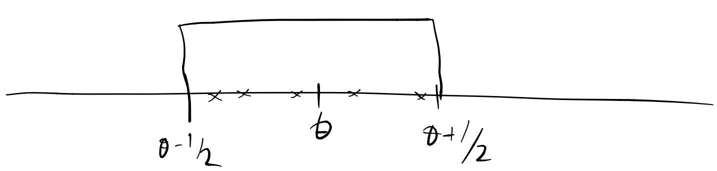
2a)



$Var(\bar{X}) = \frac{\sigma^2}{n}$ $\quad \sigma^2 = Var(X_i) \quad X_i \sim U(\theta - 1/2, \theta + 1/2)$

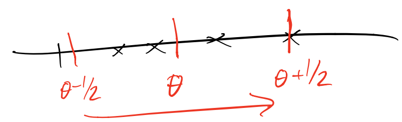
$X \sim U(0,1) \quad Var(X) = \frac{1}{12}$

So $Var(\bar{x}) = \frac{1}{12n}$

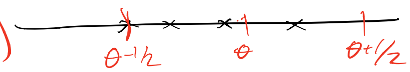
2b) $P(\theta | x_1, \ldots, x_n) \propto P(x_1 \ldots x_n | \theta) P(\theta)$

$$= \prod_{i=1}^{n} P(x_i | \theta) P(\theta)$$

$$= \prod_{i=1}^{n} I\left(\theta - \tfrac{1}{2} \le x_i \le \theta + \tfrac{1}{2}\right) \cdot 1$$



$\theta + 1/2 \ge x_{max}$
$\theta - 1/2 \le x_{min}$ $\Rightarrow \theta \in [x_{max} - 1/2, x_{min} + 1/2)$



$$P(\theta | x_1 \ldots x_n) \propto 1 \quad \text{for} \quad \theta \in [x_{max} - 1/2, x_{min} + 1/2]$$
$$= 0 \text{ otherwise.}$$

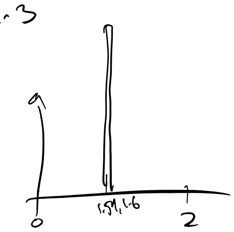
$$\theta | x_1 \ldots x_n \sim U(x_{max} - 1/2, x_{min} + 1/2)$$

2c) Frequentist : $\hat{\theta} = \bar{X} = 1.53$

$$Var(\hat{\theta}) = \frac{1}{12n} = \frac{1}{12 \cdot 3} = 0.027777$$

$$SD(\hat{\theta}) = 0.1666$$

Bayesian : $\theta | x_1 x_2 x_3 \sim U(1.59, 1.6)$



3 a) $\theta | \cancel{x}_1 \ldots x_n \sim Beta(\alpha+s, \beta+f)$

$$p(\theta|x) \propto \theta^{\alpha+s-1}(1-\theta)^{\beta+f-1} \Rightarrow \ln p(\theta|x) \propto (\alpha+s-1)\ln\theta + (\beta+f-1)\ln(1-\theta)$$

$$\frac{\partial \ln p(\theta|x)}{\partial\theta} = \frac{\alpha+s-1}{\theta} + \frac{\beta+f-1}{1-\theta}(-1)$$

$$\frac{\partial \ln p(\theta|x)}{\partial\theta} = 0 \Rightarrow \frac{\alpha+s-1}{\theta} = \frac{\beta+f-1}{1-\theta}$$

$$\Rightarrow \hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2}$$

3b) $\theta | x_1 \ldots x_n \overset{approx}{\sim} N(\hat{\theta}, -J_{\hat{\theta}|x}^{-1})$ $\quad 1-\hat{\theta} = \frac{\beta+f-1}{\alpha+\beta+n-2}$

$$\frac{\partial^2 \ln p(x|\theta)}{\partial\theta^2} = -\frac{\alpha+s-1}{\theta^2} + \frac{\beta+f-1}{(1-\theta)^2}(-1)$$

$$\left.\frac{\partial^2 \ln p(x|\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = -\left(\frac{\alpha+s-1}{\left(\frac{\alpha+s-1}{\alpha+\beta+n-2}\right)^2} + \frac{\beta-f-1}{\left(\frac{\beta+f-1}{\alpha+\beta+n-2}\right)^2}\right)$$

$$= -(\alpha+\beta+n-2)^2\left(\frac{1}{\alpha+s-1} + \frac{1}{\beta+f-1}\right)$$

$$= -(\alpha+\beta+n-2)^2\left(\frac{\alpha+\beta+n-2}{(\alpha+s-1)(\beta+f-1)}\right)$$

$$= -\frac{(\alpha+\beta+n-2)^3}{(\alpha+s-1)(\beta+f-1)}$$

$$\theta \mid x_1 \dots x_n \overset{\text{approx}}{\sim} N\left(\hat{\theta} = \frac{\alpha + s - 1}{\alpha + \beta + n - 2} \,,\, -J^{-1}_{\hat{\theta}, x} = \frac{(\alpha + s - 1)(\beta + f - 1)}{(\alpha + \beta + n - 2)^3} \right)$$

$\theta \sim \text{Beta}(\alpha + s, \beta + f)$

Check: $$\text{Var}(\theta) = \frac{(\alpha + s)(\beta + f)}{(\alpha + s + \beta + f)^2 (\alpha + s + \beta + f + 1)}$$

$$= \frac{(\alpha + s)(\beta + f)}{(\alpha + \beta + n)^2 (\alpha + \beta + n + 1)}$$

3 c) Se R-code

3 d) — " —