SOLUTIONS FOR PART III

9. PROBABILITY

9.1. *If $A \subset B$, then $P(A) \le P(B)$—TRUE.* $P(B)$ is the sum of the probabilities assigned to points in A plus the sum of the proabilities assigned to points in $B - A$.

9.2. *If $P(A)$ and $P(B)$ are not zero, and $P(A|B) = P(B|A)$, then $P(A) = P(B)$—TRUE.* The two quantities equal $P(A \cap B)/P(B)$ and $P(A \cap B)/P(A)$, so they are equal if and only if the denominators are equal.

9.3. *If $P(A)$ and $P(B)$ are not zero, and $P(A|B) = P(B|A)$, then A and B are independent—FALSE.* If A and B are the same event, then $P(A \cap B) = P(A) > P(A)P(B)$, and they are not independent.

9.4. *If $P(A) > 1/2$ and $P(B) > 1/2$, then $P(A \cap B) > 0$—TRUE.* If $P(A \cap B) = 0$, then A and B have no common points with positive probability, and $P(A \cup B) = P(A) + P(B) > 1$, which is impossible.

9.5. *If A, B are independent, then A and B^c are independent—TRUE.* $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$.

9.6. *If A, B are independent, then A^c and B^c are independent—TRUE.* $P(A^c \cap B^c) = P(A^c) - P(A^c \cap B) = 1 - P(A) - (P(B) - P(A \cap B)) = 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$.

9.7. *An event and its complement are independent if and only if their probabilities are 1 and 0.* The probability of the intersection of A and A^c is 0. If they are independent, then $0 = P(A \cap A^c) = P(A)P(A^c) = P(A)(1 - P(A))$. This requires that $P(A)$ or $1 - P(A)$ is 0.

9.8. *Restaurant menu items.* Let S denote the event that what the man orders is out of stock. Let A and F denote the events of ordering pasta and fish, respectively. We are given that $P(A) = P(F) = .5$ and $P(S|A) = P(S|F) = .5$. By direct computation, $P(S) = P(S \cap A) + P(S \cap F) = P(S|A)P(A) + P(S|F)P(F) = .5(.5) + .5(.5) = .5$.

The answer makes sense, since the probability that the dish is out of stock is .5 every day, no matter what he orders. Arbitrary values may arise for the values $P(A)$, $P(F)$, $P(S|A)$, and $P(F|A)$, in which case the conclusion is obtained by the computation above.

9.9. *Conditional probability for drawing marbles.* The contents of the containers are RR, BB, and RB. In choosing a random marble from a random container, there are six equally likely outcomes, three R and three B. The

number of outcomes in which the selected ball and the other in its container are both black is two. The condition probability of the other ball being black (event O), given that the selected ball is black (event B), is computed as $P(O|B) = P(O \cap B)/P(B) = (1/3)/(1/2) = 2/3$.

9.10. *Rolling two dice, one red and one green. There are 36 equally likely possible outcomes.*

a) Given that the red die shows a six, the probability of double-sixes is $1/6$. Given that the red die shows a six, the roll is double-sixes if and only if the green die shows a six, which occurs with probability $1/6$.

b) Given that at least one die shows a six, the probability of double-sixes may be $1/6$ or $1/11$, depending on how the information was obtained. If we saw one die and observed that it showed a six, then the probability that the other die also shows a six is $1/6$.

If someone tells us after seeing both dice that at least one shows a six, then we can compute the conditional probability. The event "at least one six" (B) consists of 11 equally likely rolls, of which exactly one is "double-six" (A). The formal computation of conditional probability is $P(A|B) = P(A \cap B)/P(B) = (1/36)/(11/36) = 1/11$.

9.11. *The TV game show problem.* The prize must be behind the contestant's door (event C) or the other unopened door (event D). We seek the conditional probabilities of these two events, given what has happened so far. Let A be the door opened by the host. For each door, the probability of being correct in the game is $1/3$. Given event C, the conditional probability of opening A is $1/2$. Given event D, the conditional probability of opening A is 1. Given that the prize is behind A, the conditional probability of opening it is 0.

Thus the conditional probability of opening A, given the door chosen by the contestant, is $(1/3)(1/2) + (1/3)1 + (1/3)0 = 1/2$. The probability that C occurs and A is opened is $(1/3)(1/2) = 1/6$. The probability that D occurs and A is opened is $(1/3)(1) = 1/3$. The probability that C occurs given that A is opened is thus $(1/6)/(1/2) = 1/3$, and the probability that D occurs given that A is opened is $(1/3)/(1/2) = 2/3$.

9.12. *Bertrand's Paradox.* In the following probability models, let p be the probability that the length of the randomaly generated chord of the unit circle exceeds $\sqrt{3}$. Note that $\sqrt{3}$ is the length of the sides of an equilateral triangle inscribed in the circle.

a) If the endpoints of the chord are generated by two random spins on the circumference of the circle, then $p = 1/3$. After spinning once, draw the inscribed triangle with a corner at the chosen point. The length of the chord will exceed $\sqrt{3}$ if and only if the second chosen point lies between the other two corners of the triangle on the circle. Since probability of lying in an arc is proportional to its length, the probability of this is 1/3.

b) If the midpoint of the chord is generated by throwing a dart at the circle, then $p = 1/4$. The chord is generated from its midpoint P by drawing the chord through P that is perpendicular to the segment from P to the center of the circle. The midpoint of a chord of length $\sqrt{3}$ has distance 1/2 from the center of the circle. Thus the length exceeds $\sqrt{3}$ if and only if the generated midpoint has distance less than 1/2 from the center. Since area is proportional to the square of the radius and probability is proportional to area, the circle of radius 1/2 has 1/4 of the probability.

c) A model where $p = 1/2$. We first pick the distance of the chord from the center; then we choose a random chord with that distance from the center. The length of the chord exceeds $\sqrt{3}$ if and only if its distance from the center is less than 1/2. We pick this distance at random from the interval [0, 1].

9.13. *Triangles on triples among n equally spaced points on a circle.* The $\binom{n}{3}$ triples are equally like. Thus we count how many yield three equal lengths, two equal lengths and one different length, and three different lengths.

When $3|n$, there are $n/3$ triples that form equilateral triangles; otherwise there are none. To generate two equal segments, we choose their common point in n ways and then their common length in $\lfloor (n-1)/2 \rfloor$ ways. When $3|n$, we must subtract from $n\lfloor (n-1)/2 \rfloor$ the $n/3$ equilateral triangles. The resulting probabilities are listed below by the congruence class of n modulo 6.

	0	1	2	3	4	5
equilateral	$\frac{2}{(n-1)(n-2)}$	0	0	$\frac{2}{(n-1)(n-2)}$	0	0
isosceles	$\frac{3n-8}{(n-1)(n-2)}$	$\frac{3}{n-2}$	$\frac{3}{n-1}$	$\frac{3n-5}{(n-1)(n-2)}$	$\frac{3}{n-1}$	$\frac{3}{n-2}$
other	$\frac{n-4}{n-1}$	$\frac{n-5}{n-2}$	$\frac{n-4}{n-1}$	$\frac{n-5}{n-2}$	$\frac{n-4}{n-1}$	$\frac{n-5}{n-2}$

9.14. *In Bertrand's Ballot Problem with outcome (a, b) where $a > b$, the probability that A is always ahead of B is $\frac{a-b}{a+b}$, and the probability that the score is tied at some point is $\frac{2b}{a+b}$.* The two probabilities must sum to 1 when $a > b$. A fails to be always ahead if and only if B has at least as much at some point. Since A winds up ahead, B has at least as much at some point if and only if there is a tie at some point. Thus it suffices to find the probability that A is always ahead.

If A is always ahead of B, then A must get the first vote, and A must get at least as many votes as B in every initial segment of the remainder. The probability that A gets the first vote is $a/(a+b)$. Given that A gets the

first vote, the probability that A never trails in the remaining portion of the election is the solution to the Ballot Problem for an election ending at $(a-1, b)$. We have computed this to be $(a-b)/a$. Since both these events must occur, the probability that A is always ahead is $\frac{a}{a+b}\frac{a-b}{a} = \frac{a-b}{a+b}$.

9.15. *a) If m 0s and n 1s are placed in some order around a circle, then there are exactly $m-n$ positions such that every arc of the circle starting at that position and moving clockwise contains more 0s than 1s.* This holds trivially when $n = 1$. For $m \ge n > 0$, there is some pair of 1s with at least one 0 between them. Let S be a set of two positions consisting of a 0 followed immediately by a 1.

Neither position in S is good; the 1 comes too soon. A position outside S is good if and only if it is good in the smaller arrangement obtained by deleting S. The number of good starting places thus equals the number of good starting places in the smaller arrangement, which by the induction hypothesis is $m-1-(n-1) = m-n$.

b) There are $\frac{1}{n+1}\binom{2n}{n}$ ballot lists of length $2n$. By part (a), every arrangement of n 1s and $n+1$ 0s can be cut in exactly one place to obtain a list in which all initial segments have more 0s than 1s. The first element (just after the cut) must be a 0. Deleting this 0 yields a ballot list, and the process is reversible. Thus the ballot lists and the cyclic arrangements of n 1s and $n+1$ 0s are equinumerous. Since n and $n+1$ are relatively prime, the number of these cyclic arrangements (and ballot lists of length $2n$) is exactly the Catalan number $\frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n} = C_n$.

9.16. *A conditional probability computation.* We are given three independent random variables X_1, X_2, X_3, each defined on $[n]$ and equally likely to take on any value in $[n]$. The variable $X_1 + X_2$ is less than 4 only for the outcomes $(1,2), (1,1), (2,1)$, which together have probability $3/n^2$. Thus $P(X_1 + X_2 \ge 4) = 1 - 3/n^2$. The events $X_1 + X_2 + X_3 \le 6$ and $X_1 + X_2 \ge 4$ both occur if and only if $X_3 = 2$ and $(X_1, X_2) \in \{(3,1), (2,2), (1,3)\}$ or $X_3 = 1$ and $(X_1, X_2) \in \{(3,1), (2,2), (1,3), (4,1), (3,2), (2,3), (1,4)\}$. Thus the probability that both occur is $\frac{1}{n}\cdot\frac{3}{n^2} + \frac{1}{n}\cdot\frac{7}{n^2} = \frac{10}{n^3}$. The desired conditional probability is $\frac{10}{n^3}/\frac{n^2-3}{n^2} = \frac{10}{n^2-3}$. The computation is valid only for $n \ge 4$.

9.17. *Application of Bayes' Theorem.* Against her four opponents, the tennis player wins with probability .6, .5, .45, .4, respectively. These are the conditional probabilities $P(A|B_i)$, where A is the event that she wins and B_i is the event that she plays the ith opponent. We want to compute $P(B_i|A)$ for each i. Since she plays 30 matches against each of the first two and 20 matches against each of the last two opponents, we have $P(B_i) = .3, .3, .2, .2$, respectively. Thus we have all the data to compute The denominator, which equals $P(A)$, is $.6\cdot .3 + .5\cdot .3 + .45\cdot .2 + .4\cdot .2 = .5$. Thus the answers are $P(A|B_i)P(B_i)\cdot 2$, which equal .36, .30, .18, .16, respectively.

9.18. *If half the females and one-third of the males in a class are smokers, and two-thirds of the students are male, then of the smokers are female.* Consider choosing a random student in the class. Let A be the event of being a smoker. Let B_1, B_2 be the events of being female or male, respectively. We seek $P(B_1|A)$. We are given $P(A|B_1) = 1/2$, $P(A|B_2) = 1/3$, and $P(B_2) = 2/3$. By Bayes' Formula, we compute

$$P(B_1 + A) = \frac{a_i b_i}{\Sigma a_j b_j} = \frac{(1/2)(1/3)}{(1/2)(1/3) + (1/3)(2/3)} = \frac{3}{7}.$$

9.19. *Simpson's paradox - it is possible for A to have a higher batting average than B in day games and night games but for B to have a higher average overall.* If the daytime hits/attempts are a/b and c/d, respectively, and the nighttime hits/attempts are w/x and y/z, respectively, then inequalities characterizing the situation described are $a/b > c/d$, $w/x > y/z$, and $(c+y)/(d+z) > (a+w)/(b+x)$. The situation described occurs if and only if these three inequalities are satisfied by eight integers also satisfying $0 \le a \le b$, $0 \le c \le d$, $0 \le w \le x$, $0 \le y \le z$. Equivalently, the three inequalities can be written as $ad > b_c$, $wz > xy$, and $(c+y)(b+x) > (a+w)(d+z)$.

Note that if $b = d$ and $x = z$, then the inequalities lead to a contradiction and the paradox cannot occur. This suggests that one way in which the paradox can occur is if the poor daytime performance of B and the good nighttime performance of A are unimportant to their season averages, as would occur if B hardly ever plays in daylight and A hardly ever plays at night. For example, the paradox occurs with the numbers $(a, b) = (3, 10)$ and $(c, d) = (0, 1)$ for daytime and $(w, x) = (1, 1)$ and $(y, z) = (7, 10)$ for nighttime. For the season, the averages are 4/11 and 7/11.

9.20. *Simpson's Paradox with university faculties of equal size.*

	Univ. H	Univ. Y				
	Total	Women	Fraction	Total	Women	Fraction
Asst. Profs.	40	25	.6	80	40	.5
Assoc. Profs.	30	5	.17	10	0	0
Full Profs.	30	5	.17	10	0	0
Total	100	35	.35	100	40	.4

9.21. *A perfect game in bowling.* If on each roll the probability of a strike is p, then the probability that 12 consecutive rolls are strikes is p^{12}. The value of p such that $p^{12} = .01$ is approximately .60. This suggests that perfect games are quite rare but not unheard of among very good bowlers.

9.22. *Probabilities for people in a line.* We have A, B, and n other people in random order, meaning that all $(n+2)!$ permuations are equally likely. To form a permutation with exactly k people between A and B, form an arbitrary permutation of the others, decide which of A and B comes first, and insert A and B so they will be separated by k. There are $n+1-k$ ways to make the insertion. Thus the probability is $\frac{n!2(n+1-k)}{(n+2)!}$, which simplifies to $\frac{2(n+1-k)}{(n+1)(n+2)}$. To check that these sum to 1, we compute

$$\sum_{k=0}^{n} \left(\frac{2}{n+2} - \frac{2k}{(n+1)(n+2)}\right) = \frac{2(n+1)}{n+2} - \frac{2}{(n+1)(n+2)} \frac{(n+1)n}{2} = \frac{2n+2-n}{n+2} = 1$$

9.23. *Probability of first player getting first Head.* The probability of heads is p on each flip. If x is the probability that the first player wins, then x is the sum of the probability of winning on the first flip and the probability of winning later. Winning later requires tails from each player on the first round, but then the rest of the game is the same as the original game. Thus $x = p + (1-p)^2 x$, which yields $x = p/[1-(1-p)^2] = 1/(2-p)$. When the coin is fair, the first player wins with probability 2/3. (Note: the probability that the first player wins on the ith try is $p(1-p)^2(i-1)$. Thus also $x = \sum_{i=1}^{\infty} p(1-p)^2(i-1)$.

9.24. *A spinner with regions* $1, 2 \ldots, n$ *and payoff* 2^k *for region* k.

a) the expected payoff per spin of the dial is $(2^{n+1}-2)/n$. The payoff is 2^k with probability $1/n$, so the expected payoff is $(1/n)\sum_{k=1}^{n} 2^k$, which equals $(2^{n+1}-2)/n$.

b) Given a coin flip to move to a neighboring region, the gambler should accept the gamble unless the current region is n. If the gambler already has the maximum payoff, then switching is a guaranteed loss. If the current payoff is the minimum, then switching is a guaranteed gain, with expected payoff $\frac{1}{2}2^n + \frac{1}{2}4$. When $2 \le k \le n-1$, the expected payoff for switching is $\frac{1}{2}2^{k+1} + \frac{1}{2}2^{k-1}$. This equals $2^k + 2^{k-2}$ and exceeds 2^k.

The expected payoff under the optimal strategy is

$$\frac{1}{n}\left(2^{n-1} + 2 + 2^n + \sum_{k=2}^{n-1}(2^k + 2^{k-2})\right) = \frac{1}{n}\left(2^{n+1} + 2^{n-1} + 2^{n-2} - 4\right)$$

9.25. *Another gambler.* The amounts in envelopes are $a_1 \le a_2 \cdots \le a_n$. For $1 \le i \le n-1$ the probability is p_i that the envelopes contain a_i and a_{i+1} dollars. He opens one envelope and has the option to switch. If he sees a_1 dollars, then he switches for a guaranteed gain. If he sees a_n dollars, then switching would be a guaranteed loss.

For $2 \le k \le n-1$, when he sees a_k the other envelope may contain a_{k-1} or a_{k+1}. Since he opened one of the two envelopes at random, the probability that he sees a_k and the other is a_{k-1} is $p_{k-1}/2$, and the probability that he sees a_k and the other is a_{k+1} is $p_k/2$. The probability that he sees a_k is the sum of these, $(p_{k-1} + p_k)/2$. The conditional probability that the other is a_{k-1} is $p_{k-1}/(p_{k-1} + p_k)$, and the conditional probability that the other is a_{k+1} is $p_k/(p_{k-1} + p_k)$.

Thus the expected payoff for switching when he sees a_k is

$$\frac{a_{k-1}p_{k-1} + a_{k+1}p_k}{p_{k-1} + p_k}.$$

He should accept the gamble to switch if this quantity exceeds a_k.

9.26. *If X is a random variable that takes values only in $[n]$, then $E(X) = \sum_{k=1}^{n} P(X \ge k)$.* Let $p_i = P(X = i)$. We compute

$$E(X) = \sum_{i=1}^{n} i p_i = \sum_{i=1}^{n} \sum_{k=1}^{i} p_i = \sum_{k=1}^{n} \sum_{i=k}^{n} p_i = \sum_{k=1}^{n} P(X \ge k).$$

Essentially, we add up in two ways the probability p_i for all pairs (k, i) such that $1 \le k \le i \le n$.

9.27. *Expected number of selections to find the desired key.*

a) Under random order without replacement, the expectation is $(n + 1)/2$. All permutations of the keys are equally likely for the selection order. Thus the right key is equally likely to be in each position. With probability $1/n$ that the key is in position k, the expected position is $(1/n)\sum_{1=i}^{n} k = (n+1)/2$.

b) Under random selection with replacement, the expectation is n. On each selection, the probability of success is $1/n$. Thus we seek the expectation for the geometric distribution with success probability $1/n$. By Proposition 9.29, the expectation is n.

9.28. *The expected number of matching pairs among k socks pulled from a pile of n pairs of socks is $\binom{k}{2}/(2n-1)$.* In each proof below, we express the number X of matching pairs as a sum of indicator variables, and then we use linearity of the expectation. In proof 1, we extract a random k-set of socks. In proofs 2 and 3, we assume that the k socks are extracted in some order. This produces every k-set in $k!$ ways, so the results are the same when we make the ordered selections equally likely as when we make the k-sets equally likely.

Proof 1. Let X_i be 1 if pair i appears; otherwise $X_i = 0$. Of the $\binom{2n}{k}$ equally likely choices for the k socks, $\binom{2n-2}{k-2}$ include pair i. Hence

$$P(X_i = 1) = \binom{2n-2}{k-2} \Big/ \binom{2n}{k} = \frac{k(k-1)}{2n(2n-1)},$$

and $E(X_i) = P(X_{i,j} = 1) = \frac{k(k-1)}{2n(2n-1)}$. Since $X = \sum_{i=1}^{n} X_i$, and since the variables $X_1, \ldots, X_n$ have the same distribution, $E(X) = nE(X_i) = \binom{k}{2}/(2n-1)$.

Proof 2. Let $X_{i,j}$ be 1 if the ith and jth socks match; otherwise $X_{i,j} = 0$. Any two socks are equally likely to be chosen ith and jth. Of the $\binom{2n}{2}$ choices of two socks, n are matched pairs. Hence $P(X_{i,j} = 1) = n/\binom{2n}{2} = 1/(2n-1)$, and $E(X_{i,j}) = P(X_{i,j} = 1) = 1/(2n-1)$. Since X is the sum of $\binom{k}{2}$ such variables, $E(X) = \binom{k}{2}/(2n-1)$.

Proof 3. Let X_j be 1 if the jth sock extracted completes a pair with an earlier sock, so $\sum_{j=1}^{k} X_j$ counts the completed pairs of socks. The probability that the mate of the jth sock appears among the first $j-1$ socks is $(j-1)/(2n-1)$. Thus $E(X_j) = (j-1)/(2n-1)$, and

$$E(X) = \sum_{j=1}^{k} E(X_j) = \frac{1}{2n-1}\sum_{j=1}^{k}(j-1) = \binom{k}{2}/(2n-1).$$

9.29. *Given a set of n men and n women, the expected number of male-female couples in a random pairing is $n^2/(2n-1)$.* Let X be the random variable giving the number of male-female couples. We express X as the sum of n^2 indicator variables. Define $X_{i,j}$ by $X_{i,j} = 1$ when the ith man is paired with the jth woman, $X_{i,j} = 0$ otherwise. By the linearity of expectation, $E(X) = n^2 E(X_{i,j})$.

By symmetry, an individual is equally likely to be paired with any one of the other $2n-1$. Thus $E(X_{i,j}) = 1/(2n-1)$, and $E(X) = n^2/(2n-1)$.

9.30. *There are 34,650 arrangements of the letters in $MISSISSIPPI$.* Among the 11 letters, there are four Is, four Ss, two Ps, and one M. Thus we compute $\binom{11}{4,4,2,1}$, which equals $\frac{11!}{4!4!2!1!}$

9.31. *Solution 9.40 overestimates the probability of hitting for the cycle.* Not all the appearances of a batter count as an at-bat. Other outcomes are possible; walks and sacrifices are not counted as at-bats.

9.32. *A polynomial p such that $p(n) = 3^n$ for $n = 0, 1, 2, 3, 4$.* The multinomial expansion of $(1+1+1)^n$ is a sum of multinomial coefficients, since always $1^{k_i} = 1$. We write the coefficients as

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n(n-1)\cdots(n-k_1-k_2+1)}{k_1!k_2!}$$

For each choice of k_1 and k_2, this expression is a polynomial in n. If we sum all such expressions that are nonzero when $0 \le n \le 4$, then the value at such n will be 3^n. There are 15 nonnegative integer solutions to $k_1 + k_2 \le 4$. Coefficients with $k_1 + k + 2 = r$ can be expressed as multiples of $n(n-1)\cdots(n-r+1)$. Summing these 15 multinomial coefficients yields the desired polynomial as

$$1+2n+2n(n-1)+\frac{4}{3}n(n-1)(n-2)+\frac{2}{3}n(n-1)(n-2)(n-3).$$

Note that evaluating at $n=0,1,2,3,4$ yields $1,2,3,9,27,81$, respectively.

9.33. *Random monomials in k variables with total degree n.* The possible lists of exponents for the k monomials are the nonnegative integer solutions to $e_1+\cdots+e_k=n$. The number of these (see Theorem 5.23) is $\binom{n+k-1}{k-1}$.

a) The probability that all k variables have positive exponent in the chosen monomial is $\prod_{i=1}^{k-1}\frac{n-i}{n+k-i}$. The outcomes in which all k variables have positive exponent correspond to the positive integer solutions to $e_1+\cdots+e_k=n$. There are $\binom{n-1}{k-1}$ of these. Dividng by $\binom{n+k-1}{k-1}$ and canceling like factors yields the probability.

b) When $(n,k)=(10,4)$, the probability that the exponents are different is $\frac{60}{143}$. Four distinct values summing to 10 can be $\{4,3,2,1\}$ or $\{5,3,2,0\}$ or $\{5,4,1,0\}$ or $\{6,3,1,0\}$ or $\{7,2,1,0\}$. Each such set can be assigned to variables in 24 ways. Hence the probability of this event is $120/\binom{13}{3}$, which simplifies to $\frac{60}{143}$. (Here 0 is allowed as an exponent.)

9.34. *Rolling six dice, each having three red faces, two green faces, and one blue face: the probability of getting the same totals over six dice is $5/36$.* The six physical dice correspond to six positions in an arrangement. The number of ways of getting three red, two green, one blue is the multinomial coefficient $\binom{6}{3,2,1}=\frac{6!}{6\cdot2\cdot1}=60$. On each die the probabilities of red, green, blue are $1/2$, $1/3$, $1/6$, respectively. Thus the probability of each of these arrangements is $(1/2)^3(1/3)^2(1/6)^1$, and the probability of the specified event is $\frac{60}{8\cdot9\cdot6}=\frac{5}{36}$.

9.35. *The coefficient of x^4y^{16} in the expansion of $(x+xy+y)^{16}$ is 1820.* Each $x+xy+y$ factor contributes 1 to the exponent of x or 1 to the exponent of y or 1 to each exponent. Since the total degree of x^4y^{16} is 20 and we have 16 factors, we must choose the term xy four times. Hence we choose the term x not at all and the term y 12 times. The coefficient is the number of ways we can arrange these choices, $\binom{16}{0,4,12}$. This equals $16\cdot15\cdot14\cdot13/(4\cdot3\cdot2\cdot1)$.

The coefficient of x^4y^8 in the expansion of $(x^2+xy+y^2)^6$ is 90. Each $x+xy+y$ factor contributes 2 to the exponent of x or 2 to the exponent of y or 1 to each exponent. We can obtain contributions to the coefficient of x^4y^8 by choosing xy from none, two, or four factors (taking xy an odd number of times yields odd total exponents). The contributions in the three cases are the numbers of ways to arrange the choices: $\binom{6}{2,0,4}$, $\binom{6}{1,2,3}$, and $\binom{6}{0,4,2}$, respectively. These have values 15, 60, and 15, respectively.

9.36. *Bijection from ballot paths to nondecreasing functions $f\colon[n]\to[n]$ such that $f(i)\le i$ for all i.* Given a ballot path P, define f by letting $f(i)$ be one more than the vertical coordinate where P steps from horizontal coordinate $i-1$ to horizontal coordinate i. Since P cannot step down and cannot be above height $i-1$ when the horizontal coordinate is $i-1$, f is nondecreasing and satisfies $f(i)\le i$ for all i. Given such a function f, the only path P that yields f by this map is the path that includes the horizontal steps from $(i-1,f(i)-1)$ to $(i,f(i)-1)$ and the vertical edges from $(i,f(i)-1)$ to $(i,f(i+1)-1)$ for all i. Hence this map is a bijection.

9.37. *The number of lattice paths of length $2n$ that never step above the diagonal is $\binom{2n}{n}$.* Each such lattice path ends at a point $(k,2n-k)$ with $k\ge n$. By Solution 9.10, the number of paths ending at (a,b) that do not step above the diagonal is $\binom{a+b}{a}-\binom{a+b}{a+1}$. Hence the answer is $\sum_{k=n}^{2n}[\binom{2n}{k}-\binom{2n}{k+1}]$. The sum telescopes, leaving only $+\binom{2n}{n}$ at the beginning and -0 at the end.

9.38. *Bijections involving ballot lists of length $2n$.*

a) From the set S of nonnegative integer lists $a_1,\ldots,a_{2n+1}$ such that consecutive entries differ by 1 and $a_1=a_{2n+1}=0$. Given a ballot list $b_1,\ldots,b_{2n}$, let a_i be the excess of the number of ones over the number of zeros among the first $i-1$ terms of b. The resulting $2n+1$-tuple a is in S. The map is easy to invert. Given $a\in S$, let $b_i=1$ if $a_{i+1}>a_i$, and $b_i=0$ if $a_{i+1}<a_i$.

b) From the set of arrangements of $2n$ people in 2 rows of length n so that heights are increasing in each row and column. Given a ballot list with n As and n Bs, let the positions correspond to the $2n$ people in increasing order of height. If the ith position in the list is the jth A, place the ith person in the jth position in the first row. If the ith position in the list is the jth B, place the ith person in the jth position in the second row. Since the people are in increasing order in the original list, the rows of the new arrangement are increasing in height. Since the jth A in a ballot list occurs before the jth B, the columns of the new arrangement are increasing in height. To obtain the unique ballot list from which a particular arrangement comes, consider the people in increasing order of height, and let position i of the list be an A if and only if the ith person is in the first row.

9.39. *Bijection from ballot lists of length $2n$ to non-crossing pairings of $2n$ points on a circle.* Given a non-crossing pairing of $2n$ points $p_1,\ldots,p_{2n}$ in order on the circle, record in position i a 1 if p_i is paired with a higher point, a 0 if p_i is paired with a lower point. For each pair of points, the resulting 1 appears in the list before the resulting 0. Hence every initial segment has at least as many 1s as 0s, and the list is a ballot list.

Given any ballot list of length $2n$, we can invert the map as follows: When we reach a 1 in position i, we leave p_i unmatched. When we reach a 0 in position i, we match p_i to the point corresponding to the most recent unmatched 1, which exists since we have seen at least as many 1s as 0s. No crossing occurs, because when the second point of the first pair in a crossing is reached, it would have been matched instead to the first point in the second pair, since that corresponded to a more recent unmatched 1.

9.40. *The Finger Game (see Example 8.20).*

a) values of x that guarantee a positive expectation for B no matter what A does. If B plays one finger with probability x, then the expected payoff to B when A plays one finger is $3 - 5x$ dollars per game. When A plays two fingers, the expectation is $7x - 4$. To guarantee positive expectation, B chooses x so that $3 - 5x > 0$ and $7x - 4 > 0$. This requires $x < 3/5$ and $x > 4/7$, so the solution is the interval $4/7 < x < 3/5$ (not very big).

b) Probability of winning. In the solution that optimizes the amount won, each player plays one finger with probability $7/12$. Since B wins when the fingers differ, the probability that B wins in this situation is $(7/12)(5/12) + (5/12)(7/12) = 70/144$. In other words, B wins slightly less than half of the games, but loses a small enough amount when losing that B comes out ahead in money in the long run.

9.41. *Paying each option with equal probability is optimal for both players in the matrix game with payoffs $\binom{a\ b}{c\ d}$ to the row player if and only if $a = d$ and $b = c$.* If the column player (B) plays each column with probability $1/2$, then the expected payoff to the row player (A) is half the row sum in each row. Thus playing the options with equal probability can be optimal for both players only if the rows have the same sum. By the analogous argument, the columns must also have the same sum.

Thus d must equal both $a + b - c$ and $a + c - b$. This yields $b = c$. Similarly, $c = b + a - d = b + d - a$ yields $a = c$. For every such matrix, the equal strategy guarantees that A has expected payoff at least $(a + b)/2$ and that B has expected loss at most $(a + b)/2$.

9.42. *The value of the game with payoff matrix $\binom{a\ b}{c\ d}$.* If any entry e is a row minimum and a column maximum (a "saddlepoint"), then the row player will play that row, guaranteeing winning at least e, and the column player will play that column, guaranteeing losing at most e. Since the column player cannot guarantee losing less than the row player can guarantee winning, these strategies are optimal, and the value of the game is e.

By playing row 1 with probability x, the row player guarantees receiving the minimum of $ax + c(1 - x) = c + (a - c)x$ and $bx + d(1 - x) = d + (b - d)x$. If the matrix has no saddlepoint, then $a - c$ and $b - d$ have opposite signs and the graphs of the two linear functions cross at some x

in the interval $[0, 1]$. Hence the row player guarantees the best return by choosing x to make these equal; this maximizes the minimum. He chooses $x = (c - d)/(b - d - a + c)$; the denominator is non-zero. Using this in the expression for the expected return in either column yields $\frac{bc-ad}{b-d-a+c}$. This is the maximum that the row player can guarantee. Similarly, the column player can guarantee losing no more than this.

10. TWO PRINCIPLES OF COUNTING

10.1. During a major league baseball season in which there are 140,000 at-bats and 35,000 hits,

a) Some player hits exactly .250—FALSE. It may happen that there are 1000 players, each with 140 at-bats, half of them having 28 hits each (batting .200) and the other half having 42 hits each (batting .300).

b) Some player hits at least .250—TRUE. If each hitter has more than four times as many at-bats as hits, then the total number of at-bats would be more than four times the total number of hits, which would contradict the given data.

c) Some player hits at most .250—TRUE. If each hitter has less than four times as many at-bats as hits, then the total number of at-bats would be less than four times the total number of hits, which would contradict the given data.

10.2. *Distinct three-person committees over 11 years.* If a set of size n has 11 distinct triples, i.e. $\binom{n}{3} \geq 11$, then $n \geq 6$.

10.3. *If S is a $(2n + 1)$-subset of $\{1, 2, \ldots, 3n\}$, then S contains three consecutive numbers, and this is not forced for a $2n$-subset.* Partition $[3n]$ into n classes, with the ith class consisting of the numbers $\{3i - 2, 3i - 1, 3i\}$. Since S has $2n + 1$ distinct numbers, by the pigeonhole principle it must have 3 numbers from some class. These three numbers are consecutive. On the other hand, the set that omits all the numbers divisible by three has $2n$ numbers but no three consecutive numbers (it has the first two numbers from each class).

10.4. *Every set of $n + 1$ numbers in $[2n]$ contains a pair of relatively prime numbers.* Any pair of consecutive numbers is relatively prime, since an integer greater than 1 cannot divide two consecutive integers. Hence it suffices to partition the set $[2n]$ into the n pairs of the form $(2i - 1, 2i)$. Since there are only n of these pairs, the pigeonhole principle guarantees that a set of $n + 1$ numbers in $[2n]$ must use two from some pair, and these are relatively prime.

The result is best possible, because in the set of n even numbers in $[2n]$, every pair has a common factor. Note that since each pair of even numbers is not relatively prime, a solution to the problem by partitioning $[2n]$ into n classes and applying the pigeonhole principle must put the n even numbers into n different classes.

10.5. *Every set of seven distinct integers contains a pair whose sum or difference is a multiple of 10.* If two numbers in our set of seven are in the same congruence class modulo 10, then we have a pair whose difference is a multiple of 10. If one is congruent to i and another is congruence to $10-i$ mod 10, then we have a pair whose sum is a multiple of 10.

We therefore define six sets, each of which is a congruence class or a pair of congruence classes that sum to $\overline{0}$. These sets are $\overline{0}, \overline{1}\cup\overline{9}, \overline{2}\cup\overline{8}, \overline{3}\cup\overline{7}, \overline{4}\cup\overline{6}, \overline{5}$. These sets partition the set of integers. Whenever we choose seven integers, the pigeonhole principle guarantees that two must lie in the same set, and then their sum or difference is a multiple of 10.

10.6. *If 1 through 10 appear in some order around a circle, then some three consecutive numbers sum to at least 17.* Each number appears in three such triples, so the total of the sums of triples is $3\sum_{i=1}^{10} i = 3\binom{11}{2} = 165$. When 165 units fall into ten triples, at least one triple receives at least 17.

10.7. *Numbers 1 through 12 on a clock face.*

For every arrangement of the numbers 1 through 12 in a circle, there are three consecutive numbers with sum at least 20. Given an arbitrary arrangement, partition the 12 numbers into 4 consecutive triples. The 78 units in the sum $\sum_{i=1}^{12} i = 13\cdot 12/2$ are distributed into these four classes, so some class receives at least the average of $78/4 = 19.5$ units. Since each class receives an integer amount, some class receives at least 20.

There are five consecutive numbers with sum at least 33. We cannot partition 12 objects into sets of size five. Nevertheless, each number belongs to five sets of five consecutive numbers. Thus the twelve sets of five consecutive numbers have sums that total to $5\sum_{i=1}^{12} i = 390$, and the largest of these sums must be at least $\lceil 390/12\rceil = 33$.

The sets of three consecutive elements cannot all have sum 19 or 20. If there is such an arrangement, then successive triples (overlapping in two positions) cannot have the same total, because this puts the same number at the beginning of the first triple and the end of the second. Thus the triples alternate between sum 19 and sum 20. We may assume that the triples starting in odd positions have sum 19. Since $x_1+x_2+x_3 = x_5+x_6+x_7 = 19$ and $x_2+x_3+x_4 = x_4+x_5+x_6 = 20$, we have $x_1 = x_4 - 1 = x_7$. Thus the diametrically opposite numbers are the same, which is forbidden.

10.8. *Given five points in a square with sides of length 1, the distance between some pair of them is at most $\sqrt{2}/2$.* Express the square as the union of four subsquares with sides of length $1/2$. The distance between any two points in a square with sides of length is at most $\sqrt{2}/2$. Therefore, every configuration that has two points in one subsquare has the desired property. Since we have only four subsquares (classes), the pigeonhole principle guarantees that the five given points have two in some class, and the distance between these is at most $\sqrt{2}/2$.

The upper bound $\sqrt{2}/2$ for the minimum distance is best possible, because the configuration with points at the four corners and one point in the center has no pair of points closer together than $\sqrt{2}/2$.

10.9. *Pigeonhole generalization: If $p_1,\ldots,p_k$ are natural numbers, then $1-k+\sum p_i$ is the minimum n such that every way of distributing n objects into classes $1,\ldots,k$ yields at least p_i objects in class i for some i.* If the ith class has fewer than p_i objects, then the total number of objects is at most $\sum(p_i-1) = -k+\sum p_i$. Thus some threshold must be met when there are $1-k+\sum p_i$ objects. A total of $-k+\sum p_i$ does not suffice, since then we can put p_i-1 objects in class i for each i.

10.10. *If ten people each mark off a 100-yard field on a 400-yard segment, then some point lies in at least four fields.* Let $x_1,\ldots,x_{10}$ be the starting points of the ten fields, with $0\le x_1\le\cdots\le x_{10}\le 300$. Consider the three disjoint intervals $[0,100)$, $[100,200)$, and $[200,300]$. With ten numbers in these three intervals, by the pigeonhole principle at least one of the three contains at least four of the starting points. The right endpoint of that interval is in the four fields having those starting points.

10.11. *Multiples of x near an integer.* Let $S = \{x, 2x, \ldots, (n-1)x\}$.

a) For $n\in\mathbb{N}$, if two numbers in S have fractional parts differing by at most $1/n$, then some number in S differs by at most $1/n$ from an integer. Let ix and jx have fractional parts differing by at most $1/n$, where $j>i$. That is, $ix = k+y$ and $jx = l+z$, where $k,l\in\mathbb{Z}$ and $|y-z|\le 1/n$. Thus $(j-i)x = (l-k)+z-y$. Since $l-k$ is an integer, we conclude that $(j-i)x$ is within $1/n$ of an integer. Also $(j-i)x\in S$, since $0<j-i<j$.

b) Some number in S is within $1/n$ of an integer. All fractional parts lie in the interval $[0,1)$. Let $I_r = [(r-1)/n, r/n)$, so $[0,1) = \bigcup_{r=1}^{n} I_r$. If some number in S has fractional part in I_1 or I_n, then it is within $1/n$ of an integer. Otherwise, the fractional parts of the $n-1$ numbers in S lie in the $n-2$ intervals $I_2,\ldots,I_{n-1}$. By the pigeonhole principle, some two of them lie in one of these intervals. Since the intervals have length $1/n$, we have two numbers in S with fractional parts that differ by at most $1/n$. Part (a) now completes the proof.

10.12. *Every set of n integers has a nonempty subset whose sum is divisible by n.* Let $S = \{b_1, \ldots, b_n\}$. We prove the stronger result that some consecutive set $\{b_{i+1}, \ldots, b_j\}$ has sum is divisible by n. The sum $\sum_{k=i+1}^{j} b_k$ is divisible by n if and only if $\sum_{k=1}^{i} b_k$ and $\sum_{k=1}^{j} b_k$ have the same remainder upon division by n. This suggests introducing the partial sums $a_j = \sum_{k=1}^{j} b_k$. We have n such sums, and we place them in the remainder classes $0, \ldots, n-1$. If any a_j is divisible by n, we are done, and $\{b_1, \ldots, b_j\}$ is the desired set. Otherwise, we are placing n partial sums in the $n-1$ remaining classes, and by the pigeonhole principle a pair of these must go in the same class. As noted above, if a_i and a_j have the same remainder under division by n, then $\{b_{i+1}, \ldots, b_j\}$ is the desired set.

The integers $1, \ldots, 1$ (or any list of $n-1$ integers congruent to 1 modulo n) are integers with no nonempty subset summing to a multiple of n.

10.13. *If S is a list of $n+1$ positive integers with sum k, for $k \le 2n+1$, then for each $i \le k$, some subset of S has sum i.* **Proof 1.** Suppose that S has largest element r and has m copies of the number 1. If $m \ge r-1$, then it is possible to add 1s successively, increasing the subset sum by one each time until reaching $r-1$. The next set is the number r alone. Then the 1s can be included one-by-one again until the next largest number in the collection can be substituted for them, and so on. To see that $m \ge r-1$, observe that the bound on the sum yields $1 \cdot r + (n-m) \cdot 2 + m \cdot 1 \le k \le 2n+1$, which simplifies to $m \ge r-1$. When $k = 2n+2$, the property fails, violated by a collection of $n+1$ 2s.

Proof 2. By induction on n; for $n = 0$, the only example is $\{1\}$, which works. Now suppose $n \ge 1$. If S contains only 1s then all sums can be achieved, so assume some integer $a \in S$ exceeds 1. Since a exceeds 1, the sum of the remaining n elements is less than $2n$, so we can apply the induction hypothesis to obtain subsets of $^{-}\{a\}$ summing to all integers from 0 to $k-a$. For $k-a+1 \le i \le k$, adding the element a to such a subset summing to $i-a$ will construct a subset of S summing to i, if $i-a \ge 0$. This requires $a \le (1+k)/2$. This holds, since $2a > k+1$ (the bad condition) and $a+n \le k$ (the sum of S) would imply $a > n+1$ and then $k \ge 2n+2$.

10.14. *Any six people contain three that pairwise know each other or three that pairwise don't know each other.* Consider an arbitrary person x among the six. Among the five other people, the pigeonhole principle guarantees that there must be at least three who know x or at least three who don't know x. By symmetry (between acquaintance and nonacquaintance), we may assume that x knows at least three others. If two who know x know each other, then we have (at least) three people who all know each other. Otherwise, we have (at least) three people (all knowing x) none of whom know each other.

10.15. *Every set of 55 numbers in* [100] *contains two numbers differing by 9, and this is best possible.* If there are 55 numbers, then some congruence class mod 9 contains at least seven of the numbers, by the pigeonhole principle. Each congruence class mod 9 contains exactly 11 numbers in [99]. If 11 numbers are placed in a row and seven are chosen, then some consecutive pair must be chosen (partition them into six classes consisting of a pair of consecutive numbers and one singleton class in the case of 11 – the pigeonhole principle implies two numbers are chosen from one such pair). This completes the proof, because when two consecutive numbers are taken from a congruence class modulo 9, their difference is exactly 9.

The proof suggests the example with 54 numbers. To avoid having two numbers differing by 9, we must take the six numbers in odd positions in each congruence class of size 11, and we can take the evens or the odds from the class of size 12. The resulting set is 1–9, 19–27, 37–45, 55–63, 73–81, 91–99. This is the only example.

10.16. *For $n = qk + r$ with $1 \le r \le k$, the largest subset of $[n]$ having no two numbers differing by k has size $(n+r)/2$ if q is even and $(n+k-r)/2$ if q is odd.* Each congruence class modulo k has q or $q+1$ elements in $[n]$, and r of them have size $q+1$. Numbers differing by k are consecutive elements of a congruence class modulo k.

If a congruence class C has s elements in $[n]$, and more than $\lceil s/2 \rceil$ elements of C lie in S, then by the pigeonhole principle two consecutive elements of C lie in S. (Explicitly, consider $\lfloor s/2 \rfloor$ disjoint classes of two consecutive elements of C, plus the last element by itself if s is odd. These sets partition C. Choosing more elements than sets forces two element into a single set, and two elements from one set are consecutive.) This bound is best possible, because the $\lceil s/2 \rceil$ odd-indexed elements are non-consecutive.

With this argument, we must have at most $\lceil q/2 \rceil$ elements in the congruence classes of size q and at most $\lceil (q+1)/2 \rceil$ elements in the congruence classes of size $q+1$. Since there are r classes of size q, our set is limited to size $r\,\lceil (q+1)/2 \rceil + (k-r)\,\lceil q/2 \rceil$. In comparison to $r(q+1)/2 + (k-r)(q/2) = n/2$, we gain $r/2$ if q is even and $(k-r)/2$ if q is odd. The bound is achieved by taking the numbers $1, \ldots, k$, then $2k+1, \ldots, 3k$, and so on, up to n.

10.17. *The Erdős-Szekeres result on monotone sublists is best possible.* The list of mn integers below has no increasing sublist of length $m+1$ and no decreasing sublist of length $n+1$. Partition the values $1, \ldots, mn$ into m sets, with the jth set being $\{(j-1)n+1, \ldots, jn\}$. Record each set in succession in the list, but write each set down in reverse order. The list is thus

$$n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, \ldots, mn, mn-1, \ldots, (m-1)n+1.$$

This list has the desired property because a decreasing sublist has elements only in a single segment, but the segments have length n, and an increasing sublist has at most one element from each segment, but there are only m segments.

10.18. *Multiple exam solutions.* In an exam with three true/false questions, in which every student answers each question, there are eight possible answer sheets: TTT, TTF, TFT, FTT, TFF, FTF, FFT, FFF.

a) At least nine students are needed to guarantee that no matter how they answer the questions, some two students agree on every question. If there are eight students, the list of eight answer sheets above has no two agreeing on every question. If there are nine or more students, then because all answer sheets are of these eight types, the pigeonhole principle implies that some two students have the same answers.

b) At least five students are needed to guarantee that no matter how they answer the questions, some two students agree on at least two questions. If two answer sheets agree in exactly two places, then they differ in the third place, and hence the parity of the number of Ts differs. The four distinct answer sheets FFF, TFT, FTT, TTF all have an even number of Ts, so no two of them agree in two places.

There are four possible answer pairs for the first two questions. With five students, some pair must agree on the first two questions.

Alternative proof. There are four possible sheets with even number of Ts and four with an odd number. Five students having the same parity of Ts would yield two identical sheets. Otherwise, two have opposite parity. They agree in two places unless they disagree in all three, but then every remaining answer sheet agrees in at least two places with one of them.

10.19. *Private club needing 990 keys.*

990 keys permit every set of 90 members to be housed. Suppose 90 members receive one key apiece, each to a different room, and the remaining 10 members receive keys to all 90 rooms. Each set of 90 members that might arrive consists of k members of the first type and $90-k$ members of the second type. When the k members of the first type go to the rooms for which they have keys, there are $90-k$ rooms remaining, and the $90-k$ members of the second type that are present have keys to those rooms.

No scheme with fewer keys works. If the number of keys is less than 990, then by the pigeonhole principle (every set of numbers has one that is at most the average) there is a room for which there are fewer than 11 keys. Since the number of keys to each room is an integer, there are at most 10 keys to this room. Hence there is a set of 90 of the 100 members that has no one with a key to this room. When this set of 90 members arrives, they have keys to at most 89 rooms among them and cannot all be housed.

10.20. *Generalization of the Chess Player Problem.* The chess player plays on d consecutive days for a total of at most b games. If k is small enough and $b \le 2d-1$, then there must be a period of consecutive days during which a total of exactly k games are played. The range for which the argument of Example 10.8 works is $k \le 2d-b$.

Let a_i be the total number of games played on days 1 through i, and set $a_0 = 0$. Then $a_j - a_i$ is the total number of games played on days $i+1$ through j. We seek i and j such that $a_i + k = a_j$. Since there is at least one game each day, the numbers in $\{a_j\colon 1 \le j \le d\}$ are distinct, as are those in $\{a_i + k\colon 0 \le i \le d-1\}$. Hence a duplication among these $2d$ numbers implies the desired result. Since $a_d \le b$, and $a_{d-1} + k \le b - 1 + k$, the pigeonhole principle yields the desired result whenever $2d > b + k - 1$. Thus the argument works whenever $k \le 2d - b$.

10.21. *An extension of Example 10.8.* We use the ideas of Exercise 10.16. The chess player plays at most 132 games over 77 days. If there is no consecutive period of days with a total of k games, then the number of partial sums congruent to r modulo k is at most $\lceil s(r)/2 \rceil$, where $s(r)$ is the number of integers in [132] that are congruent to r. Note that $s(r) = 1 + \lfloor (132-r)/k \rfloor$ for $1 \le r \le k$.

Whenever $23 \le k \le 25$ and $1 \le r \le k$, the value of $s(r)$ is 5 or 6. Thus every congruence class contains at most 3 of the partial sums if no two partial sums differ by k. This allows at most $3k$ partial sums. Since there is at least one game each day over 77 days, there are 78 partial sums. These exceeds $3k$, so there must be two that differ by k.

When $k = 26$, we can avoid this as follows: play one game each on the first 25 days, then 27 games on day 26, then one game each day for the next 25 days, then 27 games on day 52, then one game each day for the last 25 days. The total is 129 games, and no two partial sums differ by 26.

10.22. *If $m \ge 2n$ and S is a set of m points on a circle with no two diametrically opposite, then S has at most n free points, where $x \in S$ is free if fewer than n points of $S - x$ lie in the semicircle clockwise from x.* If $m > 2n$, consider a diameter of the circle not intersecting S. One of the resulting semicircles has more than n points. The first point x in this semicircle is not free, and any other point of S is free only if it is free in $S - x$. Hence $S - x$ has as many free points as S. When the remaining set has size $2n$, partition it into pairs such that between a point and its mate there are $n-1$ points in each direction on the circle. Then exactly one point in each pair is free, and the original set S has at most n free points.

10.23. *Every two-colored square grid of order at least five has a rectangle with corner of the same color, but the grid of order four has no such rectangle, so the answer is five.* Consider a subgrid of order five. Each column has a majority in some color. By the pigeonhole principle, some color is the majority in at least three of the columns. Suppose this is black. Each pair of these three columns is black in at least one common row. If they are all black in one common row, then they have at least 6 blacks in the other four rows; another row receiving two blacks completes the rectangle.

If they are not black in a common row, then the three pairs of columns have common blacks in three distinct rows. There remain at least three blacks among these columns in the remaining two rows. Thus one of these rows has at least two blacks, and this completes a black-cornered rectangle with one of the three rows described initially.

● ○ ○ ●
○ ○ ● ●
○ ● ● ○
● ● ○ ○

10.24. *There are* 3^{10} *ways to assign 10 distinct people to three distinct rooms, of which* $3^{10} - 3 \cdot 2^{10} + 3$ *ways leave no room empty.* For each person, an independent choice among three rooms must be made, so there are 3^{10} total ways to assign them (corresponding to ternary lists of length 10).

To count assignments with no room empty, apply inclusion-exclusion to the universe of all 3^{10} assignments. Let A_i be the set of assignments in which room i is empty. Note that $|\bigcap_{i\in S} A_i|$ depends only on $|S|$. The inclusion-exclusion formula counts the assignments with no rooms empty (in none of A_1, A_2, A_3) as $3^{10} - 3 \cdot 2^{10} + 3 \cdot 1^{10} - 0^{10}$, because the number of assignments with a specified k rooms empty is $(3-k)^{10}$. The answer is $3^{10} - 3 \cdot 2^{10} + 3$.

10.25. *Decimal* n*-tuples with at least one each of* $\{1, 2, 3\}$. We must avoid n-tuples that lack any of these digits. The number of n-tuples lacking k specified digits is $(10-k)^n$. The inclusion-exclusion formula for the answer is $10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n$.

10.26. *The number of decimal* m*-tuples with no digit missing.* We must avoid the ten sets corresponding to the ten possible missing digits. The number of m-tuples lacking k specified digits is $(10-k)^m$. Hence the inclusion-exclusion formula for the answer is $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(10-k)^m$.

10.27. *The probability that a bridge hand has no voids.* Among the possible hands, let A_i be the set of hands having a void in the ith suit. We want to count the hands outside all of A_1, A_2, A_3, A_4. The total number of hands is $\binom{52}{13}$. There are $\binom{39}{13}$ hands in A_i (one specified void guaranteed). There are $\binom{26}{13}$ hands in $A_i \cap A_j$ (two specified voids guaranteed). There are $\binom{13}{13}$ hands in $A_i \cap A_j \cap A_k$, and no hands belonging to all four sets. There are $\binom{4}{r}$ ways to choose r of these sets. By the inclusion-exclusion formula, there are $\binom{52}{13} - 4\binom{39}{13} + 6\binom{26}{13} - 4\binom{13}{13} + 0$ hands with no voids. Dividing by $\binom{52}{13}$ yields the probability. To obtain the probability of having at least one void, subtract this from 1 to get $[4\binom{39}{13} - 6\binom{26}{13} + 4]/\binom{52}{13}$.

10.28. *There are 16 natural numbers less than 252 that are relatively prime to 252.* Out of a universe of $252 = 2^2 \cdot 3^2 \cdot 7$ numbers, we want to count those that do not have 2 or 3 or 7 as a prime factor. Let A_1, A_2, A_3 be the numbers in [252] that are divisible by 2,3,5, respectively. Since $[n]$ has exactly $\lfloor n/k \rfloor$ numbers divisible by k, the inclusion-exclusion formula allows us to count the desired numbers by

$$252 - \left\lfloor \frac{252}{2} \right\rfloor - \left\lfloor \frac{252}{3} \right\rfloor - \left\lfloor \frac{252}{7} \right\rfloor + \left\lfloor \frac{252}{6} \right\rfloor + \left\lfloor \frac{252}{14} \right\rfloor + \left\lfloor \frac{252}{21} \right\rfloor - \left\lfloor \frac{252}{42} \right\rfloor = 72.$$

10.29. *There are natural numbers less than 200 that have no divisor in* $\{6, 10, 15\}$. Since 10 divides 200, we can include 200 in the universe without changing the answer. Out of a universe of 200 numbers, we want to count those that do not have 6 or 10 or 15 as a factor. Let A_1, A_2, A_3 be the numbers in [200] that are divisible by 6,10,15, respectively. Since $[n]$ has exactly $\lfloor n/k \rfloor$ numbers divisible by k, the inclusion-exclusion formula allows us to count the desired numbers by

$$200 - \left\lfloor \frac{200}{6} \right\rfloor - \left\lfloor \frac{200}{10} \right\rfloor - \left\lfloor \frac{200}{15} \right\rfloor + \left\lfloor \frac{200}{60} \right\rfloor + \left\lfloor \frac{200}{90} \right\rfloor + \left\lfloor \frac{200}{150} \right\rfloor - \left\lfloor \frac{200}{900} \right\rfloor = 140.$$

10.30. *Euler totient formula.*

If p, q *are prime and* $\phi(n) = \#\{i\colon 1 \le i < n \text{ and } \gcd(i, n) = 1\}$*, then* $\phi(pq) = \phi(p)\phi(q)$. If p is prime, then p has no common factor with any natural number less than p, so $\phi(p) = p - 1$. Hence the right side of the equation is $(p-1)(q-1)$. If p, q are both prime, then a number less than pq fails to be relatively prime to pq if and only if it is a multiple of p or q. There are $q - 1$ numbers less than pq that are multiples of p, also $p - 1$ numbers less than pq that are multiples of q, and no numbers less than pq that are multiples of both. Subtracting off the bad numbers, we have $\phi(pq) = pq - 1 - (p-1) - (q-1) = pq - p - q + 1$, which is the same as $(p-1)(q-1)$, as needed.

The general formula is $\phi(m) = m\prod_{p\in P(m)} \frac{p-1}{p}$*, where* $P(m)$ *is the set of prime factors of* m. The formula counts the elements of $[m]$ that are not divisible by any element of $P(m)$. For a given subset $S \subseteq P(m)$, there are

$\frac{m}{\prod_{p\in S} p}$ elements of $[m]$ that are divisible by all the elements of S. By the inclusion-exclusion formula,

$$\phi(m) = \sum_{S\subseteq P(m)} (-1)^{|S|}\frac{m}{\prod_{p\in S} p} = m \sum_{S\subseteq P(m)} \Big(\prod_{p\in P(m)-S} 1\Big)\Big(\prod_{p\in S}\frac{-1}{p}\Big) = m \prod_{p\in P(m)} \Big(1-\frac{1}{p}\Big)$$

10.31. *If $A_1,\ldots,A_n$ are sets in a universe U, and $N(T)$ is the number of elements of U that belong to the sets indexed by T but to no others among $A_1,\ldots,A_n$, then*

$$N(T) = \sum_{T\subseteq S\subseteq[n]} (-1)^{|S|-|T|}\Big|\bigcap_{i\in S} A_i\Big|.$$

Restrict U to the set U' of elements that have all properties indexed by T. Consider the $n-|T|$ properties not indexed by T. The desired elements are those in U' that contain none of these $n-|T|$ properties, and the formula above is precisely the usual inclusion-exclusion formula for counting this subset of U'.

10.32. *The number of permutations of $[n]$ that leave no odd number unmoved is $\sum_{k=0}^{\lceil n/2\rceil}(-1)^k\binom{\lceil n/2\rceil}{k}(n-k)!$.* Consider the universe U of all $n!$ permutations of $[n]$. Let the set A_i be those permutations that put $2i-1$ in position $2i-1$, for $1\le i\le \lceil n/2\rceil$. The desired answer is the number of permutations that are outside all of the sets $A_1,\ldots,A_n$. For any set S consisting of k indices, the permutations in $\bigcap_{i\in S} A_i$ are those that permute the $n-k$ elements not specified by S among themselves, arbitrarily. There are $(n-k)!$ of these. Hence the inclusion-exclusion formula to count the items outside all the A_i gives $N(\varnothing) = \sum_{k=0}^{\lceil n/2\rceil}(-1)^k\binom{\lceil n/2\rceil}{k}(n-k)!$.

10.33. *Assigning $2n$ courses to n professors.* The courses are distinct, and the professors are distinct. If each professor in turn is assigned two courses from those that remain, then the number of selections for each successive professor does not depend on *how* the previous choices were made. Thus the product rule applies, and the courses can be assigned in $\prod_{i=0}^{n-1}\binom{2n-2i}{2}$. This equals $(2n)!/2^n$, which can be obtained directly by thinking of the list of courses as $2n$ positions in a row; the answer is the number of ways to arrange two copies each of n types of markers representing the professors. Thus the answer is the multinomial coefficient $\binom{2n}{2,\ldots,2} = (2n)!/2^n$.

In the spring, the possible assignments are the same, but we exclude those where some professor has the same assignment as in the fall. Let A_i be the set of assignments in which the ith professor has the same assignment as in the fall. When k particular professors have repeat assignments, this preassigns $2k$ courses, and the ways to complete the assignment are

the ways to assign $2n-2k$ courses to $n-k$ professors without further restrictions. Thus $\left|\bigcap_{i\in S} A_i\right| = (2n-2k)!/2^{n-k}$ whenever $|S|=k$. There are $\binom{n}{k}$ sets of k professors. By the inclusion-exclusion principle, the number of assignments that belong to none of the sets A_i is thus $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2n-2k)!}{2^{n-k}}$. The probability of this event, when the courses are assigned at random, is this divided by $(2n-2k)!/2^{n-k}$.

10.34. *Selection of n objects from 5 types with no type selected more than 4 times.* There are $\binom{n+5-1}{5-1}$ ways to select n objects from 5 types with repetition allowed. These selections are our universe U. We want to eliminate the selections in which a type is picked more than 4 times, so set A_i is the subset of U where type i is picked more than 4 times. Since the objects of each type are indistinguishable, we may guarantee being in A_i by picking 5 objects of type i initially and then completing the selection. In other words, $|A_i| = \binom{(n-5)+5-1}{5-1}$. Similarly, $|A_i\cap A_j| = \binom{(n-10)+5-1}{5-1}$, since we distribute 5 to each violated type before considering arbitrary selections with repetition for the remainder. The resulting sum by the inclusion-exclusion formula is $N(\varnothing) = \sum_{k=0}^{5}(-1)^k\binom{n}{k}\binom{n-5k+4}{4}$.

10.35. *Pairings of n boys and n girls as lab partners.* For each i, the ith tallest boy must not be matched to the ith tallest girl. Let A_i be the set of pairings that violate this condition.

a) When same-sex pairs are allowed, the number of acceptable pairings is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2n-2k)!}{2^{n-k}(n-k)!}$. The universe is the set of all $\frac{(2n)!}{2^n n!}$ pairings of $2n$ people (with the people in order, arrange labels $1,1,2,2,\ldots,n,n$ to make pairs, and then divide by $n!$ becauses the pairs are not labeled). Within this universe, we count those pairings that avoid $A_1,\ldots,A_n$. Pairings that lie in A_i for all $i\in S$ in effect exclude $2|S|$ people from participation by predetermining their pairs, so the number of ways to complete the pairing is $\frac{(2n-2k)!}{2^{n-k}(n-k)!}$, where $k=|S|$. This depends only on $|S|$, so the Inclusion-Exclusion Principle yields the formula claimed.

b) When same-sex pairs are not allowed, the number of acceptable pairings is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)!$. The universe is the set of all $n!$ pairings of boys with girls. Within this universe, we count those pairings that avoid $A_1,\ldots,A_n$. The number of pairings that lie in A_i for all $i\in S$ is $(n-|S|)!$. This depends only on $|S|$, so the Inclusion-Exclusion Principle yields the formula claimed.

10.36. *Permutations of two copies each of n types of letters with no consecutive letters the same.* Let U be the universe of permutations of these letters. Let A_i be the set of permutations in which the two letters of the ith type appear consecutively. We want to count the permutations in none of these sets. The number of arrangements having k_i letters of type i is

$(\sum k_i)!/(\prod k_i!)$. Hence $|U| = (2n)!/2^n$. For permutations in A_i, the two copies of the ith letter must be next to each other (and indistinguishable), so we can view them as a single letter, and $|A_i| = (2n-1)!/2^{n-1}$. More generally, when we consider the intersection of k of these sets and force k of the pairs to appear consecutively, the number of arrangements is $(2n-k)!/2^{n-k}$. This holds for any collection S of k indices from n, and there are $\binom{n}{k}$ such subsets of indices. Hence the answer given by the inclusion-exclusion formula is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2n-k)!/2^{n-k}$.

10.37. *Spouse-avoiding circular permutations of n couples.* Let U be the set of circular permutations of the people in n couples. Let A_i be the set of circular permutations in which the ith couple occurs together. For $S \subseteq [n]$, the circular permutations in $\bigcap_{i\in S} A_i$ are counted by replacing the ith couple with a single token, for all $i \in S$, using an arbitrary circular permutation of the resulting set, and multiplying by $2^{|S|}$ to account for the orientations of the special couples. By inclusion-exclusion, the number of circular permutations with no adjacent spouses is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^k(2n-1-k)!$.

10.38. *Counting permutations.* Let E_n^k be the number of permutations of $[n]$ with k fixed points and let $D_n = E_n^0$ be the number of permutations of $[n]$ with no fixed points.

a) $E_n^k = \binom{n}{k}D_{n-k}$, We can partition the permutations with k fixed points according to which positions are fixed. The number of permutations with a particular set of k fixed points is D_{n-k}, because the remaining $n-k$ elements must be permuted among themselves without fixed points. This number D_{n-k} is the same no matter which set of k points was chosen to be fixed, and these k points can be chosen in $\binom{n}{k}$ ways, so the total value of E_n^k is $\binom{n}{k}D_{n-k}$.

b) $n! = \sum_{j=0}^{n}\binom{n}{j}D_{n-j}$. Every permutation has some number of fixed points between 0 and n; this partitions the set of permutations into disjoint sets that we count separately. In part (a) we determined the number of permutations that have exactly j fixed points, and summing this over all values of j counts all possible permutations.

10.39. $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ *when $n > 0$, by inclusion-exclusion.* This is the inclusion-exclusion formula to count the elements belonging to none of n sets $A_1,\dots,A_n$, where $|\cap_{i\in S} A_i| = 1$ for all S. In particular, this formula holds when $S = \varnothing$, so the universe has size 1, and when $|S| = 1$, so each A_i has one element. Thus the sum counts the elements in a universe of size 1 that belong to none of n sets equal to the universe. There is no such element when $n > 0$, but when $n = 0$ the answer is 1.

10.40. *Proof of $\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k} = 1$ by inclusion-exclusion.* Since the sum has the form $\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k)$, this suggests an inclusion-exclusion argument with n sets in a universe, where the size of the intersection of

k sets does not depend on which k sets are chosen, always equaling $f(k)$. For $k = 0$, the contribution is 2^n, which suggests that our universe should correspond to all subsets of $[n]$.

The resulting proof is that each side of the equality counts the subsets of $[n]$ that contain none of the elements. On the right, we know that there is exactly one such subset, the empty set. On the left, we define sets $A_i = \{T \subseteq [n]\colon i \in T\}$ within the universe of all subsets of $[n]$. Then the subsets in $\bigcap_{i\in S} A_i$ consist of all subsets of $[n]$ that contain the elements in S. There are exactly $2^{n-|S|}$ of these, so the left-side is the inclusion-exclusion sum to count the subsets of $[n]$ belonging to none of the sets A_i.

10.41. $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n-k+r-1}{r} = \binom{r-1}{n-1}$. The factor $(-1)^k\binom{n}{k}$ in the summand suggests an inclusion-exclusion computation. The quantity $\binom{n-k+r-1}{r}$ is the number of ways to choose r elements from $n-k$ types. Here r remains fixed. The term $k = 0$ suggests making the universe be the set of selections of r elements from n types, with repetition allowed. When we select r elements from $n-k$ types, we are leaving some types unused.

The right side is the number of selections of r elements from n types in which every type is used at least once. The left side counts the same set using the Inclusion-Exclusion Principle.

11. GRAPH THEORY

11.1. *The graph with vertex set* $[12]$ *in which vertices are adjacent if and only if the numbers are relatively prime has 45 edges.* We count edges by summing the vertex degrees and dividing by 2. All of $\{1,7,11\}$ have degree 11. All of $\{2,4,8\}$ are adjacent precisely to the odd numbers, for degree 6. Vertices 3 and 9 are adjacent to the 8 numbers not divisible by 3. Vertex 5 is adjacent to all but itself and 10. Vertices 6 and 12 are adjacent to $\{1,5,7,11\}$, and 10 is adjacent to $\{1,3,7,9,11\}$. The degrees (in order for vertices 1 through 12) are thus 11, 6, 8, 6, 10, 4, 11, 6, 8, 5, 11, 4. They sum to 90, so there are 45 edges.

11.2. *In the graph G with vertex set $\mathbb{Z}_n$ in which vertices u and v are adjacent if and only if they differ by 6, the number of components is* $\gcd(n,6)$ *when $n > 12$.* Arrange the vertices in a circle. Each is adjacent to the vertices six positions away in each direction. Since $n > 12$, G is regular of degree 2 and the components are cycles.

When n is divisible by 6, G consists of six cycles with $n/6$ vertices each; each cycle goes once around the circle. When n is divisible by 3 but not 2, G consists of three cycles with $n/3$ vertices each, going twice around the circle. When n is divisible by 2 but not 3, G consists of two cycles with $n/2$

vertices each, going thrice around the circle. When n is relatively prime to 6, there is a single cycle through all the vertices.

When $n = 12$, the graph consists of six disjoint edges. When $n < 12$, no pairs differ by 6, and the graph consists of n isolated vertices.

11.3. *Eulerian with even order and odd size.* An Eulerian graph with an even number of vertices and an odd number of edges can be formed as the union of an even cycle and an odd cycle that share one vertex.

11.4. *If G is a connected non-Eulerian graph, then the minimum number of trails that together traverse each edge of G exactly once is half the number of vertices* having odd degree. The assumption of connectedness is necessary, because the conclusion is not true for a graph with two components in which one component is Eulerian.

Suppose G is a connected non-Eulerian graph, so G has $2k$ vertices of odd degree for some positive integer k. A trail contributes even degree to every vertex, except odd degree to its endpoints when they are not the same vertex. Therefore, a partition of the edges into trails must have some trail ending at each vertex of odd degree. Since each trail has (at most) two ends, at least k trails are required. There are several ways to prove that k trails are always enough.

Proof 1 (using Eulerian circuits). Pair up the vertices of odd degree in G arbitrarily and form G' by adding a copy of each pair as an edge. The resulting graph G' is connected and has even degree at every vertex. Therefore it has an Eulerian circuit, since the characterization of Eulerian graphs allows multiple edges. As we traverse the circuit, we start a new trail in G after each time we traverse an edge of $G' - E(G)$. This yields k pairwise edge-disjoint trails partitioning $E(G)$.

Proof 2 (using Eulerian circuits). Form G' by adding a new vertex w and making it adjacent to all vertices of G that have odd degree. The resulting graph G' is connected and has even degree at every vertex. Therefore it has an Eulerian circuit. As we traverse the circuit, we start a new trail in G after each time we leave w and arrive at a vertex of G. Since w has degree $2k$, the circuit visits it k times, so we obtain k pairwise edge-disjoint trails partitioning $E(G)$.

Proof 3 (induction on k). If $k = 1$, we add an edge between the two odd vertices, obtain an Eulerian circuit, and delete the added edge to obtain a single trail covering all edges of G. If $k > 1$, let P be a path between two vertices of odd degree. The graph $G' = G - E(P)$ has $2k - 2$ vertices with odd degree since we have changed degree parity only at the ends of P. We apply the induction hypothesis to each component of G' that has vertices with odd degree. Every component not having vertices of odd degree has an Eulerian circuit that contains a vertex of P; we splice it into P to avoid having an additional trail. Altogether, we have used the desired number of trails to partition $E(G)$.

Proof 4 (induction on the number of edges). With only one edge, $G = K_2$, and we have one trail. With more edges, if G has a vertex x of even degree adjacent to a vertex y of odd degree, then $G' = G - xy$ has the same number of vertices of odd degree as G. The trail decomposition of G' guaranteed by the induction hypothesis has one trail ending at x and no trail ending at y. Add xy to the trail ending at x to obtain the desired decomposition of G. If G has no vertex of even degree adjacent to a vertex of odd degree, then every vertex of G has odd degree. In this case, deleting an edge xy reduces k, and we can add xy as a trail of length one to the decomposition of $G - xy$ guaranteed by the induction hypothesis.

11.5. *The vertices of a simple graph cannot have distinct degrees.* In a simple graph with n vertices, every vertex degree belongs to the set $\{0, \ldots, n-1\}$. If fewer than n values occur, then the pigeonhole principle yields the claim. Otherwise, both $n-1$ and 0 occur as vertex degrees. This is impossible; if one vertex is adjacent to all others, then there can be no isolated vertex.

11.6. *In a league with two divisions of 11 teams each, there is no schedule with each team playing 7 games against teams within its division and 4 games against teams in the other division.* In the language of graph theory, the scheduled games can be viewed as edges in a graph with 22 vertices. We are asking for a graph that is regular of degree 11, but we are also asking for the subgraph induced by the 11 teams in one division to be regular of degree 7. Since 7 and 11 are both odd, this is impossible, because every graph has an even number of vertices of odd degree.

11.7. *A connected graph in which every vertex degree is even has no edge whose deletion disconnects it.*

Proof 1 (degree-sum). Let G be a connected graph with all vertices having even degree, and suppose G has a disconnecting edge. Deleting this edge leaves a graph with vertices of odd degree in distinct components. A component containing one of these vertices is a subgraph having an odd number of vertices of odd degree, which cannot exist.

Proof 2 (Eulerian circuits). A connected graph with even degrees is Eulerian. For any pair of vertices u, v, we can start at a visit to u on an Eulerian circuit and traverse the circuit in two opposite directions until it first reaches v. This gives a pair of edge-disjoint u, v-trails. Deleting an single edge leaves one of these trails intact, and it contains a u, v-path. Since u, v were chosen as arbitrary vertices, no deletion of a single edge can make any vertex unreachable from any other.

11.8. *If l, m, n are nonnegative integers with $l + m = n$, then there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree if and only if m is even, except for $(l, m, n) = (2, 0, 2)$.* Since every graph has an even number of vertices of odd degree, and the only simple connected graph with two vertices has both degrees odd, the condition is necessary. To prove sufficiency, we construct such a graph G. If $m = 0$, let $G = C_l$ (except $G = K_1$ if also $l = 0$). If $m > 0$, begin with $K_{1,m-1}$, which has m vertices of odd degree, and then add a path of length l beyond one of the leaves. (Many other constructions also work.)

11.9. *Deleting a vertex of maximum degree cannot increase the average degree, but deleting a vertex of minimum degree can reduce the average degree.* Deleting any vertex of a regular graph reduces the average degree. For the first statement, suppose G has n vertices and m edges, and a, a' are the average degrees of G and $G - x$. Since $G - x$ has $m - d(x)$ edges and degree sum $2m - 2d(x)$, we have $a' = \frac{na-2d(x)}{n-1} \le \frac{(n-2)a}{n-1} < a$ if $d(x) \ge a$.

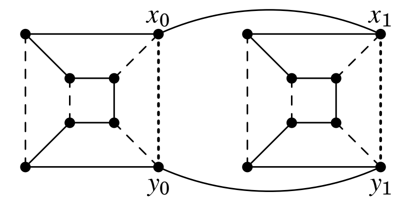
11.10. *Inductive aspects of the d-dimensional cube Q_d.*

Inductively, Q_d is constructed from the disjoint union of two copies of Q_{d-1} by adding a matching that pairs up corresponding vertices. The vertex set of Q_d is the set of d-tuples with entries in $\{0, 1\}$. Two vertices are adjacent if they differ in exactly one place. Suppose $d \ge 1$. The subgraph of Q_d consisting of all the vertices with 0 in the last place and all the edges of Q_d joining vertices in this set is isomorphic to Q_{d-1}; call this G_0. The subgraph of Q_d consisting of all the vertices with 1 in the last place and all the edges of Q_d joining vertices in this set is isomorphic to Q_{d-1}; call this G_1. The edges of Q_d that do not appear in these two subgraphs are those between a vertex of G_0 and a vertex of G_1 differing only in the last place. These three sets of edges are pairwise disjoint and account for all edges.

Q_d has $d2^{d-1}$ edges. Let a_d be the number of edges in Q_d. By the inductive construction, $a_d = 2a_{d-1} + 2^{d-1}$ for $d \ge 1$. Also $a_0 = 0$. We prove by induction on d that $a_d = d2^{d-1}$. For $d = 0$, we have $a_0 = 0 = 0 \cdot 2^{-1}$. For $d > 0$, we use the recurrence relation and then the induction hypothesis to obtain $a_d = 2a_{d-1} + 2^{d-1} = 2((d-1)2^{d-2}) + 2^{d-1} = d2^{d-1}$.

Q_d has a spanning cycle, for all $d \ge 2$. Proof by induction on d. For $d = 2$, the graph Q_2 is itself a 4-cycle. The inductive step is illustrated by the picture below (going from Q_3 to Q_4). For the inductive step, suppose the claim holds for Q_d. Let C be a spanning cycle in Q_d, and let xy be an edge in C. Let G_i be the copy of Q_d in Q_{d+1} in which all the vertex labels end with $i \in \{0, 1\}$, with x_i, y_i being the vertices obtaining by appending i to the labels of x and y. Let C_i be the copy of C in G_i obtained by appending i to the label of each vertex in C, listing the vertices in the same order as on C. Then deleting x_0y_0 from C_0 and x_1y_1 from C_1 and replacing them with

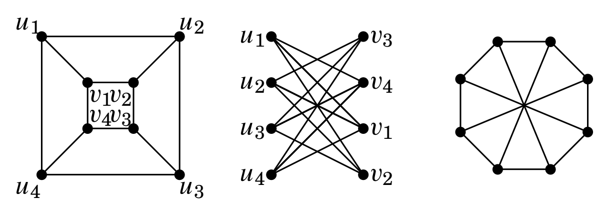
the edges x_0x_1 and y_0y_1 yields a spanning cycle of Q_{d+1}. It traverses all the vertices of G_0 from x_0 to y_0, then crosses to G_1, then traverses all vertices of G_1 ending at x_1, then returns to x_0 to complete the cycle.



11.11. *The d-dimensional cube has $\binom{d}{2}2^{d-2}$ 4-cycles and $\binom{d}{3}2^{d+1}$ 6-cycles.* Traversing a 4-cycle must change two coordinates and then unchange them. Thus a 4-cycle is determined by choosing two coordinates to change and choosing a fixed value in each of the other $d - 2$ coordinates.

Traversing a 6-cycle similarly must change and unchange exactly three coordinates, since only 4 vertices are available if $d - 2$ coordinates remain fixed. For each of the $\binom{d}{3}2^{d-3}$ choices for the $d - 3$ fixed coordinates, the number of ways to complete a 6-cycle is the number of 6-cycles in Q_3. For each pair of adjacent vertices and for each pair of complementary vertices in Q_3, there is exactly one 6-cycle through the remaining six vertices. The other types of nonadjacent pairs leave no 6-cycle when deleted. Thus there are 12+4=16 6-cycles in Q_3, which completes the computation.

11.12. *Isomorphism.* Using the correspondence indicated below, the first two graphs are isomorphic; the graphs are bipartite, with $u_i \leftrightarrow v_j$ if and only if $i \ne j$. The third graph contains odd cycles and hence is not isomorphic to the others.



These are the 3-dimensional cube and the graph obtained by deleting four disjoint edges from $K_{4,4}$. The vertices of the cube are the binary 3-tuples; the *parity* of a vertex is the number of 1s in the 3-tuple. Each vertex of the cube is adjacent to every vertex of the opposite parite except its complement. Since the complement of the complement of a binary list is the original list, this describes Q_3 as formed by deleting four disjoint edges from $K_{4,4}$.

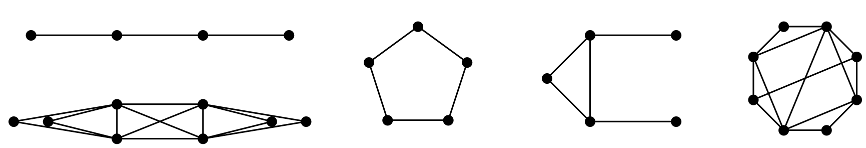
11.13. *$G \cong H$ if and only if $\overline{G} \cong \overline{H}$.* If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and nonadjacency, and hence f preserves non-adjacency and adjacency in $\overline{G}$ and is an isomorphism from $\overline{G}$ to $\overline{H}$. The same argument applies for the converse, since the complement of $\overline{G}$ is G.

11.14. *The smallest number of vertices for which nonisomorphic simple graphs exist with the same vertex degrees is 5.* For $n \le 3$, there is only one isomorphism class with each number of edges. As listed in Example 11.26, there are 11 isomorphism classes of simple graphs with four vertices. When the number of edges is in $\{0, 1, 5, 6\}$, there is only one isomorphism class. With two edges, there are two isomorphism classes, but they have different degree lists (similarly for four edges). With three edges, the three isomorphism classes have degree lists 3111, 2220, and 2211, all different. Hence a nonisomorphic pair of simple graphs with the same vertex degrees must have at least five vertices. With five vertices, the 5-vertex path and the disjoint union of a triangle and an edge are nonisomorphic graphs with degree lists 22211.

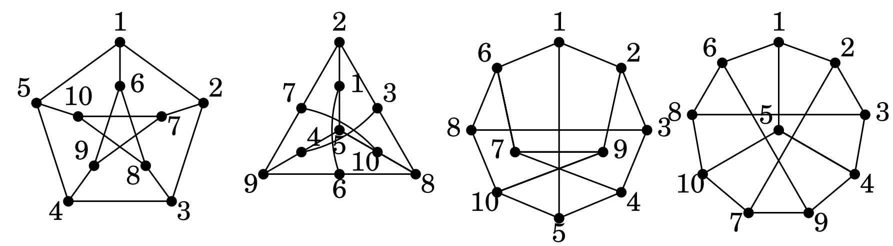
11.15. *There are two isomorphism classes of 4-regular simple graphs with 7 vertices.* Simple graphs G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic, because an isomorphism $\phi\colon V(G) \to V(H)$ is also an isomorphism from $\overline{G}$ to $\overline{H}$, and vice versa. Hence it suffices to count the isomorphism classes of 2-regular simple graphs with 7 vertices. Every component of a finite 2-regular graph is a cycle. In a simple graph, each cycle has at least three vertices. Hence each class it determined by partitioning 7 into integers of size at least 3 to be the sizes of the cycles. The only two graphs that result are C_7 (one cycle) and $C_3 + C_4$ (two cycles).

11.16. *The number of vertices in a graph isomorphic to its complement is congruent to 0 or* $1 \pmod 4$. If G and $\overline{G}$ are isomorphic, then they have the same number of edges, but together they have $\binom{n}{2}$ edges (with none repeated), so the number of edges in each must be $n(n-1)/4$. Since this number must be an integer and the factors in the numerator *cannot both* be divisible by 2, one factor in the numerator must be divisible by 4. Hence the number of vertices is a multiple of 4 or one more than a multiple of 4.

Below are planar graphs with 4, 5, and 8 vertices that are isomorphic to their complements.



11.17. *Isomorphisms for the Petersen graph.* Isomorphism is proved by giving an adjacency-preserving bijection between the vertex sets. For pictorial representations of graphs, this is equivalent to labeling the two graphs with the same vertex labels so that the adjacency relation is the same in both pictures; the labels correspond to a permutation of the rows and columns of the adjacency matrices to make them identical. There various drawings of the Petersen graph below illustrate its symmetries; the labelings indicate that these are all "the same" (unlabeled) graph. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself.

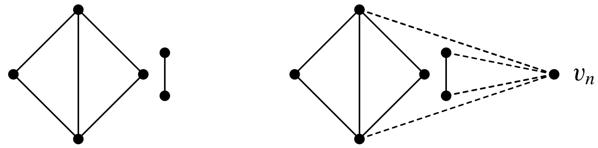


With vertices named as above, the adjacency relation is the same in each graph. This represents the disjointness relation on $\binom{[5]}{2}$ via the correspondence below.

1	2	3	4	5	6	7	8	9	10
$\{1,2\}$	$\{3,4\}$	$\{5,1\}$	$\{2,3\}$	$\{4,5\}$	$\{3,5\}$	$\{5,2\}$	$\{2,4\}$	$\{4,1\}$	$\{2,4\}$

Alternatively, one can name the vertices using four disjoint sets of labels and establish isomorphisms from each graph to the next. Transitivity of the isomorphism relation then implies that each two graphs chosen from this set are isomorphic.

11.18. *There are exactly $2^{\binom{n-1}{2}}$ simple graphs with vertex set $v_1, \ldots, v_n$ in which every vertex has even degree.* If we do not care about the degrees of the vertices, then there are $\binom{n}{2}$ simple graphs, since each pair of vertices may form an edge or not. Similarly, there are $2^{CH(n-1,2)}$ simple graphs with vertex set $\{v_1, \ldots, v_{n-1}\}$. We establish a bijection to that set of graphs. Given a simple graph G with vertex set $\{v_1, \ldots, v_{n-1}\}$, we form a new graph G' by adding a vertex v_n and making it adjacent to each vertex that has odd degree in G, as illustrated below. The vertices with odd degree in G have even degree in G'. Also, v_n itself has even degree because the number of vertices of odd degree in G is even. Conversely, deleting the vertex v_n from a graph with vertex set $\{v_1, \ldots, v_n\}$ having even vertex degrees produces a graph with vertex set $\{v_1, \ldots, v_{n-1}\}$, and this is the inverse of the first procedure. We have established a one-to-one correspondence between the sets; hence they have the same size.



G $\qquad$ G'

11.19. *Combinatorial proofs with graphs.*

a) For $0 \le k \le n$, $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$. Consider the complete graph K_n, which has $\binom{n}{2}$ edges. If we partition the vertices of K_n into a k-set and an $n-k$-set, then we can count the edges as those within one block of the partition and those choosing a vertex from each. Hence the total number of edges is $\binom{k}{2} + \binom{n-k}{2} + k(n-k)$.

b) If $\sum n_i = n$, *then* $\sum \binom{n_i}{2} \le \binom{n}{2}$. Again consider the edges of K_n, and partition the vertices into sets S_i with sizes $\{n_i\}$. The left side of the inequality counts the edges in K_n having both ends in the same S_i, which is at most all of $E(K_n)$.

11.20. *a) If x and y are nonadjacent vertices of degree at least $(n+k-2)/2$ in a simple graph G with n vertices, then x and y have at least k common neighbors.* Let X and Y be the sets of neighbors of x and y. Since x and y are nonadjacent, $|X \cup Y| \le n-2$. Using the degree bound and identities for set operations, we have

$$|X \cap Y| = |X| + |Y| - |X \cup Y| \ge \frac{n+k-2}{2} + \frac{n+k-2}{2} - (n-2) = k.$$

b) Every simple n-vertex graph G with minimum degree at least $\lfloor n/2 \rfloor$ is connected. It suffices to prove the stronger result that every two nonadjacent vertices have a common neighbor. Let $N(x)$ and $N(y)$ denote the neighbor sets of x and y; each has size at least $\lfloor n/2 \rfloor$, so the sizes sum to at least $n-1$. If x and y are nonadjacent, then $N(x)$ and $N(y)$ are contained in $V(G) - \{x, y\}$, and hence $|N(x) \cup N(y)| \le n-2$. Since $|N(x) \cap N(y)| + |N(x) \cup N(y)| = |N(x)| + |N(y)|$, we have

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \ge (n-1) - (n-2) = 1.$$

Thus every two nonadjacent vertices have a common neighbor.

To show that the requirement on minimum degree is best possible, consider the n-vertex graph consisting of two disjoint cliques with $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ vertices and no additional edges. For $n \ge 2$, this is a disconnected graph with minimum degree $\lfloor n/2 \rfloor - 1$.

11.21. *A graph G is connected if and only if for every partition of $V(G)$ into non-empty sets S, T, there is an edge xy with $x \in S$ and $y \in T$.* When G is connected and S, T is a nontrivial partition, we choose $u \in S$ and $v \in T$. Since G is connected, G has a u, v-path, and this path must leave S along some edge xy. When G is not connected, we let S be the vertex set of a component of G and let T be the remainder of the vertex set; since S is a component, there is no edge from S to T.

11.22. *Measuring k gallons using buckets with integer capacities $l > m > n$ in gallons.* We create vertices for the possible states of our system. Each vertex is a triple (a, b, c) listing the integer numbers of gallons in the three buckets. Our initial vertex is $(l, 0, 0)$. Since we are only allowed to pour from one gallon into another, the sum of the three integers is always l.

The operation of pouring from one bucket into another that already has some water is not reversible; thus we need directed edges, like those used in functional digraphs. Directed edges would also allow us to model the operation of spilling out a bucket, if that were allowed.

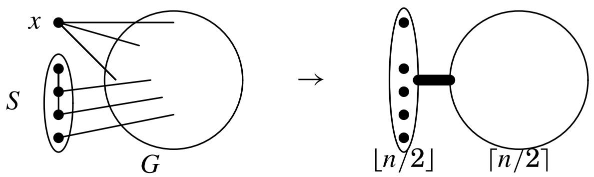
Given this directed graph as a model, the problem is to determine whether it has a path (following directed edges in the forward direction) from the vertex $(l, 0, 0)$ to a vertex in which k is one of the integers.

11.23. *If G is a finite simple graph with minimum degree k, and $k \ge 2$, then G contains a path of length at least k and a cycle of length at least $k+1$.* Because $V(G)$ is finite, every path in G is finite, so G has a maximal path P. Let u be an endpoint of P. Because P cannot be extended, every neighbor of u belongs to P. Since u has at least k neighbors, its neighbor v that is farthest along P from u is the other end of a path P' of length at least k, and uv completes a cycle of length at least $k+1$ with P'.

11.24. *If G is a connected graph, then any two paths of maximum length in G have a common vertex.* Let P and Q be the paths, with Q at least as long as P. The proof is by contradiction; if $V(P)$ and $V(Q)$ are disjoint, then we find another path that is longer than P. Since G is connected, G have a path from $V(P)$ to $V(Q)$. Let R be a shortest such path, with endpoints $x \in V(P)$ and $y \in V(Q)$. By the choice of R, the internal vertices of R (if there are any) are neither in P nor Q. Hence we can form a new path S by traveling from one end of P to x along P, then to y along R, then to an end of Q along Q. We use the end of P farther from x and the end of Q farther from y, choosing arbitrarily if x or y is in the middle of its path. If p, q, r are the lengths of P, Q, R, respectively, then the length of S is at least $p/2 + r + q/2$. Since $q \ge p$ and $r > 0$, S is longer than P.

11.25. *The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$.* Suppose G is an n-vertex triangle-free simple graph. Let

k be the maximum of the vertex degrees, and let x be a vertex of degree k in G. Since G has no triangles, there are no edges between neighbors of x. Hence summing the degrees of x and vertices in the set S of non-neighbors of x counts at least one endpoint of every edge, and the total is at least $e(G)$. Since we are summing over $n-k$ vertices, each having degree at most k, we obtain $e(G) \le k(n-k)$. By the Arithmetic-Geometric Mean Inequality (Proposition 1.1), the product of two numbers is at most the square of their average, so $e(G) \le k(n-k) \le n^2/4$. To prove that the bound is best possible, we exhibit $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ as a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges.



11.26. *Every n-vertex graph G with $n-k$ edges has at least k components.*

Proof 1 (properties of spanning trees). Let l be the number of components of G, and let n_i be the number of vertices in the ith component. The ith component has a spanning tree with $n_i - 1$ edges. Hence the total number of edges is at least $\sum_{i=1}^{l}(n_i - 1)$, which equals $n-l$. The total number of edges equals $n-k$, and $n-k \ge n-l$ yields $l \ge k$.

Proof 2 (properties of cut-edges). Let $m = n-k$. We begin with n isolated vertices and no edges, and we add the edges of the graph one by one. At first there are n components. Each addition of an edge of the graph reduces the number of components of the current graph by at most one. Thus when m edges have been added there are at least $n-m=k$ components.

11.27. *If G is a graph with n vertices and $n-1$ edges, then G is connected if and only if G has no cycles.*

Proof 1 (properties of trees and connected graphs). If G is connected, then we can delete edges from cycles, one by one, until we obtain a graph G' with no cycles. Since at each step we deleted an edge of a cycle, G' is connected (Lemma 11.22). Since G' is connected and has no cycles, it is a tree and therefore has $n-1$ edges (Theorem 11.25). Since G has $n-1$ edges, we did not delete any, and $G = G'$.

Conversely, if G has no cycles, then every component of G is a tree. Suppose the components are $G_1, \ldots, G_k$, with G_i having n_i vertices, for each i. By Theorem 11.25, G_i has $n_i - 1$ edges. Summing the number of edges from each component yields $\sum_{i=1}^{k}(n_i - 1) = (\sum n_i) - k = n-k$ edges for G. Since G has $n-1$ edges, $k=1$, and G is connected.

Proof 2 (induction on n). A graph with 1 vertex and no edges is connected and has no cycles, so the statement holds for $n=1$. For the induction step, suppose $n>1$. By the degree-sum formula, $\sum_{v \in V(G)} d(v) = 2n-2$, which is less than $2n$. Hence some vertex v has degree less than 2, by the pigeonhole principle. If G is connected, then this vertex v has degree 1. Deleting v and its incident edge leaves a graph G' with $n-1$ vertices and $n-2$ edges. Since a vertex of degree 1 cannot be an internal vertex on a path, all paths in G connecting vertices of G' remain in G', and hence G' is also connected. By the induction hypothesis, G' has no cycles. Since a vertex of degree 1 cannot belong to a cycle, G also has no cycles.

Conversely, suppose G has no cycles. Since G has fewer edges than vertices, some component H of G has fewer edges than vertices. By the degree-sum formula, the pigeonhole principle, and the connectedness of H, there is a vertex v in H that has degree 1 (in G). Deleting v and its incident edge leaves a graph G' with $n-1$ vertices and $n-2$ edges. Deleting a vertex cannot create a cycle, so G' has no cycles. By the induction hypothesis, G' is connected. Since paths to or from the neighbor of v in G can be extended to become paths to or from v, G is connected. Since G has n vertices and $n-1$, the sum of the degrees of

11.28. *A graph G is a tree if and only if for all $x, y \in V(G)$, there is exactly one x,y-path in G. Necessity:* If G is a tree, then G is connected and has an x,y-path. If G has two x,y-paths, let H be their union, and let e be an edge belonging to only of the paths. By the transitivity of the connection relation, $H-e$ is connected. By Lemma 11.36, e thus belongs to a cycle in H, which is contained in G. This contradicts the absence of cycles in G.

Sufficiency: The existence of x,y-paths makes G connected, and the uniqueness of paths forbids cycles, since every pair u,v in a cycle is connected by two paths along the cycle. Thus G must be a tree.

11.29. *If G is a tree with maximum degree k, then G has at least k leaves.*

Proof 1a (maximal paths). Deleting a vertex x of degree k produces a forest of k subtrees, and x has one neighbor w_i in the ith subtree G_i. Let P_i be a maximal path starting at x along the edge xw_i. The other end of P_i must be a leaf of G and must belong to G_i, so these k leaves are distinct. (Note: choosing a maximal path is cleaner than "follow paths from the maximum degree vertex through each of its neighbors until you get stuck, which must be at distinct leaves since there are no cycles".)

Proof 1b (leaves in subtrees). Deleting a vertex x of degree k produces a forest of k subtrees. Each subtree is a single vertex, in which case the vertex is a leaf of G, or it has at least two leaves, of which at least one is not a neighbor of x. In either case we obtain a leaf of the original tree in each subtree.

Proof 2 (counting two ways). Count the degree sum by edges and by vertices. By edges, it is $2n-2$. Let k be the maximum degree and l the number of leaves. The remaining vertices must have degree at least two each, so the degree sum when counted by vertices is at least $k+2(n-l-1)+l$. The inequality $2n-2 \geq k+2(n-l-1)+1$ simplifies to $l \geq k$. (Note: The same idea is used in noting that degree $2(n-1)-k$ remains for the vertices other than the one of maximum degree. Since all degrees are 1 or at least 2, there must be at least k vertices of degree 1.)

Proof 3 (induction on the number of vertices). For $n \leq 3$, we inspect the unique tree with n vertices. For $n > 3$, delete a leaf u. If $\Delta(T-u) = \Delta(T)$, then by induction $T-u$ has at least k vertices, and replacing u adds a leaf at the expense of at most one leaf from $T-u$. Otherwise $\Delta(T-u) = \Delta(T)-1$, which happens only if the neighbor of u is the only vertex of maximum degree in T. Now induction guarantees at least $k-1$ leaves in $T-u$, and adding u gives another, since the vertex of maximum degree in T cannot be a leaf in $T-u$ (this is the reason for $n=3$ in the basis).

11.30. *A connected graph with n vertices has exactly one cycle if and only if it has exactly n edges.* Let G be a connected graph with n vertices.

Necessity: If G has one cycle, then deleting an edge e of that cycle leaves a connected graph with no cycle. By definition, $G-e$ is a tree, and every tree with n vertices has $n-1$ edges. Hence $G-e$ has $n-1$ edges and G has n edges.

Sufficiency: Let G have exactly n edges. If G has no cycles, then by definition G is a tree and has $n-1$ edges. Therefore G has at least one cycle. If G has more than one cycle, then we can find edges e, f such that each belongs to a cycle that does not contain the other (otherwise, all the cycles have the same edges). Hence $G-e-f$ is connected. Continue deleting edges from cycles until all are gone; at most $n-2$ edges remain. We now have an acyclic connected graph with no cycles, since deleting edges on cycles never disconnects a graph. Such a graph is a tree and hence has $n-1$ edges, which is more edges than remain. Hence our assumption that G has more than one cycle is impossible, and G must have exactly one cycle.

11.31. *If $d_1, \ldots, d_n$ are n natural numbers (with $n \geq 2$), then there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n-2$.* First we prove the condition is necessary. If a tree T has these degrees, then each $d_i \geq 1$ because T is connected. Also, $\sum d_i$ must equal twice the number of edges in T, and every n-vertex tree has $n-1$ edges.

We prove sufficiency by induction on n. If $n=2$, then the only list of two natural numbers summing to 2 is $(1,1)$, and this is realized as the degrees of the tree with two vertices. For $n > 2$, we assume that this condition is sufficient for a list of $n-1$ natural numbers. Let $d_1, \ldots, d_n$ be a list of n natural numbers satisfying the condition. Since the sum is only $2n-2$, the numbers cannot all be at least 2, so there is at least one 1. Also $2n-2 > n$, so there must be at least one number greater than 1. Suppose $d_i = 1$ and $d_j > 1$. Form a new list of numbers by deleting d_i and reducing d_j by 1. This reduces the sum by 2 and produces a set of $n-1$ natural numbers summing to $2(n-1)-2$. By the induction hypothesis, there is a tree with these degrees. Find a vertex x with degree $d_j - 1$ in such a tree, and add a new vertex joined only to x. The resulting graph is a tree (connected and acyclic) with degrees $d_1, \ldots, d_n$.

11.32. *If T is a tree with m edges and G is a simple graph with $\delta(G) \geq m$, then T is a subgraph of G.* We use induction on m. Basis step ($m=0$): every simple graph contains K_1. Induction step ($m>0$): suppose that the claim holds for trees with fewer than m edges. Since $m>0$, Lemma 11.39 allows us to choose a leaf v with neighbor u in T and consider the smaller tree $T' = T - v$. By the induction hypothesis, there is a subgraph H of G isomorphic to T', since $\delta(G) \geq m > m-1$. Let x be the vertex in H that represents u in T'. Since T' has only $m-1$ vertices other than u, x has a neighbor y in G that does not appear in H. Adding the edge xy to represent uv expands this copy of T' in G into a copy of T.

11.33. *Inductive proof that K_n is a union of k bipartite graphs if and only if $n \leq 2^k$.* For $k=1$, one airline can handle at most two cities, because K_3 contains an odd cycle; this is the basis case. For $k>1$, suppose $n > 2^k$, and consider a schedule without odd cycles. If the subgraph colored by the kth airline is bipartite, then we can partition the cities into two sets with no edges of color k within them. One of these sets has more than 2^{k-1} cities. Since the pairs within this set are covered by $k-1$ subgraphs, the induction hypothesis guarantees that some airline among the first $k-1$ can offer an odd cycle. For the construction when $n = 2^k$, partition the cities into two sets S, T of equal size. Use copies of the construction for $k-1$ on each of S, T, and give all edges between S and T to airline k. If $n < 2^k$, discard cities from this construction.

11.34. *If a graph G has no cycles of even length, then every edge of G appears in at most one cycle.* If G has three pairwise internally-disjoint w, z-paths, then the lengths of two of them must have the same parity (by the pigeonhole principle), and the union of these two paths forms a cycle through w, z of even length. Hence we can prove the claim by proving that if G has an edge in two cycles, then G has a pair of points joined by three pairwise internally-disjoint paths.

Let uv be an edge in distinct cycles C_1 and C_2. Continuing along C_2 from v, let xy be the first edge of C_2 that does not appear in C_1; note that $x \in V(C_1)$. Let z be the first vertex of C_2 after x that appears in C_1 (this may be y. Now $C_1 \cup C_2$ has three pairwise internally-disjoint x, z-paths, two along C_1 and the third on C_2.

11.35. *Every tree T has at most one perfect matching.*

Proof 1 (contradiction). Suppose M and M' are two complete matchings in the tree. Form the symmetric difference of the edge sets, $M \triangle M'$. Since the matchings are complete, each vertex has degree 0 or 2 in the symmetric difference, so every component is an isolated vertex or a cycle. Since the tree has no cycle, every vertex must have degree 0 in the symmetric difference, which means that the two matchings are the same. (The symmetric difference idea is quite powerful.)

Proof 2 (induction). For the basis step, a tree with one vertex has no perfect matching; a tree with two vertices has one. For the induction step, consider an arbitrary tree T on $n > 2$ vertices, and consider a leaf v. In any perfect matching, v must be matched to its neighbor u. The remainder of any matching is a matching in $T - \{u, v\}$. Since each perfect matching in T must contain the edge uv, the number of perfect matchings in T equals the number of perfect matchings in $T - \{u, v\}$. Each component of $T - \{u, v\}$ is a tree; by the induction hypothesis, each component has at most one perfect matching. The number of perfect matchings in a graph is the product of the number of perfect matchings in each component, so the original T has 0 or 1 perfect matching.

11.36. *a) If $k \in \mathbb{N}$, then a k-regular bipartite graph has the same number of vertices in each partite set.* We count the edges in two ways. If the partite sets are X, Y, then counting the edges by their endpoints in X gives $k\,|X|$, and counting them by their endpoints in Y gives $k\,|Y|$. Hence $k\,|X| = k\,|Y|$, which implies $|X| = |Y|$ since $k > 0$.

b) For $k \geq 1$, every k-regular bipartite graph has a perfect matching. Suppose the bipartition is X, Y. Counting the edges by endpoints in X and endpoints in Y shows that $k\,|X| = k\,|Y|$, so $|X| = |Y|$. Hence it suffices to verify Hall's condition; a matching saturating X will be a perfect matching. Consider $S \subseteq X$, and suppose there are m edges between S and $N(S)$. Since G is k-regular, we have $m = k\,|S|$. Since these m edges are incident to $N(S)$, we have $m \leq k\,|N(S)|$. Hence $k\,|S| \leq k\,|N(S)|$, which yields $|N(S)| \geq |S|$. Having chosen $S \subseteq X$ arbitrarily, we have established Hall's condition.

11.37. *$K_{n,n}$ has $n!$ perfect matchings and $n!(n-1)!/2$ cycles of length $2n$.* Matchings are pairings of $x_1, \ldots, x_n$ with $y_1, \ldots, y_n$ in some order; thus the number of matchings is the number of permutations of $[n]$.

In $K_{n,n}$, we can list the vertices in order on a cycle (alternating between the partite sets), in $2(n!)^2$ ways. Each cycle arises exactly $2n$ times in this way, since we obtain the same subgraph no matter where we start the cycle and no matter which direction we follow.

11.38. *The wheel with n vertices has chromatic number 3 when n is odd, 4 when n is even.* The cycle with $n-1$ vertices has chromatic number 2 when $n-1$ is even, 3 when $n-1$ is odd. In every proper coloring, the coloring used on the remaining vertex of the wheel must be different from all colors used on the cycle, since it is adjacent to all those vertices, so one additional color is needed.

11.39. *If G does not have two disjoint odd cycles, then $\chi(G) \leq 5$.* **Proof 1** (direct). If G has no odd cycle, then $\chi(G) \leq 2$, so we may assume that G has an odd cycle C. If $\chi(G - V(C)) \geq 3$, we have an odd cycle disjoint from C. Hence $\chi(G - V(C)) \leq 2$, and we can combine this with a 3-coloring of C to obtain a 5-coloring of G.

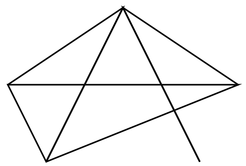
Proof 2 (contrapositive). If $\chi(G) \geq 6$, consider an optimal coloring. The subgraph induced by vertices colored 1,2,3 coloring must have an odd cycle, else it would be bipartite and we could replace these three colors by two. Similarly, the subgraph induced by vertices colored 4,5,6 in the optimal coloring has an odd cycle, and these two odd cycles are disjoint.

11.40. *A graph with maximum degree k has chromatic number at most $k+1$.* Such a graph can be colored with $k+1$ colors using a "greedy" algorithm. Place the vertices in an arbitrary order $v_1, \ldots, v_n$. Taking each vertex in turn, assign it the least-indexed color in $\{1, \ldots, k+1\}$ that has not already been used on one of its neighbors. Since it has at most k neighbors, there is some color left in $\{1, \ldots, k+1\}$ available to use on it.

Equality holds for the complete graph with $k+1$ vertices. *Comment:* among connected graphs, complete graphs and odd cycles are the only graphs where the chromatic number exceeds the maximum degree.

11.41. *The segment graph of a collection of lines in the plane with no three intersecting at a point is 3-colorable.* The vertices of G are the points of intersection of a family of lines; the edges are the segments on the lines joining two points of intersection. By tilting the picture slightly, we can insure that no pair of vertices has the same x-coordinate.

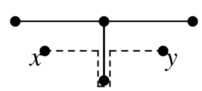
Of the neighbors that v has on a single line, one has larger x-coordinate and one has smaller x-coordinate. Hence if we index $V(G)$ in increasing order of x-coordinates, each vertex will have at most two earlier neighbors. Applying the greedy algorithm to this ordering produces a coloring using at most three colors. Alternatively, if $H \subseteq G$, the vertex of H with largest x-coordinate has degree at most 2 in H, so $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H) \leq 3$. The configuration below illustrates that the bound does not hold when more than two lines are allowed to meet at a point; this configuration can arise using six lines, and it yields a subgraph with chromatic number 4.



11.42. *If $n = k(k-1)-1$, then $\chi(G_{n,k}) > k+1$, where $G_{n,k}$ is the "generalized cycle" graph of Example 11.52.* If only $k+1$ colors are available, then by the pigeonhole principle some color must be used $k-1$ times, since $(k-2)(k+1) = k(k-1)-2$. Following the n steps around the circle, the minimum separation between consecutive appearances among the $k-1$ appearances of this color is less than k, since the total distance is $k(k-1)-1$, again by the pigeonhole principle. Since vertices at most $k-1$ apart are adjacent, this prohibits a proper $k+1$-coloring.

11.43. *The sum of the coefficients of $\chi(G; k)$ is 0 unless G has no edges.* The sum of the coefficients of a polynomial is its value when the argument is 1. The value of $\chi(G; 1)$ is the number of proper 1-colorings of G. This is 0 unless G has no edges.

11.44. *A plane graph that is a tree has one face.* We use induction on the number of vertices. With one vertex, every two points other than the point for the vertex are joined by a segment, except that pairs collinear with the vertex need a path of two segments. For a tree T with more vertices, consider two points x, y not in the embedding of T. Deleting a leaf of T and its incident edge produces a tree T'. By the induction hypothesis, there is a polygonal x, y-path avoiding T'. This path also avoids T unless it crosses the extra edge. Since the path is formed of finitely many segments, it crosses this edge finitely often. If it crosses the edge more than once, we can shortcut between two such crossings to reduce the number of crossings until we have a polygonal x, y-path that crosses T once on this edge and nowhere else. Now we can cut the path just before and after this crossing and replace the crossing by a path around the leaf, as suggested below.



11.45. *The complement of each simple planar graph with at least 11 vertices is nonplanar.* A planar graph with n vertices has at most $3n-6$ edges. Hence each planar graph with 11 vertices has at most 27 edges. Since K_{11} has 55 edges, the complement of each planar subgraph has at least 28 edges and is non-planar. In fact, there is also no planar graph on 9 or

10 vertices having a planar complement, but the easy counting argument here is not strong enough to prove that.

11.46. *Every simple planar graph has a vertex of degree at most 5.* Theorem 11.65 states that every simple planar graph G with n vertices has at most $3n-6$ edges. By the degree-sum formula, the degrees of our graph G thus sum to at most $6n-12$. By the pigeonhole principle, some vertex has degree at most the average, which equals $6-12/n$. Thus some vertex has degree at most 5.

Every planar graph has chromatic number at most 6. Since multiple edges do not affect chromatic number, we need only consider simple planar graphs. We use induction on the number of vertices; every graph with at most 6 vertices has chromatic number at most 6. When $n > 6$, we know that a simple planar graph G with n vertices has a vertex x of degree at most 5. Let $G' = G-x$. Since G' is planar, the induction hypothesis implies that G' has chromatic number at most 6. Since at most five of the colors are used on the neighbors of x, there is some color available to use on x to extend this to a proper 6-coloring of G.

11.47. *A simple n-vertex planar graph with no cycle of length less than k has at most $(n-2)k/(k-2)$ edges.* We may assume that the graph is connected by adding edges if it is not. Thus Euler's Formula applies. Consider any embedding of the graph in the plane. Each face has at least k edges on its boundary, and each edge contributes twice to boundaries of faces. Therefore, counting the appearances of edges in faces grouped according to the e edges or according to the f faces yields $2e \geq kf$.

Since the graph is connected, we can apply Euler's formula, $n-e+f = 2$. Substituting for f in the inequality yields $2e \geq k(2-n+e)$, which rearranges to $e \leq k(n-2)/(k-2)$. The formula makes sense also when $k = 2$, since the availability of multiple edges means there is no limit on the total number of edges.

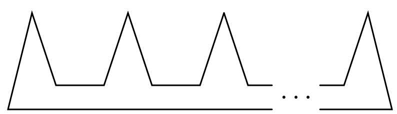
The Petersen graph has 10 vertices, 15 edges, and no cycle of length less than 5. Since 15 exceeds $5(8)/3$, this inequality implies that the Petersen graph is not planar.

11.48. *The maximum number of edges in a simple outerplanar graph of order n is $2n-3$.* For the lower bound, we provide a construction. A simple cycle on n vertices together with the chords from one vertex to the $n-3$ vertices not adjacent to it on the cycle forms an outerplanar graph with $2n-3$ edges.

For the upper bound, consider an arbitrary simple outerplanar graph G. We may assume G is connected, else we can add an edge to join two components while still having every vertex on the unbounded face F. Now, since every vertex appears on F, it takes at least n edges to traverse the boundary of F (note that if G is a tree, then F has length $2n-2$). Since the face lengths sum to $2e$ and every bounded face has length at least 3, we have $2e = \sum l(F_i) \ge n + 3(f-1)$. Substituting for f in Euler's formula yields $e \le 2n-3$.

Comments: 1) if G has no 3-cycles, the bound becomes $e \le (3n-4)/2$. 2) The upper bound can also be proved by a graph transformation, adding a new vertex inside F and an edge from it to each vertex of G. This produces an $n+1$-vertex planar graph G' with n more edges than G. Since $e(G') \le 3(n+1)-6$ edges, we have $e(G) \le 3(n+1)-6-n = 2n-3$.

11.49. *There are art galleries with n walls that require $\lfloor n/3 \rfloor$ guards.* Three vertices are used in forming each "alcove" in the art gallery below. No guard at one point can see into more than one of them, so at least $\lfloor n/3 \rfloor$ guards are needed.



12. RECURRENCE RELATIONS

12.1. *If $a_n = 3a_{n-1} - 2$ for $n \ge 1$, with $a_0 = 1$, then $a_n = 1$ for $n \ge 0$.* Iterating the recurrence suggests this answer, which is easily proved by induction. The basis step is given in the initial condition. The induction step assumes that $a_{n-1} = 1$ and computes $a_n = 3a_{n-1} - 2 = 3 \cdot 1 - 2 = 1$.

12.2. *If $a_n = a_{n-1} + 2a_{n-2}$ for $n \ge 2$, with $a_0 = 1$ and $a_1 = 8$, then $a_n = 3 \cdot 2^n - 2(-1)^n$ for $n \ge 0$.* The characteristic equation is $x^2 - x - 2 = 0$, with solutions $x = 2$ and $x = -1$. The general solution is $a_n = A2^n + B(-1)^n$.

We choose A and B to satisfy the initial conditions at $n = 0$ and $n = 1$. This requires $1 = A + B$ and $8 = 2A - B$. Thus $A = 3$ and $B = -2$, which yields the solution claimed.

12.3. *If $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \ge 2$, with $a_0 = a_1 = 1$, then $a_n = [3^n + (-1)^n]/2$ for $n \ge 0$.* The characteristic equation is $x^2 - 2x - 3 = 0$, with solutions $x = 3$ and $x = -1$. The general solution is $a_n = A3^n + B(-1)^n$.

We choose A and B to satisfy the initial conditions at $n = 0$ and $n = 1$. This requires $1 = A + B$ and $1 = 3A - B$. Thus $A = 1/2$ and $B = 1/2$, which yields the solution claimed.

12.4. *If $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \ge 2$, with $a_0 = 1$ and $a_1 = 3$, then $a_n = 3^n$ for $n \ge 0$.* The characteristic equation is $x^2 - 5x + 6 = 0$, with solutions $x = 3$ and $x = 2$. The general solution is $a_n = A3^n + B2^n$.

We choose A and B to satisfy the initial conditions at $n = 0$ and $n = 1$. This requires $1 = A + B$ and $3 = 3A + 2B$. Thus $A = 1$ and $B = 0$, which yields the solution claimed.

12.5. *If $a_n = 3a_{n-1} - 1$ for $n \ge 1$, with $a_0 = 1$, then $a_n = (3^n + 1)/2$ for $n \ge 0$.* The characteristic root is 3 and the inhomogenous term is a constant, so the solution has the form $A3^n + c$. For a particular solution involving the inhomogeneous term, we require $c = 3c - 1$ to obtain $c = 1/2$. Now the initial condition yields $1 = A \cdot 3^0 + 1/2$, so $A = 1/2$. Thus the solution is $a_n = (3^n + 1)/2$.

12.6. *Analysis of the recurrence $a_n = -a_{n-1} + \lambda^n$.* The characteristic root of the homogeneous part is -1. When $\lambda \ne -1$, a particular solution $b_n = \frac{1}{\lambda+1}\lambda^{n+1}$ is found by solving $C\lambda^n = -C\lambda^{n-1} + \lambda^n$ for C. The general solution is then $a_n = A(-1)^n + \frac{1}{\lambda+1}\lambda^{n+1}$. This is unbounded for $|\lambda| > 1$.

When $\lambda = -1$, a particular solution $b_n = n\lambda^n$ is found by solving $Cn\lambda^n = -C(n-1)\lambda^{n-1} + \lambda^n$ for C after dividing by λ and equating corresponding coefficients. The linear term confirms that $\lambda = -1$, and the constant term yields $C = -\lambda = 1$. The general solution is $a_n = A(-1)^n + n(-1)^n$ for constant A, and again this is unbounded.

12.7. *Recurrence relations with solution formula $a_n = n^3$.* Since the solution has no nontrivial exponential part, we seek a first-order relation of the form $a_n = a_{n-1} + f(n)$. It suffices to set $f(n) = n^3 - (n-1)^3 = 3n^2 - 3n + 1$.

Every homogeneous constant-coefficient first-order linear recurrence has the form $a_n = ca_{n-1}$, with general solution Ac^n. The constant A cannot be chosen to make Ac^n be n^3.

12.8. *Recurrence relation to count the pairings of $2n$ people.* The last person can be paired with $2n-1$ others, leaving a smaller instance of the same problem to complete the arrangement, so $a_n = (2n-1)a_{n-1}$, is the recurrence, with $a_0 = 1$ (or $a_1 = 1$).

12.9. *If $n > 0$ and no three circles meet at a point, then n pairwise intersecting circles cut the plane into $n^2 - n + 2$ regions.* Let a_n be the number of regions in such a configuration with n circles. Suppose $n > 1$, and consider adding the nth circle to the configuration consisting of the first $n-1$ circles. The nth circle intersects each earlier circle exactly twice. The $2(n-1)$ intersection points partition the new circle into $2(n-1)$ arcs, each of which cuts an old region into two new regions. Other regions remain unchanged. Hence $a_n = a_{n-1} + 2(n-1)$ for $n \ge 2$. This recurrence is not valid when $n = 1$; we have the initial condition $a_1 = 2$. The solution of a recurrence of the form $a_n = a_{n-1} + f(n)$ is $a_n = a_1 + \sum_{i=2}^{n} f(i)$, so we have $a_n = 2 + \sum_{i=1}^{n} 2(i-1) = 2 + 2\sum_{i=0}^{n-1} i = n^2 - n + 2$.

12.10. *Regions determined by a configuration of n pairwise non-parallel lines in the plane, when no three line meet at a point.* Add a circle that encloses all the points of intersection, thereby placing a bound on the unbounded regions. Discarding the unbounded portions of the lines yields a planar graph with one more region than the original configuration. The graph has $\binom{n}{2}+2n$ vertices, one for each pair of lines and two added on the outer circle. It has n segments from each line, plus $2n$ arcs on the outer circle, making a total of n^2+2n edges. By Euler's Formula, $\binom{n}{2}+2n-n^2-2n+f=2$; thus the graph has $n(n+1)/2+2$ regions and the original configuration has $n(n+1)/2+1$ regions, which agrees with Example 12.11.

12.11. *Recurrence for savings account.* To obtain the next amount, add \$100 at the start of the year and then 5 percent at the end, so $a_n = 1.05(a_{n-1}+100)$.

To solve the recurrence when $a_0=0$, we rewrite it as $a_n = 1.05a_{n-1}+105$. The form of the solution is $A(1.05)^n + b_0$, where b_0 is a constant. The recurrence yields $b_0 = 1.05b_0 + 105$, and hence $b_0 = -2100$. Satisfying the initial condition $a_0 = 0$ now requires $0 = A(1.05)^0 - 2100$, which yields $A = 2100$. Thus the sequence is $a_n = 2100(1.05)^n - 2100$.

12.12. *Mortgage: Recurrence for \$50,000 mortgage with interest per year at 5 percent of the unpaid amount, followed by payment of \$5,000 to end the year.* If a_n is the amount outstanding at the end of the nth year, then $a_n = (1.05)a_{n-1} - 5000$, with $a_0 = 50000$ (add 5 percent to the previous amount and then subtract 5000). Iterating this recurrence by calculator shows that the payment is completed in the 15th year.

12.13. *Bounded regions formed by chords among n points on a circle.* Let a_n be the number of regions.

a) $a_n = a_{n-1} + f(n)$ *for* $n \geq 1$, *where* $f(n) = n-1+\sum_{i=1}^{n-1}(i-1)(n-1-i)$, *with initial condition* $a_0 = 1$. With no points, there is one region. When we add an nth point, its chords to the $n-1$ other points add one region for each region they cut. As they leave the new point, each is cutting a region (this can be done successively for the $n-1$ chords), which accounts for the term $n-1$. The other term counts the crossings made by the chords as they continue; after each crossing, a new region is cut. The chord to the ith vertex separates $i-1$ vertices from $n-1-i$ vertices, and thus it cuts the $(i-1)(n-1-i)$ old chords crossing it.

b) Solution to the recurrence. The summation in the formula for $f(n)$ is an instance of both Exercise 9.11 and Exercise 5.30 (where the value of the summand should be $(i-1)(n-i)$; the value of the sum is $\binom{n-1}{3}$. This can also be computed by rearranging $(i-1)(n-1-i)$ into terms to

which the Summation Identity can be applied. By the Summation Identity, $\sum_{k=1}^{n}(k-1+\binom{k-1}{3}) = \binom{n}{2}+\binom{n}{4}$. Thus $a_n = a_0 + \sum_{k=1}^{n} f(n) = 1+\binom{n}{2}+\binom{n}{4}$.

12.14. *Solution of* $a_n = ca_{n-1} + f(n)\beta^n$, *where* f *is a polynomial of degree* d *and* β *is a constant.* Define b_n by setting $a_n = \beta^n b_n$. Substituting into the recurrence for $\langle a \rangle$ and canceling β^n yields $b_n = (c/\beta)b_{n-1} + f(n)$. If $c \neq \beta$, then Theorem 12.16 yields $b_n = Ac^n\beta^{-n} + p(n)$, where p is a polynomial of degree d. If $c = \beta$, then Theorem 12.16 yields $b_n = p(n)$, where p is a polynomial of degree $d+1$. Multiplying by β^n yields $a_n = Ac^n + p(n)\beta^n$ and $a_n = p(n)\beta^n$ in the two cases.

12.15. *Recurrence for moving n spaces.* With the options 1,2,3 at each move, the recurrence is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

12.16. *The number of ways to park the three types of cars in the lot of length n satisfies* $a_n = a_{n-1} + 2a_{n-2}$, *with* $a_0 = 1$ *and* $a_1 = 1$. The last car may be Type 1 (length 1) or Type 2 (length 2) or Type 3 (length 2). In the first Type, there are a_{n-1} ways to complete the list. In the other two Types, there are a_{n-2} ways to complete the list.

12.17. *Domino tilings of two-by-n checkerboard are counted by the Fibonacci numbers.* When the nth column is a vertical domino, there are a_{n-1} ways to complete the tiling. Otherwise, it must be filled by two horizontal dominos, and then there are a_{n-2} ways to complete the tiling. Thus $a_n = a_{n-1} + a_{n-2}$. For the initial conditions, $a_0 = a_1 = 1$. Combinatorially, every such tiling consists of units of 1 vertical tile or two horizontal tiles, which establishes a bijection to the 1,2-lists with sum n.

12.18. *Shopkeeper making change.* The number of ways that the shopkeeper can make change for n cents is the number of lists of 1s, 5s, and 10s that sum to n. If $n > 0$, every such list has a last element. The last element can be 1, 5, or 10, and in these three cases the number of ways to complete the rest of the list is a_{n-1}, a_{n-5}, or a_{n-10}, respectively. Hence we obtain the recurrence $a_n = a_{n-1} + a_{n-5} + a_{n-10}$, with the initial conditions $a_0 = 1$ (there is one such list that sums to 0, the empty list) and $a_n = 0$ for $n < 0$. Alternatively, one can compute 10 values in a row to start the list; starting with $n = 0$, the values of a_n are 1, 1, 1, 1, 1, 2, 3, 4, 5, 6.

12.19. *Identities for Fibonacci numbers.* Let F_n denote the nth Fibonacci number, where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

a) $\sum_{i=0}^{n} F_i^2 = F_nF_{n+1}$ *for* $n \geq 0$. Proof by induction on n. For $n = 0$, we have $\sum_{i=0}^{n} F_i^2 = F_0^2 = 1 = F_0F_1$. For the inductive step, suppose the claim is true for n, where $n \geq 0$. Using the induction hypothesis after isolating the last term of the sum, we have

$$\sum_{i=0}^{n+1} F_i^2 = F_{n+1}^2 + \sum_{i=0}^{n} F_i^2 = F_{n+1}^2 + F_n F_{n+1}$$
$$= F_{n+1}(F_{n+1} + F_n) = F_{n+1}F_{n+2},$$

where the last step uses the recurrence $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$.

b) $\sum_{i=0}^{n} F_{2i} = F_{2n+1}$. For $n = 0$, both sides equal 1. For the induction step, $F_{2n+1} = F_{2n} + F_{2n-1} = F_{2n} + \sum_{i=0}^{n-1} F_{2i} = \sum_{i=0}^{n} F_{2i}$. Note that the induction hypothesis is used only for one preceding value.

c) $\sum_{i=0}^{2n-1} (-1)^i F_{2n-i} = F_{2n-1}$.

12.20. *Bijection from the set of 1,2-lists that sum to n to the set of 0,1-lists of length $n-1$ that have no consecutive 1s.* Since the n units of the sum become $n-1$ binary positions, we want each 1 or 2 in the sum to contribute 0 or 1 bits, except that one bit will be lost. Convert each 1 to a 0 and each 2 to a 10, and drop the last bit, which is always a 0 and provides no information. The 0 following each 1 ensures that there are no consecutive 1s, the length is as desired, and the process is reversible.

12.21. *For the Fibonacci numbers, $1 + \sum_{i=0}^{n} F_i = F_{n+2}$.*

Proof 1 (induction on n). Since $F_0 = 1$ and $F_2 = 2$, we have $1 + \sum_{i=0}^{0} F_i = 1 + 1 = 2 = F_2$, so the claim holds for $n = 0$. For the induction step, suppose $n > 0$. We compute $1 + \sum_{i=0}^{n} F_i = 1 + (\sum_{i=0}^{n-1} F_i) + F_n = 1 + F_{n+1} + F_n = 1 + F_{n+2}$. In the induction step, we only use the induction hypothesis $\sum_{i=0}^{n-1} F_i = F_{n+1}$ for the one most recent value of the induction parameter. Also the Fibonacci recurrence $F_{n+1} + F_n$ is valid for all $n \geq 0$. Hence the proof of the induction step is valid for $n \geq 1$, and we need only check one value in the basis step.

Proof 2 (combinatorial argument). The right side of the identity counts lists of 1s and 2s that sum to $n+2$. One such list has no 2s; let that be the list counted by the isolated 1 on the left side. The other lists have at most n 1s. The number of lists having $n-i$ 1s after their last 2 is the number of lists that sum to i in the portion before that 2, which is F_i. Summing over all choices for i counts the lists that have a 2.

12.22. *For the Fibonacci numbers, $F_n = \sum_{i=0}^{n} \binom{n-i}{i}$.*

Proof 1 (induction on n). When $i > n-i$, the binomial coefficient is 0, so we can write the sum as $\sum_{i\geq 0} \binom{n-i}{i}$ without changing it. When n is 0 or 1, the sum has one nonzero term equal to 1 and the formula holds. For the induction step, suppose $n > 1$. We use the recurrence for F_n, the induction hypothesis giving the formula for F_{n-1} and F_{n-2}, and the recurrence relation $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ for the binomial coefficients. The resulting computation is

$$F_n = F_{n-1} + F_{n-2} = \sum_{i\geq 0} \binom{n-1-i}{i} + \sum_{i\geq 0} \binom{n-2-i}{i}$$
$$= \binom{n-1}{0} + \sum_{i\geq 1} \binom{n-1-i}{i} + \sum_{i\geq 1} \binom{n-1-i}{i-1}$$
$$= \binom{n}{0} + \sum_{i\geq 1} \binom{n-i}{i} = \sum_{i\geq 0} \binom{n-i}{i}.$$

Proof 2 (combinatorial argument). Recall that F_n is the number of lists of 1s and 2s that add up to n. In a list of 1s and 2s summing to n, if there are i 2s, then there must be $n-2i$ 1s. Hence there are $n-i$ terms all together, and choosing the i positions for the 2s specifies the list. Summing over all the possibilities for the number of 2s counts all the lists.

12.23. *Combinatorial proof of $F_{n+m} = F_n F_m + F_{n-1}F_{m-1}$.* Consider the 1, 2-lists summing to $n+m$. These are formed by concatenating a sum to n with a sum to m (in $F_n F_m$ ways) or by concatenating a sum to $n-1$ with a sum to $m-1$ separated by a 2 (in $F_{n-1}F_{m-1}$ ways). Since every sum doesn't or does have a 2 crossing from the first n to the last m, this counts all the sums.

F_{n-1} divides F_{kn-1}. Certainly this holds for $k = 1$. For $k > 1$, we write

$$F_{kn-1} = F_{(k-1)n-1+n} = F_{(k-1)n-1}F_n + F_{(k-1)n-2}F_{n-1}.$$

By the induction hypothesis, both terms on the right are divisible by F_{n-1}, so F_{n-1} also divides their sum.

12.24. *Every natural number can be written as a sum of distinct Fibonacci numbers.* Every natural number can be expressed as a sum of distinct powers of 2. More generally, if $\{a_n\}$ is a increasing sequence of natural numbers such that $a_1 = 1$ and $a_n \leq 2a_{n-1}$ for $n > 1$, then every natural number can be expressed as a sum of distinct terms. The Fibonacci numbers for $n \geq 1$ have this property, since $F_{n-1} < F_{n-1} + F_{n-2} = F_n$ and $F_n = F_{n-1} + F_{n-2} < 2F_{n-1}$.

To prove the general statement, note first that an increasing sequence of natural numbers is unbounded, since any potential bound $M \in \mathbb{N}$ is exceeded after at most M elements of the sequence. Hence for any natural number n there is a largest index k such that $a_k \leq n$. We now prove by strong induction on n that any $n \in \mathbb{N}$ can be expressed as a sum of distinct a_is. Since $a_1 = 1$, the number 1 can be expressed. Now assume that $n > 1$ and numbers less than n can be expressed as a sum of distinct a_is. Let k be the largest index such that $a_k \leq n$. Since $n < a_{k+1} \leq 2a_k$, we have $2a_k > n$. This means $n - a_k < a_k \leq n$. By the strong induction hypothesis, we know that $n - a_k$ can be expressed as a sum of distinct values in the sequence.

Since $n - a_k < a_k$, it cannot be that a_k is one of the values used. Hence we can add a_k to the expression for $n - a_k$ to obtain an expression for n as a sum of distinct a_is. This completes the induction step.

12.25. *Solution of the Fibonacci recurrence: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = F_1 = 1$.*

a) By the characteristic equation method. The characteristic polynomial of this recurrence is $x^2 - x - 1$. Its roots, by the quadratic formula, are $(1 \pm \sqrt{5})/2$. Let $\alpha = (1+\sqrt{5})/2$ and let $\beta = (1-\sqrt{5})/2$. For every $A, B \in \mathbb{R}$, the sequence with terms $A\alpha^n + B\beta^n$ is a solution of the recurrence, and we must select A, B to match the initial conditions. From $F_0 = 1$, we require $A + B = 1$. From $F_1 = 1$, we require $A\alpha + B\beta = 1$. Using the first equation in the second and simplifying leads to $A - B = 1/\sqrt{5}$. With the first equation, this yields $A = (1 + 1/\sqrt{5})/2$ and $B = (1 - 1/\sqrt{5})/2$. Factoring out $1/\sqrt{5}$ yields $A = \lambda_1/\sqrt{5}$ and $B = -\lambda_2/\sqrt{5}$. Thus

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}$$

b) By the generating function method. Multiplying the recurrence by x^n and summing over the region of validity ($n \geq 2$) yields $\sum_{n\geq 2} F_n x^n = x\sum_{n\geq 2} F_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2}x^{n-2}$. Letting $F(x) = \sum_{n\geq 0} F_n x^n$, we rewrite the equation as $F(x) - 1 - x = x(F(x) - 1) + x^2 F(x)$. Solving for $F(x)$ yields $F(x) = 1/(1 - x - x^2)$.

The denominator $1 - x - x^2$ factors as $(1 - \alpha x)(1 - \beta x)$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Using partial fractions, we write $F(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$. Clearing fractions yields $1 = A(1 - \beta x) + B(1 - \alpha x)$. Equating corresponding coefficients of x yields $1 = A + B$ and $0 = -A\beta - B\alpha$. Thus $A = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})$ and $B = -\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})$, and formula for F_n is as in part (a).

12.26. *Efficiency of the Euclidean algorithm.* If the Euclidean algorithm takes k steps on the pair (a_0, a_1), (where $k \geq 2$), then it takes $k - 1$ steps on the pair (a_1, a_2) that comes next in the sequence and $k - 2$ steps on the pair (a_2, a_3). We prove by induction on k that when the algorithm takes $k \geq 2$ steps, the sum of the numbers in the input pair is at least F_{k+2}.

Basis step: we verify the claim for $k = 2$ and $k = 3$, because we will use in the induction step the hypothesis that the claim is true for $k = n$ and $k = n - 1$. We have $F_4 = 5$ and $F_5 = 8$. We verify (the contrapositive) that any pair with sum less than 5 can be done in fewer than 2 steps, and any pair with sum less than 8 can be done in fewer than 3 steps. Here the contrapositive converts the verification to checking finitely many cases. A pair involving 1 moves to $(1, 0)$ in one step. There are six other pairs of natural numbers with sum less than 8. Of these, $(2, 2)$, $(4, 2)$, $(3, 3)$ take only one step. The pairs taking two steps are $(3, 2)$, $(5, 2)$, and $(4, 3)$, with $(3, 2) \to (2, 1) \to (1, 0)$, $(5, 2) \to (2, 1) \to (1, 0)$, and $(4, 3) \to (3, 1) \to (1, 0)$.

Induction step: we prove the claim for $k = n+1$, assuming that it holds for $k = n$ and $k = n-1$. If the algorithm takes $n+1$ steps on the pair (a_0, a_1), then it takes n steps on (a_1, a_2) and $n - 1$ steps on (a_2, a_3). By the induction hypothesis, $a_1 + a_2 \geq F_{n+2}$ and $a_2 + a_3 \geq F_{n+1}$. Hence $(a_1 + a_2) + (a_2 + a_3) \geq F_{n+3}$. It suffices to prove that $a_0 \geq a_1 + a_2$ and $a_1 \geq a_2 + a_3$. The reason for this is that when the Euclidean algorithm replaces (a_{i-1}, a_i) with (a_i, a_{i+1}), the numbers satisfy $a_{i-1} = ka_i + a_{i+1}$, where $a_{i+1} < a_i$. This requires $k \geq 1$, and hence $a_{i-1} \geq a_i + a_{i+1}$. This completes the proof by induction.

To prove that the bound on $a_0 + a_1$ is best possible, it suffices to construct an example (a_0, a_1) taking k steps whose sum is only F_{k+2}. Let $(a_0, a_1) = (F_{k+1}, F_k)$. These numbers have the desired sum. To show that the pair takes the proper number of steps, note that (F_{k+1}, F_k) moves to (F_k, F_{k-1}) in the first step. Hence we need only verify that $(F_3, F_2) = (3, 2)$ takes two steps, which we checked in the basis step.

12.27. *Deterministic prefix reversals end after at most $F_n - 1$ flips.* Our pile is a permutation of 1 through n. When the top card is m, we reverse the order of the first m cards. The largest card that ever appears will go to its proper level, and nothing larger can make it come up again. The largest card that appears later likewise can only appear once later. Thus the process always stops at some step with 1 at the top.

Let a_k be the maximum number of flips given that at most k distinct cards appear at the top during the process. Let T be the number of the flip on which the largest card m that appears reaches the top. Next it moves to position m and never returns. Afterwards, at most $k - 1$ different cards can appear, so there are at most $1 + a_{k-1}$ flips after T.

Before flip T, neither m nor 1 reaches the top, and neither does the card that starts in position m; call it l. Switch cards 1 and m in the original stack S to form a stack S'. The moves before T will be the same in S and S', meaning that the card reaching the top on corresponding flips is the same for both. Thus 1 reaches the top in T flips for S'. If $1 \neq l$, then l and m never reach the top in flipping S'; thus $T \leq a_{k-2}$ and $a_k \leq a_{k-2} + a_{k-1} + 1$ in this case. If $1 = l$, then $k - 1$ values may reach the top in flipping S', but now the flip after flip T in stack S brings 1 to the top, and in total there are at most $a_{k-1} + 1$ flips. Thus in each case $a_k \leq a_{k-2} + a_{k-1} + 1$. Since $a_1 = 0$ and $a_2 = 1$, we obtain $a_k \leq F_k - 1$ by induction. Finally, at most n different cards appear at the top when processing a stack with n cards.

12.28. *If $\langle a \rangle$ satisfies a kth-order homogeneous linear recurrence with constant coefficients, and the characteristic polynomial of the recurrence factors* *as* $p(x) = \prod_{i=1}^{r}(x-\alpha_i)^{d_i}$ *for distinct* $\alpha_1, \ldots, \alpha_r$*, then the solution of the recurrence has the form* $a_n = \sum_{i=1}^{r} q_i(n)\alpha_i^n$*, where each* q_i *is a polynomial of degree less than* d_i*. The* k *coefficients of these polynomials are determined by the initial values.*

A direct proof without the generating function method uses differentiation of the characteristic polynomial (which we do not introduce until Chapter 16) and linear independence of sequences. To avoid these, we use the generating function method discussed in this chapter. Indeed, this solution is sketched in Application 12.37.

Let the recurrence be $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ for $n \geq k$, so that the characteristic polynomial is also expressed as $p(x) = x^k - \sum_{i=1}^{k} c_i x^{k-i}$. Let $A(x) = \sum_{n=0}^{\infty}$; this is the (formal power series) generating function for the sequence $\langle a\rangle$. We show first that $A(x) = r(x)/p(x)$ for some polynomial r of degree less than k. Let $C_i(x) = \sum_{j=0}^{k-1-i} a_j x^j$ for $0 \leq i \leq k$ (with $C_k(x) = 0$). When the recurrence is multiplied by x^n and summed over $n \geq k$, the term $c_i a_{n-i}$ becomes $c_i \sum_{n\geq k} a_{n-i}x^n$. After extracting x^i, the summation is the expansion of $A(x)$, missing the terms before x^{k-i} that form $C_i(x)$. Thus

$$A(x) - C_0(x) = \sum_{i=1}^{k} c_i x^i [A(x) - C_i(x)].$$

We obtain $p(x)A(x) = r(x)$, where $r(x)$ is a polynomial in x of degree less than k. The coefficients in $r(x)$ depend on the initial conditions $a_0, \ldots, a_{k-1}$ and the constants $c_1, \ldots, c_k$.

Using the factorization $p(x) = \prod_{i=1}^{r}(x-\alpha_i)^{d_i}$, we rewrite $r(x)/p(x)$ using partial fraction expansion. We obtain $A(x) = \sum_{i=1}^{r} q_i(x)/(1-\alpha_i x)^{d_i}$, where q_i is a polynomial of degree less than d_i. The polynomials q_i are determined by the initial values for $\langle a\rangle$.

Because $(1-\alpha x)^{-d} = \sum_{n=0}^{\infty}\binom{n+d-1}{d-1}\alpha^n x^n$ (Theorem 5.23), and $\binom{n+d-1}{d-1}$ is a polynomial in n with degree less than d, we obtain a formula for a_n of the form claimed.

12.29. *If* $\langle b\rangle$ *and* $\langle d\rangle$ *are solutions to the inhomogeneous linear* k*th-order recurrences* $x_n = f(n) + \sum_{i=1}^{k} h_i(n)x_{n-i}$ *and* $x_n = g(n) + \sum_{i=1}^{k} h_i(n)x_{n-i}$*, then* $\langle b\rangle + \langle d\rangle$ *is a solution to the recurrence* $x_n = f(n) + g(n) + \sum_{i=1}^{k} h_i(n)x_{n-i}$*).* Set $x_n = b_n$ in the first recurrence and $x_n = d_n$ in the second. Summing the two resulting equations yields $\langle b\rangle + \langle d\rangle$ as a solution to the third equation.

12.30. *If* $\langle a\rangle$ *is a solution of* $x_n = c_1x_{n-1} + c_2x_{n-2} + c\alpha^n$*, where* $c_1, c_2, c, \alpha \in \mathbb{R}$*, then* $\langle a\rangle$ *and* $C\alpha^n$ *are solutions of the homogeneous third-order recurrence* $x_n = (c_1+\alpha)x_{n-1} + (c_2-\alpha c_1)x_{n-2} - \alpha c_2 x_{n-3}$*.*

Every solution to the recurrence satisfies $x_n = c_1x_{n-1} + c_2x_{n-2} + c\alpha^n$ for all n (beyond the initial conditions), and hence it also satisfies $x_{n-1} = c_1x_{n-2} + c_2x_{n-3} + c\alpha^{n-1}$. Subtracting α times the second equation from the first yields the desired third-order recurrence. Hence every solution to the original recurrence also satisfies the new recurrence.

For $x_n = C\alpha^n$, we show that the sequence satisfies the desired recurrence by explicit computation.

$$\begin{aligned}&(c_1+\alpha)C\alpha^{n-1} + (c_2-\alpha c_1)C\alpha^{n-2} - \alpha c_2 C\alpha^{n-3}\\ &\qquad = C\alpha^n + C\left[c_1\alpha^{n-1} - \alpha c_1\alpha^{n-2} + c_2\alpha^{n-2} - \alpha c_2\alpha^{n-3}\right] = C\alpha^n\end{aligned}$$

12.31. Consider the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n > 3$.

If $a_i = 1$ *for* $i \in \{1,2,3\}$*, then* $a_n \leq 2^{n-2}$ *for* $n \geq 2$. We use induction on n. Basis step: $2 \leq n \leq 4$. For $n = 2$ and $n = 3$, we cite the initial conditions. Since the claim fails for $n = 1$, instead we use $a_4 = 1+1+1 = 3 \leq 2^2$.

Induction step: $n > 4$. We have $a_n = a_{n-1} + a_{n-2} + a_{n-3} \leq 2^{n-3} + 2^{n-4} + 2^{n-5} = (4+2+1)2^{n-5} < 2^{n-2}$.

If $a_i = i$ *for* $i \in \{1,2,3\}$*, then* $a_n < 2^n$ *for* $n \geq 1$. We use induction on n. Basis step: $1 \leq n \leq 3$. This is given in the initial conditions.

Induction step: $n > 4$. We have $a_n = a_{n-1} + a_{n-2} + a_{n-3} < 2^{n-1} + 2^{n-2} + 2^{n-3} = (4+2+1)2^{n-3} < 2^n$.

12.32. *The solution of* $a_n = \frac{2}{3}(1+\frac{2}{3^n+1})a_{n-1}$ *for* $n \geq 1$*, with* $a_0 = 1$*, is* $a_n = 2^{n+1}/(3^n+1)$. Letting $b_n = (3^n+1)a_n$, we substitute into the recurrence to obtain $\frac{1}{3^n+1}b_n = \frac{2}{3}(1+\frac{2}{3^n+1})\frac{1}{3^{n-1}+1}b_{n-1}$. Multiplying by 3^n+1 and simplifying yields $b_n = 2b_{n-1}$. Note that $b_0 = 2a_0 = 2$. Thus the solution is $b_n = 2^{n+1}$, which yields $a_n = 2^{n+1}/(3^n+1)$.

12.33. *Algorithm for finding the largest and smallest in a set of* n *numbers.* If $n = 2$, compare the two numbers. If $n > 2$, find the largest and smallest in sets of sizes $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$ (recursively), and compare the largests and the smallests in these sets.

Letting a_n denote the number of comparisons made for a set of size n, we have $a_n = a_{\lceil n/2\rceil} + a_{\lfloor n/2\rfloor} + 2$ for $n \geq 4$, with $a_1 = 0$, $a_2 = 1$, and $a_3 = 3$.

When $n = 2^k$, let $b_k = a_n$. We obtain $b_k = 2b_{k-1} + 2$ for $k \geq 2$, with $b_1 = 1$. To solve this recurrence, the particular solution C is found by $C = 2C + 2$, yielding $C = -2$, and the general solution is $b_k = A2^k - 2$. Since $b_1 = 1$, we have $b_k = (3/2)2^k - 2$. In terms of n, this yields $a_n = (3/2)n - 2$.

12.34. *Alternative solution of the recurrence* $D_n = (n-1)(D_{n-1} + D_{n-2})$ *(with* $D_0 = 1$*).* Subtracting nD_{n-1} from both sides yields $D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2}$. Sustituting $f_n = D_n - nD_{n-1}$ yields $f_n = -f(n-1)$. Since $D_0 = 1$ and $D_1 = 0$, we have $f_1 = -1$. Thus the solution of the recurrence for f is $f(n) = (-1)^n$.

Substituting this yields the first-order recurrence $D_n = nD_{n-1} + (-1)^n$. We use this to prove by induction on n that $D_n = n!\sum_{k=0}^{n}(-1)^k/k!$. Basis step ($n = 0$): $D_n = 1 = 0!(-1)^0/0!$.

Induction step ($n > 0$):

$$D_n = nD_{n-1} + (-1)^n = n(n-1)!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!} + (-1)^n$$

$$= n!\sum_{k=0}^{n-1}\frac{(-1)^k}{k!} + n!\frac{(-1)^n}{n!} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

12.35. *The number B_n of equivalence relations on n elements satisfies the recurrence $B_n = \sum_{k=1}^{n}\binom{n-1}{k-1}B_{n-k}$, with the initial condition $B_0 = 1$.* The combinatorial argument to prove the recurrence is as follows. Consider the element x; it appears in an equivalence class of size k, for some k with $1 \le k \le n$. The elements equivalent to x can be chosen in $\binom{n-1}{k-1}$ ways, and then the partition is completed by using some partition of the remaining $n-k$ elements. By summing over k, we obtain all the partitions.

12.36. *Congruence classes of triangles with perimeter n and sides with integer length.* Let a_n be the number of sets $\{x, y, z\} \subseteq \mathbb{N}$ such that x, y, z are the lengths of the sides of a triangle with perimeter n.

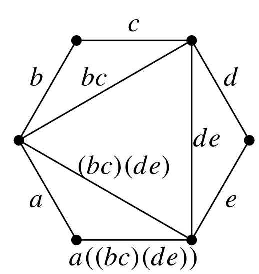
Option 1. If $y+z \ge x+2$, we can subtract 1 from each to get a triangle on $n-3$. For any triangle on $n-3$, we can add 1 to each side to get a triangle on n satisfying $y+z \ge x+2$. If n is even, every triangle on n satisfies this, so we get $a_n = a_{n-3}$. If n is odd, there are $\lfloor \lceil n/2 \rceil /2 \rfloor$ triangles with $y+z = \lceil n/2 \rceil$ and $x = \lfloor n/2 \rfloor$ that are not counted by a_{n-3}. By considering the possible congruences classes, we have $a_n = a_{n-3} + \lfloor (n+1)/4 \rfloor$ when n is odd. The initial conditions are $a_n = 0$ for $n \le 0$. This can be written as a single recurrence as $a_n = a_{n-3} + 1/2(1-(-1)^n)\lfloor (n+1)/4 \rfloor$.

Option 2. Triangles with $y \ne z$ arise by adding 1 to the two smaller parts of a triangle on $n-2$. There are $\lfloor n/3 \rfloor - \lfloor n/4 \rfloor$ triangles with $y = z$, since there is one such triangle for each y with $n/4 < y \le n/3$. Hence $a_n = a_{n-2} + \lfloor n/3 \rfloor - \lfloor n/4 \rfloor$.

12.37. *Parenthesizations of a product of $n+1$ elements.* Let a_0 be the number of parenthesizations. Observe that $a_0 = 1$. The final product in the parenthesization combines the initial k numbers with the final $n+1-k$ numbers, for some k. There are a_{k-1} ways to parenthesize the initial k elements, and there are a_{n-k} ways to parenthesize the final $n+1-k$ elements. Each way to do the first part can combine with each way to do the final part. summing over k yields all parenthesizations, so $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$. This is the same recurrence and initial condition satisfied by the Catalan sequence, so $a_n = C_n$ for all n.

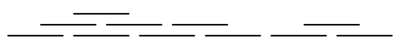
12.38. *A bijection from the set of parenthesizations of $n+1$ distinct elements to the set of triangulations of a convex $n+2$-gon, as suggested by the figure below.* Choose one edge of the $(n+2)$-gon as a root, and associate the $n+1$ distinct elements with the other edges, in order. From a parenthesization, we obtain a triangulation. Write the parenthesization on the root edge. The outermost pairing combines some parenthesization of the first k elements with some parenthesization of the last $n+1-k$, for some k. Create a triangle using edges from the root edge to the vertex at the end of the edge for the kth element. Write the parenthesization of the first k elements on the chord cutting those edges off, and write the parenthesization of the last $n+1-k$ elements on the other chord. We now have a $(k+1)$-gon for the first group and an $(n+2-k)$-gon for the second group. Proceed recursively by the same method to produce the triangulations in those polygons.

We prove inductively that the function is a bijection. When $n = 0$, the root edge and the edge for the element are the same; both get this element as label. This is the only parenthesization, and the only triangulation is the empty region. For larger n, the map is a bijection because the final pairing can be retrieved by knowing the triangle that contains the root edge, and then the induction hypothesis provides the inverse to retrieve the parenthesizations of the two groups that are combined in the final pairing.



12.39. *Catalan numbers count noncrossing pairings of $2n$ people around a circular table.* A particular individual must pair with someone, and this pair must leave an even number of intervening people on both sides. If $2k-2$ is the number of intervening people on the left, then the pairing can be completed in $a_{k-1}a_{n-k}$ ways. Thus $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$. There is one pairing when $n = 0$. Thus $\langle a \rangle$ satisfies the Catalan recurrence, and the solution is the sequence of Catalan numbers. (Exercise 9.26 requests a bijective proof.)

12.40. *Catalan numbers count piles of pennies.* When the base has length n, let k be the largest initial number of pennies above which the next row is occupied; $k = 4$ in the illustration. the illustration. the illustration. the illustration. the illustration. To complete such a pile, we complete above the inital base of length $k-1$ in row 2 and above the remaining base of length $n-k$ in the base row. Thus the recurrence for the number of piles is $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$. There is one pile with a base of no pennies. Thus the solution is the Catalan numbers.



12.41. *The nth difference of a polynomial of degree n is the constant $n!$ times the leading coefficient of the polynomial.* We prove that the first difference of a polynomial of degree d with leading coefficient a is a polynomial of degree $d-1$ with leading coefficient da. Since the nth difference is obtained by applying the first difference n times, this yields the claimed result.

By the definition of first difference, the first difference of a sum of polynomials is the sum of their first differences, and the first difference of c times a polynomial f is $c\Delta f$. Thus it suffices to prove the claim for the pure powers.

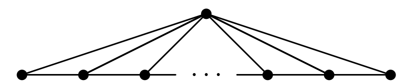
We have $\Delta x^d = (x+1)^d - x^d$. Expanding $(x+1)^d$ by the Binomial Theorem shows that the result is a polynomial of degree $d-1$ with leading coefficient d, as desired.

12.42. *The number $s(n,k)$ of surjective functions from an n-element set A to a k-element set B satisfies the recurrence $s(n,k) = k(s(n-1,k-1)+s(n-1,k)$ when $n > 1$, with initial conditions $s(1,1) = 1$ and $s(1,k) = 0$ for $k > 1$.* If $|A| = 1$, then the function covers all of B if and only if $|B| = 1$; this verifies the initial conditions. Now suppose $n > 1$, $A = \{a_1,\dots,a_n\}$, $B = \{b_1,\dots,b_k\}$, and $f\colon A \to B$ is surjective. If $f(a_n) = b_i$, then we can complete a surjective function in two types of way; either no other element of A has image b_i under f, or some other element of A has image b_i under f. In the first case, we can complete the function in $s(n-1,k-1)$ ways, by combining $f(a_n) = b_i$ with a surjective function from $A - \{a_n\}$ to $B - \{b_i\}$. In the second case, we can complete the function in $s(n-1,k)$ ways, by combining $f(a_n) = b_i$ with a surjective function from $A - \{a_n\}$ to B. We have $s(n-1,k-1) + s(n-1,k)$ ways to build a surjective function for each of the k possible choices for $f(a_n)$, so the total number of surjective functions is $s(n,k) = k(s(n-1,k-1) + s(n-1,k)$.

12.43. For the graph G_n sketched below, consisting of a path with n vertices and one additional vertex adjacent to every vertex of the path, let a_n be the number of spanning trees.

a) $a_n = a_{n-1} + \sum_{i=0}^{n-1} a_i$ *for* $n \ge 2$, *where* $a_0 = a_1 = 1$. Let $x_1,\dots,x_n$ be the vertices of the path in order, and let z be the vertex off the path. There are a_{n-1} spanning trees not using the edge zx_n; they combine the edge $x_{n-1}x_n$ with a spanning tree of $K_1 \vee P_{n-1}$. Among trees containing zx_n, let i be the highest index such that all of the path $x_{i+1},\dots,x_n$ appears in the tree. For each i, there are a_i such trees, since the specified edges are combined with a spanning tree of $K_1 \vee P_i$. The term 1 corresponds to $i = 0$; here the entire tree is $P_n \cup zx_n$. This exhausts all spanning trees.

b) $a_n = 3a_{n-1} - a_{n-2}$ *for* $n \ge 3$. We dispose of the summation by subtracting one instance of the recurrence formula from the next. For $n \ge 3$, we subtract $a_{n-1} = a_{n-2} + \sum_{i=0}^{n-2} a_i$ from $a_n = a_{n-1} + \sum_{i=0}^{n-1} a_i$ to obtain $a_n - a_{n-1} = a_{n-1} - a_{n-2} - a_{n-1}$, which yields the desired recurrence.



12.44. *For the graph G_n with $2n$ vertices and $3n-2$ edges pictured below, the chromatic polynomial is*
$\chi(G_n;k) = (k^2-3k+3)^{n-1}k(k-1)$.

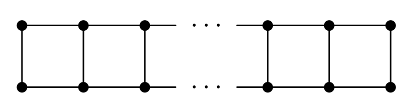
Proof 1 (induction on n). Since G_1 is a 2-vertex tree, $\chi(G_1;k) = k(k-1)$. For $n > 1$, let u_n, v_n be the two rightmost vertices of G_n. The proper colorings of G_n are obtained from proper colorings of G_{n-1} by assigning colors also to u_n and v_n. Each proper coloring f of G_{n-1} satisfies $f(u_{n-1}) \ne f(v_{n-1})$. Thus each such f extends to the same number of colorings of G_n.

There are $(k-1)^2$ ways to specify $f(u_n)$ and $f(v_n)$ so that $f(u_n) \ne f(u_{n-1})$ and $f(v_n) \ne f(v_{n-1})$. Of these extensions, $k-2$ give u_n and v_n the same color, and we delete them. Since $(k-1)^2 - (k-2) = k^2 - 3k + 3$, the induction hypothesis yields

$$\chi(G_n;k) = (k^2-3k+3)\chi(G_{n-1};k) = (k^2-3k+3)^{n-1}k(k-1).$$

Proof 2 (induction plus chromatic recurrence). Again $\chi(G_1;k) = k(k-1)$. Let $e = u_nv_n$. For $n > 1$, observe that $\chi(G_n - e;k) = \chi(G_{n-1};k)(k-1)^2$ and $\chi(G_n \cdot e;k) = \chi(G_{n-1};k)(k-2)$, by counting the ways to extend each coloring of G_{n-1} to the last column. Thus

$$\begin{aligned}\chi(G_n;k) &= \chi(G_n - u_nv_n;k) - \chi(G_n \cdot u_nv_n;k)\\ &= \chi(G_{n-1};k)[(k-1)^2 - (k-2)] = (k^2-3k+3)^{n-1}k(k-2).\end{aligned}$$



12.45. *Gambler's Ruin recurrence.* Let $a_n(r,s)$ be the probability that A goes broke on the nth flip. If all of n,r,s are positive, then $a_n(r,s) = \frac{1}{2}a_{n-1}(r+1,s-1) + \frac{1}{2}a_{n-1}(r-1,s+1)$. If at least one of n,r,s is 0, then $a_n(r,s) = 0$, except when $n = r = 0$, in which case $a_0(0,s) = 1$.

12.46. *Recurrence for partitions of integer n with k parts.* The partitions of n with k parts consist of those that have 1 as a part and those that do not. Those with 1 as a part correspond to partitions of $n-1$ with $k-1$ parts. Those without 1 as a part still have k positive parts when we reduce each part by 1, so these correspond to partitions of $n-k$ with k parts. Thus $p_{n,k} = p_{n-1,k-1} + p_{n-k,k}$. This argument requires $k \ge 1$ and $n \ge k$. To enable computation of values, we need initial conditions $p_{0,0} = 1$, $p_{n,0} = 0$ for $n > 0$, and $p_{n,k} = 0$ for $n < k$.

12.47. *Solution of the Catalan recurrence.* Suppose that $a_0 = 1$ and that $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n \ge 1$. Rewrite the recurrence as $a_n = \sum_{k=0}^{n-1} a_k a_{n-1-k}$ for $n \ge 1$. The generating function method yields the equation $A(x) - 1 = x[A(x)]^2$, where $A(x) = \sum_{k\ge 0} a_k x^k$. By the quadratic formula, $A(x) = (1 \pm \sqrt{1-4x})/(2x)$. We choose the negative sign on the square root because A is a formal power series in x, and thus the coefficient of x^{-1} must be zero.

The extended binomial theorem yields $\binom{1/2}{k}(-4)^k$ as the coefficient of x^k in $\sqrt{1-4x}$. The Catalan number for $n \ge 1$ is the coefficient of x^n in $-\frac{1}{2}(1-4x)^{1/2}/x$. Thus

$$C_n = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} = \frac{-2^n}{(n+1)!}(-2)^{n+1}\prod_{i=0}^{n}(\frac{1}{2} - i)$$
$$= \frac{-2^n}{(n+1)!}\prod_{i=0}^{n}(2i-1) = \frac{1}{n+1}\frac{\prod_{i=1}^{n}(2i-1)}{n!}\frac{\prod_{i=1}^{n}(2i)}{\prod_{i=1}^{n} i} = \frac{1}{n+1}\binom{2n}{n}$$

12.48. *Direct combinatorial deriviation of the Fibonacci generating function.* We enumerate the 1,2-lists, with the coefficient of x^n counting the lists with sum n. At each position in the list, we can choose 1 or 2, with the exponent recording the contribution to the sum. Hence our factor for each position is $x^1 + x^2$. When we form lists of length k, the generating function is $(x+x^2)^k$. Since we allow lists of all lengths, the full generating function is $\sum_{k\ge 0}(x+x^2)^k$. This is a formal geometric series, and the sum is $1/(1-(x+x^2))$, which is $1/(1-x-x^2)$.

12.49. If $\langle a\rangle$ satisfies the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, with initial values a_0, a_1, then the generating function for $\langle a\rangle$ is $\frac{(a_1 - c_1 a_0)x + a_0}{1 - c_1 x - c_2 x^2}$.

Multiplying the recurrence by x^n and then summing over the region of validity ($n \ge 2$) yields $\sum_{n\ge 2} a_n x^n = c_1 x \sum_{n\ge 2} a_{n-1}x^{n-1} + c_2 x^2 \sum_{n\ge 2} a_{n-2}x^{n-2}$.

Letting $A(x) = \sum_{n\ge 0} a_n x^n$, we obtain $A(x) - a_0 - a_1 x = c_1 x[A(x) - a_0] + c_2 x^2 A(x)$. Solving for $A(x)$ yields $A(x) = \frac{(a_1 - c_1 a_0)x + a_0}{1 - c_1 x - c_2 x^2}$.

12.50. *Generating function for selections from five types of coins with between 2 and 6 coins of each type.* For each type of coin, we make a choice, independently. We model this choice with $x^2 + x^3 + x^4 + x^5 + x^6$. The generating function for the full problem is the fifth power of this, one factor for each type of coin. The number of such selections with n coins is the coefficient of x^n in $(x^2 + \cdots x^6)^5$.

12.51. *Proof of $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ by generating functions.* Consider $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$. This equals the left side and is the coefficient of x^n in the product of $\sum_{k\ge 0}\binom{n}{k}x^k$ and $\sum_{l\ge 0}\binom{n}{l}x^l$. Thus the sum is the coefficient of x^n in $(1+x)^n(1+x)^n$. This coefficient is $\binom{2n}{n}$.

12.52. *Proofs of convolution identities using generating functions.*

a) $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$. The left side is the coefficient of x^k in the product of the generating functions $\sum_{r\ge 0}\binom{m}{r}x^r$ and $\sum_{r\ge 0}\binom{n}{r}x^r$. Since these generating functions are $(1+x)^m$ and $(1+x)^n$, the product is $(1+x)^{m+n}$. The right side is indeed the coefficient of x^k in the product.

b) $\sum_{k=-m}^{n}\binom{m+k}{r}\binom{n-k}{s} = \binom{m+n+1}{r+s+1}$. Shifting the index of summation on the left yields $\sum_{k=0}^{m+n}\binom{k}{r}\binom{m+n-k}{s}$. This is now the coefficient of x^{m+n} in the product of the generating functions $\sum_{j\ge 0}\binom{j}{r}x^j$ and $\sum_{j\ge 0}\binom{j}{s}x^j$. We discard zeros and then shift the index to obtain $\sum_{j\ge 0}\binom{j}{r}x^j = \sum_{j\ge r}\binom{j}{r}x^j = x^r\sum_{j\ge 0}\binom{j+r}{r}x^j$. Theorem 12.35 now yields $x^r(1-x)^{-(r+1)}$ as this generating function. Similarly, $\sum_{j\ge 0}\binom{j}{s}x^j = x^s(1-x)^{-(s+1)}$. Hence the product is $x^{r+s}(1-x)^{-(r+s+2)}$.

By Theorem 12.35, the expansion of this generating function is $\sum_{j\ge 0}\binom{j+r+s+1}{r+s+1}x^{j+r+s}$. To obtain the coefficient of x^{m+n}, we set $j = m+n-r-s$ in the summand and extract the coefficient $\binom{m+n+1}{r+s+1}$, as desired.

c) $\sum_{i=0}^{k}\binom{m+k-i-1}{k-i}\binom{n+i-1}{i} = \binom{m+n+k-1}{k}$. We rewrite the identity as $\sum_{i=0}^{k}\binom{i+n-1}{n-1}\binom{k-i+m-1}{m-1} = \binom{k+m+n-1}{m+n-1}$. The left side is the coefficient of x^k in the product of the generating functions $\sum_{r\ge 0}\binom{r+m-1}{m-1}x^r$ and $\sum_{r\ge 0}\binom{r+n-1}{n-1}x^r$. By Theorem 12.35, these generating functions are $(1-x)^{-m}$ and $(1-x)^{-n}$. The product is $(1-x)^{-(m+n)}$. By Theorem 12.35, the coefficient of x^k in its expansion is $\binom{k+m+n-1}{m+n-1}$, as desired.

12.53. *Generating function for summing initial terms.* The expression $b_n = \sum_{k=0}^{n} a_k$ is a convolution; we have $b_n = \sum_{k=0}^{n} a_k \cdot 1$. Hence the generating function $B(x) = \sum_{n\ge 0} b_n x^n$ is the product of the generating functions $A(x) = \sum_{n\ge 0} a_n x^n$ and $(1-x)^{-1} = \sum_{1\ge .} x^n$. Thus $B(x) = A(x)/(1-x)$.

12.54. *The generating function for repeated rolls of a six-sided die by total outcome is* $\frac{x-x^7}{1-2x+x^7}$. When the die is rolled r times, the generating function is $(\sum_{i=1}^{6} x^i)^r$. Since r may be any natural number, we sum over these possibilities. The result is a geometric series missing its initial term. This simplifies as below:

$$\sum_{r\ge 1}\left[x\left(\frac{1-x^6}{1-x}\right)\right]^r = \left(\frac{1}{1-x\frac{1-x^6}{1-x}}\right) - 1 = \frac{1-x}{1-x-x(1-x^6)} - 1$$
$$= \frac{(1-x)-(1-2x+x^7)}{1-2x+x^7} = \frac{x-x^7}{1-2x+x^7}$$

12.55. *Generating function for partitions into parts of size at most k.* We have a factor in the generating for the parts of each size; for each size, we choose the number of parts. When the size i is as most k, we are allowed to use any number of parts and have the factor $1+x^i+x^{2i}+\cdots$. When i exceeds k, we must choose none and have the factor 1. Hence the generating function is $\prod_{i=1}^{k}(1-x^i)^{-1}$.

12.56. *Generating functions for partitions of integers.* The generating function for partitions into distinct parts is $\prod_{i=1}^{\infty}(1+x^i)$, since each size i can be used 0 times or one time. Each choice of distinct positive integers summing to n contributes 1 to the coefficient of x^n in the product.

The generating function for partitions into odd parts is $\prod_{i=1}^{\infty}(1-x^{2i-1})^{-1}$, since each odd size $2i-1$ can be used any number of times. Using k copies of $2i-1$ corresponds to choosing $(x^{2i-1})^k$ in the expansion of $1/(1-x^{2i-1})$. Each choice yielding exponents that sum to n counts one partition of n into odd parts and contributes one to the coefficient of x^n in the formal power series.

12.57. *The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.* (Thus the generating functions of Exercise 12.56 are equal.) From a partition into odd parts, we form a partition into distinct parts by iteratively combining two identical parts until no identical parts remain. From a partition into distinct parts, we retrieve the unique partition into odd parts that maps to it by splitting each part $\lambda_i = j_i 2^{r_i}$ into 2^{r_i} copies of the odd number j_i. This works because every positive integer λ has a unique expression as an odd number times a power of 2.