HIP1 - SSY230

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Problem 10)

Solution:

Given a linear model with observations:

$$y(t) = \phi^T(t)\theta + \epsilon(t), \tag{1}$$

where $\epsilon(t)$ are independently and identically distributed as $\mathcal{N}(0, \sigma^2)$ for $t = 1, \ldots, N$. The least-squares estimate $\hat{\theta}$ is given by:

$$\hat{\theta} = \left(\sum_{t=1}^{N} \phi(t)\phi^T(t) \right)^{-1} \left(\sum_{t=1}^{N} \phi(t)y(t) \right), \tag{2}$$

where $\phi(t)$ are the regressors and $\epsilon(t)$ are the error terms.

Assumptions according to lemmas B1 and B3:

B.1 states that for a discrete-time stationary process with finite variance, the sample mean converges almost surely and in mean square to the expected value as $N \to \infty$. This implies that the sample moments of such processes will converge to their true moments as the sample size increases. B.3 states that if $z(t) = \phi^T(t)\epsilon(t)$ forms a zero-mean stationary process and if the limit

$$P = \lim_{N\to\infty} E[z_N z_N^T] \tag{3}$$

exists and is nonsingular, then

$$X_N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} z(t) \tag{4}$$

is asymptotically Gaussian distributed.

Asymptotic Normality of $\hat{\theta}$

To account for the potential stochastic nature of $\phi(t)$, we must consider $\phi(t)$ as a random process. The random nature of $\phi(t)$ contributes to the variability of $\hat{\theta}$. Under the condition that $\phi(t)$ is stationary with finite variance and that $\epsilon(t)$ are i.i.d. normal, the Central Limit Theorem ensures that the normalized sum

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} \phi(t)\epsilon(t) \tag{5}$$

converges in distribution to a normal distribution as N grows large. By Slutsky's theorem, the product of a matrix that converges in probability, such as

$$\left(\sum_{t=1}^{N} \phi(t)\phi^T(t) \right)^{-1} \tag{6}$$

and a vector that converges in distribution, such as

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} \phi(t)\epsilon(t)$$

will also converge in distribution. Therefore, $\hat{\theta}$ is asymptotically normally distributed. The covariance matrix of the asymptotic distribution of $\hat{\theta}$, denoted as $\Sigma_{\hat{\theta}}$, can then be expressed by:

$$\Sigma_{\hat{\theta}} = \sigma^2 \left(\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \phi(t)\phi^T(t) \right)^{-1} \tag{7}$$

assuming the limit exists and is nonsingular (invertible) as indicated by Lemma B.3. The least-squares estimator $\hat{\theta}$ is therefore asymptotically normally distributed with mean θ and covariance matrix $\Sigma_{\hat{\theta}}$ as N becomes large, ensuring the conditions of Lemmas B.1 and B.3 are satisfied.