# Bayesian Statistics - Lecture 4

## Lecture 4: Predictions. Decisions.

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# Lecture overview

- **Prediction**
  - Normal model
  - More complex examples
- **Decision theory**
  - The elements of a decision problem
  - The Bayesian way
  - Point estimation as a decision problem

- Posterior predictive density for future $\tilde{y}$ given observed $\mathbf{y}$

$$p(\tilde{y}|\mathbf{y}) = \int_\theta p(\tilde{y}|\theta, \mathbf{y})p(\theta|\mathbf{y})d\theta$$

- If $p(\tilde{y}|\theta, \mathbf{y}) = p(\tilde{y}|\theta)$ [not true for time series], then

$$p(\tilde{y}|\mathbf{y}) = \int_\theta p(\tilde{y}|\theta)p(\theta|\mathbf{y})d\theta$$

- Parameter uncertainty in $p(\tilde{y}|\mathbf{y})$ by **averaging over** $p(\theta|\mathbf{y})$.

# Prediction - Normal data, known variance

- Under the uniform prior $p(\theta) \propto c$, then

$$
\begin{aligned}
p(\tilde{y}|\mathbf{y}) &= \int_\theta p(\tilde{y}|\theta)p(\theta|\mathbf{y})d\theta \\
\theta|\mathbf{y} &\sim N(\bar{y}, \sigma^2/n) \\
\tilde{y}|\theta &\sim N(\theta, \sigma^2)
\end{aligned}
$$

- Under the uniform prior $p(\theta) \propto c$, then

$$p(\tilde{y}|\mathbf{y}) = \int_\theta p(\tilde{y}|\theta)p(\theta|\mathbf{y})d\theta$$
$$\theta|\mathbf{y} \sim N(\bar{y}, \sigma^2/n)$$
$$\tilde{y}|\theta \sim N(\theta, \sigma^2)$$

Simulation algorithm:

1. Generate a **posterior draw** of $\theta$ ($\theta^{(1)}$) from $N(\bar{y}, \sigma^2/n)$
2. Generate a **predictive draw** of $\tilde{y}$ ($\tilde{y}^{(1)}$) from $N(\theta^{(1)}, \sigma^2)$
3. Repeat Steps 1 and 2 $N$ times to output:
    - Sequence of posterior draws: $\theta^{(1)}, ...., \theta^{(N)}$
    - Sequence of predictive draws: $\tilde{y}^{(1)}, ..., \tilde{y}^{(N)}$.

- $\theta^{(1)} = \bar{y} + \varepsilon^{(1)}$, where $\varepsilon^{(1)} \sim N(0, \sigma^2/n)$. (Step 1).
- $\tilde{y}^{(1)} = \theta^{(1)} + \upsilon^{(1)}$, where $\upsilon^{(1)} \sim N(0, \sigma^2)$. (Step 2).
- $\tilde{y}^{(1)} = \bar{y} + \varepsilon^{(1)} + \upsilon^{(1)}$.
- $\varepsilon^{(1)}$ and $\upsilon^{(1)}$ are independent.
- The sum of two normal random variables is normal so

$$\begin{aligned} E(\tilde{y}|\mathbf{y}) &= \bar{y} \\ V(\tilde{y}|\mathbf{y}) &= \frac{\sigma^2}{n} + \sigma^2 = \sigma^2\left(1 + \frac{1}{n}\right) \end{aligned}$$

$$\tilde{y}|\mathbf{y} \sim N\left[\bar{y}, \sigma^2\left(1 + \frac{1}{n}\right)\right]$$

- Easy to see that the predictive distribution is normal.
- The mean

$$E_{\tilde{y}|\theta}(\tilde{y}) = \theta$$

and then remove the conditioning on $\theta$ by averaging over $\theta$

$$E(\tilde{y}|\mathbf{y}) = E_{\theta|\mathbf{y}}(\theta) = \mu_n \text{ (Posterior mean of } \theta).$$

- The predictive variance of $\tilde{y}$ (total variance formula):

$$\begin{aligned} V(\tilde{y}|\mathbf{y}) &= E_{\theta|\mathbf{y}}[V_{\tilde{y}|\theta}(\tilde{y})] + V_{\theta|\mathbf{y}}[E_{\tilde{y}|\theta}(\tilde{y})] \\ &= E_{\theta|\mathbf{y}}(\sigma^2) + V_{\theta|\mathbf{y}}(\theta) \\ &= \sigma^2 + \tau_n^2 \\ &= \text{(Population variance + Posterior variance of } \theta). \end{aligned}$$

- In summary:

$$\tilde{y}|\mathbf{y} \sim N(\mu_n, \sigma^2 + \tau_n^2).$$

- **Autoregressive process**

$$y_t = \mu + \phi_1(y_{t-1} - \mu) + \ldots + \phi_p(y_{t-p} - \mu) + \varepsilon_t, \ \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

**Simulation algorithm**. Repeat $N$ times:

1. Generate a **posterior draw** of $\theta^{(1)} = (\phi_1^{(1)}, \ldots, \phi_p^{(1)}, \mu^{(1)}, \sigma^{(1)})$ from $p(\phi_1, \ldots, \phi_p, \mu, \sigma | \mathbf{y}_{1:T})$.
2. Generate a **predictive draw** of future time series by:
   2.1 $\tilde{y}_{T+1} \sim p(y_{T+1} | y_T, y_{T-1}, \ldots, y_{T-p}, \theta^{(1)})$
   2.2 $\tilde{y}_{T+2} \sim p(y_{T+2} | \tilde{y}_{T+1}, y_T, \ldots, y_{T-p}, \theta^{(1)})$
   2.3 $\tilde{y}_{T+3} \sim p(y_{T+3} | \tilde{y}_{T+2}, \tilde{y}_{T+1}, y_T, \ldots, y_{T-p}, \theta^{(1)})$
   2.4 ...

- Problem: Predicting the auctioned price in eBay coin auctions.

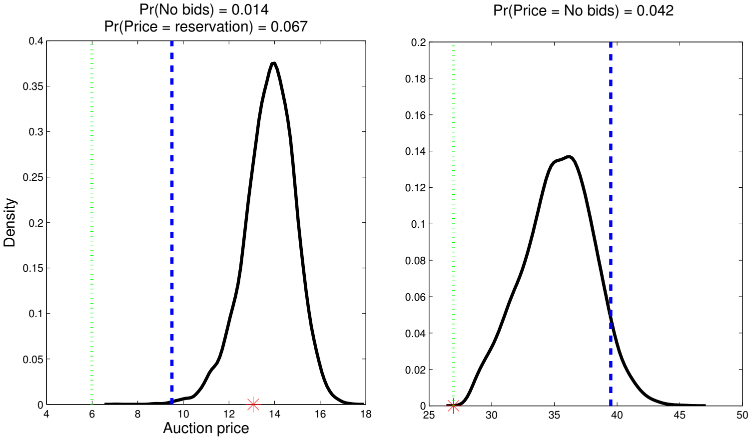
- Data: Bid from 1000 auctions on eBay.
    - The highest bid is not observed.
    - The lowest bids are also not observed because of the seller's reservation price.

- Covariates: auction-specific, e.g. Book value from catalog, seller's reservation price, quality of sold object, rating of seller, powerseller, verified seller ID etc

- Buyers are strategic. Their bids does not fully reflect their valuation. Game theory. **Very** complicated likelihood.

- Simulate from **posterior predictive distibution** of the **price** in a new auction:

1. Simulate a draw $\theta^{(i)}$ from the posterior $p(\theta|\text{historical bids})$
2. Simulate the number of bidders conditional on $\theta^{(i)}$ (Poisson)
3. Simulate the bidders' valuations, $\mathbf{v}^{(i)}$
4. Simulate all bids, $\mathbf{b}^{(i)}$, conditional on the valuations
5. For $\mathbf{b}^{(i)}$, return the next to largest bid (proxy bidding).


Pr(No bids) = 0.014
Pr(Price = reservation) = 0.067
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
4
6
8
10
12
14
16
18
Density
Auction price
Pr(Price = No bids) = 0.042
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
25
30
35
40
45
50

- Let $\theta$ be an **unknown quantity**. **State of nature**. Examples: Future inflation, Global temperature, Disease.
- Let $a \in \mathcal{A}$ be an **action**. Ex: Interest rate, Energy tax, Surgery.
- Choosing action $a$ when state of nature is $\theta$ gives **utility**

$$U(a, \theta)$$

- Alternatively loss $L(a, \theta) = -U(a, \theta)$.
- Loss table:

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $a_1$ | $L(a_1, \theta_1)$ | $L(a_1, \theta_2)$ |
| $a_2$ | $L(a_2, \theta_1)$ | $L(a_2, \theta_2)$ |

- Example:

| | Rainy | Sunny |
|---|---|---|
| Umbrella | 20 | 10 |
| No umbrella | 50 | 0 |

- Example **loss functions** when both $a$ and $\theta$ are continuous:
  - Linear: $L(a,\theta) = |a-\theta|$
  - Quadratic: $L(a,\theta) = (a-\theta)^2$
  - Lin-Lin:

$$L(a,\theta) = \begin{cases} c_1 \cdot |a-\theta| & \text{if } a \leq \theta \\ c_2 \cdot |a-\theta| & \text{if } a > \theta \end{cases}$$

- Example:
  - $\theta$ is the number of items demanded of a product
  - $a$ is the number of items in stock
  - Utility

$$U(a,\theta) = \begin{cases} p \cdot \theta - c_1(a-\theta) & \text{if } a > \theta \text{ [too much stock]} \\ p \cdot a - c_2(\theta - a)^2 & \text{if } a \leq \theta \text{ [too little stock]} \end{cases}$$

# Optimal decision

- Ad hoc decision rules:
  - *Minimax*. Minimizes the maximum loss.
  - *Minimax-regret* ... bla bla bla ...
- **Bayesian theory**: maximize the **posterior expected utility**:

$$a_{bayes} = \mathrm{argmax}_{a \in \mathcal{A}} \, E_{p(\theta|y)}[U(a,\theta)],$$

  where $E_{p(\theta|y)}$ denotes the posterior expectation.
- Using simulated draws $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(N)}$ from $p(\theta|y)$ :

$$E_{p(\theta|y)}[U(a,\theta)] \approx N^{-1} \sum_{i=1}^{N} U(a, \theta^{(i)})$$

- **Separation principle**:

1. First obtain $p(\theta|y)$
2. then form $U(a,\theta)$ and finally
3. choose $a$ that maximes $E_{p(\theta|y)}[U(a,\theta)]$.

- Choosing a **point estimator** is a decision problem.
- Which to choose: posterior median, mean or mode?
- It depends on your loss function:
  - **Linear loss** $\rightarrow$ Posterior median
  - **Quadratic loss** $\rightarrow$ Posterior mean
  - **Zero-one loss** $\rightarrow$ Posterior mode
  - **Lin-Lin loss** $\rightarrow$ $c_2/(c_1 + c_2)$ quantile of the posterior