SSY130 - Hand in Problem 1

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In this task a continuous time signal is passed through a low-pass filter before sampling. The goal is to be able to reconstruct the signal after sampling. To do this our sampling frequency has to full fill two criterion. The following information is available for the signals, noise and filter:

$$H(\omega) = \tfrac{1}{1+j\omega/\omega_0}$$

$$X_s = \begin{cases} 1, & |\omega| < 250\pi \times 10^3 \\ 0, & |\omega| > 250\pi \times 10^3 \end{cases}$$

$$N = \begin{cases} 0.1, & 360\pi \times 10^3 \le |\omega| < 700\pi \times 10^3 \\ 0, & |\omega| < 360\pi \times 10^3 \\ 0, & |\omega| > 700\pi \times 10^3 \end{cases}$$

$$x(t) = x_s(t) + n(t)$$

(a) Determine the minimum sample rate so that the continuous time signal $x_s(t)$ can be perfectly reconstructed if the noise signal is not present.

To determine the sampling rate that results in a perfect reconstruction, the Nyquist criterion should be considered. It reads:

If $|X(\omega)| = 0$ for all $|\omega| < \omega_s/2$, then $x(t)$ can be perfectly reconstructed.

It our case we have that the highest ω the desired signal can have while still having the value 0 is $< 250\pi \times 10^3$. The Nyquist criterion then gives:

$$\omega_s > 2\omega_{max}$$
$$\omega_s > 500\pi \times 10^3 \text{ rad/s}$$

(b) Determine the minimum sample rate so that the magnitude of the filtered and sampled noise signal for $|\omega| < 250\pi \times 10^3$ is at least 20 times lower than the magnitude of the filtered and sampled desired signal at $\omega = 0$.

First we should examine the magnitude of the filtered and sampled desired signal at $\omega = 0$. We get:

$$|H(0)X_s(0)| = 1$$

When it comes to the noise signal we should assume the worst case scenario which means that the noise gets phase shifted by the sampling rate in to the area where the magnitude of the noise equals 0.1. We get:

$$|H(\omega + \omega_s)| \cdot |N|_{max}$$
$$=$$
$$|H(\omega + \omega_s)| \cdot 0.1$$

Now we can apply the condition that the magnitude of the noise should be at least 20 times lower than the magnitude of the desired signal. we get:

$$|H(\omega + \omega_s)| \cdot |N|_{max} \leq \tfrac{1}{20} \cdot |H(0)X_s(0)|$$
or
$$|H(\omega + \omega_s)| \cdot 0.1 \leq \tfrac{1}{20}$$

Which can be expressed as:

$$\frac{1}{\sqrt{1^2 + \frac{(\omega+\omega_s)^2}{\omega_0^2}}} \cdot 0.1 \leq \frac{1}{20}$$

Resulting in:

$$\omega + \omega_s \geq \sqrt{3} \cdot \omega_0$$

Once again assuming the worst we assume that ω is at the lower end of its boundary ($\omega = -250\pi \times 10^3$). Inserting this in the condition above we get the minimum sample rate that guarantees that the magnitude of the filtered and sampled noise signal for $|\omega| < 250\pi \times 10^3$ is at least 20 times lower than the magnitude of the filtered and sampled desired signal at $\omega = 0$.

$$\omega_s \geq \sqrt{3} \cdot \omega_0 - \omega$$
$$=$$
$$\omega_s \geq \sqrt{3} \cdot 300\pi \times 10^3 + 250\pi \times 10^3 = 769\pi \times 10^3 \text{ rad/s}$$

Considering the criteria in both (a) and (b) we can conclude that the minimum sampling rate that full fills both cases is $\omega_s = 769\pi \times 10^3$ rad/s

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In this task we are going to construct a continuous signal from a discrete signal using the zero order hold method. The discrete signal in question is:

$$x_d(n) = \sin(2\pi n f_0/f_s)$$
$$\text{where } f_0 = 5 \text{ kHz and } f_s = 30 \text{ KHz}$$
$$\text{resulting in } x_d(n) = \sin(n\pi/3)$$

The reconstruction of the signal will not be ideal, and as such will contain sinusoidal components with higher frequencies than the fundamental component. The task is to determine the frequencies and the magnitudes of the first three of these sinusoidal components.

To solve these tasks we need to, as stated above, construct an continuous time signal from our discrete time signal. From the lecture notes we get the formula:

$$X(\omega) = H_{ZOH} X_d(\omega)$$

The two components on the right hand side are as follows:

$$H_{ZOH} = \Delta t e^{-j\Omega/2} \cdot \frac{\sin(\Omega/2)}{\Omega/2}, \text{ where } \Omega = \frac{2\pi f}{f_s} \text{ and}$$
$$X_d = \frac{\pi j}{\Delta t} \sum_{n=-\infty}^{\infty} (\delta(\Omega - \pi/3 - 2\pi n) + \delta(\Omega + \pi/3 - 2\pi n))$$

We will find our sought after sinusoidal harmonics at the ω where the delta functions equals 1. Analysing for $n = 0, 1, 2$ we will find the frequencies for the fundamental component and the three first harmonics. The frequencies where calculated to be:

$$\Omega_{\text{fundamental component}} = \pi/3$$
$$\Omega_{\text{first harmonic}} = 5\pi/3$$
$$\Omega_{\text{second harmonic}} = 7\pi/3$$
$$\Omega_{\text{third harmonic}} = 11\pi/3$$

Transferring back from our relative frequency Ω we get $f = \frac{\Omega}{2\pi} f_s$. This grants us our results:

$$f_{\text{fundamental component}} = 5 \text{ kHz}$$
$$f_{\text{first harmonic}} = 25 \text{ kHz}$$
$$f_{\text{second harmonic}} = 35 \text{ kHz}$$
$$f_{\text{third harmonic}} = 55 \text{ kHz}$$

The amplitudes with respect to the different frequencies can be calculated in Matlab using the following formula:

$$|H_{ZOH}| \cdot |X_d|$$
$$=$$
$$|\Delta t e^{-j\Omega/2} \cdot \frac{\sin(\Omega/2)}{\Omega/2}| \cdot \frac{\pi}{\Delta t}$$

These calculations gives us the result:

$$A_{\text{fundamental component}} = 3$$
$$A_{\text{first harmonic}} = 0.6$$
$$A_{\text{second harmonic}} = 0.429$$
$$A_{\text{third harmonic}} = 0.273$$