Bayesian Statistics I

Lecture 1 - The Bayesics and Bernoulli data

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Course overview

- Course webpage. Course syllabus.
- Modes of teaching:
  - Lectures (Mattias Villani)
  - Mathematical exercises (Oscar Oelrich)
  - Computer labs (Oscar Oelrich)
- Modules:
  - The **Bayesics**, single- and multiparameter models
  - **Regression** and **Classification models**
  - **Advanced models** and **Posterior Approximation** methods
  - **Model Inference and evaluation** and **Variable Selection**
- Examination
  - Lab reports
  - Home exam

Lecture overview

- The likelihood function
- Bayesian inference
- Bernoulli model

Likelihood function - Bernoulli trials

- **Bernoulli trials**:

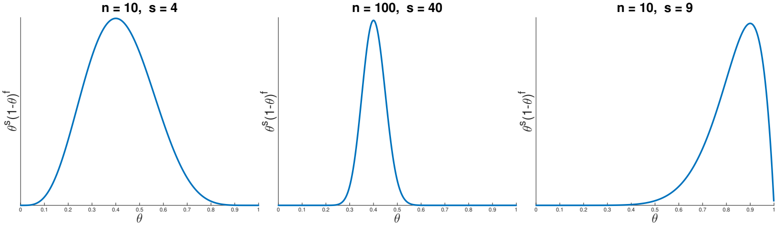
$$X_1, ..., X_n|\theta \overset{iid}{\sim} Bern(\theta).$$

- **Likelihood** from $s = \sum_{i=1}^{n} x_i$ successes and $f = n - s$ failures.

$$p(x_1, ..., x_n|\theta) \;\; = \;\; p(x_1|\theta) \cdots p(x_n|\theta) = \theta^s (1-\theta)^f$$

- **Maximum likelihood estimator** $\hat{\theta}$ maximizes $p(x_1, ..., x_n|\theta)$.

- Given the data $x_1, ..., x_n$, plot $p(x_1, ..., x_n|\theta)$ as a function of θ.



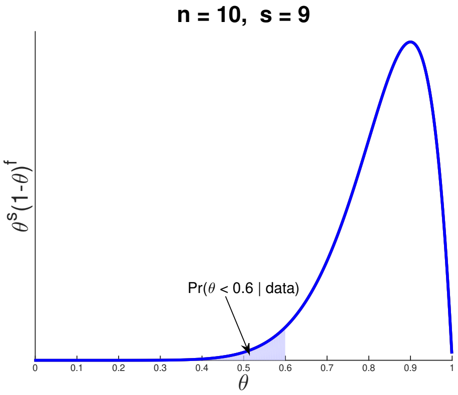
The likelihood function

- Say it out loud:

  The likelihood function is
  the probability of the observed data
  considered as a function of the parameter.

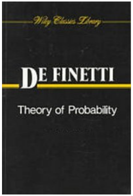
- The symbol $p(x_1, ..., x_n|\theta)$ plays two different roles:
- **Probability distribution** for the data.
  - The data $\mathbf{x} = (x_1, ..., x_n)$ are random.
  - θ is fixed.
- **Likelihood function** for the parameter
  - The data $\mathbf{x} = (x_1, ..., x_n)$ are fixed.
  - $p(x_1, ..., x_n|\theta)$ is function of θ.

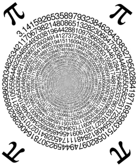
Probabilities from the likelihood?



Uncertainty and subjective probability

- $\Pr(\theta < 0.6|\text{data})$ only makes sense if θ is random.
- But θ may be a fixed natural constant?
- **Bayesian: doesn't matter if θ is fixed or random.**
- Do **You** know the value of θ or not?
- $p(\theta)$ reflects Your knowledge/**uncertainty** about θ.
- **Subjective probability**.
- The statement $\Pr(10\text{th decimal of } \pi = 9) = 0.1$ makes sense.





Bayesian learning

- Bayesian learning about a model parameter θ:
  - state your **prior** knowledge as a probability distribution $p(\theta)$.
  - collect **data** $\mathbf{x}$ and form the **likelihood** function $p(\mathbf{x}|\theta)$.
  - **combine** prior knowledge $p(\theta)$ with data information $p(\mathbf{x}|\theta)$.
- How to combine the two sources of information?

Bayes' theorem

Learning from data - Bayes' theorem

- How to update from prior $p(\theta)$ to posterior $p(\theta|Data)$?
- Bayes' theorem for events A and B

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}.$$

- Bayes' Theorem for a model parameter θ

$$p(\theta|Data) = \frac{p(Data|\theta)p(\theta)}{p(Data)}.$$

- It is the prior $p(\theta)$ that takes us from $p(Data|\theta)$ to $p(\theta|Data)$.
- A probability distribution for θ is extremely useful. Predictions. Decision making.
- No prior - no posterior - no useful inferences - no fun.

Medical diagnosis

- A = {Very rare disease}, B ={Positive medical test}.
- $p(A) = 0.0001$. $p(B|A) = 0.9$. $p(B|A^c) = 0.05$.
- Probability of being sick when test is positive:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|A^c)p(A^c)} \approx 0.0018.$$

- Probably not sick, but 18 times more probable now.

- Morale: If you want $p(A|B)$ then $p(B|A)$ does not tell the whole story. The prior probability $p(A)$ is also very important.

"You can't enjoy the Bayesian omelette without breaking the Bayesian eggs"
Leonard Jimmie Savage

The normalizing constant is not important

- Bayes theorem

$$p(\theta|Data) = \frac{p(Data|\theta)p(\theta)}{p(Data)} = \frac{p(Data|\theta)p(\theta)}{\int_\theta p(Data|\theta)p(\theta)d\theta}.$$

- Integral $p(Data) = \int_\theta p(Data|\theta)p(\theta)d\theta$ can make you cry.
- **$p(Data)$ is just a constant** so that $p(\theta|Data)$ integrates to one.
- Example: $x \sim N(\mu, \sigma^2)$

$$p(x) = (2\pi\sigma^2)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right].$$

- We may write

$$p(x)| \propto \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right].$$

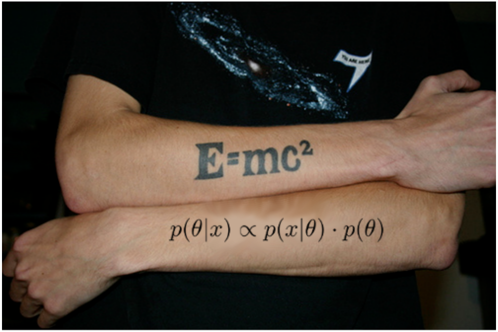
Great theorems make great tattoos

- All you need to know:

$$p(\theta|Data) \propto p(Data|\theta)p(\theta)$$

or

$$\text{Posterior} \propto \text{Likelihood} \cdot \text{Prior}$$



Bernoulli trials - Beta prior

- **Model**

$$x_1, ..., x_n|\theta \overset{iid}{\sim} \mathrm{Bern}(\theta)$$

- **Prior**

$$\theta \sim \mathrm{Beta}(\alpha, \beta)$$

$$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \text{ for } 0 \le \theta \le 1.$$

- **Posterior**

$$\begin{aligned} p(\theta|x_1, ..., x_n) &\propto p(x_1, ..., x_n|\theta)p(\theta) \\ &\propto \theta^s(1-\theta)^f\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \theta^{s+\alpha-1}(1-\theta)^{f+\beta-1}. \end{aligned}$$

- Posterior is proportional to the $\mathrm{Beta}(\alpha+s, \beta+f)$ density.
- The prior-to-posterior mapping:

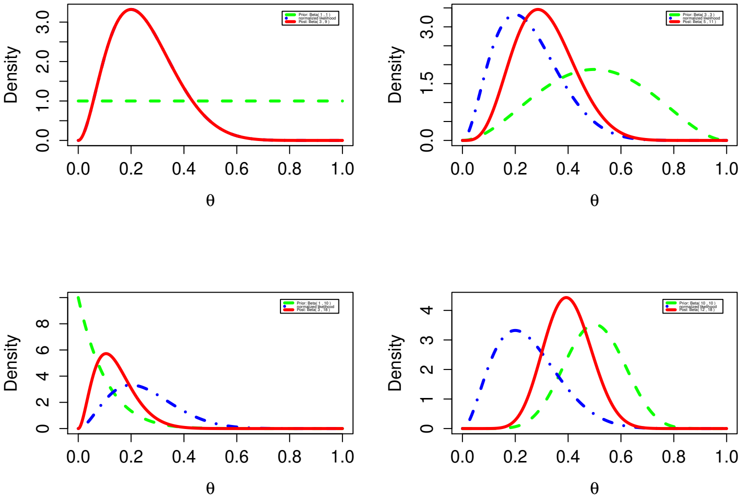
$$\theta \sim \mathrm{Beta}(\alpha, \beta) \overset{x_1,...,x_n}{\Longrightarrow} \theta|x_1, ..., x_n \sim \mathrm{Beta}(\alpha+s, \beta+f)$$

Bernoulli example: spam emails

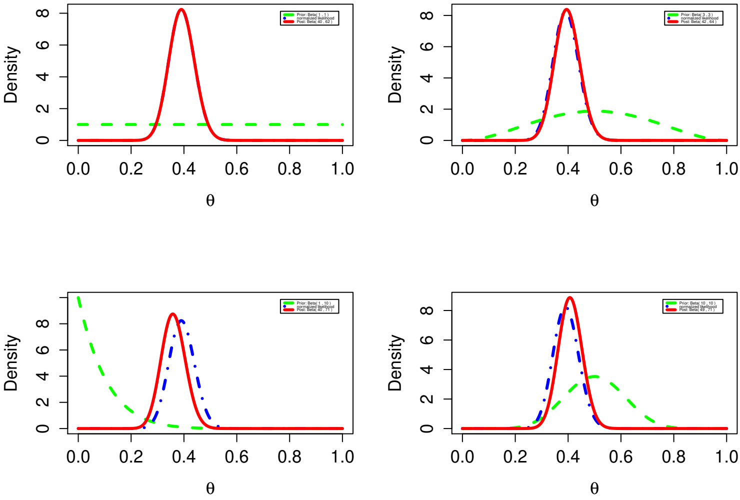
- George has gone through his collection of 4601 e-mails.
- He classified 1813 of them to be spam.
- Let $x_i = 1$ if i:th email is spam.
- **Model**: $x_i|\theta \overset{iid}{\sim} \text{Bern}(\theta)$
- **Prior**: $\theta \sim \text{Beta}(\alpha, \beta)$.
- **Posterior**

$$\theta|x \sim \text{Beta}(\alpha + 1813, \beta + 2788)$$

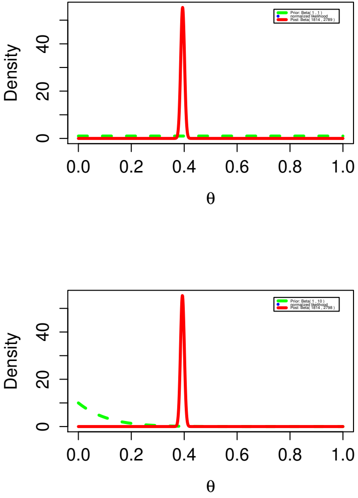
Spam data (n=10) - Prior is influential



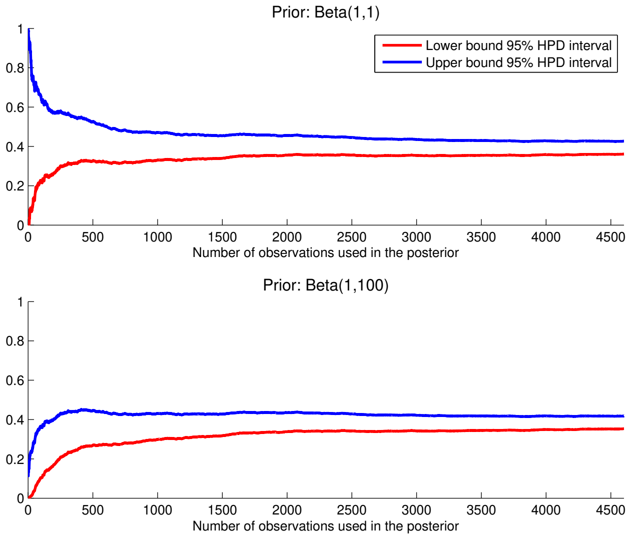
Spam data (n=100) - Prior is less influential



Spam data (n=4601) - Prior does not matter



Spam data - posterior convergence



Bayes respects the Likelihood Principle

- **Bernoulli trials with order**:
  $x_1 = 1, x_2 = 0, ..., x_4 = 1, ..., x_n = 1$

$$p(\mathbf{x}|\theta) = \theta^s (1-\theta)^f$$

- **Bernoulli trials without order**. n fixed, s random.

$$p(s|\theta) = \binom{n}{s} \theta^s (1-\theta)^f$$

- **Negative binomial sampling**: sample until you get s successes. s fixed, n random.

$$p(n|\theta) = \binom{n-1}{s-1} \theta^s (1-\theta)^f$$

- The **posterior distribution is the same** in all three cases.
- Bayesian inference respects the **likelihood principle**.