1 Problem Description

The objective of the HW3 is to solve the problem of Oscillatory Motion and Chaos and also to solve Poisson equation. The Oscillatory problem can be solved using Euler-Cromer or Runge-Kutta method, And to solve the Poisson equation we can use Jacobi or gauss-seidel. Detailed explanation can be found below:

2 Solution to Pendulum Problem

2.1 Calculate analytically at what (approximate) value of Ω_D the resonance occurs. Do you expect the small-angle (linear) approximation to be good?

$$\frac{d^2\theta}{dt^2} = -g\theta - 2\gamma\frac{d\theta}{dt} + \alpha_D \sin(\Omega_D t) \tag{1}$$

$$\theta'' + 2\gamma\theta' + \omega_0^2\theta = F(t) \quad F(t) = \alpha_D \sin(\Omega_D t) \tag{2}$$

Assuming $F(t) = 0$, such that $\theta(t) \sim e^{rt}$, where r is a root.

$$r^2 + 2\gamma r + \omega_0^2 = 0 \quad r_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{3}$$

$$\theta(t)_{\text{Homogeneous}} = (Ae^{rt} + Be^{-rt})e^{-\gamma t}(\text{if } \gamma \neq \omega_0) \tag{5.1}$$

$$\theta(t)_{\text{Homogeneous}} = (A + Bt)e^{-\gamma t}(\text{if } \gamma = \omega_0) \tag{5.2}$$

Condition: steady-state (ss)

$$\theta(t)_{\text{ss}} = \theta_P \sin(\Omega_D t - \phi) \tag{6}$$

Puting eqn (6) into (2) with $F(t) = \alpha D \sin(\Omega_D t)$ results in:

$$\alpha_D \sin(\Omega_D t) = \frac{\omega_0^2 - \Omega_D^2}{\alpha} \sin(\Omega_D t - \phi) + 2\gamma\Omega_D \cos(\Omega_D t - \phi) \tag{7}$$

As,

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{8}$$

$$\sin(\alpha + \beta) = \sin(\alpha)(\frac{\theta_P}{\alpha_D}(\omega_0^2 - \Omega_D^2)) + \cos(\alpha)(\frac{2\gamma\Omega_D\theta_P}{\alpha_D}) \tag{9}$$

Therefore:

$$\cos(\phi) = \frac{\theta_P}{\alpha_D}(\omega_0^2 - \Omega_D^2) \tag{10}$$

$$\sin(\phi) = \frac{2\gamma\Omega_D\theta_P}{\alpha_D} \tag{11}$$

Finding, $\theta_P(\Omega_D)$ and $\phi(\Omega_D)$ from $\sin(\phi)$ and $\cos(\phi)$

$$\theta_P(\Omega_D) = (\frac{\alpha_D}{\sqrt{(\omega_0^2 - \Omega_D^2)^2 + 4\gamma^2\Omega_D^2}})(12)$$

$$\phi(\Omega_D) = \arctan\left(\frac{2\gamma\Omega_D}{\omega_0^2 - \Omega_D^2}\right) \tag{13}$$

To find Ω_{Res} analytically, we can equate the first derivative of to zero:

$$\frac{d\theta_P}{d\Omega} = 0 \tag{13}$$

On solving we will get

$$\Omega_{\text{Res}} = \sqrt{\omega_0^2 - 2\gamma^2} \tag{14}$$

On taking,

$$\omega_0 = 1sec^{-1} \tag{15}$$

$$\gamma = 0.25sec^{-1} \tag{16}$$

We get,

$$\Omega_{\text{Res}} = 0.935 \text{ rad/sec} \tag{17}$$

2.2 Calculate $\theta(t)$ using the Euler-Cromer and the Runge-Kutta 4th order methods. Plot $\theta(t)$ and $\omega(t) = \frac{d\theta}{dt}$ over a sufficiently long time . Plot $\theta_0(\Omega_D)$ and $\phi(\Omega_D)$.Finally calculate the full-width at half maximum (FWHM) of the resonance curve and compare it to γ.

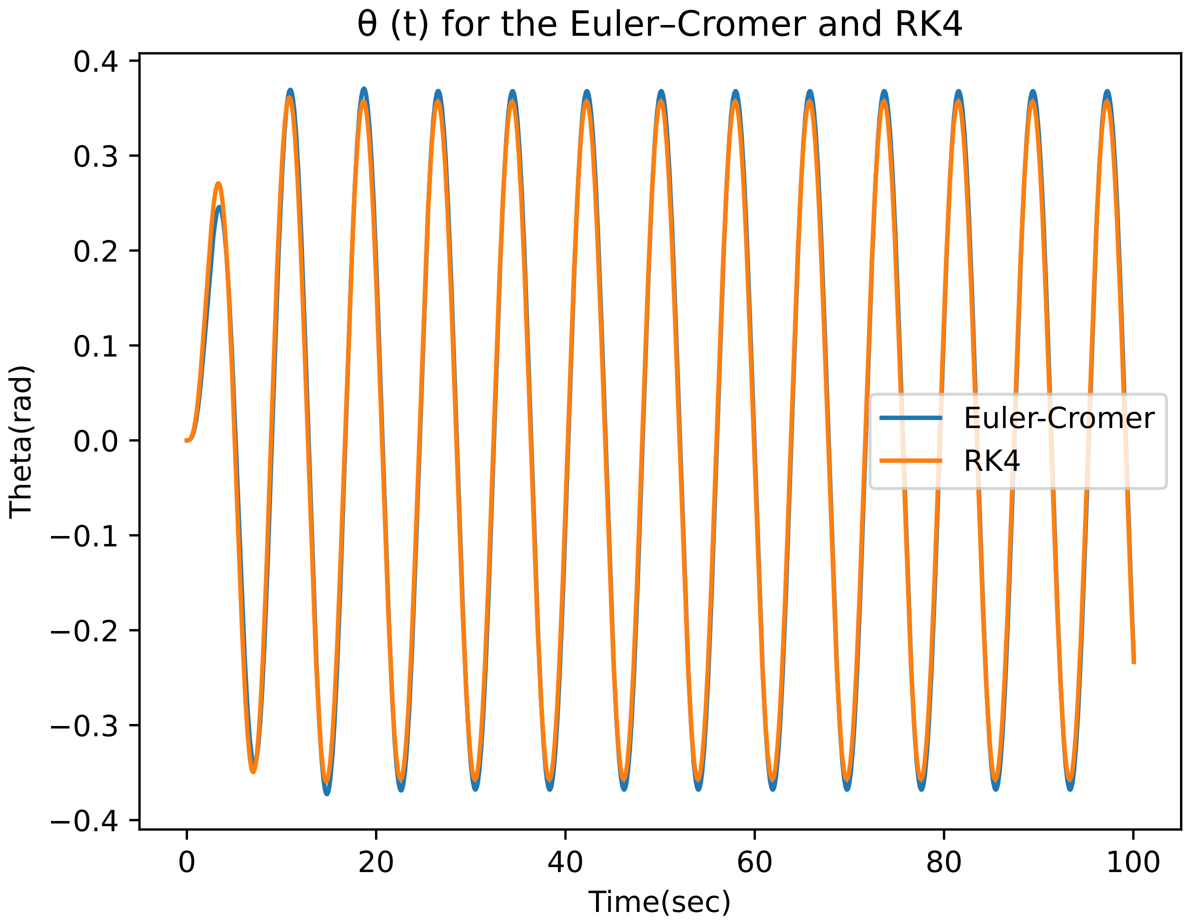

Figure 1: (t) for the Euler–Cromer and RK4

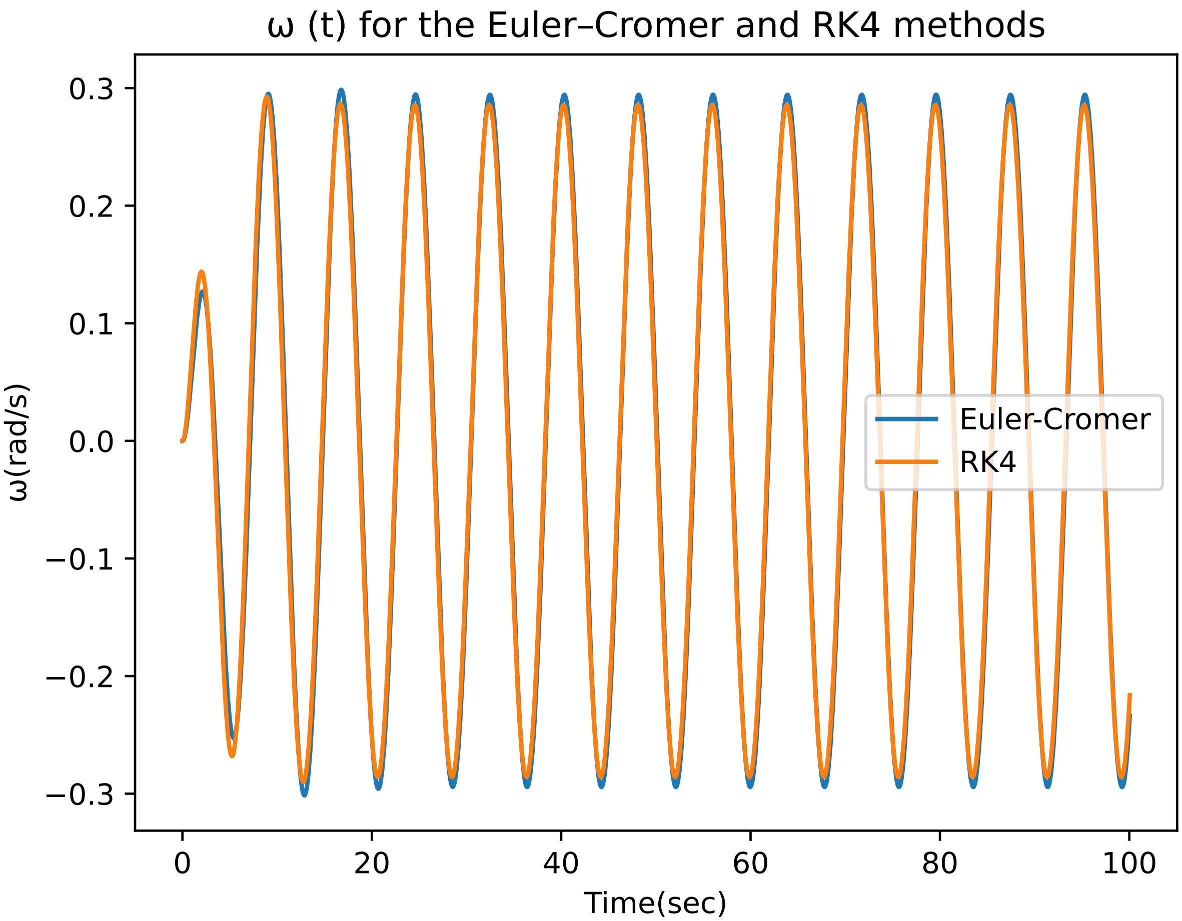

Figure 2: (t) for the Euler–Cromer and RK4 methods

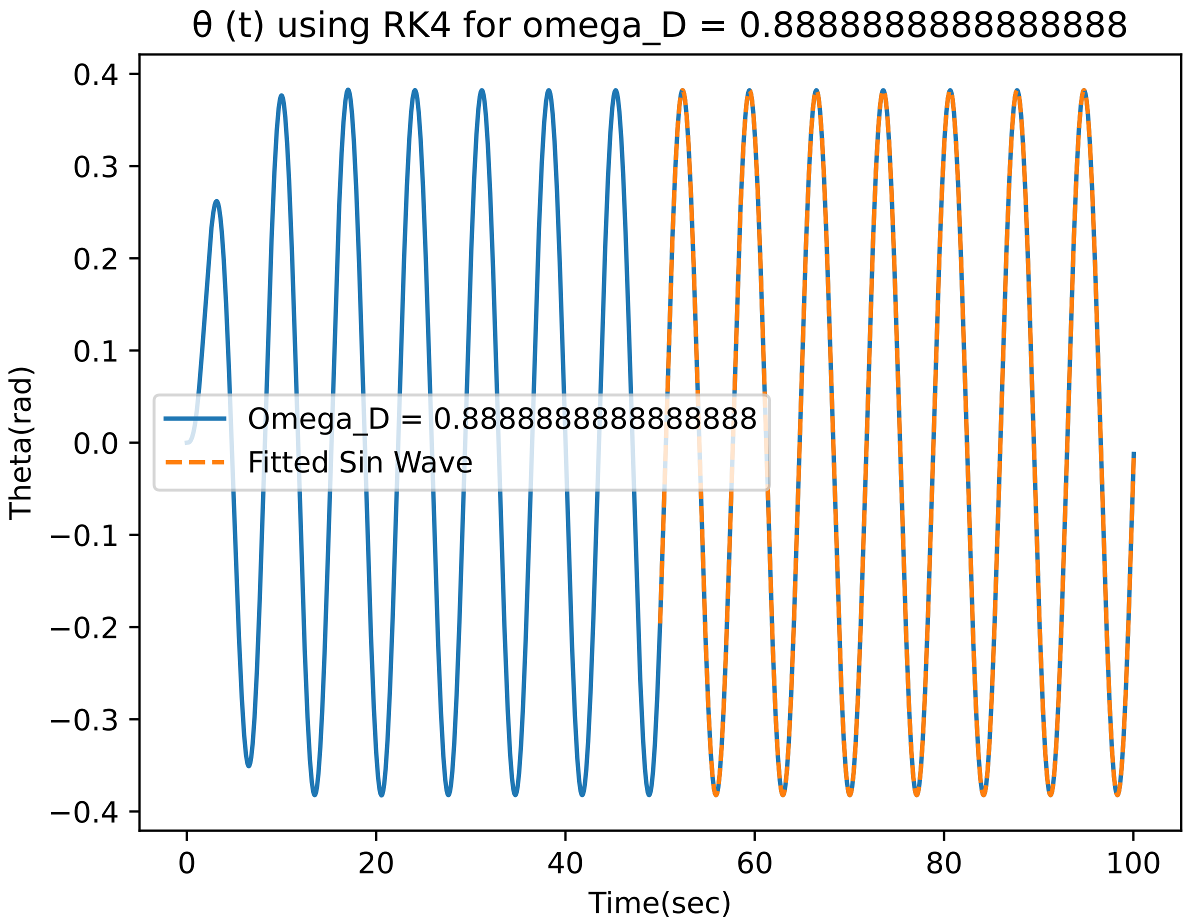

Figure 3: Fitted sin wave for omega$_D$ = $(0.88 rad/sec)$

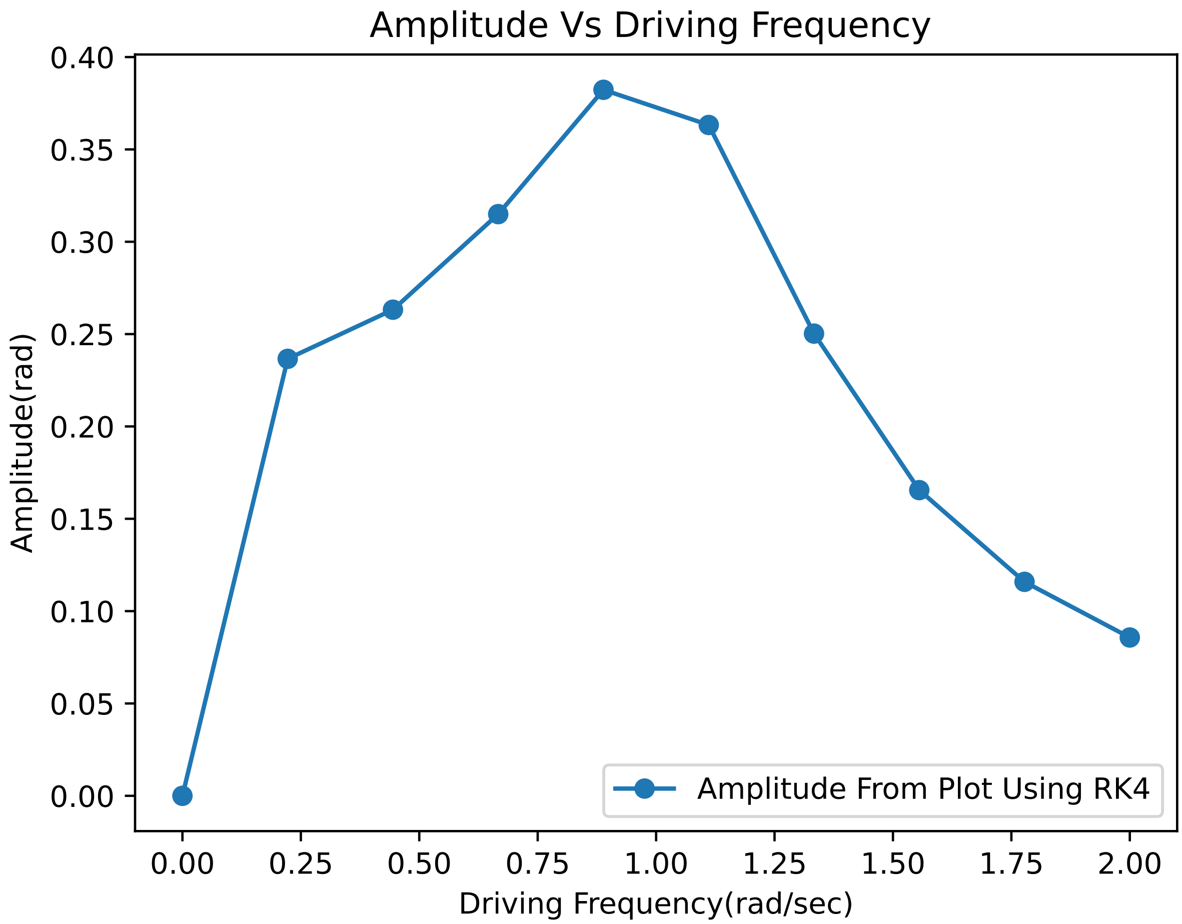

Figure 4: Amplitude Vs Driving Frequency

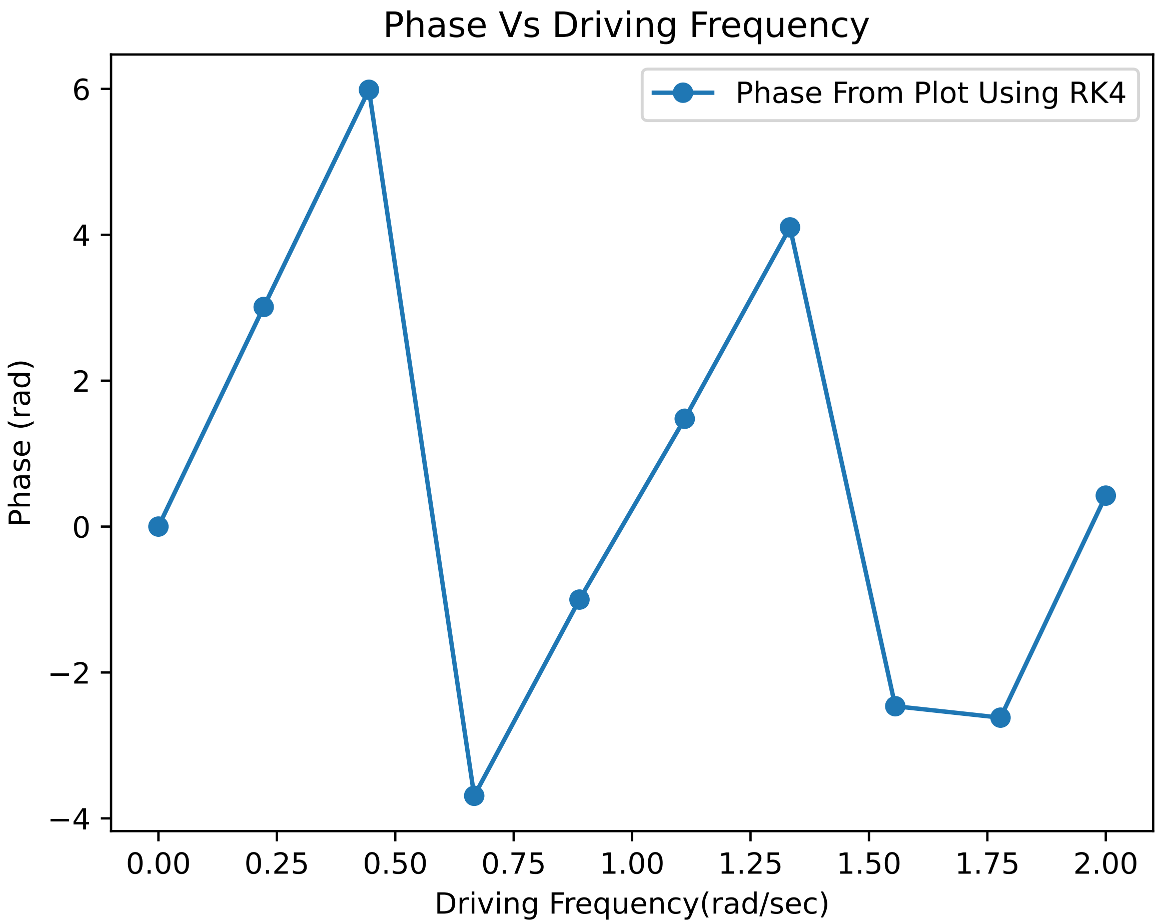

Figure 5: Phase Vs Driving Frequency

Figure 1 and Figure 2, shows that there is almost no difference between Euler-Cromer and RK4. From Figure 4 we can calculate the full-width at half maximum (FWHM), by taking 0.4 as a peak (approximatly) and then taking the value of x-axis intersecting with 0.2(y-axis coordinate). FWHM = (1.45 - 0.18) = 1.27.

$$\frac{FWHM}{\gamma} = \frac{1.27}{0.25} = 5.08 \tag{18}$$

The FWHM is coming closer to the 5 times of the γ, it can be further improved if we overlap some kind of a Gaussian curve.

2.3 For a driving frequency close to resonance, compute potential, kinetic and total energies and plot them in the same graph over circa 10 periods.

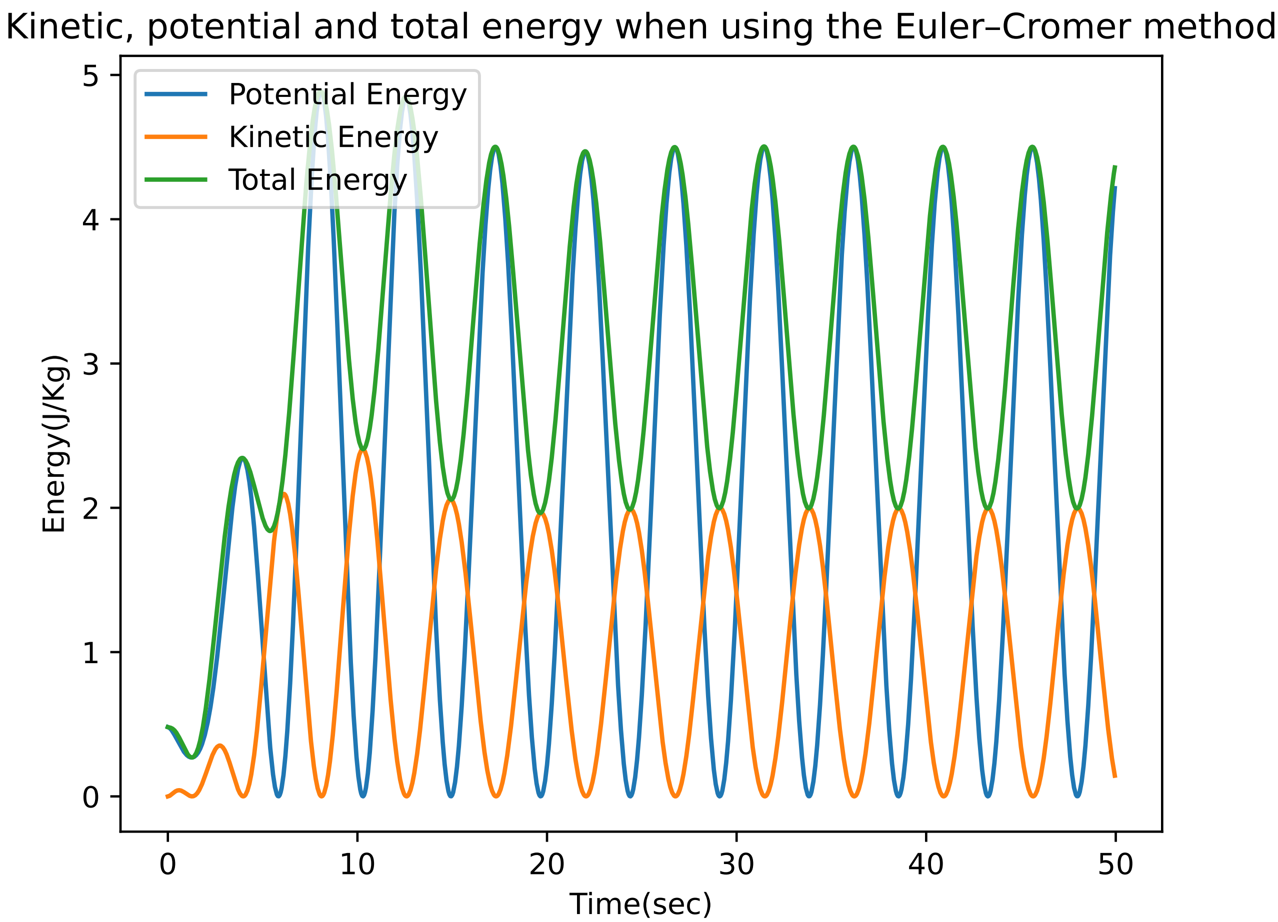

Figure 6: Kinetic, potential and total energy when using the Euler–Cromer method

Figure 6, is showing how the potential and kinetic energy vary with time. At t = 0, kinetic energy is 0

2.4 Switch non-linear effect and compare with previous results for Θ(t) and ω(t) using Ω_D close to resonance. Increase α_D to 1.2 rad/s^2

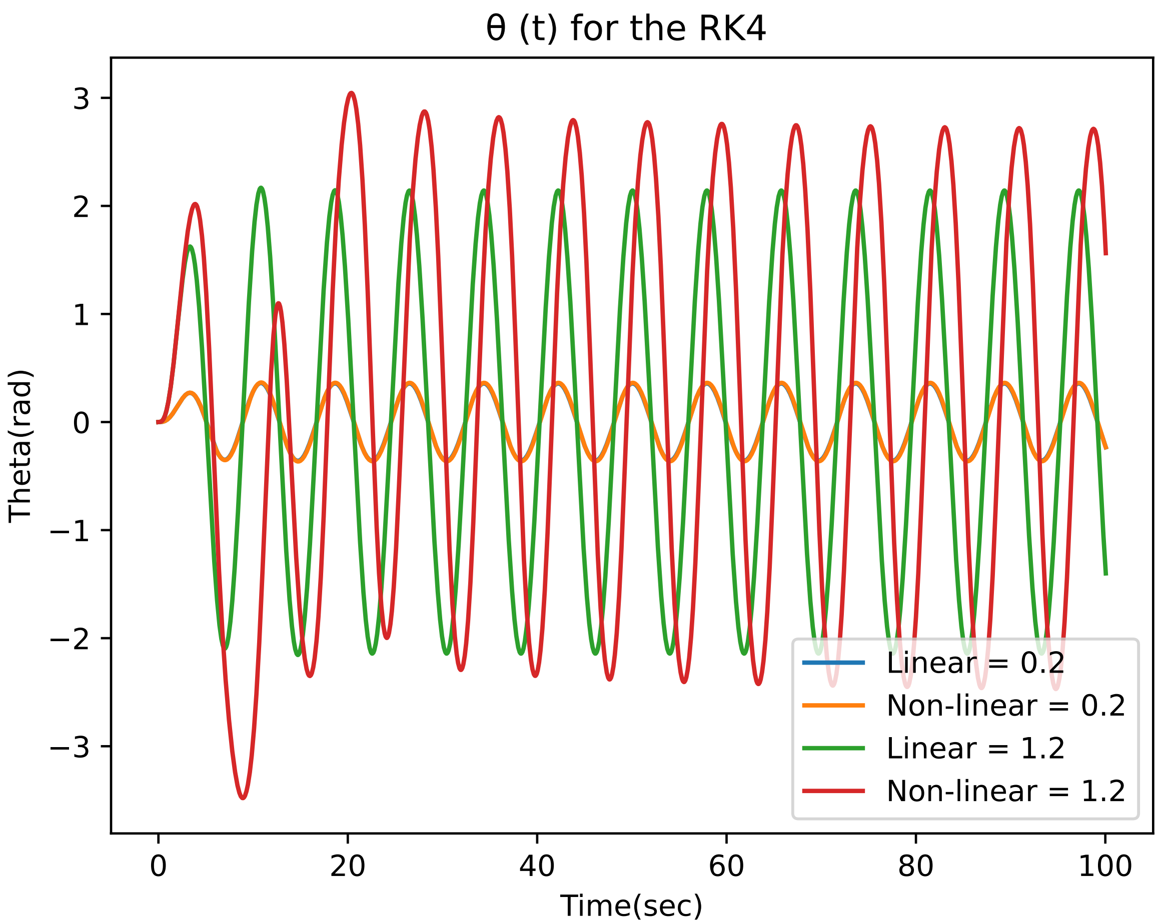

Figure 7: (t) for the RK4 [Non Linear vs Linear

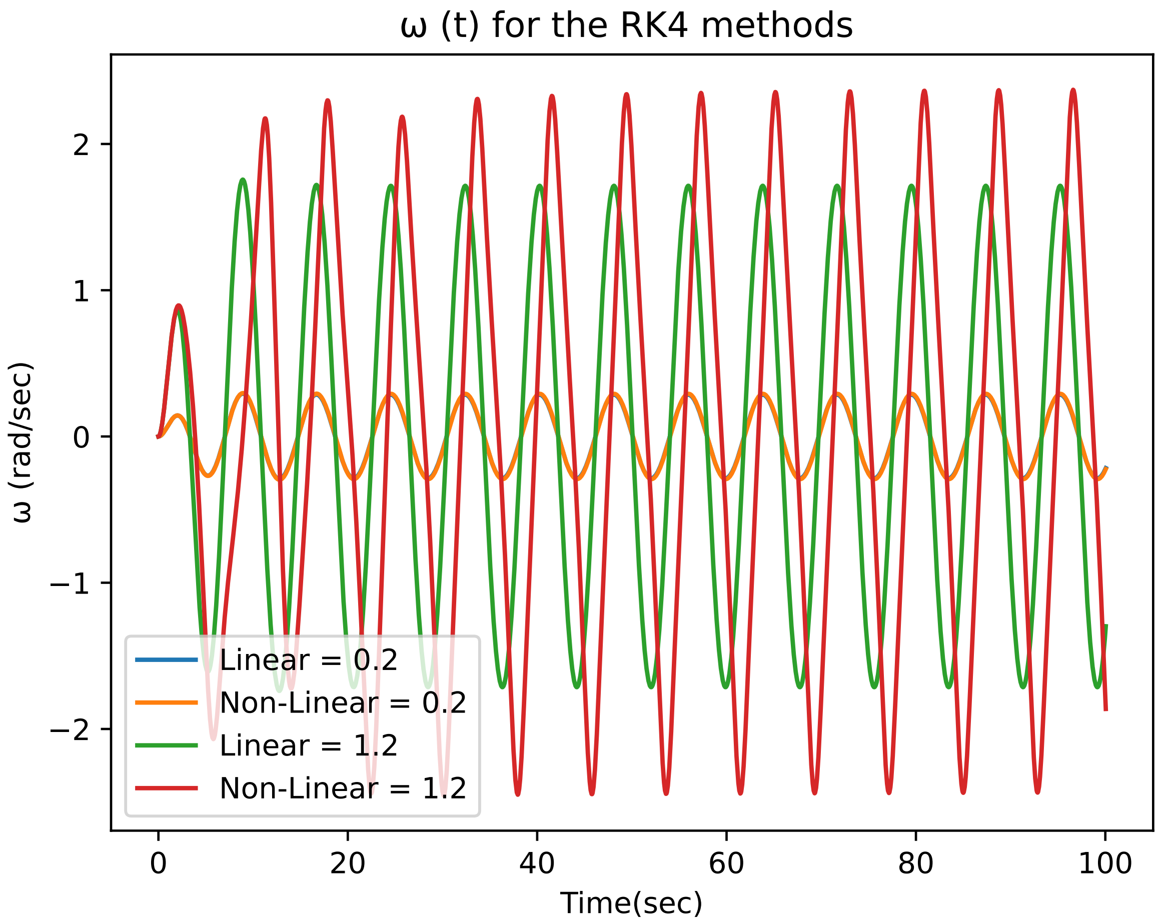

Figure 8: (t) for the RK4 methods [Non Linear vs Linear]

Figure 7 and Figure 8, are the plots when we switch on the non-linear effect. For smaller α_D, we see no change but for higher α_D, there is a difference in the θ.

2.5 Take non-linear pendulum of part (4) with D = $0.666s^1$ and values of α_D = 0.2, 0.5 and 1.2 rad/s^2 to compute $\Delta\Theta(t)$ for several trajectories with slightly different initial angle ($\Delta\Theta_i n$ = 0.001 rad). And find Lyapunov exponent

From the Figure 9 to Figure 17, we can calulate the average value of Lyapunov exponent close to $-0.24s^{-1}$. This means that the system is operation at non-chaotic regime, and the system is not sensitive to changes in initial conditions

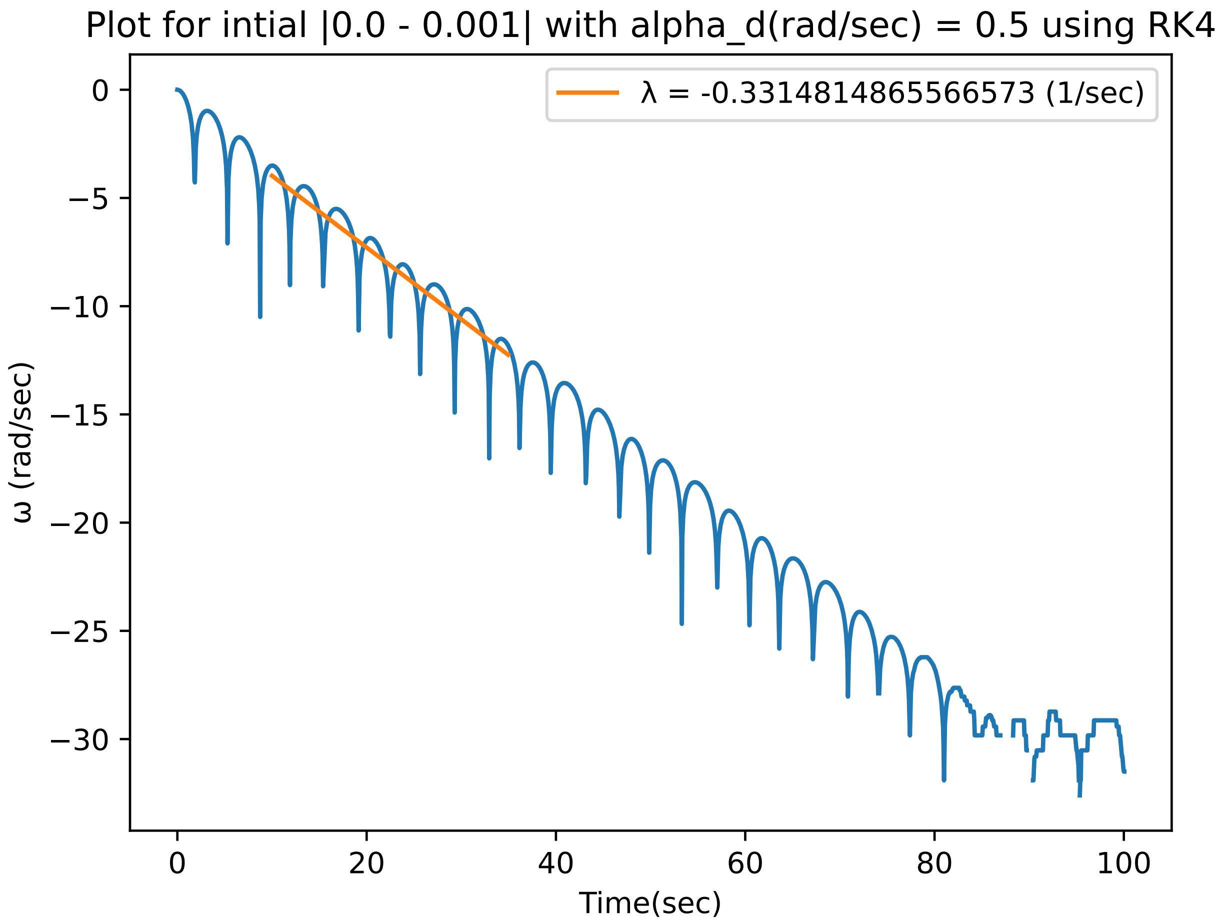

Figure 9: $\Delta\theta$ = (0.0 - 0.001) with diving frequency = 0.5

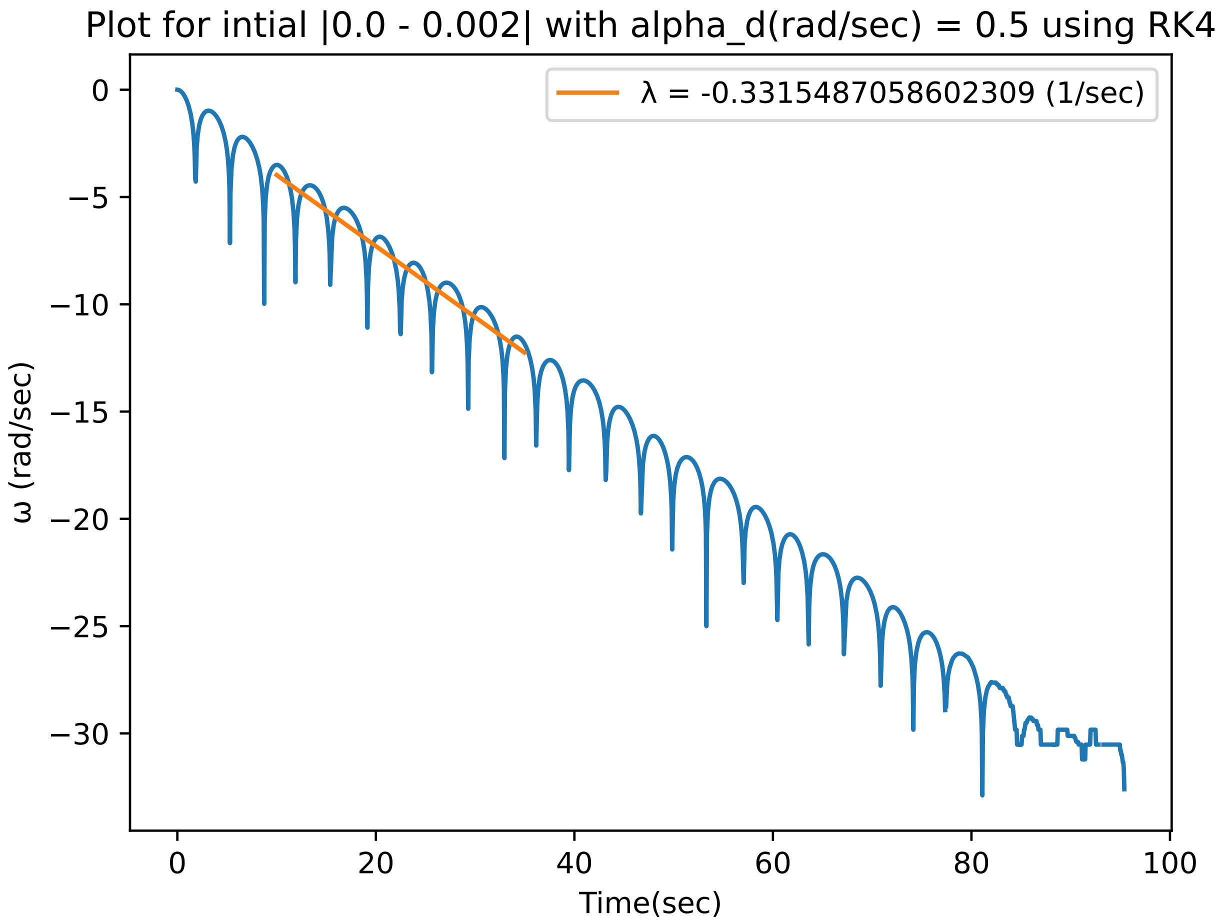

Figure 10: $\Delta\theta$ = (0.0 - 0.002) with diving frequency = 0.5

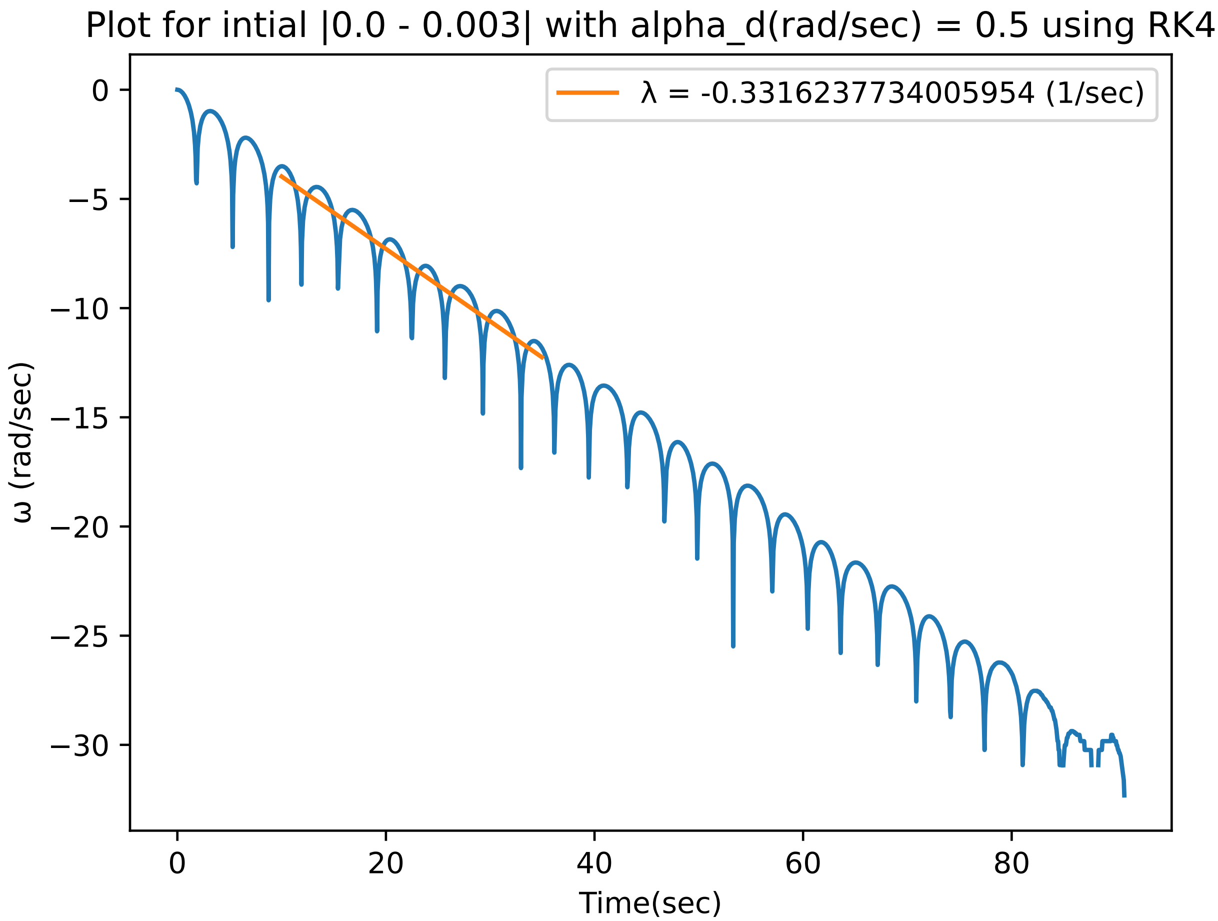

Figure 11: $\Delta\theta$ = (0.0 - 0.003) with diving frequency = 0.5

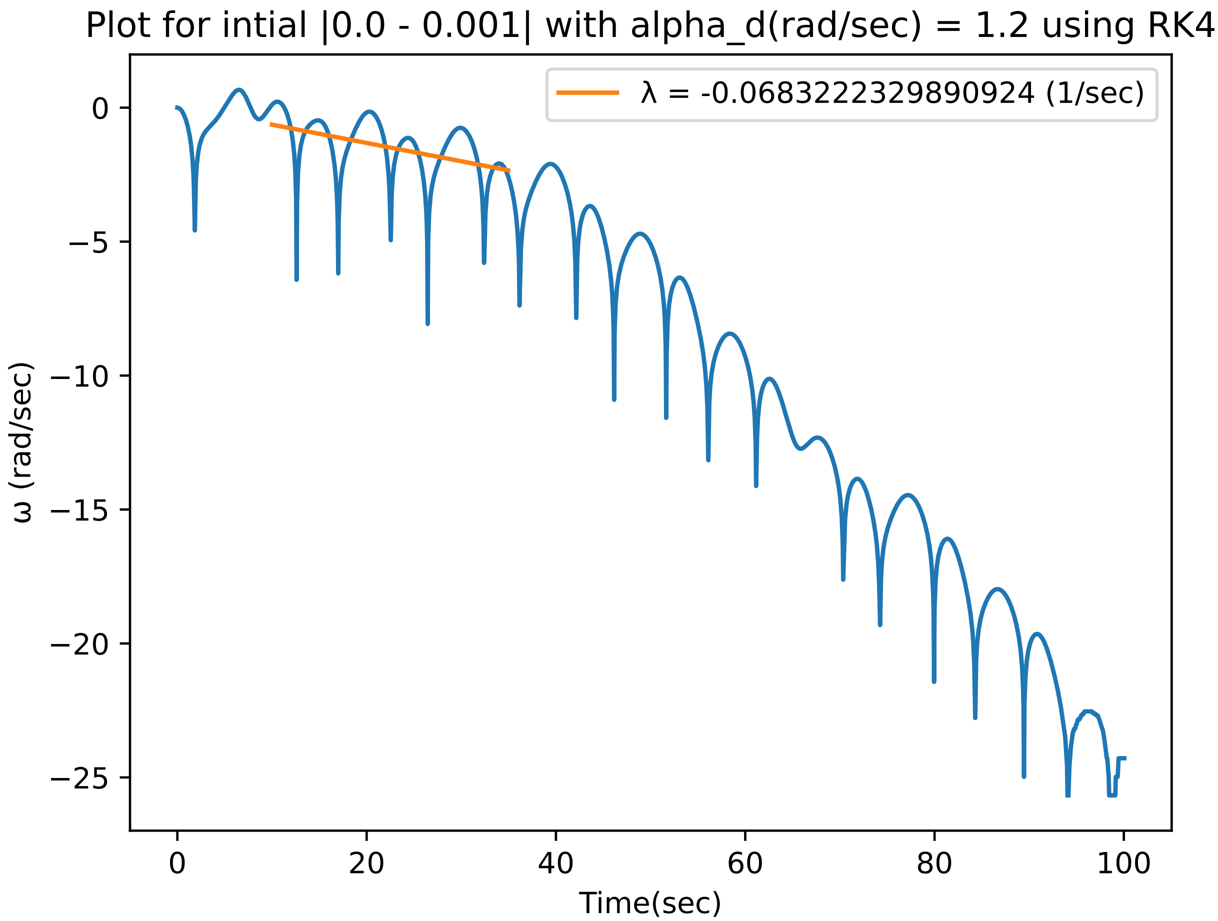

Figure 12: $\Delta\theta$ = (0.0 - 0.001) with diving frequency = 1.2

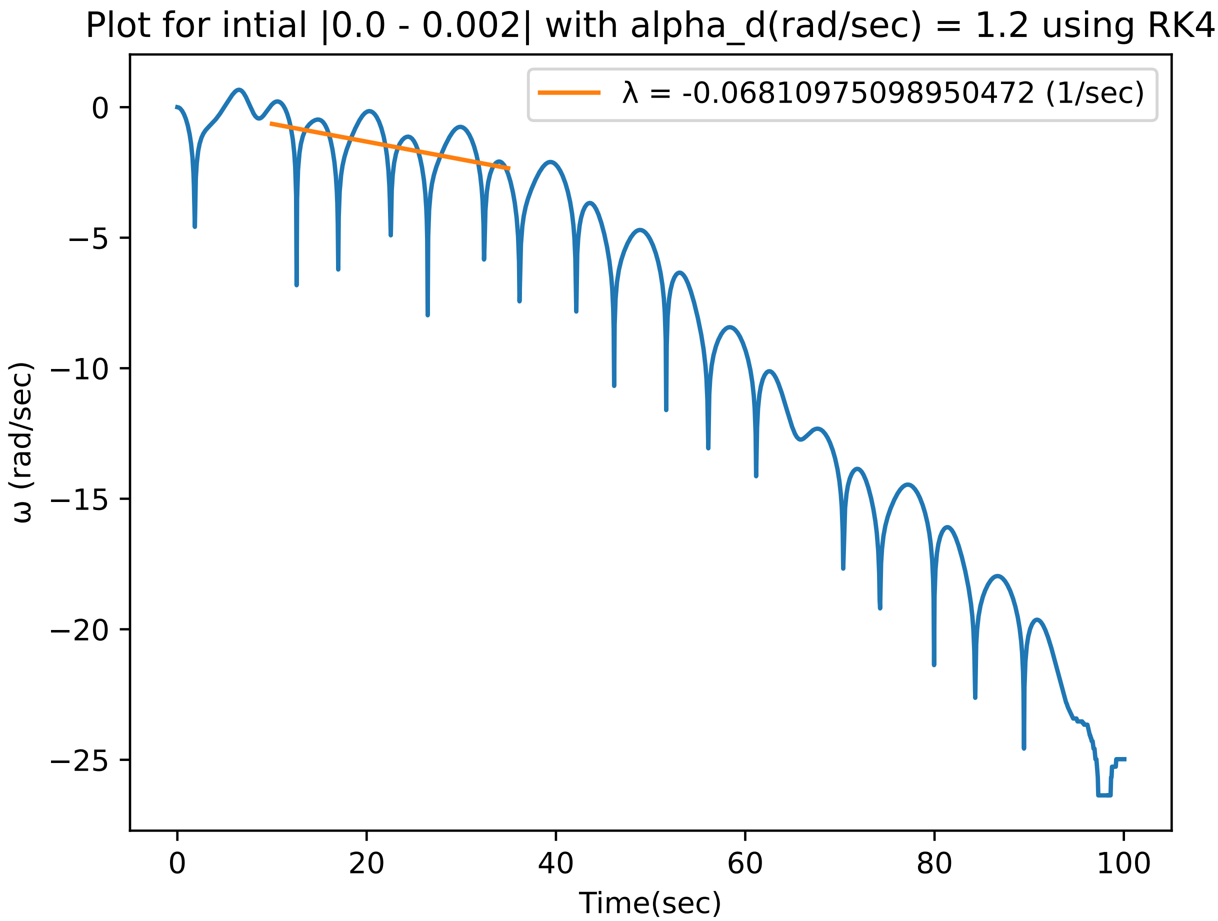

Figure 13: $\Delta\theta$ = (0.0 - 0.002) with diving frequency = 1.2

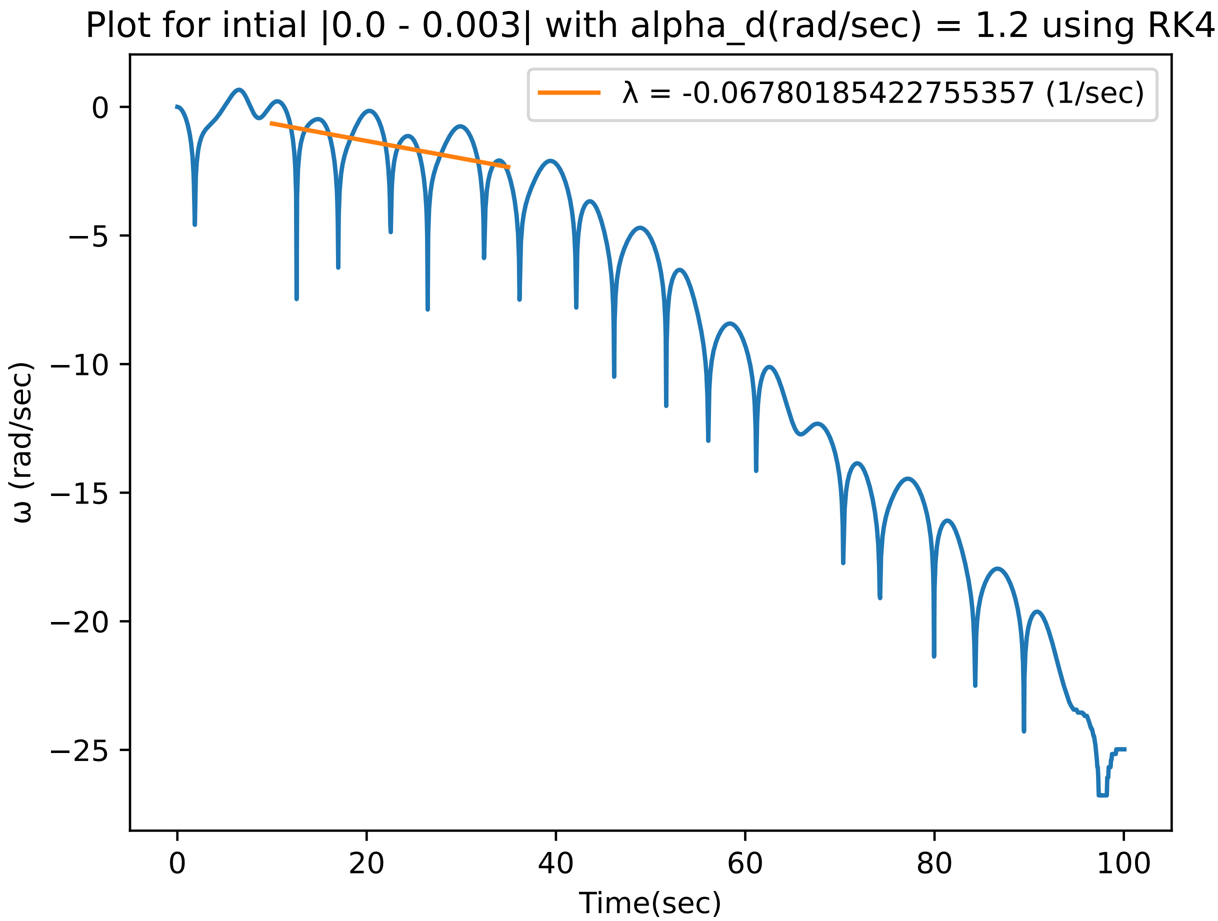

Figure 14: $\Delta\theta$ = (0.0 - 0.003) with diving frequency = 1.2

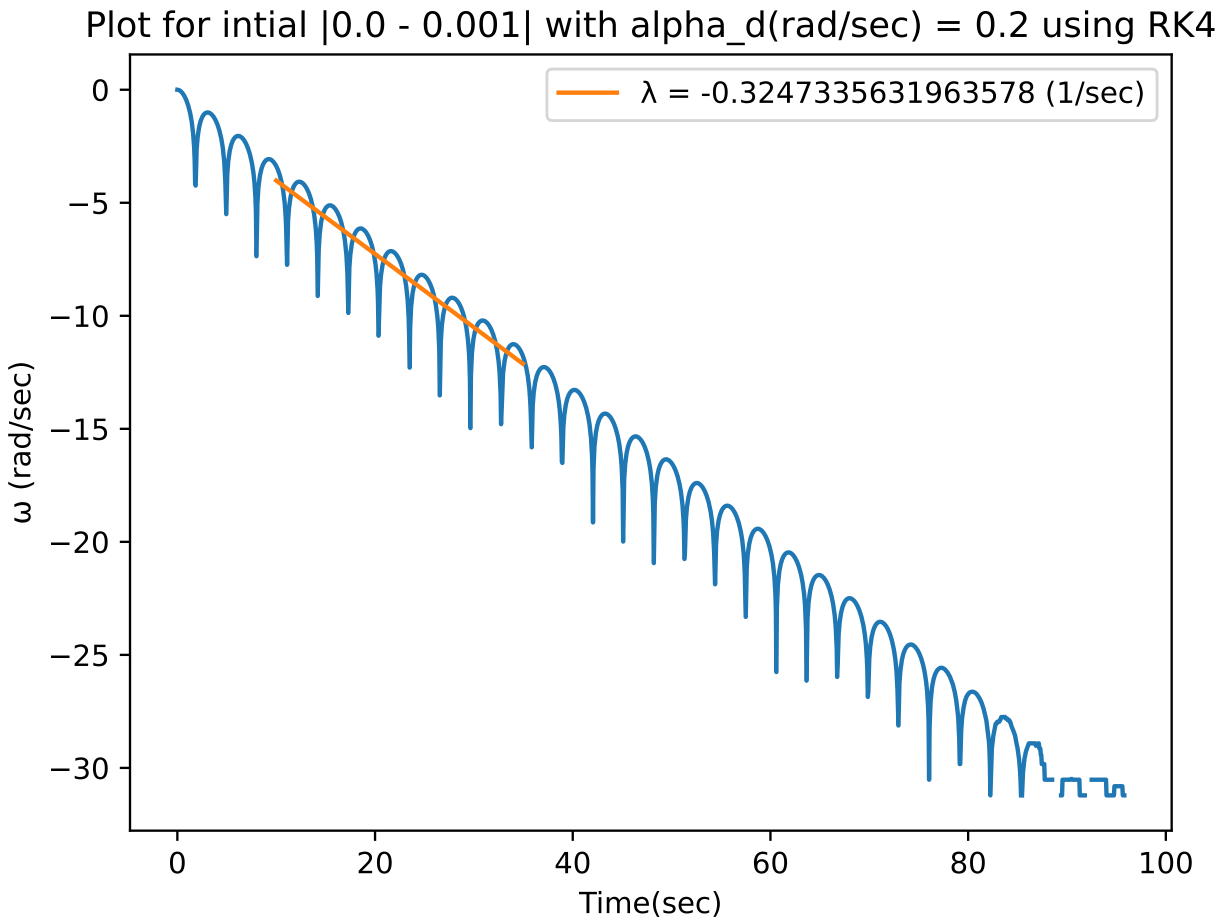

Figure 15: $\Delta\theta$ = (0.0 - 0.001) with diving frequency = 0.2

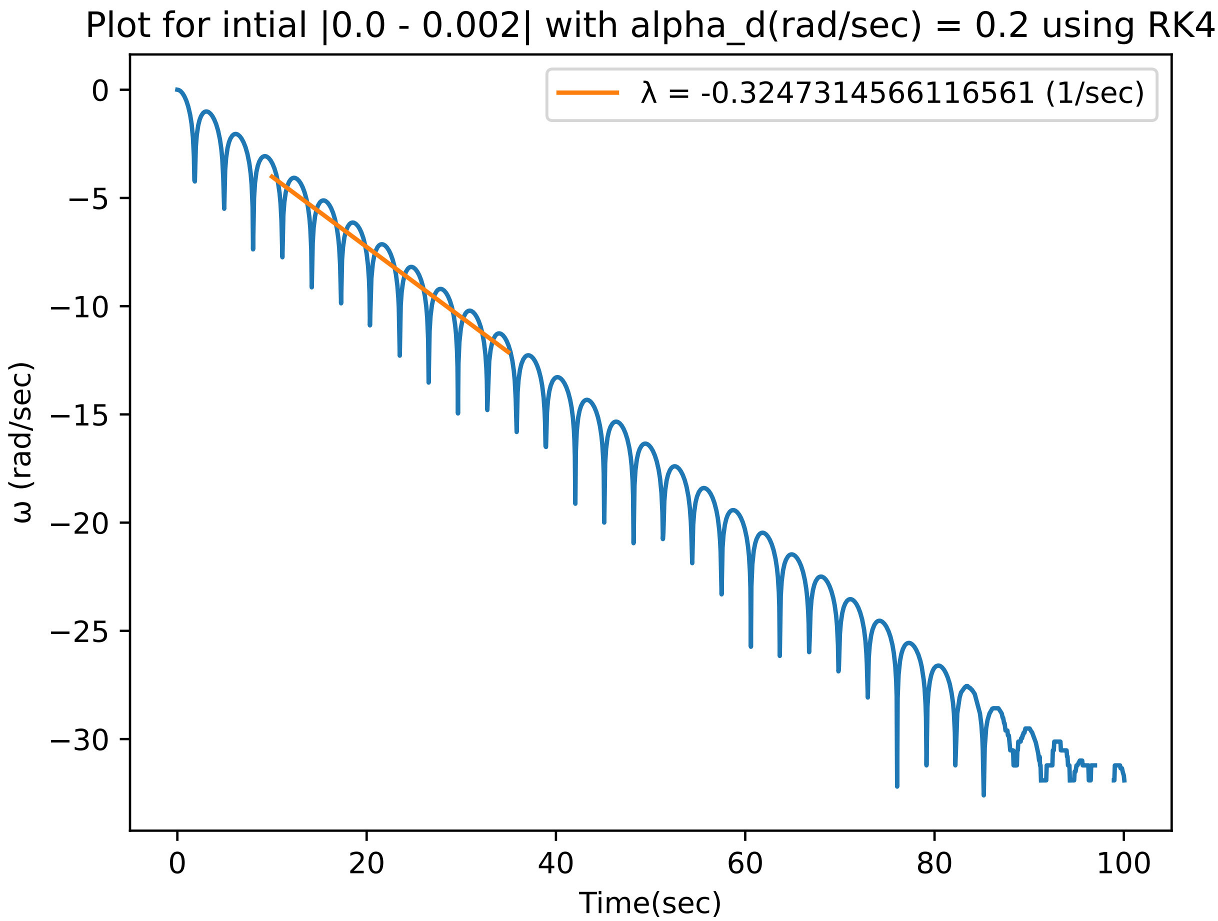

Figure 16: $\Delta\theta$ = (0.0 - 0.002) with diving frequency = 0.2

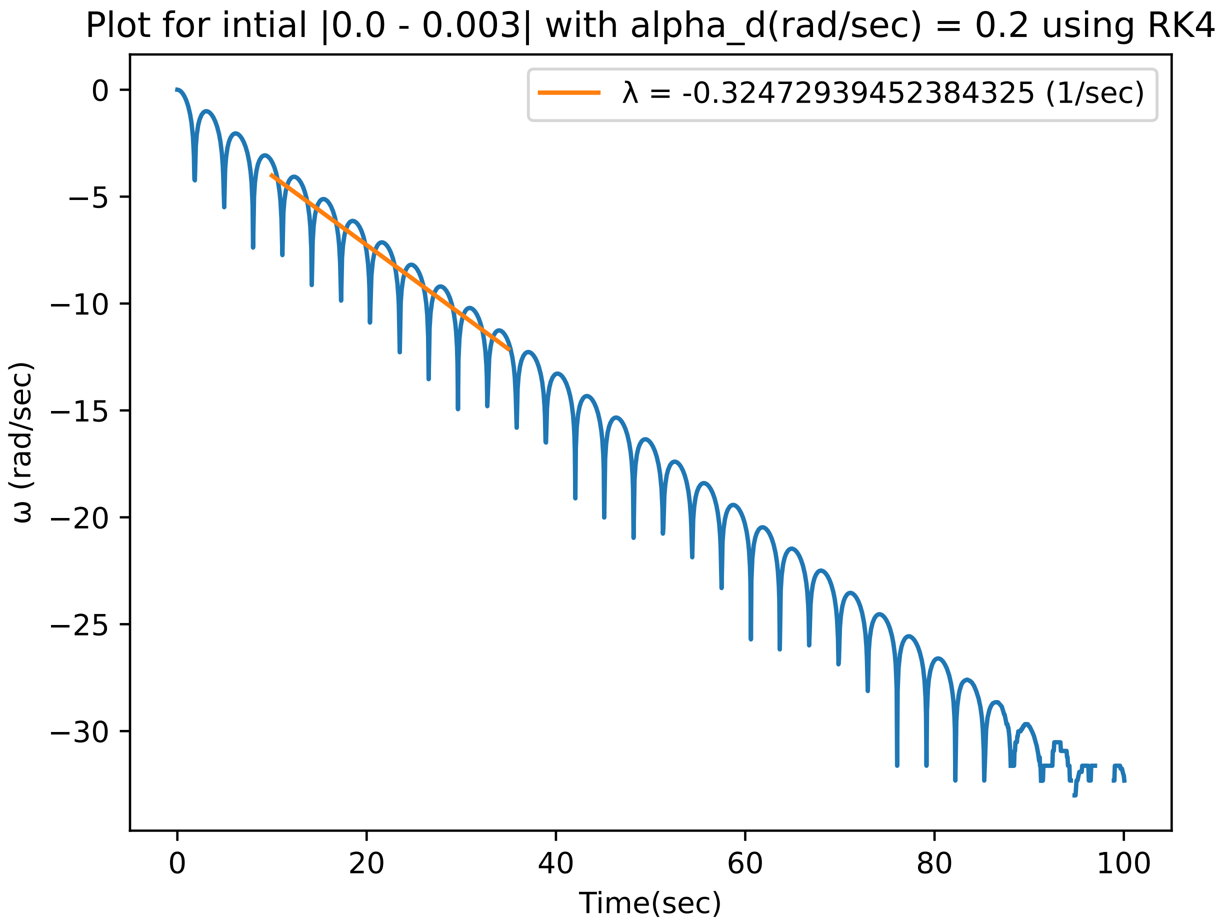

Figure 17: $\Delta\theta$ = (0.0 - 0.003) with diving frequency = 0.2

3 Solution to electric dipole

3.1 Plot V(r) (r is the distance from the origin)

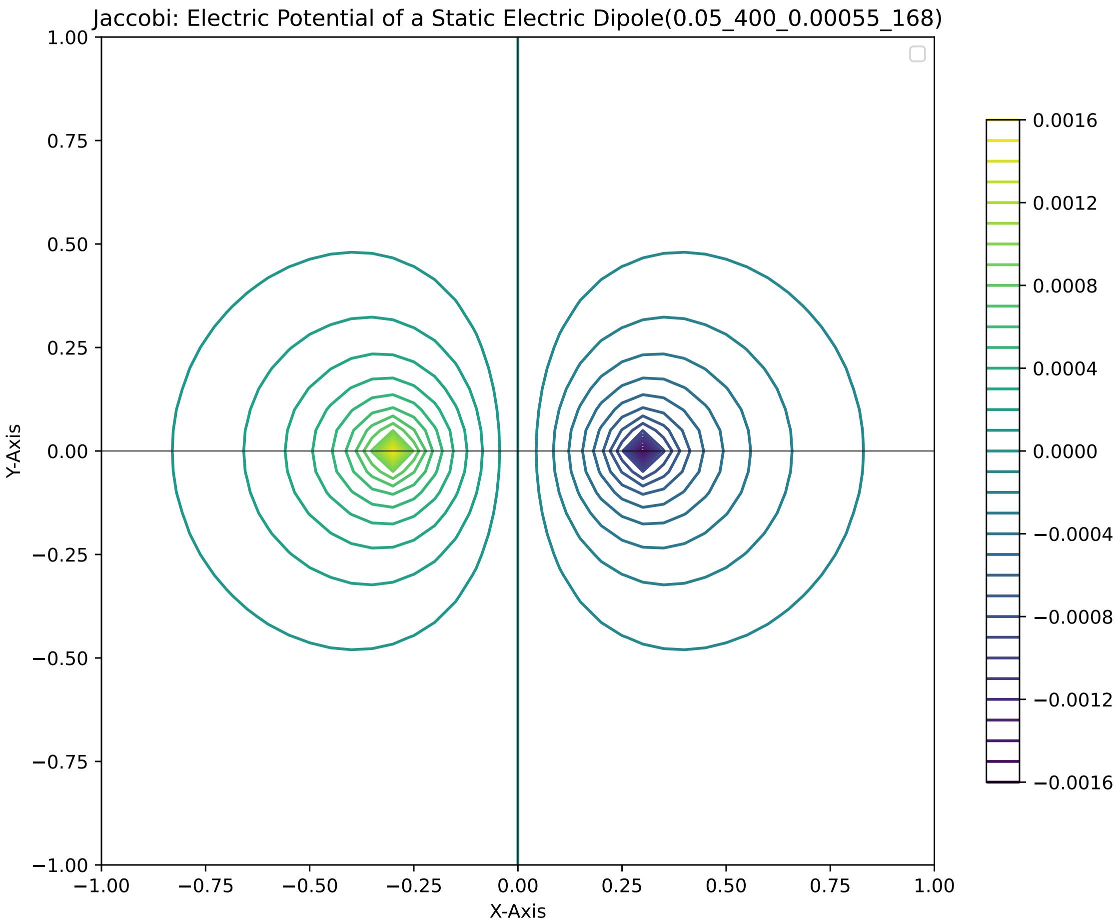

Figure 18: Electric Potential of a Static Electric Dipole

For the literature the electric potential will have and asymptotic behaviour.

3.2 How the number of required iteration steps, , increases with reducing the tolerance (error) limit

From Table 1, we can see the number of iterations is increasing exponentially if if reduce the tolerance values.

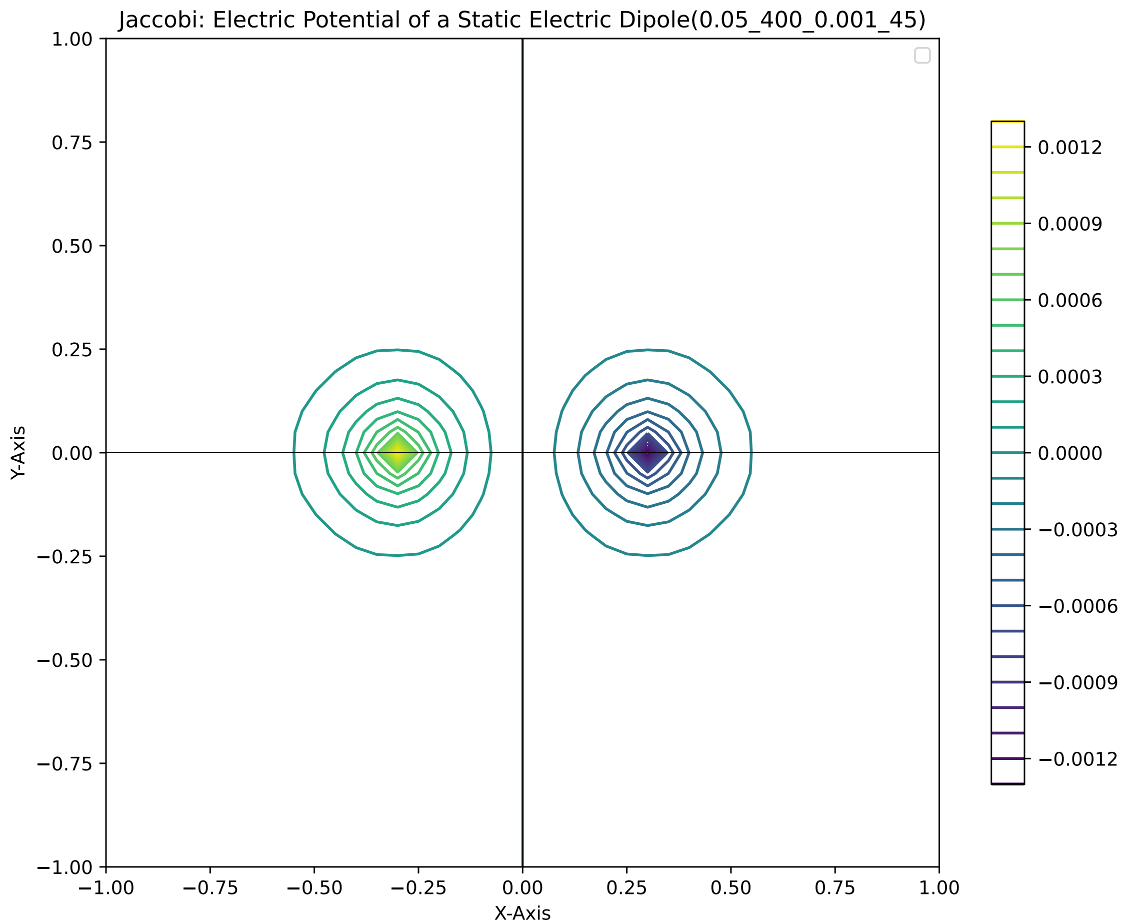

Figure 19: Electric Potential of a Static Electric Dipole - tolerance = 0.001

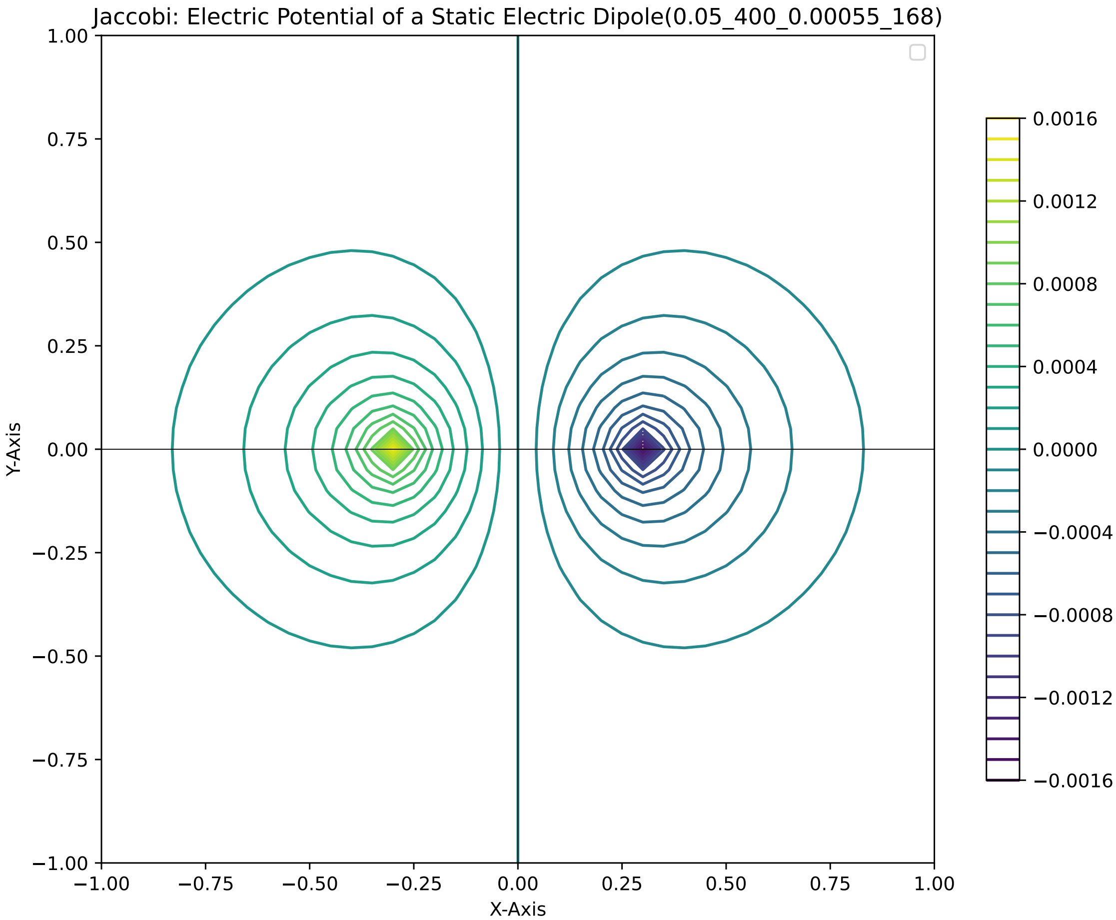

Figure 20: Electric Potential of a Static Electric Dipole - tolerance = 0.00055

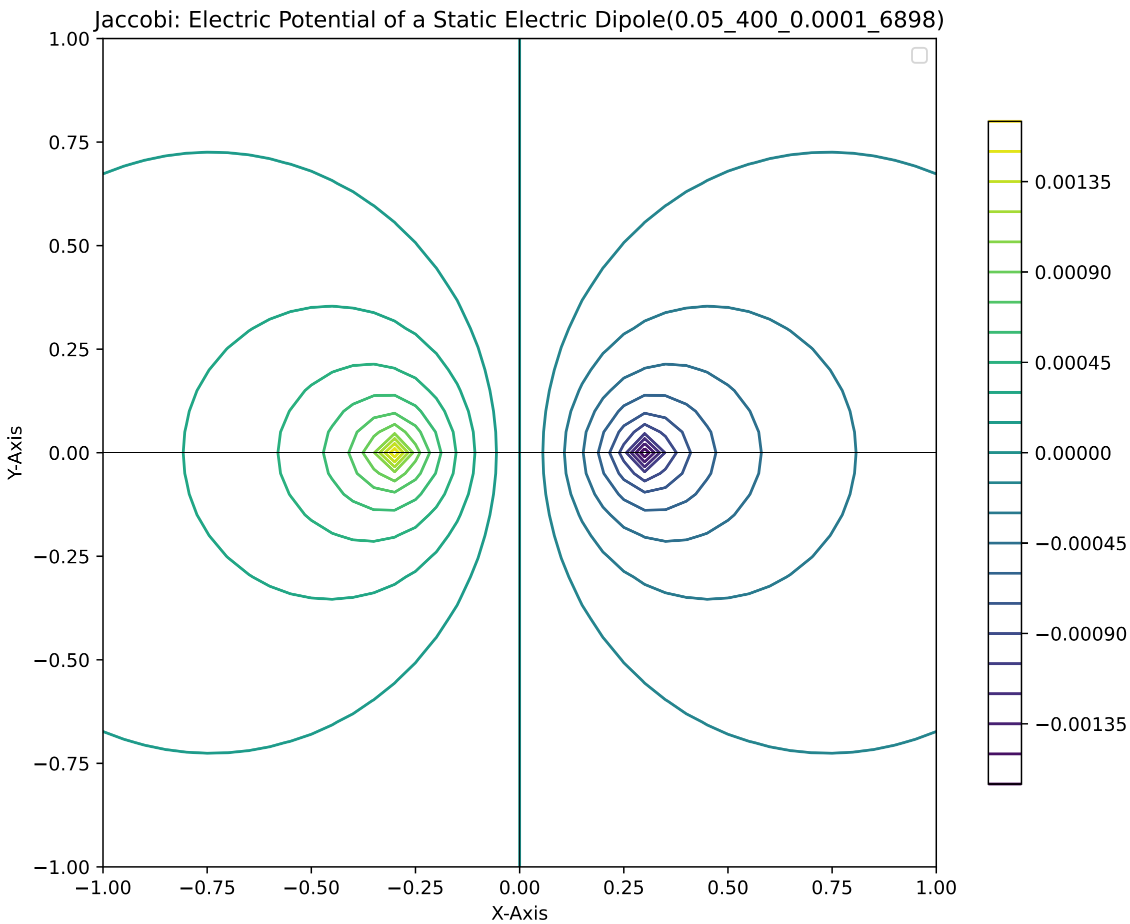

Figure 21: Electric Potential of a Static Electric Dipole - tolerance = 0.0001

Tolerance	No. of Iterations
0.001	45
0.00055	168
0.0001	6898

Table 1: Table of Tolerance and Number of Iterations

3.3 Simultaneous Over-Relaxation Method (SOR) & investigate (and plot) how the number of iteration steps, depends on the number n of grid points

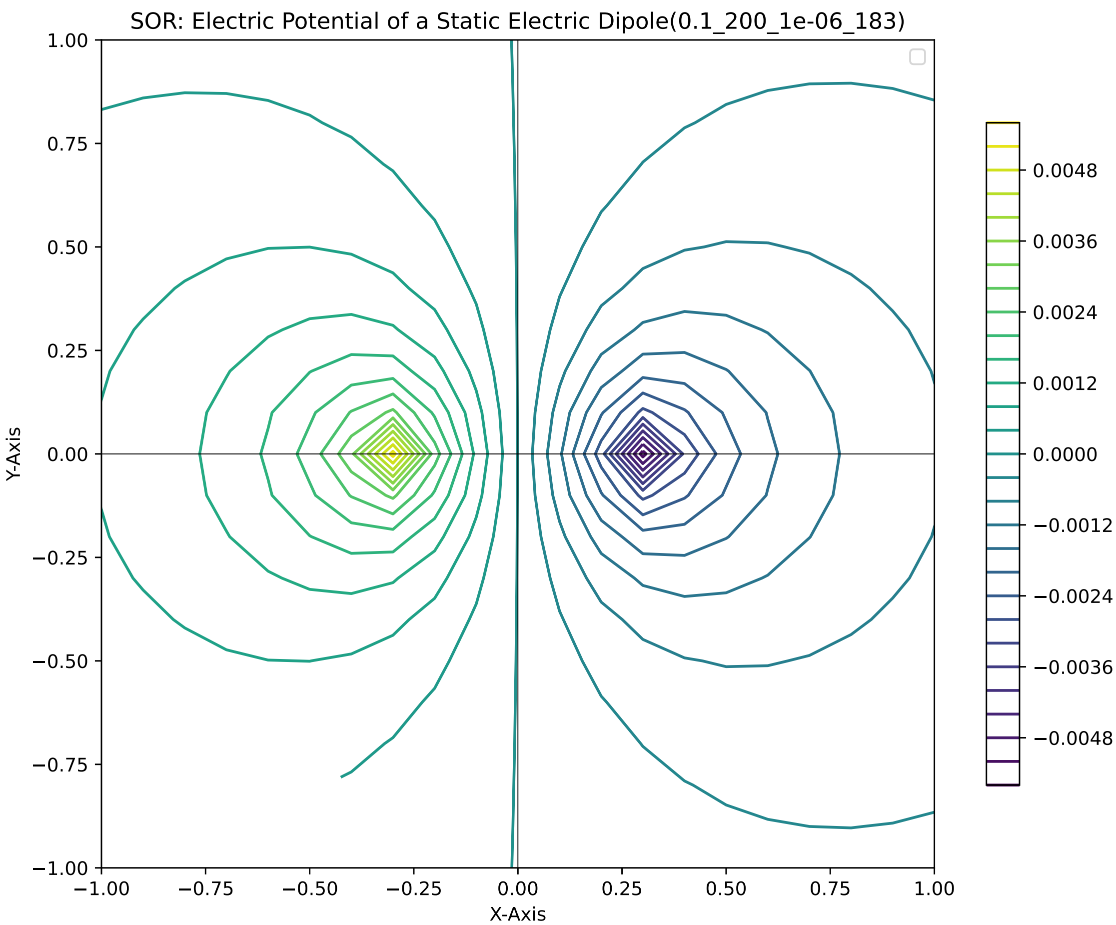

Figure 22: Electric Potential of a Static Electric Dipole - fixed accuracy = 1e-06 ; n = 200

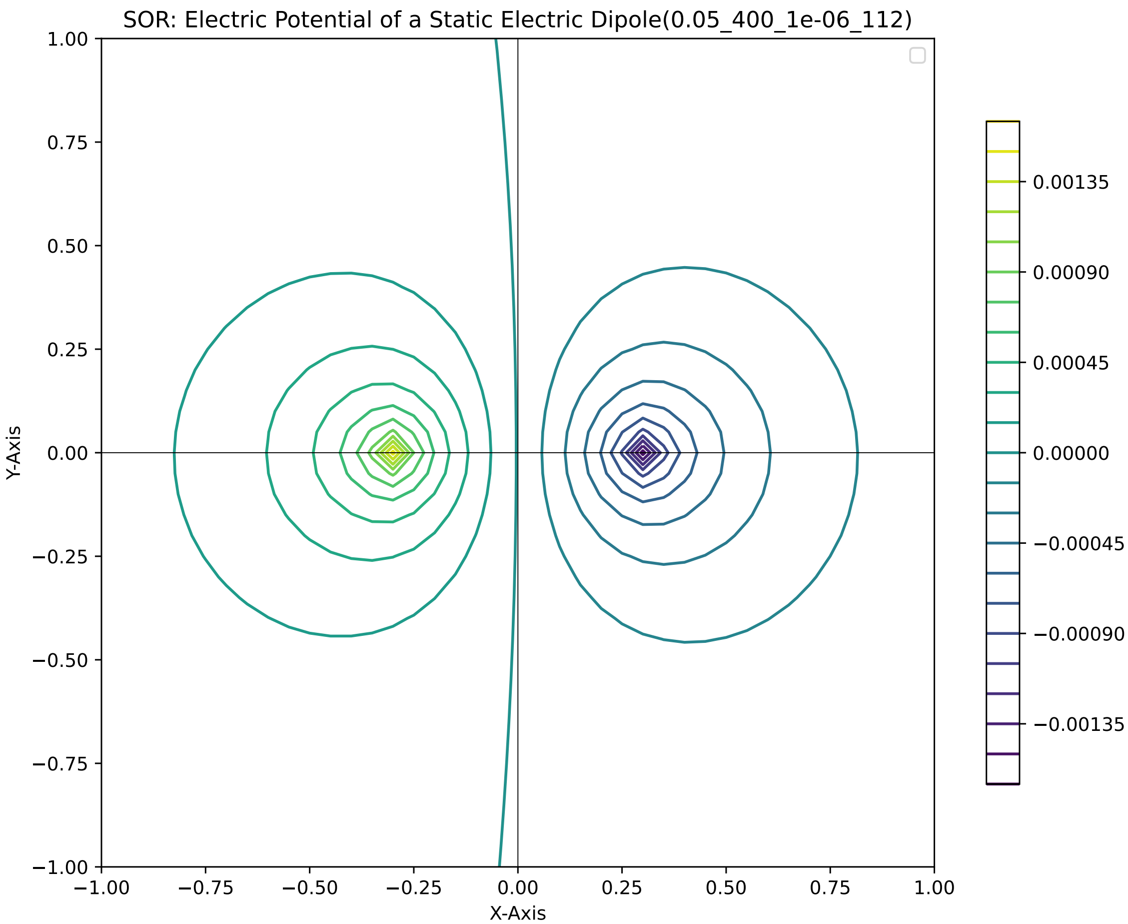

Figure 23: Electric Potential of a Static Electric Dipole - fixed accuracy = 1e-06 ; n = 400

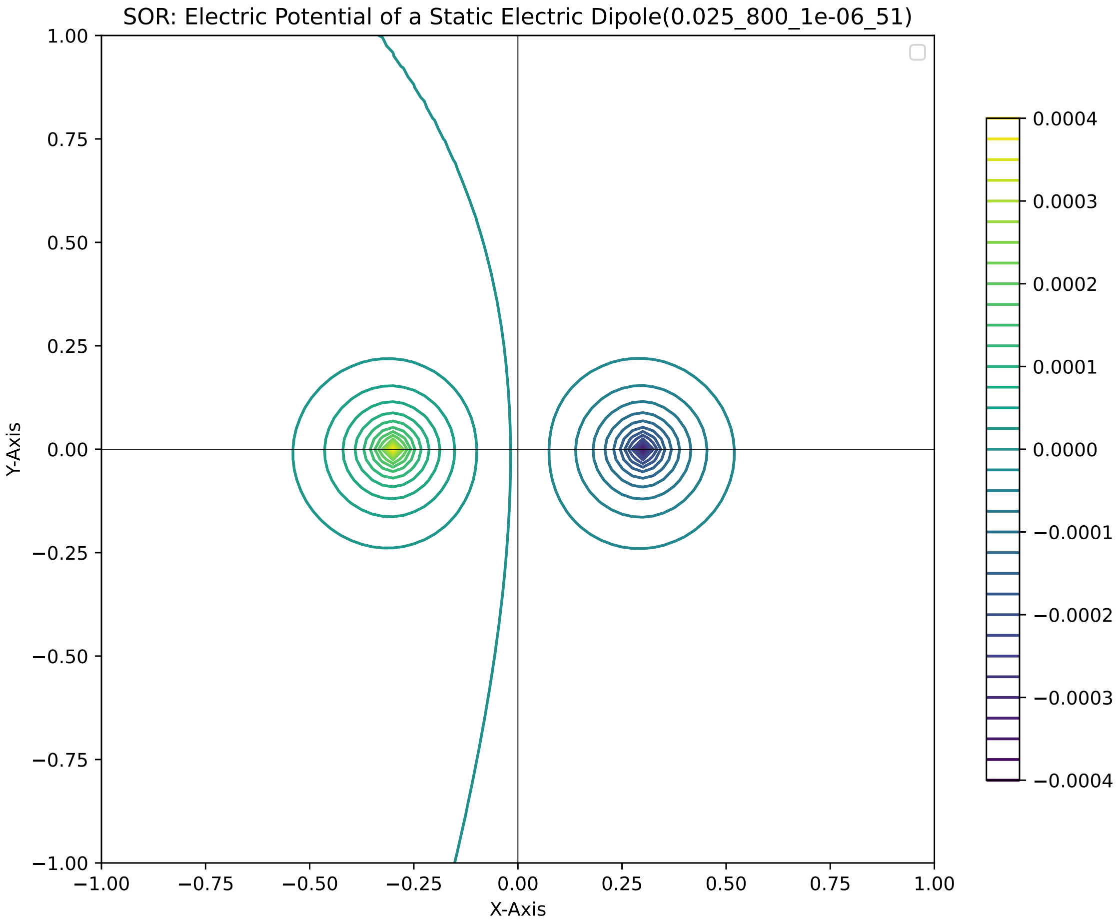

Figure 24: Electric Potential of a Static Electric Dipole - fixed accuracy = 1e-06 ; n = 800

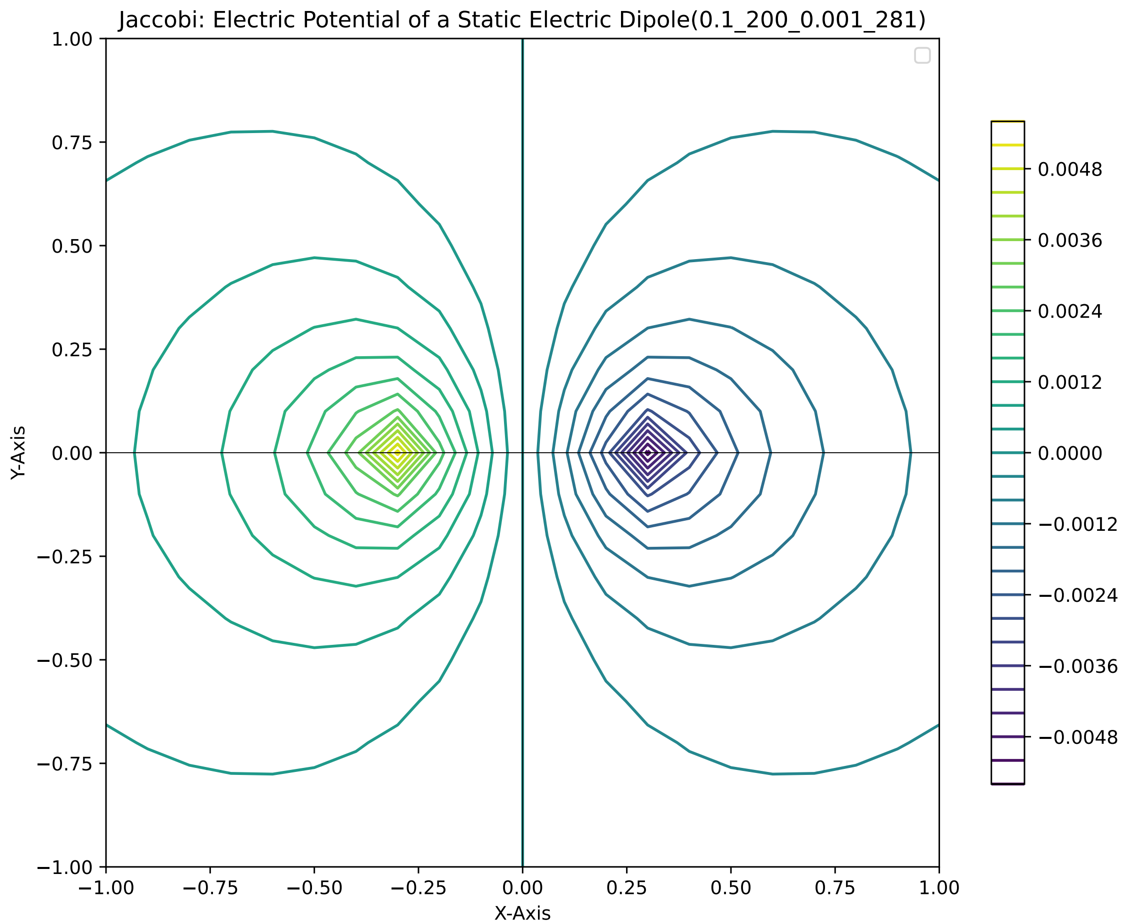

Figure 25: Electric Potential of a Static Electric Dipole - Tolerance = 0.001 ; n = 200

4 Contributions References

References

[1] Electrostatics with partial differential equations – A numerical example. Available at: http://ael.cbnu.ac.kr/lectures/graduate/adv-em-1/2018-1/lecture-notes/electrostatics-1/ch4-electrostatics-1-appendix-C-numerical-sol-poisson.pdf

[2] Numerical Methods and the Dampened, Driven Pendulum. Available at: https://webhome.phy.duke.edu/~mbe9/data_science/pendulum/numerical_methods_and_the_dampened_driven_pendulum.pdf

[3] How do I fit a sine curve to my data with pylab and numpy? Available at: https://stackoverflow.com/questions/16716302/how-do-i-fit-a-sine-curve-to-my-data-with-pylab-and-numpy

[4] Remarks on the static dipole-dipole potential at large distances. Available at: `https://journals.aps.org/prd/abstract/10.1103/PhysRevD.92.096007`