LINEAR ALGEBRA II
SPRING 2024 - HKU

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These notes will be updated as the semester progresses. Their goal is to present the material from the textbook and the class in a more concise form. I will often try to give slightly different phrasing (and/or proofs) than the one provided in the textbook. The intent is to make you think the concepts through and to work the concepts by yourself.

You are strongly encouraged to do the exercises as you read. They will help you parse the definitions, examples, and concepts used in the proofs of the theory. Some of these exercises will be assigned as Homework or will be discussed in class.

Points in red and blue are still being edited.

I would appreciate any comments. If you find mistakes, which are probably present, please let me know too. I normally revise part of the notes after the class in which we discussed the material, so please refer frequently to the website for the most up-to-date version.

1. Jan. 15, 2024

1.1. **Fields.** In your previous linear algebra class (Math 2101) you defined a vector space over the real numbers. The very same definition works in a slightly more general context, we start by introducing some terminology for that.

Definition 1. A *field* is a triple $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set, and we have operations (i.e. functions):

- addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$,
- multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$;

satisfying the following list of axioms:

(a) addition and multiplication are associative;

(b) addition and multiplication are commutative;

(c) there exists $0 \in \mathbb{F}$, such that $a + 0 = 0 + a = a$, for all $a \in \mathbb{F}$;

(d) there exists $1 \in \mathbb{F}$, such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in \mathbb{F}$;

(e) $0 \neq 1$;

(f) every $a \in \mathbb{F}$ has an *additive inverse*, i.e. an element $b \in \mathbb{F}$ such that $a + b = b + a = 0$;

(g) every $a \in \mathbb{F}\backslash\{0\}$ has a multiplicative inverse;

(h) distributivity, i.e. for every $a, b, c \in \mathbb{F}$ one has: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Notation 1. We will omit the $\cdot$ when writing the multiplication operation, i.e. for any $a, b \in \mathbb{F}$ we will write ab for $a \cdot b$.

Example 1. (i) The real numbers $\mathbb{R}$ form a field with usual addition and multiplication.

(ii) The complex numbers $\mathbb{C}$ form a field with usual addition and multiplication.

(iii) The *rational numbers* $\mathbb{Q} := \{\frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{Z}\backslash\{0\}\}$ are a field.

Exercise 1. Write out explicitly what conditions (a-b) and (g) above are and check them in one of the examples in Example 1.

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Exercise 2. Let p be a prime number and consider $\mathbb{F}_p = \{0, 1 \ldots, p-1\}$, then for $a, b, c \in \mathbb{F}_p$ we define:

$$a + b := c \text{ if } (a + b - c) \text{ is a multiple of } p, \qquad a \cdot b := c \text{ if } (a \cdot b - c) \text{ is a multiple of } p.$$

(i) Check that $+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ and $\cdot : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$ are well-defined.

(ii) Prove that $\mathbb{F}_p$ is a field.

Exercise 3. Can you come up with another example of a field?

1.2. **Vector spaces.** In a previous Linear Algebra class you probably approached vector spaces by concrete examples. The main point of this class is to develop the theory from an abstract point of view focused on proofs, mostly basis-free, and applicable to general fields of characteristic zero, until later results that might require $\mathbb{F}$ to be the real or complex numbers.

Let $\mathbb{F}$ be a field.

Definition 2. A *vector space* over $\mathbb{F}$ is the data of

(i) a set V;

(ii) an operation $+ : V \times V \to V$;

(iii) a *scalar multiplication* operation $\cdot : \mathbb{F} \times V \to V$.

These are subject to the following axioms:

(a) the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and inverse;

(b) the operation $\cdot$ is associative;

(c) for every $v \in V$ one has $1 \cdot v = v$;

(d) scalar multiplication distributes over vector addition (i.e. the operation $+$ on V) and vector addition distributes over scalar multiplication[1].

Example 2. (i) The set $\{0\}$ is a vector space over any field $\mathbb{F}$.

(ii) Given a set S consider $\mathbb{F}^S$ the set of functions $f : S \to \mathbb{F}$. The operations are defined by pointwise addition and multiplication, i.e. given $f, g \in \mathbb{F}^S$ and $a \in \mathbb{F}$ we let:

$$(f + g)(s) := f(s) + g(s), \qquad (a \cdot f)(s) := a \cdot f(s),$$

and $0_{\mathbb{F}^S}$ is the zero function.

(iii) For any $n \geq 1$, the set $\mathbb{F}^n$ is a vector space, where the operations are defined as follows. Let $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$, $w = (w_1, \ldots, w_n) \in \mathbb{F}$, and $a \in \mathbb{F}$, then:

$$v + w := (v_1 + w_1, \ldots, v_n + w_n), \qquad a \cdot v := (av_1, \ldots, av_n),$$

and $0_{\mathbb{F}^n} := (0, \ldots, 0)$.

(iv) For any $n, m \geq 1$ the set $M_{m\times n}(\mathbb{F})$ of $m \times n$ matrices with coefficients in $\mathbb{F}$ equipped with matrix addition and scalar multiplication is a vector space over $\mathbb{F}$.

(v) The set $\mathbb{F}^{\mathbb{N}}$ of sequences with value in $\mathbb{F}$ is a vector space with termwise addition and scalar multiplication.

Remark 1. A set G equipped with an operation $+ : G \times G \to G$ satisfying condition (a) above is an *Abelian group*. These objects are very important in algebra and are studied in more detail in an abstract algebra course, e.g. Math3301 (Algebra I).

Lemma 1. *Let V be a vector space over $\mathbb{F}$.*

(1) Given $v \in V$ such that $v + w = w$ for all $w \in V$, then $v = 0_V$.

[1] See Exercise 1 in Worksheet 1 for an example where this fails.

(2) The additive inverse is unique.

(3) For every $v \in V$, we have $0 \cdot v = 0_V$.

(4) For every $a \in \mathbb{F}$, we have $a \cdot 0_V = 0_V$.

(5) For every $v \in V$ we have $v + (-1) \cdot v = 0$, i.e. the additive inverse of v is given by $-v := (-1) \cdot v$.

Proof. (1) We have $v = v + 0_V = 0_V$, where the first equality follows from Definition 2 (a) and the second from the assumption.

(2) Assume there exists $u_1, u_2 \in V$ such that $u_1 + v = 0_V = u_2 + v$. Then we have: $u_1 = u_1 + 0_V = u_1 + u_2 + v = u_2 + u_1 + v = u_2 + 0_V = u_2$.

(3) Notice $v + 0 \cdot v = (1+0) \cdot v = 1 \cdot v = v$. Thus by (1), we have $0 \cdot v = 0_V$.

(4) For any $a \in \mathbb{F}$, we have: $a \cdot 0_V = a \cdot (0_V + 0_V) = a \cdot 0_V + a \cdot 0_V$. By (1), we have $a \cdot 0_V = 0_V$.

(5) Notice $v + (-1) \cdot v = (1 + (-1)) \cdot v = 0 \cdot v = 0_V$, where in the last step we used (3).

□

Notation 2. (1) Notice that for $a, b \in \mathbb{F}$ and $v \in V$ we have:

$$(ab) \cdot v = a \cdot (b \cdot v)$$

by Defintition 2 (b). Thus, we can omit the $\cdot$ for the operation of scalar multiplication as we omitted it for multiplication in a field (see) without causing ambiguity.

(2) We will denote the additive inverse of v by $-v$.

(3) We will denote 0_V simply by 0. This should not be confused with $0 \in \mathbb{F}$ the identity of the operation $+$ in $\mathbb{F}$, as these live in different sets, except when $V = \mathbb{F}$, in which case the notation is consistent.

Remark 2. (i) The empty set $\emptyset$ is not a vector space. Namely, it fails condition (a) from Definition 2.

(ii) Condition (a) from Definition 2 can be substituted by

(a)' the operation $+$ is associative, commutative, it admits an identity $0_V \in V$ and (3) from Lemma 1 holds.

Indeed, assume (a)', then we have $0_V = 0 \cdot w = (1 + (-1)) \cdot w = w + (-1)w$ for every $w \in V$. Thus (a) holds.

Example 3. Let V be a vector space over $\mathbb{R}$. We can define a vector space over the complex numbers $V_{\mathbb{C}}$, called the *complexification* of V as follows:

- as a set we let $V_{\mathbb{C}} := V \times V$;
- $+ : V_{\mathbb{C}} \times V_{\mathbb{C}} \to V_{\mathbb{C}}$ is given by $(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$;
- scalar multiplication is defined as $(a + bi) \cdot (u_1, v_1) = (au_1 - bv_1, bu_1 + av_1)$.

The reader should check that $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.

Exercise 4. Universal property of complexification. Let V be a vector space over $\mathbb{R}$ and W a vector space over $\mathbb{C}$. Notice that W can be seen as a vector space over $\mathbb{R}$, where $a \cdot w := (a + i0) \cdot w$, i.e. using the natural inclusion of $\mathbb{R}$ into $\mathbb{C}$. Let $\mathrm{Hom}_{\mathbb{R}}(V, W)$ denote the set of linear operators between V and W, where W is seen as a vector space over $\mathbb{R}$ and let $\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W)$ denote the set of linear operator between $V_{\mathbb{C}}$ and W as vector spaces over $\mathbb{C}$. Prove that there exists a bijection:

$$\mathrm{Hom}_{\mathbb{R}}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W).$$

2. Jan. 18, 2024

2.1. Subspaces.

Definition 3. Let V be a vector space, a subset $U \subseteq V$ is said to be a *subspace* if:

(a) $0 \in U$;

(b) the restrictions $+_U : U \times U \to V$ and $\cdot_U : \mathbb{F} \times U \to V$ factors as:

$$\begin{tikzcd} U \times U \arrow[r, "+'_U"] \arrow[dr, "+_U"'] & U \arrow[d] \\ & V \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F} \times U \arrow[r, "\cdot'_U"] \arrow[dr, "\cdot_U"'] & U \arrow[d] \\ & V \end{tikzcd}.$$

Given a subspace $U \subseteq V$ we will simply write $+ : U \times U \to U$ and $\cdot : \mathbb{F} \times U \to U$ for $+'_U$ and $\cdot'_U$, respectively.

Exercise 5. (i) Check that Definition 3 agrees with Definition (1.33) from the textbook.

(ii) Show that only requiring condition (b) in Definition 3 would not agree with the notion as defined in the textbook.

Example 4. (i) let $U \subset \mathbb{F}^n$ defined as $U := \{(v_1, \dots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}$;

(ii) let $p \in \mathbb{F}[x, y, z]$ be a polynomial of the form $p(x, y, z) = ax + by + cz$, for some constants $a, b, c \in \mathbb{F}$, then $U := \{(v_1, v_2, v_3) \in \mathbb{F}^3 \mid p(v_1, v_2, v_3) = 0\}$ is a subspace;

(iii) the subset of functions $f : [0, 1] \to \mathbb{R}$ which are continuous is a subspace of all the functions from $[0, 1]$ to $\mathbb{R}$;

(iv) let $U \subset \mathbb{F}[x]$ denote the subset of polynomials p such that $p(0) = p'(0) = \cdots = p^{(k)}(0) = 0$;

(v) the set of all sequences of complex numbers whose limit is 0 is a subspace of $\mathbb{C}^{\mathbb{N}}$.

Exercise 6. (i) Let $p \in \mathbb{R}[x, y, z]$ be a polynomial of degree 1 and define the subset:

$$U_p := \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid p(v_1, v_2, v_3) = 0\}.$$

Show that U_p is a subspace if and only if p is of the form taken in (ii) of Example 4.

(ii) With the notation as in (i), assume that $p_1, p_2 \in \mathbb{R}[x, y, z]$ are polynomials of degree 1 with no constant term, prove that

$$U_{p_1} \cup U_{p_2} = U_{p_1 p_2}.$$

Is $U_{p_1 p_2}$ a subspace of $\mathbb{R}^3$? What about $U_{p_1} \cap U_{p_2}$?

(iii) Can you guess which types of polynomials $p \in \mathbb{R}[x, y, z]$ have the property that U_p is a subspace of $\mathbb{R}^3$.

Exercise 7. Let $\mathbb{F}^{\mathbb{N}}$ be the vector space of sequences over $\mathbb{F}$. For an integer $p \geq 1$, we define the subset $S_p \subset \mathbb{F}^{\mathbb{N}}$ of sequences $(a_n)_{n \geq 1}$ satisfying:

$$\sum_{n=1}^{\infty} |a_n|^p < \infty.$$

Proof or disproof S_p is a subspace for every integer $p \geq 1$.

Definition 4. Given $U_1, U_2 \subseteq V$ two subspaces of V we define the *sum* $U_1 + U_2 \subseteq V$ as the subset of elements $v \in V$ such that there exist $u_1 \in U_1$ and U_2 such that $u_1 + u_2 = v$. For $U_1, \dots, U_k$ a collection of k subspaces of V, we inductively define:[2]

$$U_1 + \cdots + U_k := U_1 + (U_2(\cdots + U_k)).$$

Example 5. (i) let $U_i = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_j = 0 \text{ for } j \neq i\}$ for $i = 1, 2, 3, 4$. Then

$$U_2 + U_3 + U_4 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}, \qquad U_1 + U_2 + U_3 + U_4 = \mathbb{F}^4.$$

[2] In fact, this definition is independent of the choice of parenthesization, hence justifying the notation.

(ii) let $U_1 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_3 + v_4 = 0 \text{ and } v_1 + v_2 = 0\}$, let $U_2 = \{(v_1, v_2, v_3, v_4) \in \mathbb{F}^4 \mid v_1 = 0\}$, then $U_1 + U_2 = \mathbb{F}^4$.

Exercise 8. In the condition in Definition 4 are the vectors u_1 and u_2 uniquely determined? Compare (i) and (ii) in Example 4.

Definition 5. Given $U_1, U_2 \subseteq V$ two subspaces of V we say that $U_1 + U_2$ is a *direct sum* if u_1 and u_2 are uniquely determined. In this case, we use the notation $U_1 \oplus U_2$[3]. Similarly, given subspaces $U_1, \ldots, U_k \subset V$ we say that $U_1 + \cdots + U_k$ is a direct sum if any vector $v \in U_1 + \cdots + U_k$ can be written in an unique way as $v = u_1 + \cdots u_k$, where $u_j \in U_j$ for $1 \le j \le k$. We denote the direct sum by $U_1 \oplus \cdots \oplus U_k$.

Exercise 9. Let $V = \mathbb{F}^4$. Provide three distinct subspaces $U_1, U_2, U_3 \subseteq V$ such that:

$$V_1 + V_2 = V_1 \oplus V_2, V_2 + V_3 = V_2 \oplus V_3, \text{ but } V_1 + V_2 + V_3 \neq V_1 \oplus V_2 \oplus V_3.$$

Remark 3. Let $U_1, \ldots, U_k \subseteq V$ be a family of subspaces, then we have $U_1 + \cdots + U_k$ is a direct sum if and only if for every $j \in \{1, \ldots k\}$ we have $U_j \cap (\sum_{i=1, i\neq j}^n U_i) = \{0\}$, where $\sum_{i=1, i\neq j}^n U_i$ denotes the sum of $U_1, \ldots, U_k$ where we omit U_j.

2.2. **Span and linear dependence.**

Definition 6. Given a subset $S \subseteq V$ we define $\operatorname{Span} S$ the *span of* S to be the subset of V consisting of vectors $v \in V$ such that

$$v = a_1 v_1 + \ldots + a_k v_k$$

for some $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{F}$, and $v_1, \ldots, v_k \in S$. It is convenient to define $\operatorname{Span} \emptyset = \{0\}$. If $\operatorname{Span} S = V$ we say that S *spans* V.

Remark 4. It is clear that $\operatorname{Span} S$ is a vector space and that it contains S. We claim that $\operatorname{Span} S$ is the smallest subspace of V containing S. Consider a subspace $U \subseteq V$ such that $S \subseteq U$, we claim that $\operatorname{Span} S \subseteq U$. Indeed, given $v \in \operatorname{Span} S$ we have $v = v = a_1 u_1 + \ldots + a_k u_k$ for some $a_1, \ldots, a_k \in \mathbb{F}$, and $v_1, \ldots, v_k \in S$. Since $u_1, \ldots, u_k \in U$ we have $v \in U$. Thus, it follows that $\operatorname{Span} S$ belongs to the intersection of all subspaces of V containing $\operatorname{Span} S$.

Example 6. (i) Consider $\{e_1, \ldots, e_n\} \subseteq \mathbb{F}^n$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 is in the ith position. Then $\operatorname{Span}\{e_1, \ldots, e_n\} = \mathbb{F}^n$.

(ii) Let $a, b, c \in \mathbb{F}$ and consider $S = \{(b, -a, 0), (0, c, -b)\}$, then we have $\operatorname{Span} S = U_p$, where U_p is defined as in Example 4 (ii).

Exercise 10. Let e_i be as in Example 6 (i) and consider the set $S = \{je_i - ie_j\}_{1 \le i < j \le n}$, then

$$\operatorname{Span} S = \{(v_1, \ldots, v_n) \in \mathbb{F}^n \mid v_1 + 2v_2 + \cdots + nv_n = 0\}.$$

Also notice that S is not a basis in general. How could you change it to be a basis?

Definition 7. A vector space U is *finite-dimensional* if there exists a finite subset $S \subseteq U$ such that $\operatorname{Span} S = U$.

Example 7. (i) $\mathbb{F}^n$ is finite-dimensional;

(ii) the set $\mathcal{P}_n(\mathbb{F})$ of polynomials of degree at most n;

(iii) for any S a finite set $\mathbb{F}^S$ is a finite-dimensional vector space.

Exercise 11. Check which of the examples of vector spaces defined so far are finite-dimensional.

Definition 8. (1) A *polynomial* with coefficients in $\mathbb{F}$ is a function $p : \mathbb{F} \to \mathbb{F}$ such that

$$p(x) = a_0 + a_1 x + \cdots a_n x^n \tag{1}$$

for some $n \in \mathbb{N}$ and $a_i \in \mathbb{F}$.

(2) We let $\mathbb{F}[x]$ denote the set of polynomials in $\mathbb{F}$, notice that the textbook uses the notation $\mathcal{P}(\mathbb{F})$.

[3]At the moment this notation might seem unmotivated, but it will be clearer when we consider this operation on vector spaces.

(3) Given a polynomial $p \in \mathbb{F}[x]$ the *degree* of p is the smallest natural number $n \in \mathbb{N}$ such that p can be written as (1). By convention, we set the degree of the zero polynomial to be $-\infty$.

(4) Let $\mathcal{P}_n(\mathbb{F})$ denote the set of polynomials of degree at most n.

Exercise 12. Check that $\mathcal{P}_n(\mathbb{F})$ forms a vector space.

Exercise 13. Assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $p : \mathbb{F} \to \mathbb{F}$ be a function. Check that $p \in \mathcal{P}_n(\mathbb{F})$ if and only if $p^{(n+1)} = 0$.

Exercise 14. The set $\mathbb{F}[x] = \mathcal{P}(\mathbb{F})$ is a vector space. Think about how we can formally define this.

Definition 9. Let V be a vector space over $\mathbb{F}$. Given a finite subset $S = \{v_1, \dots, v_n\} \subset V$ we say that S is *linearly independent* if

$$a_1v_1 + \cdots + a_nv_n = 0 \implies a_1 = \cdots = a_n = 0,$$

where $a_1, \cdots, a_n \in \mathbb{F}$. By convention, we declare that $S = \emptyset$ is linearly independent. We say that a subset $S \subset V$ is linearly dependent if it is not linearly independent.

Example 8. (i) For every $k \in \{1, \dots, n\}$, the set $S = \{e_1, \dots, e_k\} \subset \mathbb{F}^n$, where e_i's are defined as in Example 6 (i), is linearly independent.

(ii) For any $k \geq 0$ the set $S_k := \{1, x, \dots, x^k\} \subset \mathbb{F}[x]$ is linearly independent.

(iii) Given $\{v, w\} \subset V$, then $\{v, w\}$ is linearly independent if and only if $v \neq aw$ for every $a \in \mathbb{F}$ and $bv \neq w$ for every $b \in \mathbb{F}$.

Exercise 15. Given a finite subset $S \subset V$. Prove that S if $0 \in S$ then S is linearly dependent.

Example 9. (1) the subset $S = \{e_1 - e_2, e_2 - e_3, e_3 - e_1\} \subset \mathbb{F}^3$ is linearly dependent.

(2) the subset $S = \{x^2, x^2 - 2x, 3x\} \subset \mathbb{F}[x]$ is linearly dependent.

Exercise 16. Given $S = \{(2, 3, 1), (1, -1, 2), (7, 3, c)\} \subset \mathbb{F}^3$. Check that S is linearly dependent if and only if $c = 8$.

Exercise 17. Given $\{v_1, v_2, v_3, v_4\} \subset V$ a linearly independent set. Prove that $\{v_1, v_1+v_2, v_1+v_2+v_3, v_1+v_2+v_3+v_4\}$ is a linearly independent set.

The next result is extremely useful in many future proofs since it allows one to make a linearly dependent set smaller.

Lemma 2. *Let $\{v_1, \dots, v_n\} \subset V$ be a linearly dependent subset of a vector space V. Then there exists $k \in \{1, \dots, n\}$ such that*

$$v_k \in \operatorname{Span}\{v_1, \dots, v_{k-1}\},$$

when $k = 1$ the right-hand side above should be interpreted as $\operatorname{Span}\emptyset$. Moreover, one has:

$$\operatorname{Span}\{v_1, \dots, v_n\} = \operatorname{Span}\{v_1, \dots, v_n\} \backslash \{v_k\}.$$

Proof. Since $\{v_1, \dots, v_n\}$ is linearly dependent there exists $a_1, \dots, a_n \in \mathbb{F}$ not all zero such that

$$a_1v_1 + \cdots + a_nv_n = 0.$$

Thus, let $k \in \{1, \dots, n\}$ such that $a_k \neq 0$, then we have:

$$v_k = -a_k^{-1}(a_1v_1 + \cdots + a_{k-1}v_{k-1} + a_{k+1}v_{k+1} + \cdots + a_nv_n),$$

where the expression on the right above works for $k \in \{2, \dots, n-1\}$, we leave it to the reader to write the correct expression for the edge cases. To prove the last assertion we notice that clearly $\operatorname{Span}\{v_1, \dots, v_n\} \backslash \{v_k\} \subseteq \operatorname{Span}\{v_1, \dots, v_n\}$. Now suppose that $w \in \operatorname{Span}\{v_1, \dots, v_n\}$ and let $w = a_1v_1 + \cdots + a_nv_n$. Since $v_k \in \operatorname{Span}\{v_1, \dots, v_{k-1}\}$, there exists $b_1, \dots, b_{k-1} \in \mathbb{F}$ such that $v_k = b_1v_1 + \cdots + b_{k-1}v_{k-1}$. Then

$$w = (a_1 + b_1)v_1 + \cdots + (a_{k-1} + b_{k-1})v_{k-1} + \sum_{i=k+1}^{n} a_iv_i,$$

so $w \in \operatorname{Span}\{v_1, \dots, v_n\} \backslash \{v_k\}$. This finishes the proof. $\square$

Definition 10. Let $T \subseteq V$ be a subset of a vector space. We say that T is a *spanning set of* V if $\operatorname{Span} T = V$.

Lemma 3. *Let V be a finite-dimensional vector space. Consider $S, T \subseteq V$ subsets of a vector space V. Suppose that $\operatorname{Span} S = V$ and that T is a linearly independent subset. Then $|T| \leq |S|$.*

Proof. Let $v_1 \in T$ and consider $S \cup \{v_1\}$. Since $\operatorname{Span} S = V$ we have that $v_1 \in \operatorname{Span} S$, so $S \cup \{v_1\}$ is linearly dependent. By Lemma 2 there exists $u_1 \in S$ such that $\operatorname{Span} S \cup \{v_1\} = \operatorname{Span} S \cup \{v_1\} \backslash \{u_1\}$. Now let $T_1 := T \backslash \{v_1\}$, $S_1' := S \backslash \{u_1\}$, and $S_1 := S_1' \cup \{v_1\}$.

Let $v_2 \in T_1$ and consider $S_1 \cup \{v_2\}$, as argued in the previous paragraph we can find $u_2 \in S_1$ such that $\operatorname{Span} S_1 = \operatorname{Span}(S_1 \cup \{v_2\} \backslash \{u_2\})$. Then we let $T_2 := T_1 \backslash \{v_2\}$, $S_2' := S_1' \backslash \{u_1, u_2\}$, and $S_2 := S_1 \cup \{v_2\}$. Notice that we can repeat this process k times, where $k = |T|$ to obtain two sequences:

$$\emptyset \subset T_k \subset \cdots \subset T_1 \subset T, \qquad \text{and} \qquad S_k' \subset \cdots \subset S_1' \subset S$$

where $\operatorname{Span} S_i = \operatorname{Span} S$, $|S_i'| = |S| - i$ and $|T_i| = |T| - i$ for every $i \in \{1, \ldots, k\}$. This implies that $|S| \geq k = |T|$.

This is the same argument as in (2.22) in the textbook. $\square$

Corollary 1. *Let $U \subseteq V$ be a subset of a finite-dimensional vector space V, then U is finite-dimensional.*

Proof. We do an induction on the number of vectors necessary to span U. The base case is $U = \operatorname{Span} \emptyset = \{0\}$, in which case U is finite-dimensional. Assume that $U \neq \{0\}$ and let $v_1 \in U$ be a non-zero vector. Then if $U = \operatorname{Span} v_1$ we are done, otherwise there exists $v_2 \in U$ such that $v_2 \notin \operatorname{Span} v_1$ and we can consider $\operatorname{Span}\{v_1, v_2\}$. We claim that repeating this step k times gives $\operatorname{Span}\{v_1, \ldots, v_k\} = U$ for some $k \in \mathbb{N}$. Indeed, let $S \subset V$ be a finite set such that $\operatorname{Span} S = V$. Such a set exists since V is finite-dimensional. Then consider $\{v_1, \ldots, v_k\} \subseteq \{v_1, \ldots, v_k\} \cup S$. Since $S \subseteq \{v_1, \ldots, v_k\} \cup S$ spans V, we have that $k \leq |S|$, thus k is finite. $\square$

3. Jan. 22, 2024

3.1. **Basis.** The following concept is extremely important in linear algebra. One could say that the main difference between this course and Math2101 is that in Math2101 one is choosing a basis for every vector space that is considered by default, whereas in Math2102 we are not.

Definition 11. A subset $S \subset V$ is a *basis* if it satisfies:

(a) $\operatorname{Span} S = V$;

(b) S is linearly independent.

Example 10. (i) The set $\{e_1, \ldots, e_n\}$ as defined in Example 6 (i) is a basis of $\mathbb{F}^n$.

(ii) The set $\{1, \ldots, x^4\}$ is a basis of $\mathcal{P}_4(\mathbb{F})$ the vector space of polynomials of degree at most 4.

(iii) The sets $\{(7,5), (-4,9)\}$ and $\{(1,2), (3,5)\}$ are both basis of $\mathbb{F}^2$.

Remark 5. A subset $S = \{v_1, \ldots, v_n\} \subset V$ is a basis of V if and only if every element $u \in V$ can be written as:

$$u = a_1 v_1 + \cdots + a_n v_n$$

for an unique choice of $a_1, \ldots, a_n \in \mathbb{F}$. Indeed, suppose that there are two n-uples $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{F}^n$ such that

$$u = a_1 v_1 + \cdots + a_n v_n, \qquad u = b_1 v_1 + \cdots + b_n v_n,$$

and $a_i \neq b_i$ for some $i \in \{1, \ldots, n\}$. Then we have:

$$\begin{aligned} 0 = u - u &= (a_1 v_1 + \cdots + a_n v_n) - (b_1 v_1 + \cdots + b_n v_n) \\ &= (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n. \end{aligned}$$

Since S is linearly independent, we have that $a_i = b_i$ for all $i \in \{1, \ldots, n\}$.

One of the consequences of Lemma 2 is that any finite spanning set contains a subset which is a basis.

Lemma 4. *Let $T \subset V$ be a finite spanning subset of V. Then there exists $S \subseteq T$ such that S is a basis.*

Proof. We proceed by downward induction. If T is linearly independent we are done. If T is linearly dependent, by Lemma 2 there exists $v \in T$ such that $\operatorname{Span} T \backslash \{v\} = \operatorname{Span} T = V$ and $|T \backslash \{v\}| < |T|$. Since T is finite this process stops and we obtain a basis. □

We get two immediate consequences:

Corollary 2. *(1) Every finite-dimensional vector space V admits a basis.*

(2) Any linearly independent subset $S = \{v_1, \dots, v_k\} \subset V$ extends to a basis.

Proof. For (1) let T be a finite set such that $\operatorname{Span} T = V$. By Lemma 4 there exists $S \subseteq T$ such that S is a basis of V.

For (2) let $T = \{w_1, \dots, w_n\}$ be a finite set such that $\operatorname{Span} T = V$. Then $\operatorname{Span} S \cup T = V$. Order the set $T \cup S$ as follows $\{v_1 < v_2 < \cdots < v_k < w_1 < \cdots < w_n\}$, then running the argument in the proof of Lemma 4 we notice that we obtain a subset $R \subset V$ such that:

$$S \subseteq R \subset S \cup T \qquad \text{and} \qquad \operatorname{Span} R = V.$$

□

The following result is interesting because it uses that $\mathbb{F}$ is a field in a serious way. In other words, certain concepts so far would make sense for more general objects as (commutative) rings, i.e. the integers $\mathbb{Z}$, however, the following result is on of the first to fail.

Lemma 5. *Let V be a finite vector space and consider a subspace $U \subseteq V$. Then there exists a subspace $W \subseteq V$ such that $U \oplus W = V$.*

Proof. Notice that U is also finite-dimensional. Let T be a basis for U (it exists by Corollary 2 (1)). By Corollary 2 (2) we can find $T \subset R$ such that R is a basis of V. We claim that $W := \operatorname{Span} R \backslash T$ satisfies $U \oplus W = V$. Indeed, it is clear that $U + W = V$, by Lemma **??** we need to check that $U \cap W = \{0\}$. We give names to the elements of $U = \{v_1, \dots, v_k\}$ and $W = \{v_{k+1}, \dots, v_n\}$. Assume by contradiction that there exists a non-zero vector $v \in U \cap W$, then we have

$$v = a_1 v_1 + \cdots a_k v_k = a_{k+1} v_{k+1} + \cdots + a_n v_n.$$

Thus, $a_1 v_1 + \cdots a_k v_k - (a_{k+1} v_{k+1} + \cdots + a_n v_n) = 0$, and since $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is linearly independent, we have that $a_i = 0$ for all $i \in \{1, \dots, n\}$. So we get a contradiction with $U \cap W \neq \{0\}$. This finishes the proof. □

3.2. **Dimension.** The notion of dimension is rather intuitive. The next result justifies that one can define it in a naïve way.

Lemma 6. *Given T, S two basis of a vector space V, we have $|S| = |T|$.*

Proof. Notice that S and T are both linearly independent sets and spanning sets for V. Thus, Lemma 3 implies that $|S| \leq |T|$ and $|T| \leq |S|$. □

Definition 12. The *dimension* of a vector space V, denoted by $\dim V$, is the size of any basis of V.

Exercise 18. Go through all the examples of finite-dimensional vector spaces we had so far and find out their dimension.

Here are a couple of easy consequences of the defintion.

Lemma 7. *Assume that V is finite-dimensional.*

(1) For any subspace $U \subseteq V$, we have $\dim U \leq \dim V$.

(2) Let $S \subseteq V$ be a linearly independent set, if $|S| = \dim V$, then S is a basis.

(3) Given a subspace $U \subseteq V$ such that $\dim U = \dim V$, then $U = V$.

(4) Let $S \subseteq V$ such that $\operatorname{Span} S = V$ and $|S| = \dim V$, then S is a basis.

Proof. (1) Let $S \subset U$ be a basis of U. Notice that $S \subset V$ is also linearly independent. Now Lemma 3 implies that $|S| \leq |T|$ for any basis T of V, i.e. $|S| \leq \dim V$.

(2) Assume that S is not a basis, then by Corollary 2 (2) there exists $S \subset S'$ such that S' is a basis. However, this would imply that $\dim V = |S| < |S'|$ where S' is a basis of V, which is a contradiction.

(3) Let $S \subset U$ be a basis of U. Since $S \subset V$ is linearly independent in V and $|S| = \dim U = \dim V$, by (2) we have S is a basis of V. Thus, $U = \operatorname{Span} S = V$.

(4) Assume that S is not a basis, i.e. S is linearly dependent, then by Lemma 2 there exist $v \in S$ such that $\operatorname{Span} S \backslash \{v\} = V$. This gives $\dim V \leq |S| - 1$, which is a contradiction. ☐

Now we investigate how the notion of dimension interacts with sums of subspaces.

Lemma 8. *Let $U_1, U_2 \subset V$ be subspaces of V. Then we have:*

$$\dim U_1 + U_2 = \dim U_1 + \dim U_2 - \dim U_1 \cap U_2.$$

Proof. Let S_{12} be a basis of $U_1 \cap U_2$. By Corollary 2 (2) there exists $S_{12} \subset S_1$ and $S_{12} \subset S_2$ such that S_i is a basis of U_i, for $i = 1, 2$. We claim that $S_1 \cup S_2$ is a basis of $U_1 + U_2$. Clearly, we have $S_1 \cup S_2 \subset U_1 + U_2$, this gives that $\operatorname{Span} S_1 \cup S_2 \subset U_1 + U_2$. Since $U_1 \subseteq \operatorname{Span} S_1 \cup S_2$ and $U_2 \subseteq \operatorname{Span} S_1 \cup S_2$, we obtain $\operatorname{Span} S_1 \cup S_2 = U_1 + U_2$.

Now, we need to check that $S_1 \cup S_2$ is linearly independent. For this we actually need to give names to the elements of S_{12}, S_1 and S_2. Let $S_{12} = \{u_1, \ldots, u_i\}$, $S_1 \backslash S_{12} = \{v_1, \ldots, v_j\}$ and $S_2 \backslash S_{12} = \{w_1, \ldots, w_k\}$. Suppose we have an equation:

$$a_1 u_1 + \cdots + a_i u_i + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0,$$

for some $a_1, \ldots, a_i, b_1, \ldots, b_j, c_1, \ldots, c_k \in \mathbb{F}$. Then solving for $w := c_1 w_1 + \cdots + c_k w_k$ we have that $w \in U_1$. But this also gives that $w \in U_2$. Thus, there exist some scalars $d_1, \ldots, d_i$ such that

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots d_i u_i.$$

Now, since $\{w_1, \ldots, w_k\} \cup \{u_1, \ldots, u_i\} = S_2$ is a linearly independent set, we get that all c_ℓ's vanish. Thus we have that $a_1 u_1 + \cdots + a_i u_i + b_1 v_1 + \cdots + b_j v_j = 0$. Since $S_1 = \{v_1, \ldots, v_j\} \cup \{u_1, \ldots, u_i\}$ is linearly independent, we get that all a_ℓ's and b_ℓ's also vanish. This finishes the proof. ☐

The previous result is an example of how questions about vector subspaces can be reduced to set-theoretic questions by using bases. We will return to this in later sections.

Exercise 19. Let V be a ten-dimensional vector space.

(1) Suppose that $U_1, U_2 \subset V$ are subspaces of dimension 6. Prove that there exists two vectors $u_1, u_2 \in U_1 \cap U_2$ such that neither is a scalar multiple of the other.

(2) Suppose that $U_1, U_2, U_3 \subset V$ are subspaces such that $\dim U_1 = \dim U_2 = \dim U_3 = 7$, prove that $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

4. Jan. 25, 2024

4.1. **Linear Maps.** Most objects we encounter in mathematics only have "real" meaning when compared in an appropriate way to other objects of the same type. For instance, when study sets we are naturally lead to studying functions and comparing sets using them.

Vector spaces, a more structure kind of set, need to be compared with each other using a more structured kind of function. We introduce that now:

Definition 13. Let V and W be two vector spaces. A *linear map*, sometimes also called a *linear transformation*, is a function $T : V \to W$ satisfying:

(a) (additivity) $T(u + v) = T(u) + T(v)$ for every $u, v \in U$;

(b) (homogeneity) $T(au) = aT(u)$ for every $a \in \mathbb{F}$ and $u \in U$.

We will let $\mathcal{L}(V, W)$ denote the set of linear maps between V and W, and simply write $\mathcal{L}(V) := \mathcal{L}(V, V)$ for the set of linear maps from V to itself. Also notice that for any linear map T, one has $T(0) = 0$.

Example 11. Let V be a vector space.

(i) The zero map $0 : V \to V$, where $0(V) := 0_V = 0$ is a linear map.

(ii) The identity map $\mathrm{Id}_V : V \to V$ given by $\mathrm{Id}_V(v) = v$.

(iii) Given any $a \in \mathbb{F}$ then $T_a(v) := a \cdot v$ is a linear map.

(iv) Differentiation $D : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear map, i.e. $D(p) := p'$.

(v) Integration $T_{[0,1]} : \mathbb{R}[x] \to \mathbb{R}$ given by $T_{[0,1]}(p) := \int_0^1 p(x)dx$.

(vi) Let $q \in \mathbb{R}[x]$ then $T_q : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $p(x) \mapsto p(q(x))$ is a linear map.

(vii) Let consider a collection of scalars $a_{i,j} \in \mathbb{F}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. $a_{1,1}, \dots, a_{n,1}, a_{1,2}, a_{2,2}, \dots, a_{n,2}, \dots, a_{1,m}, \dots, a_{n,m} \in \mathbb{F}$, then

$$T(v_1, \dots, v_n) := (a_{1,1}v_1 + \cdots + a_{1,n}v_n, \dots, a_{m,1}v_1 + \cdots + a_{m,n}v_n)$$

is a linear map.

Exercise 20. Prove that any linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ is of the form given in Example 11.(vii).

Lemma 9. *Let V and W be finite-dimensional vector spaces and $\{v_1, \dots, v_n\} \subset V$ be a basis of V. Given any subset $\{w_1, \dots, w_n\} \subset W$ there exists an unique linear map $T : V \to W$ such that*

$$T(v_i) = w_i \qquad \textit{for } i \in \{1, \dots, n\}. \tag{2}$$

Proof. We first define T. Given any $v \in V$ can be written as $v = a_1v_1 + \cdots + a_nv_n$ we let $Tv := c_1w_1 + \cdots + c_nw_n$. Notice that this is well-defined, since there is only one single way of written v as above and that it satisfies the conditions required. It is clear that it is a linear operator, we leave the details to be checked to the reader. Finally, assume that there exists $T' \in \mathcal{L}(V, W)$ satisfying equations (2). Then for any $a_i \in \mathbb{F}$ we have $T'(a_iv_i) = a_iw_i$, thus for any $v \in V$, which can be written uniquely as $v = a_1v_1 + \cdots + a_nv_n$ we obtain:

$$T'(v = a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n = T(v).$$

This finishes the proof of uniqueness.

We leave the details to check that T is a linear map to the reader. □

We now observe that the set $\mathcal{L}(V, W)$ can be naturally endowed with the structure of a vector space.

Lemma 10. *For any two vector spaces V, W the set $\mathcal{L}(V, W)$ is a vector space with addition and scalar multiplication defined as:*

$$(T_1 + T_2)(v) := T_1(v) + T_2(v) \qquad \textit{and} \qquad (a \cdot T)(v) := a \cdot T(v).$$

Proof. The details are left to the reader. □

Exercise 21. Given U, V, and W vector spaces we can consider the composition operation:

$$\begin{aligned} (-) \circ (-) : \mathcal{L}(V, W) \times \mathcal{L}(U, V) &\to \mathcal{L}(U, W) \\ (S, T) &\mapsto S \circ T(v) := S(T(v)). \end{aligned}$$

(i) Check that $\circ$ defined above is a linear map.

(ii) Check that the operation $\circ$ is associative and that it has identity elements. Part of the exercise is making sense of what that means.

Remark 6. This is an abstract remark and can be skipped as we will not use this concept in this course.. A mathematical concept that is really helpful in organizing certain mathematical objects is that of a category. You can look up its definition here. We essentially just showed that vector spaces together with linear maps form a category Vect. In fact, the category Vect has many nice properties.

Exercise 22. Let V be a vector space, such that $\dim V > 1$. Prove that there exists $T, S \in \mathcal{L}(V)$ such that $ST \neq TS$.

4.2. **Null spaces and ranges.** In this subsection we define subspaces that are naturally associated to a linear operator.

Definition 14. Let $T : V \to W$ be a linear map.

(1) the *null space* of T:
$$\operatorname{null} T := \{v \in V \mid Tv = 0\};$$

(2) the *range* of T:
$$\operatorname{range} T := \{w \in W \mid Tv = w, \text{ for some } v \in V\}.$$

We check that $\operatorname{null} T$ and $\operatorname{range} T$ are in fact subspaces of V and W, respectively. Let $u, v \in \operatorname{null} T$ we have
$$T(u + v) = T(u) + T(v) = 0, \qquad \text{and} \qquad T(a \cdot u) = a \cdot T(u) = a \cdot 0 = 0.$$

Assume that $w_1, w_2 \in \operatorname{range} T$, then there exist $u_1, u_2 \in U$ such that $T(u_i) = w_i$, for $i = 1, 2$. Then we have
$$T(u_1 + u_2) = w_1 + w_2, \qquad \text{and} \qquad T(a \cdot u_1) = a \cdot w_1,$$
thus $\operatorname{range} T$ is a vector space.

Remark 7. The null space is sometimes also called *kernel* of T. The range is sometimes called *image* of T.

The next results shows how the kernel and range related to the notions of injective and surjective.

Lemma 11. *Let $T : V \to W$ be a linear operator.*

(i) T is injective if and only if $\operatorname{null} T = \{0\}$*;*

(ii) T is surjective if and only if $\operatorname{range} T = W$*.*

Proof. For (i), first assume that T is injective. Assume that there exists a non-zero vector $v \in \operatorname{null} T$. However, this is a contradiction with $Tv \neq 0$, since $v \neq 0$. Now assume that T is injective, and suppose that $Tv = 0$. Since $Tv = T(0) = 0$ and T is injective, we get $v = 0$. For (ii) there is nothing to check. □

Theorem 1 (Fundamental theorem of linear maps)**.** *Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V, W)$. Then*
$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof. Let $\{u_1, \dots, u_n\}$ be a basis of $\operatorname{null} T$. By Lemma 2 (2) we can extend it to $\{u_1, \dots, u_n, u_{n+1}, \dots, u_{n+m}\}$ a basis of V. We claim that $\{Tu_{n+1}, \dots, Tu_{n+m}\}$ is a basis of $\operatorname{range} T$. Indeed, let $w \in W$, then there exist $v \in V$ such that $Tv = w$. Since $\{u_1, \dots, u_n, u_{n+1}, \dots, u_{n+m}\}$ is a basis of V, there exists scalars a_i's such that $v = a_1 u_1 + \cdots + a_{n+m} u_{n+m}$ and we have:
$$w = T(v) = T(a_1 u_1 + \cdots + a_{n+m} u_{n+m}) = a_1 T(u_1) + \cdots a_n T(u_n) + a_{n+1} T(u_{n+1}) + \cdots + a_{n+m} T(u_{n+m}).$$
Since the first n terms vanish, we get $w = a_{n+1} T(u_{n+1}) + \cdots + a_{n+m} T(u_{n+m})$. This shows that $\{Tu_{n+1}, \dots, Tu_{n+m}\}$ is a spanning set. We now check that it is also linearly independent. Assume that there exists scalars $b_1, \dots, b_m \in \mathbb{F}$, not all zero, such that we have
$$b_1 T(u_{n+1}) + \cdots + b_m T(u_{n+m}) = 0$$
this implies that $b_1 u_{n+1} + \cdots + b_m u_{n+m} \in \operatorname{null} T$. However, since $\{u_1, \dots, u_n\}$ is a basis of $\operatorname{null} T$, it means there are scalars $c_1, \cdots, c_n \in \mathbb{F}$, not all zero, such that:
$$c_1 u_1 + \cdots + c_n u_n = b_1 u_{n+1} + \cdots + b_m u_{n+m}.$$
However, this is a contradiction with $\{u_1, \dots, u_{n+m}\}$ being a basis of V.

This finishes the proof. □

Here are a couple of easy consequences of the previous result.

Corollary 3. *Let V and W be finite-dimensional vector spaces.*

(1) Assume that $\dim V > \dim W$*, then any linear map $T : V \to W$ is not injective.*

(2) *Assume that* $\dim V < \dim W$*, then any linear map* $T : V \to W$ *is not surjective.*

Proof. (1) Assume by contradiction that there exist $T : V \to W$ an injective linear map. By Lemma 11.(1) we have $\operatorname{null} T = 0$ and Theorem 1 implies $\dim V = \dim rangeT \leq \dim W$, which is a contradiction. (2) Assume by contradiction that there exist $T : V \to W$ a surjective linear map. Then Lemma 11.(2) and Theorem 1 implies that $\dim W = \dim \operatorname{range} T \leq \dim T$, since $\dim \operatorname{null} T \geq 0$, which is also a contradiction. □

In fact, we can deduce some statements you will be familiar with from Math 2101 from Corollary 3.

Remark 8. (1) Any homogeneous system of linear equations with more variables than equations has a nonzero solution. Consider $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$, a system of linear equations is given by an $m \times n$ matrix A, which gives a linear operator $T_A : \mathbb{F}^n \to \mathbb{F}^m$, by Exercise 20. By Corollary 3(1) we get that there exists $v \in \mathbb{F}^n$ such that $Tv = 0$.

(2) Any inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms. Again consider $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ and a linear transformation $T : V \to W$ encoding the system of linear equations. If $m > n$, then by Corollary 3.(2) there exists $w \in W$ such that $Tv = w$ has no solution.

4.3. **Matrices.** Given two positive natural numbers $n, m \geq 1$ and a field $\mathbb{F}$ an *m-by-n matrix* with coefficients in $\mathbb{F}$ is a list $(a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ sometimes denoted by:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$$

We let $\mathbb{F}^{m,n}$ denote the vector space of m by n matrices. You should make sure you understand why this is a vector space.

Definition 15. Given $T \in \mathcal{L}(V, W)$ a linear map between two vector spaces V, W. Let $B_V = \{v_1, \dots, v_n\}$ be a basis of V and $B_W = \{w_1, \dots, w_m\}$ be a basis of W. The *matrix A associated to T and these basis* is defined by:

$$Tv_j = a_{1,j} w_1 + \cdots + a_{m,j} w_m \qquad \text{for } 1 \leq j \leq n.$$

Sometimes we emphasize the dependence on the basis by denoting $A := \mathcal{M}(T, B_V, B_W)$.

Example 12. (i) Let $T : \mathcal{P}_3(\mathbb{F}) \to \mathcal{P}_2(\mathbb{F})$ be the linear map associated to the differentiation. Consider the bases $B_3 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$ of $\mathcal{P}_3(\mathbb{F})$ and $\mathcal{P}_2(\mathbb{F})$, respectively. Then we have:

$$\mathcal{M}(T, B_3, B_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(ii) For T as in (i) if we take the basis $B_3 = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ and $B_2 = \{1, x, x^2\}$ for $\mathcal{P}_3(\mathbb{F})$ and $\mathcal{P}_2(\mathbb{F})$, respectively. Then we obtain a *different* matrix:

$$\mathcal{M}(T, B_3, B_2) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(iii) Consider $T : \mathbb{R}^4 \to \mathbb{R}^4$ the linear map given by $T(v_1, v_2, v_3, v_4) = (v_1 + v_2, v_3, 0, v_2)$. On the basis $B_V = \{e_1, e_2, e_3, e_4\}$ for the source $V = \mathbb{R}^4$ and $B_w = \{e_1, e_2, e_3, e_4\}$ for the target, we have:

$$\mathcal{M}(T, B_V, B_W) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

However, if we take the basis $B_W^{(2)} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4\}$ then we have:

$$\mathcal{M}(T, B_V, B_W^{(2)}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 23. (i) Make sure that you understand how to obtain the matrices in the Example above.

(ii) Can you come up with a basis of $\mathbb{R}^4$ in Example 12.(iii) where all of the columns are non-zero?

(iii) Can you come you with a basis B_W such that the set of column vectors of the matrix in Example 12.(iii) are linearly independent?

Remark 9. The construction from Definition 15 sends the operations of addition and scalar multiplication of linear maps to the corresponding operations between matrices as defined in Math 2101. Indeed, given two linear maps $T, S \in \mathcal{L}(V, W)$, a scalar $a \in \mathbb{F}$, and two bases B_V of V and B_W of W. Then we have:

$$\mathcal{M}(T+S, B_V, B_W) = \mathcal{M}(T, B_V, B_W) + \mathcal{M}(T, B_V, B_W) \quad \text{and} \quad \mathcal{M}(aT, B_V, B_W) = a\mathcal{M}(T, B_V, B_W).$$

Question: When did you learn matrix multiplication? Have you every thought what was the *meaning* behind the rule of how to multiply matrices?

Lemma 12. *Consider U, V, and W three vector spaces and suppose that we picked bases B_U, B_V and B_W, respectively. Consider $T : U \to V$ and $S : V \to W$ two linear maps. Then*

$$\mathcal{M}(S \circ T, B_U, B_W) = \mathcal{M}(S, B_V, B_W)\mathcal{M}(T, B_U, B_V), \tag{3}$$

where on the righthand side above we consider the multiplication of matrices.

Proof. This is done on page 73 of the textbook. We leave the details to the reader. □

Problem 1. Does the formula fail if we take different bases for V, i.e. do we have

$$\mathcal{M}(S \circ T, B_U, B_W) = \mathcal{M}(S, B'_V, B_W)\mathcal{M}(T, B_U, B_V)$$

for B'_V different than B_V?

Let's recall a couple of concepts from Math 2101.

Definition 16. Given a matrix A represented as follow:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}. \tag{4}$$

(1) The *column space of* A, denoted $\mathrm{col}(A)$, is defined as the span of $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^m$, where $v_i = (a_{1,i}, \ldots, a_{m,i})$ for $1 \le i \le n$.

(2) The *row space of* A is the span of $\{w_1, \ldots, w_m\}$ in $\mathbb{R}^n$ spanned by $w_j = (a_{j,1}, \ldots, a_{j,n})$ for $1 \le j \le m$.

Lemma 13. *For any $m \times n$ matrix A we have*

$$\dim \mathrm{col}(A) = \dim \mathrm{row}(A).$$

Proof. **TODO:** Write this. □

5. Jan. 29, 2024

5.1. **Isomorphisms.** The following notion is going to allow us to compare vector spaces and identify when there are "essentially" the same for all purposes of linear algebra.

Definition 17. A linear map $T : V \to W$ is *invertible* if there exists a map $S : W \to V$ such that

$$S \circ T = \mathrm{Id}_V \quad \text{and} \quad T \circ S = \mathrm{Id}_W.$$

Remark 10. Notice that if an inverse exists it is unique. Indeed, assume that S_1 and S_2 are inverses of T. Then we have $S_1 = S_1 \circ T \circ S_2 = S_2$. Thus, we will denote by T^{-1} the uniquely determied inverse, if it exists.

Example 13. Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v_1, v_2, v_3) = (v_1 + v_2 + v_3, v_2, v_3)$. Then $T^{-1}(u_1, u_2, u_3) = (u_1 - u_2 - u_3, u_2, u_3)$.

Exercise 24. Determine which ones of the linear maps that we consider so far are invertible and determine their inverses.

Lemma 14. *Let $T : V \to W$ be a linear map. The following are equivalent:*

(1) T is invertible;

(2) T is injective and surjective;

(3) $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = W$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) is Lemma 11. Assume (1). Let $v, u \in V$ such that $Tv = Tu$ then we have $v = T^{-1}Tv = T^{-1}Tu = u$, thus T is injective. Now let $w \in W$ be any vector, then $T^{-1}w \in V$ satisfies $TT^{-1}w = w$, so T is surjective.

Assume (2). For any $w \in W$ we let $S(w) = v$ for any $v \in V$ such that $Tv = w$. This is well-defined, since by the injectivity of T, there is only one v satisfying $Tv = w$. We claim that S is an inverse of T. Indeed, we have $TSw = w$ and $STv = v$ for every $w \in W$ and $v \in V$. $\square$

In the case where our vector space is finite-dimensional, then we have a stronger version of Lemma 14.

Corollary 4. *Assume that V and W are finite-dimensional vector space and that $\dim V = \dim W$ and consider $T \in \mathcal{L}(V, W)$. The following are equivalent:*

(1) T is invertible;

(2) T is injective;

(2)' $\operatorname{null} T = \{0\}$;

(3) T is surjective;

(3)' $\operatorname{range} T = W$.

Proof. The implications (1) $\Rightarrow$ (2)/(2)$'$ and (1) $\Rightarrow$ (3)/(3)$'$ are clear.

The equivalences (2) $\Leftrightarrow$ (2)$'$ and (3) $\Leftrightarrow$ (3)$'$ were establishes in Lemma 11.

Assume (2)', then by Theorem 1 we have $\dim V = \dim \operatorname{range} T = \dim W$, so we have (3)'. Now (2)' and (3)' imply (1) by Lemma 14.

Assume (3)', then by Theorem 1 we have $\dim V = \dim \operatorname{range} T - \dim \operatorname{null} T = \dim W - \dim \operatorname{null} T$. Thus, $\dim \operatorname{null} T = 0$, so we have (2)'. Now (2)' and (3)' imply (1) by Lemma 14. $\square$

Remark 11. It is important to notice that the assumption that V and W are finite-dimensional in Corollary 4 is crucial. Indeed, consider the linear map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ given by

$$T(f)(n) = \begin{cases} 0 \text{ if } n = 0, \\ f(n+1) \text{ else.} \end{cases}$$

This is injective but not an isomorphism.

Exercise 25. Write an example of a linear map which is surjective but not an isomorphism.

Exercise 26. Prove that there exists a polynomials $p \in \mathbb{R}[x]$ such that $((x^2 + 5x + 7)p)'' = q$ for any $q \in \mathbb{R}[x]$.

Exercise 27. Consider $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(W, V)$ two linear maps. Assume that $\dim V = \dim W < \infty$. Prove that $ST = \mathrm{Id}_W$ if and only if $TS = \mathrm{Id}_V$.

The next concept plays the role for vector spaces of what bijections are for sets.

Definition 18. Two vector spaces V and W are said to be *isomorphic* if there exists an invertible linear map $T : V \to W$, equivalently if there exists an invertible linear map $S : W \to T$. In this case either morphism $T : V \to W$ or $S : W \to V$ are called *isomorphisms*.

Notation 3. We will sometimes simply write $V \simeq W$ to say that V and W are isomorphic.

It turns out that it is rather easy to determine if two finite-dimensional vector spaces are isomorphic or not as the next result shows.

Lemma 15. *Let V and W be two finite-dimensional vector spaces. The following are equivalent:*

(1) $V \simeq W$.

(2) $\dim V = \dim W$.

Proof. Assume (1) and let $T : V \to W$ be an invertible linear map. By Theorem 1 we have:

$$\dim T = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{range} T = \dim W,$$

where the second and third equalities above follow from Corollary 4.

Assume (2) and let $\{v_1, \dots, v_n\}$ be a basis of V and $\{w_1, \dots, w_n\}$ be a basis of W. Let $T(c_1v_1 + \cdots + c_nv_n) := c_1w_1 + \cdots + c_nw_n$, which is well-defined as argued in the proof of Lemma 9. We claim T is injective. Indeed, assume that $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n = 0$ for some non-zero combination of c_i's, then linear independence of w_i's imply that all c_i's are zero which gives that $c_1v_1 + \cdots + c_nv_n = 0$. Thus, by Corollary 4 we are done. □

The next result justify the idea that on can always think of a linear map between finite-dimensional vector spaces as a matrix. Notice however that this depends on the choice of bases.

Lemma 16. *Let V and W be finite-dimensional vector spaces of dimensions $n = \dim V$ and $m = \dim W$. Then $\mathcal{L}(V, W) \simeq \mathbb{F}^{m,n}$.*

Proof. Let $B_V = \{v_1, \dots, v_n\}$ be a basis of V and $B_W = \{w_1, \dots, w_m\}$ be a basis of W. We claim that

$$\mathcal{M} := \mathcal{M}(-, B_V, B_W) : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$$

as defined in Definition 15 is an isomorphism. First notice that $\mathcal{M}$ is a linear map by Remark 9. First we prove that $\mathcal{M}$ is injective. Indeed, assume that $\mathcal{M}(T) = 0$, then $Tv = 0$ for every $v \in B_V$ since B_V is a basis we get $Tu = 0$ for every $u \in V$, thus $T = 0$. Now we prove that $\mathcal{M}$ is surjective. Let $A \in \mathbb{F}^{m,n}$, represented as equation (4). Let $Tv_i := a_{1,i}w_1 + \cdots + a_{m,i}w_m$ for $1 \le i \le n$. This is a well-defined linear operator and it is clear that $\mathcal{M}(T) = A$. □

Here is a nice consequence of the discussion so far:

Corollary 5. *For any finite-dimensional vector spaces V and W, we have $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$.*

Proof. By Lemma 16 and Lemma 15 we have that

$$\dim \mathcal{L}(V, W) = \mathbb{F}^{\dim W, \dim V}.$$

The result now follows from calculating the dimension of the space of $\dim V$ by $\dim W$ matrices. □

The following is variation on Definition 15.

Definition 19. Let V be a vector space of dimension n, given a basis $B_V = \{v_1, \dots, v_n\}$ we let $\mathcal{M}(V, B_V) : V \to \mathbb{F}^n$ denote the linear map determined as follows

$$\mathcal{M}(V, B_V)(v) = (a_1, \dots, a_n) \text{ if } a_1v_1 + \cdots + a_nv_n = v.$$

You should check this is well-defined and indeed a linear map.

Remark 12. Notice that $\mathcal{M}(V, B_V)$ is always an isomorphism. Indeed, the same argument as in the proof of Lemma 16 works, we simply take W to be $\mathbb{F}$.

Remark 13. This remark is a bit abstract, but the reason we need it is to obtain certain matrix and vector multiplication compatibility without doing many calculations. Given a vector $v \in V$ in a finite-dimensional vector space V, we can think of v as a linear operator $L_v : \mathbb{F} \to V$ given as $L_v(1) := v$. Notice that there was a choice in determining this linear operator, namely a basis of $\mathbb{F}$, in this case $\{1\} \subset \mathbb{F}$[4]. Now we claim that:

$$\mathcal{M}(L_v, \{1\}, B_V) = \mathcal{M}(V, B_V)(v).$$

[4] So there are other linear operators associated to v, but in a sense picking $\{1\} \subset \mathbb{F}$ as a basis of $\mathbb{F}$ over itself is rather natural, i.e. it makes sense for any field.

Indeed, let $\{v_1, \ldots, v_n\} = B_V$, then $\mathcal{M}(L_v, \{1\}, B_V) = (a_{i,1})_{1\le i\le n}$ as defined in Definition 15 is given by:

$$L_v(1) = v = a_{1,1}v_1 + \cdots + a_{n,1}v_n,$$

whereas $\mathcal{M}(V, B_V)(v) = (b_{i,1})_{1\le i\le n}$, as defined in Defintion 19, is the set $(b_{1,1}, \ldots, b_{n,1})$ such that

$$v = b_{1,1}v_1 + \cdots + b_{n,1}v_n.$$

The following is a consequence of Remark 13.

Corollary 6. *Let $T : V \to W$ be a linear map between finite-dimensional vector spaces and B_V and B_W bases, respectively. Then for any $v \in V$ we have:*

$$\mathcal{M}(T, B_V, B_W)\mathcal{M}(V, B_V)(v) = \mathcal{M}(W, B_W)(Tv).$$

Proof. Indeed, we compute:

$$\begin{aligned}\mathcal{M}(T, B_V, B_W)\mathcal{M}(V, B_V)(v) &= \mathcal{M}(T, B_V, B_W)\mathcal{M}(L_v, \{1\}, B_V)\\ &= \mathcal{M}(T \circ L_v, \{1\}, B_W)\\ &= \mathcal{M}(Tv, \{1\}, B_W)\\ &= \mathcal{M}(W, B_W)(Tv).\end{aligned}$$

□

The next result relates a notion from Math 2101 which was defined using a basis, namely the column space of a matrix, with the range of a linear map which didn't depend on a basis to be defined.

Proposition 1. *Let $T : V \to W$ be a linear map between finite-dimensional vector spaces and let $B_V = \{v_1, \ldots, v_n\}$ and $B_W = \{w_1, \ldots, w_n\}$ be basis of V and W, respectively. The restriction of $\mathcal{M}(W, B_W)$ to* $\operatorname{range} T$ *has the following factorization:*

$$\begin{tikzcd} \operatorname{range} T \arrow[r, "\varphi"] \arrow[d, "\subset"'] & \operatorname{col}(\mathcal{M}(T, B_V, B_W)) \arrow[d, "\subset"] \\ W \arrow[r, "\mathcal{M}(W,B_W)"'] & \mathbb{F}^{\dim W} \end{tikzcd}$$

and φ is an isomorphism.

Proof. TODO: maybe this proof becomes simpler after the previous Remark.

Let $(a_{i,j})_{1\le i\le m; 1\le j\le n}$ denote the entries of the matrix $\mathcal{M}(T, B_V, B_W)$. Given $w \in \operatorname{range} T$ there exists $b_1, \ldots, b_n \in \mathbb{F}$ such that $w = T(b_1v_1 + \cdots + b_nv_n)$, we have

$$w = b_1T(v_1) + \cdots + b_nT(v_n).$$

Now we notice that $\mathcal{M}(W, B_V)(T(v_j)) = (a_{j,1}, \ldots, a_{j,m}) \in \mathbb{F}^m$ for $j \in \{1, \ldots, n\}$. (Check this!). By definition we have

$$\operatorname{col}(\mathcal{M}(T, B_V, B_W)) = \operatorname{Span}(\{(a_{1,1}, \ldots, a_{1,m}), \ldots, (a_{n,1}, \ldots, a_{n,m})\}).$$

Thus, linearity of $\mathcal{M}(W, B_W)$ implies

$$\mathcal{M}(W, B_W)(w) \in \operatorname{Span}\{\mathcal{M}(W, B_W)(Tv_1), \ldots, \mathcal{M}(W, B_W)(Tv_n)\} = \operatorname{col}(\mathcal{M}(T, B_V, B_W))$$

as required. This shows that φ factors as claimed.

We now check that φ is an isomorphism. We first notice that it is injective since it is the restriction of an injective linear map. Finally, let $w \in \operatorname{col}(\mathcal{M}(T, B_V, B_W))$, then it can be written as $w = d_1\mathcal{M}(W, B_W)(Tv_1) + \cdots + d_n\mathcal{M}(W, B_W)(Tv_n)$ for some $d_1, \ldots, d_n \in \mathbb{F}$. By the linearity of $\mathcal{M}(W, B_W)$ we have

$$w = \mathcal{M}(W, B_W)(d_1Tv_1 + \cdots d_nTv_n)$$

and it is clear that $d_1Tv_1 + \cdots d_nTv_n \in \operatorname{range} T$. This finishes the proof. □

The proposition immediately imply:

Corollary 7. *Let V, W be finite-dimensional vector spaces, $T : V \to W$ a linear map and B_V and B_W bases, then*

$$\dim \operatorname{range} T = \dim \operatorname{col}(\mathcal{M}(T, B_V, B_W)) = \dim \operatorname{row}(\mathcal{M}(T, B_V, B_W)).$$

We emphasize that in Corollary 7 the quantity $\dim \operatorname{range} T$ is defined without any appeal to a basis either of V or W.

We end this subsection stating a useful formula for change of bases. Before it we introduce some notation to simplify the formula. Let V be a finite-dimensional vector space and consider $T : V \to V$ a linear map and B_V a basis of V, then we simply write:

$$\mathcal{M}(T, B_V) := \mathcal{M}(T, B_V, B_V).$$

Lemma 17. *Let V be a finite-dimensional vector space and consider $T : V \to V$ a linear map and B_V^1 and B_V^2 two bases of V. Then we have:*

$$\mathcal{M}(T, B_V^1) = \mathcal{M}(\mathrm{Id}_V, B_V^1, B_V^2)^{-1}\mathcal{M}(T, B_V^2)\mathcal{M}(\mathrm{Id}_V, B_V^1, B_V^2).$$

Proof. This is (3.84) from the textbook. Write details. □

6. Feb. 1, 2024

6.1. **Products and Quotients of Vector spaces.** In this section we introduce some constructions that define a new (abstract) vector space from a number of other vector spaces.

Definition 20. Let $n \geq 1$ be a natural number. Given a collection of vector spaces $\{V_1, \ldots, V_n\}$ the *product* $V_1 \times \cdots \times V_n$ of $\{V_1, \ldots, V_n\}$ is the vector space whose:

- set is the Cartesian product $V_1 \times \cdots \times V_n$;
- $+ : (V_1 \times \cdots \times V_n) \times (V_1 \times \cdots \times V_n) \to V_1 \times \cdots \times V_n$ is defined coordinate-wise, i.e. $(v_i) + I + (w_i)_I := (v_i + w_i)_I$, where $I = \{1, \ldots, n\}$;
- $\cdot : \mathbb{F} \times (V_1 \times \cdots \times V_n) \to V_1 \times \cdots \times V_n$ is also given coordinate-wise.

Remark 14. The following is the universal property of the product. Given another vector space W and a collection of linear maps $f_i : W \to V_i$, then there exists an *unique* linear map $\varphi : W \to V_1 \times \cdots \times V_n$ such that for every $i \in I$ the following diagrams commute:

$$\begin{tikzcd} W \arrow[r, dashed, "\varphi"] \arrow[dr, "f_i"'] & V_1 \times \cdots \times V_n \arrow[d, "p_i"] \\ & V_i \end{tikzcd},$$

where $p_i : V_1 \times \cdots \times V_n \to V_i$ sends $(v_j)_{j \in I}$ to v_i. You will prove this for $I = \{1, 2\}$ in Homework 1.

Example 14. (i) Let $V_1 = \mathbb{C}^2$ and $V_2 = \mathbb{C}^3$, then we have $V_1 \times V_2 = \{((v_1, v_2), (v_3, v_4, v_5)) \in \mathbb{C}^2 \times \mathbb{C}^3\}$. Notice that $\mathbb{C}^2 \times \mathbb{C}^3 \neq \mathbb{C}^5$, because of the parenthesization. We however normally simplify this to $\mathbb{C}^5$ because $\mathbb{C}^2 \times \mathbb{C}^3$ is isomorphic to $\mathbb{C}^5$.

(ii) Let $V_1 = \mathcal{P}(\mathbb{R})$ and $V_2 = \mathbb{R}^{m,n}$, then we have $V_1 \times V_2 = \mathcal{P}(\mathbb{R}) \times \mathbb{R}^{m,n}$.

Remark 15. In fact, it is more natural, and sometimes convenient, to take Remark 14 as the definition of product. More precisely, we say that the product U of vector spaces $\{V_1, \ldots, V_n\}$, if it exists, is a vector space U satisfying the condition of Remark 14. Then we check two things:

(i) the construction given in Definition 20 solves the question asked in Remark 14;

(ii) any solution to Remark 14 is isomorphic to $V_1 \times \cdots \times V_n$.

Actually, one can do even better than (ii), we can prove that for any solution U, there exists an isomorphism $\psi : U \to V_1 \times \cdots \times V_n$ and that ψ is unique, if we require it to be compatible with the morphisms p_i for $V_1 \times \cdots \times V_n$ and p_i' for U.

Remark 16. In fact, $V_1 \times \cdots \times V_n$ also satisfies a dual universal property. Namely, let W be a vector space and $g_i : V_i \to W$ be a collection of linear maps. Then there exists an unique linear $\psi : V_1 \times \cdots \times V_n \to W$ such that for every $i \in I$ the following diagrams commute:

$$\begin{tikzcd} V_i \arrow[r, "\imath_i"] \arrow[dr, "f_i"'] & V_1 \times \cdots \times V_n \arrow[d, dashed, "\psi"] \\ & W \end{tikzcd},$$

where $\imath_i : V_i \to V_1 \times \cdots \times V_n$ sends v_i to $(0, \ldots, 0, v_i, 0, \ldots, 0)$. You will also prove this for $I = \{1, 2\}$ in Homework 1. A similar comment as in Remark 15 applies to the object that would satisfy this condition, it is called the coproduct of $\{V_1, \ldots, V_n\}$.

Proposition 2. *Let S be a (not necessarily) finite set, and $\{V_s\}_S$ a collection of vector spaces. Consider:*

(1) $\prod_{s\in S} V_s$ the subset of functions $f : S \to \bigcup_{s\in S} V_s$ satisfying $f(s) \in V_s$ for every $s \in S$.

(2) $\oplus^{\text{ext}}_{s\in S} V_s$ the subset of $\prod_{s\in S} V_s$ such that $f(s) \neq 0$ only for finitely many $s \in S$.

Then:

(i) $\prod_S V_s$ and $\oplus^{\text{ext}}_S V_s$ are vector spaces.

(ii) $\prod_S V_s$ together with the morphisms $p_s : \prod_S V_s \to V_s$ given by $p_s(f) := f(s)$ satisfies the conditions of Remark 14.

(iii) $\oplus^{\text{ext}}_S V_s$ together with the morphisms $\imath_s : V_s \to \oplus^{\text{ext}}_S V_s$ given by $\imath(v)(t) = v$ for $t = s$ and 0 otherwise, satisfies the conditions of Remark 16.

Proof. We define $(a \cdot f + g)(s) := a \cdot f(s) + g(s)$ for $a \in \mathbb{F}$ and $f, g \in \prod_S V_s$. This clearly makes $\prod_S V_s$ into a vector space.

Let's specify the morphisms $p_s : \prod_S V_s \to V_s$. Given $f \in \prod_S V_s$ we let $p_s(f) := f(s)$.

We prove (ii). Let $\{g_s : W \to V_s\}$ be a collection of linear maps, then we define $h : W \to \prod_S V_s$ by

$$h(w)(s) := g_s(w).$$

We notice that $p_s \circ h(w) = h(w)(s) = g_s(w)$ as required. Assume we are given $h' : W \to \prod_S V_s$ another linear map satisfying

$$p_s \circ h' = g_s \text{ for all } s \in S.$$

Then we have that for any $w \in W$ and $s \in S$:

$$h'(w)(s) = p_s \circ h'(w) = g_s(w) = p_s \circ h(w) = h(w)(s),$$

so $h' = h$.

The argument for (iii) is analogous to (ii) and we leave the details for the reader. □

Here are some immediate properties of the product of vector spaces.

Lemma 18. *(1) Let $V_1, \ldots, V_n$ be finite-dimensional vector spaces, then we have $\dim(V_1 \times \cdots \times V_n) = (\dim V_1) + \cdots + (\dim V_n)$.*

(2) Suppose that $V_1, \ldots, V_n$ are subspaces of an ambient vector space W. Then we have a linear map:

$$\Gamma : V_1 \times \cdots \times V_n \to V_1 + \cdots + V_n$$

given by $\Gamma(v_1, \ldots, v_n) := v_1 + \cdots + v_n$. Then $V_1 + \cdots + V_n$ is a direct sum if and only if Γ is injective.

(3) Given subspaces $V_1, \ldots, V_n$ of a finite-dimensional vector space W, then $V_1 + \cdots + V_n$ is a direct sum if and only if $\dim(V_1 + \cdots + V_n) = \sum_{i=1}^n \dim V_i$.

Proof. (1) can be proved by induction. We notice that given $\{v_1, \dots, v_n\}$ a basis of V_1 and $\{u_1, \dots, u_m\}$ a basis of V_2, then $\{(v_1, 0), \dots, (v_n, 0), (0, u_1), \dots, (0, u_m)\}$ is a basis of $V_1 \times V_2$. Indeed, assume we are given $(v, u) \in V_1 \times V_2$, then there exist unique $a_1, \dots, a_n$ in $\mathbb{F}$ such that $v = \sum_{i=1}^n a_i v_i$ and there are unique $b_1, \dots, b_m$ in $\mathbb{F}$ such that $u = \sum_{j=1}^m b_j u_j$. Thus we get:

$$(u, v) = \sum_{i=1}^n a_i(v_i, 0) + \sum_{j=1}^m b_j(0, u_j).$$

that is $a_1, \dots, a_n, b_1, \dots, b_m$ are unique coefficients writing (v, u) as a linear combination of the basis vectors.

(2) Again by induction it is enough to consider the case $n = 2$. Now recall that $\dim V_1 + V_2 = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$ and that $V_1 + V_2$ is a direct sum if and only if $\dim V_1 \cap V_2 = 0$. If we assume Γ is injective then Corollary 4 implies Γ is an isomorphism, thus $\dim V_1 + V_2 = \dim(V_1 \times V_2)$, which gives that $V_1 + V_2$ is a direct sum. Conversely, if $\dim V_1 \cap V_2 = 0$, then $\dim \operatorname{null} \Gamma = \dim(V_1 \times V_2) - \dim \operatorname{range} T$. But Γ is surjective, thus $\dim \operatorname{range} T = \dim V_1 + V_2 = \dim V_1 + \dim V_2$, which gives that $\operatorname{null} \Gamma = 0$.

(3) is essentially a simple restatement of (2). □

7. Feb. 5, 2024

7.1. **Quotient spaces.** The motivation for this session is the following. Suppose we are given a subspace $U \subseteq V$ and linear map $L : V \to W$ to another vector space such that $L|_U : U \to W$ is the identically 0 linear map. Can we describe the data of L in "smaller" terms, i.e. is there some vector space Q and a map $T : Q \to W$ such that we recover L from T. Notice that if V is finite-dimensional, one answer would be to pick a subspace $U' \subset V$ such that $U' \oplus U = V$, which exists by Lemma 5 and consider $T := L|_{U'} : U' \to W$. There are however two problems with this solution:

(1) we had to assume that V is finite-dimensional;

(2) we had to make a *choice* of U', which we know, by Exercise 5 from Worksheet 1, is *not* unique–so T is not unique either.

To solve these two problems we need some preparatory discussion.

Let $U \subset V$ be a subspace, the consider the relation $R_U \subseteq V \times V$ on V defined by

$$(v_1, v_2) \in R_U \text{ if } v_1 - v_2 \in U.$$

Exercise 28. For any subspace $U \subseteq V$ the relation R_U is an equivalence relation[5].

Let V/U denote the set of equivalence classes of R_U.

Lemma 19. *There exists an unique structure of vector space on V/U such the natural map:*

$$\begin{aligned} \pi : V &\to V/U \\ v &\mapsto \pi(v) \end{aligned}$$

is a linear map.

Proof. First we give a concrete description of $\pi(v)$. Notice that $w \in \pi(v)$ if and only if $w - v \in U$, i.e. $w \in v + U := \{w \in V \mid w - v \in U\}$. Now we define $\cdot' : \mathbb{F} \times V/U \to V/U$ as follows:

$$a \cdot' \bar{v} := \pi(a \cdot v'), \text{ for any } v' \in V \text{ such that } \pi(v') = \bar{v}.$$

Similarly, we define $+' : V/U \times V/U \to V/U$ as follows:

$$\bar{v_1} +' \bar{v_2} := \pi(v_1' + v_2'), \text{ for any } v_1', v_2' \in V \text{ such that } \pi(v_1') = \bar{v}_1 \text{ and } \pi(v_2') = \bar{v}_2.$$

Let's check that these are well-defined. Let v' and v'' be such that $\pi(v') = \pi(v'')$, i.e. there exist $u \in U$ such that $v' + u = v''$. Then we have

$$a \cdot \pi(v') = \pi(a \cdot v') = \pi(a \cdot v' + a \cdot u)) = \pi(a \cdot (v' + u)) = \pi(a \cdot v'') = a \cdot \pi(v'').$$

The check that $+'$ is well-defined is similar. We leave it to the reader to check that π is a linear map. □

[5]If you don't remember this definition from set theory, see here.

Definition 21. Given a subspace $U \subset V$ the quotient set V/U with the structure from Lemma 19 is called the *quotient space of* V *by* U.

Remark 17. Notice that $\operatorname{null} \pi = U$. Indeed, for any $u \in U$ we have $\pi(u) = \pi(u - u) = \pi(0) = 0$. Thus, Theorem 1 imply that $\dim V/U = \dim V - \dim U$, whenever V is finite-dimensional.

We now show that this construction solves the problem that motivated us in the begining of this subsection.

Lemma 20. *Given a subspace $U \subseteq V$ and a linear map $L : V \to W$ such that $L|_U = 0$. Then there exist an unique $T : V/U \to W$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "L"'] & V/U \arrow[d, dashed, "T"] \\ & W \end{tikzcd},$$

i.e. $T \circ \pi = L$.

Proof. We let $T(\bar{v}) := L(v')$ for any $v' \in V$ such that $\pi(v') = \bar{v}$. We check this is well-defined. Let v', v'' such that $\pi(v') = \pi(v'')$ then $v' = v'' + u$, which gives

$$L(v') = L(v'' + u) = L(v'').$$

We leave it to the reader to check that T is linear and uniquely determined. $\square$

Remark 18. Notice that given a subspace $U \subseteq V$ of a finite-dimensional vector and a linear map $L : V \to W$ we can always modify L on U to make it satisfy the conditions of Lemma 20. Indeed, consider $L - \widetilde{L|_U}$, where $\widetilde{L|_U} : V \to W$ is the extension of $L|_U$ defined as follows: let $W \subseteq V$ be any subspace such that $W \oplus U = V$, then $\widetilde{L|_U} := 0|_W \oplus \widetilde{L|_U}$. Notice that $\widetilde{L|_U}$ does not depend on W. Explain the notation $0|_W \oplus \widetilde{L|_U}$ and why it doesn't depend on U.

Corollary 8. *Let $L : V \to W$ be a linear map, then one has a canonical map:*

$$\bar{L} : V/\operatorname{null} T \to W$$

satisfying the following:

(i) $\bar{L} \circ \pi = L$;

(ii) $\bar{L}$ *is injective;*

(iii) $\operatorname{range} \bar{L} = \operatorname{range} L$;

(iv) $V/\operatorname{null} T \simeq \operatorname{range} T$.

Proof. The existence of $\bar{L}$ and (i) follow directly from Lemma 20, by noticing that $L|_{\operatorname{null} L} = 0$

For (ii), suppose that there exist $\bar{v} \in V/\operatorname{null} L$ such that $\bar{L}(\bar{v}) = 0$, but $\bar{v} \neq \pi(0)$. This implies that there exist $v' \in V$ such that $\pi(v') = \pi(0)$, i.e. $v' \in \operatorname{null} L$ and $\bar{L}(\pi(v')) = L(v') = 0$, which is a contradiction.

For (iii), we notice that $\operatorname{range} \bar{L} \subseteq \operatorname{range} L$ by set-theoretic consideration. Now assume that $w \in \operatorname{range} L$ and let $v \in V$ such that $L(v) = w$, then $\bar{L}(\pi(v)) = w$.

For (iv), we simply notice that $\bar{L}$ factors as:

$$\begin{tikzcd} V/\operatorname{null} T \arrow[r, "\varphi"] \arrow[dr, "\bar{L}"'] & \operatorname{range} \bar{L} \arrow[d, dashed, "\imath"] \\ & W \end{tikzcd},$$

where by (ii) and (iii) φ is injective and surjective, hence an isomorphism by Lemma 14. $\square$

8. Feb. 8, 2024

8.1. **Duality.** The next definition associates a vector space to given vector space that is very useful.

Definition 22. Given a vector space V the vector space $V^* := \mathcal{L}(V, \mathbb{F})$ is called its *dual space*[6]. Elements $\lambda \in V^*$ are called *linear functionals* on V.

Example 15. (1) Let $\mathcal{P}_3(\mathbb{R})$, then $\varphi : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$ given by $\varphi(p) := \int_0^1 p(x)dx$ is a linear functional, i.e. $\varphi \in \mathcal{P}_3(\mathbb{R})^*$.

(2) For any $a := (a_1, \dots, a_n) \in \mathbb{F}^n$ we have $\lambda_a : \mathbb{F}^n \to \mathbb{F}$ given by $\lambda_a(v_1, \dots, v_n) := \sum_{i=1}^n a_i v_i$.

(3) Let $V = \mathbb{R}[[x]]$ denote the vector space of Taylor series, i.e. $f : \mathbb{R} \to \mathbb{R}$ given by a series $f(x) = \sum_{i \geq 0} a_i x^i$. Then $V^* \simeq \mathbb{R}[x]$. Give details.

Here are a couple of properties on the finite-dimensional case.

Lemma 21. *Given a finite-dimensional vector space V.*

(1) One has $\dim V = \dim V^*$.

(2) A choice of basis B_V for V determines a basis B_{V^*} for V^*.

Proof. By Corollary 5 we know that $\dim \mathcal{L}(V, \mathbb{F}) = \dim V \cdot \dim \mathbb{F} = \dim V$.

Let $B_v = \{v_1, \dots, v_n\}$ be a basis of V, then we define $B_{V^*} := \{\lambda_1, \dots, \lambda_n\} \subset V^*$ as follows:

$$\lambda_j(v_i) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ else} \end{cases} .$$

Notice that if there exists a collection of scalars $a_1, \dots, a_n \in \mathbb{F}$ not all zero such that $\lambda := \sum_i^n a_i \lambda_i = 0$, then $\lambda(v_1 + \cdots v_n) = 0$ would contradict $\{v_1, \dots, v_n\}$ being linearly independent. Thus, B_{V^*} is linearly independent. By Lemma 7 (2) B_{V^*} is a basis. □

Remark 19. The set B_{V^*} is called the *dual basis*. It has the following property:

$$v = \sum_{i=1}^n \lambda_i(v) v_i,$$

with the notation as in the proof of Lemma 21.

Definition 23. Given a linear map $T \in \mathcal{L}(V, W)$ its dual is the linear map $T^* \in \mathcal{L}(W^*, V^*)$ defined as follows:

$$T^*(\lambda) := \lambda \circ T \text{ for every } \lambda \in W^*.$$

Example 16. Let $D : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear map given by differentiation. Then $D^* : \mathbb{R}[x]^* \to \mathbb{R}[x]^*$ is given by $D^*(\varphi) = \varphi \circ D$. For example, if $\varphi \in \mathbb{R}[x]^*$ is given by $\varphi(p) = \int_0^1 p(x)dx$, then $D^* \circ \int_0^1 p(x)dx = p(1) - p(0)$ by the fundamental theorem of Calculus.

Here are some properties of the construction from Definition 23.

Lemma 22. *Let V and W be two vector spaces. Then $(-)^* : \mathcal{L}(V, W) \to \mathcal{L}(W^*, V^*)$ is a linear operation. Moreover, we have $(T \circ S)^* = S^* \circ T^*$.*

Proof. Let $T, T' \in \mathcal{L}(V, W)$ then $(aT + T')^* : W^* \to V^*$ is given on an element $\lambda \in W^*$ by

$$(aT + T')^*(\lambda) = \lambda \circ (aT + T') = \lambda \circ aT + \lambda \circ T' = a\lambda \circ T + \lambda \circ T'$$

by Exercise 21. The second equation is left as an exercise. □

Exercise 29. Is the operation linear map that sends $T \in \mathcal{L}(V, W)$ to $T^* \in \mathcal{L}(W^*, V^*)$ an isomorphism? Prove or give a counter-example.

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[6]The textbook uses the notation V' for the dual of V and similarly to other dual concepts. We adopt a notation that is consistent with Wikipedia.

We now study how the notion of dual vector space interacts with the concepts of null space and range. First we introduce the following:

Definition 24. Let $U \subset V$ be a subspace, the *annihilator of* U is:

$$U^0 := \{\lambda \in V^* \mid \lambda(u) = 0 \text{ for all } u \in U\}.$$

We notice that $U^0 \subset V^*$ is a subspace. Indeed, let $\lambda, \mu \in U^0$, then we have

$$(a \cdot \lambda + \mu)(v) = a \cdot \lambda(v) + \mu(v) = \lambda(a \cdot v) + \mu(v) = 0 + 0,$$

if $v \in U$, since this implies $a \cdot v \in U$ as well.

Example 17. (1) Let $\mathcal{P}_3(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ be the subspace of degree at most 3 polynomials.

(2) Let $\mathbb{R}[x] \subset \mathbb{R}[[x]]$ be the subspace of polynomials. Then $\mathbb{R}[x]^0$ What is this?

Lemma 23. *Let $T : V \to W$ be a linear map, then:*

(i) $\operatorname{null} T^* = (\operatorname{range} T)^0$*;*

(ii) $\operatorname{range} T^* \subseteq (\operatorname{null} T)^0$ *is a subspace.*

Proof. For (i) let $\lambda \in \operatorname{null} T^*$ this implies that $\lambda \circ T : V \to \mathbb{F}$ vanishes. Consider $w \in \operatorname{range} T$, i.e. $w = Tv$ for some $v \in V$, then

$$\lambda(w) = \lambda(Tv) = 0.$$

So $\operatorname{null} T^* \subseteq (\operatorname{range} T)^0$. Now let $\lambda \in (\operatorname{range} T)^0$ then for any $w = Tv \in W$ we have

$$\lambda(w) = \lambda(Tv) = \lambda \circ T(v) = T^*(\lambda)(v) = 0.$$

Notice that the above equation holds for any v, thus $T^*(\lambda) = 0$, which gives $(\operatorname{range} T)^0 \subseteq \operatorname{null} T^*$.

For (ii), consider $\lambda \in \operatorname{range} T^*$, i.e. $\lambda = \mu \circ T$ for some $\mu \in V^*$. Let $v \in (\operatorname{null} T)^0$, then we have

$$\lambda(v) = \mu \circ T(v) = 0.$$

This implies that $\operatorname{range} T^* \subseteq (\operatorname{null} T)^0$ as sets. We leave it to the reader to check that one is a subspace of the other. □

Warning 1. Notice that in Lemma 23 (ii) we don't always have that $(\operatorname{null} T)^0 \subseteq \operatorname{range} T^*$. Can you give an example where this does *not* happen?

Remark 20. Let $U \subseteq V$ be a subspace of a finite-dimensional vector space V. Let $T : U \to W$ be a linear map, then there exist an *extension of* T *to* V, i.e. a linear map $\tilde{T} : V \to W$ such that $\tilde{T}\Big|_U = T$. Indeed, let $\{u_1, \ldots, u_n\}$ be a basis of U, which can be extended to $\{u_1, \ldots, u_n, u_{n+1}, \ldots, w_m\}$ a basis of V. Then we define:

$$\tilde{T}(u_i) := \begin{cases} T(u_i) & \text{for } i \leq n; \\ 0 & \text{for } i \geq n+1. \end{cases}$$

Notice that this is well-defined and satisfy the required condition to be an extension.

We collect some properties of the annihilator when the ambient vector space is finite-dimensional.

Lemma 24. *Let $U \subset V$ be a subspace of a finite-dimensional vector space V.*

(i) $\dim U^0 = \dim V - \dim U$*;*

(ii) $U^0 = V$ *if and only if* $U = \{0\}$*;*

(iii) $U^0 = \{0\}$ *if and only if* $U = V$*.*

Proof. For (i) let $\imath : U \to V$ denote the inclusion linear map. Then we get a linear map $\imath^* : V^* \to U^*$ and Theorem 1 implies that:

$$\dim \operatorname{range} \imath^* + \dim \operatorname{null} \imath^* = \dim V^*.$$

Notice that $\operatorname{null} \imath^* = U^0$, by Lemma 23(i), since $\operatorname{range} \imath = U$.

Now we claim that $\operatorname{range} \imath^* = U^*$. Indeed, let $\lambda : U \to \mathbb{F}$ be a linear map, by Remark 20 there exist $\tilde{\lambda} : V \to \mathbb{F}$ such that $\tilde{\lambda} \circ \imath = \lambda$. Thus, $U^* \subseteq \operatorname{range} \imath^*$, which implies the equality, since we have $\operatorname{range} \imath^* \subseteq U^*$ by definition. Thus, we obtain:

$$\begin{aligned}
\dim V &= \dim V^* \\
&= \dim \operatorname{range} \imath^* + \dim \operatorname{null} \imath^* \\
&= \dim U^* + \dim U^0 \\
&= \dim U + \dim U^0,
\end{aligned}$$

where in the first and last equalities we used that $V \simeq V^*$ and $U \simeq U^*$ for finite-dimensional vector spaces.

For (ii) we notice that $U^0 \subseteq V$ being a subset, we have that $U^0 = V$ if and only if $\dim U^0 = \dim V - \dim U = \dim V$, that is $U = 0$.

Since (iii) is proved similarly, we leave the details to the reader. □

Similar to Lemma 24 we collect some properties of the dual of a linear map when the vector spaces involved are finite-dimensional.

Lemma 25. *Suppose that V and W are finite-dimensional and let $T : V \to W$ be a linear map. Then:*

(i) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$*;*

(ii) $\dim \operatorname{range} T^* = \dim \operatorname{range} T$*;*

(iii) $\operatorname{range} T^* = (\operatorname{null} T)^0$*;*

(iv) T^* is injective if and only if T is surjective;

(v) T^* is surjective if and only if T is injective.

Proof. For (i) notice that by Lemma 23 we have $\dim \operatorname{null} T^* = \dim (\operatorname{range} T)^0$. By the fundamental theorem of linear maps we have:

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Notice that by Lemma 24 (1) we have $\dim (\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T$. Thus, substituting back we obtain:

$$\dim V = \dim \operatorname{null} T + \dim W - \dim (\operatorname{range} T)^0 \qquad = \dim \operatorname{null} T + \dim W - \dim \operatorname{null} T^*.$$

For (ii) we compute:

$$\begin{aligned}
\dim \operatorname{range} T^* &= \dim W^* - \dim \operatorname{null} T^* \\
&= \dim W - \dim (\operatorname{range} T)^0 \\
&= \dim \operatorname{range} T,
\end{aligned}$$

where the first equality is the fundamental theorem of linear maps applied to T^*, the second uses that W is finite-dimensional, so $\dim W = \dim W^*$ and Lemma 23(i). The last equality follows from Lemma 24(i).

For (iii), by Lemma 23(ii) we have that $\operatorname{range} T^* \subseteq (\operatorname{null} T)^0$ is a subspace. We will be done if we prove that they have the same dimension. Notice that:

$$\begin{aligned}
\dim \operatorname{range} T^* &= \dim \operatorname{range} T \\
&= \dim V - \dim \operatorname{null} T \\
&= \dim (\operatorname{null} T)^0,
\end{aligned}$$

where the first equality comes from (ii), the second from the fundamental theorem of linear maps, and the third from Lemma 24(i).

(iv) We have the following sequence of logical equivalences:

$$\begin{aligned} T^* \text{ is injective } &\Leftrightarrow \operatorname{null} T^* = 0 \\ &\Leftrightarrow (\operatorname{range} T)^0 = 0 \\ &\Leftrightarrow \operatorname{range} T = W \\ &\Leftrightarrow T \text{ is surjective }. \end{aligned}$$

Here the first equality is Lemma 11(i), the second is Lemma 23(i), the third is Lemma 24(ii) and the last is Lemma 11(ii).

The argument for (v) is similar to that for (iv) and we leave it as an exercise. □

Finally, we remark on how passing to the dual linear map interacts with associating a matrix to a linear map. Consider the function:

$$\begin{aligned} (-)^{\mathrm{t}} : \mathbb{F}^{m,n} &\to \mathbb{F}^{n,m} \\ (a_{ij}) &\mapsto (a_{ji}) \end{aligned}$$

that sends a matrix to its transpose, i.e. we swap the indices of its terms.

Lemma 26. *Let V and W be two finite-dimensional vector spaces with bases B_V and B_W. These determine bases B_{V^*} and B_{W^*} of V^* and W^*, respectively, as explained in Remark 19. The following diagram commutes:*

$$\begin{tikzcd} \mathcal{L}(V,W) \arrow[r, "\mathcal{M}(-,B_V,B_W)"] \arrow[d, "(-)^*"'] & \mathbb{F}^{m,n} \arrow[d, "(-)^{\mathrm{t}}"] \\ \mathcal{L}(W^*,V^*) \arrow[r, "\mathcal{M}(-,B_{W^*},B_{V^*})"'] & \mathbb{F}^{n,m} \end{tikzcd},$$

where $n = \dim V$ and $m = \dim W$.

Proof. Write this! □

Here is a nice consequence of the previous result.

Corollary 9. *Let $A \in \mathbb{F}^{m,n}$ be an m by n matrix. Then* $\dim \operatorname{col}(A) = \dim \operatorname{row}(A)$.

Proof. Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be the linear operator corresponding to A for B_n and B_m the standard bases of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively. That is we have $\mathcal{M}(T, B_n, B_m) = A$. Then we have:

$$\begin{aligned} \dim \operatorname{col}(A) &= \dim \operatorname{range} T \\ &= \dim \mathit{range} T^* \\ &= \dim \operatorname{col}(\mathcal{M}(T^*, B_m^*, B_n^*)) \\ &= \dim \operatorname{col}(A^{\mathrm{t}}) \\ &= \dim \operatorname{row}(A). \end{aligned}$$

□

8.2. **Polynomials.** This subsection is a digression that collects some facts about polynomials that we will need in the next topic.

Recall that $\mathbb{C}$ is the field of complex numbers. As a set one has $\mathbb{C} \simeq \mathbb{R}\times\mathbb{R}$ the addition and multiplication are given by:

$$(a,b) + (a',b') = (a+a', b+b') \quad \text{and} \quad (a,b)\cdot(a',b') = (aa' - bb', ba' + ab').$$

These are normally denoted by $a + ib := (a,b)$. We have functions:

- $\overline{(-)} : \mathbb{C} \to \mathbb{C}$ given by $\overline{(a+ib)} = a - ib$ called *complex conjugation*;
- $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ given by $\operatorname{Re}(a+ib) = a$ called the *real part*;

- $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$ given by $\operatorname{Im}(a + ib) = b$ called the *imaginary part*;
- $|-| : \mathbb{C} \to \mathbb{R}$ given by $|a + ib| = \sqrt{a^2 + b^2}$ called the *absolute value.*

Here is a list of properties they satisfy:

Lemma 27. *For any two complex numbers $z, w \in \mathbb{C}$ we have:*

(i) $z + \overline{z} = 2 \operatorname{Re} z$;

(ii) $z - \overline{z} = 2 \operatorname{Im} zi$;

(iii) $z\overline{z} = |z|^2$;

(iv) $\overline{z + w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{zw}$;

(v) $\overline{\overline{z}} = z$;

(vi) $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq z$;

(vii) $|\overline{z}| = |z|$;

(viii) $|zw| = |z||w|$;

(ix) $|z + w| \leq |z| + |w|$.

Proof. Left as an exercise. □

Definition 25. Given a polynomial $p \in \mathbb{F}[x]$ a *zero of* p is an element $\alpha \in \mathbb{F}$ such that $p(\alpha) = 0$.

Lemma 28. *Let $p \in \mathbb{F}[x]$ be a polynomial then the following are equivalent:*

(1) α *is a zero of* p;

(2) there exists a polynomial $q \in \mathbb{F}[x]$ such that $p(x) = (x - \alpha)q(x)$.

Proof. Assume (1) and let $p(z) = \sum_{i=0}^m a_i z^i$ for some coefficients $a_i \in \mathbb{F}$. Then we have:

$$p(z) - p(\alpha) = \sum_{i=1}^{m} a_i(z^i - \alpha^i).$$

Recall that each $z^i - \alpha^i$ can be factored as:

$$z^i - \alpha^i = (z - \alpha) \sum_{j=1}^{i} z^{i-j}\alpha^{j-1}.$$

Thus, we obtain:

$$\begin{aligned} \sum_{i=1}^{m} a_i(z^i - \alpha^i) &= \sum_{i=1}^{m} a_i(z - \alpha) \sum_{j=1}^{i} z^{i-j}\alpha^{j-1} \\ &= (z - \alpha) \sum_{i=1}^{m} a_i \sum_{j=1}^{i} z^{i-j}\alpha^{j-1}, \end{aligned}$$

that is we can take $q(z) = \sum_{i=1}^m a_i \sum_{j=1}^i z^{i-j}\alpha^{j-1}$. Notice that this is not unique.

Now assume (2), then clearly we get $p(\alpha) = (\alpha - \alpha)q(\alpha) = 0$. □

Here is a basic result about polynomials that follows from the previous Lemma.

Corollary 10. *Let p be a polynomial of degree $m \geq 0$. Then p has at most m zeros.*

Proof. We proceed by induction on m. For $m = 0$ we have $p(z) = a_0$ where $a_0 \neq 0$[7]. Thus, $p(\alpha) = a_0$ for every $\alpha \in \mathbb{F}$, so p has no zeros.

Assume we proved the result for all polynomials of degree up to $m - 1$. Let p have degree m. If it has no zeros, there is nothing to prove. Assume that p has a zero α, then $p(z) = (z - \alpha)q(z)$ by Lemma 28. Since $\deg p = \deg(z - \alpha)\deg(q)$ we obtain that q has degree $m - 1$, so it has at most $m - 1$ zeros by the inductive hypothesis. This finishes the proof. □

[7] Recall that we posed that $\deg 0 = -\infty$, which avoids us having to deal with the exception of the zero polynomial, which possibly has infinitely many zeros, e.g. if the field $\mathbb{F}$ is infinite.