Problem 2a)

Model: $X_1, \dots, X_n \mid \theta \overset{iid}{\sim} N(\theta, \sigma^2)$ $\quad \sigma^2$ known

Prior: $P(\theta) \propto$ constant

Posterior:
$$p(\theta \mid x_1, \dots, x_n) \propto p(x_1, \dots, x_n \mid \theta) \cdot p(\theta)$$
$$= p(x_1, \dots, x_n \mid \theta) \qquad [\text{prior is constant}]$$
$$= \prod_{i=1}^{n} p(x_i \mid \theta) \qquad [\text{Independence}]$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_i - \theta)^2\right)$$
$$\propto \prod_{i=1}^{n} \exp\left(-\frac{1}{2\sigma^2}(x_i - \theta)^2\right)$$
$$= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{\theta})^2\right)$$

Now, $\sum_{i=1}^{n}(x_i - \theta)^2 = \sum_{i=1}^{n}\left((x_i - \bar{x}) - (\theta - \bar{x})\right)^2$

$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{n}(\theta - \bar{x})^2 - 2(\theta - \bar{x})\sum_{i=1}^{n}(x_i - \bar{x})$$
$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\theta - \bar{x})^2 - 2(\theta - \bar{x})\underbrace{\left(\sum_{i=1}^{n} x_i - n\bar{x}\right)}_{0}$$
$$= \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\theta - \bar{x})^2$$

So, $p(\theta | x_1, \dots, x_n) \propto \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i - x)^2 - n(\theta - \bar{x})^2\right)\right)$

$$= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) \cdot \exp\left(-\frac{1}{2\sigma^2} n(\theta - \bar{x})^2\right)$$

$$\propto \exp\left(-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right)$$

$$\propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

Problem 2b)

<u>prior:</u> $\theta \sim N(\mu_0, \tau_0^2)$

<u>posterior:</u> $p(\theta | x_1, \dots, x_n) \propto p(x_1, \dots, x_n | \theta) \cdot p(\theta)$

$$\propto \exp\left(-\frac{1}{2(\sigma^2/n)}(\theta - \bar{x})^2\right) \cdot \exp\left(-\frac{1}{2\tau_0^2}(\theta - \mu_0)^2\right)$$

$$= \exp\left(-\frac{1}{2}\left(\underbrace{\frac{1}{\sigma^2/n}(\theta - \bar{x})^2 + \frac{1}{\tau_0^2}(\theta - \mu_0)^2}\right)\right)$$

$$\frac{n}{\sigma^2}\left(\theta^2 + \bar{x}^2 - 2\theta\bar{x}\right) + \frac{1}{\tau_0^2}\left(\theta^2 + \mu_0^2 - 2\theta\mu_0\right)$$

$$= \text{const} + \theta^2\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right) - 2\left(\frac{n}{\sigma^2}\bar{x} + \mu_0\right)\theta$$

Not important. It will end up in the normalization constant.

we want this to be of the form

$\frac{1}{\tau_n^2}(\theta - \mu_n)^2$ since then the posterior will be $N(\mu_n, \tau_n^2)$

this is acheived if

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2}$$ (since this is the coefficient on θ^2 above)

and

$$\frac{\mu_n}{\tau_n^2} = \frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}$$ (since this is the coefficient on -2θ above)

So, $$\mu_n = \tau_n^2 \left(\frac{n}{\sigma^2}\bar{x} + \frac{1}{\tau_0^2}\mu_0 \right)$$

$$= \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}} \left(\frac{n}{\sigma^2}\bar{x} + \frac{1}{\tau_0^2}\mu_0 \right)$$

$$= w\bar{x} + (1-w)\mu_0$$

with $$w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}$$

Pheew!!!

Problem 4b)

Model: $x_1, \ldots, x_n \sim \text{Poisson}(\theta)$

Prior: $\theta \sim \text{Gamma}(\alpha, \beta)$

Posterior: $p(\theta \mid x_1 \ldots, x_n) \propto p(x_1, \ldots x_n \mid \theta) \cdot p(\theta)$

$$\propto \prod_{i=1}^{n} \underbrace{\frac{\theta e^{-\theta x_i}}{x_i}}_{\text{Poisson distribution}} \cdot \underbrace{\theta^{\alpha-1} e^{-\beta\theta}}_{\text{proportional to Gamma}(\alpha,\beta)\text{ density.}}$$

$$\propto \theta^{n} e^{-\theta \sum_{i=1}^{n} x_i} \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \theta^{n+\alpha-1} e^{-\left(\sum_{i=1}^{n} x_i + \beta\right)\theta}$$

$$\propto \text{Gamma}\left(\alpha+n, \beta + \sum_{i=1}^{n} x_i\right)$$