(1 - \frac{1}{k-1})(n - x)^2/2 + x(n - x)$, with equality only when $m_i = (n - x)/(k - 1)$ for $i < k$.

We choose x to maximize this function. Differentiating with respect to x yields $-(1 - \frac{1}{k-1})(n - x) + (n - x) - x$. Setting this to 0 yields $x = n/k$. Thus the candidates for the extreme are at $x \in \{0, n/k, n\}$. The corresponding values for the function are $\frac{k-2}{k-1}\frac{n^2}{2}$, $\frac{k-1}{k}\frac{n^2}{2}$, and 0. Hence the function is maximized only by setting $x = n/k$, and also $m_i = (n - n/k)/(k - 1) = n/k$ for all other summands. The value of the maximum is $f_k(n) = (1 - \frac{1}{k})\frac{n^2}{2}$.

b) Combinatorial proof for optimization over integers; the optimum occurs precisely when the numbers in $\{m_1, \ldots, m_k\}$ differ by at most one. The sum $\sum_{i<j} m_i m_j$ counts the edges in a simple graph where the vertices are partitioned into independent sets of sizes $m_1, \ldots, m_k$, and two vertices are adjacent if and only if they belong to distinct sets.

If the sizes of two of the sets differ by more than one, then we claim that moving one vertex from the larger set (size m) to the smaller set (size m') increases the number of edges. All edges are unchanged except those involving the moved vertex. It loses m' neighbors (those in the smaller set), and it gains $m - 1$ neighbors (the other vertices in the larger set). Since $m > m' + 1$, the net gain $m - 1 - m'$ is positive.

With this claim, the function is maximized only when the values $m_1, \ldots, m_k$ differ by at most one, which means that all equal $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

16.27. *Two differentiable functions on an interval (a, b) have the same derivative if and only if they differ by a constant.* The difference h of the two functions is differentiable, and its derivative is the difference of their derivatives. The two functions thus have the same derivative if and only if h' is identically 0, which is true if and only if h is constant, which is true if and only if the functions differ by a constant.

16.28. *Mean Value Theorem from Cauchy Mean Value Theorem.* To obtain the Mean Value Theorem for a particular f, a, b, we let $g(x) = x$ and use the Cauchy Mean Value Theorem for this f, g, a, b.

16.29. *If $f(x) = x^3$, $g(x) = x^2$, $a = 0$, and $b = 1$, then the value c guaranteed by the Cauchy Mean Value Theorem is* $2/3$. Since $f(b) = g(b) = 1$ and $f(a) = g(a) = 0$, the Cauchy Mean Value Theorem yields the equation $g'(c) = f'(c)$, or $2c = 3c^2$. Since $c \neq 0$, we obtain $c = 2/3$.

16.30. *If f is differentiable on $[a, b]$ and $f'(a) < y < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = y$.* For fixed y, the function g_y defined by $g_y(x) = f(x) - yx$ is also differentiable. Hence g_y is continuous on $[a, b]$ and attains a maximum and a minimum on $[a, b]$. Since g_y is differentiable, the derivative is 0 at a local extremum. Since $g_y'(x) = f'(x) - y$, attaining $g_y'(c) = 0$ yields $f'(c) = y$, as desired. (Comment: This is the Intermediate

Value Property for f'. It does not require that f' be continuous.) (Note: One must also exclude the case that g_y has no local minimum.)

16.31. *If f is differentiable and $f'(x) < 1$ for all x, then f has at most one fixed point.* We use contradiction. If $f(a) = a$ and $f(b) = b$, then $\frac{f(b)-f(a)}{b-a} = 1$. By the Mean Value Theorem, there exists c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 1$, which contradicts the hypothesis that $f'(c) < 1$.

16.32. *If f is differentiable, then f' is nonnegative everywhere if and only if f is nondecreasing.* If f is nondecreasing, then the difference quotient is always nonnegative, which means that its limit must be nonnegative.

If f is not nondecreasing, then there are numbers $a, b \in \mathbb{R}$ with $a < b$ and $f(a) > f(b)$. By the Mean Value Theorem, there is a number $c \in [a, b]$ such that $f'(c) = [f(b) - f(a)]/(b - a) < 0$.

16.33. (•) Let f be differentiable, with $f'(0) > 0$. Suppose also that f is not monotone in any neighborhood of 0. Explain why f' must be discontinuous at 0. Construct an example of such a function f. (Hint: Modify Example 16.19).

16.34. *If f is differentiable and f, f' are positive on $\mathbb{R}$, then the function $g = f/(1 + f)$ is bounded and increasing.* Since f never equals -1, g is defined and differentiable wherever f is. Using the quotient rule,

$$g' = \left(\frac{f}{1+f}\right)' = \frac{(1+f)f' - f \cdot f'}{(1+f)^2} = \frac{f'}{(1+f)^2}.$$

Since $f' > 0$, we conclude that $g' > 0$. Thus g is increasing. Also, $0 < f(x)/(1 + f(x)) < 1$ for all x, so g is bounded.

16.35. *If $f(a) = f(b) = 0$, and f is differentiable on $[a, b]$, and f' is nonnegative on $[a, b]$, then f is identically 0 on $[a, b]$.* Since f' is nonnegative, Exercise 16.32 implies that f is nondecreasing. Since also $f(a) = f(b)$, we conclude that f must be constant on $[a, b]$.

16.36. *A monotone differentiable function f on an interval S has an inverse, with $\frac{d(f^{-1}(y))}{dy} = \frac{1}{f'(f^{-1}(y))}$.* If two points x and y have the same image, then f is not monotone, hence f is an injection to its image T and therefore has an inverse function defined on T.

To compute the derivative, we apply the chain rule to differentiate the identity $(f \circ f^{-1})(y) = y$. We obtain $f'(f^{-1}(y))\frac{d}{dy}f^{-1}(y) = 1$, from which the claimed formula follow immediately.

16.37. *Behavior of an operator on differentiable functions.* The image of the function f under the operator A is the function A_f whose value at x is $\lim_{t\to 1} \frac{f(tx)-f(x)}{tf(x)-f(x)}$. (If $f(x) = 0$, then A_f is not defined at x.)

a) If f is continuously differentiable, then $A_f(x) = xf'(x)/f(x)$. We apply l'Hôpital's Rule, differentiating the numerator and denominator with respect to t and then taking the limit of the ratio as $t \to 1$.

b) Instances of A_f. When $f(x) = x^n$, we have $A_f(x) = xnx^{n-1}/x^n = n$. When $f(x) = e^x$, we have $A_f(x) = xe^x/e^x = x$.

c) The formula of part (a) holds even when f' is not continuous. The computation using l'Hôpital's Rule is not valid. If $x = 0$, then $A_f(x) = 0$ unless $f(0) = 0$, so we may assume that $x \neq 0$. Let $t = 1 + h$, and let $u = hx$. We obtain

$$\begin{aligned}\lim_{t\to 1}\frac{f(tx)-f(x)}{tf(x)-f(x)} &= \lim_{h\to 0}\frac{f(x+hx)-f(x)}{hf(x)} \\ &= \lim_{u\to 0}\frac{f(x+u)-f(x)}{u[f(x)/x]} = \frac{f'(x)}{f(x)/x} = \frac{xf'(x)}{f(x)}\end{aligned}$$

16.38. *l'Hôpital's Rule, weak form: If f and g are differentiable in a neighborhood of a, and $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.* Using the definition of the derivative as a linear approximation, we have $f(x) = f(a) + f'(a)(x - a) + e_1(x)$ and $g(x) = g(a) + g'(a)(x - a) + e_2(x)$, where $\lim_{x\to a} \frac{e_i(x)}{x-a} = 0$. Since $f(a) = g(a) = 0$, we have

$$\frac{f(x)}{g(x)} = \frac{f'(a)(x-a)+e_1(x)}{g'(x)(x-a)+e_2(x)} = \frac{f'(a)+e_1(x)/(x-a)}{g'(a)+e_2(x)/(x-a)}$$

Since e_1 and e_2 are error functions at a, the limit as $x \to a$ is $f'(a)/g'(a)$.

16.39. *If f and g are differentiable, $\lim f(x) = \infty$, $\lim g(x) = \infty$, $\lim f(x)/g(x) = L \neq 0$, and $\lim f'(x)/g'(x) = M$ (with all limits as $x \to a$), then $L = M$.* Since $\lim f(x)$ and $\lim g(x)$ are infinite, we have $\lim \frac{1}{f(x)} = \lim \frac{1}{g(x)} = 0$. Thus, although we are given $\lim f(x)/g(x) = L$, we can rewrite this and compute it by applying l'Hôpital's Rule:

$$\begin{aligned}\lim \frac{1/g(x)}{1/f(x)} &= \lim \frac{-g'(x)/(g(x))^2}{-f'(x)} f(x))^2 \\ &= \lim\left(\frac{g'(x)}{f'(x)} \cdot \frac{f^2(x)}{g^2(x)}\right) = \frac{1}{M}L^2\end{aligned}$$

We thus have $L = L^2/M$, we requires $L = 0$ or $L = M$. Since we are given $L \neq 0$, we obtain $L = M$.

16.40. (+) Prove Theorem 16.42, using the Cauchy Mean Value Theorem.

16.41. (•) The *first forward difference* of a function $f\colon \mathbb{R} \to \mathbb{R}$ is the function Δf defined by $\Delta f(x) = f(x + 1) - f(x)$. The *kth forward difference* of f is defined by $\Delta^k f(x) = \Delta^{k-1} f(x + 1) - \Delta^{k-1} f(x)$.

a) Prove that $\Delta^k f(x) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} f(x+j)$.

b) Prove that $f^{(k)}(x) = \lim_{h\to 0} \frac{1}{h} \sum_{j=0}^{k}(-1)^j \binom{k}{j} f(x+jh)]$ when the limit exists.

16.42. *If f is smooth, then f is a polynomial of degree at most k if and only if $f^{(k+1)}(x) = 0$ for all x.* We use induction on k. For $k = 0$, the derivative f' is identically 0 if and only if f is constant, that is, if and only if f is a polynomial of degree 0.

For $k > 0$, we apply the induction hypothesis to f'. We conclude that $f^{(k+1)}(x) = 0$ if and only if f' is a polynomial of degree $k-1$. Thus it suffices to show that f' is a polynomial of degree $k-1$ if and only if f is a polynomial of degree k. Certainly the latter implies the former. Conversely, from a polynomial f' of degree $k-1$ we can write down, term by term, a polynomial g of degree k such that $g' = f'$. By Exercise 16.27, two functions have the same derivative if and only if they differ by a constant. Hence g and f differ by a constant, which makes f also a polynomial of degree k.

16.43. *If f is smooth, $f(0) = 0$, f has a local minimum at 0, and $f^{(k)}(0) \neq 0$ for some natural number k, then the smallest such k is even.* Suppose that $f^{(k)}(0) \neq 0$, but $f^{(j)}(0) = 0$ for $0 \le j \le k-1$. We first claim that $f(x) = x^k[f^{(k)}(0)/k! + A(x)]$, where $\lim_{x\to 0} A(x) = 0$. To prove this, let $p_n(x) = \sum_{j=0}^{n} f^{(j)}(0)x^j/j!$. We claim that p_n approximates f so well near 0 that the difference tends to 0 faster than x^n; that is $\lim_{x\to 0} \frac{f(x)-p_n(x)}{x^n} = 0$.

Since f is smooth, we have $\lim_{x\to 0} f^{(j)}(x) = f^{(j)}(0)$ for all j. For $j \le n$, we also have $\lim_{x\to 0} p_n^{(j)}(x) = f^{(j)}(0)$. Hence n successive applications of l'Hôpital's Rule yields

$$\lim_{x\to 0} \frac{f(x) - p_n(x)}{x^n} = \frac{f^{(n)}(x) - p_n^{(n)}(x)}{n!} = \frac{1}{n!}(f^{(n)}(0) - f^{(n)}(0) = 0.$$

To obtain the desired function A, let $A(x) = (f(x) - p_k(x))/x^k$ for $x \neq 0$, and let $A(0) = 0$. We have shown that $\lim_{x\to 0} A(x) = 0$. By the hypothesis, we have $p_k(x) = f^{(k)}(0)x^k/k!$, so $f(x)$ has the desired form.

Since $A(x) \to 0$, we may choose δ so that $|x| < \delta$ implies $|A(x)| < \left|f^{(k)}(0)/k!\right|$. This means that $f(x)/x^k$ and $f^{(k)}(0)$ have the same sign for $|x| < \delta$. Since f has a local minimum at 0, we conclude that x^k has only one sign when $|x| < \delta$, and hence k is even.

A smooth function f such that $f(x) = 0$ if and only if $x = 0$, and $f^{(j)}(0) = 0$ for all $j \in \mathbb{N}$. Such a function is defined by $f(0) = 0$ and $f(x) = e^{-1/x^2}$ for $x \neq 0$.

Consider $h(y) = p(y)/e^y$, where p is a polynomial in y of degree k. As $y \to \infty$, we have $p(y) \to \infty$ and $e^y \to \infty$. Hence the second form of l'Hôpital's Rule (Theorem 16.42), applied k times, yields $h(y) \to 0$. Substituting $x = 1/y$ yields $\lim_{x\to 0} p(1/x)e^{-1/x} = 0$.

Now, let $g(x) = 0$ for $x \le 0$ and $g(x) = e^{-1/x}$ for $x > 0$. By the remark above, $\lim_{x\to 0} g(x) = 0$, so g is continuous at 0. By induction on j, we show that $g^{(j)}(0)$ exists and is 0; this is true for $j = 0$.

For any polynomial p, the derivative of $p(1/x)e^{-1/x}$ for $x > 0$ is $[(1/x)^2 p(1/x) + p'(1/x)]e^{-1/x}$; the expression in brackets is another polynomial in $1/x$. Hence the derivative tends to 0 as $x \to 0$. By induction on j, the jth derivative of g has this form for $x > 0$.

At $x = 0$, applying the induction hypothesis to the difference quotient yields

$$g^{(j+1)}(0) = \lim_{x\to 0} \frac{g^{(j)}(x) - g^{(j)}(0)}{x - 0} = \lim_{x\to 0} \frac{1}{x} g^{(j)}(x) = 0.$$

Finally, consider f. We have $f(x) = g(q(x))$, where $q(x) = x^2$.

16.44. (•) Suppose that f and g are smooth. Compute the kth derivative of $f \circ g$, for $1 \le k \le 5$. Describe the form of the expression for general k. (Comment: The sum of the coefficients of terms where f is differentiated j times is known as the *Stirling number* $S(k, j)$. It equals the number of ways to partition a set of k elements into j nonempty subsets.)

16.45. (–) Using an initial guess of 1 for the solution, apply Newton's Method to seek a solution to the equation $x^5 = 33$ and compute the first four iterations. Repeat this with an initial guess of 2. (Use a calculator.)

16.46. *A quadratic function f for which the recurrence generated by Newton's Method is $x_{n+1} = \frac{1}{2}(x_n - 1/x_n)$ is given by $f(x) = x^2 + 1$.* The recurrence is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Since $\frac{f(x)}{f'(x)} = \frac{x^2+1}{2x}$, the formula is as desired.

The recurrence can only converge to a zero of f. Since f has no zero, the recurrence cannot converge.

16.47. (•) Find a differentiable function f and a sequence $\langle x\rangle$ such that $x_n \to 0$, $f'(x_n) \to \infty$, and $f(x_n) = 1$ for every n. Determine $\lim[x_n - f(x_n)/f'(x_n)]$. What does this exercise say about Proposition 16.44?

16.48. (!) Given a differentiable function f, let $g(x) = x - \frac{f(x)}{f'(x)}$ when $f'(x) \neq 0$. The function g is the function that generates x_{n+1} from x_n in Newton's Method.

a) Verify that $g(x) = x$ if and only if $f(x) = 0$.

b) When $f(x) = x^2 - 2$, verify that $g(x) - \sqrt{2} = \frac{1}{2x}(x - \sqrt{2})^2$.

c) Use (b) to show that when Newton's Method is applied to $x^2 - 2$ with $x_0 = 1$, the value of x_5 is within 2^{-31} of $\sqrt{2}$.

d) What can be said about Newton's Method in general when a is a zero of f and $|g(x) - a| \le c\,|x - a|^2$ for some constant c and for x near a?

16.49. *Convex functions.* The convexity inequality is $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}$ and all $t \in [0, 1]$.

If f and g are convex, then $f + g$ is convex. Evaluating $f + g$ at $tx + (1-t)y$, the value is $f(tx + (1-t)y) + g(tx + (1-t)y)$. Because each is convex, this is less than or equal to $tf(x) + (1-t)f(y) + tg(x) + (1-t)g(y)$, which equals $t(f+g)(x) + (1-t)(f+g)(y)$.

If f and g are convex, then fg need not be convex. Consider the example $f(x) = x$ and $g(x) = 1 - x$. Each of these is convex; in fact, equality always holds in the convexity inequality for each of them. However, $x(1-x)$ is not convex, because this function is 0 at $x = 0$ and $x = 1$ and positive in $(0, 1)$, but the convexity inequality would require it to be non-positive for any point between 0 and 1.

If f is convex and $c \in \mathbb{R}$, then cf is convex if and only if $c \ge 0$ or equality always holds in the convexity inequality. $(cf)(tx + (1-t)y) = c\cdot[f(tx+(1-t)y)] \le c\cdot[tf(x)+(1-t)f(y)] = t(cf)(x)+(1-t)(cf)(y)$, where the inequality holds if $c \ge 0$. If $c < 0$, then the inequality is reversed, which means that if there is ever a strict inequality in the convexity inequality for f, then cf is not convex if $c < 0$.

16.50. *If f is convex on the interval $[a, b]$, then the maximum of f on $[a, b]$ is $f(a)$ or $f(b)$.* If not, then there is some $x \in (a, b)$ such that $f(x) > f(a)$ and $f(x) > f(b)$. Since $a < x < b$, there exist t such that $0 < t < 1$ and $x = (1-t)a + tb$. Since f is convex, $f(x) \le (1-t)f(a) + tf(b)$. However, the right side is less than $(1-t)f(x) + tf(x)$, which yields $f(x) < f(x)$. The contradiction implies that there is no such x.

16.51. *If f is twice differentiable and f'' is nonnegative everywhere, then $f((a+b)/2) \le (f(a) + f(b))/2$, achieved when f is linear between a and b.* Since f'' is nonnegative everywhere, f is convex, from which the claimed inequality follows by the definition of convex function.

16.52. *The only polynomials of odd degree that are convex on $\mathbb{R}$ are the polynomials of degree 1.* Since polynomials are smooth, polynomials are twice differentiable. Hence convex polynomials have nonnegative second derivatives. The second derivative of a polynomial f of odd degree is also a polynomial of odd degree if f has degree greater than 1. However, polynomials of odd degree are unbounded above and below and cannot be nonnegative everywhere. Hence this occurs only when f has degree 1.

16.53. *The fourth degree polynomials that are convex on $\mathbb{R}$.* It is necessary and sufficient that the second derivative be everywhere nonnegative. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. We have $f''(x) = 12ax^2 + 6bx + 2c$. The quadratic formula yields the possible zeros of f'' as $(-3b \pm \sqrt{9b^2 - 24ac})/12a$. If $9b^2 - 24ac < 0$, then f'' never changes sign,

and in this case f is convex if and only if also $c \ge 0$, which makes f'' always nonnegative. If $9b^2 - 24ac = 0$, then f'' is 0 at one point, and it will be always nonnegative if and only if $a > 0$. If $9b^2 - 24ac > 0$, then f'' takes on values of both signs, and f cannot be convex.

16.54. (•) Let Y be a random variable that takes only values in $\{y_1, \dots, y_n\}$, with corresponding probabilities $p_1, \dots, p_n$. Suppose that $-1 \le y_i \le 1$ for all i. Suppose also that the expectation of Y is 0 and that f is convex. Prove that the expectation of $f(Y)$ is at most $[f(1) + f(-1)]/2$.

16.55. *A continuous midpoint-convex function is convex.* We assume that f satisfies the convexity inequality when $t = 1/2$ for all $x, y \in \mathbb{R}$, and we must show that it satisfies the inequality for all $t \in [0, 1]$. We first prove it for every fraction t with denominator 2^n, by induction on n. The basis step ($n = 1$) is the hypothesis. For the induction step ($n > 1$), suppose that the inequality holds for fractions with denominator 2^{n-1}. Let $z = tx + (1-t)y$. We want to compare $f(z)$ with $tf(x) + (1-t)f(y)$. If $t \le 1/2$, then let $w = x/2 + y/2$; we have $z = (2t)w + (1-2t)y$. Since $2t$ is a fraction with denominator 2^{n-1}, we can apply the induction hypothesis to conclude that

$$\begin{aligned} f(z) &\le 2tf(w) + (1-2t)f(y) = 2tf(x/2 + y/2) + (1-2t)f(y) \\ &\le 2tf(x)/2 + 2tf(y)/2 + (1-2t)f(y) = tf(x) + (1-t)f(y). \end{aligned}$$

When $t > 1/2$, apply the same argument with x and y interchanged and $t' = 1 - t$. In other words, $z = t'y + (1-t')x$ and $z = (2t')w + (1-2t')x$.

We use the continuity of f to extend the claim to all $t \in [0, 1]$. With x, y, t fixed, let $\langle a\rangle$ be a sequence converging to t such that a_n is a fraction with denominator 2^n ($\langle a\rangle$ can be obtained from the binary expansion of t). By the preceding paragraph, $f(a_n x + (1-a_n)y) \le a_n f(x) + (1-a_n)f(y)$ for all f. By the continuity of f and the properties of limits, $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

16.56. *Convergence to pth root.* Graphing the function $f(x) = x^p$ and viewing it geometrically suggests that f is convex. Since $f''(x) = p(p-1)x^{p-2}$ exists and is positive for all $x > 0$, the function is indeed convex. We define a sequence that converges to $a^{1/p}$, where the proof of convergence uses convexity and theorems about real numbers.

Beginning with $x_0 = a$, define a sequence $\langle x\rangle$ by $x_{n+1} = (1-1/p)x_n + (1/p)(a/x_n^{p-1})$ for $n \ge 0$. This recurrence expresses x_{n+1} as a convex combination of two positive numbers. This will help us prove that $a^{1/p} \le x_{n+1} \le x_n$ if $x_n > a^{1/p}$. Because $f(x) = x^p$ is monotone increasing and $a/x_n^{p-1} = x_n a/x_n^p \le x_n$, we conclude that $f(a/x_n^{p-1}) \le f(x_n)$. Since $x_{n+1} = (1-1/p)x_n + (1/p)(a/x^{p-1})$, the convexity of f yields $f(x_{n+1}) \le$

$(1-1/p)f(x_n) + (1/p)f(a/x_n^{p-1}) \le (1-1/p)f(x_n) + (1/p)f(x_n) = f(x_n)$. Now the monotonicity of f implies $x_{n+1} \le x_n$.

Next we use the convexity inequality again to prove $x_{n+1} \ge a^{1/p}$ if $x_n \ge a^{1/p}$. First some preliminary computation; let $\alpha = x_n^{1/p}$ and $\beta = a^{1/p}x_n^{(1-p)/p}$. The hypothesis implies $\alpha/\beta \ge 1$, or $\alpha \ge \beta$. Hence $(1-1/p)\alpha + (1/p)\beta \ge \beta$. Now we write $x_n = f(\alpha)$ and $a/x^{p-1} = f(\beta)$ and apply the recurrence and then the convexity inequality (and monotonicity of f) to compute

$$\begin{aligned} x_{n+1} &= (1-1/p)f(\alpha) + (1/p)f(\beta) \ge f[(1-1/p)\alpha + (1/p)\beta] \\ &\ge f(\beta) = ax_n^{1-p} \ge a(a^{1/p})^{1-p} = a^{1/p}. \end{aligned}$$

We now have a sequence that is decreasing and bounded below by $a^{1/p}$. By the Monotone Convergence Theorem, it must have a limit L. Taking the limit on the two sides of the recurrence that are names for the terms in the same sequence, we find that L must satisfy the equation. $L = (1 - 1/p)L + (1/p)(a/L^{p-1})$, which simplifies to $L^p = a$. Hence we have proved that this sequence converges to $a^{1/p}$, so it is a procedure for computing approximations to $a^{1/p}$.

16.57. $(+)$ Consider the polynomial f defined by $f(x) = (x-a)(x-b)(x-c)(x-d)$ with $a < b < c < d$. Describe the set of starting points x_0 such that Newton's Method converges to a zero of f. (Hint: Draw very careful pictures. The set of starting points x_0 that fail is an uncountable set.)

16.58. *If $f_n \to f$ uniformly and $g_n \to g$ uniformly on an interval I, then $f_n + g_n \to f + g$ uniformly on I.* The hypothesis imply that for all $\varepsilon > 0$, there exist N_1 and N_2 such that $n \ge N_1$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all x and $n \ge N_2$ implies $|g_n(x) - g(x)| < \varepsilon/2$ for all x. If $n \ge \max\{N_1, N_2\}$, then

$$|(f_n + g_n)(x) - (f+g)(x)| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f_n + g_n$ satisfies the definition of uniform convergence.

Sequences f_n and g_n such that each converges pointwise but not uniformly, and yet $f_n + g_n$ does converge uniformly. We may choose any f_n that converges pointwise but not uniformly and set $g_n = -f_n$. For example, let $f_n = x^n$ on $[0, 1]$.

16.59. *Uniform convergence of a particular series.*

a) $\sum_{n=0}^{\infty} e^{-nx}/2^n$ converges uniformly for $x \in \mathbb{R}$—FALSE. When $x = -1$, the series is $\sum_{n=0}^{\infty}(e/2)^n$, which diverges.

b) $\sum_{n=0}^{\infty} e^{-nx}/2^n$ converges uniformly for $x \ge 0$—TRUE. Since $\left|e^{-nx}/2^n\right| \le 2^{-n}$, the series converges uniformly by the Weierstrass M-test.

16.60. *Pointwise versus uniform convergence.* Define $f_n \colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = n^2/(x^2 + n^2)$, and define $f = \lim f_n$.

f_n converges to the function that is 1 everywhere. For each n, $f_n(0) = 1$. For $x \ne 0$, $f_n(x) = 1/(1 + x^2/n^2)$, so $\lim_{n\to\infty} f_n(x) = 1$.

$\{f_n\}$ does not converge uniformly to f. For $x = n$, we have $f_n(n) = 1/2$. Uniform convergence requires, for each $\varepsilon > 0$, a single threshold N such that every x satisfies $|f_n(x) - f(x)| < \varepsilon$ when $n \ge N$. Choose $\varepsilon = 1/2$. Since $f_n(n) = 1/2$, for every N there are values of x such that $n \ge N$ does not imply $|f_n(x) - f(x)| < 1/2$.

16.61. *Pointwise versus uniform convergence.* Let $f_n(x) = x^2/(x^2 + n^2)$.

a) f_n converges pointwise to 0 everywhere on $\mathbb{R}$. For fixed x, let $y_n = f_n(x)$. Various results from Chapter 14 yield $y_n \to 0$.

b) f_n does not converge uniformly to 0 on $\mathbb{R}$. For $x = n$, we have $f_n(n) = 1/2$. Choose $\varepsilon = 1/2$. Since $f_n(n) = 1/2$, for every N there are values of x such that $n \ge N$ does not imply $|f_n(x) - f(x)| < 1/2$.

16.62. $(\bullet)$ Recall that $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. Define g_n by $g_n(x) = x^n/n!$.

a) Prove that $\sum_{n=0}^{\infty} g_n$ converges uniformly to $\exp(x)$ on any bounded interval I (and hence $\exp(x)$ is continuous).

b) Prove that $\exp(x+y) = \exp(x)\exp(y)$.

c) Determine $\lim_{h\to 0}(\exp(h) - 1)/h$. (Comment: l'Hôpital's rule cannot be applied here, since we do not yet know that $\exp(x)$ defines a differentiable function. The series definition of $\exp(h)$ must be used.)

d) Use (b) and (c) to prove that $(d/dx)(\exp(x)) = \exp(x)$.

16.63. *If $a > 0$ and $f(x) = \exp(-ax^2)$, then f is convex when $|x| \ge \sqrt{a/2}$.* Since f is smooth, f is convex on those intervals where f'' is nonnegative. By the chain rule, we have $f''(x) = (4a^2x^{-}2a)e^{-}ax^2$. Since the second factor is always positive, the sign of $f''(x)$ is the sign of $4a^2x^2 - 2a$, which is the sign of $4ax^2 - 2$, since $a > 0$. Thus f is convex when $|x| \ge \sqrt{a/2}$.

Below we sketch the graph of f (***to be added***)

16.64. *If $|x| < 1$, then $\sum_{n=0}^{\infty} n^2x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x}$.* From $(1-x)^{-1} = \sum_{n\ge 0} x^n$, we differentiate twice to obtain $(1-x)^{-2} = \sum_{n\ge 0}(n+1)x^n$ and $2(1-x)^{-3} = \sum_{n\ge 0}(n+2)(n+1)x^n$. Since $n^2 = (n+2)(n+1) - 3(n+1) + 1$, we have the desired result.

In general, when q is a polynomial of degree k, we can express $q(n)$ as $a_k\binom{n}{k} + \cdots + a_0\binom{n}{0}$. Since $\sum_{n\ge 0}\binom{n}{j}x^n = x^j/(1-x)^{j+1}$ (Application 12.34), we then have the expression $\sum_{n\ge 0} q(n)x^n = \sum_{j=0}^{k} a_jx^j/(1-x)^{j+1}$.

16.65. *Expected number of runs for singles hitters and home run hitters,* where the singles hitters hit singles with probability p and the home run hitters hit home runs with probability $p/4$ (and otherwise strike out).

The home run hitters (probability $p/4$) score exactly k runs if and only if there are exactly k home runs before the third out. This has probability $a_k = \binom{k+2}{2}(p/4)^k(1-p/4)^3$. The expectation is

$$\sum_{k=0}^{\infty} ka_k = \sum_{k=0}^{\infty} \frac{1}{2}(k+2)(k+1)k(\frac{p}{4})^k(1-\frac{p}{4})^3 = \frac{3p}{4-p}.$$

Here we have used

$$\sum_{k=0}^{\infty}(k+2)(k+1)kx^k = x(\sum_{k=0}^{\infty} x^k)''' = x(\frac{1}{1-x})''' = 6x(1-x)^{-4}.$$

The singles hitters score exactly $k > 0$ runs if and only if there are exactly $k+2$ singles before the third out. This has probability $b_k = \binom{k+4}{2}p^{k+2}(1-p)^3$. The expectation is

$$\sum_{k=1}^{\infty} kb_k = \frac{1}{2}p^2(1-p)^3 \sum_{k=1}^{\infty}(k+4)(k+3)kp^k.$$

To sum the series, we use $(k+4)(k+3)k = (k+3)(k+2)(k+1) + (k+2)(k+1) - 2(k+1) - 6$ to obtain

$$\sum_{k=1}^{\infty}(k+4)(k+3)kx^k = \sum_{k=0}^{\infty}(k+4)(k+3)kx^k = (\frac{1}{1-x})'''+(\frac{1}{1-x})''-2(\frac{1}{1-x})'-6\frac{1}{1-x}$$

$$= \frac{6}{(1-x)^4} + \frac{2}{(1-x)^3} - \frac{2}{(1-x)^2} - \frac{6}{1-x}$$

Setting $x = p$ and inserting this into the formula for the expectation yields $p^2(\frac{3}{1-p} + 1 - (1-p) - 3(1-p)^2)$.

As p approaches 1, the home run hitters score about .75 runs per inning, while the expectation for the singles hitters grows without bound. When p is very small, the home run hitters expect about $3p/4$ runs per inning, while the singles hitters expect only about $10p^3$. When $p > .279$ (approximately), the singles hitters do better.

16.66. *$\sum_{k=0}^{n} kx^k$ is a ratio of two polynomials in x.* We compute

$$\sum_{k=0}^{n} kx^k = x\sum_{k=0}^{n} kx^{k-1} = x\frac{d}{dx}(\sum_{k=0}^{n} x^k)$$
$$= x\frac{d}{dx}(\frac{1-x^{n+1}}{1-x}) = \frac{x(n_x^{n+1} - (n+1)x^n + 1)}{(1-x)^2}.$$

16.67. If q is a polynomial, then $\sum_{k=0}^{\infty} q(k)x^k$ is the ratio of two polynomials in x (for $|x| < 1$).

Proof 1 (differentiation). It suffices to prove that the statement holds when $q(k) = k^m$, because the general case is a finite sum of multiples of such answers. We prove by induction on m that $\sum_{k=0}^{\infty} k^m x^k = \frac{p_m(x)}{(1-x)^{m+1}}$ for some polynomial p_m. When $m = 0$, we have the geometric series $\sum_{k=0}^{\infty} x^k = 1/(1-x)$.

For the induction step, suppose that $m > 0$ and that the claim holds for $m-1$. We compute

$$\sum_{k=0}^{\infty} k^m x^k = x\frac{d}{dx}\left(\sum_{k=0}^{\infty} k^{m-1}x^k\right) = x\left(\frac{d}{dx}p_{m-1}(x)(1-x)^m\right)$$
$$= x\frac{mp_{m-1}(x) + (1-x)p'_{m-1}(x)}{(1-x)^{m+1}} = \frac{p_m(x)}{(1-x)^{m+1}},$$

where $p_m(x) = x[mp_{m-1}(x) + (1-x)p'_{m-1}(x)]$. Since p_{m-1} is a polynomial, also p_m is a polynomial.

Proof 2 (generating functions). It suffices to prove that the statement holds when $q(k) = \binom{k}{m}$, because every polynomial is a finite sum of multiples of such binomial coefficients (see Solution 5.29). By Theorem 12.35, $\sum_{k=0}^{\infty}\binom{k}{m}x^k = x^m/(1-x)^{m+1}$.

16.68. *The random variable X defined for nonnegative integer n by* Prob $(X = n) = p(1-p)^n$, *where* $0 < p < 1$.

a) The probability generating function ϕ for X is given by $\phi(t) = p[1-(1-p)t]^{-1}$. By definition, $\phi(t) = \sum_{n=0}^{\infty}$ Prob $(X = n)t^n$. Using the geometric series, we compute $\phi(t) = \sum p(1-p)^n t^n = p/(1-(1-p)t)$. Since $\phi(1)$ sums all the probability, we have $\phi(1) = p/(1-(1-p)) = 1$, as desired.

b) $E(X) = (1-p)/p$. We compute

$$E(x) = \sum n\text{Prob}\,(X = n) = \phi'(1) = \frac{p(1-p)}{(1-(1-p)t)^{-2}}\Big|_{t=1} = \frac{1-p}{p}.$$

c) Prob $(X \le 20) = 1 - (1-p)^{21}$. We compute

$$\text{Prob}\,(X \le 20) = p\sum_{n=0}^{20}(1-p)^n = p\frac{1-(1-p)^{21}}{1-(1-p)} = 1-(1-p)^{21}.$$

16.69. *Curvature of $y(x) = x^n$, where $n \ge 2$.*

a) Curvature at the point $(x, y(x))$. From the analysis on page 283, we obtain curvature = $\kappa(x) = \left|y''(x)\right| / (1 + (y'(x))^2)^{3/2}$.

b) Equation for where the curvature is maximized. We may assume that $x > 0$, so that $y''(x) > 0$. Hence κ will be maximized when $\kappa'(x) = 0$. Computing κ' by the quotient rule and chain rule yields

$$\begin{aligned} \kappa' = 0 &\Leftrightarrow (1 + (y')^2)^{3/2} y''' = y'' 3(1 + (y')^2) y' y'' \\ &\Leftrightarrow (1 + (y')^2) y''' = 3y'(y'')^2. \end{aligned}$$

c) The curvature of $y(x) = x^3$ *is maximized at* $x = (1/45)^{1/4}$. When $y(x) = x^3$, we have $y' = 3x^2$, $y'' = 6x$, and $y''' = 6$. The equation of part (b) becomes $(1 + 9x^4)6 = 9x^2 \cdot 36x^2$, which simplifies to $1 + 9x^4 = 54x^4$. Thus $x = (1/45)^{1/4}$.

16.70. (−) Check the computations in Solution 16.71.

16.71. *Critical exponent for uniform convergence.*

a) If $f_n(x) = x^n(1 - x)$, *then* $f_n \to 0$ *uniformly on* $[0, 1]$. It suffices to show that $\lim_{n\to\infty} \max\{f_n(x): 0 \le x \le 1\} = 0$. Since f_n is 0 at the endpoints, the maximum occurs where $f_n'(x) = 0$, which is where $nx^{n-1} = (n + 1)x^n$, or $x = \frac{n}{n+1}$. We have $f_n(\frac{n}{n+1}) = (\frac{n}{n+1})^n \frac{1}{n+1} \to 0$.

b) If $f_n(x) = n^2 x^n(1 - x)$, *then* f_n *converges to 0 pointwise but not uniformly on* $[0, 1]$. Since n^2 is a constant factor in f_n, the maximum occurs at the same point as in part (a). Since $f_n(\frac{n}{n+1}) = n^2(\frac{n}{n+1})^n \frac{1}{n+1} \to \infty$, the convergence cannot be uniform.

For pointwise convergences, note that $f_n(1) = 0$ for all n. For $0 \le x < 1$, we have $f_n(x) = x^n(1 - x) \to 0$, since $x^n \to 0$.

c) If $f_n(x) = n^\alpha x^n(1-x)$, *where* $\alpha \ge 0$, *then* $f_n \to 0$ *uniformly on* $[0, 1]$ *if and only if* $\alpha < 1$. The convergence is uniform if and only if $\max |f_n| \to 0$; that is, $f_n(\frac{n}{n+1}) \to 0$. Since $f_n(\frac{n}{n+1}) = n^\alpha(\frac{n}{n+1})^n \frac{1}{n+1} = (1 - \frac{1}{n+1})^{n+1} n^{\alpha-1}$ and $(1 - 1/k)^k \to e^{-1}$, this sequence converges to 0 if and only if $\alpha < 1$.

16.72. *If* g *is bounded and differentiable function on* $\mathbb{R}$ *such that* $\lim_{x\to\infty} g'(x)$ *does not exist, then the sequence* $\{f_n\}$ *defined by* $f_n(x) = \frac{1}{n} g(nx)$ *converges uniformly on* $\mathbb{R}$, *but* f_n' *does not even converge.* Since g is bounded, we have M such that $|g(x| \le M$ for all x. The uniform convergence of $\{f_n\}$ follows from $|f_n(x)| = \left|\frac{1}{n} g(nx)\right| \le M/n \to 0$. Since this is independent of x, the convergence is uniform.

For each n, we have $f_n'(x) = g'(nx)$. Since $\lim_{x\to\infty} g'(x)$ does not exist, there exists x for which $\lim_{n\to\infty} g'(nx)$ does not exist.

16.73. (•) Prove that the sequence defined in Example 16.74 is uniformly Cauchy.

16.74. (•) Consider the proof that the function g in Example 16.75 is nowhere differentiable. Compute the difference quotient $[g(x + h_m) - g(x)]/h_m$ for all m in the following two cases: $x = 0$ and $x = .1496$.

16.75. *The function* $f: \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \sum_{n=0}^{\infty} \frac{\sin(3^n x)}{2^n}$ *is continuous on* $\mathbb{R}$ *and is not differentiable at 0.* Since g_n defined by $g_n(x) = \sin(3^n x)/2^n$ is continuous, each partial sum of the series for $f(x)$ is continuous. We have $|\sin(3^n x)/2^n| < 1/2^n$, and $\sum_{n\ge0} 1/2^n = 2$. By the Weierstrass M-test (Corollary 16.62), the series converges uniformly to f. Since each g_n is continuous and the convergence of $\sum_{n\ge0} g_n$ to f is uniform, Corollary 16.67 implies that f is continuous.

Next we show that f is not differentiable at 0 (in fact, f is differentiable *nowhere*). Since $f(0) = 0$, the difference quotient simplifies to $f(h)/h$. Hence we study $q(h) = \sum_{n\ge0} \sin(3^n h)/(2^n h)$.

Suppose that $\lim_{h\to0} q(h)$ exists. Let $h_m = \pi 3^{-m}$. By sequential continuity, $\lim_{m\to\infty} q(h_m)$ exists. However, $q(h_m) = \sum_{n\ge0} \frac{\sin(3^{n-m}\pi)3^m}{2^n\pi} = \sum_{n=0}^{m-1} \frac{\sin(3^{n-m}\pi)3^m}{2^n\pi}$, since $\sin(3^{n-m}\pi) = 0$ for $n \ge m$.

Let $a_m = \sum_{n=0}^{m-1} \frac{\sin(3^{n-m}\pi)3^m}{2^n\pi}$. Since $\sin(x) > 0$ for $0 < x < \pi$, each term in the sum for a_m is positive. The last term is $\frac{\sin(\pi/3)}{\pi/3}(3/2)^m$; let this be b_m. We have $a_m > b_m$, and yet $b_m \to \infty$. Hence $a_m \to \infty$, and $\langle a \rangle$ does not converge. Thus $\lim_{m\to\infty} q(h_m)$ does not exist, and $\lim_{h\to0} q(h)$ does not exist.

16.76. *Functions differentiable at exactly one point.* Suppose that f is continuous and nowhere differentiable and that $a \in \mathbb{R}$. Let $g(x) = (x - a)^2 f(x)$. For $x \ne a$, $f(x) = g(x)/(x - a)^2$, so differentiability of g at x would require differentiability of f at x. Thus g is not differentiable for $x! = a$. As a pointwise product of continuous functions, g is continuous everywhere.

The differentiability of g at a (with $g'(a) = 0$) follows from

$$\lim_{h\to0} \frac{g(a + h) - g(a)}{h} = \lim_{h\to0} \frac{h^2 f(a + h)}{h} = 0,$$

where the last equality holds because the continuity of f and a implies that $\lim_{h\to0}(hf(a + h)) = 0$.

17. INTEGRATION

17.1. $\int_0^2 \min\{x, 2 - x\}\,dx$ *and* $\int_0^2 \max\{x, 2 - x\}\,dx$. The integrals can be stated with simple integrands by breaking the interval of integration at 1. The first becomes

$$\int_0^2 \min\{x, 2-x\}\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)dx$$
$$= \frac{1}{2}x^2\Big|_0^1 + (2x - \frac{1}{2}x^2)\Big|_1^2 = \frac{1}{2} - 0 + 4 - 2 - 2 + \frac{1}{2} = 1.$$

The second becomes

$$\int_0^2 \max\{x, 2-x\}\,dx = \int_0^1 (2-x)\,dx + \int_1^2 x\,dx$$
$$= (2x - \frac{1}{2}x^2)\Big|_0^1 + \frac{1}{2}x^2\Big|_1^2 = 2 - \frac{1}{2} - 0 + 0 + 2 - \frac{1}{2} = \frac{1}{2}.$$

17.2. *Yields on bank accounts.*

6% simple interest produces a 6% yield.

6% interest compounded daily on beginning balance b produces the account value $(1 + .06/365)^365b$ at the end of one year. Since $(1 + .06/365)^365 = 1.0618313$, this is 6.18313% yield.

6% interest compounded continuously on beginning balance b produces the account value $\lim_{n\to\infty}(1 + .06/n)^n b = e^{.06}b$ at the end of one year. since $e^{.06} = 1.0618365$, this is 6.18365% yield.

17.3. (–) How many years does it take to double the value of a bank account paying 4% simple interest? How many years if the interest rate is p%?

17.4. (•) "If f is bounded and nonconstant on [0, 1], then for each partition P of the interval, $L(f, P) < U(f, P)$."

17.5. (•) Give a proof or a counterexample: "If f is continuous and nonconstant on [0, 1], then for each partition P of the interval, $L(f, P) < U(f, P)$."

17.6. *If f and g are bounded real-valued functions on a set S, then* $\sup_S(f + g) \le \sup_S f + \sup_S g$. Let $M = \sup_S(f + g)$. Let $\langle x\rangle$ be a sequence in S such that $f(x_n) + g(x_n) \to M$. We have $f(x_n) \le \sup_S f$ and $g(x_n) \le \sup_S g$ for all n. If $b_n \le L$ and $\lim b_n$ exists, then $\lim b_n \le L$. Hence $M \le \sup_S f + \sup_S g$.

An example where the two sides differ. Let $S = [0, 1]$, $f(x) = x$, and $g(x) = 1 - x$. Now $\sup_S(f + g) = 1$ and $\sup_S f + \sup_S g = 2$.

17.7. (–) Let $f(x) = x^2$, and let P_n be the partition of [1, 3] into n subintervals of equal length. Compute formulas for $L(f, P_n)$ and $U(f, P_n)$ in terms of n. Verify that they have the same limit. Determine how large n must be to ensure that $U(f, P_n)$ is within .01 of $\int_1^3 f(x)\,dx$.

17.8. *a) If $f\colon [a, b] \to \mathbb{R}$ and R is a refinement of a partition P of $[a, b]$, then* $L(f, P) \le L(f, R) \le U(f, R) \le U(f, P)$. Each interval I in P is broken into one or more intervals $I_1, \dots, I_k$ in R. Always $\inf_{I_j} f \le \inf_I f \le \sup_I f \le \sup_{I_j} f$. Weighting this inequality by the length of I_j and summing over all intervals in R yields the desired inequality.

b) If $f\colon [a, b] \to \mathbb{R}$ and P, Q are partitions of $[a, b]$, then $L(f, P) \le U(f, Q)$. Let R be a partition of $[a, b]$ that is both a refinement of P and a refinement of Q; for example, let the set of breakpoints of R be the union of the sets of breakpoints for P and Q. By part (a), $L(f, P) \le L(f, R) \le U(f, R) \le U(f, Q)$.

17.9. *If f is integrable on $[a, b]$, then $-f$ is integrable on $[a, b]$, with* $\int_a^b(-f) = -\int_a^b f$. For a partition P of $[a, b]$, $L(-f, P) = -U(f, P)$, since $\inf_S(-f) = -\sup_S(f)$. Similarly, $U(-f, P) = -L(f, P)$. Since f is integrable on $[a, b]$, for every $\varepsilon > 0$ there is a partition R of $[a, b]$ such that $U(f, R) - L(f, R) < \varepsilon$. Since $U(-f, R) - L(-f, R) = -L(f, R) + U(f, R) < \varepsilon$, this condition holds for $-f$, and hence $-f$ is integrable (condition Proposition 17.9b). The value of the integral is

$$\sup_P L(-f, P) = \sup_P(-U(f, P)) = -\inf_P U(f, P) = \int_a^b f.$$

The geometric interpretation of $\int_a^b(f - g) = \int_a^b f - \int_a^b g$ is that the (signed) area between the graphs of f and g can be obtained by subtracting the area under the graph of g from the area under the graph of f.

17.10. *If f is integrable on $[a, b]$, and $a < c < b$, then f is integrable on $[a, c]$ and on $[c, b]$.* By Proposition 17.9c, there is a sequence $\langle R\rangle$ of partitions such that $\lim_{n\to\infty} L(f, R_n) = \lim_{n\to\infty} U(f, R_n)$. Define R_n' by adding c as a breakpoint to the set of breakpoints of R (this has the effect of breaking at c the interval that crosses c in R). By considering inf f and sup f on the pieces of the broken interval, we have $L(f, R_n) \le L(f, R_n') \le U(f, R_n') \le U(f, R_n)$.

Now, R_n' consists of a partition P_n of $[a, c]$ and a partition Q_n of $[c, b]$. We have $L(f, P_n) \le U(f, P_n)$ and $L(f, Q_n) \le U(f, Q_n)$. Also, $L(f, R_n') = L(f, P_n) + L(f, Q_n)$ and $U(f, R_n') = U(f, P_n) + U(f, Q_n)$.

From all our inequalities, we conclude

$$0 \le U(f, P_n) - L(f, P_n) \le U(f, R_n') - L(f, R_n') \le U(f, R_n) - L(f, R_n) \text{ and}$$
$$0 \le U(f, Q_n) - L(f, Q_n) \le U(f, R_n') - L(f, R_n') \le U(f, R_n) - L(f, R_n).$$

Hence $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$ and $\lim_{n\to\infty} L(f, Q_n) = \lim_{n\to\infty} U(f, Q_n)$, and f is integrable on $[a, c]$ and $[c, b]$.

17.11. *If $f\colon [0, 1] \to [0, 1]$ is defined by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational, then f is not integrable.* Let P be a partition of [0, 1]. Since each interval I that has distinct endpoints contains both rational and irrational numbers, $\sup_I(f) = 1$ and $\inf_I(f) = 0$. Therefore $U(f, P) =$

1 and $L(f, P) = 0$, and these values cannot be made close by choosing appropriate partitions.

17.12. *A function f such that $|f|$ is integrable on* $[0, 1]$ *but f is not integrable on* $[0, 1]$. Let $f(x) = 1$ for $x \in \mathbb{Q}$, and $f(x) = -1$ for $x \notin \mathbb{Q}$. By the argument in the preceding exercise, f is not integrable over any interval, but $|f|$ is integrable over each bounded interval.

17.13. *(Mean Value Theorem for integrals) If f is continuous on* $[a, b]$*, then there exists $c \in [a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f$.* Consider first the special case when $\int_a^b f = 0$. We prove by contradiction that $f(c) = 0$ for some $c \in [a, b]$. Otherwise, Rolle's Theorem implies that f is always positive or always negative on $[a, b]$, which contradicts $\int_a^b f = 0$.

For the general case, let $g(x) = f(x) - \frac{1}{b-a}\int_a^b f$. The function g is continous, and $\int_a^b g = \int_a^b f - \int_a^b f = 0$. Thus there exists c with $g(c) = 0$, which yields $f(c) = \frac{1}{b-a}\int_a^b f(t)d_t$.

Alternatively, let $F(x) = \int_a^x f(t)d_t$. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Thus F is differentiable on $[a, b]$. Applying the Mean Value Theorem to F yields $c \in [a, b]$ such that $f(c) = F'(c) = \frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b f$.

17.14. *Monotone functions are integrable.* When f is increasing on an interval $I = [c, d]$, we have $\sup_I f = f(d)$ and $\inf_I f = f(c)$. Thus if f is increasing on $[a, b]$ and P_n is the partition of $[a, b]$ into n intervals of equal length, we have $U(f, P_n) = \frac{b-a}{n}\sum_{i=1}^n f(x_i)$ and $L(f, P_n) = \frac{b-a}{n}\sum_{i=1}^n f(x_{i-1})$. Thus $U(f, P_n) - L(f, P_n) = \frac{b-a}{n}(f(b) - f(a))$, because the sums telescope. As $n \to 0$, $U(f, P_n) - L(f, P_n) \to 0$; thus f is integrable on $[a, b]$.

17.15. *Properties of integration.* Let f be a continuous function on the interval $[a, b]$.

a) If $f(x) \ge 0$ for $x \in [a, b]$ and f is not everywhere zero on $[a, b]$, then $\int_a^b f(x)dx > 0$. Choose $c \in [a, b]$ such that $f(c) > 0$. Since f is continuous, there is an interval I about c on which $f(x) \ge \frac{1}{2}f(c) > 0$ (taking I to be one-sided if $c \in \{a, b\}$). Let m be the length of I. Then

$$\int_a^b f(x)dx \ge \int_I f(x)dx \ge m\frac{1}{2}f(c) > 0.$$

b) If $\int_a^b f(t)g(t)d_t = 0$ for every continuous function g on $[a, b]$, then $f(x) = 0$ for $a \le x \le b$. When we choose $g(t) = f(t)$ for $a \le t \le b$, we obtain the nonnegative continuous integrand f^2. The contrapositive of part (a) now yield $f^2(t) = 0$ for $a \le t \le b$, and hence also $f(t) = 0$.

17.16. *The function g defined by $g(x) = \int_0^x (1+t^2)^{-1}dt + \int_0^{1/x}(1+t^2)^{-1}dt$ is constant.* Since the integrands are continuous, the Fundamental Theorem of Calculus implies that g is differentiable. Furthermore,

$$g'(x) = \frac{1}{1+x^2} + \frac{-1}{x^2}\frac{1}{1+(1/x)^2} = 0.$$

Since $g'(x) = 0$ for all x, g is constant.

17.17. *If $f\colon [0, 1] \to [0, 1]$ is a bijection with $f(0) = 1$ and $f(1) = 0$, then $\int_0^1 f(x)dx = \int_0^1 f^{-1}(y)dy$.* Both sides compute the area under the graph of f.

17.18. $\int_0^1 (1 - x^a)^{1/b}\,dx = \int_0^1 (1 - x^b)^{1/a}\,dx$. Let $f(x) = (1 - x^a)^{1/b}$. If $y = (1 - x^a)^{1/b}$, then $y^b = 1 - x^a$, and hence $x = (1 - y^b)^{1/a}$. Hence substituting y for x in the second integral rewrites the desired equality as $\int_0^1 f(x)dx = \int_0^1 f^{-1}(y)dy$.

By Exercise 17.17, the equality holds when $f\colon [0, 1] \to [0, 1]$ is a bijection with $f(0) = 1$ and $f(1) = 0$. Indeed, evaluating $(1 - x^a)^{1/b}$ at $x = 0$ and $x = 1$ yields the values 1 and 0, and solving for x in terms of y as above for all $y \in [0, 1]$ shows that f is a bijection.

When a and b are reciprocals of positive integers A and B, the integral becomes $I = \int_0^1 (1 - x^{1/A})^B\,dx$. Change variables by setting $x = y^A$, so $dx = Ay^{A-1}dy$; this yields $I = \int_0^1 (1 - y)^B Ay^{A-1}\,dy$. We integrate by parts $A - 1$ times, with the boundary terms vanishing, to obtain

$$I = \int_0^1 \frac{(1-y)^{B+1}}{B+1}A(A-1)y^{A-2}\,dy = \cdots$$
$$= \int_0^1 \frac{A!(1-y)^{A+B-1}}{(B+1)(B+2)\cdots(B+A-1)}\,dy = \frac{A!B!}{(A+B)!}.$$

Comment: For general a, b, the integral can be evaluated in terms of the gamma function, defined in Exercise 17.52.

17.19. (!) For $x > 0$, determine $\lim_{h\to 0}\left(\frac{1}{h}\ln(\frac{x+h}{x})\right)$.

17.20. (!) Evaluate $\frac{1}{n}\sum_{k=1}^n \ln(k/n)$ as a function of n. Interpret the sum as a lower sum of an improper integral, and evaluate its limit as $n \to \infty$.

17.21. (!) Let N be a positive integer, and let $a_n = \sum_{j=n+1}^{(N+1)n}(1/j)$.

a) Consider the lower sum $L(f, P)$ where $f(x) = 1/x$ and P is the partition of $[1, N + 1]$ into Nn pieces. Change the index of summation to obtain $L(f, P) = a_n$.

b) Evaluate $\lim a_n$.

17.22. *If $x > 0$, then $\ln(\frac{x+1}{x}) > \frac{1}{x+1}$.* Since $\ln(\frac{x+1}{x} = \int_1^{1+1/x} \frac{1}{t}\,dt$, we obtain a lower bound by taking a lower sum for the integral. We use just one interval, so the lower sum is the length of the interval times the minimum of the integrand. Hence $\ln(\frac{x+1}{x} \geq \frac{1}{x}\frac{1}{1+1/x} = \frac{1}{x+1}$. (Hint: Use a lower sum for an appropriate integral.)

17.23. *The function f defined by $f(x) = (1+1/x)^x$ for $x > 0$ is increasing.* We compute $\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln f(x) = \ln(1+1/x) - \frac{x}{x^2+x}$, Since f is always positive, we have $f'(x) > 0$ if and only if $\ln(\frac{x+1}{x}) > \frac{1}{x+1}$. The latter inequality is the result of Exercise 17.22.

17.24. *Proof of the product rule for differentiation by considering by considering $(d/dx)(\ln(fg))$.* By the chain rule, $(d/dx)(\ln(fg)) = (fg)'/(fg)$. Since $\ln(fg) = \ln(f) + \ln(g)$, the left side is $f'/f + g'/g$. Equating the two expressions for the derivative yields $(fg)'/(fg) = f'/f + g'/g$. Multiplying by fg yields the product rule $(fg)' = f'g + g'f$.

17.25. *If $n \in \mathbb{N}$ and $b \geq 1$, then $\ln(b) \geq n(1 - b^{-1/n})$.* We treat $\ln b$ as the integral $\int_1^b (1/t)\,dt$. For each partition P_n of $[1, b]$ into n intervals, $\ln b \geq L(f, P_n)$, where $f(t) = 1/t$. Since f is decreasing, the infimum on each interval occurs at its right endpoint.

Let P_n be the partition with breakpoints at $b^{j/n}$ for $0 \leq j \leq n$. We have

$$L(f, P_n) = \sum_{j=1}^n (b^{j/n} - b^{(j-1)/n})\frac{1}{b^{j/n}} = \sum_{j=1}^n (1 - b^{-1/n}) = n(1 - b^{-1/n}).$$

17.26. *Integration by parts for $\ln(x)$ and $\tan^{-1}(x)$.*

$$\int \ln(x)dx = x\ln(x) - \int x\frac{1}{x}dx = x\ln(x) - x.$$

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}dx = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1).$$

These can also be obtained immediately from Theorem 17.26.

17.27. *The logarithm function is unbounded.* The property $\ln(xy) = \ln x + \ln y$ implies that $\ln(x^n) = n\ln(x)$ for all $n \in \mathbb{N}$. When $\ln(x) > 0$, $\{n\ln(x)\}$ is unbounded above. When $\ln(x) < 0$, $\{n\ln(x)\}$ is unbounded below.

17.28. *If $f(x) = \int_0^x f(t)d_t + c$ and f is continuous, then $f(x) = f(0)e^x$.* Assuming that f is continous, the Fundamental Theorem of Calculus yields $f'(x) = f(x)$, which has the solution $f(x) = ce^x$ with $c = f(0)$.

17.29. *Properties of exponentiation.*

a) $e^{x+y} = e^x e^y$. Using $\ln ab = \ln a + \ln b$ and the definition of the exponential function as the inverse of the logarithm, we have $\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y})$. Since ln is strictly increasing, it is injective, and hence $e^x e^y = e^{x+y}$.

b) Computation of $(d/dx)x^\alpha$. Since $x^\alpha = e^{\alpha\ln x}$, we use the chain rule to compute $(d/dx)x^\alpha = e^{\alpha\ln x}(\alpha/x) = \alpha x^{\alpha-1}$.

c) $(d/dx)a^x = a^x\ln a$. Let $f(x) = a^x$, so $\ln f(x) = x\ln a$. Differentiating both sides (using the chain rule on the left) yields $f'(x)/f(x) = \ln a$. Thus $f'(x) = f(x)\ln a = a^x\ln a$.

17.30. *Solutions to $x^a = a^x$ for $a, x > 0$.* Taking logarithms of both sides yields $a\ln x = x\ln a$, or $\frac{\ln a}{a} = \frac{\ln x}{x}$. Consider the function $f(x) = \frac{\ln x}{x}$. The derivative is $\frac{1-\ln x}{x^2}$, which is 0 only at $x = e$. Also $\lim_{x\to\infty} f(x) = 0$, $f(1) = 0$, and $f(x) < 0$ for $x < 1$. Hence f is monotone decreasing from e^{-1} to 0 (injective) for $x \geq e$, and f is monotone increasing from 0 to e^{-1} (injective) for $1 \leq x \leq e$.

From this we conclude that for each x greater than e, there is exactly one a between 1 and e such that $x^a = a^x$. Note, for example, that 4 matches with 2. In general, it is difficult to determine the corresponding a exactly in terms of x.

17.31. *Computation of $\sum_{k=0}^n k^p$ using the exponential function.* The geometric sum yields $\sum_{k=0}^n e^{xk} = \frac{1-e^{n+1}x}{1-e^x}$. The pth derivative of the sum is $\sum_{k=0}^n k^p e^{kx}$; evaluated at $x = 0$, it equals $\sum_{k=0}^n k^p$. Hence the finite sum can be computed by taking the pth derivative of $\frac{1-e^{n+1}x}{1-e^x}$ and setting $x = 0$ in the resulting expression.

17.32. (+) Use trapezoids to obtain upper and lower bounds on $\int_0^n x^k\,dx$. Use this to prove Theorem 5.31.

17.33. *For the functions defined by $f_n(x) = ae^{-anx} - be^{-bnx}$, where a, b are real constants with $0 < a < b$, $\sum_{n=1}^\infty \int_0^\infty f_n(x)dx = 0$ and $\int_0^\infty \sum_{n=1}^\infty f_n(x)dx = \ln(b/a)$.*

First compute $\int_0^\infty f_n(x)\,dx = \lim_{t\to\infty}\int_0^t (ae^{-anx} - be^{-bnx} = \frac{1}{n}\lim_{t\to\infty}(-e^{-anx} + e^{-bnx})\big|_0^t = \frac{1}{n}\lim_{t\to\infty}(e^{-bnt} - e^{-ant}) = 0$. Hence $\sum_{n=1}^\infty \int_0^\infty f_n(x)dx = 0$

Next, $\int_0^\infty \sum_{n=1}^\infty f_n(x)dx = \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}$, by the geometric series. To perform the integral, we take the limit, as $t \to \infty$ and $\varepsilon \to \infty$, of $\int_\varepsilon^t \left(\frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}\right)dx$. The integral evaluates to $\left(\ln(1-e^{-ax}) - \ln(1-e^{-bx})\right)\big|_\varepsilon^t$, which converges to $-\ln\left(\frac{1-e^{-a\varepsilon}}{1-e^{-b\varepsilon}}\right)$ as $t \to \infty$.

To take the limit of this as $\varepsilon \to 0$, we use l'Hôpital's Rule to compute $\lim_{\varepsilon\to 0}\frac{1-e^{-a\varepsilon}}{1-e^{-b\varepsilon}} = \lim_{\varepsilon\to 0}\frac{ae^{-a\varepsilon}}{be^{-b\varepsilon}} = a/b$. Since ln is continuous at a/b, our final answer is $-\ln(a/b)$, or $\ln(b/a)$.

17.34. (•) Let $f\colon \mathbb{R} \to \mathbb{R}$ be monotone increasing. Suppose that $0 \leq a \leq b$, $s = f(a)$, $t = f(b)$, and $0 \leq s \leq t$. The proof of ?? shows that

$$\int_a^b f(x)\,dx = yf^{-1}(y)\Big|_s^t - \int_s^t f^{-1}(y)\,dy.$$

Prove that this formula still holds when the requirements $0 \le a \le b$ and $0 \le s \le t$ are weakened to $a \le b$ and $s \le t$. This completes the proof of ?? . (Hint: Use substitution to reduce to the case where a and s are positive.)

17.35. (•) By the Fundamental Theorem of Calculus, $\int_0^1 e^x\,dx = e-1$. The steps below evaluate the integral as a limit of sums.

a) Write down the lower sum $L(f, P_n)$, where $f(x) = e^x$ and P_n is the partition of [0, 1] into n equal parts.

b) Use a finite geometric sum to evaluate the sum in part (a).

c) Verify directly that $\lim_{n\to\infty} L(f, P_n) = e-1$. (What properties of the exponential function does this use?)

17.36. $\lim_{n\to\infty}\sum_{k=1}^n (n^2+nk)^{-1/2} = 2(\sqrt{2}-1)$. We express the summation as the nth lower sum for a definite integral, so the limit is the value of the integral. Let $f(x) = 1/\sqrt{1+x}$. Note that f is decreasing for $x \ge 0$, so the infimum of f on an interval occurs at the right endpoint. If P_n partitions [0, 1] into n equal parts, then $L(f, P_n) = \frac{1}{n}\sum_{k=1}^n f(k/n) = \frac{1}{n}\sum_{k=1}^n 1/\sqrt{1+k/n} = \sum_{k=1}^n (n^2+nk)^{-1/2}$. Hence the limit is the value of $\int_0^1 f(x)\,dx$, which is $2(1+x)^{1/2}\big|_0^1$, equal to $2(\sqrt{2}-1)$.

17.37. *Limits by l'Hôpital's Rule.*

$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0.$

$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0.$

17.38. (+) Let $\langle x\rangle$ be the sequence defined by $x_1 = \sqrt{2}$ and $x_{n+1} = (\sqrt{2})^{x_n}$ for $n \ge 1$. Prove that $\langle x\rangle$ converges and determine the limit.

17.39. *The minimum of the function f defined by $f(x) = x/\ln x$ for $x > 1$ is e.* The derivative of f is $f'(x) = \frac{\ln(x)-1}{(\ln x)^2}$. Thus f' is negative for $1 < x < e$, positive for $x > e$, and 0 for $x = e$. Hence f is minimized uniquely at e, where $f(x) = e$.

For $x \ne e$, we have proved that $x/e > \ln x$. Exponentiating yields $e^{x/e} > x$, and raising to the power e yields $e^x > x^e$ for $x \ne e$. Thus $e^\pi > \pi^e$. The actual values are about 23.14 and 22.46.

17.40. *If $f(x) = u(x)\prod_{i=1}^n (x-a_i)$, where $a_i \ne 0$ for all i, and u is differentiable and never zero, then $\sum\frac{1}{a_i} = \frac{u'(0)}{u(0)} - \frac{f'(0)}{f(0)}$.* We compute $f'(x) = u(x)\sum_j\prod_{i\ne j}(x-a_i) + u'(x)\prod_{i=1}^n(x-a_i)$. Thus $\frac{f'(x)}{f(x)} = \sum_j\frac{1}{x-a_j} + \frac{u'(x)}{u(x)}$. Setting $x = 0$ yields $\frac{f'(0)}{f(0)} = -\sum\frac{1}{a_j} + \frac{u'(0)}{u(0)}$ and hence the formula claimed.

Comment: The same result can be obtained by differentiating $\ln f$. An alternative formula is $\sum(1/a_i) = -(\ln(f/u))'(0)$.

17.41. (+) Suppose that $h\colon\mathbb{R}\to\mathbb{R}$ and that $h(x^n) = h(x)$ for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$.

a) Prove that if h is continuous at $x = 1$, then h is constant.

b) Show that without this assumption h need not be constant.

c) Suppose that $f(x^n) = nx^{n-1}f(x)$ for all $x > 0$ and all $n\in\mathbb{N}$. Suppose also that $\lim_{x\to 1} f(x)/\ln(x)$ exists. What does this imply about f?

17.42. *Differentiation of f^g.* With f and g differentiable, let $y = f^g$. Differentiating $\ln y = g\ln f$ yields $y'/y = gf'/f + g'\ln f$. Thus $(f^g)' = y' = gf^{g-1}f' + g'f^g\ln f$.

17.43. (•) *AGM Inequality.*

a) Prove that $y^a z^{1-a} \le ay + (1-a)z$ for all positive y, z and $0 \le a \le 1$. Determine when equality can hold.

b) Let $x_1,\dots,x_n$ be a list of n positive real numbers. Prove that $(\sum_{i=1}^n x_i)/n \ge (\prod_{i=1}^n x_i)^{1/n}$, with equality only when $x_1 = \cdots = x_n$. (Hint: Part (a) can be applied to give a proof by induction on n.)

c) Let $a_1,\dots,a_n$ be nonnegative real numbers. Find the maximum of $\prod_{i=1}^n x_i^{a_i}$ subject to $\sum x_i = 1$.

d) Use part (c) to give a different proof of part (b).

17.44. *Divergence of $\sum_{n=1}^\infty 1/n$.* Let P_N denote the partition of the interval $[1, N+1]$ into intervals of length 1. Now $\sum_{n=1}^N = U(f, P_N)$, where $f(x) = 1/x$. Since $U(f, P_N) \ge \int_1^{N+1} f$, we have $\sum_{n=1}^N 1/n \ge \ln(N+1)$. Since $\ln(N+1)\to\infty$ as $N\to\infty$, the series diverges.

17.45. *Natural logarithms and Stirling's Formula.*

a) For $0 < \varepsilon < 1$,

$$\int_\varepsilon^1 \ln(x)dx = x(\ln x) - x\big|_\varepsilon^1 = -1 - \varepsilon(\ln\varepsilon) + \varepsilon.$$

b) $\int_0^1 \ln(x)dx = \lim_{\varepsilon\to 0}[-1-\varepsilon(\ln\varepsilon)+\varepsilon] = -1$.

c) Since $\ln x$ increases with x, the partition P_n with n equal parts yields $U(f, P_n) = \frac{1}{n}\sum_{k=1}^n \ln(k/n)$. This also equals $\frac{1}{n}\ln(\prod_{k=1}^n(k/n)) = \frac{1}{n}\ln(n!/n^n)$. By part (b), the limit of the expression as $n\to\infty$ is -1.

d) Exponentiating both sides of the expression in part (c) yields $\lim_{n\to\infty}(n!^{1/n}/n) = e^{-1}$.

17.46. *Radius of convergence R and series behavior when $|x| = R$.* The radius of convergence is 1 in each case except the last.

a) $\sum\frac{x^n}{n^2}$. The ratio test applies. The series converges if $1 > \lim\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim\left|\frac{n^2}{(n+1)^2}\right||x| = |x|$. It diverges if $|x| > 1$. When $|x| = 1$, the series reduces to $\sum(1/n^2)$ or $\sum((-1)^n/n^2)$, both of which converge. Hence the series converges if and only if $-1 \le x \le 1$.

b) $\sum \frac{x^n}{n}$. The ratio test gives convergence for $|x| < 1$ and divergence for $|x| > 1$ as above. Since $\sum(1/n)$ diverges and $\sum((-1)^n/n)$ converges, the series converges if and only if $1 \le x < 1$.

c) $\sum \frac{x^{2n}}{2n}$. Again we apply the ratio test, with $a_n = (x^2)^n/(2n)$. The limit of the ratio is x^2. Hence the series converges if $x^2 < 1$ and diverges if $x^2 > 1$. When $x = \pm 1$, we have $\sum x^{2n}/(2n) = \sum 1/(2n)$, which diverges. Hence the series converges if and only if $-1 < x < 1$.

d) $\sum \frac{x^n n^n}{n!}$. After canceling common factors, the ratio of absolute value of successive terms is $|x|\,(\frac{n+1}{n})^n$. To apply the ratio test, we compute $\lim |x|\,(\frac{n+1}{n})^n = |x| \lim(1+1/n)^n = |x|\,e$. Thus the series converges if $|x|\,e < 1$ and diverges if $|x|\,e > 1$.

When $x = 1/e$, the series is $\sum(n/e)^n/n!$. By the limit comparison test, and Stirling's Formula, this is comparable to $\sum 1/\sqrt{2\pi n}$, which diverges. When $x = -1/e$, the series converges. We write the series as $\sum(-1)^n(n/e)^n/n!$. An alternating series converges if the terms are decreasing in magnitude and converge to 0. Let $a_n = (n/e)^n/n!$.

To show that $a_n \to 0$, we use Stirling's approximation for $n!$ (see Exercise 17.45). This yields $a_n \sim 1/\sqrt{2\pi n}$, and hence $a_n \to 0$.

By canceling like factors, we have $a_{n+1} < a_n$ if and only if $(n+1)^n < n^n e$, or in other words $(1+1/n)^n < e$. We have shown previously that $(1+1/n)^n$ is an increasing sequence (Exercise 13.31, Exercise 17.23). By l'Hôpital's Rule (as applied in Theorem 17.31), the limit of $\ln a_n$ is 1, so $a_n \to e$. An increasing sequence with limit e is bounded above by e, so the desired inequality holds for all n.

17.47. *If f is continous on $[a, b]$, then* $\lim_{n\to\infty}(\int_a^b |f|^n)^{1/n} = \max\{|f(x)| : a \le x \le b\}$. If $\max|f| = 0$, the result is immediate. Otherwise, let $g(x) = f(x)/\max|f|$. It thus suffices to prove that $\lim(\int_a^b |g|^n)^{1/n} = 1$ when $\max|g| = 1$. Let $h_n = (\int_a^b |g|^n)^{1/n}$.

Since $|g(x)| \le 1$, we have $h_n \le (b-a)^{1/n} \to 1$. Thus it suffices to show that $h_n \ge c_n$ for some sequence $\langle c\rangle$ converging to 1.

Since g attains its maximum on $[a, b]$, there exists c such that $|g(x_0)| = 1$. Choose $\varepsilon > 0$. Since $|g|$ is continuous, we can choose δ such that $|x - c| < \delta$ implies $|g(x)| \ge 1 - \varepsilon/2$. (Note: if the maximum occurs at an endpoint, we use a one-sided interval.) Now

$$h_n^n \ge \int_{c-\delta}^{c+\delta} |g|^n \ge 2\delta(1-\varepsilon/2)^n.$$

Thus $h_n \ge (2\delta)^{1/n}(1-\varepsilon) \to 1 - \varepsilon/2$. Since h_n is at least a quantity

converging to $1-\varepsilon/2$, we have $h_n > 1-\varepsilon$ for sufficiently large n. This holds for all $\varepsilon > 0$, so indeed $h_n \to 1$.

17.48. ($\bullet$) Let $\langle a\rangle$ be a bounded sequence, and suppose that $\lim b_n = 1$. Prove that $\limsup a_n b_n = \limsup a_n$.

17.49. (!) Let f be continuous and nonnegative on $[0, \infty)$.

a) Prove that $\int_0^\infty f(x)\,dx$ exists if $\lim_{x\to\infty} \frac{f(x+1)}{f(x)}$ exists and is less than 1.

b) Prove that $\int_0^\infty f(x)\,dx$ exists if $\lim_{x\to\infty}(f(x))^{1/x}$ exists and is less than 1.

c) In parts (a) and (b), prove that the integrals do not exist if the specified limits exist but exceed 1.

17.50. (!) Let x, y, t be positive real numbers.

a) Prove that $t^2+t(x+y)+(\frac{x+y}{2})^2 \ge t^2+t(x+y)+xy \ge t^2+2t\sqrt{xy}+xy$.

b) After taking reciprocals of the expressions in part (a), integrate from 0 to ∞ with respect to t to prove that

$$\frac{x+y}{2} \ge \frac{x-y}{\ln(x)-\ln(y)} \ge \sqrt{xy}.$$

c) For $u \in \mathbb{R}$, use part (b) to show that $\frac{1}{2}(e^u + e^{-u}) \ge \frac{1}{2u}(e^u - e^{-u}) \ge 1$.

d) Prove part (c) directly using power series.

17.51. *If $n \in \mathbb{N}$, then $n! = \int_0^\infty e^{-x}x^n dx$.* When $n = 0$, the value of the integral is 1. In general, let $I_n = \int_0^\infty e^{-x}x^n dx$. Integration by parts (with $u = x^{n+1}$ and $dv = e^{-x}dv$ yields $I_{n+1} = -x^{n+1}e^{-x}\Big|_0^\infty - \int_0^\infty (n+1)x^n(-e^{-x}dx) = (n+1)I_n$. By induction, $I_n = n!$ for $n \in \mathbb{N} \cup \{0\}$.

17.52. ($\bullet$) The function Γ defined by $\Gamma(y) = \int_0^\infty e^{-x}x^{y-1}\,dx$ for $y > 0$ extends the notion of factorial to real arguments, with $\Gamma(n+1) = n!$.

a) Prove that the improper integral defining $\Gamma(y)$ converges when $y \ge 1$. (Hint: Use Exercise 17.49a.)

b) (+) When $0 < y < 1$, the integral defining $\Gamma(y)$ is also improper at the endpoint 0. Prove that this improper integral also converges.

c) Prove that $\Gamma(y+1) = y\Gamma(y)$.

d) Given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, evaluate $\int_0^\infty e^{-x^2}\,dx$.

e) (+ +) Prove that $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}\,dx = \sqrt{\pi}$.

18. THE COMPLEX NUMBERS

18.1. (−) Prove that $\mathbb{C}$ is a group under addition, with identity $(0, 0)$.

18.2. (−) *Multiplication of complex numbers.*

a) Prove that $(1, 0)$ is an identity for multiplication.

b) Prove that if $a^2+b^2 \neq 0$, then $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}) \cdot (a,b) = (1,0)$. (Comment: This proves that $z^{-1} = \overline{z}/|z|^2$.)

c) Prove that $\mathbb{C} - \{0\}$ is a group under multiplication.

18.3. (−) Prove that addition and multiplication of complex numbers are associative and commutative and satisfy the distributive law.

18.4. *Solutions to* $x^2+y^2=0$. In order to have $x^2=-y^2$, we must have $x = \pm iy$. When x and y must be real, this requires $x=y=0$. When they may be complex, the solution set is $\{(z,iz): z \in \mathbb{C}\} \cup \{(z,-iz): z \in \mathbb{C}\}$.

18.5. *Properties of complex numbers.* Consider $z, w \in \mathbb{C}$.

a) $|zw| = |z|\,|w|$. It suffices to show that $|zw|^2 = |z|^2|w|^2$. Since $|u|^2 = u\overline{u}$,

$$|zw|^2 = zw\overline{zw} = zw\overline{z}\,\overline{w} = z\overline{z}w\overline{w} = |z|^2|w|^2.$$

One can also write $z = x+iy$ and $w = a+ib$ and multiply out both sides.

b) $|z+w|^2 = |z|^2+|w|^2+2R_e(z\overline{w})$. Again using $|u|^2 = u\overline{u}$, we have

$$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+w\overline{z}+z\overline{w}+w\overline{w} = |z|^2+|w|^2+w\overline{z}+z\overline{w}$$

Since $u+\overline{u} = 2R_e(u)$, the result follows.

18.6. *If w_1 and w_2 are distinct points in $\mathbb{C}$, then the $\{z: |z-w_1| = |z-w_2|\}$ forms the perpendicular bisector of the segment joining w_1 and w_2.* The set described is the set of points equidistant from w_1 and w_2.

18.7. *Elementary properties of complex conjugation.* These all follow by writing $z = x+iy$, $w = a+ib$, and computing.

$$a)\overline{zw} = (\overline{ax-by+i(ay+bx)}) = (ax-by)-i(ay+bx) = \overline{z}\,\overline{w}.$$

$$b)\overline{z+w} = (\overline{a+x+i(b+y)}) = (a+x)-i(b+y) = \overline{z}+\overline{w}.$$

$$c)|\overline{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|.$$

18.8. (−) Suppose $z = (x,y) \in \mathbb{C}$. Prove that $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$.

18.9. *Cube roots of 1.* Suppose that $z^3 = 1$. In the form $z = re^{i\theta}$, we obtain $1 = z^3 = r^3e^{3i\theta}$. This yields $r=1$ and $3i\theta = 2n\pi i$. Thus $\theta = 2n\pi/3$. Taking $n = 0,1,2$ yields the roots $z \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. Since $e^{i\theta} = \cos\theta + i\sin\theta$, we set $\theta = 2\pi/3$ and $4\pi/3$ to obtain $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ as the roots other than 1.

18.10. *For sequences of complex numbers, $z_n \to A$ if and only if $R_e(z_n) \to R_e(A)$ and $I_m(z_n) \to I_m(A)$.* Write $A = a+ib$. If $z_n \to A$, then for any ε we

have $|z_n - A|^2 < \varepsilon^2$ for sufficiently large n. This yields $(x_n-a)^2+(y_n-b)^2 < \varepsilon^2$, which implies both $|x_n-a| < \varepsilon$ and $|y_n-b| < \varepsilon$. Hence both $R_e(z_n)$ and $I_m(z_n)$ converge, and $R_e(z_n) \to R_e(A)$ and $I_m(z_n) \to I_m(A)$.

Conversely, if both of these converge, for $\varepsilon > 0$ we can choose N so that $n > N$ imples that $|x_n-a| < e/\sqrt{2}$ and $|y_n-b| < e/\sqrt{2}$. Thus

$$|z_n-A|^2 = |x_n+iy_n-(a+ib)|^2 = +x_n-a|^2+|y_n-b|^2 < \varepsilon^2/2+\varepsilon^2/2 = \varepsilon^2.$$

We have $|z_n - A| < \varepsilon$ for $n > N$, as desired.

Since sequences of real numbers converge if and only if they are Cauchy, we obtain as a corollary that $\langle z \rangle$ converges if and only if it is Cauchy. Here we use that $|z_n - z_m|$ yields both $|x_n - xm| < \varepsilon$ and $|y_n - y_m| < \varepsilon$, and that $|x_n - x_m| < \varepsilon/\sqrt{2}$ and $|y_n - y_m| < \varepsilon/\sqrt{2}$ together yield $|z_n - z_m| < \varepsilon$.

18.11. $|z+w| = |z|+|w|$ *if and only if $z\overline{w}$ is real and nonnegative.* We have $|z+w|^2 = |z|^2+|w|^2+2\mathrm{Re}(z\overline{w})$, as in Exercise 18.5. Hence we have $|z+w| = |z|+|w|$ if and only if $2\mathrm{Re}(z\overline{w}) = 2\,|z|\,|w|$. Using $|z|\,|w| = |zw|$, the requirement becomes that the real part of $z\overline{w}$ must equal its magnitude, which requires $z\overline{w}$ to be real and nonnegative.

Application to Exercise 8.27. In terms of complex numbers, (a,b,c) is a Pythagorean triple if and only if $|a+ib|^2 = |c|^2$. If this also holds for (α,β,γ) and their sum, then $|\alpha+i\beta| = |\gamma|^2$ and $|(a+\alpha)+i(b+\beta)|^2 = |c+\gamma|^2$. We obtain $|(a+ib)+(\alpha+i\beta)| = |c+\gamma|$. Since c and γ are real, we have $|c+\gamma| = |c|+|\gamma| = |a+ib|+|\alpha+i\beta|$.

Hence we can apply the first part of the problem to conclude that $(a+ib)(\alpha-i\beta)$ is real and nonnegative. Being real requires that $b\alpha - a\beta = 0$ (the real part $a\alpha+b\beta$ is always positive, since a,b,α,β are positive integers). The condition $a\beta = b\alpha$ yields $a/\alpha = b/\beta = \lambda$ for some λ, and hence $(a,b,c) = \lambda(\alpha,\beta,\gamma)$.

18.12. *Trigonometry and the exponential function.*

a) If $z, w \in \mathbb{C}$, then $e^ze^w = e^{z+w}$. The term z^nw^m in the expansion of $e^{z+w} = \sum(z+w)^k/k!$ arises when $k = m+n$. By the binomial theorem, the coefficient is $\binom{m+n}{m}\frac{1}{(m+n)!} = \frac{1}{m!}\frac{1}{n!}$. This is precisely the coefficient that arises for z^nw^m in the expansion of $e^ze^w = \sum_n z^n/n! \sum_m w^m/m!$.

b) $\cos(n\theta)$ *and* $\sin(n\theta)$ *are polynomials in the variables* $\cos\theta$ *and* $\sin\theta$. From the defining formula $e^{i\theta} = \cos\theta + i\sin\theta$, part (a) yields $(e^{i\theta})^n = e^{in\theta}$. Hence $\cos(n\theta)+i\sin(n\theta) = (\cos\theta+i\sin\theta)^n$. Expanding the right side by the Binomial Theorem and collecting the real and imaginary parts yields polynomials for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of the variables $\cos\theta$ and $\sin\theta$.

c) $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ *and* $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$. Examining part (b) with $n = 3$ yields

$$\cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

With $\cos^2\theta + \sin^2\theta = 1$, the real part of the display yields $\cos(3\theta) = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$, and the imginary part yields $\sin(3\theta) = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$.

18.13. *Conjugation is a continuous function.* Given $\varepsilon > 0$, let $\delta = \varepsilon$. Now $|z-a| < \delta$ implies $|\overline{z} - \overline{a}| = |z-a| < \varepsilon$. Hence by definition the conjugation function is continuous.

$\exp(\overline{z}) = \overline{\exp(z)}$. By Exercise 18.7, $(\overline{z})^k = \overline{z^k}$ and $\sum_{k=0}^n \overline{w_k} = \overline{\sum_{k=0}^n w_k}$. Hence $\exp(\overline{z}) = \lim_{n\to\infty}\sum_{k=0}^n (\overline{z})^k/k! = \lim_{n\to\infty}(\overline{\sum_{k=1}^n z^k/k!})$. Since the conjugation function is continuous, we can bring the limit inside the application of the function to obtain $\exp(\overline{z}) = \overline{\lim_{n\to\infty}\sum_{k=0}^n z^k/k!} = \overline{\exp(z)}$.

18.14. $\left|e^{i\theta}\right| = 1$ *for all* $\theta \in \mathbb{R}$ *and* $e^{2\pi i} = 1$. Since $e^{i\theta} = \cos\theta + i\sin\theta$, we have $\left|e^{i\theta}\right|^2 = |\cos\theta|^2 + |\sin\theta|^2 = 1$ and $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$.

18.15. *Trigonometric integrals.*

a) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ *and* $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. In general, $\mathrm{Re} z = \frac{1}{2}(z+\overline{z})$ and $\mathrm{Im} z = \frac{1}{2i}(z-\overline{z})$ (Exercise 18.8). Also, $\overline{\exp(z)} = \exp(\overline{z})$ (Exercise 18.13). Applied to $\cos\theta = \mathrm{Re}\, e^{i\theta}$ and $\sin\theta = \mathrm{Im}\, e^{i\theta}$, these properties provide the computation.

b) $\int_0^{2\pi}(\cos\theta)^{2n}d\theta = \binom{2n}{n}\pi/2^{2n-1} = \int_0^{2\pi}(\sin\theta)^{2n}d\theta$. If k is a nonzero integer, then $\int_0^{2\pi} e^{ik\theta}d\theta = \frac{1}{ik}e^{ik\theta}\Big|_0^{2\pi} = 0$, but the integral is 2π when $k = 0$. Using this and part (a), we have

$$\int_0^{2\pi}(\cos\theta)^{2n}d\theta = \int_0^{2\pi}\frac{1}{2^{2n}}(e^{i\theta} + e^{-i\theta})^{2n}$$
$$= \frac{1}{2^{2n}}\int_0^{2\pi}\left(\sum_{r=0}^{2n}\binom{2n}{r}e^{(2n-2r)\theta}\right)d\theta = \frac{1}{2^{2n}}\binom{2n}{n}2\pi.$$

The computation using the sine function is quite similar.

c) Evaluation of $\int_0^{2\pi}(\cos\theta)^{2n}(\sin\theta)^{2m}d\theta$. Using part (b), we compute

$$\int_0^{2\pi}(\cos\theta)^{2n}(\sin\theta)^{2m}d\theta = \int_0^{2\pi}(\cos\theta)^{2n}(1-\cos^2\theta)^m d\theta$$
$$= \int_0^{2\pi}(\cos\theta)^{2n}\sum_{k=0}^m\binom{m}{k}(-1)^k\cos^{2k}(\theta)d\theta$$
$$= \sum_{k=0}^m(-1)^k\binom{m}{k}\int_0^{2\pi}\cos^{2(n+k)}\theta\ d\theta$$
$$= \sum_{k=0}^m(-1)^k\binom{m}{k}2\pi\binom{2n+2k}{n+k}\frac{1}{2^{2n+2k}}$$

Comment: The value of this integral can be expressed more simply and symmetrically using the gamma function defined in Exercise 17.52. Using the substitution $u = \sin^2\theta$, we have

$$\int_0^{2\pi}(\cos\theta)^{2n}(\sin\theta)^{2m}d\theta = 4\int_0^{\pi/2}(\cos\theta)^{2n}(\sin\theta)^{2m}d\theta = 4\int_0^1(1-u)^n u^m\frac{du}{2\sqrt{u(1-u)}}$$
$$= 2\int_0^1(1-u)^{m-1/2}u^{n-1/2}du = 2\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(m+n+1)}$$

The final step here takes considerable effort. To prove the identity $\int_0^1(1-u)^{a-1}u^{b-1}du = \Gamma(a)\Gamma(b)/\Gamma a + b$, one multiplies the definitions of $\Gamma(a)$ and $\Gamma(b)$ from Exercise 17.52, writes the result as a double integral, changes variables twice (once using polar coordinates), and obtains the product of two integrals recognizable as $\Gamma(a+b)$ and $\int_0^1(1-u)^{a-1}u^{b-1}du$. These operations are beyond what has been discussed in this text.

18.16. (•) Suppose z is a nonzero complex number and m is a positive integer. Prove that $w^m = z$ has the m distinct solutions $w = |z|^{1/m}e^{i(\theta+2k\pi)/m}$ for $0 \le k \le m-1$. Plot these solutions for the case $m = 8$ and $z = 256i$.

18.17. (•) Define $f\colon \mathbb{C} \to \mathbb{C}$ by $f(z) = iz$. Describe the functional digraph of f.

18.18. *For* $z \in \mathbb{C}$ *and* $z \ne 1$, $\sum_{k=0}^{n-1} z^k = (1-z^n)/(1-z)$. The formula holds if and only if $(1-z)\sum_{k=0}^{n-1} z^k = 1-z^n$. The left side is $\sum_{k=0}^{n-1} z^k - \sum_{k=0}^{n-1} z^{k+1}$, which simplifies to $1-z^n$ since the other terms cancel.

When $z^n = 1$, the identity states that $1 + z + z^2 + \cdots + z^{n-1} = 0$. We think of these summands, equally spaced around the unit circle, as forces exerted on the origin. The symmetry implies that the total force is 0.

18.19. *Formula for* $\prod_{k=0}^{n-1} z^k$ *when* $z^n = 1$. We have $\prod_{k=0}^{n-1} z^k = z^{\sum_{k=0}^{n-1} k} = z^{n(n-1)/2}$. If n is odd, then $(n-1)/2$ is an integer, and $z^{n(n-1)/2} = (z^n)^{(n-1)/2} =$

$1^{(n-1)/2} = 1$. If n is even, then $z^{n/2} \in \{+1, -1\}$. Now $z^{n(n-1)/2} = (\pm 1)^{(n-1)} = \pm 1$, with the same sign as $z^{n/2}$ since $n-1$ is odd.

18.20. ($\bullet$) Prove that the set of nth roots of 1, under multiplication, form a group "isomorphic" to $\mathbb{Z}_n$.

18.21. ($\bullet$) Use the characteristic equation method (Theorem 12.22) to solve the recurrence $a_n = -a_{n-2}$ for $a_0 = 2$ and $a_1 = 4$, giving a single formula for a_n.

18.22. ($\bullet$) Fill in the details of the proof in Theorem 18.17 that every closed rectangle in $\mathbb{C}$ is compact.

18.23. *For $w \in \mathbb{C}$ and $r \in \mathbb{R}$, the set $\{z \in \mathbb{C}\colon\ |z - w| \le r\}$ is closed.* We show that the complement is open. The complement of $\{z :\ |z - w| \le r\}$ is $A = \{z :\ |z - w| > r\}$. For $z \in A$, let $\varepsilon = |z - w| - r$, which is positive. For all $u \in \mathbb{C}$ such that $|u - z| < \varepsilon$, we have $|u - w| \ge |z - w| - |u - z| = r + \varepsilon - |u - z| > r$. Therefore $|u - z| < \varepsilon$ implies that $u \in A$, so A is open and its complement is closed.

18.24. ($\bullet$) Suppose $f\colon \mathbb{C} \to \mathbb{C}$. Prove that the following statements are equivalent:

A) f is continuous.
B) for every open set T, $I_f(T)$ is open.
C) for every closed set T, $I_f(T)$ is closed.
D) for each sequence $\langle z \rangle$, $z_n \to w$ implies $f(z_n) \to f(w)$.

18.25. ($\bullet$) Suppose f is a real-valued function that is defined and continuous on a compact subset S of $\mathbb{C}$. Prove that S contains an element at which f achieves its minimum on S. (Comment: this completes the proof of Theorem 18.22.)

18.26. ($\bullet$) Prove that on $\mathbb{C}$, every polynomial is continuous, the absolute value function is continuous, and the composition of continuous functions is continuous. Conclude that $|p|$ is continuous when p is a complex polynomial.

18.27. ($\bullet$) (Root test) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series. Let $L = limsup|a_n|^{1/n}$. Prove that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < 1/L$ and diverges if $|z| > 1/L$.

18.28. *Properties of the function f defined by $f(t) = (1 + it)^2/(1 + t^2)$.*

a) f maps $\mathbb{R}$ into the unit circle, minus the point -1. We compute

$$|f(t)|^2 = \left|\frac{1 - t^2}{1 + t^2}\right|^2 + \left|\frac{2t}{1 + t^2}\right|^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{(1 + t^2)^2} = 1.$$

Thus f maps into the unit circle. Note that $I_m(f(t)) = 0$ only when $t = 0$, in which case $f(t) = 1$, so the point -1 is not in the image. Similarly, images of positive t lie in the upper half-plane, images of negative t in the bottom half-plane, and increasing t^2 from 0 leads to all values of $R_e(f(t))$ between 1 and -1, so the rest of the unit circle lies in the image and the point -1 is approached as $t \to \pm\infty$.

The trigonometric relationship is $f(\tan(\theta/2)) = e^{i\theta}$. The triangle formed by the points $f(t)$, -1, and $R_e(f(t))$ is a right triangle with angle $\theta/2$. Its base has length $1 + \frac{1-t^2}{1+t^2}$, and its altitude has length $\frac{2t}{1+t^2}$.

18.29. (+) Let $2r + 1$ be an odd positive integer, and let ω be a complex number such that $\omega^{2r+1} = 1$ but that $\omega^n \ne 1$ when n is a natural number less than $2r + 1$. Obtain explicit formulas (as rational numbers times binomial coefficients) for the nonzero coefficients in the polynomial g defined by

$$g(x, y) = 1 - \prod_{j=0}^{2r}(1 - \omega^j x - \omega^{2j} y).$$

A. FROM N TO R

A.1. ($\bullet$) Establish a bijection between $\mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$ and the set of equivalence classes of $\mathbb{N} \times \mathbb{N}$ under the relation $\sim$ defined by putting $(a, b) \sim (c, d)$ if $a + d = b + c$.

A.2. ($\bullet$) Write an inductive definition of exponentiation by a natural number and prove that $x^{m+n} = x^m x^n$ when $x, m, n \in \mathbb{N}$.

A.3. ($\bullet$) Complete the proof of Theorem A.12 by verifying that multiplication in $\mathbb{Z}$ is well-defined, has identity element $[(n, n + 1)]$, and is commutative and associative.

A.4. *Multiplication of integers.*

The product of nonzero integers is nonzero, The definition of multiplication by k is $m_k(x) = x$ if $k = 1$ and $m_{k+1}(x) = m_k(x) + x$ if $k > 1$, and $m_{-k}(x) = -m_k(x)$ if $k > 0$, where $x \in \mathbb{Z}$.

First we prove by induction that if $k \in \mathbb{N}$, then $m_k(x)$ has the same sign as x (and hence is nonzero if x is nonzero). Basis: For $k = 1$, $m_k(x) = x$, which has the same sign as x. Inductive step: For $k \ge 1$, $m_{k+1}(x) = m_k(x) + x$, which by the induction hypothesis is the sum of two numbers with the same sign. If $y, x \in \mathbb{N}$, then $y + x \in \mathbb{N}$, and if $-y, -x \in \mathbb{N}$, then $-y - x \in \mathbb{N}$. Hence the sum has the same sign as x, and the claim holds also for $k + 1$, which completes the induction. Finally, if $k \in \mathbb{N}$, then $m_{-k}(x) =$

$-m_k(x)$, which by the preceding paragraph has opposite sign to x and hence is nonzero if x is nonzero.

Multiplication by a nonzero integer is an injective function from $\mathbb{Z}$ *to* $\mathbb{Z}$. If $m_k(x) = m_k(y)$, then the properties of subtraction tell us $m_k(x) - m_k(y) = 0$, which becomes $0 = kx - ky = k(x - y)$ by the distributive property. Since the product of any two nonzero integers is nonzero, this requires $x = y$ if k is nonzero, making m_k injective.

A.5. (•) Prove that multiplication by a natural number is an order-preserving function from $\mathbb{Z}$ to $\mathbb{Z}$ ($x > y$ implies $f(x) > f(y)$), and use this to prove that multiplication by a nonzero integer is an injective function from $\mathbb{Z}$ to $\mathbb{Z}$.

A.6. $\mathbb{Q}$ *is an ordered field.*

a) $\mathbb{Q}$ *is a group under addition, with* $0/1$ *being the identity element and* $-a/b$ *being the additive inverse of* a/b. **Closure**: this amounts to the statement that addition of rational numbers is well-defined. **Associative property**: $(a/b+c/d)+e/f = [(ad+bc)/(bd)]+e/f = [(ad+bc)f+bde]/(bdf) = (adf+bcf+bde)/(bdf) = [adf+b(cf+de)]/(bdf) = a/b+[(cf+de)/(df)] = a/b+(c/d+e/f)$, with each step valid by the definition of addition of rationals or the properties of arithmetic of integers. **Commutative property**: $(a/b + c/d) = (ad + bc)/bd = (cb + da)/db = c/d + a/b$, by the commutative properties of integer addition and integer multiplication. **Identity**: $a/b + 0/1 = (a1 + b0)/(b1) = a/b$, and similarly for addition in the other order. **Inverse**: $a/b + (-a/b) = [ab + b(-a)]/(bb) = 0/b^2 = 0/1$, by the properties of integer arithmetic and definition of equivalence of fractions.

b) $\mathbb{Q} - \{0\}$ *is a group under multiplication, with* $1/1$ *being the identity element and* d/c *being the multiplicative inverse of* c/d. **Closure**: this amounts to the statement that multiplication of rational numbers is well-defined. **Associative property**: $[(a/b)(c/d)](e/f) = [(ac)/(bd)](e/f) = (ace)/(bdf) = (a/b)[(ce)/(df)] = (a/b)[(c/d)(e/f)]$, by the definition of multiplication of rationals and the properties of arithmetic of integers. **Commutative property**: follows from the definition of rational multiplication by the commutative property of integer multiplication. **Identity**: $(a/b)(1/1) = (a1)/(b1) = a/b$, and similarly for multiplication in the other order. **Inverse**: $(a/b)(b/a) = (ab)/(ba) = 1/1$, by the properties of integer arithmetic and definition of equivalence of fractions.

c) The distributive law holds for rational arithmetic. We use the definitions of rational operations and the properties (such as the distributive law) of integer operations. We also use the cancelation property, which follows from the injectivity of multiplication by a nonzero integer. The cases where one or more of $\{a/b, c/d, e/f\}$ is zero should be handled separately; we omit these. $(a/b)(c/d + e/f) = (a/b)[(cf + de)/df] = a(cf + de)/bdf = (acf + ade)/bdf = acf/bdf + ade/bdf = ac/bd + ae/bf = (a/b)(c/d) + (a/b)(e/f)$.

A.7. (•) Define $\langle a\rangle$ by $a_1 = 2$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ for $n \in \mathbb{N}$. Prove that $\langle a\rangle$ is a Cauchy sequence of rational numbers. Prove that $\langle a\rangle$ has no limit in $\mathbb{Q}$. What does this say about Lemma A.23?

A.8. *The sum and difference of two Cauchy sequences are Cauchy sequences.* If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then we can choose N_1 such that $|a_m - a_n| < 1/(2M)$ for all $m, n > N_1$, and we can choose N_2 such that $|b_m - b_n| < 1/(2M)$ for all $m, n > N_2$. Let $N = \max\{N_1, N_2\}$. Then for all $m, n > N$, we have $|(a_m + b_m) - (a_n + b_n)| = |(a_m - a_n) + (b_m - b_n)| \le |a_m - a_n| + |b_m - b_n| < 1/(2M) + 1/(2M) = 1/M$. Since this argument holds for arbitrary $M \in \mathbb{N}$, the sequence $\{a_n + b_n\}$ satisfies the definition of Cauchy sequence. To modify this for $\{a_n - b_n\}$, the computation becomes $|(a_m - b_m) - (a_n - b_n)| = |(a_m - a_n) + (b_n - b_m)|$ and continues as above.

(•) Closure under multiplication and scalar multiplication.

A.9. (•) Prove that if a Cauchy sequence of rational numbers has a convergent subsequence, then $\langle a\rangle$ also converges and has the same limit.

A.10. (•) Prove that multiplication of real numbers is commutative and that addition and multiplication of real numbers are associative.

A.11. (•) Prove that **0** is an identity element for addition and that **1** is an identity element for multiplication of real numbers. Given $\alpha \in \mathbb{R}$ with $\alpha \neq \mathbf{0}$, prove that $\alpha + (-\alpha) = \mathbf{0}$ and that $\alpha \cdot \alpha^{-1} = \mathbf{1}$. Prove that $\mathbf{0} < \mathbf{1}$.

A.12. (•) Prove that the sum and the product of positive real numbers are positive.

A.13. (•) Prove that the limit of any convergent sequence of upper bounds for S is an upper bound for S.

A.14. (•) Suppose that $|a_{n+1} - a_n| \le M/2^n$ for some constant $M > 0$. Prove that $\langle a\rangle$ is a Cauchy sequence. (Hint: estimate $|a_m - a_n|$ by using a telescoping sum, and use the convergence of $\sum_{k=0}^{\infty} 1/2^k$.)

A.15. (•) Prove that the function f constructed in Theorem A.36 preserves the order relation on $\mathbb{Q}$.

A.16. (•) Prove that the function f constructed in Theorem A.36 is a bijection and preserves addition, multiplication and positivity on $\mathbb{R}$.

A.17. (•) Use the axioms of a complete order field (Definitions 1.39–1.41) to prove (some of) the properties that follow from them (Propositions 1.43–1.46).