Linear Algebra

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Outline

- Motivation
- Vector spaces
- Linear feature extraction and prediction
- Linear pre processing
- References

Motivation

Pre Processing

- Pre processing methods simplify feature extraction and prediction
- Understanding linear transformations is a key to understanding many popular pre processing methods
- Example pre processing methods
    - Discrete Fourier transform
    - Principal component analysis

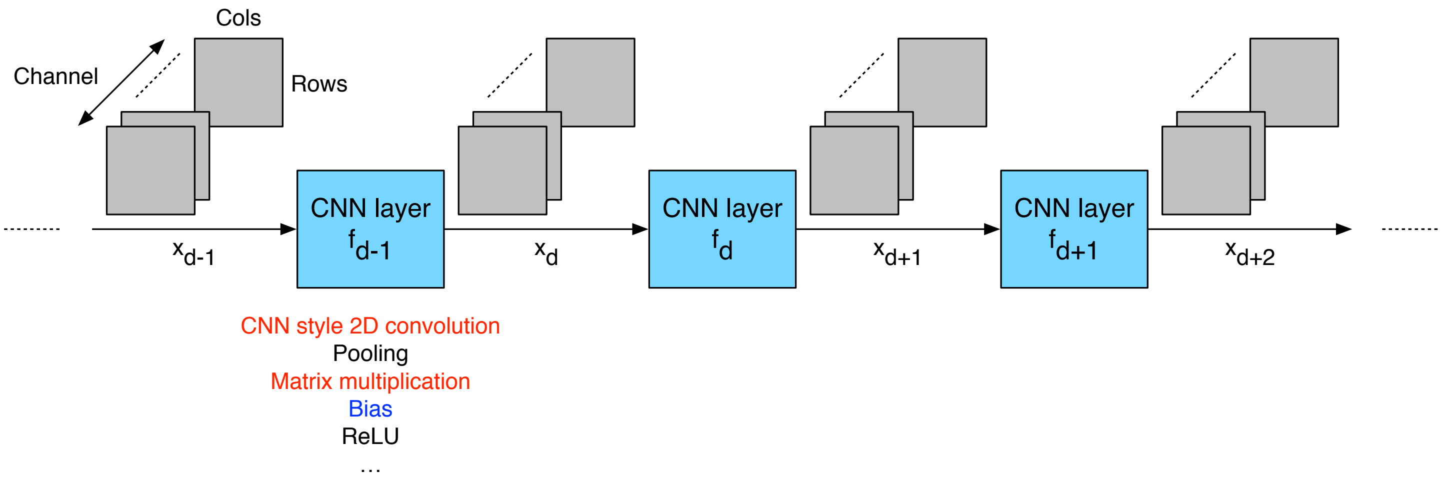
Feature Extraction And Prediction

- CNNs are compositions of nonlinear functions (layers)

$$\mathbf{y} = f_{D-1}(\ldots(f_2(f_1(f_0(\mathbf{x}, h_0), h_1), h_2), \ldots), h_{D-1})$$

- Layers transform from data space to feature space to information space
- Examples layers
  - Fully connected layers with single and multiple inputs
  - CNN style 2D convolution layers

Feature Extraction And Prediction



- A key part of these 2 layers are linear transformations
- Understanding linear transformations is a key to the design and implementation of xNNs

Vector Spaces

Preliminaries

- Notation
  - Scalars are not bold
  - Vectors are bold lower case
  - Matrices are bold upper case
  - Indices start at 0 and go from 0, ..., size - 1

Set

- A collection of distinct objects

Field

- A set with well defined addition and multiplication operations
  - Associativity: $a + (b + c) = (a + b) + c$ and $a (b c) = (a b) c$
  - Commutativity: $a + b = b + a$ and $a b = b a$
  - Additive identity: $a + 0 = a$
  - Additive inverse: $a + (-a) = 0$
  - Multiplicative identity: $1 a = a$
  - Multiplicative inverse: $a a^{-1} = 1$
  - Distributivity: $a (b + c) = (a b) + (a c)$
- Elements of fields are generally referred to as scalars
- Examples: R (real scalars), C (complex scalars)

Vector

To do: add an example of a vector

- K tuple of scalars, always columns
- F^K
- Examples: R^K and C^K

Matrix

To do: add an example of a matrix

- M x K tuple of scalars
- Collection of K vectors of size M x 1 arranged in columns
  - Leads to column space and right null space
    - What can matrix vector multiplication reach and what can it not
  - Visualize using outer product of matrix vector multiplication
- Collection of M vectors of size K x 1 transposed and arranged as rows
  - Leads to row space and left null space
    - What can vector matrix multiplication reach and what can it not
  - Visualize using outer product of vector matrix multiplication

Tensor

- K_0 x ... x K_{D-1} array of scalars
- Ordering
  - Last dimension is contiguous in memory
  - Working from right to left goes from closest to farthest spacing in memory
  - Feature maps: batch x channel x row x column
  - Filter coefficients: output channel x input channel x row x col

Function

- Mapping f: $X \rightarrow Y$ from domain to co domain
  - Injective: one to one; each y produced by at most one x
  - Surjective: onto; each y produced by at least one x
  - Bijective: one to one and onto
    - Bijective functions are invertible

- An infinite set is
  - Countably infinite if there's a bijection between the natural numbers and elements of the set
  - Un countably infinite if there's not

Vector Space

- Set of vectors and linear combinations of those vectors
- Satisfy
  - Associativity: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
  - Commutativity: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
  - Additive identity: $\mathbf{x} + \mathbf{0} = \mathbf{x}$
  - Additive inverse: $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  - Multiplicative compatibility: $a\,(b\,\mathbf{x}) = b\,(a\,\mathbf{x})$
  - Multiplicative identity: $1\,\mathbf{x} = \mathbf{x}$
  - Distributivity: $(a + b)(\mathbf{x} + \mathbf{y}) = a\,\mathbf{x} + a\,\mathbf{y} + b\,\mathbf{x} + b\,\mathbf{y}$
- Examples: R^K, C^K, $R^{K\_0 \times \ldots \times K\_D-1}$

Normed Vector Space

- A vector space with a notion of distance
- A norm maps an element of the vector space to a scalar
- Satisfies
  - Non negativity: $||\mathbf{x}|| \geq 0$ and $||\mathbf{x}|| = 0$ iff $\mathbf{x} = 0$
  - Absolute scalability: $||a\,\mathbf{x}|| = |a|\;||\mathbf{x}||$
  - Triangle inequality: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$
- Example: l_p norm (common p = 1, 2 and ∞)

$$||\mathbf{x}||_p = (\Sigma_n(|x(n)|^p))^{1/p},\ p \geq 1$$

Inner Product Space

- A vector space with a notion of distance and angle
- An inner product maps 2 elements of a vector space to a scalar
- Satisfies
  - Positive definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$
  - Conjugate symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathrm{conj}(\langle \mathbf{y}, \mathbf{x} \rangle)$
  - Linearity: $\langle a\, \mathbf{x}, \mathbf{y} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- Inner products induce norms on a vector space
  - But not all norms have associated inner products (e.g., l_∞)
- Example: dot product

$$\langle x, y \rangle = x \bullet y = x^H y = \Sigma_n (\mathrm{conj}(x(n))\, y(n)) = ||\mathbf{x}||_2\, ||\mathbf{y}||_2 \cos(\theta)$$

Linear Feature Extraction And Prediction

Specifically

- Linear transformations used within fully connected and CNN style 2D convolution layers for feature extraction and prediction
    - Matrix vector multiplication
    - Matrix matrix multiplication
    - CNN style 2D convolution

Matrix Vector Multiplication

- Notation: M (output dimension), K (input dimension)
    - Setting up for BLAS notation

$$\begin{bmatrix} y(0) \\ \vdots \\ y(M-1) \end{bmatrix} = \begin{bmatrix} H(0,0) & \cdots & H(0,K-1) \\ \vdots & & \vdots \\ H(M-1,0) & \cdots & H(M-1,K-1) \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(K-1) \end{bmatrix}$$

- Mechanics
    - Inner product of matrix row and vector input to produce each output

Matrix Vector Multiplication

- A traditional neural network is composed of repeated fully connected layers
  - Output vector of features = pointwise nonlinearity (feature extraction matrix * input vector of data + bias vector)
  - Repeat with output of current layer as input to next layer

$$\mathbf{y} = \mathrm{ReLU}(\mathbf{H}\ \mathbf{x} + \mathbf{b})$$

Matrix Vector Multiplication

- Matrix vector multiplication is a linear transformation
  - Every linear map can be represented as a matrix
  - Every matrix represents a linear map
  - So matrix vector multiplication in a neural network is doing linear transformations
- Multiple linear transformations can be composed into a single linear transformation
  - A reason why nonlinearities are included in xNNs
  - Otherwise there would be no depth

$$\mathbf{y} = \mathbf{H}_{D-1} \ldots \mathbf{H}_1 \mathbf{H}_0 \mathbf{x} = \mathbf{H} \mathbf{x}$$

Matrix Vector Multiplication

- You design a network to accomplish a goal
  - Don’t ever lose sight of this
  - Network design is not arbitrary
  - So it always makes sense to stop and think how the operations you’re including help you accomplish that goal
  - Why do neural networks include these layers?
  - How do these layers map from data to features to predictions?
  - Pay attention to how the input features are combined to generate output features
- The purpose of the next few slides is to help build intuition on how matrix vector multiplication help layers map from data to features to classes

Matrix Vector Multiplication

- Intuition of feature extraction and prediction for a fully connected layer
  - Note: sometimes linear classification is viewed as template matching where each row is a different template and the predicted class is the maximum output

- Inner product depends on magnitude and angle
  - y(m) = **H**(m, :) **x**
  - y(m) is the extracted feature or prediction
  - **H**(m, :) is the feature extractor or predictor
  - **x** is the input

Matrix Vector Multiplication

- How strong or important is a feature extractor? $||\mathbf{H}(m, :)||_2$
  - Note that the input mag contributes the same to each extracted feature $||\mathbf{x}||_2$
  - So here input magnitude only matters relative to bias
  - But input magnitude will also matter for network structures with branches that come together
  - Input magnitude will also matter when the same feature extractor is applied to different inputs

- How aligned is the feature extractor with the input? θ
  - In same direction: positive feature
  - Orthogonal: 0 feature
  - In opposite direction: negative feature

Matrix Vector Multiplication

- Intuition of bias
  - Affine transformation
  - Allows the dividing line to shift
  - Implementation of rank 1 outer product
  - Will use bias in a constructive variant of the universal approximation proof

- Intuition of ReLU
  - Removes negatively aligned features or predictions
  - Allows depth
  - Subsequent layers combine positively aligned extracted features

Matrix Vector Multiplication

- Intuition of size of K and M
    - Small K to large M
        - Different combinations of a small number of features to predict a large number of classes
    - Large K to small M
        - 1 feature or a combination of features to predict a small number of classes is now possible
    - Example: ImageNet classification and final fully connected layer size

Matrix Vector Multiplication

- In general, for the final classification layer, it's better to have K > M
  - Classification goal is to create matrix and bias that takes K features and makes the correct 1 of the M elements at the output much larger (closer to + ∞) than all the others
  - To get a feel for this consider 2 extreme cases
    - 2 features K linearly combined + bias then ReLU to predict 200 classes M
    - 200 features K linearly combined + bias then ReLU to predict 2 classes M
  - Create example features, matrices and biases for both cases
  - Look at sensitivity of the prediction to errors in the features for each
    - Can relate to matrix condition number
    - Show the condition number of K > M is always less than that of K < M
    - Now vary K and M and show diminishing returns after some point too
    - Generalize to considerations in a hierarchical head design
  - Start of a project idea

Matrix Vector Multiplication

- Arithmetic intensity
  - Compute = MK (MACs = multiply accumulates)
  - Data movement = K + MK + M (elements)
  - Ratio = (MK)/(K + MK + M) (consider M and K large)
    ≈ 1 (memory wall)

- If you want to make matrix vector multiplication run fast, you need to build a fast memory subsystem
  - Typically not an efficient thing to do from an operation per power perspective

Matrix Matrix Multiplication

- Notation: M (output dimension), K (input dimension), N (number of inputs and outputs)
  - BLAS M, N, K notation for **Y** = **H X**
  - Will try and conform to this throughout
  - Matrix vector multiplication is a special case with N = 1

$$\begin{bmatrix} Y(0,0) & \cdots & Y(0,N-1) \\ \vdots & & \vdots \\ Y(M-1,0) & \cdots & Y(M-1,N-1) \end{bmatrix} = \begin{bmatrix} H(0,0) & \cdots & H(0,K-1) \\ \vdots & & \vdots \\ H(M-1,0) & \cdots & H(M-1,K-1) \end{bmatrix} \begin{bmatrix} X(0,0) & \cdots & X(0,N-1) \\ \vdots & & \vdots \\ X(K-1,0) & \cdots & X(K-1,N-1) \end{bmatrix}$$

Matrix Matrix Multiplication

- Application of same matrix transformation to multiple input vectors
  - Stack all the inputs next to each other
  - Get matrix matrix multiplication
- Lots of matrix operations and decompositions
  - Transpose is highlighted here because it will come up later
  - Transpose swaps matrix element indices
  - When applied to products of matrices remember socks then shoes, shoes then socks (or just remember the formula)

$$\mathbf{C}^T = (\mathbf{A}\ \mathbf{B})^T = \mathbf{B}^T\ \mathbf{A}^T$$

Matrix Matrix Multiplication

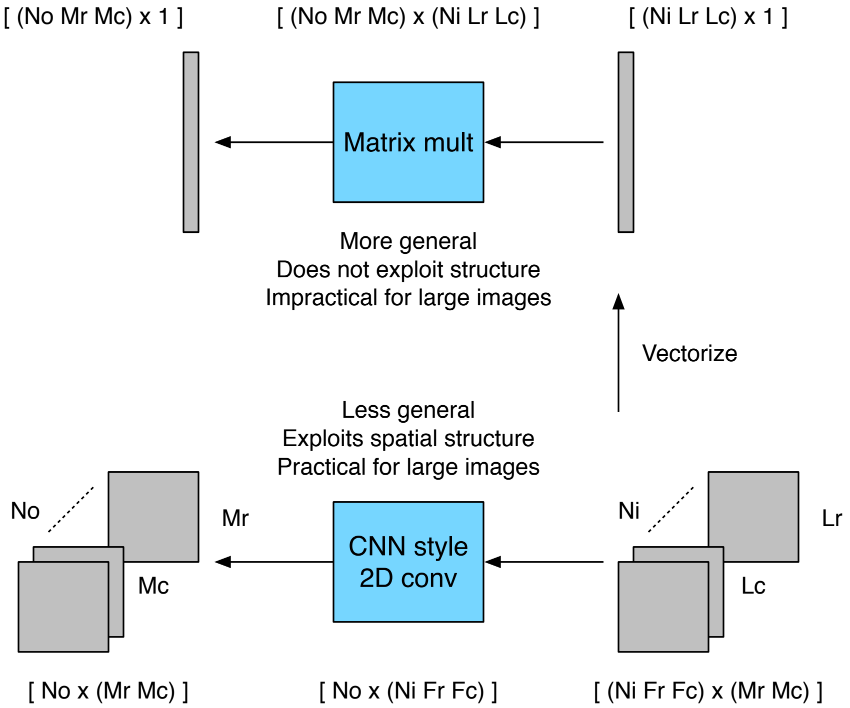
- Arithmetic intensity
  - Compute = MNK (MACs)
  - Data movement = KN + MK + MN (elements)
  - Ratio = (MNK)/(KN + MK + MN) (cube in num, squares in den)
    = $N^3/(3*N^2)$ (special case M = N = K)
    = N/3 (ratio maxed with sq matrix)

- If you want to make matrix matrix mult run fast, if it's possible choose a large matrix size such that you get multiple ops per element of data moved

- Why are bubbles spherical? Min surface area per volume enclosed
  - Think of surface area as data movement and volume as MACs

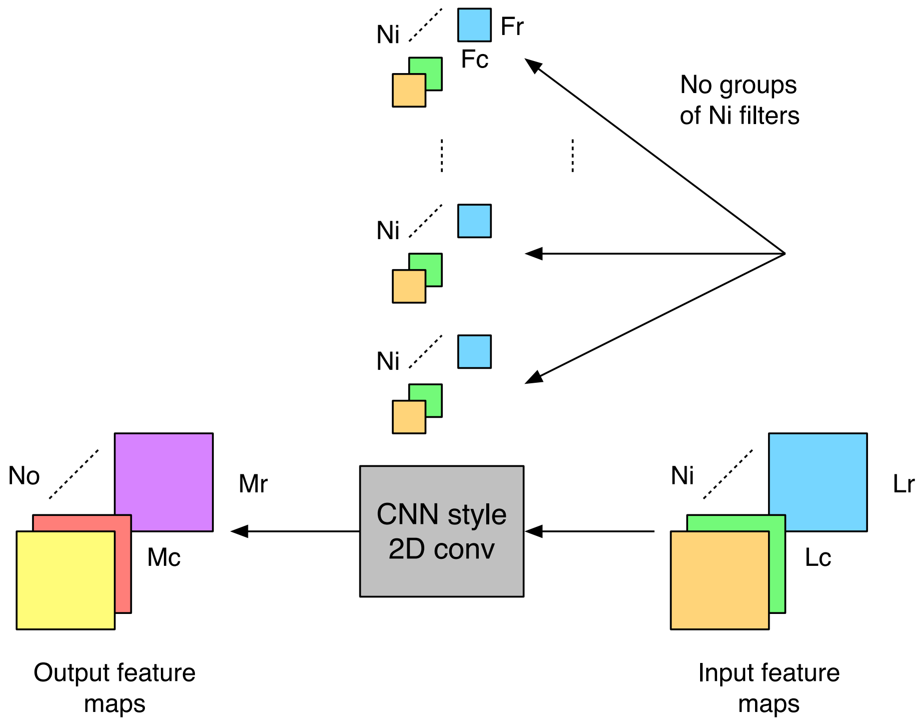
CNN Style 2D Convolution

- Consider applying a standard neural network to an image
  - Dimensions / memory / arithmetic intensity make it unreasonable to apply normal neural network linear layer to large images
- Use CNN style 2D convolution layer instead
  - It's a less general transformations but if the input / problem has translational invariance then perhaps the loss of generality is ok (and for many applications it is)
  - Very high (but not unreasonable) memory and compute for modern hardware



CNN Style 2D Convolution

- Input feature maps
  - 3D tensor
  - N_i inputs x L_r rows x L_c cols
- Filter coefficients
  - 4D tensor
  - N_o outputs x N_i inputs x F_r rows x F_c cols
- Output feature maps
  - 3D tensor
  - N_o outputs x M_r rows x M_c cols



CNN Style 2D Convolution

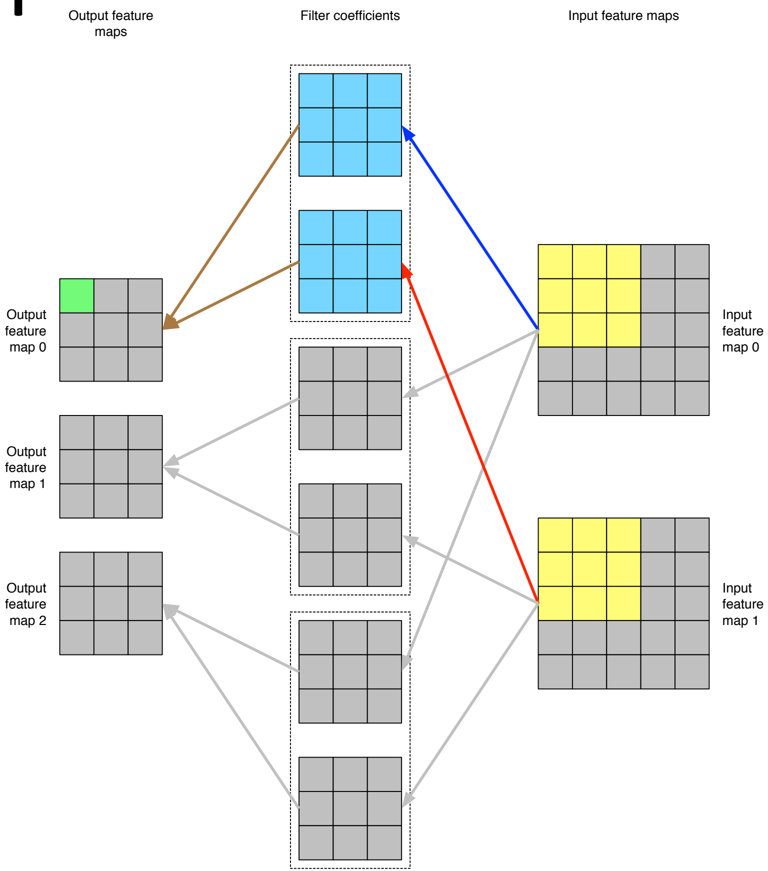
- A brief comment on a complete CNN style 2D convolution layer before returning to the CNN style 2D convolution operation at it's core

- A traditional CNN is composed of multiple CNN style 2D convolution layers (plus some other layers that will be discussed layer)
  - $\circledast$ is used here to notate CNN style 2D convolution
  - The notation for the bias is meant to indicate that the same bias value V(no) is added to every Mr x Mc element of output feature map no
  - ReLU is applied pointwise

$$\mathbf{Y}^{\text{No x Mr x Mc}} = \text{ReLU}(\mathbf{H}^{\text{No x Ni x Fr x Fc}} \circledast \mathbf{X}^{\text{Ni x Lr x Lc}} + \mathbf{V}^{\text{No}})$$

CNN Style 2D Convolution

- An example showing CNN style 2D convolution is matrix matrix multiplication using 2 x 5 x 5 input feature maps, 3 x 2 x 3 x 3 filters and 3 x 3 x 3 output feature maps
- This sequence of figures illustrates the specific set of inputs and filter coefficients used to generate each output



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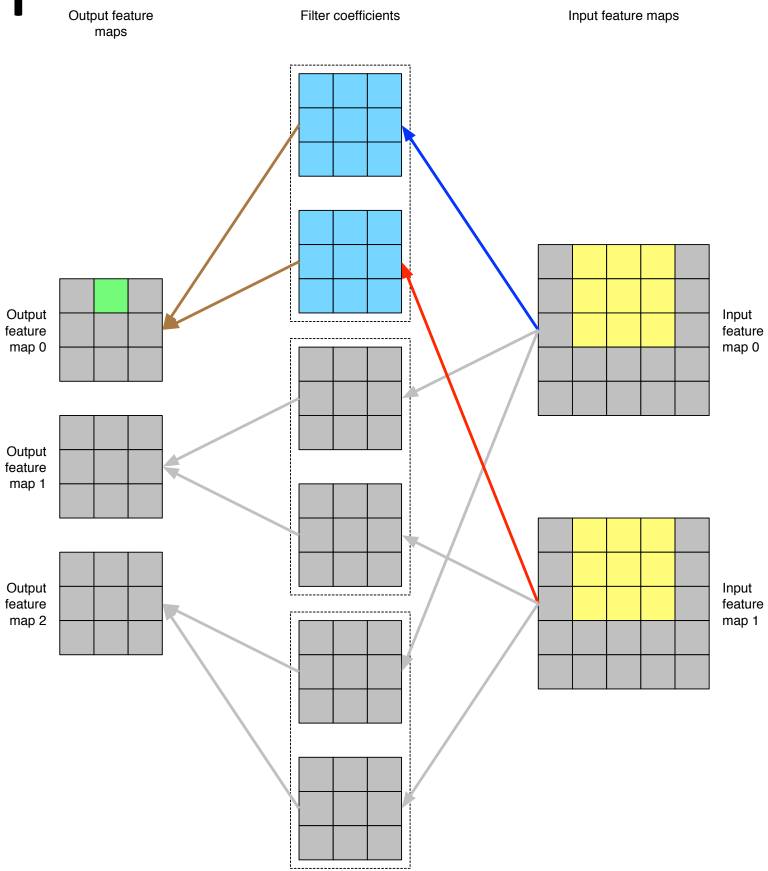

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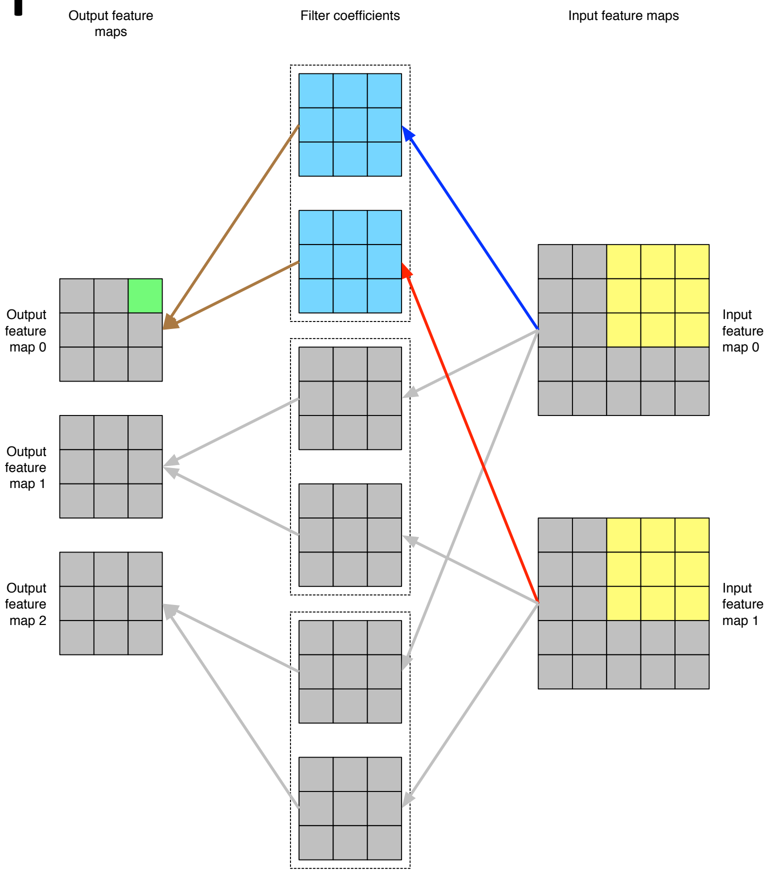

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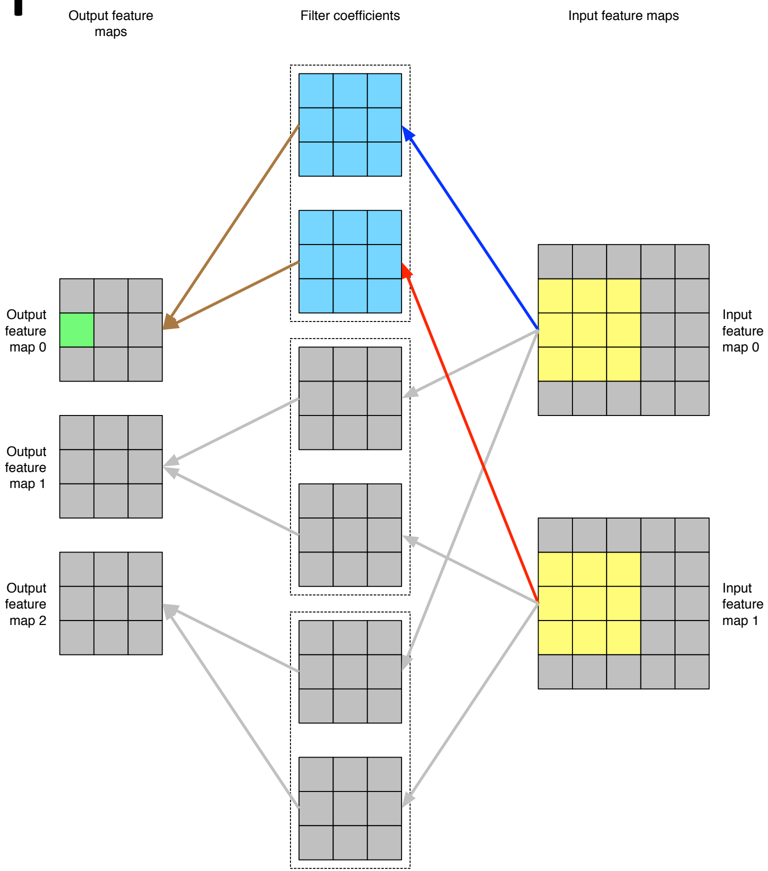

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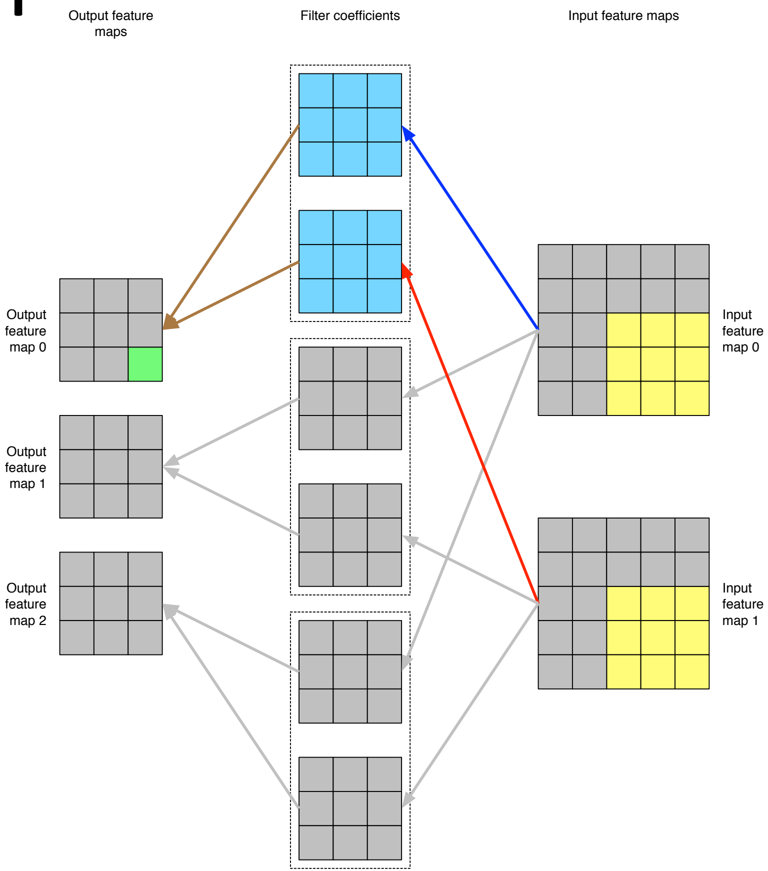

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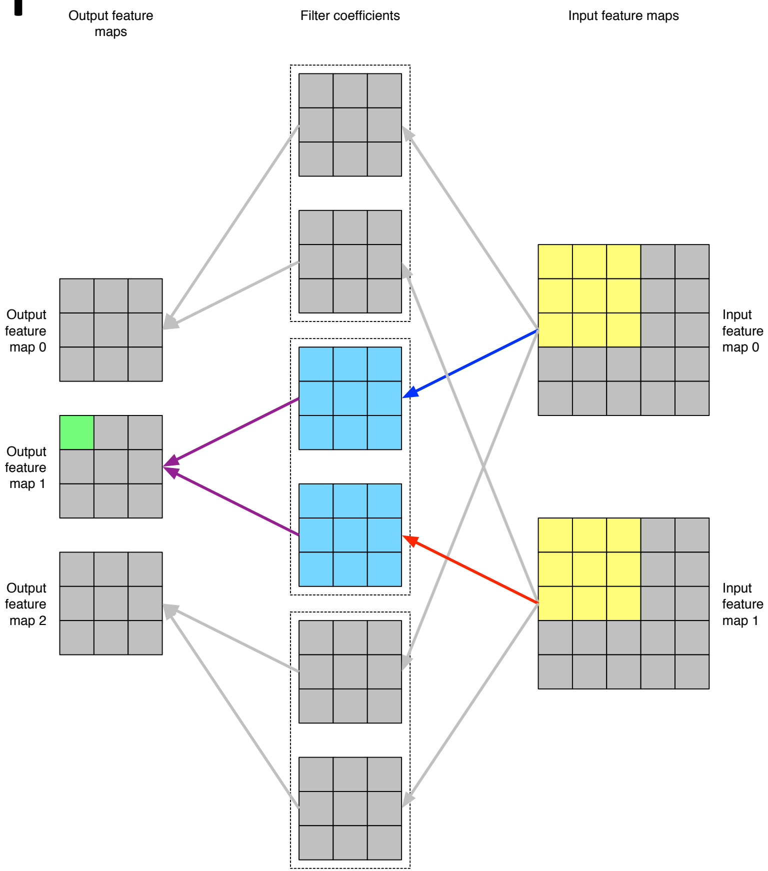

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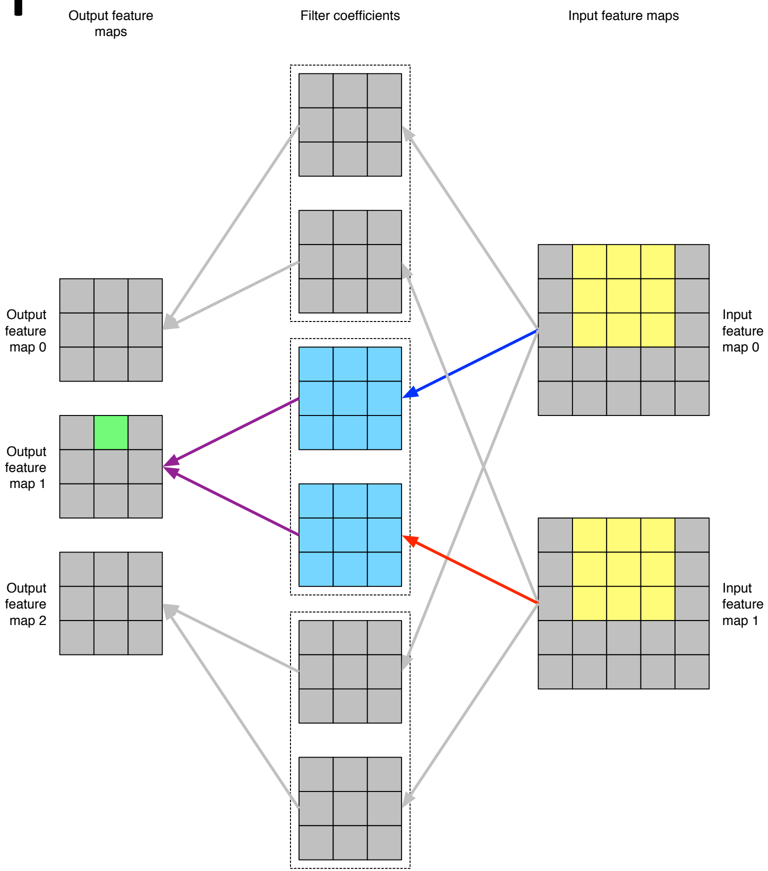

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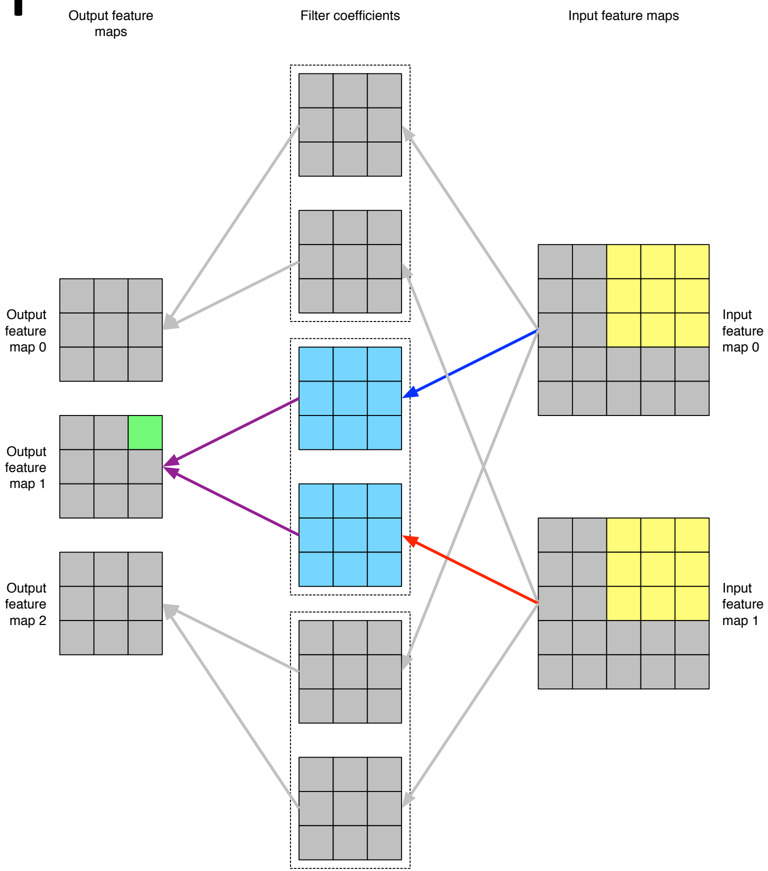

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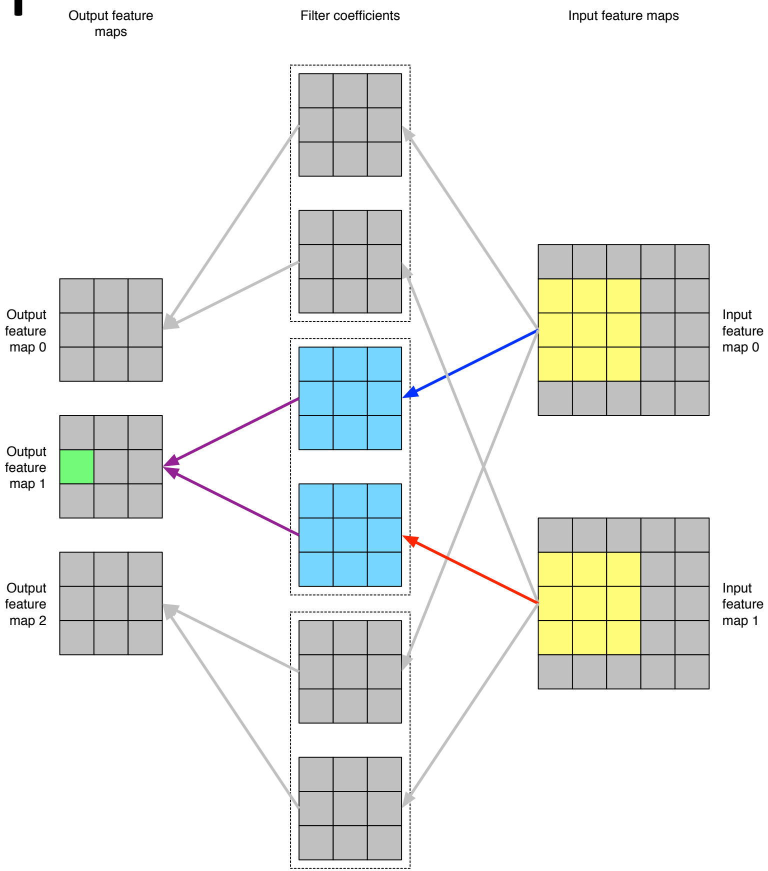

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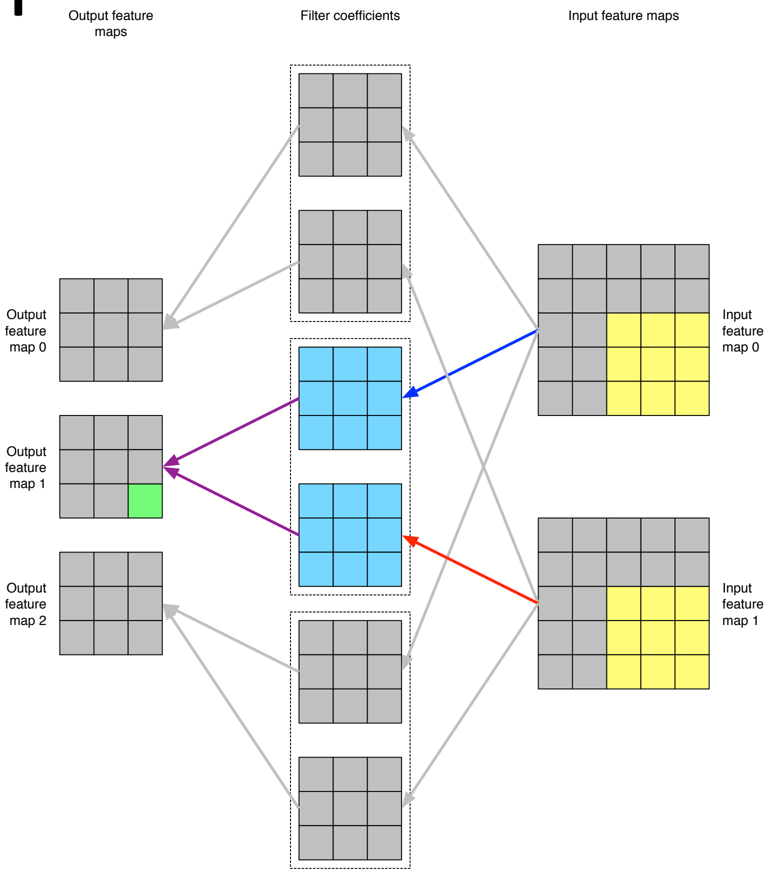

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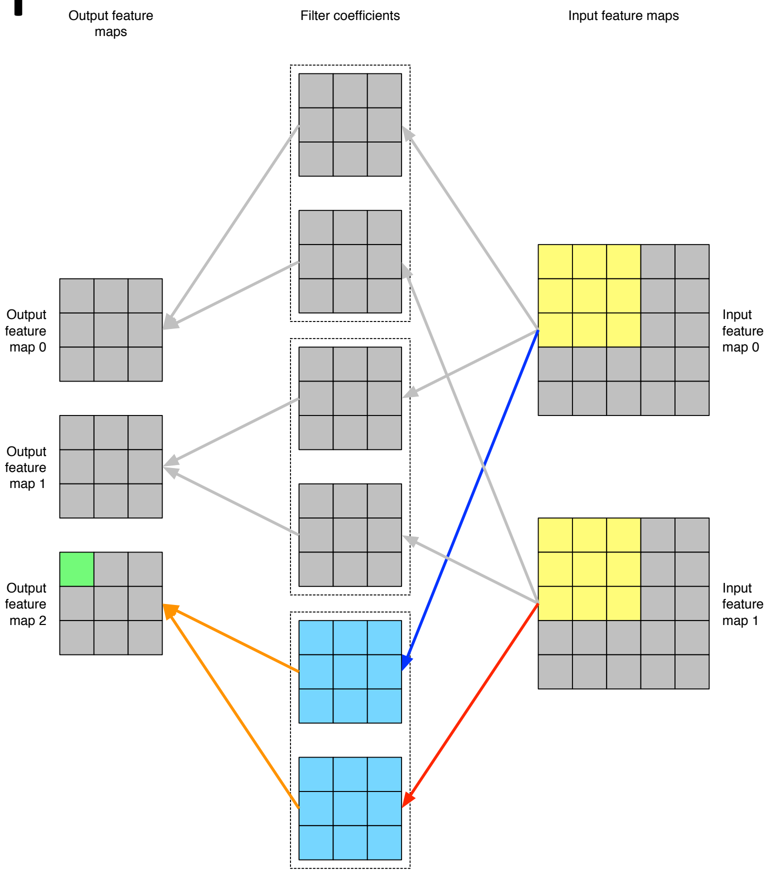

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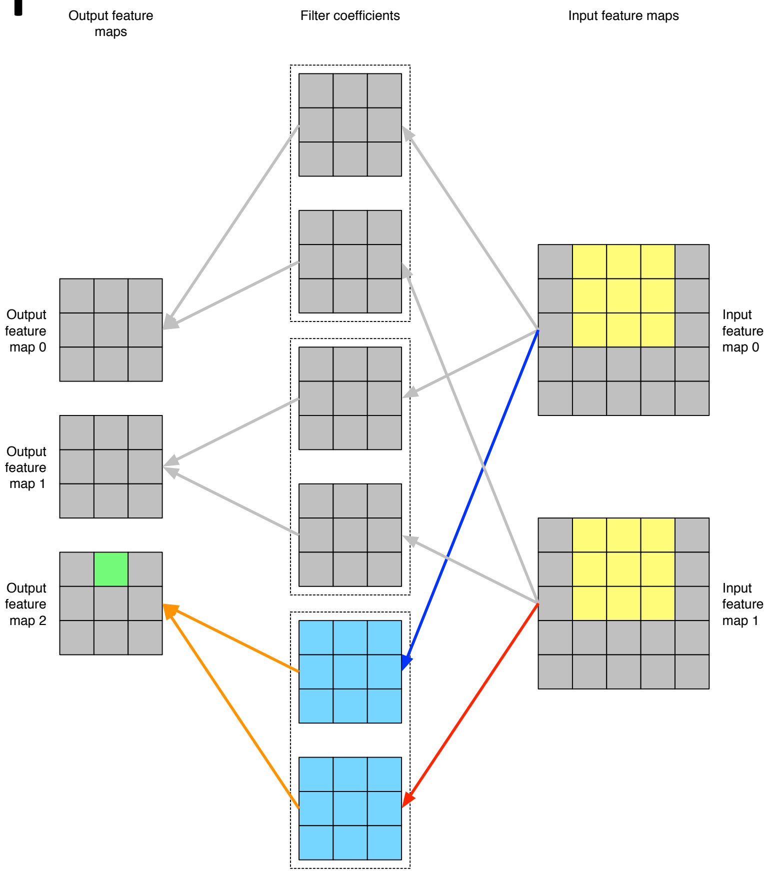

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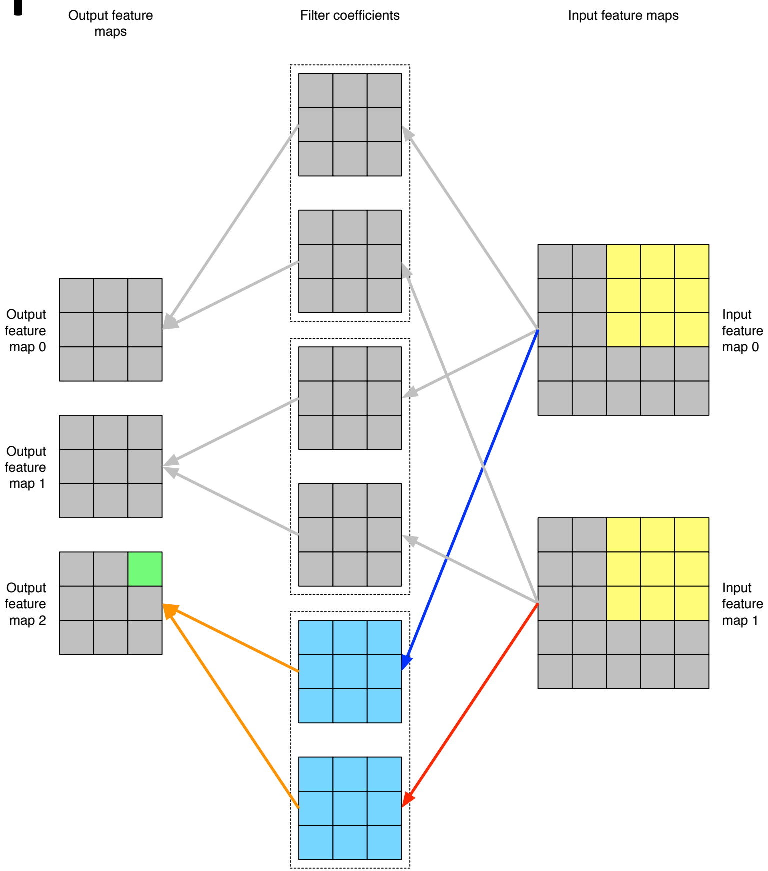

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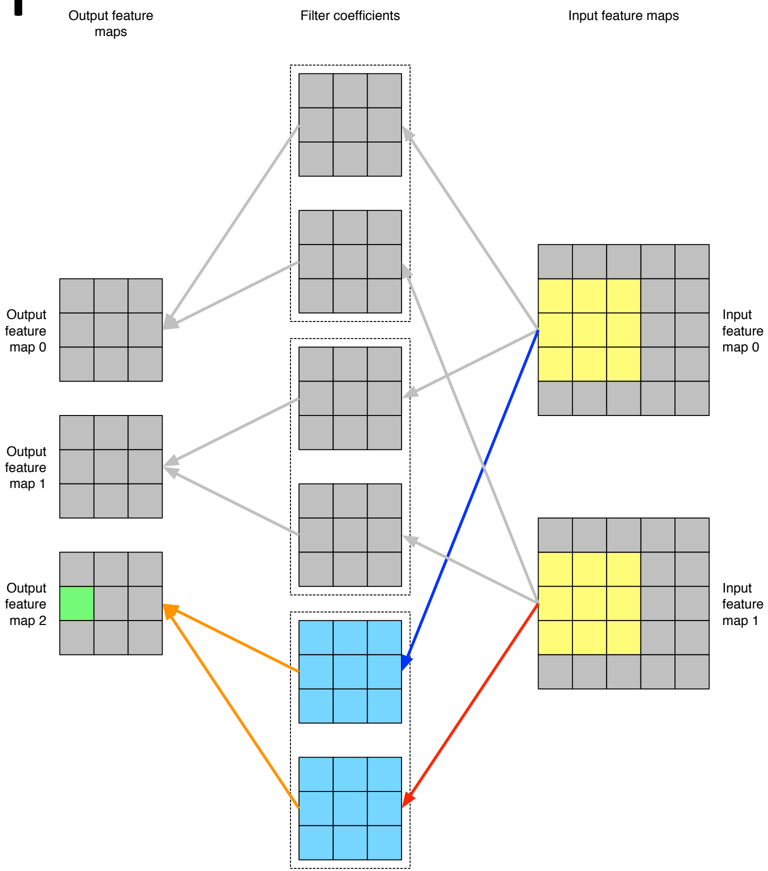

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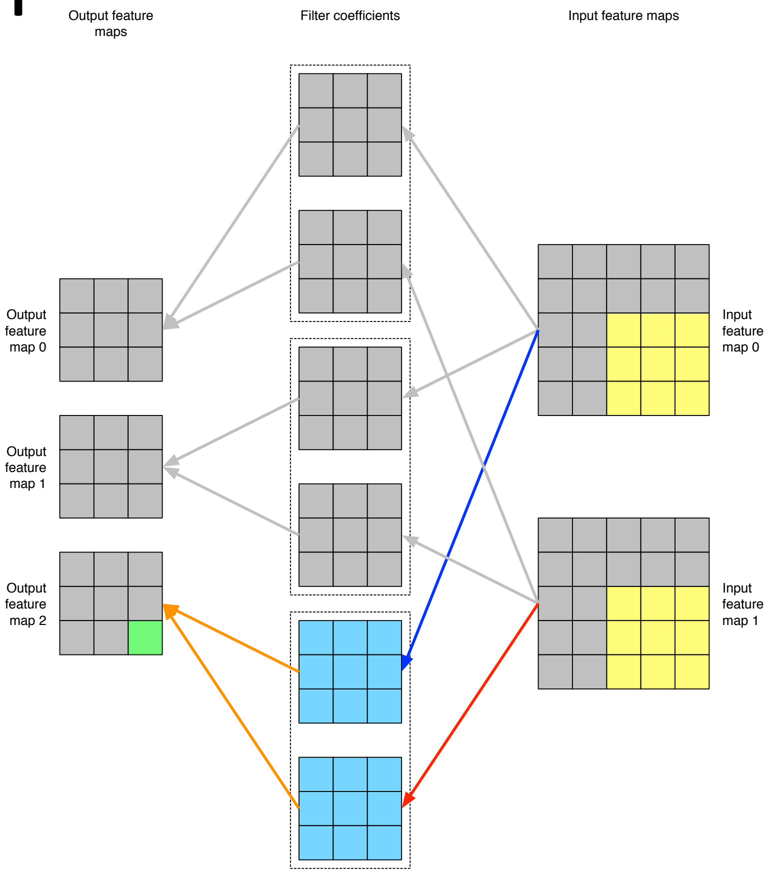


CNN Style 2D Convolution

- Mathematically it's 6 loops (listed from common outer to inner)
  - Output feature map channel $n_o = 0, ..., N_o - 1$
  - Output feature map row $m_r = 0, ..., L_r - F_r = M_r - 1$
  - Output feature map col $m_c = 0, ..., L_c - F_c = M_c - 1$
  - Input feature map channel $n_i = 0, ..., N_i - 1$
  - Filter row $f_r = 0, ..., F_r - 1$
  - Filter col $f_c = 0, ..., F_c - 1$
- For each n_o, m_r and m_c

$$Y(n_o, m_r, m_c) = \Sigma_{ni} \Sigma_{fr} \Sigma_{fc} H(n_o, n_i, f_r, f_c) X(n_i, m_r + f_r, m_c + f_c)$$

CNN Style 2D Convolution

- For each n_o, m_r and m_c

$$Y(n_o, m_r, m_c) = \Sigma_{ni}\ \Sigma_{fr}\ \Sigma_{fc}\ H(n_o, n_i, f_r, f_c)\ X(n_i, m_r + f_r, m_c + f_c)$$

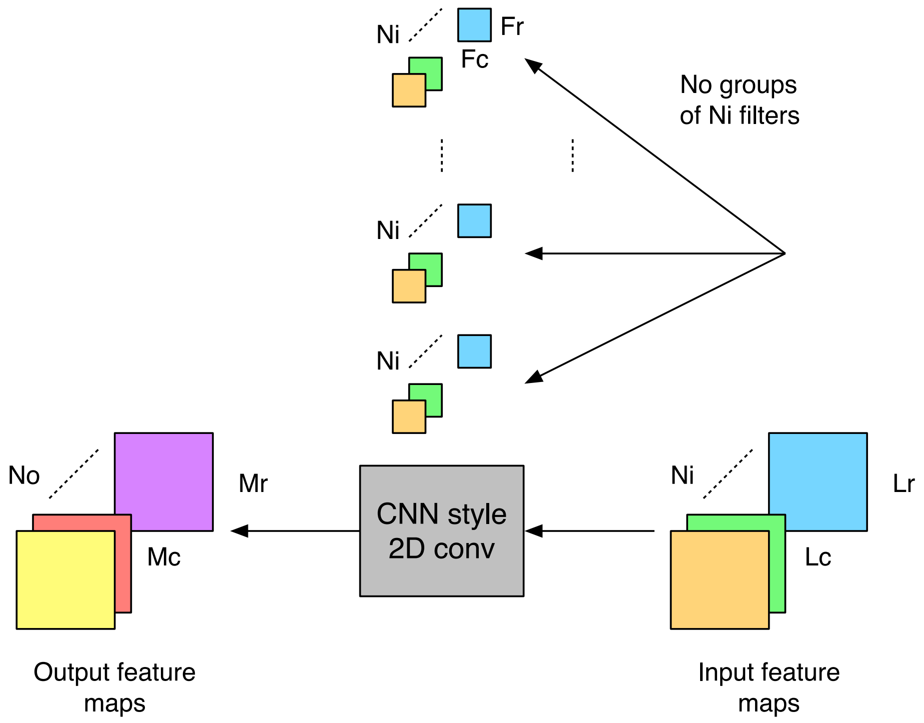
- Can be viewed as an inner product (by expanding the summations)
  - Of a vector formed from N_i F_r F_c filter coefficients
  - With a vector formed from F_r F_c elements of each N_i input feature maps
  - To produce a single output at the corresponding row col of an output feature map
- Repeated
  - For all row col values of the output feature map using the same filter coefficients
  - For all output feature map channels using different filter cxs for each output feature map channel
- Using 2D correlation instead of 2D convolution
  - Which is equivalent with a flip of the filter and indexing change

CNN Style 2D Convolution

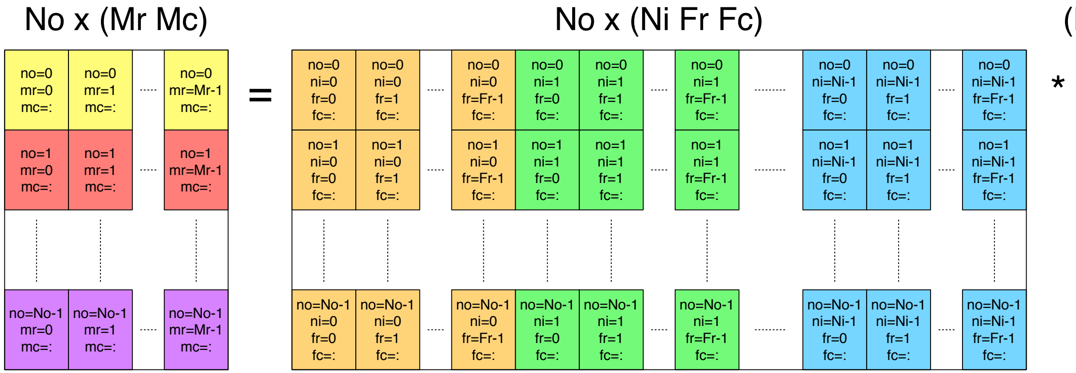
- High performance implementations of CNN style 2D convolution do not explicitly use 6 loops (but compute the same thing)
- Key realization is that CNN style 2D convolution is matrix matrix multiplication: $\mathbf{Y}^{2D} = \mathbf{H}^{2D}\,\mathbf{X}^{2D}$
  - $\mathbf{H}^{2D}$ = reshape 4D filter coefficient tensor to 2D matrix
    - Trivial, nothing actually needs to be reshaped in practice
  - $\mathbf{X}^{2D}$ = form 3D input feature map tensor into 2D Toeplitz style filtering matrix
    - This is the key
    - Will generate blocks of this on the fly as each input is repeated ~ F_r F_c times
  - $\mathbf{Y}^{2D}$ = compute 2D matrix of output feature maps
    - Matrix matrix multiplication is efficient on hardware
    - Trivial to reshape to 3D output feature map tensor, nothing actually needs to be done in practice

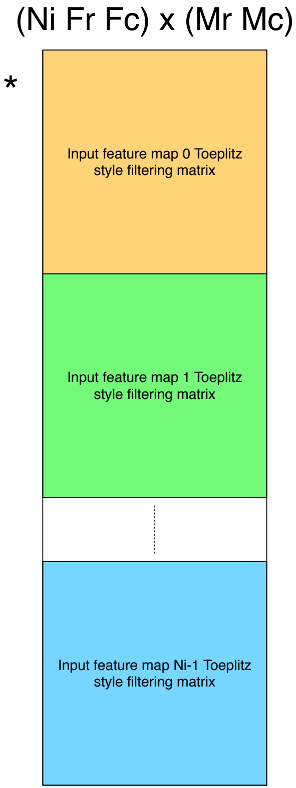
CNN Style 2D Convolution

- Starting point / reminder
- Input feature maps
  - 3D tensor
  - N_i inputs x L_r rows x L_c cols
- Filter coefficients
  - 4D tensor
  - N_o outputs x N_i inputs x F_r rows x F_c cols
- Output feature maps
  - 3D tensor
  - N_o outputs x M_r rows x M_c cols



CNN Style 2D Convolution

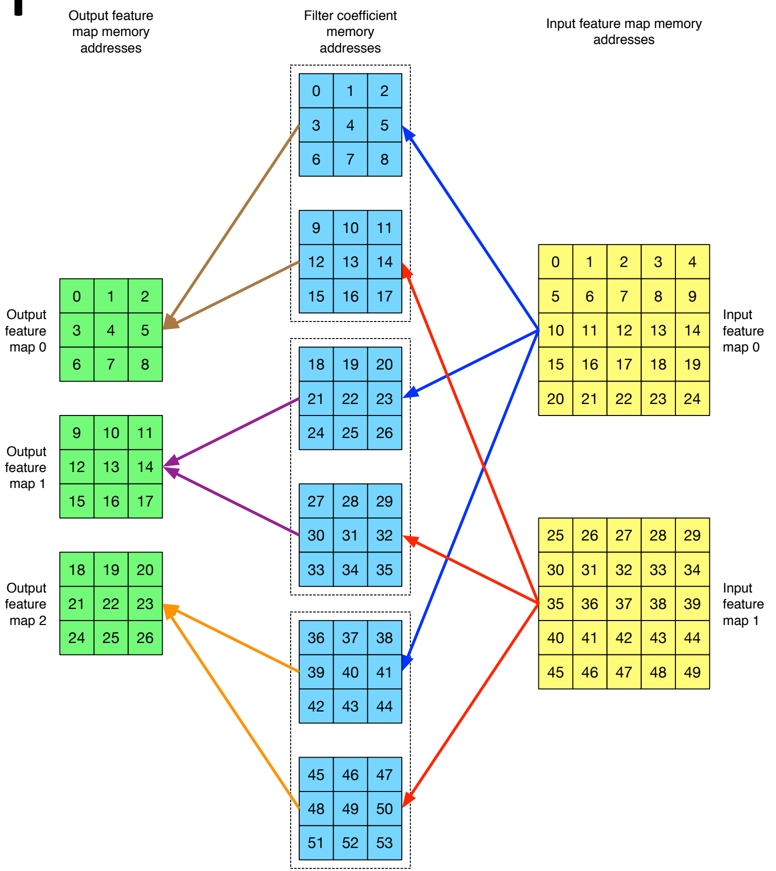




- CNN style 2D convolution written as matrix matrix multiplication
  - Output feature maps (each box is 1 x M_c elements)
  - Filter coefficients (each box is 1 x F_c elements)
  - Input feature maps (ordering not shown)

CNN Style 2D Convolution

- An example showing CNN style 2D convolution is matrix matrix multiplication using 2 x 5 x 5 input feature maps, 3 x 2 x 3 x 3 filters and 3 x 3 x 3 output feature maps
- This figure illustrates memory addresses (specifically offsets to the initial pointer for each array)
- The next page shows where the memory addresses go in matrix matrix multiplication



CNN Style 2D Convolution

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26

Output feature map memory addresses (note vectorization)

=

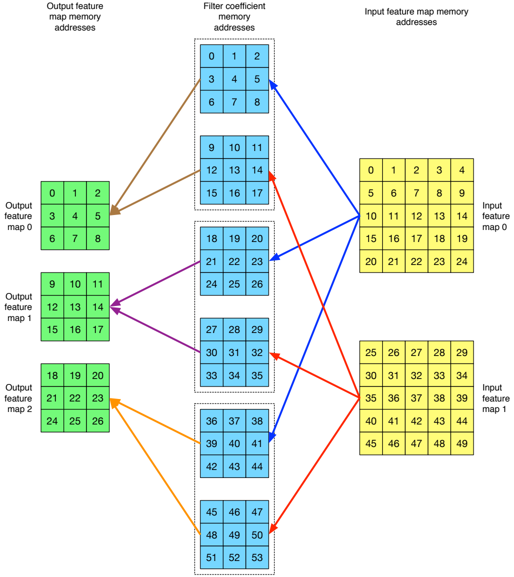
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53

Filter coefficient memory addresses (note vectorization)

0	1	2	5	6	7	10	11	12
1	2	3	6	7	8	11	12	13
2	3	4	7	8	9	12	13	14
5	6	7	10	11	12	15	16	17
6	7	8	11	12	13	16	17	18
7	8	9	12	13	14	17	18	19
10	11	12	15	16	17	20	21	22
11	12	13	16	17	18	21	22	23
12	13	14	17	18	19	22	23	24
25	26	27	30	31	32	35	36	37
26	27	28	31	32	33	36	37	38
27	28	29	32	33	34	37	38	39
30	31	32	35	36	37	40	41	42
31	32	33	36	37	38	41	42	43
32	33	34	37	38	39	42	43	44
35	36	37	40	41	42	45	46	47
36	37	38	41	42	43	46	47	48
37	38	39	42	43	44	47	48	49

Input feature map memory addresses (note Toeplitz filtering matrix structure)

- Main figure is matrix form
- Small figure is convolution form from previous page for reference



CNN Style 2D Convolution

- Limiting cases illustrated via depth wise separable convolution
  - Depth wise separable convolution splits a CNN style 2D conv layer into 2 layers
    - Traditional 2D convolution
    - CNN style 2D convolution with 1 x 1 filters
    - Less generality of either vs original, but 1 extra level of depth
  - Traditional 2D convolution to mix across space ($N_i = N_o = 1$)
    - Can also get small values of N_i and N_o via grouping
    - Equivalent to vector matrix multiplication
    - Note that K dimension reduces from (N_i F_r F_c) to (F_r F_c)
  - CNN style 2D convolution with 1 x 1 filters to mix across channel
    - Equivalent to standard matrix matrix multiplication
    - Note that K dimension reduces from (N_i F_r F_c) to N_i

CNN Style 2D Convolution

0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26

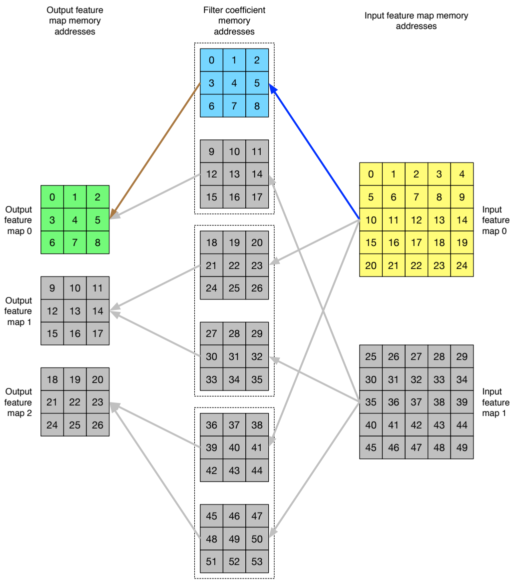
Output feature map memory addresses
(note vectorization)

=

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53

Filter coefficient memory addresses
(note vectorization)

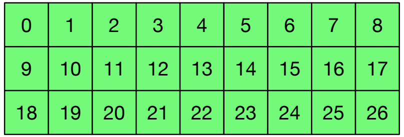
- Traditional convolution
- Equivalent to vector matrix multiplication



Input feature map memory addresses (note Toeplitz filtering matrix structure)

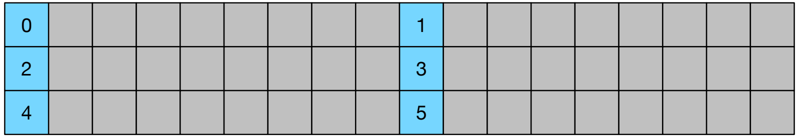
0	1	2	5	6	7	10	11	12
1	2	3	6	7	8	11	12	13
2	3	4	7	8	9	12	13	14
5	6	7	10	11	12	15	16	17
6	7	8	11	12	13	16	17	18
7	8	9	12	13	14	17	18	19
10	11	12	15	16	17	20	21	22
11	12	13	16	17	18	21	22	23
12	13	14	17	18	19	22	23	24
25	26	27	30	31	32	35	36	37
26	27	28	31	32	33	36	37	38
27	28	29	32	33	34	37	38	39
30	31	32	35	36	37	40	41	42
31	32	33	36	37	38	41	42	43
32	33	34	37	38	39	42	43	44
35	36	37	40	41	42	45	46	47
36	37	38	41	42	43	46	47	48
37	38	39	42	43	44	47	48	49

CNN Style 2D Convolution

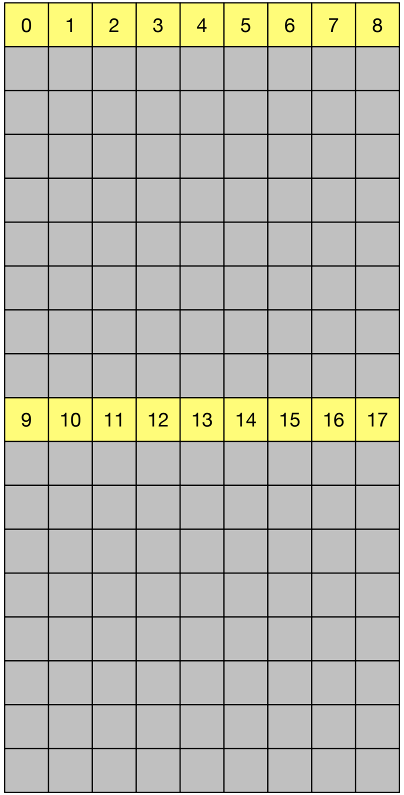


Output feature map memory addresses
(note vectorization)

=

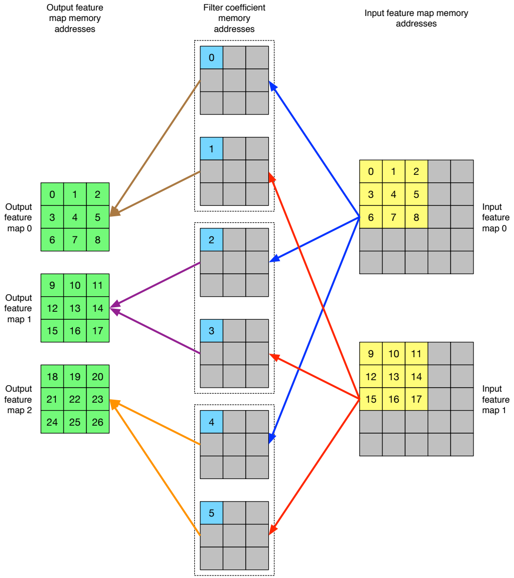


Filter coefficient memory addresses
(note vectorization)



Input feature map memory addresses (note Toeplitz filtering matrix structure)

- CNN style convolution with 1x1 filters
- Equivalent to pure matrix matrix multiplication



CNN Style 2D Convolution

- This is important

- Intuition of feature extraction
  - CNN style 2D convolution is a linear transformation
  - Output feature maps matrix = filter coefficient matrix * input feature maps filtering matrix
  - Matrix vector multiplication as used in a fully connected layer of a neural network had the intuition of matching features to inputs over channel
  - For CNN style 2D convolution have the intuition of matching features to inputs over channel and space
    - How far can it see in space? For 1 layer? For repeated layers?
    - How many features does it work over?

CNN Style 2D Convolution

- Intuition of bias
    - Add a constant to all elements in an output feature map
    - Can be a different constant for each output feature map
    - Affine transformation
    - Allows the dividing line to shift
    - Implementation using a rank 1 outer product

- Intuition of ReLU
    - Removes negatively aligned features or predictions
    - Allows depth
    - Subsequent layers combine positively aligned extracted features

CNN Style 2D Convolution

- Memory
  - Formulas
    - Input feature maps: $N_i\ L_r\ L_c$
    - Output feature maps: $N_o\ M_r\ M_c$
    - Filter coefficients: $N_i\ N_o\ F_r\ F_c$
  - Early in the network feature map memory tends to dominate
  - Deeper in the network filter coefficient memory tends to dominate

CNN Style 2D Convolution

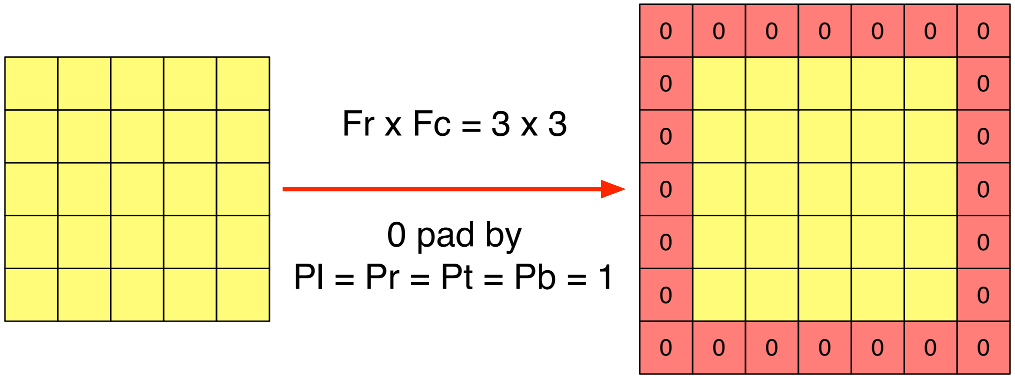
- Compute
  - Formulas
    - $(N_o)\ (M_r\ M_c)\ (N_i\ F_r\ F_c)$ (MACs)
      $= (N_o\ M_r\ M_c)\ (N_i\ F_r\ F_c)$
      = (number of outputs) (number of input MACs per output)
  - Tends to be highest in the beginning of the network
    - If $(M_r\ M_c)$ is more aggressively reduced than $(N_i\ N_o)$ is increased
  - Scaling the input size by 1/2 in rows and cols ~ reduces compute by 1/4

CNN Style 2D Convolution

- Arithmetic intensity
  - Compute = $N_i N_o F_r F_c M_r M_c$ (MACs)
  - Data movement = $N_i L_r L_c + N_o M_r M_c + N_i N_o F_r F_c$ (elements)
  - Ratio = compute / data movement

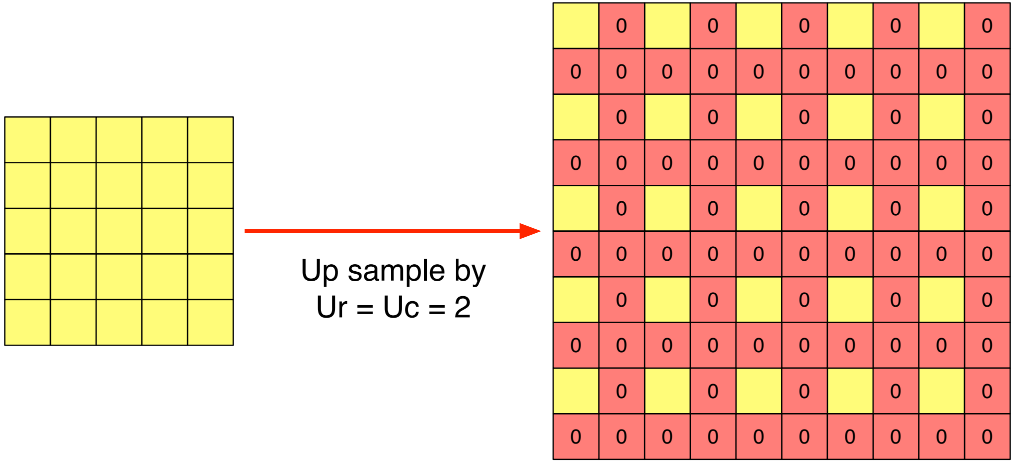
CNN Style 2D Convolution

- Variant: input feature map 0 padding
  - P_l left, P_r right, P_t top, P_b bottom
  - Typically $P_l + P_r = F_c – 1$ and $P_t + P_b = F_r – 1$
  - Used for same size input / output feature maps
  - Implementation key is efficient 0 insert



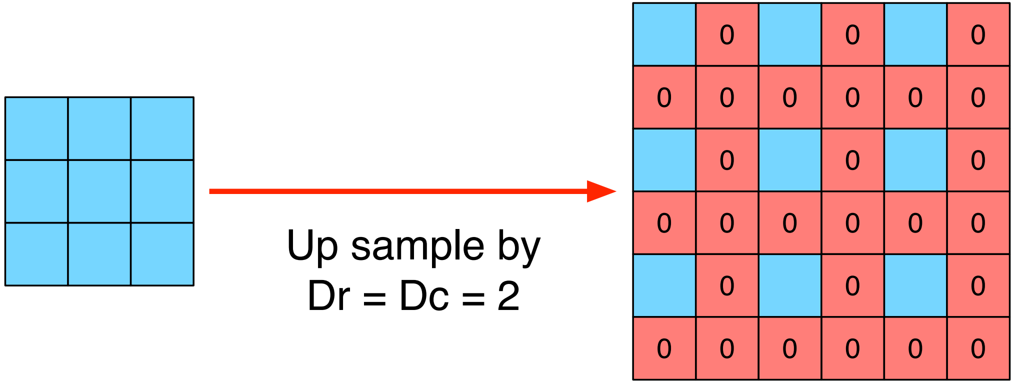
CNN Style 2D Convolution

- Variants: input feature map up sampling
  - U_r rows, U_c cols
  - Typically called deconvolution
  - Used in decoder style head designs
  - Implementation key is input memory reuse
  - Alternatives are bilinear and nearest neighbor interpolation



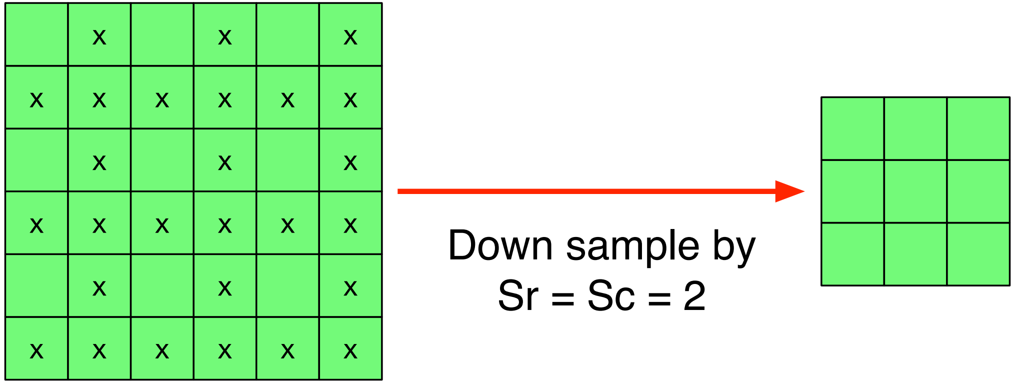
CNN Style 2D Convolution

- Variants: filter coefficient up sampling
  - D_r rows, D_c cols
  - Typically called dilated or Atrous convolution
  - Used to maintain spatial resolution with large receptive field
  - Implementation key is input feature map filtering matrix row removal



CNN Style 2D Convolution

- Variants: output feature map down sampling
  - S_r rows, S_c cols
  - Typically called strided convolution
  - Used to reduce spatial resolution
  - Implementation key is input feature map filtering matrix column removal
  - Alternative is pooling



CNN Style 2D Convolution

To do: add a pic showing this in graph format to connect to back propagation

- Alternate view of CNN style 2D convolution
  - Add together Fr x Fc CNN style 2D convolutions with 1x1 filter size
    - Reminder: CNN style convolution with 1x1 filters is pure matrix matrix multiplication
    - Input size is reduced to Mr x Mc for each with an appropriate shift / offset of the original input
  - Tradeoffs
    - Advantage of simpler input feature map matrix structure and associated data movement logic
    - Drawback of additional input feature map memory movement
  - Side benefit: useful for understanding back propagation through a CNN style 2D convolution layer

Linear Pre Processing

Discrete Fourier Transform

- A linear transformation from domain to 1/domain via a projection onto a complex exponential basis
  - Example: time to 1/time = frequency
  - Index ranges
    - k = 0, ..., K – 1
    - n = 0, ..., K – 1

$$y(k) = (1/\mathrm{sqrt}(K)) \Sigma_n x(n) e^{-i(2\pi/K)nk}$$

Discrete Fourier Transform

- Can be written as a K x K DFT matrix $\mathbf{F}_K$ that transforms input vectors **x** to output vectors **y**
  - $\mathbf{y} = \mathbf{F}_K \mathbf{x}$ where $F_K(a, b) = (1/\text{sqrt}(K))\ e^{-i(2\pi/K)ab}$
  - $\mathbf{F}_K$ is a unitary matrix so it's invertible (conj transpose = inverse, called the IDFT)
  - Output is typically circular complex Gaussian (will discuss implications later)
  - Fast implementations possible
    - O(K log K) for fast Fourier transform (FFT)
    - Vs $O(K^2)$ for DFT

Discrete Fourier Transform

- Data transformation
  - Sometimes it's easier to do feature extraction in the frequency domain vs time domain
    - Common example of this is speech to text
  - Unitary so invertible (no information lost (until you read the next slide))
    - Effectively lets the network decide what data to keep and what data to throw away

Discrete Fourier Transform

- Dimensionality reduction
  - The DFT frequently concentrates the majority of information in naturally occurring signals to L < K basis components
  - A common dimensionality reduction strategy is to keep the L main components and get rid of the rest
  - General comment: often want to keep at much information as possible while getting of as much data as possible
    - Difficulty: knowing that it’s really ok to get rid of data

Principal Component Analysis

- Note
  - A little bit of this is dependent on probability for parts of the understanding

- Setup
  - M x K data matrix **X**
  - Each row is a different trial (ex: point in time)
  - Each column is a different measurement from that trial (ex: different stock)
  - Columns are normalized to 0 mean
  - Columns are potentially linearly correlated

Principal Component Analysis

- Goal
  - Linearly transform to a new M x K matrix **Y** via a K x K matrix **Q**

$$\mathbf{Y} = \mathbf{X}\,\mathbf{Q}$$

  - Where **Q** is chosen such that columns of **Y** are orthogonal and ordered from largest to smallest variance
  - For dimensionality reduction keep first L < K columns

Principal Component Analysis

- Mechanics for finding **Q**
  - Note that $\mathbf{X} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$ via SVD
    - **U** is a M x M orthogonal matrix of eigenvectors of $\mathbf{X}\,\mathbf{X}^T$
      - An eigenvector **v** of matrix **A** is a nonzero vector **v** that satisfies $\mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v}$
      - Intuition of matrix scaling eigenvectors inputs by the eigenvalue λ
      - Side note: multicarrier modulation exploits this
    - **S** is a M x K diagonal matrix of singular values
    - $\mathbf{V}^T$ is a K x K orthogonal matrix of eigenvectors of $\mathbf{X}^T\,\mathbf{X}$
  - Select $\mathbf{Q} = \mathbf{V}$

$$\mathbf{Y} = \mathbf{X}\,\mathbf{Q} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T\,\mathbf{V} = \mathbf{U}\,\mathbf{S}$$

Principal Component Analysis

- Example
  - Statistical arbitrage (e.g., SPY, MDY and IJR)
  - Stock 0 time series in col 0, stock 1 time series in col 1, …, stock K – 1 time series in col K – 1
  - 0 th principal component for trend trading (you would keep this for feature extract)
  - K – 1 th principal component for stat arb (throw away for feature extract)

- Project idea (while on the topic of finance)
  - Information extraction project
  - Prediction of the pmf of various stocks / ETFs at different times into the future
  - Show that it's possible testing out of sample and taking transaction friction into account to profit

References

List

- Convolutional neural networks: theory, implementation and application (chapter 3 linear algebra)
  - https://github.com/arthurredfern/UT-Dallas-CS-6301-CNNs/blob/master/References/ConvolutionalNeuralNetworks.pdf
- Linear algebra
  - https://www.math.ucdavis.edu/~linear/linear-guest.pdf
- A guide to convolution arithmetic for deep learning
  - https://arxiv.org/abs/1603.07285