Math 347: Lecture 4 - Worksheet

September 5, 2018

1) What is the relation between the two following statements?

$$(\forall x \in A)(\exists y \in B)P(x,y) \quad \text{and} \quad (\exists y \in B)(\forall x \in A)P(x,y).$$

Find examples of A and B that justify your claim.

Those statements are not equivalent. Consider $A = B = \mathbb{R}$ and $P(x,y) : x^2 = y$. The first is true, the second is false.

2) Is the following statement true or false?

$$(\exists a, b \in \mathbb{R})(\forall x \in \mathbb{R})(ax^2 + bx \neq a).$$

How does this relate to Example 2.10 from the book?

The negation of the above statement is

$$(\forall a, b \in \mathbb{R})(\exists x \in \mathbb{R})(ax^2 + bx = a),$$

which is stating that for a, b real numbers a solution to the equation

$$ax^2 + bx - a = 0$$

exists in $\mathbb{R}$. By considering the quadratic formula one always has a solution in $\mathbb{C}$, however we notice that the determinant of the above polynomial is

$$\Delta = b^2 + 4a^2 \geq 0.$$

Thus, the solution always belongs to $\mathbb{R}$ and we are done.

3) A function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* if it satisfies the following

$$(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})(|f(x)| \leq M).$$

We say that f is unbounded if the above statement is false. Prove that if f is unbounded then

$$(\forall n \in \mathbb{N})(\exists x_n \in \mathbb{R})(|f(x_n)| > n).$$

Again we negate the statement in the first line to obtain:

$$(\forall M \in \mathbb{R})(\exists x \in \mathbb{R})(|f(x)| > M).$$

So a f is unbounded if the above is true. Now we notice that $\mathbb{N} \subset \mathbb{R}$, thus the following is also true

$$(\forall M \in \mathbb{N})(\exists x \in \mathbb{R})(|f(x)| > M).$$

That's exactly what we needed to prove.

4) Show that the following statement is false: "If a and b are integers, then there are integers m and n such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of the statement to make it true?

We notice by solving the system of equations that m and n are given by

$$m = \frac{a+b}{2} \quad \textit{and} \quad n = \frac{a-b}{2}.$$

This makes it clear that for instance for $a = 1$ and $b = 0$, we have no integer solution for m and n. On the other hand we notice that if $a + b$ and $a - b$ are both divisible by 2, then we have a solution. In other words, if a and b are both even or both odd. .

5) Let $P(x)$ be the statement "x is odd", and $Q(x)$ the statement "$x^2 - 1$ is divisible by 8". Determine whether the following are true:

(i) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$,

(ii) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

(i) Notice that $P(x)$ by definition is

$$(\exists k \in \mathbb{Z})(x = 2k + 1).$$

Thus, if $P(x)$ holds, we compute

$$x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4k(k+1).$$

However, for any $k \in \mathbb{Z}$ the product $k(k+1)$ is divisible by 2. Thus $x^2 - 1$ is divisible by 8, that is $Q(x)$ is true. (ii) We will prove that this assertion is true. Assume by contradiction that $Q(x)$ is true and $P(x)$ is not true. Since we can factor

$$x^2 - 1 = (x-1)(x+1),$$

and we notice that either $x - 1$ and $x + 1$ are both divisible by 2 or both are not divisible by 2. If the later case, we will have that two odd numbers multiply to a multiple of 8, which, in particular, is even. This is a contradiction. Hence $x - 1$ is even, that is x is odd. That is a contradiction with $P(x)$ being false. Thus we proved the assertion $Q(x) \to P(x)$ to be true for all $x \in \mathbb{Z}$.

6) For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ show that (i) and (ii) have different meanings:

(i) $(\forall \epsilon > 0)(\exists \delta > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \epsilon)]$

(ii) $(\exists \delta > 0)(\forall \epsilon > 0)[(|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \epsilon)]$

Can you come up with examples of a and f that satisfy or do not satisfy the above?

This is analogous to question 1). Consider the function $f(x) = x$, defined on the real numbers. The first assertion is true, indeed given $\epsilon > 0$ consider $\delta(\epsilon) = \frac{\epsilon}{2}$[1]*. Then one has*

$$|x - a| < \frac{\epsilon}{2} \;\Rightarrow\; 2|x - a| < \epsilon,$$

[1] Notice that δ depends on ϵ and that is fine, because we fixed ϵ first.

since $|x-a| \geq 0$, *one has*

$$|x-a| \leq 2|x-a|$$

that implies the inequality we wanted. Note that $a \in \mathbb{R}$ *is arbitrary. This just proved that the function* $f(x) = x$ *is continuous at all points* $a \in \mathbb{R}$.

In the second case we notice that if one chooses $\epsilon = \delta/2$ *the same computation as above will lead to*

$$|x-a| < \delta \Rightarrow |x-a| < \delta/2.$$

If one considers $x = a + \frac{2\delta}{3}$, *this gives a contradiction, i.e.* $\frac{2\delta}{3} < \frac{\delta}{2}$.

7) Prove the following identities about sets.

(i) $(A \cup B)^c = A^c \cap B^c$ (de Morgan's law);

(ii) $A \cap [(A \cap B)^c] = A - B$;

(iii) $A \cap [(A \cap B^c)^c] = A \cap B$;

(iv) $(A \cup B) \cap A^c = B - A$.

(i) We have

$$x \in (A \cup B)^c \Leftrightarrow \neg(x \in A \cup B) \Leftrightarrow \neg((x \in A) \vee (x \in B)) \Leftrightarrow (\neg(x \in A) \wedge \neg(x \in B)) = (x \in A^c \cap B^c).$$

(ii) By applying the above exercise to $(A \cap B)^c$ *one can rewrite the lefthand side as*

$$A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c),$$

where we used 2) (ii) from Homework 1 for the above equality. Finally we notice that

$$A \cap A^c = \emptyset,$$

and that

$$A \cap B^c = A - B$$

essentially by definition, i.e. $x \in A \cap B^c$ *is*

$$(x \in A) \wedge \neg(x \in B) \Leftrightarrow (x \in A) \wedge (x \notin B).$$

(iii) By applying (ii) one has

$$A \cap [(A \cap B^c)^c] = A - B^c.$$

So we just need to prove that $A - B^c = A \cap B$. *However,* $x \in A - B^c$ *exactly if*

$$(x \in A) \wedge \neg(x \notin B) \Leftrightarrow (x \in A) \wedge (x \in B) \Leftrightarrow (x \in A \cap B).$$

(iv) We notice that by Exercise 3 from Homework 1

$$(A \cup B) \cap A^c = (A \cup A^c) \cap (B \cup A^c).$$

Notice that $A \cup A^c = U$, *the universe that we are considering. And* $U \cap S = S$ *for any set* S[2]. *Thus we are left to check that*

$$B \cup A^c = B - A,$$

that exactly the last line of (ii).

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[2] If one want this can be taken as the definition of universe in our context.