Bayesian Statistics I

Lecture 7 - Monte Carlo. Gibbs sampling.

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Lecture overview

- Monte Carlo simulation
- Gibbs sampling
- Data augmentation
  - Mixture models
  - Probit regression
- Regularized regression

Monte Carlo sampling

- If $\theta^{(1)}, \ldots, \theta^{(N)}$ is an iid sequence from $p(\theta)$, then

$$\bar{\theta} = \frac{1}{N}\sum_{t=1}^{N} \theta^{(t)} \quad \rightarrow \quad E(\theta)$$

$$\bar{g}(\theta) = \frac{1}{N}\sum_{t=1}^{N} g(\theta^{(t)}) \quad \rightarrow \quad E[g(\theta)]$$

  for some function $g(\theta)$ of interest.
- $\mathbb{V}(\bar{g}(\theta)) = \frac{c}{\sqrt{N}}$ for some constant c.
- Easy to compute tail probabilities $\Pr(\theta \leq c)$ by letting

$$g(\theta) = I\,(\theta \leq c)$$

  and

$$\frac{1}{N}\sum_{t=1}^{N} g(\theta^{(t)}) = \frac{\#\ \theta\text{-draws smaller than } c}{N}.$$

Direct sampling by the inverse CDF method

- Let $F(x)$ be the CDF of X. **Inverse CDF method**:
  1. Generate u from the uniform distribution on $[0, 1]$.
  2. Compute $x = F^{-1}(u)$.

- **Exponential distribution**:

$$u = F(x) = 1 - \exp(-\lambda x)$$

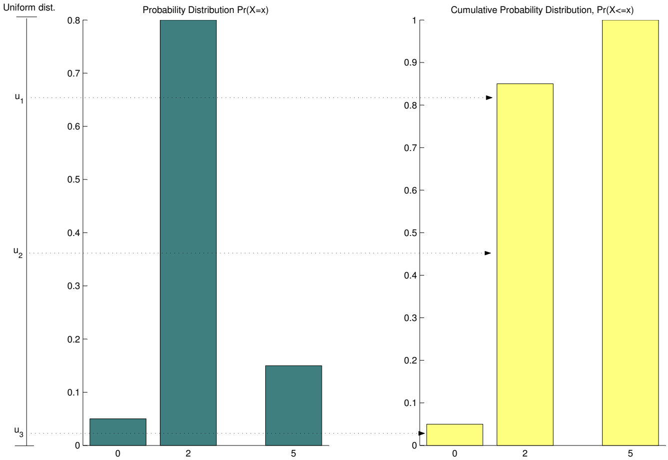
  Inverting gives

$$x = -\ln(1 - u)/\lambda$$

- So, if $u \sim U(0, 1)$ then

$$x = -\ln(1 - u)/\lambda \sim Expon(\lambda)$$

Inverse CDF method, discrete case



Direct sampling by the inverse CDF method

- **Cauchy distribution**:

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\frac{1}{1+x^2} \\
u &= F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)
\end{aligned}
$$

Inverting ...

$$x = \tan[\pi(u - 1/2)].$$

- Use **relations**:

$$y, z \text{ are indep } N(0,1) \Rightarrow \frac{y}{z} \sim \text{Cauchy}(0,1)$$

- **Chi-square**. If $x_1, \ldots, x_v \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^{v} x_i^2 \sim \chi_v^2$.

Gibbs sampling

- Easily implemented methods for **sampling from multivariate distributions**, $p(\theta_1, ..., \theta_k)$.
- Requirements: Easily sampled **full conditional distributions**:
  - $p(\theta_1|\theta_2, \theta_3..., \theta_k)$
  - $p(\theta_2|\theta_1, \theta_3, ..., \theta_k)$
  - $\vdots$
  - $p(\theta_k|\theta_1, \theta_2, ..., \theta_{k-1})$
- Gibbs sampling is a special case of **Metropolis-Hastings** (see Lecture 8).
- Metropolis-Hastings is a **Markov Chain Monte Carlo** (**MCMC**) algorithm.

The Gibbs sampling algorithm

- Choose initial values $\theta_2^{(0)}, \theta_3^{(0)}, ..., \theta_k^{(0)}$.
- Repeat for $j = 1, ..., N$:
  - Draw $\theta_1^{(j)}$ from $p(\theta_1 | \theta_2^{(j-1)}, \theta_3^{(j-1)}, ..., \theta_k^{(j-1)})$
  - Draw $\theta_2^{(j)}$ from $p(\theta_2 | \theta_1^{(j)}, \theta_3^{(j-1)}, ..., \theta_k^{(j-1)})$

    $\vdots$
  - Draw $\theta_k^{(j)}$ from $p(\theta_k | \theta_1^{(j)}, \theta_2^{(j)}, ..., \theta_{k-1}^{(j)})$
- Return draws: $\theta^{(1)}, ..., \theta^{(N)}$, where $\theta^{(j)} = (\theta_1^{(j)}, ..., \theta_k^{(j)})$.

Gibbs sampling, cont.

- Gibbs draws $\theta^{(1)}, \ldots, \theta^{(N)}$ are **dependent**, but

$$\bar{\theta} = \frac{1}{N}\sum_{t=1}^{N}\theta_j^{(t)} \quad \rightarrow \quad E(\theta_j)$$

$$\bar{g}(\theta) = \frac{1}{N}\sum_{t=1}^{N} g(\theta^{(t)}) \quad \rightarrow \quad E[g(\theta)]$$

- $\theta^{(1)}, \ldots., \theta^{(N)}$ **converges in distribution** to the target $p(\theta)$.
- $\theta_j^{(1)}, \ldots, \theta_j^{(N)}$ converges to the marginal distribution of θ_j.
- **Dependent draws** $\rightarrow$ **less efficient** than iid sampling.
- **IID samples**: $\theta^{(1)}, \ldots., \theta^{(N)}$: $\mathrm{Var}(\bar{\theta}) = \frac{\sigma^2}{N}$.
- **Autocorrelated samples**: $\mathrm{Var}(\bar{\theta}) = \frac{\sigma^2}{N}\left(1 + 2\sum_{k=1}^{\infty}\rho_k\right)$, where ρ_k is the autocorrelation at lag k.
- **Inefficiency factor**: $1 + 2\sum_{k=1}^{\infty}\rho_k$.

Gibbs sampling bivariate normal

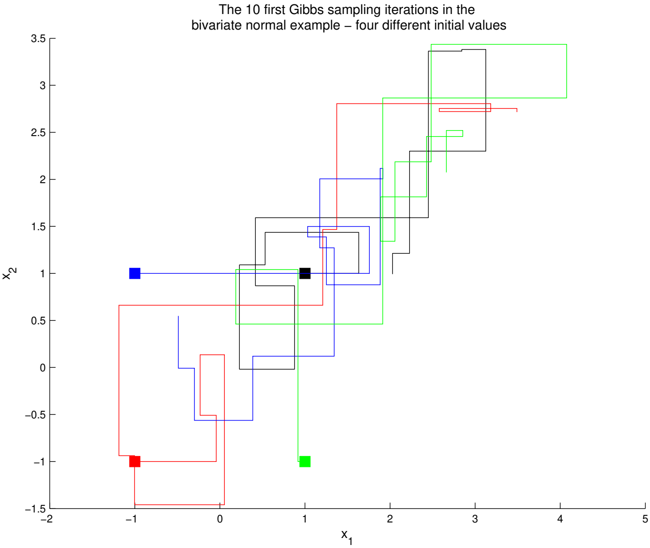
- **Joint distribution**

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$

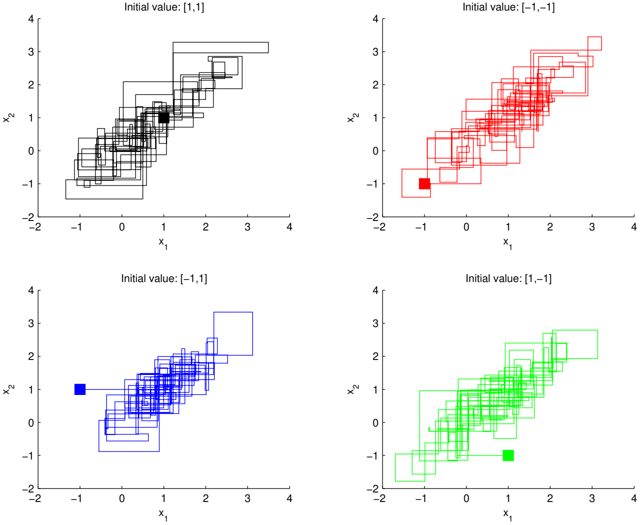
- **Full conditional posteriors**

$$\begin{aligned} \theta_1|\theta_2 &\sim N[\mu_1 + \rho(\theta_2 - \mu_2), 1 - \rho^2] \\ \theta_2|\theta_1 &\sim N[\mu_2 + \rho(\theta_1 - \mu_1), 1 - \rho^2] \end{aligned}$$

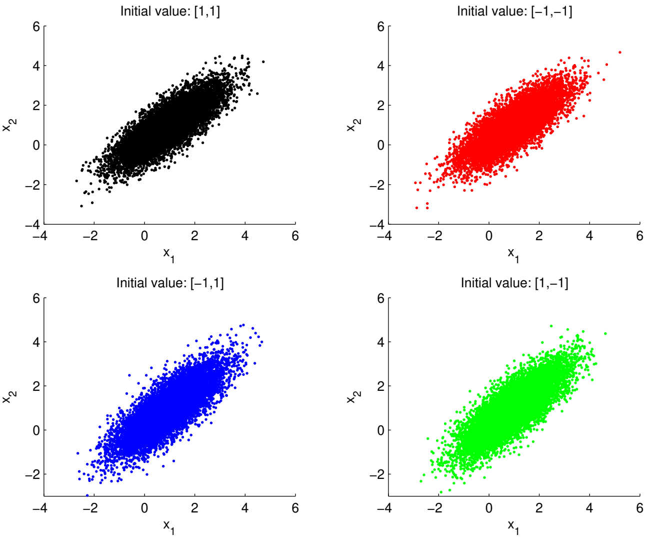
Gibbs sampling - Bivariate normal



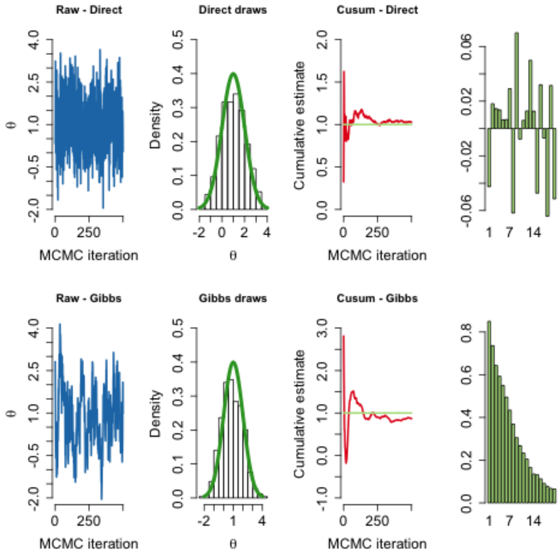
Gibbs sampling - Bivariate normal



Gibbs sampling - Bivariate normal



Direct sampling vs Gibbs sampling



Estimating $Pr(\theta_1 > 0, \theta_2 > 0)$

- Joint probability by counting:

$$Pr(\theta_1 > 0, \theta_2 > 0) \approx N^{-1} \sum_{i=1}^{N} 1(\theta_1^{(i)} > 0, \theta_2^{i)} > 0)$$

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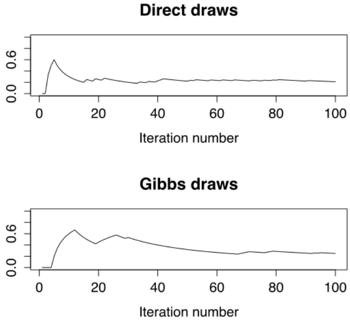


Normal model with conditionally conjugate prior

- **Normal model** with **conditionally conjugate prior**

$$\mu \sim N(\mu_0, \tau_o^2)$$
$$\sigma^2 \sim Inv - \chi^2(\nu_0, \sigma_0^2)$$

- **Full conditional posteriors**

$$\mu|\sigma^2, x \sim N\left(\mu_n, \tau_n^2\right)$$
$$\sigma^2|\mu, x \sim Inv - \chi^2\left(\nu_n, \frac{\nu_0\sigma_0^2 + \sum_{i=1}^n (x_i - \mu)^2}{n + \nu_0}\right)$$

with μ_n and τ_n^2 defined the same as when σ^2 is known.

Gibbs sampling for AR processes

- **AR(p) process**

$$x_t = \mu + \phi_1(x_{t-1} - \mu) + \ldots + \phi_p(x_{t-p} - \mu) + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2).$$

- Let $\phi = (\phi_1, \ldots, \phi_p)'$.
- **Prior**:
  - $\mu \sim$Normal
  - $\phi \sim$Multivariate Normal
  - $\sigma^2 \sim$Scaled Inverse χ^2.
- The **posterior** can be simulated by **Gibbs sampling**[1]:
  - $\mu|\phi, \sigma^2, x \sim$ Normal
  - $\phi|\mu, \sigma^2, x \sim$ Multivariate Normal
  - $\sigma^2|\mu, \phi, x \sim$ Scaled Inverse χ^2

[1] Villani (2009). Steady State Priors for Vector Autoregressions. *Journal of Applied Econometrics*.

Data augmentation - Mixture distributions

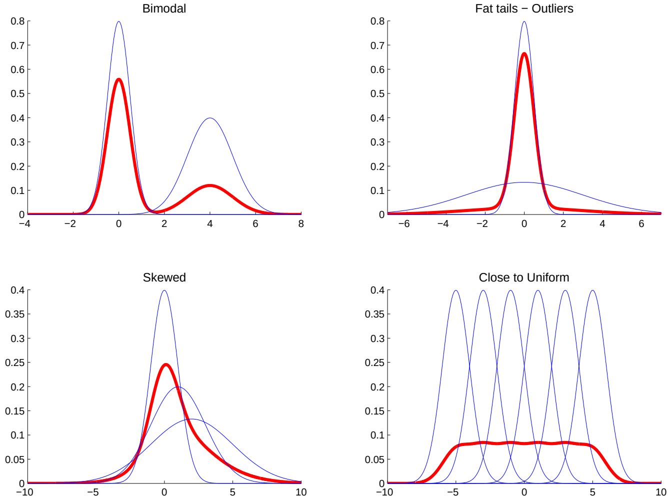
- Let $\phi(x|\mu,\sigma^2)$ denote the **PDF** of $x \sim N(\mu,\sigma^2)$.

- Two-component **mixture of normals** [MN(2)]

$$p(x) = \pi \cdot \phi(x|\mu_1,\sigma_1^2) + (1-\pi) \cdot \phi(x|\mu_2,\sigma_2^2)$$

- **Simulate** from a MN(2):
  - Simulate a **membership indicator** $I \in \{1,2\}$: $I \sim Bern(\pi)$.
  - If $I = 1$, simulate x from $N(\mu_1,\sigma_1^2)$
  - If $I = 2$, simulate x from $N(\mu_2,\sigma_2^2)$.

Illustration of mixture distributions



Mixture distributions, cont.

- The **likelihood** is a product of sums. **Messy** to work with.
- **Assume** that we know where each observation comes from

$$I_i = \begin{cases} 1 \text{ if } x_i \text{ came from Density 1} \\ 2 \text{ if } x_i \text{ came from Density 2} \end{cases}.$$

- Given $I_1, ..., I_n$ it is easy to estimate π, $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ by separating the sample according to the I's.
- But we do **not** know $I_1, ..., I_n$!
- **Data augmentation**: add $I_1, ..., I_n$ as unknown parameters.
- **Gibbs sampling**:
  - Sample π, $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$ **given** $I_1, ..., I_n$
  - Sample $I_1, ..., I_n$ **given** π, $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$

Gibbs sampling for mixture distributions

- Prior: $\pi \sim Beta(\alpha_1, \alpha_2)$. Conjugate prior for (μ_j, σ_j^2), see L5.
- Define: $n_1 = \sum_{i=1}^{n}(I_i = 1)$ and $n_2 = n - n_1$.
- Gibbs sampling:
  - $\pi \mid \mathbf{I}, \mathbf{x} \sim Beta(\alpha_1 + n_1, \alpha_2 + n_2)$
  - $\sigma_1^2 \mid \mathbf{I}, \mathbf{x} \sim Inv\text{-}\chi^2(\nu_{n_1}, \sigma_{n_1}^2)$ and $\mu_1 | \mathbf{I}, \sigma_1^2, \mathbf{x} \sim N\left(\mu_{n_1}, \frac{\sigma_1^2}{\kappa_{n_1}}\right)$
  - $\sigma_2^2 \mid \mathbf{I}, \mathbf{x} \sim Inv\text{-}\chi^2(\nu_{n_2}, \sigma_{n_2}^2)$ and $\mu_2 | \mathbf{I}, \sigma_2^2, \mathbf{x} \sim N\left(\mu_{n_2}, \frac{\sigma_2^2}{\kappa_{n_2}}\right)$
  - $I_i \mid \pi, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \mathbf{x} \sim Bern(\theta_i)$, $i = 1, ..., n$,

$$\theta_i = \frac{(1-\pi)\phi(x_i; \mu_2, \sigma_2^2)}{\pi\phi(x_i; \mu_1, \sigma_1^2) + (1-\pi)\phi(x_i; \mu_2, \sigma_2^2)}.$$

Gibbs sampling for mixture distributions

- K-component mixture of normals

$$p(x) = \sum_{k=1}^{K} \pi_k \phi(x; \mu_k, \sigma_k^2)$$

- Multi-class indicators: $I_i = k$ if x_i comes from component k.
- Gibbs sampling
  - $(\pi_1, ..., \pi_K) \mid \mathbf{I}, \mathbf{x} \sim \textit{Dirichlet}(\alpha_1 + n_1, \alpha_2 + n_2, ..., \alpha_K + n_K)$
  - $\sigma_k^2 \mid \mathbf{I}, \mathbf{x} \sim \textit{Inv-}\chi^2$ and $\mu_k | \mathbf{I}, \sigma_k^2, \mathbf{x} \sim \textit{Normal}$, *for* $k = 1, ..., K$,
  - $I_i \mid \pi, \mu, \sigma^2, \mathbf{x} \sim \textit{Multinomial}(\theta_{i1}, ..., \theta_{iK})$, *for* $i = 1, ..., n$,

$$\theta_{ij} = \frac{\pi_j \phi(x_i; \mu_j, \sigma_j^2)}{\sum_{r=1}^{k} \pi_r \phi(x_i; \mu_r, \sigma_r^2)}.$$

- Gibbs sampling is very powerful for missing data problems.
- Semi-supervised learning.

Data augmentation - Probit regression

- Probit regression:

$$\Pr(y_i = 1 \mid x_i) = \Phi(x_i^T \beta)$$

- Random utility formulation:

$$\begin{aligned} u_i &\sim N(x_i^T \beta, 1) \\ y_i &= \begin{cases} 1 & \text{if } u_i > 0 \\ 0 & \text{if } u_i \leq 0 \end{cases}. \end{aligned}$$

- Check: $\Pr(y_i = 1 \mid x_i) = \Pr(u_i > 0) = 1 - \Pr(u_i \leq 0) = 1 - \Pr(u_i - x_i^T \beta < -x_i^T \beta) = 1 - \Phi(-x_i^T \beta) = \Phi(x_i^T \beta)$.
- Given $u = (u_1, ..., u_n)$, β can be analyzed by linear regression.
- u is not observed. Gibbs sampling to the rescue![2]

[2] Albert and Chib (1993). Bayesian Analysis of Binary and Polychotomous Response Data. *JASA*.

Gibbs sampling for the Probit regression

- Simulate from **joint posterior** $p(u, \beta|y)$ by iterating between
  - $p(\beta|u, y)$ is multivariate normal (linear regression)
  - $p(u_i|\beta, y)$, $i = 1, ..., n$.
- The **full conditional** posterior distribution of u_i

$$p(u_i|\beta, y) \propto p(y_i|\beta, u_i)p(u_i|\beta)$$
$$= \begin{cases} N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (-\infty, 0] \text{ if } y_i = 0 \\ N(u_i|x_i'\beta, 1) & \text{truncated to } u_i \in (0, \infty) \text{ if } y_i = 1 \end{cases}$$

- Histogram of β-draws approximates the marginal posterior of β

$$p(\beta|y) = \int p(u, \beta|y)du$$

Gibbs sampling for Regularized regression

- Recap: The joint posterior of β, σ^2 and λ is

$$\beta|\sigma^2,\lambda,\mathbf{y},\mathbf{X} \sim N\left(\mu_n, \Omega_n^{-1}\right)$$
$$\sigma^2|\lambda,\mathbf{y},\mathbf{X} \sim Inv-\chi^2\left(\nu_n,\sigma_n^2\right)$$
$$p(\lambda|\mathbf{y},\mathbf{X}) \propto \sqrt{\frac{|\Omega_0|}{|X'X+\Omega_0|}}\left(\frac{\nu_n\sigma_n^2}{2}\right)^{-\nu_n/2}\cdot p(\lambda)$$

- This is the **conditional-marginal decomposition**

$$p(\beta,\sigma^2,\lambda|\mathbf{y},\mathbf{X}) = p(\beta|\sigma^2,\lambda,\mathbf{y},\mathbf{X})p(\sigma^2|\lambda,\mathbf{y},\mathbf{X})p(\lambda|\mathbf{y},\mathbf{X})$$

- **Gibbs sampling** can instead be used:
  - Sample $\beta|\sigma^2,\lambda,\mathbf{y},\mathbf{X}$ from Normal
  - Sample $\sigma^2|\beta,\lambda,\mathbf{y},\mathbf{X}$ from Inv-χ^2
  - Sample $\lambda|\beta,\sigma^2,\mathbf{y},\mathbf{X}$ from Gamma
- λ is **easy** to simulate **conditional on** β and σ^2.

Gibbs sampling for Regularized regression

- Assume a Gamma prior for λ (same as $\lambda^{-1} \sim \mathrm{Inv} - \chi^2$)

$$\lambda \sim \mathrm{Gamma}\left(\frac{\eta_0}{2}, \frac{\eta_0}{2\lambda_0}\right).$$

- $\mathbb{E}(\lambda) = \frac{\eta_0/2}{\eta_0/(2\lambda_0)} = \lambda_0$ and $\mathbb{V}(\lambda) = \frac{\eta_0/2}{(\eta_0/(2\lambda_0))^2} = \frac{1}{2\eta_0\lambda_0^2}$.

- Using Bayes' theorem twice:

$$\begin{aligned} p\left(\lambda|\beta, \sigma^2, \mathbf{y}\right) &\propto p\left(\mathbf{y}|\beta, \sigma^2, \lambda\right) p\left(\lambda|\beta, \sigma^2\right) \\ &\propto p\left(\beta|\sigma^2, \lambda\right) p\left(\lambda\right) \end{aligned}$$

- Note:
  - likelihood $p\left(\mathbf{y}|\beta, \sigma^2, \lambda\right)$ does not depend on λ.
  - prior $p\left(\lambda|\sigma^2\right)$ is assumed to not depend on σ^2.

Gibbs sampling for Regularized regression

- Full conditional posterior

$$
\begin{aligned}
p\left(\lambda|\beta,\sigma^2,\mathbf{y}\right) &\propto p\left(\beta|\sigma^2,\lambda\right)p\left(\lambda\right)\\
&\propto \prod_{i=1}^{m}\frac{1}{\sqrt{2\pi\sigma^2/\lambda}}\exp\left(-\frac{\beta_i^2}{2\sigma^2/\lambda}\right)\cdot\lambda^{\eta_0/2-1}\exp\left(-\lambda\frac{\eta_0}{2\lambda_0}\right)\\
&\propto \lambda^{m/2}\exp\left(-\frac{\lambda}{2\sigma^2}\sum_{i=1}^{m}\beta_i^2\right)\cdot\lambda^{\eta_0/2-1}\exp\left(-\lambda\frac{\eta_0}{2\lambda_0}\right)\\
&\propto \lambda^{(m+\eta_0)/2-1}\exp\left(-\lambda\left(\frac{\sigma^{-2}\sum_{i=1}^{m}\beta_i^2+\eta_0/\lambda_0}{2}\right)\right)
\end{aligned}
$$

- This shows that

$$
\lambda|\beta,\sigma^2,\mathbf{y}\sim\text{Gamma}\left(\frac{m+\eta_0}{2},\frac{\sigma^{-2}\sum_{i=1}^{m}\beta_i^2+\eta_0/\lambda_0}{2}\right).
$$

- $\mathbb{E}(\lambda|\beta,\sigma^2,\mathbf{y})=\frac{m+\eta_0}{\sigma^{-2}\sum_{i=1}^{m}\beta_i^2+\eta_0/\lambda_0}$, so λ is learned from variability of the β_i. Large m helps!

Improving the efficiency of the Gibbs sampler

- **Efficient blocking**. Correlated parameters should ideally be included in the same updating block.

- **Reparametrization**. Convergence can improve dramatically in alternative parametrizations.

- **Data augmentation**.
  - Augment with latent variables to make **full conditional posteriors more easily sampled** (Probit, Mixture models).
  - But typically **increases the autocorrelation** between draws.