Bayesian Statistics - Lecture 2

Lecture 2: Normal. Poisson. Prior elicitation.

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- The **Normal model** with known variance
- The **Poisson model**
- **Conjugate priors**
- **Prior elicitation**

- **Model**

$$x_1, \ldots, x_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2).$$

- **Prior**

$$p(\theta) \propto c \text{ (a constant)}$$

- **Likelihood**

$$\begin{aligned} p(x_1, \ldots, x_n | \theta, \sigma^2) &= \Pi_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}(x_i - \theta)^2\right] \\ &\propto \exp\left[-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right]. \end{aligned}$$

- **Posterior**

$$\theta | x_1, \ldots, x_n \sim N(\bar{x}, \sigma^2/n)$$

- **Prior**

$$\theta \sim N(\mu_0, \tau_0^2)$$

- **Posterior**

$$\begin{aligned} p(\theta|x_1, ..., x_n) &\propto p(x_1, ..., x_n|\theta, \sigma^2)p(\theta) \\ &\propto N(\theta|\mu_n, \tau_n^2), \end{aligned}$$

where

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2},$$

$$\mu_n = w\bar{x} + (1 - w)\mu_0,$$

and

$$w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}.$$

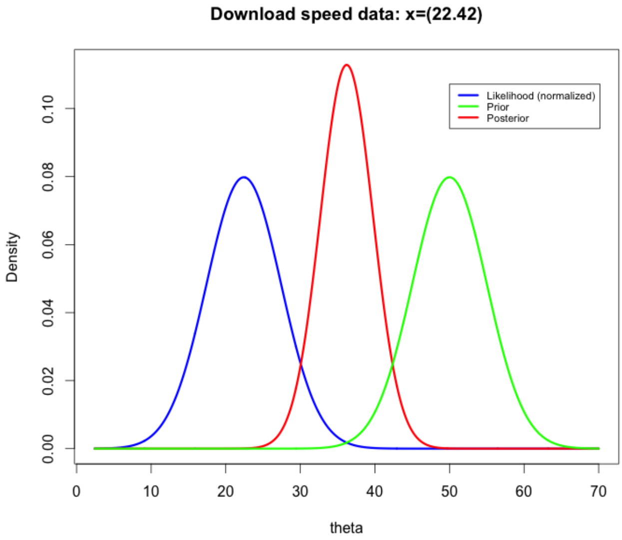
$$\theta \sim N(\mu_0, \tau_0^2) \overset{x_1,\ldots,x_n}{\Longrightarrow} \theta | x \sim N(\mu_n, \tau_n^2).$$

Posterior precision = Data precision + Prior precision

Posterior mean =

$$\frac{\text{Data precision}}{\text{Posterior precision}}(\text{Data mean}) + \frac{\text{Prior precision}}{\text{Posterior precision}}(\text{Prior mean})$$

- **Data**: $x = (22.42, 34.01, 35.04, 38.74, 25.15)$ Mbit/sec.
- **Model**: $X_1, ..., X_5 \sim N(\theta, \sigma^2)$.
- Assume $\sigma = 5$ (measurements can vary $\pm$10MBit with 95% probability)
- My **prior**: $\theta \sim N(50, 5^2)$.


Download speed data: x=(22.42)
Likelihood (normalized)
Prior
Posterior
Density
0.00
0.02
0.04
0.06
0.08
0.10
0
10
20
30
40
50
60
70
theta

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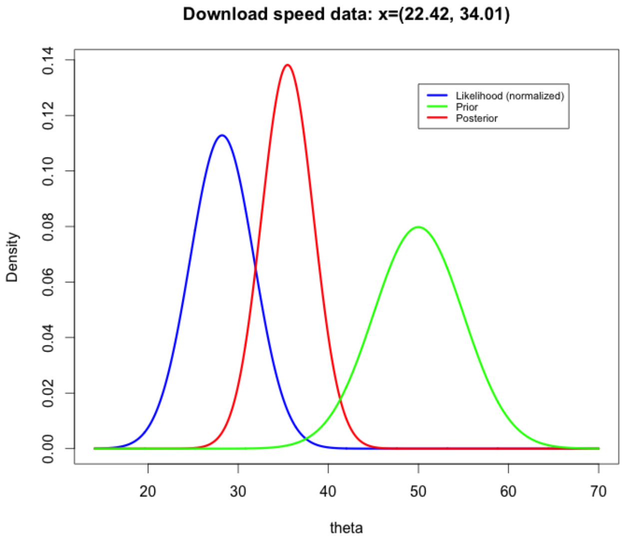

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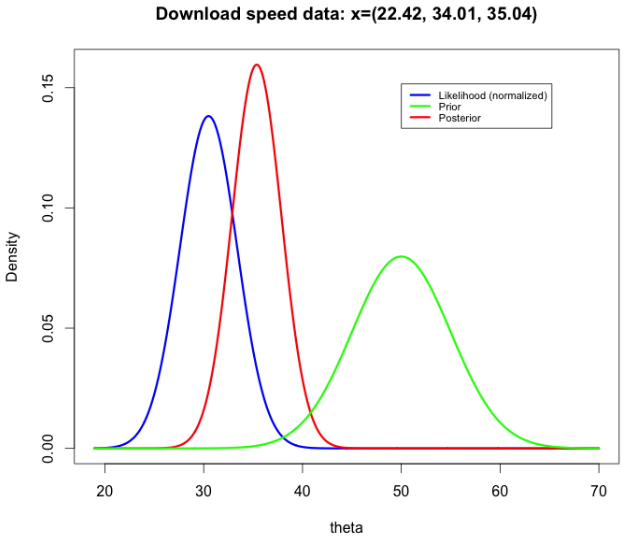

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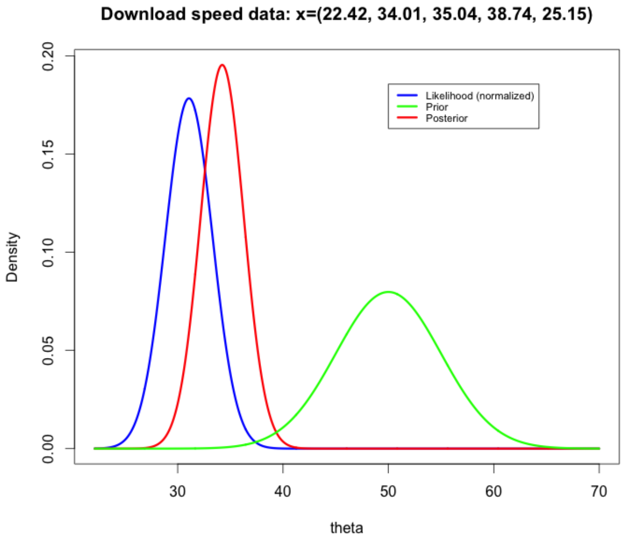
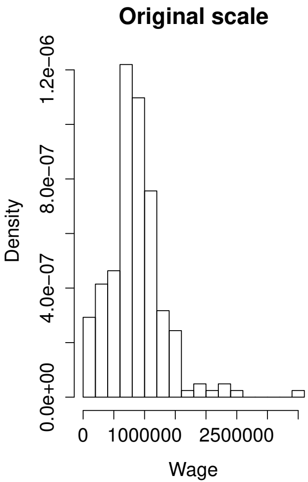


- Data on wages for 205 Canadian workers.



- **Model**

$$X_1, \ldots, X_n|\theta \sim N(\theta, \sigma^2),\ \sigma^2 = 0.4$$

- **Prior**

$$\theta \sim N(\mu_0, \tau_0^2),\ \mu_0 = 12 \text{ and } \tau_0 = 10$$

- **Posterior**

$$\theta|x_1, \ldots, x_n \sim N\left(\mu_n, \tau_n^2\right),$$

where $\mu_n = w\bar{x} + (1 - w)\mu_0$.

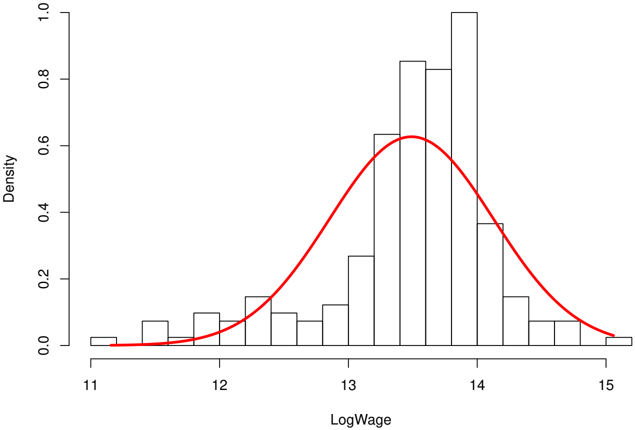
- For the Canadian wage data:

$$w = \frac{\sigma^{-2}n}{\sigma^{-2}n + \tau_0^{-2}} = \frac{2.5 \cdot 205}{2.5 \cdot 205 + 1/100} = 0.999.$$

$$\mu_n = w\bar{x} + (1 - w)\mu_0 = 0.999 \cdot 13.489 + (1 - 0.999) \cdot 12 \approx 13.489$$

$$\tau_n^2 = (2.5 \cdot 205 + 1/100)^{-1} = 0.00195$$

Canadian wages data - model fit



- **Model**

$$y_1, \ldots, y_n | \theta \overset{iid}{\sim} Pois(\theta)$$

- **Poisson distribution**

$$p(y) = \frac{\theta^y e^{-\theta}}{y!}$$

- **Likelihood** from iid Poisson sample $y = (y_1, \ldots, y_n)$

$$p(y|\theta) = \left[\prod_{i=1}^{n} p(y_i|\theta)\right] \propto \theta^{(\sum_{i=1}^{n} y_i)} \exp(-\theta n),$$

- ***Prior***

$$p(\theta) \propto \theta^{\alpha-1} \exp(-\theta\beta) \propto Gamma(\alpha, \beta)$$

which contains the info: $\alpha - 1$ counts in β observations.

- **Posterior**

$$
\begin{aligned}
p(\theta|y_1,...,y_n) &\propto \left[\prod_{i=1}^{n} p(y_i|\theta)\right] p(\theta) \\
&\propto \theta^{\sum_{i=1}^{n} y_i} \exp(-\theta n)\theta^{\alpha-1}\exp(-\theta\beta) \\
&= \theta^{\alpha+\sum_{i=1}^{n} y_i - 1}\exp[-\theta(\beta+n)],
\end{aligned}
$$

which is proportional to the $Gamma(\alpha + \sum_{i=1}^{n} y_i, \beta + n)$ distribution.

- **Prior-to-Posterior mapping**

$$\text{Model: } y_1,...,y_n|\theta \overset{iid}{\sim} Pois(\theta)$$

$$\text{Prior: } \theta \sim Gamma(\alpha, \beta)$$

$$\text{Posterior: } \theta|y_1,...,y_n \sim Gamma(\alpha + \sum_{i=1}^{n} y_i, \beta + n).$$

$n = 576$, $\sum_{i=1}^{n} y_i = 229 \cdot 0 + 211 \cdot 1 + 93 * 2 + 35 * 3 + 7 * 4 + 1 \cdot 5 = 537$.

Average number of hits per region=$\bar{y} = 537/576 \approx 0.9323$.

$$p(\theta|y) \propto \theta^{\alpha+537-1} \exp[-\theta(\beta + 576)]$$

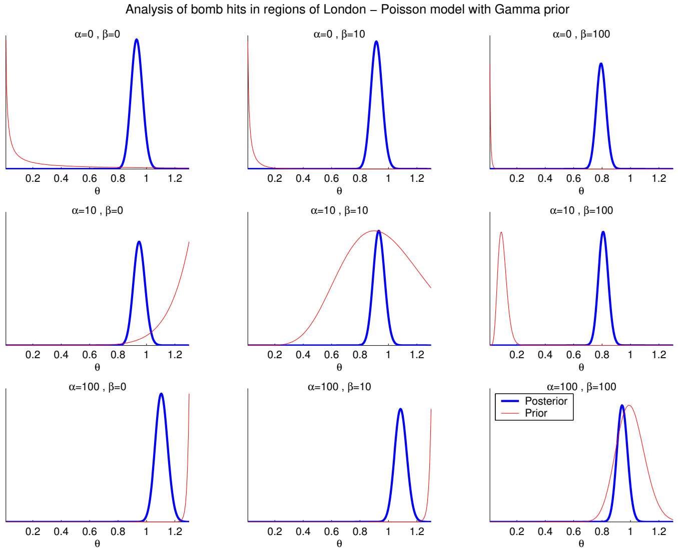
$$E(\theta|y) = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} \approx \bar{y} \approx 0.9323,$$

and

$$SD(\theta|y) = \left(\frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta + n)^2}\right)^{1/2} = \frac{(\alpha + \sum_{i=1}^{n} y_i)^{1/2}}{(\beta + n)} \approx \frac{(537)^{1/2}}{576} \approx 0.0402.$$

if α and β are small compared to $\sum_{i=1}^{n} y_i$ and n.

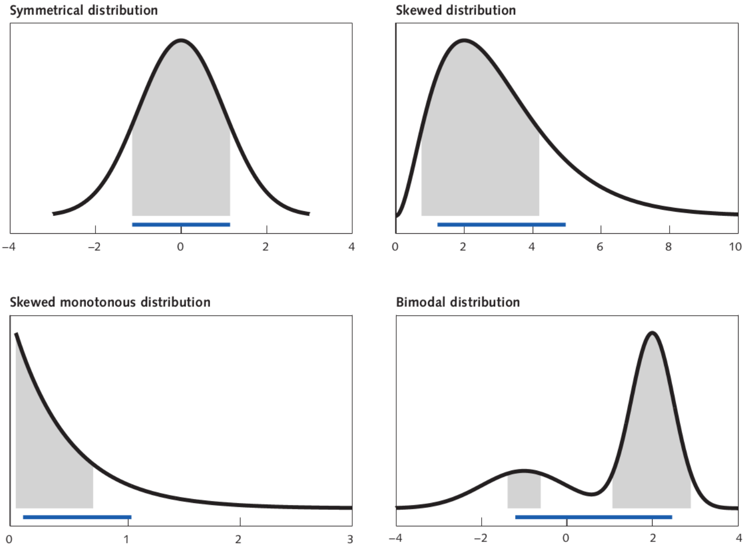
Poisson bomb hits in London



- **Bayesian 95% credible interval**: the probability that the unknown parameter θ lies in the interval is 0.95.
- Approximate 95% **credible interval** for θ (for small α and β):

$$E(\theta|y) \pm 1.96 \cdot SD(\theta|y) = [0.8535; 1.0111]$$

- An exact 95% **equal-tail interval** is $[0.8550; 1.0125]$ (assuming $\alpha = \beta = 0$)
- **Highest Posterior Density** (**HPD**) interval contains the θ values with highest pdf.
- An exact Highest Posterior Density (HPD) interval is $[0.8525; 1.0144]$. Obtained numerically, assuming $\alpha = \beta = 0$.


Symmetrical distribution
−4
−2
0
2
4
Skewed distribution
0
2
4
6
8
10
Skewed monotonous distribution
0
1
2
3
Bimodal distribution
−4
−2
0
2
4

- Normal likelihood: Normal prior$\rightarrow$Normal posterior.
- Bernoulli likelihood: Beta prior$\rightarrow$Beta posterior.
- Poisson likelihood: Gamma prior$\rightarrow$Gamma posterior.

- **Conjugate priors**: A prior is conjugate to a model if the prior and posterior belong to the same distributional family.

- Formal definition: Let $\mathcal{F} = \{p(y|\theta), \theta \in \Theta\}$ be a class of sampling distributions. A family of distributions $\mathcal{P}$ is **conjugate** for $\mathcal{F}$ if

$$p(\theta) \in \mathcal{P} \Rightarrow p(\theta|x) \in \mathcal{P}$$

holds for all $p(y|\theta) \in \mathcal{F}$.

- The prior should be determined (**elicited**) by an **expert**. Typically, expert$\neq$statistician.
- Elicit the prior on **a quantity that the expert knows well**. Convert afterwards.
- **Ask probabilistic questions** to the expert:
  - $E(\theta) = ?$
  - $SD(\theta) = ?$
  - $Pr(\theta < c) = ?$
  - $Pr(y > c) = ?$
- **Show some consequences** of the elicitated prior to the expert.
- Beware of **psychological effects**, such as anchoring.

- **Autoregressive process** or order p

$$y_t = \phi_1(y_{t-1} - \mu) + \dots + \phi_p(y_{t-p} - \mu) + \varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

- Informative prior on the unconditional mean: $\mu \sim N(\mu_0, \tau_0^2)$.
- "Noninformative" prior on σ^2: $p(\sigma^2) \propto 1/\sigma^2$
- Assume $\phi_i \sim N(\mu_i, \psi_i), i = 1, \dots, p$ are independent a priori.
- Prior on $\phi = (\phi_1, \dots, \phi_p)$ centered on persistent AR(1) process: $\mu_1 = 0.8, \mu_2 = \dots = \mu_p = 0$
- $Var(\phi_i) = \frac{c}{i^\lambda}$. "Longer" lags are more likely to be zero a priori.

Different types of prior information

- Real **expert information**. Combo of previous studies and experience.
- **Vague prior** information.
- **Reporting priors**. Easy to understand the information they contain.
- **Smoothness priors**. Regularization. Shrinkage. Big thing in modern statistics/machine learning.