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HOME EXAM NO. 1 IN BAYESIAN STATISTICS I

This is the first of two exams on the master course Bayesian Statistics I. You should be able to solve all problems without programming, but you may find a program like Excel or OpenOffice Spreadsheet handy, and programming languages are allowed.

You are supposed to **solve these problems individually without any co-operation**.

(1) Comment on the following statements:
  (a) 'To a Bayesian, even the speed of light can be a random variable. This is clearly wrong since the speed of light is a natural constant'.
  (b) 'Bayesianism has no place in science. Everything about it is subjective, everything is allowed.'
  (c) 'Noninformative priors do not exist'
  (d) 'A Bayesian analysis requires a fully specified likelihood. The Bayesian approach is therefore not applicable when some observations are missing'.

(2) Consider a random sample from the exponential distribution: $x_1, ..., x_n|\theta \overset{iid}{\sim} Exponential(\theta)$, where $E(x) = 1/\theta$.
  (a) What is the natural conjugate prior for θ?
  (b) Derive the posterior distribution of θ.
  (c) Derive Jeffreys' prior for θ.

(3) Consider a sample from the uniform model: $x_1, ..., x_n|\theta \overset{iid}{\sim} U(0, \theta)$, where $\theta > 0$.
  (a) Show that the Pareto(α, β) density is the natural conjugate prior for θ.
  (b) Derive the posterior distribution of θ.
  (c) Derive Jeffreys' prior for θ.
  (d) Derive the predictive distribution of x_{n+1} given $x_1, ..., x_n$, using the natural conjugate prior.

(4) Late train arrivals has been a nuisance in Sweden in recent years. Let θ be the probability that the morning train at 8.15 am from Nyköping arrives more than an hour late to Stockholm Central.
   (a) Assume a Beta(α, β) prior for θ. Let one of your friends play the role of an expert. Elicit you friend's values of α and β by letting him/her state his/her prior mean and standard deviation of θ.
   (b) During the last 100 days, the train has never been more than an hour late. Update the prior from 4a) to a posterior distribution for θ based on these data.
   (c) When the train is more than an hour late, the railway company (SJ) needs to refund the ticket cost to every passenger in the train. The total cost for being late on a given day is approximately $296,000$ Krona. SJ can avoid delays by servicing the train more frequently. The monthly service cost is $108,000$ Krona, and would guarantee that the morning train would never be more than an hour late in that month. According to a Bayesian analysis, should SJ pay the extra service cost?

(5) Let $y_1, ..., y_n$ be a random sample from the $N(\theta, \sigma^2)$ model, where both θ and σ^2 are unknown. Let $n = 25$, $\bar{y} = 33.75$ and $s^2 = 5^2$, where s is the sample standard deviation.
   (a) Compute the posterior distribution of θ based on some conjugate prior.
   (b) Investigate the sensitivity of the posterior to variations in the prior.
   (c) Suppose that the data has two outlying (extreme) observations. Discuss how you would deal with this?

MAY THE FORCE BE WITH YOU.
- MATTIAS