SSY230 - HIP2
Prediction Error Method

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Exercise 16)

Given N observations of $y(t)$ and $\phi(t)$, the model is described by:

$$y(t) = \theta\phi(t) + e(t)$$

where $e(t)$ is iid according to a Rayleigh distribution:

$$f(e) = \frac{e}{\sigma^2} e^{-\frac{e^2}{2\sigma^2}}, \quad e \geq 0.$$

Objective

Formulate the Maximum-Likelihood estimates of σ^2 and θ.

Likelihood Function

The likelihood function for the given model, assuming that $e(t) = y(t) - \theta\phi(t)$ and substituting into the Rayleigh distribution, is:

$$L(\theta, \sigma^2) = \prod_{t=1}^{N} \frac{y(t) - \theta\phi(t)}{\sigma^2} \exp\left(-\frac{(y(t) - \theta\phi(t))^2}{2\sigma^2}\right)$$

Log-Likelihood Function

Applying the logarithmic identity $\log(ab) = \log(a) + \log(b)$, the log-likelihood function is derived as follows:

$$\log L(\theta, \sigma^2) = \sum_{t=1}^{N} \left[\log\left(\frac{y(t) - \theta\phi(t)}{\sigma^2}\right) - \frac{(y(t) - \theta\phi(t))^2}{2\sigma^2}\right]$$

Using $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$, it simplifies to:

$$\log L(\theta, \sigma^2) = \sum_{t=1}^{N} \left[\log(y(t) - \theta\phi(t)) - \log(\sigma^2) - \frac{(y(t) - \theta\phi(t))^2}{2\sigma^2} \right]$$

Maximization of the Log-Likelihood

To find the estimates of θ and σ^2, we take the derivatives of $\log L(\theta, \sigma^2)$ with respect to θ and σ^2, set them to zero, and solve:

Derivative with Respect to θ

Applying the quotient rule for derivatives:

$$\frac{\partial}{\partial\theta} \log L(\theta, \sigma^2) = \sum_{t=1}^{N} \left[\frac{-\phi(t)}{y(t) - \theta\phi(t)} + \frac{\phi(t)(y(t) - \theta\phi(t))}{\sigma^2} \right] = 0$$

Derivative with Respect to σ^2

Using the chain rule and the identity for the derivative of a logarithm:

$$\frac{\partial}{\partial\sigma^2} \log L(\theta, \sigma^2) = \sum_{t=1}^{N} \left[-\frac{1}{\sigma^2} + \frac{(y(t) - \theta\phi(t))^2}{2(\sigma^2)^2} \right] = 0$$

Newton-Raphson method can be used, for example, to solve the numerical equations.