LINEAR ALGEBRA II - LECTURE NOTES
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I will write these notes as the semester go along. Their goal is to present the material from the textbook and the class in a more concise form. I will often try to give slightly different phrasing than the one provided in the textbook. The intend is to make you think the concepts through and to work the concepts by yourself.

You are strongly encourage to do the exercises as you read. They intend to help you parse the definitions and examples that preceded. Some of these exercises will be assigned as Homework or will be discussed in class.

Points in red and blue are still being edited.

I will appreciate any comments. If you find mistakes, which are probably present, please let me know too.

1. Jan. 15: Vector spaces

Plan:

- Introduce the word field, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- Define vector spaces over $\mathbb{F}$.
- Examples: $\mathbb{F}^S$ for S a set.
- Proposition with basic properties: uniquess of additive identity and inverse, multiplication by 0 gives the zero vector, the zero vector is stable and -1 multiplication gives the additive inverse.

1.1. **Fields.** In your previous linear algebra class (Math 2101) you defined a vector space over the real numbers. The very same definition works in a slightly more general context, we start by introducing some terminology for that.

Definition 1. A *field* is a triple $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set, and we have operations (i.e. functions):

- addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$,
- multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$;

satisfying the following list of axioms:

(a) addition and multiplication are associative;
(b) addition and multiplication are commutative;
(c) there exists $0 \in \mathbb{F}$, such that $a + 0 = 0 + a = a$, for all $a \in \mathbb{F}$;
(d) there exists $1 \in \mathbb{F}$, such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in \mathbb{F}$;
(e) $0 \neq 1$;
(f) every $a \in \mathbb{F}$ has an *additive inverse*, i.e. an element $b \in \mathbb{F}$ such that $a + b = b + a = 0$;
(g) every $a \in \mathbb{F} \backslash \{0\}$ has a multiplicative inverse;
(h) distributivity, i.e. for every $a, b, c \in \mathbb{F}$ one has: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Notation 1. We will omit the $\cdot$ when writing the multiplication operation, i.e. for any $a, b \in \mathbb{F}$ we will write ab for $a \cdot b$.

Example 1. (i) The real numbers $\mathbb{R}$ form a field with usual addition and multiplication.
(ii) The complex numbers $\mathbb{C}$ form a field with usual addition and multiplication.
(iii) The *rational numbers* $\mathbb{Q} := \{\frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{Z}\backslash\{0\}\}$ are a field.

Exercise 1. Write out explicitly what conditions (a-b) and (g) above are and check them in one of the examples in Example 1.

Exercise 2. Let p be a prime number and consider $\mathbb{F}_p = \{0, 1 \ldots, p-1\}$, then for $a, b, c \in \mathbb{F}_p$ we define:

$$a + b := c \text{ if } (a + b - c) \text{ is a multiple of } p, \qquad a \cdot b := c \text{ if } (a \cdot b - c) \text{ is a multiple of } p.$$

(i) Check that $+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ and $\cdot : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ are well-defined.
(ii) Prove that $\mathbb{F}_p$ is a field.

Exercise 3. Can you come with another example of a field?

1.2. **Vector spaces.** In a previous Linear Algebra class you probably approached vector spaces by concrete examples. The main point of this class is to develop the theory from an abstract point of view focused on proofs, mostly basis-free, and applicable to general fields of characteristic zero, until later results that might require $\mathbb{F}$ to be the real or complex numbers.

Let $\mathbb{F}$ be a field.

Definition 2. A *vector space* over $\mathbb{F}$ is the data of

(i) a set V;
(ii) an operation $+ : V \times V \to V$;
(iii) a *scalar multiplication* operation $\cdot : \mathbb{F} \times V \to V$.

These are subject to the following axioms:

(a) the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and inverse;
(b) the operation $\cdot$ is associative;
(c) for every $v \in V$ one has $1 \cdot v = v$;
(d) scalar multiplication distributes over vector addition (i.e. the operation $+$ on V).

Example 2. (i) The set $\{0\}$ is a vector space over any field $\mathbb{F}$.

(ii) Given a set S consider $\mathbb{F}^S$ the set of functions $f : S \to \mathbb{F}$. The operations are defined by pointwise addition and multiplication, i.e. given $f, g \in \mathbb{F}^S$ and $a \in \mathbb{F}$ we let:
$$(f+g)(s) := f(s) + g(s), \qquad (a \cdot f)(s) := a \cdot f(s),$$
and $0_{\mathbb{F}^S}$ is the zero function.

(iii) For any $n \geq 1$, the set $\mathbb{F}^n$ is a vector space, where the operations are defined as follows. Let $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$, $w = (w_1, \ldots, w_n) \in \mathbb{F}$, and $a \in \mathbb{F}$, then:
$$v + w := (v_1 + w_1, \ldots, v_n + w_n), \qquad a \cdot v := (av_1, \ldots, av_n),$$
and $0_{\mathbb{F}^n} := (0, \ldots, 0)$.

Remark 1. A set G equipped with an operation $+ : G \times G \to G$ satisfying condition (a) above is an *Abelian group*. These objects are very important in algebra and are studied in more details in an abstract algebra course, e.g. Math3301 (Algebra I).

Lemma 1. *Let V be a vector space over $\mathbb{F}$.*

(1) Given $v \in \mathbb{F}$ such that $v + w = w$ for all $w \in \mathbb{F}$, then $v = 0_V$.
(2) The additive inverse is unique.
(3) For every $v \in V$, we have $0 \cdot v = 0_V$.
(4) For every $a \in \mathbb{F}$, we have $a \cdot 0_V = 0_V$.
(5) For every $v \in V$ we have $v + (-1) \cdot v = 0$, i.e. the additive inverse of v is given by $-v := (-1) \cdot v$.

Proof. (1) We have $v = v + 0_V = 0_V$. First equality follows from Definition 2 (a) the second from the assumption of the Lemma.

(2) Assume there exists $u_1, u_2 \in V$ such that $u_1 + v = 0_V = u_2 + v$. Then we have: $u_1 = u_1 + 0_V = u_1 + u_2 + v = u_2 + u_1 + v = u_2 + 0_V = u_2$.

(3) Notice $v + 0 \cdot v = (1 + 0) \cdot v = 1 \cdot v = v$. Thus by (1), we have $0 \cdot v = 0_V$.

(4) For any $a \in \mathbb{F}$, we have: $a \cdot 0_V = a \cdot (0_V + 0_V) = a \cdot 0_V + a \cdot 0_V$. By (1), we have $a \cdot 0_V = 0_V$.

(5) Notice $v + (-1) \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0_V$, where in the last step we used (3).

□

Notation 2. (1) Notice that for $a, b \in \mathbb{F}$ and $v \in V$ we have:
$$(ab) \cdot v = a \cdot (b \cdot v)$$
by Defintition 2 (b). Thus, we can omit the $\cdot$ for the operation of scalar multiplication as we omitted it for multiplication in a field (see) without causing ambiguity.

(2) We will denote the additive inverse of v by $-v$.

(3) We will denote 0_V simply by 0. This should not cause confusion with $0 \in \mathbb{F}$ the identity of the operation $+$ in $\mathbb{F}$, as these live in different sets, except when $V = \mathbb{F}$, in which case the notation is consistent.

Remark 2. (i) The empty set $\emptyset$ is not a vector space. Namely, it fails condition (a) from Definition 2.

(ii) Condition (a) from Definition 2 can be substituted by (a)' the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and (3) from Lemma 1 holds.

Indeed, assume (a)', then we have $0_V = 0 \cdot w = (1 + (-1)) \cdot w = w + (-1)w$ for every $w \in V$. Thus (a) holds.

Example 3. Let V be a vector space over $\mathbb{R}$. We can define a vector space over the complex numbers $V_{\mathbb{C}}$, called the *complexification* of V as follows:

- as a set we let $V_{\mathbb{C}} := V \times V$;
- $+ : V_{\mathbb{C}} \times V_{\mathbb{C}} \to V_{\mathbb{C}}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
- scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

The reader should check that $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.

Exercise 4. Universal property of complexification. Let V be a vector space over $\mathbb{R}$ and W a vector space over $\mathbb{C}$. Notice that W can be seen as a vector space over $\mathbb{R}$, where $a \cdot w := (a + i0) \cdot w$, i.e. using the natural inclusion of $\mathbb{R}$ into $\mathbb{C}$. Let $\mathrm{Hom}_{\mathbb{R}}(V, W)$ denote the set of linear operators between V and W, where W is seen as a vector space over $\mathbb{R}$ and let $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ denote the set of linear operator between $V_{\mathbb{C}}$ and W as vector spaces over $\mathbb{C}$. Prove that there exists a bijection:

$$\mathrm{Hom}_{\mathbb{R}}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W).$$

2. Jan. 18, 2024

Plan:

- define subspaces and give plenty of examples;
- define sum and direct sum of vector subspaces and prove their properties;
- define span, example of degree of polynomial and definition of finite-dimensional vector space;
- linear independence with properties;
- subspaces of finite-dimensional vector spaces are finite-dimensional.
- define basis, how to construct basis and every finite-dimensional vector space has a basis.

2.1. **Subspaces.**

Definition 3. Let V be a vector space, a subset $U \subseteq V$ is said to be a *subspace* if:

(a) $0 \in U$;

(b) the restrictions $+_U : U \times U \to V$ and $\cdot_U : \mathbb{F} \times U \to V$ factors as:

$$\begin{tikzcd} U \times U \arrow[r, "+'_U"] \arrow[dr, "+_U"'] & U \arrow[d] \\ & V \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \times U \arrow[r, "\cdot'_U"] \arrow[dr, "\cdot_U"'] & U \arrow[d] \\ & V \end{tikzcd} .$$

Given a subspace $U \subseteq V$ we will simply write $+ : U \times U \to U$ and $\cdot : \mathbb{F} \times U \to U$ for $+'_U$ and $\cdot'_U$, respectively.

Exercise 5. (i) Check that Definition 3 agrees with Definition (1.33) from the textbook.

(ii) Show that only requiring condition (b) in Definition 3 would not agree with the notion as defined in the textbook.

Example 4. (i) let $U \subset \mathbb{F}^n$ defined as $U := \{(v_1, \ldots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}$;

(ii) let $p \in \mathbb{F}[x, y, z]$ be a polynomial of the form $p(x, y, z) = ax + by + cz$, for some constants $a, b, c \in \mathbb{F}$, then $U := \{(v_1, v_2, v_3) \in \mathbb{F}^3 \mid p(v_1, v_2, v_3) = 0\}$ is a subspace;

(iii) the subset of functions $f : [0, 1] \to \mathbb{R}$ which are continuous is a subspace of all the functions from $[0, 1]$ to $\mathbb{R}$;

(iv) let $U \subset \mathbb{F}[x]$ denote the subset of polynomials p such that $p(0) = p'(0) = \cdots = p^{(k)}(0) = 0$;

(v) the set of all sequences of complexes numbers whose limit is 0 is a subspace of $\mathbb{C}^\infty$.

Exercise 6. (i) Let $p \in \mathbb{R}[x, y, z]$ be a polynomial of degree 1 and define the subset:

$$U_p := \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid p(v_1, v_2, v_3) = 0\}.$$

Show that U_p is a subspace if and only if p is of the form taken in (ii) of Example 4.

(ii) With the notation as in (i), assume that $p_1, p_2 \in \mathbb{R}[x,y,z]$ are polynomials of degree 1 with no constant term, prove that
$$U_{p_1} \cap U_{p_2} = U_{p_1 p_2}$$
is a subspace of $\mathbb{R}^3$.

(iii) Can you guess which types of polynomials $p \in \mathbb{R}[x,y,z]$ have the property that U_p is a subspace of $\mathbb{R}^3$.

Definition 4. Given $U_1, U_2 \subseteq V$ two subspaces of V we define the *sum* $U_1 + U_2 \subseteq V$ as the subset of elements $v \in V$ such that there exist $u_1 \in U_1$ and U_2 such that $u_1 + u_2 = v$. For $U_1, \dots, U_k$ a collection of k subspaces of V, we inductively define:[1]
$$U_1 + \cdots + U_k := U_1 + (U_2(\cdots + U_k)).$$

Example 5. (i) let $U_i = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_i \neq 0\}$ for $i = 1,2,3,4$. Then
$$U_2 + U_3 + U_4 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}, \qquad U_1 + U_2 + U_3 + U_4 = \mathbb{F}^4.$$

(ii) let $U_1 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_3 + v_4 = 0 \text{ and } v_1 + v_2 = 0\}$, let $U_2 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}$, then $U_1 + U_2 = \mathbb{F}^4$.

Exercise 7. In the condition in Definition 4 are the vectors u_1 and u_2 uniquely determined? Compare (i) and (ii) in Example 4.

Definition 5. Given $U_1, U_2 \subseteq V$ two subspaces of V we say that $U_1 + U_2$ is a *direct sum* if u_1 and u_2 are uniquely determined. In this case we use the notation $U_1 \oplus U_2$[2]. Similarly, given subspaces $U_1, \dots, U_k$ we define:
$$U_1 \oplus \cdots \oplus U_k := U_1 \oplus (U_2(\cdots \oplus U_k)).$$

Lemma 2. *Given subspaces $U_1, \dots, U_k$, then $U_1 + \cdots + U_k$ is a direct sum if and only if $U_i \cap U_j = \{0\}$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$.*

Proof. We start with $k = 2$.

First we prove that the condition is necessary. Let $U_1 + U_2 = U_1 \oplus U_2$ and consider $v \in U_1 + U_2$, written as $v = u_1 + u_2$ for some $u_1 \in u_1$ and $u_2 \in u_2$. Assume by contradiction that $U_1 \cap U_2 \neq \{0\}$. Then there exists a non-zero vector $w \in U_1 \cap U_2$ such that
$$v = (u_1 + w) + (-w + u_2), \text{ where } u_1 + w \in U_1 \text{ and } -w + u_2 \in U_2.$$
This shows that u_1, u_2 are not unique, so we get a contradiction.

Now assume that $U_1 \cap U_2 = \{0\}$. Again by contradiction suppose that there exists $v \in U_1 + U_2$ which can be written as:
$$v = u_1 + u_2 \text{ and } v = u_1' + u_2',$$
for $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$, such that $u_1 \neq u_1'$ and $u_2 \neq u_2'$. Then consider $w = u_1 - u_1'$. Notice that $w \in U_1$ and, since $w = u_2' - u_2$, we have that $w \in U_2$. As $w \neq 0$ we obtain a contradiction with $U_1 \cap U_2 = \{0\}$.

The general case follows by induction, we leave the details to the reader. ☐

Exercise 8. Let $V = \mathbb{F}^4$. Provide three distinct subspaces $U_1, U_2, U_3 \subseteq V$ such that:
$$V_1 + V_2 = V_1 \oplus V_2, V_2 + V_3 = V_2 \oplus V_3, \text{ but } V_1 + V_2 + V_3 \neq V_1 \oplus V_2 \oplus V_3.$$

2.2. **Span and linear dependence.**

Definition 6. Given a subset $S \subseteq V$ we define $\operatorname{Span} S$ the *span of* S to be the subset of V consisting of vectors $v \in V$ such that
$$v = a_1 u_1 + \ldots + a_k u_k$$
for some $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{F}$, and $v_1, \dots, v_k \in S$. It is convenient to define $\operatorname{Span} \emptyset = \{0\}$.

Remark 3. It is clear that $\operatorname{Span} S$ is a vector space and that it contains S. We claim that $\operatorname{Span} S$ is the smallest subspace of V containing S. Consider a subspace $U \subseteq V$ such that $S \subseteq U$, we claim that $\operatorname{Span} S \subseteq U$. Indeed, given $v \in \operatorname{Span} S$ we have $v = v = a_1 u_1 + \ldots + a_k u_k$ for some $a_1, \dots, a_k \in \mathbb{F}$, and $v_1, \dots, v_k \in S$. Since $u_1, \dots, u_k \in U$ we have $v \in U$. Thus, it follows that $\operatorname{Span} S$ is in the intersection of all subspaces of V containing $\operatorname{Span} S$.

[1] In fact, this definition is independent of the choice of parenthesization, hence justifying the notation.

[2] At the moment this notation is not totally motivated, it will be clearer when we consider this operation on vector spaces.

Example 6. (i) Consider $\{e_1, \dots, e_n\} \subseteq \mathbb{F}^n$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 is in the ith position. Then $\operatorname{Span}\{e_1, \dots, e_n\} = \mathbb{F}^n$.

(ii) Let $a, b, c \in \mathbb{F}$ and consider $S = \{(b, -a, 0), (0, c, -b)\}$, then we have $\operatorname{Span} S = U_p$, where U_p is defined as in Example 4 (ii).

Exercise 9. Let e_i be as in Example 6 (i) and consider the set $S = \{je_i - ie_j\}_{1=i<j=n}$, then

$$\operatorname{Span} S = \{(v_1, \dots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}.$$

Definition 7. A vector space U is *finite-dimensional* if there exists a finite subset $S \subseteq U$ such that $\operatorname{Span} S = U$.

Example 7. (i) $\mathbb{F}^n$ is finite-dimensional.

Exercise 10. Check which of the examples of vector spaces defined so far are finite-dimensional.

Definition 8. (1) A *polynomial* with coefficients in $\mathbb{F}$ is a function $p : \mathbb{F} \to \mathbb{F}$ such that

$$p(x) = a_0 + a_1 x + \cdots a_n x^n \tag{1}$$

for some $n \in \mathbb{N}$ and $a_i \in \mathbb{F}$.

(2) We let $\mathbb{F}[x]$ denote the set of polynomials in $\mathbb{F}$[3].

(3) Given a polynomial $p \in \mathbb{F}[x]$ the *degree* of p is the smallest natural number $n \in \mathbb{N}$ such that p can be written as (1). By convention we set the degree of the zero polynomial to be $-\infty$.

(4) Let $\mathcal{P}_n(\mathbb{F})$ denote the set of polynomials of degree at most n.

Exercise 11. Check that $\mathcal{P}_n(\mathbb{F})$ forms a vector space.

Exercise 12. Assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $p : \mathbb{F} \to \mathbb{F}$ be a function. Check that $f \in \mathcal{P}_n(\mathbb{F})$ if and only if $p^{(n+1)} = 0$.

Exercise 13. The set $\mathbb{F}[x] = \mathcal{P}(\mathbb{F})$ is a vector space. Think about how we can formally define this.

Definition 9. Let V be a vector space over $\mathbb{F}$. Given a finite subset $S = \{v_1, \dots, v_n\} \subset V$ we say that S is *linearly independent* if

$$a_1 v_1 + \cdots + a_n v_n = 0 \implies a_1 = \cdots = a_n = 0,$$

where $a_1, \cdots, a_n \in \mathbb{F}$. By convention we declare that $S = \emptyset$ is linearly independent. We say that a subset $S \subset V$ is linearly dependent if it is not linearly independent.

Example 8. (i) For every $k \in \{1, \dots, n\}$, the set $S = \{e_1, \dots, e_k\} \subset \mathbb{F}^n$, where e_i's are defined as in Example 6 (i), is linearly independent.

(ii) For any $k \geq 0$ the set $S_k := \{1, x, \dots, x^k\} \subset \mathbb{F}[x]$ is linearly independent.

(iii) Given $\{v, w\} \subset V$, then $\{v, w\}$ is linearly independent if and only if $n \neq aw$ for every $a \in \mathbb{F}$.

Exercise 14. Given a finite subset $S \subset V$. Prove that S if $0 \in S$ then S is linearly dependent.

Example 9. (1) the subset $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_1\} \subset \mathbb{F}^3$ is linearly dependent.

(2) the subset $S = \{x^2, x^2 - 2x, 3x\} \subset \mathbb{F}[x]$ is linearly dependent.

Exercise 15. Given $S = \{(2, 3, 1), (1, -1, 2), (7, 3, c)\} \subset \mathbb{F}^3$. Check that S is linearly independent if and only if $c = 8$.

Exercise 16. Given $\{v_1, v_2, v_3, v_4\} \subset V$ a linearly independent set. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.

The next result is extremely useful in many future proofs since it allows one to make a linearly dependent set smaller.

Lemma 3. *Let $\{v_1, \dots, v_n\} \subset V$ be a linearly dependent subset of a vector space V. Then there exists $k \in \{1, \dots, n\}$ such that*

$$v_k \in \operatorname{Span}\{v_1, \dots, v_{k-1}\}.$$

In this case, one has:

$$\operatorname{Span}\{v_1, \dots, v_n\} = \operatorname{Span}\{v_1, \dots, v_n\} \backslash \{v_k\}.$$

[3] The textbook uses the notation $\mathcal{P}(\mathbb{F})$.

Proof. Since $\{v_1, \ldots, v_n\}$ is linearly dependent there exists $a_1, \ldots, a_n \in \mathbb{F}$ not all zero such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Thus, let $k \in \{1, \ldots, n\}$ such that $a_k \neq 0$, then we have:

$$v_k = -a_k^{-1}(a_1 v_1 + \cdots + a_{k-1} v_{k-1} + a_{k+1} v_{k+1} + \cdots + a_n v_n),$$

where the expression on the right above works for $k \in \{2, \ldots, n-1\}$, we leave it to the reader to write the correct expression for the edge cases. To prove the last assertion we notice that clearly $\operatorname{Span}\{v_1, \ldots, v_n\} \backslash \{v_k\} \subseteq \operatorname{Span}\{v_1, \ldots, v_n\}$. Now suppose that $w \in \operatorname{Span}\{v_1, \ldots, v_n\}$ and let $w = a_1 v_1 + \cdots + a_n v_n$. Since $v_k \in \operatorname{Span}\{v_1, \ldots, v_{k-1}\}$, there exists $b_1, \ldots, b_{k-1} \in \mathbb{F}$ such that $v_k = b_1 v_1 + \cdots + b_{k-1} v_{k-1}$. Then

$$w = (a_1 + b_1) v_1 + \cdots + (a_{k-1} + b_{k-1}) v_{k-1} + \sum_{i=k+1}^{n} a_i v_i,$$

so $w \in \operatorname{Span}\{v_1, \ldots, v_n\} \backslash \{v_k\}$. This finishes the proof. □

Definition 10. Let $T \subseteq S \subset V$ be two finite subsets of a vector space. We say that T is a *spanning subset of* S if

$$\operatorname{Span} T = \operatorname{Span} S.$$

Lemma 4. *Consider $R, T \subseteq S \subset V$ finite subsets of a vector space V. Suppose that $\operatorname{Span} T = \operatorname{Span} S$ and that R is linearly independent. Then $|R| \leq |T|$.*

Proof. See the textbook (2.22). □

Corollary 1. *Let $U \subseteq V$ be a subset of a finite-dimensional vector space V, then U is finite-dimensional.*

Proof. We do an induction on the number of vectors necessary to span U. The base case is $U = \operatorname{Span} \emptyset = \{0\}$, in which case U is finite-dimensional. Assume that $U \neq \{0\}$ and let $v_1 \in U$ be a non-zero vector. Then if $U = \operatorname{Span} v_1$ we are done, otherwise there exists $v_2 \in U$ such that $v_2 \notin \operatorname{Span} v_1$ and we can consider $\operatorname{Span}\{v_1, v_2\}$. We claim that repeating this step k times gives $\operatorname{Span}\{v_1, \ldots, v_k\} = U$ for some $k \in \mathbb{N}$. Indeed, let $S \subset V$ be a finite set such that $\operatorname{Span} S = V$. Such set exists since V is finite-dimensional. Then consider $\{v_1, \ldots, v_k\} \subseteq \{v_1, \ldots, v_k\} \cup S$. Since $S \subseteq \{v_1, \ldots, v_k\} \cup S$ spans V, we have that $k \leq |S|$, thus k is finite. □

2.3. **Basis.**