These slides have not yet been updated for the Spring 2019 semester

Probability

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Feb 04, 2019
Feb 06, 2019

Outline

- Motivation
- Probability spaces
- Random variables
- Random processes
- Information theory
- References

Motivation

Information

- Probability is the math that describes information
  - This course uses xNNs to extract information from data
  - This course uses xNNs to generate data from information

- Examples
  - Understanding machine learning as information extraction from training data to apply to the problem of information extraction from testing data
  - Understanding the flow of information through the network and implications of network design
  - Weight initialization as the application of known information
  - Error functions to quantify how well the information extraction process worked
  - Compressing filter coefficients and feature maps towards an information bound

Probability Spaces

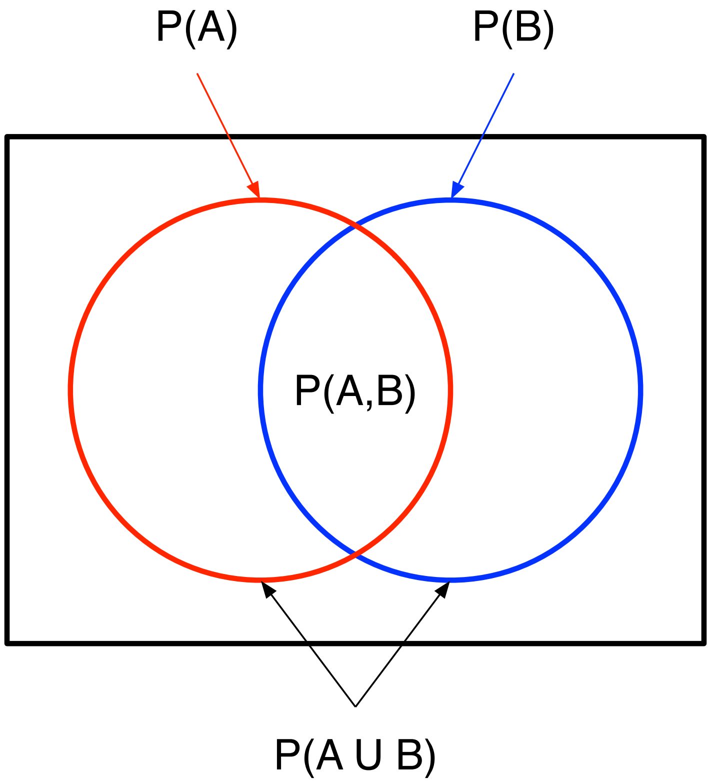
Probability Space Definition (S, E, P)

- A sample space S of all possible outcomes
  - Think: S is all possible outcomes of an experiment
  - Ex: flipping a coin 2x and recording heads (H) or tails (T) for each flip
  - S = {HH, HT, TH, TT}
- An event space E where each event is a set of 0 or more outcomes from the sample space
  - Think: E is all possible subsets of the sample space S (including nothing and everything)
  - E = { $\emptyset$, // null subset
    {HH}, {HT}, {TH}, {TT}, // all subsets of 1 outcome
    {HH, HT}, {HH, TH}, {HH, TT}, {HT, TH}, {HT, TT}, {TH, TT}, // all subsets of 2 outcomes
    {HH, HT, TH}, {HH, HT, TT}, {HH, TH, TT}, {HT, TH, TT} // all subsets of 3 outcomes
    {HH, HT, TH, TT} } // sample space subset
- A probability measure function P: E → [0, 1] that satisfies
  - $P(A) \in R$ and $P(A) \geq 0$ for all events $A \in E$
  - P(S) = 1
  - $P(\cup_i A_i) = \Sigma_i P(A_i)$ for mutually exclusive events A_i
  - Think: P is a function that assigns probabilities to subsets of the sample space

Events

- Notation
  - 1 event A, 2 events A and B
  - K events $\{A_0, \ldots, A_{K-1}\}$
- Single
  - The probability of an event occurring — $P(A) \in [0, 1]$
  - The probability of an event not occurring — $P(A^c) = 1 - P(A)$ — A^c denoting not A
- Joint
  - The probability of events A and B occurring — $P(A, B)$ — also written as $P(A \cap B)$
  - If $A \subseteq B$ — $P(A, B) = P(A)$
  - If A and B are independent — $P(A, B) = P(A)\ P(B)$
- Union
  - The probability of event A or B occurring — $P(A \cup B) = P(A) + P(B) - P(A, B)$
  - If A and B are mutually exclusive — $P(A \cup B) = P(A) + P(B)$

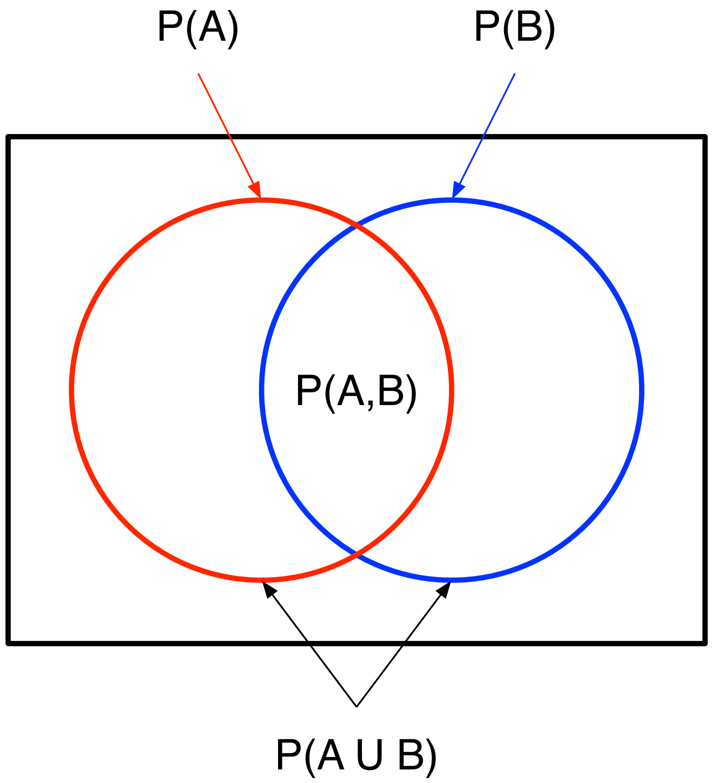
Events



Events

- Conditional
  - The probability of event A given event B $P(A|B) = P(A, B) / P(B)$
    If A and B are independent $P(A|B) = P(A)$
  - Bayes' theorem $P(A|B) = P(B|A)\, P(A) / P(B)$
  - Chain rule of probability $P(A_0, \ldots, A_{K-1})$
    $= P(A_0|A_1, \ldots, A_{K-1})P(A_1, \ldots, A_{K-1})$
  - Can recursively apply to 2nd term on RHS

Events



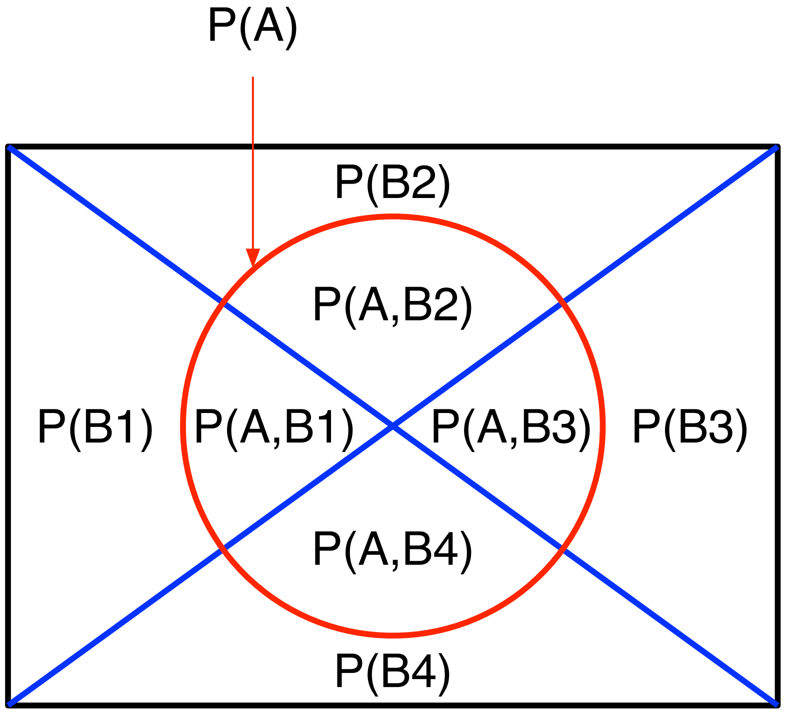
$$P(A \mid B) = P(A, B) / P(B)$$
$$P(B \mid A) = P(A, B) / P(A)$$

Events

- Law of total probability
  - Let $\{B_0, ..., B_{K-1}\}$ be a set of disjoint events whose union is the full event space
  - Let A be an event in the same event space
  - Marginal probability of A

$$P(A) = \Sigma_k P(A, B_k) = \Sigma_k P(A|B_k) P(B_k)$$

Events



$$P(A) = \text{sum}_k\ P(A, B_k)$$
$$= \text{sum}_k\ P(A \mid B_k)\ P(B_k)$$

Random Variables

Discrete

- A discrete random variable is a function X with a finite or countably infinite range that maps outcomes s from the sample space S to numbers $x \in R$

$$X(s) = x_k$$

- x_k is a realization of X
- Note that a random variable is not random and it's not a variable
  - The outcome of the experiment s is random
  - The mapping $X(s) = x_k$ by the random variable (function) to a real number is deterministic

Discrete

- A discrete random variable is described by it's probability mass function that specifies the probability that it takes on a specific value or it's cumulative distribution function that specifies the probability that it's value falls within an interval
- Probability mass function
  - Single $p_X(x_k) = P(X(s) = x_k)$
    where $\Sigma_k\, p_X(x_k) = 1$
  - Joint and conditional $p_{X,Y}(x_j, y_k) = p_{X|Y}(x_j|y_k)\, p_Y(y_k) = p_{Y|X}(y_k|x_j)\, p_X(x_j)$
  - Marginal $p_X(x_j) = \Sigma_k\, p_{X,Y}(x_j, y_k) = \Sigma_k\, p_{X|Y}(x_j|y_k)\, p_Y(y_k)$
  - Independent X and Y $p_{X,Y}(x_j, y_k) = p_X(x_j)\, p_Y(y_k)$
    $p_{X|Y}(x_j|y_k) = p_X(x_j)$
- Cumulative distribution function
  - Single $F_X(x_k) = P(X(s) \le x_k) = \Sigma_{xj \le xk}\, p_X(x_j)$

Continuous

- A continuous random variable is a function X with an uncountably infinite range that maps outcomes s from the sample space S to numbers $x \in R$

$$X(s) = x$$

- x is a realization of X
- Note that a random variable is (still) not random and it's (still) not a variable
    - The outcome of the experiment s is random
    - The mapping X(s) = x by the random variable (function) to a real number is deterministic

Continuous

- A continuous random variable is described by it's cumulative distribution function that specifies the probability that it's value falls within an interval
  - If the cumulative distribution function is absolutely continuous then it also has a probability density function (this set of slides will assume this is true so we don't have to use the word measure and weird looking integrals)
- Probability density function
  - Single: $\int_a^b p_X(x)\, dx = P(a \le X(s) \le b)$ where $\int p_X(x)\, dx = 1$
  - Joint and conditional: $p_{X,Y}(x, y) = p_{X|Y}(x|y)\, p_Y(y) = p_{Y|X}(y|x)\, p_X(x)$
  - Marginal: $p_X(x) = \int p_{X,Y}(x, y)\, dy = \int p_{X|Y}(x|y)\, p_Y(y)\, dy$
  - Independent X and Y: $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$, $p_{X|Y}(x|y) = p_X(x)$
- Cumulative distribution function
  - Single: $F_X(x) = \int^x p_X(t)\, dt$

Expected Value

- Expected value is a linear operator that maps functions of random variables to a probability weighted average of all events (shown here for a discrete random variable)

$$E[f(X(s))] = \Sigma_k \, p_X(x_k) \, f(x_k)$$

- Scalar examples
  - Mean $\mu_X = E[X(s)]$
  - Variance $\sigma_X^2 = E[(X(s) – \mu_X)^2]$
  - Standard deviation σ_X
  - nth order moment about the mean $E[(X(s) – \mu_X)^n]$
  - Covariance (units of X(s)*Y(s)) $cov(X(s), Y(s)) = E[(X(s) – \mu_X) (Y(s) – \mu_Y)] = E[X(s) Y(s)] – \mu_X \mu_Y$
  - Correlation ([-1, 1]) $corr(X(s), Y(s)) = cov(X(s), Y(s)) / (\mu_X \mu_Y)$
  - Independent X(s) and Y(s) $cov(X(s), Y(s)) = corr(X(s), Y(s)) = 0$

    $cov(X(s), Y(s)) = corr(X(s), Y(s)) = 0$ does not imply independent X(s) and Y(s)

Expected Value

- Vector examples
  - Notation $\mathbf{x} = [X_0(s), \ldots, X_{K-1}(s)]^T$
  - Mean vector $\boldsymbol{\mu}_\mathbf{x} = E[\mathbf{x}] = [E[X_0(s)], \ldots, E[X_{K-1}(s)]]^T$
- Matrix examples
  - Covariance matrix $\boldsymbol{\Sigma}_{\mathbf{x},\mathbf{x}} = E[(\mathbf{x} - \boldsymbol{\mu}_\mathbf{X}) (\mathbf{x} - \boldsymbol{\mu}_\mathbf{X})^T]$
    $\boldsymbol{\Sigma}_{\mathbf{x},\mathbf{x}}(m, k) = E[(X_m(s) - \mu_{Xm}) (X_k(s) - \mu_{Xk})]$

Expected Value

- Linear regression
  - $\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{e}$, $\mathbf{e}$ is a 0 mean vector random variable representing measurement error
  - $\mathbf{e} = \mathbf{y} - \mathbf{A}\,\mathbf{x}$
  - $\min_{\mathbf{x}} \mathbf{e}^T \mathbf{e} = (\mathbf{y} - \mathbf{A}\,\mathbf{x})^T (\mathbf{y} - \mathbf{A}\,\mathbf{x}) = \mathbf{x}^T \mathbf{A}^T \mathbf{A}\,\mathbf{x} - 2\,\mathbf{y}^T \mathbf{A}\,\mathbf{x} + \mathbf{y}^T \mathbf{y}$
  - $\mathbf{x}^{hat} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}$
- Estimator mean
  - $$\begin{aligned} E[\mathbf{x}^{hat}] &= E[(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}] \\ &= E[(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T (\mathbf{A}\,\mathbf{x} + \mathbf{e})] \\ &= E[(\mathbf{A}^T \mathbf{A})^{-1} (\mathbf{A}^T \mathbf{A})\,\mathbf{x}] + E[(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{e}] \\ &= E[\mathbf{x}] + (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T E[\mathbf{e}] \\ &= \mathbf{x} \end{aligned}$$
- Estimator covariance
  - Substitute $\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{e}$ into the $\mathbf{x}^{hat}$ formula to get $\mathbf{x}^{hat} = \mathbf{x} + (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{e}$ or $\mathbf{x}^{hat} - \mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{e}$
  - $E[(\mathbf{x}^{hat} - \mathbf{x})(\mathbf{x}^{hat} - \mathbf{x})^T] = E[((\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{e})((\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{e})^T] = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T E[\mathbf{e}\,\mathbf{e}^T]\,\mathbf{A}\,(\mathbf{A}^T \mathbf{A})^{-1} = \sigma_e^2\,(\mathbf{A}^T \mathbf{A})^{-1}$

Examples Of Discrete PMFs

- Bernoulli
  - $p_X(x_k)$ = $1 - p$, $x_k = 0$, $p \in [0, 1]$
    = p, $x_k = 1$
    = 0, elsewhere
  - Expectations
    - Mean = p
    - Variance = $p (1 - p)$
- Uniform
  - $p_X(x_k)$ = $1 / N$, $x_k \in \{a, a + 1, ..., b\}$, $b - a + 1 = N$
    = 0, elsewhere
  - Expectations
    - Mean = $(a + b) / 2$
    - Variance = $(N^2 - 1) / 12$

Examples Of Continuous PDFs

- Uniform
  - $p_X(x)$ = 1 / (b – a), $x \in [a, b]$, a and b finite
    = 0, elsewhere
  - Expectations
    - Mean = (a + b) / 2
    - Variance = $(b - a)^2 / 12$ Side note: this leads to a famous SNR formula for quantizers
- Gaussian (or normal)
  - $p_X(x) = (1 / (2\pi\sigma^2)^{1/2}) \exp(-(x - \mu_x)^2 / 2\sigma_x^2)$
  - Expectations
    - Mean = μ_x
    - Variance = σ_x^2
- xNN use: filter coefficient initialization
  - For initialization with a Gaussian distribution it's frequently truncated (limited)

Experiment

- A class generated discrete probability mass function

- Experiment
  - Think of a 2 digit number between 10 and 50
  - Both digits are odd
  - Both digits are different from each other

Experiment

- A class generated discrete probability mass function
- Experiment
  - Think of a 2 digit number between 10 and 50
  - Both digits are odd
  - Both digits are different from each other
- How many people thought of the number 37?

Normalization

- Purpose
  - Take a random variable with an arbitrary distribution and normalize it to 0 mean and unit variance
    - Note that other variations of normalization exist
  - This is used by batch norm layers in CNNs to improve training
  - Note: CNNs use the word norm and normalization a lot for different operations
    - Input data normalization (a variant of what is described here)
    - Normalization layer (operates across feature maps, famous in AlexNet, rarely used now)
    - Batch normalization layer (a variant of what is described here, used in many places to improve training, can frequently be absorbed into convolution for deployment)
    - Group normalization layer (similar purpose to batch normalization, different operation)
    - ...
- Normalization
  - $Y(s) = (X(s) - \mu_x) / \sigma_x$
  - $E[Y(s)] = E[(X(s) - \mu_x) / \sigma_x] = (1 / \sigma_x)(E[X(s)] - \mu_x) = (1 / \sigma_x)(\mu_x - \mu_x) = 0$
  - $E[(Y(s))^2] = E[((X(s) - \mu_x) / \sigma_x)^2] = (1 / \sigma_x^2) E[(X(s) - \mu_x)^2] = (1 / \sigma_x^2) \sigma_x^2 = 1$

Law Of Large Numbers

- Let $X_0(s)$, $X_1(s)$, ... be a sequence of independent identically distributed random variables with $E[X_i(s)] = \mu_X$ and let the sample average be

$$X_{0:K-1}^{bar}(s) = (X_0(s) + \ldots + X_{K-1}(s))/K$$

- $X_{0:K-1}^{bar}(s)$ converges to μ_X as $K \to \infty$
  - In probability for the weak law (unlikely outcome probability reduces as $K \to \infty$)
  - Almost surely for the strong law (pointwise)
- Variants exist that replace the independence constraint with a variance constraint
- The law of large numbers allows the expected value of a random variable with a finite mean to be estimated from it's sample average
  - Note that the sample average is a random variable

Central Limit Theorem

- The central limit theorem describes the distribution of the sample average on the previous slide (a random variable) about μ as $K \rightarrow \infty$
- Let $\{X_0(s), \ldots, X_{K-1}(s)\}$ be a set of independent identically distributed random variables each with mean μ_X and finite variance σ_X^2, then

$$K^{1/2} (X_{0:K-1}^{bar}(s) - \mu_X) \rightarrow N(0, \sigma_X^2)$$

- $N(0, \sigma^2)$ is 0 mean σ^2 variance Gaussian distribution
  - So $X_{0:K-1}^{bar}(s)$ is "close" to $N(\mu_X, \sigma_X^2 / K)$
- Convergence is in distribution (the cdf converges as $K \rightarrow \infty$)
- Variants exist that replace the independent identically distributed condition

Central Limit Theorem

- A few places where the central limit theorem sort of sometimes comes up
  - Viewing the inner product in matrix vector or matrix matrix multiplication as a weighted sum of random variables
  - Viewing the DFT operation as a (rotated) sum of random variables

- Why this matters
  - Input can have ~ arbitrary distribution, maybe nicely bounded
  - But the output of the operation starts to look Gaussian
  - Gaussian random variables have long tails
  - With finite precision arithmetic this affects accuracy

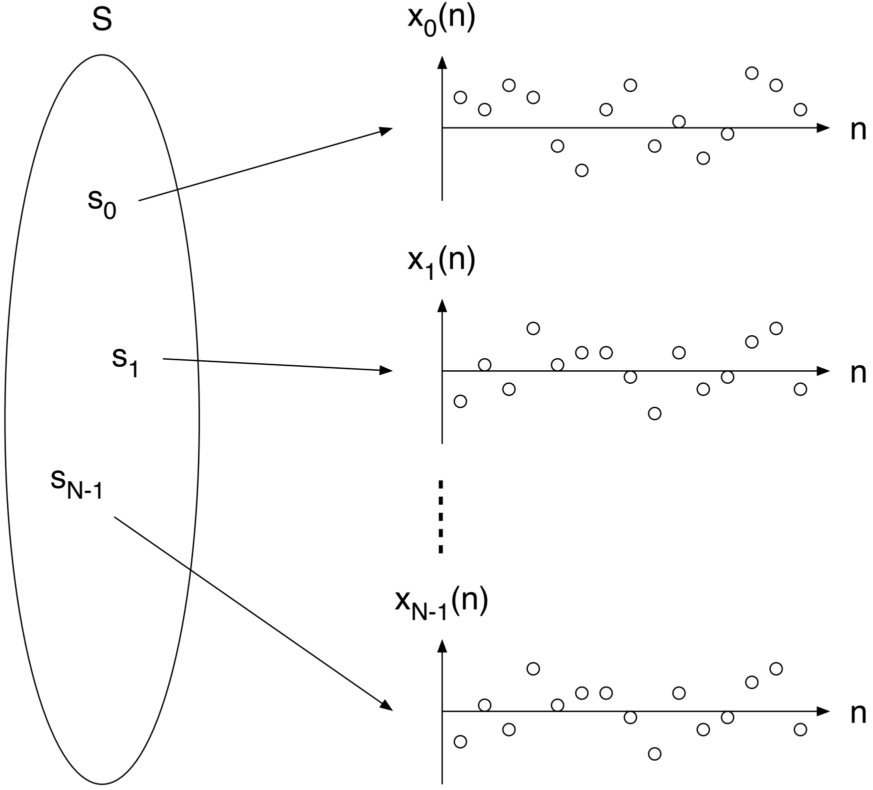
- More on precision when CNN performance and implementation is discussed

Random Processes

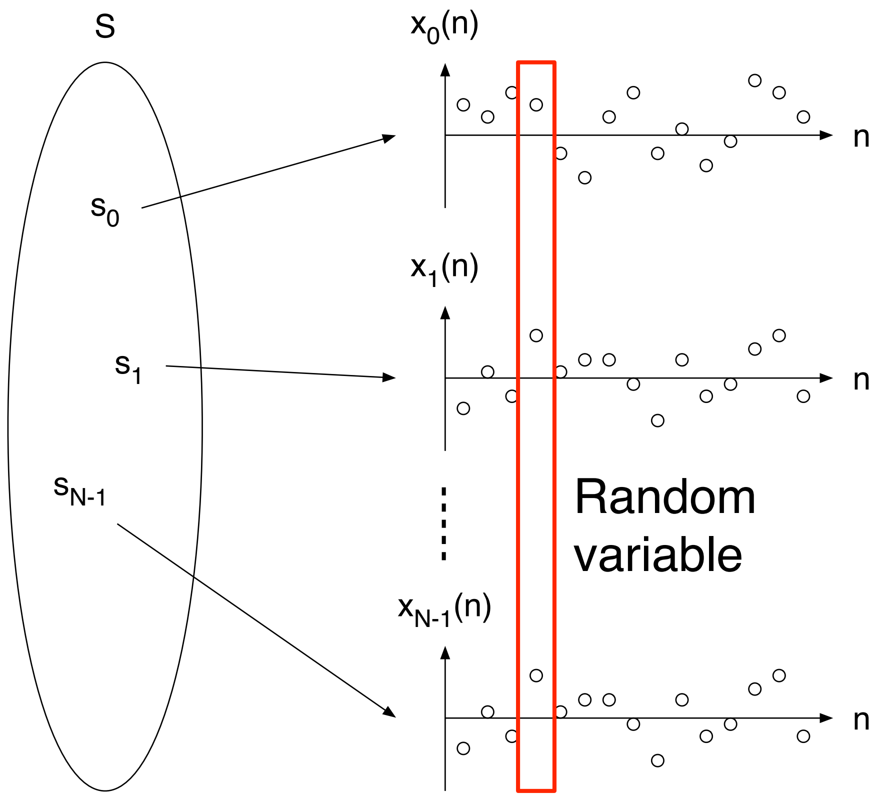
Definition

- A random process X(s, n) maps events s from the sample space S to functions x(n) where the domain of the function is the index set and the range of the function is the state space
  - X(s, n) is a random variable at a fixed n
    - By considering all times n this leads to the observation that a random process can be considered a collection of random variables $\{X(s, n_0), \ldots, X(s, n_{N-1})\}$
  - X(s, n) is a deterministic function of n for a fixed s
    - This is referred to as a realization of the random process
    - The set of all possible functions is referred to as the ensemble
  - X(s, n) is a number for a fixed s at a fixed n
- Names
  - If n refers to time then X(s, n) is called a random process
  - If n has multiple dimensions like width and height of an image then X(s, n) is called a random field

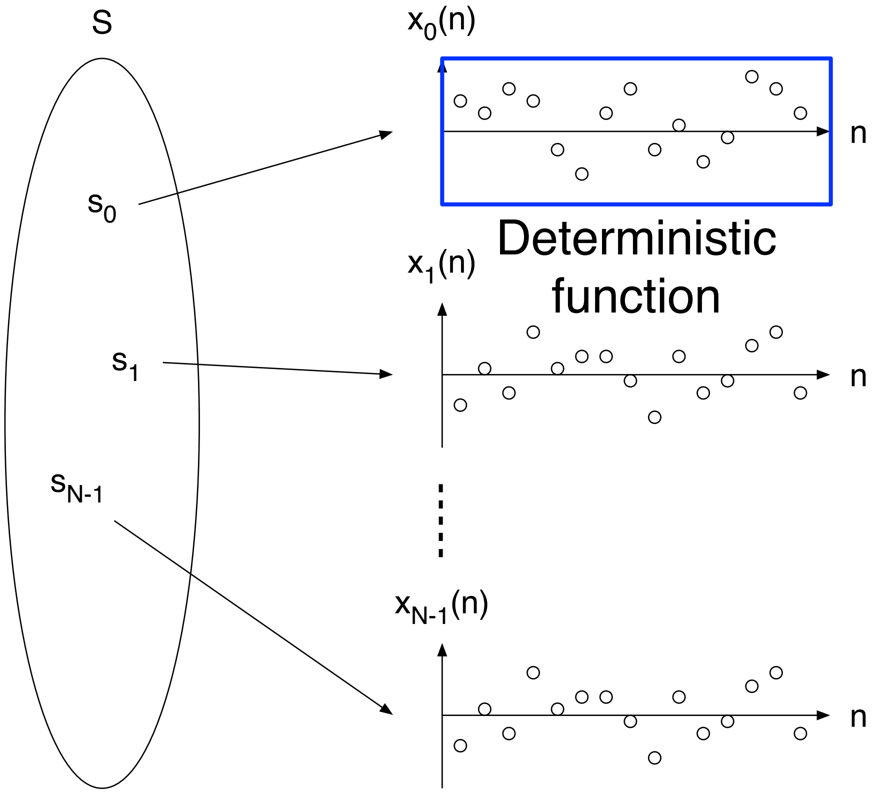
Definition



Definition



Definition



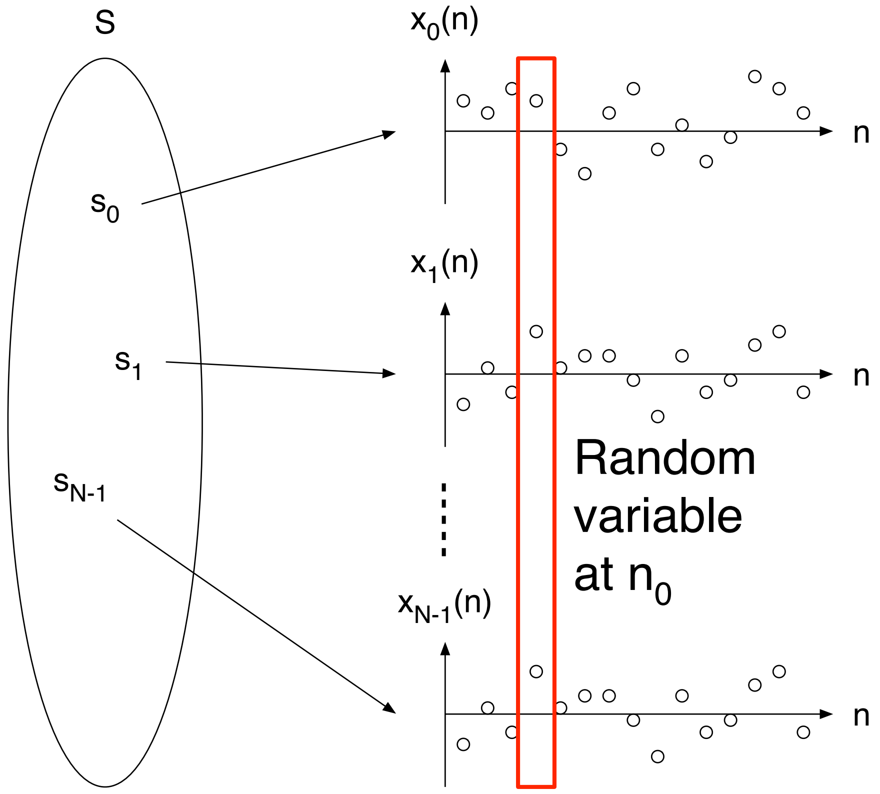
Stationarity

- Non stationary
  - Using the view of a random process as a collection of random variables, a random process is defined by is joint CDF $F_{X\_0,\ldots,X\_\{N-1\}}(x_{n\_0}, \ldots, x_{n\_(N-1)})$ which in general is a function of n_k
  - Informally, a non stationary random process has a CDF that changes with n (and doesn't fit neatly into 1 of the less restrictive stationary categorizations)
- (Strictly) stationary
  - Random processes X(s, n) for which the joint CDF does not change with time

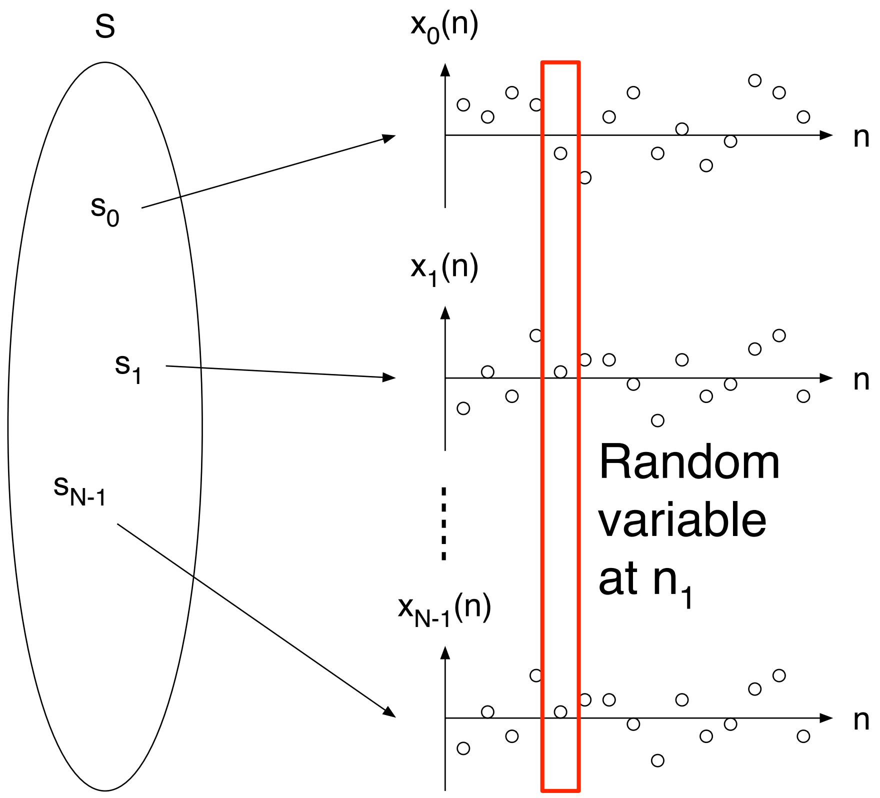
$$F_{X\_0,\ldots,X\_\{N-1\}}(x_{n\_0+\tau}, \ldots, x_{n\_(K-1)+\tau}) = F_{X\_0,\ldots,X\_\{N-1\}}(x_{n\_0}, \ldots, x_{n\_(K-1)}) \text{ for all K, n and } \tau$$

- Weakly (wide sense or second order) stationary
  - Random processes X(s, n) for which the mean and auto covariance do not change with time
  - Autocorrelation only depends on time difference $\tau = n_1 - n_2$
- Other types of stationarity exist (e.g., cyclostationary)

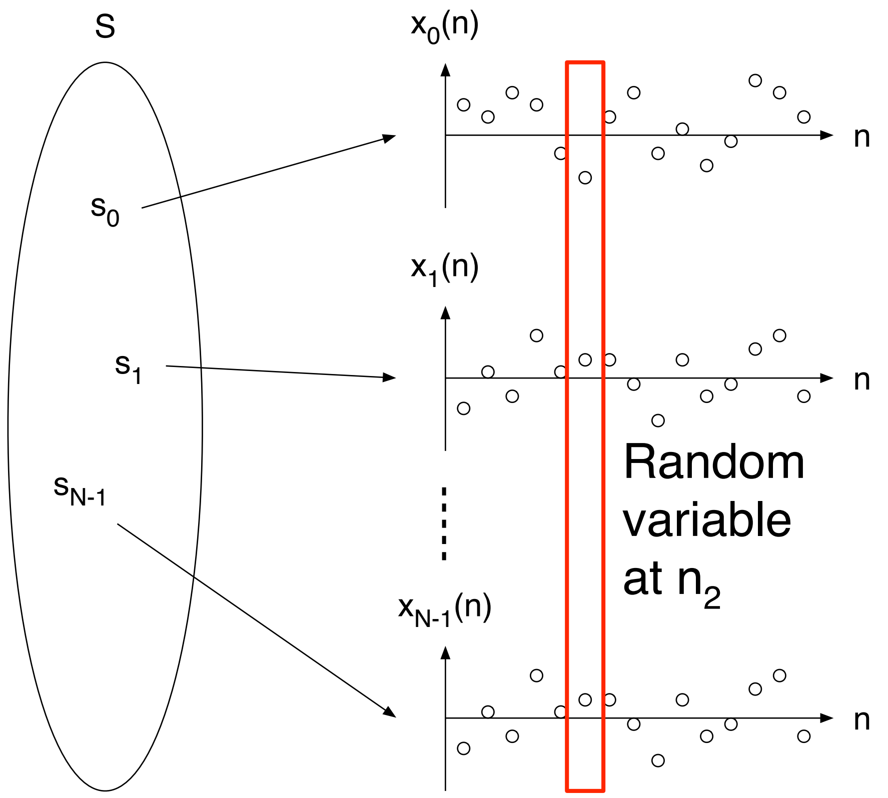
Stationarity



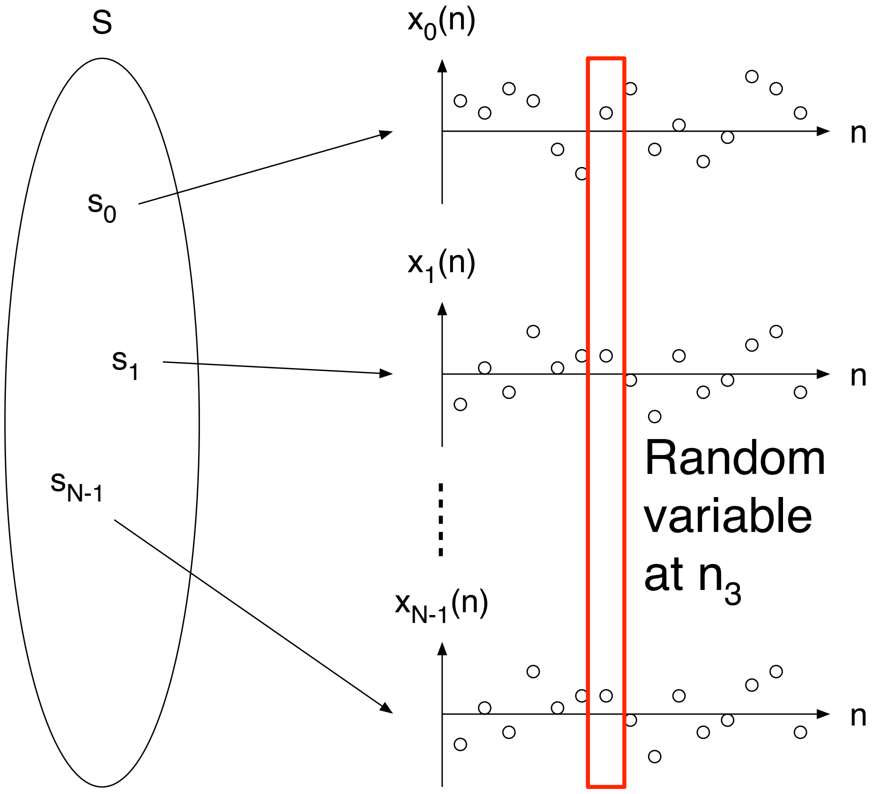
Stationarity



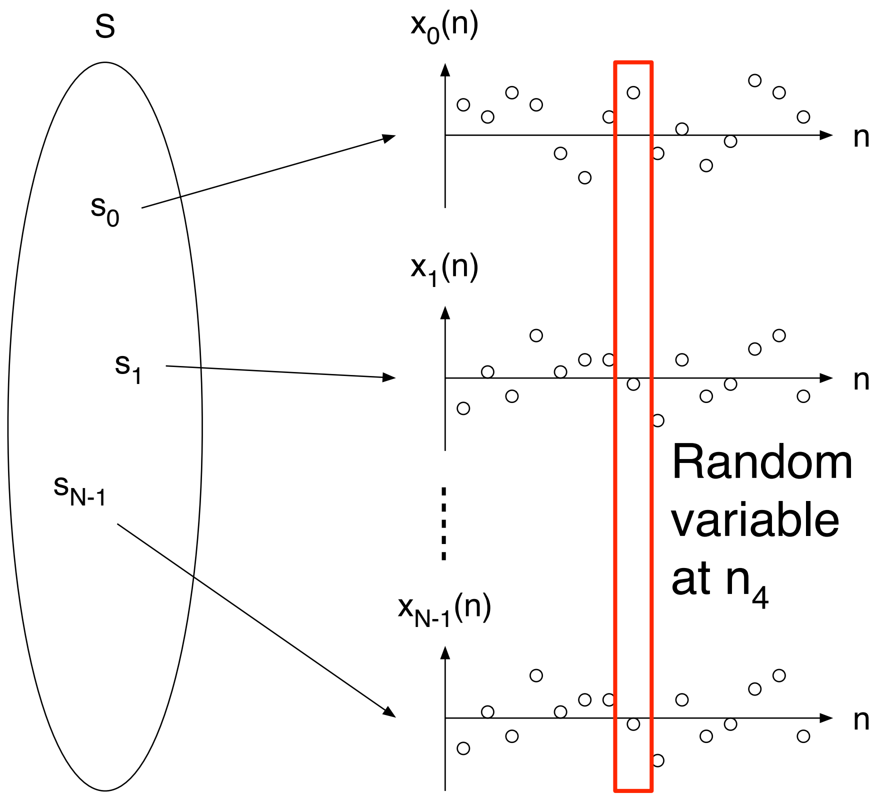
Stationarity



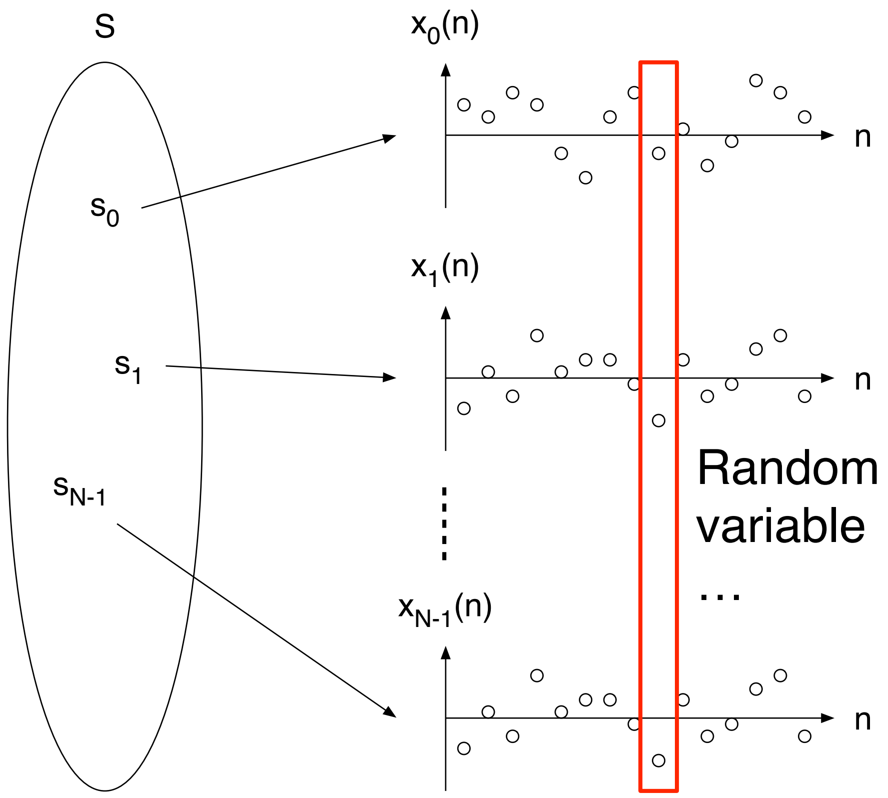
Stationarity



Stationarity



Stationarity



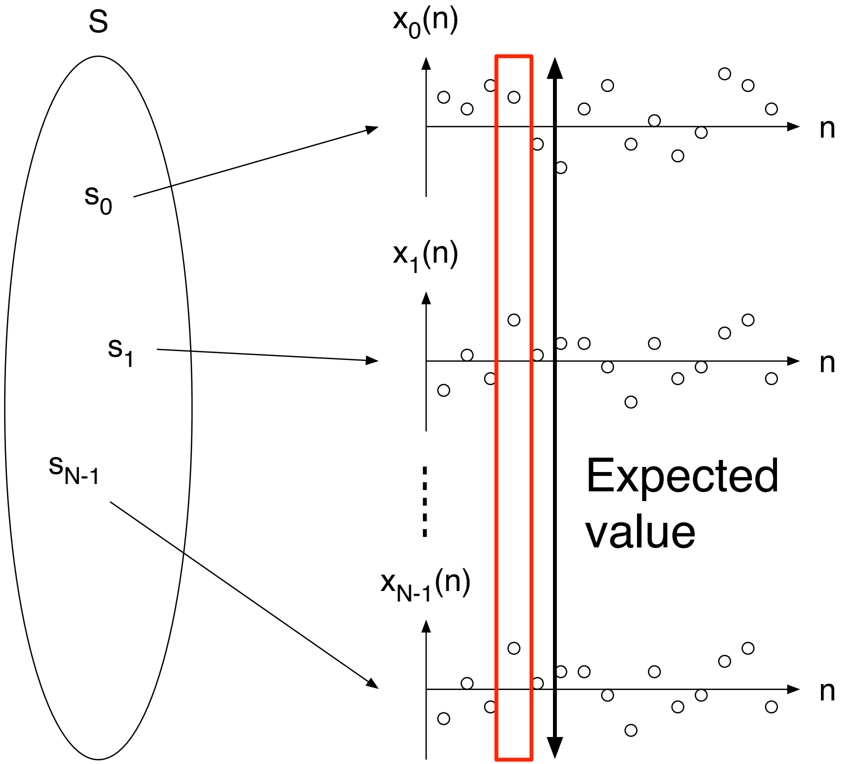
Expected Value

- The expected value of a random process is found by viewing the random process as a random variable at a fixed n and applying the expected value operator as before
  - Conceptually, it operates across many realizations s of a random process at a single n
  - Mean, variance and higher order moments are defined as in the case of a random variable
- Let $p_X(x_i, n) = P(X(s, n) = x_i)$ at a fixed n, then

$$E[f(X(s, n))] = \Sigma_i \; p_X(x_i, n) \; f(x_i)$$

- Which in general is a function of n

Expected Value

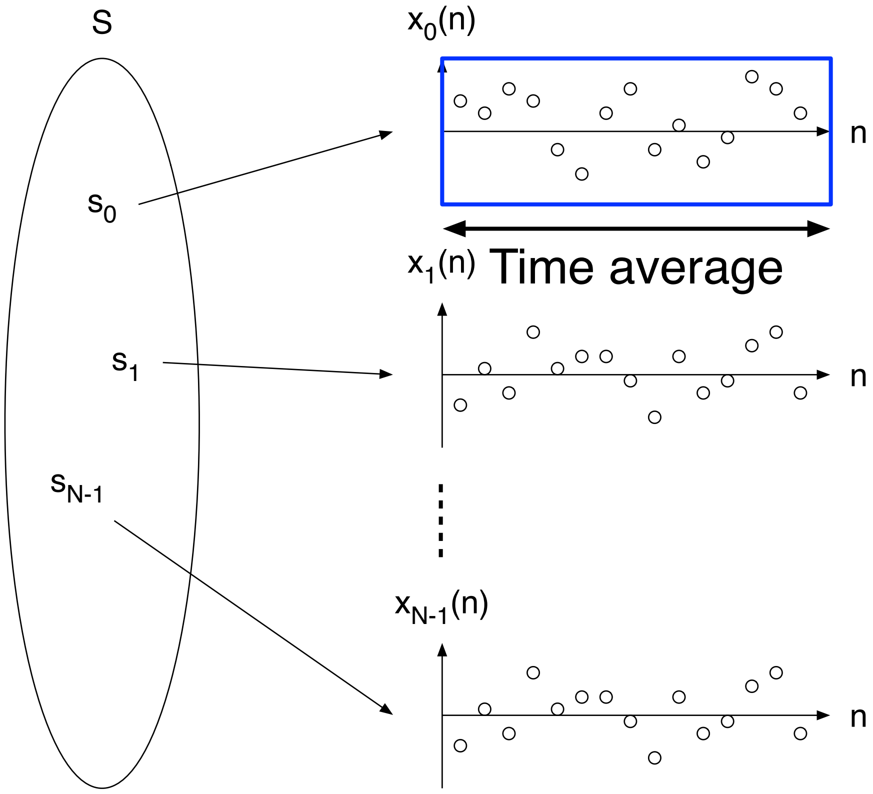


Time Average

- The time average of a random process is found by viewing the random process as a deterministic function for a fixed s and applying the time average operator
  - Conceptually, it operates across 1 realization s of a random process at many points n
  - Different time averages are defined similar to the expected value of a random variable
  - The time average itself is a random variable as it depends on the chosen s

$$\langle f(X(s, n)) \rangle = 1/N\ \Sigma_n\ f(X(s, n))$$

Time Average



Ergodicity

- Ergodicity: when time averages converge to expectations
  - In some sense (e.g., mean square)
  - For some orders of moments for which the process is stationary

- Example: mean ergodic
  - $\langle X(s, n) \rangle$ converges in the mean and in the mean square sense to $E[X(s, n)]$
    - $\lim_{N \to \infty} E[\ ((1/N\ \Sigma_n\ f(X(s, n))) - \mu_X)\ ] = 0$
    - $\lim_{N \to \infty} E[\ ((1/N\ \Sigma_n\ f(X(s, n))) - \mu_X)^2\ ] = 0$

Information Theory

1 Word Definition

- Information is surprise

Before Formalities

- How many fingers do you think an alien has on their hand?
  - My favorite question in Cover and Thomas' book Elements of Information Theory
- Why do slides with lots of equations on them have 0 information during their presentation?
- Claude Shannon and communication system design
  - Inner and outer encoders and decoders with a noisy channel in the middle
  - Remove redundancy for compression, add redundancy for coding
  - Linear algebra, calculus and probability

Entropy

- Purpose
  - A way to mathematically quantify information

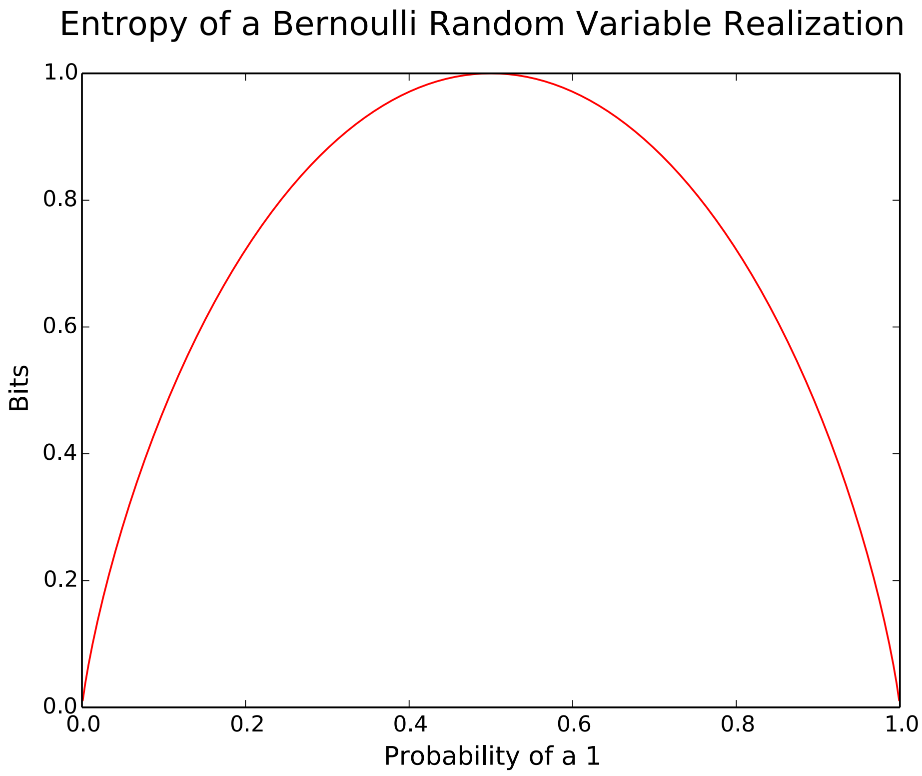
- Example
  - Consider a 1 bit message that can take on 2 values $x_k = \{0, 1\}$
    - Re: Bernoulli random variable
  - A transmitter sends a message to a receiver containing 1 bit of data
  - How much information is contained in the message?
    - If $p_X(0) = 1$ and $x_k = 0$ is received? Is there any surprise?
    - If $p_X(1) = 1$ and $x_k = 1$ is received? Is there any surprise?
    - If $p_X(0) = p_X(1) = 0.5$ and $x_k = 0$ or $x_k = 1$ is received? Is there any surprise?
    - Some other $p_X(0) = 1 – p$, $p_X(1) = p$ split?

Entropy

Note: this definition and most subsequent slides will consider entropy in the context of discrete random variables

- Definition
  - Informally, entropy is the information in a realization of a random variable
  - $H(X(s)) = -\Sigma_k\, p_X(x_k) \log_2(p_X(x_k))$
  - Units of bits because of log base 2 choice
- Revisiting the example on the previous slide
  - p: $H(X(s)) = -((1-p)\log_2(1-p)) - (p \log_2(p))$, general formula for example
  - p = 0.0: $H(X(s)) = -1 \log_2(1) = 0$ bits, no surprise, no information
  - p = 1.0: $H(X(s)) = -1 \log_2(1) = 0$ bits, no surprise, no information
  - p = 0.5: $H(X(s)) = -0.5 \log_2(0.5) - 0.5 \log_2(0.5) = 1$ bit, max information
- Information and data are not the same thing
  - In the example there was always 1 bit of data
  - But the information H(X(s)) varied based on the value of p (more generally the PMF)

Entropy


Entropy of a Bernoulli Random Variable Realization
Bits
Probability of a 1
0.0
0.2
0.4
0.6
0.8
1.0

Entropy

- What distribution maximizes entropy under what constraints
  - $x_k \in \{a, a + 1, ..., b\}$: discrete uniform distribution
  - $x \in [a, b]$: continuous uniform distribution
  - $x \in (-\infty, \infty)$, $E[X(s)] = \mu_x$, $E[(X(s) - \mu_x)^2] = \sigma_x^2$: Gaussian distribution with mean μ_x and variance σ_x^2

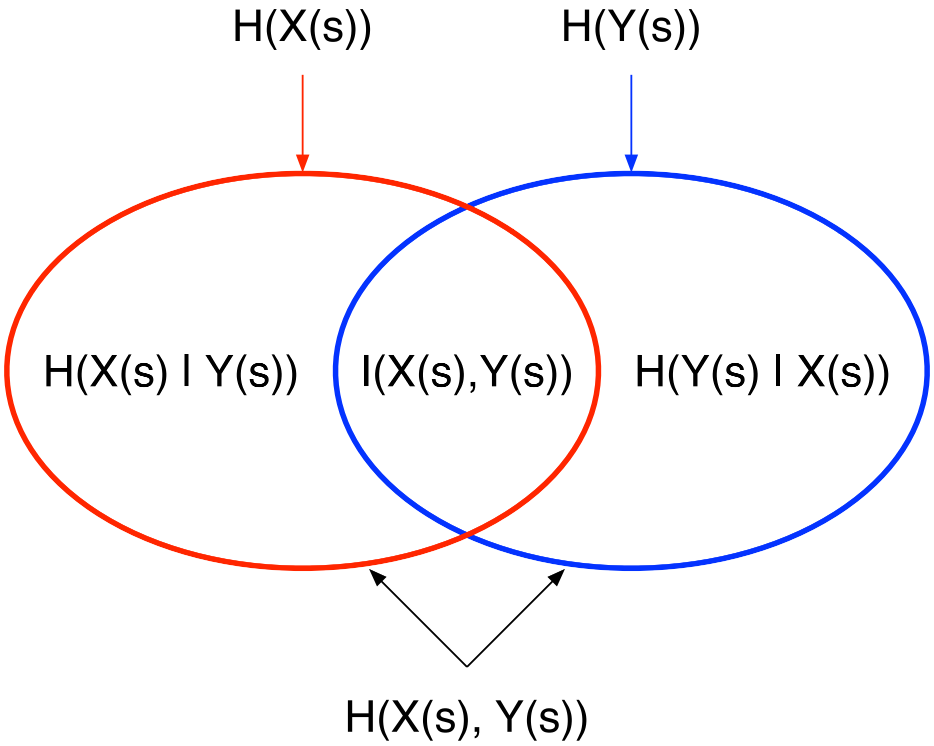
- For most success stories of CNNs, the input to the network is not an entropy maximizing distribution
  - Actually, it's just the opposite
  - And that's a good thing that's as it's implicitly exploited by the network
    - Natural images have a certain look to them
    - Human voice has a certain tone to it
    - Language has a certain structure to it
    - ...
  - Think of it from the perspective of a network doing function approximation
    - Having a smaller domain to map to a finite set makes the mapping easier

Joint Entropy

- Definition
  - Informally, the information in a realization of 2 random variables
  - $H(X(s), Y(s)) = -\Sigma_j\Sigma_k\ p_{X,Y}(x_j, y_k)\ \log_2(p_{X,Y}(x_j, y_k))$

- Properties
  - Symmetry $H(X(s), Y(s)) = H(Y(s), X(s))$
  - Greater than or equal to largest $H(X(s), Y(s)) \geq \max\{H(X(s)), H(Y(s))\}$
  - Less than or equal to sum $H(X(s), Y(s)) \leq H(X(s)) + H(Y(s))$
  - Independent X(s) and Y(s) $H(X(s), Y(s)) = H(X(s)) + H(Y(s))$

Joint Entropy

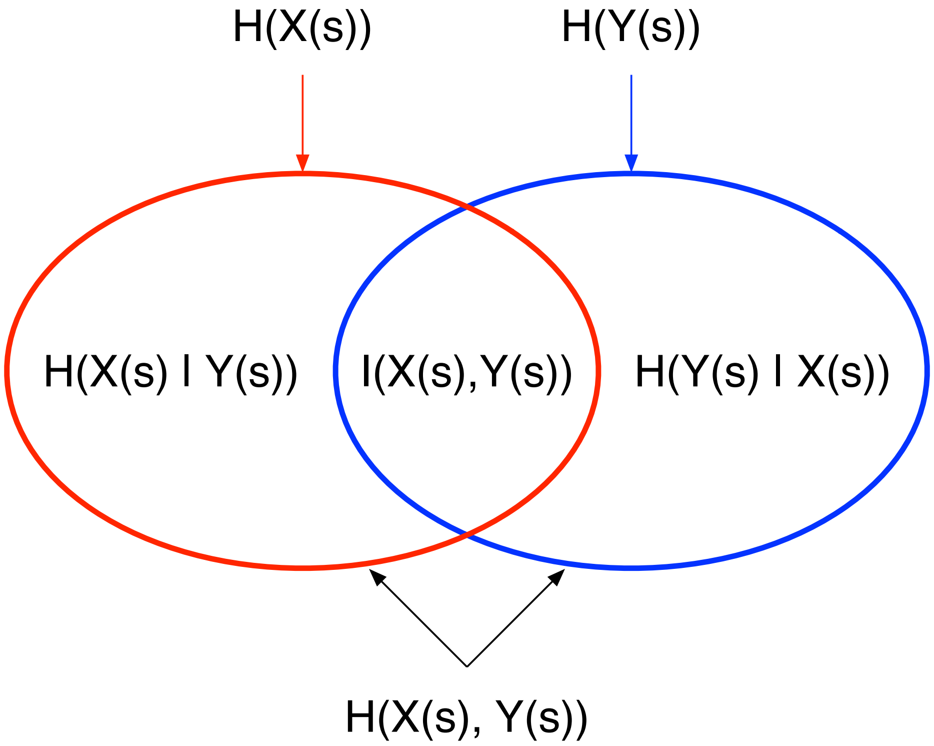


Conditional Entropy

- Definition
  - Informally, the information in the realization of 1 random variable conditioned on all possible values of another random variable
  - H(X(s) | Y(s)) $= \Sigma_k \, p_Y(y_k) \, H(X(s) \mid Y(s) = y_k)$
    $= - \Sigma_j \Sigma_k \, p_{X,Y}(x_j, y_k) \log_2(p_{X|Y}(x_j \mid y_k))$

- Properties
  - Information reduction — $H(X(s) \mid Y(s)) \le H(X(s))$
  - X(s) is completely determined by Y(s) — $H(X(s) \mid Y(s)) = 0$
  - Independent X(s) and Y(s) — $H(X(s) \mid Y(s)) = H(X(s))$
  - Chain rule — $H(X(s) \mid Y(s)) = H(X(s), Y(s)) - H(Y(s))$
  - Bayes' rule — $H(X(s) \mid Y(s)) = H(Y(s) \mid X(s)) - H(Y(s)) + H(X(s))$

Conditional Entropy



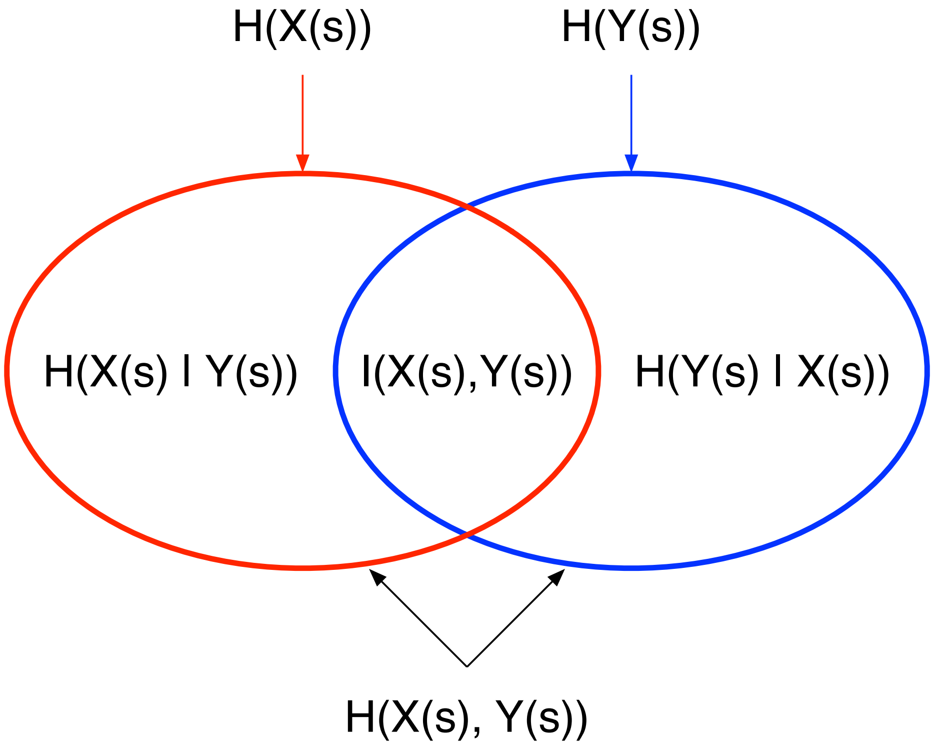
Mutual Information

- Definition
  - Informally, the information obtained about the realization of 1 random variable through the observation of the realization of another random variable; the shared information between realizations of 2 random variables
  - $I(X(s), Y(s)) = \Sigma_j \Sigma_k \; p_{X,Y}(x_j, y_k) \log_2(p_{X,Y}(x_j, y_k) / (p_X(x_j)\; p_Y(y_k)))$

- Properties
  - Self $\quad I(X(s), X(s)) = H(X(s))$
  - Symmetry $\quad I(X(s), Y(s)) = I(Y(s), X(s))$
  - Non negativity $\quad I(X(s), Y(s)) \geq 0$
  - Independent X(s) and Y(s) $\quad I(X(s), Y(s)) = 0$
  - Conditional and joint relationship

$$
\begin{aligned}
I(X(s), Y(s)) &= H(X(s)) - H(X(s) \mid Y(s)) \\
&= H(Y(s)) - H(Y(s) \mid X(s)) \\
&= H(X(s)) + H(Y(s)) - H(X(s), Y(s)) \\
&= H(X(s), Y(s)) - H(X(s) \mid Y(s)) - H(Y(s) \mid X(s))
\end{aligned}
$$

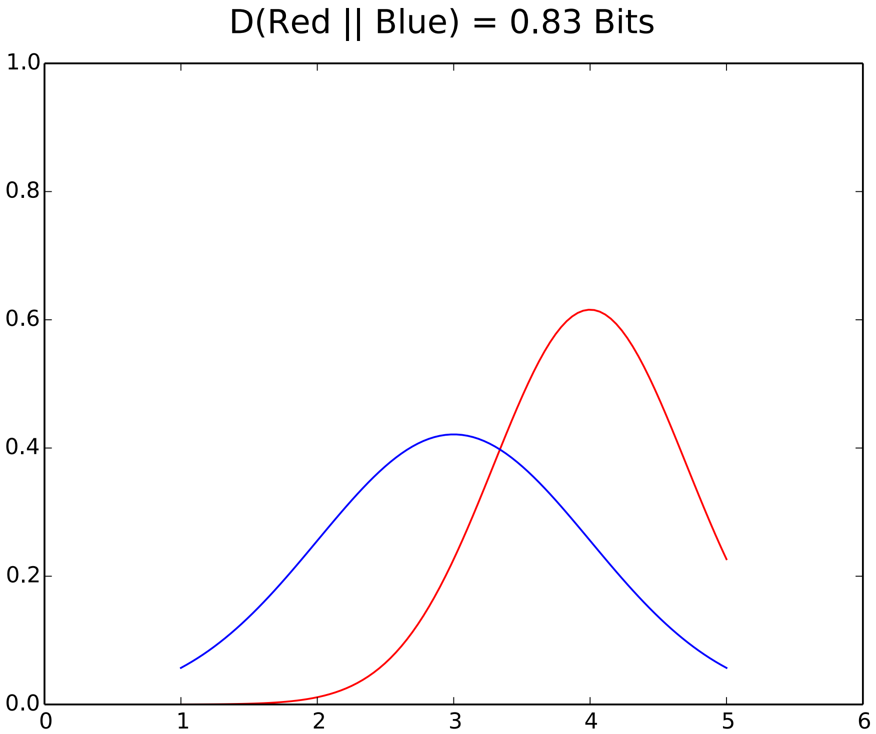
Mutual Information



Kullback Leibler (KL) Divergence

- xNN use: Error calculation for classification networks
- Definition
  - Informally, a non symmetric distance (i.e., divergence) between 2 probability distributions; the amount of information lost when 1 distribution is used to approximate another (nice for an info extracting network to minimize); the expected value of the log difference between 2 distributions
  - $D(X(s) \,||\, Y(s))$
    $= -\Sigma_k\, p_X(x_k) \log_2(p_Y(x_k) / (p_X(x_k)))$, if $p_Y(x_k) = 0$ only when $p_X(x_k) = 0$
    $= -\Sigma_k\, p_X(x_k) (\log_2(p_Y(x_k)) - \log_2(p_X(x_k)))$
    $= -\Sigma_k\, p_X(x_k) \log_2(p_Y(x_k)) + \Sigma_k\, p_X(x_k) \log_2(p_X(x_k))$
    $= H_{ce}(X(s), Y(s)) - H(X(s))$, $H_{ce}(X(s), Y(s))$ is cross entropy
- Notes
  - $D(X(s) \,||\, Y(s)) = 0$ iff $p_X(x_k) = p_Y(x_k)$
  - **For a 1 hot probability mass function $p_X(x_k)$, entropy $H(X(s)) = 0$ and** $D(X(s) \,||\, Y(s))$ = **$H_{ce}(X(s), Y(s))$**
  - An option for making it symmetric, define $D(X(s), Y(s)) = (D(X(s) \,||\, Y(s)) + D(Y(s) \,||\, X(s))) / 2$
  - Alternatives for comparing distributions: optimal transport

Kullback Leibler (KL) Divergence



Data Processing Inequality

- xNN use: network design guidelines for information extraction
  - Think of a realization of a random variable as a network input containing new information
  - Think of trained filter coefficients as a network input containing past information
  - Processing the input by the network can only lose information (from the data processing inequality)
  - A key in good network design is not to create any fundamental bottlenecks of information mapping from input to output that lose significant amounts / important information (consider the extreme example of a layer zeroing out all feature maps)
  - Note that bottlenecks in residual layers are not fundamental bottlenecks because of the parallel direct path (will discuss later)

- Inequality
  - Let $Y(s)$ be a function of $X(s)$ and $Z(s)$ be a function of $Y(s)$ such that $X(s) \rightarrow Y(s) \rightarrow Z(s)$
  - $I(X(s), Z(s)) \leq I(X(s), Y(s))$
  - In words: $Z(s)$ cannot have more information about $X(s)$ than $Y(s)$ has about $X(s)$
  - You never gain information by processing data (you just make the information that's already there easier to extract)

- Proof
  - $I(X(s), Z(s)) = H(X(s)) - H(X(s) \mid Z(s)) \leq H(X(s)) - H(X(s) \mid Y(s), Z(s)) = H(X(s)) - H(X(s) \mid Y(s)) = I(X(s), Y(s))$

Compression

- xNN uses
  - Minimize the amount of data that needs to be moved around to improve performance (data movement can easily take more power than computation)
  - Minimize or simplify the amount of data that needs to be processed while keeping as much information as possible

- Define
  - Lossless compression: x → compression → y → decompression → x
  - Lossy compression: x → compression → y → decompression → x + error

- Limits
  - Question: How much lossless compression of data is possible (how small can y be)?
  - Answer: The entropy (information) of the data defines the limit
  - Intuition: What remains after removing all redundancy from the data is information
    But it's not possible to throw away information and exactly recover the original data

Lossy Compression

- Frequently data type / application specific for the largest gains
  - General strategy of hiding reconstruction errors (information loss) in areas that are less noticeable to the user / consumer
  - Examples
    - Audio coding formats
    - Image coding formats
    - Video coding formats

- We’ve already considered some pre processing methods that can be considered data compression on the input data to the network
  - DFT and keeping $L < K$ basis elements (throwing away the other basis elements)
  - PCA with $L < K$ (throwing away columns)

Lossy Compression

- Project idea
  - Would be incredibly amazing if solved
  - But there's a high probability of failure
  - Information bits << data bits for many applications of interest
    - Ex: video
  - CNN processing complexity is ~ proportional to input size
  - Project idea: design a compression method and associated network capable of processing an input in the compressed domain
    - Achieve similar levels of accuracy as a network processing an uncompressed input
    - Do so at a massive complexity reduction
    - Make complexity proportional to information rate vs data rate

Lossless Compression

- 2 examples of redundancy
  - Redundancy within a symbol: non uniform symbol distribution
  - Redundancy across symbols: dependencies (e.g., underlying model, correlation, ...)

- 3 examples of how to remove redundancy
  - First remove redundancy within a symbols to create new symbols, then remove redundancy across the new symbols
  - First remove redundancy across symbols to create new symbols, then remove redundancy within the new symbols
  - Remove redundancy within and across symbols at the same time

- Entropy codes are common for removing redundancy within a symbol
  - Huffman coding
  - Arithmetic coding

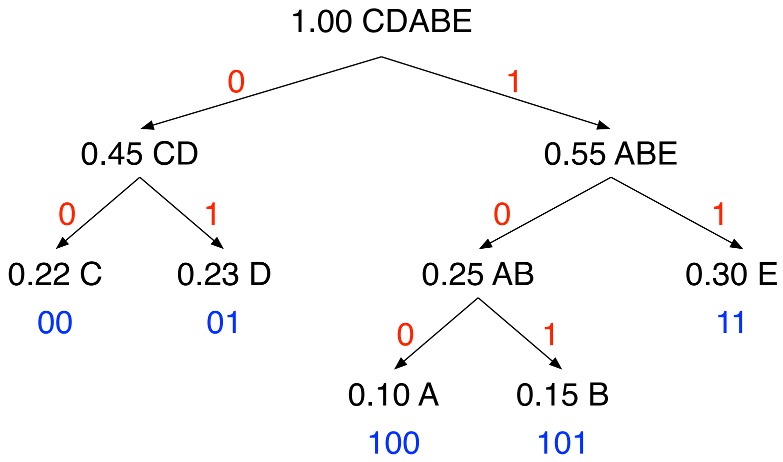
- Run length codes are common for removing redundancy across symbols
  - We’ll skip this in these slides

Huffman Coding

- Strategy
  - Record symbol probabilities
  - Build a min heap tree bottoms up (this is the key)
  - Traverse the tree top down and assign 0 / 1 to left / right branches
  - Codes for leaves = branch path are a prefix code
  - Simple table lookup for encoding and state machine for decoding
  - Close to entropy bound for many distributions of interest for independent symbols

Huffman Coding

0.10 A	0.22 C	0.25 AB	0.45 CD	1.00 CDABE
0.15 B	0.23 D	0.30 E	0.55 ABE	
0.22 C	0.25 AB	0.45 CD		
0.23 D	0.30 E			
0.30 E				



Huffman Coding

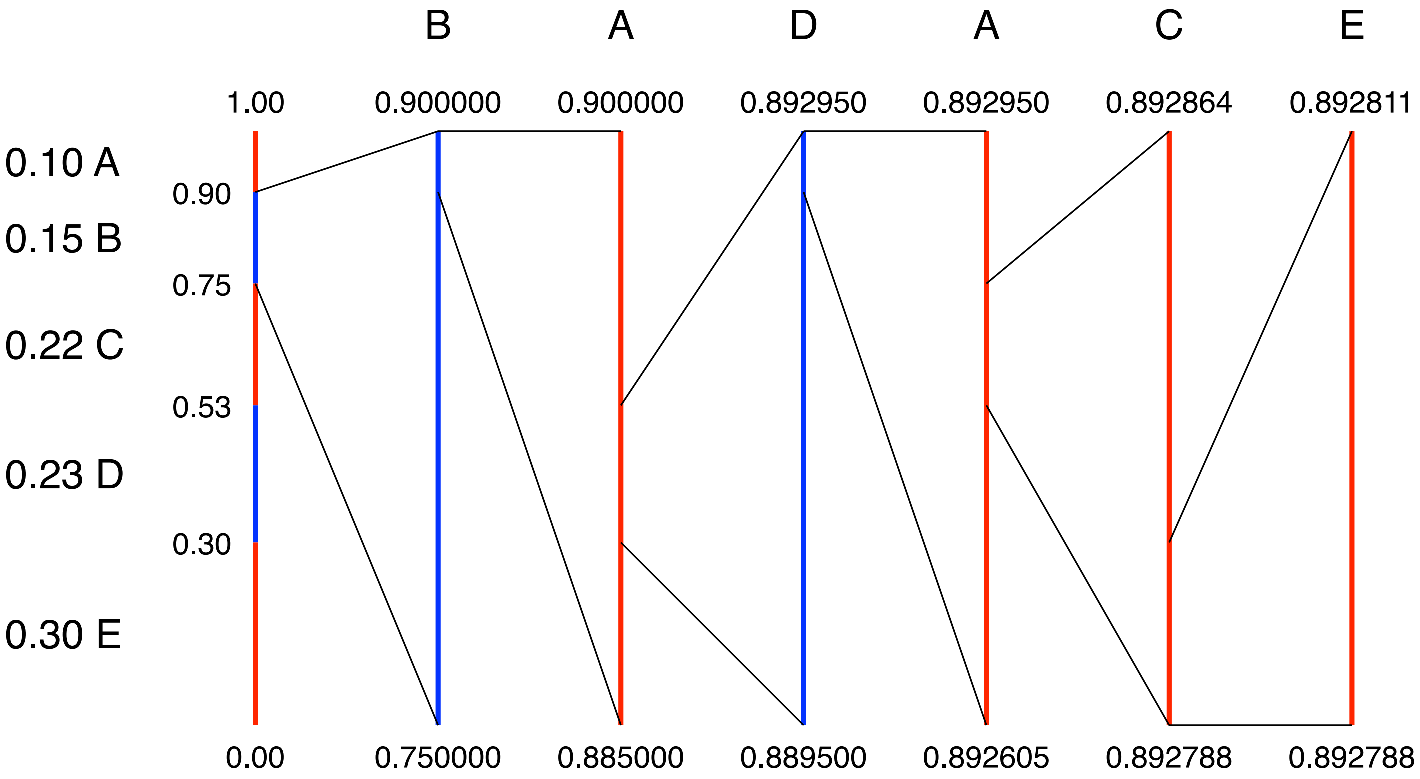
0.10	A	100
0.15	B	101
0.22	C	00
0.23	D	01
0.30	E	11

B	A	D	A	C	E
101	100	01	100	00	11

Arithmetic Coding

- Strategy
  - Encode complete message to a single real number
  - Start with intervals proportional to symbol probabilities
  - Rescale top and bottom limit of the interval based on symbol to encode
  - Slightly more complex arithmetic for encoding and decoding (depending on hardware)
  - Optimal in the sense that it achieves the entropy bound for independent symbols

Arithmetic Coding



Project Idea

- What is a bad ace?
    - Say you're playing relatively deep 9 handed 1/2 NLH
    - You're dealt Ah Kh late position, make it 12 pre flop and get 5 callers (i.e., you're at WinStar)
    - The flop comes out As 7s 4s, it's checked to you, you make it 20 and get 3 callers
        - There's a descent chance your A with K kicker is the best hand at the present time
        - Given the pre flop action someone else could have a big A, 77 or 44 (pairs less likely but very bad for you)
        - The A on the flop and bet probably chased out 2 people, maybe 1 with a connected hand and 1 with a par that missed
        - So why are the 3 people hanging around? For at least 1 of them it's because there are 3 spades on the board
    - The turn comes out 9s
        - You're going to lose this hand to a flush
        - Your ace is no good, it's a bad ace
        - If you don't get a free card fold to a bet
- Project idea: train a network to play a 9 handed 1/2 NLH ring game using reinforcement learning

Discussion

- Revisiting the motivating examples
  - Understanding machine learning as information extraction from training data to apply to the problem of information extraction from testing data
  - Understanding the flow of information through the network and implications of network design
  - Weight initialization as the application of known information
  - Error functions to quantify how well the information extraction process worked
  - Compressing filter coefficients and feature maps towards an information bound

- Project idea
  - Entropy / information analysis of CNN designs
    - Flow of information and feature maps
    - Filter coefficients

References

List

- Random
  - http://www.randomservices.org/random/index.html
- StatLect
  - https://www.statlect.com
- Lecture notes on probability, statistics and linear algebra
  - http://www.math.harvard.edu/~knill/teaching/math19b_2011/handouts/chapters1-19.pdf
- A mathematical theory of communication
  - http://math.harvard.edu/~ctm/home/text/others/shannon/entropy/entropy.pdf
- Joint entropy, conditional entropy, relative entropy, and mutual information
  - http://octavia.zoology.washington.edu/teaching/429/lecturenotes/lecture3.pdf
- Visual information theory
  - http://colah.github.io/posts/2015-09-Visual-Information/
- Computational optimal transport
  - https://arxiv.org/abs/1803.00567
- Mathematics of deep learning
  - https://arxiv.org/abs/1712.04741