1 Problem Description

The objective of the HW3 is to solve the problem of Oscillatory Motion and Chaos and also to solve Poisson equation. The Oscillatory problem can be solved using Euler-Cromer or Runge-Kutta method, And to solve the Poisson equation we can use Jacobi or gauss-seidel. Detailed explanation can be found below:

2 Solution to Pendulum Problem

2.1 Calculate analytically at what (approximate) value of Ω_D the resonance occurs. Do you expect the small-angle (linear) approximation to be good?

$$\frac{d^2\theta}{dt^2} = -g\theta - 2\gamma\frac{d\theta}{dt} + \alpha_D \sin(\Omega_D t) \tag{1}$$

$$\theta'' + 2\gamma\theta' + \omega_0^2\theta = F(t) \quad F(t) = \alpha_D \sin(\Omega_D t) \tag{2}$$

Assuming $F(t) = 0$, such that $\theta(t) \sim e^{rt}$, where r is a root.

$$r^2 + 2\gamma r + \omega_0^2 = 0 \quad r_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{3}$$

$$\theta(t)_{\text{Homogeneous}} = (Ae^{rt} + Be^{-rt})e^{-\gamma t} \quad \text{if } \gamma \neq \omega_0 \tag{5.1}$$

$$\theta(t)_{\text{Homogeneous}} = (A + Bt)e^{-\gamma t} \quad \text{if } \gamma = \omega_0 \tag{5.2}$$

Condition: steady-state (ss)

$$\theta(t)_{\text{ss}} = \theta_P \sin(\Omega_D t - \phi) \tag{6}$$

Puting eqn (6) into (2) with $F(t) = \alpha D \sin(\Omega_D t)$ results in:

$$\alpha_D \sin(\Omega_D t) = \frac{\omega_0^2 - \Omega_D^2}{\alpha} \sin(\Omega_D t - \phi) + 2\gamma\Omega_D \cos(\Omega_D t - \phi) \tag{7}$$

As,

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{8}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\left(\frac{\theta_P}{\alpha_D}(\omega_0^2 - \Omega_D^2)\right) + \cos(\alpha)\left(\frac{2\gamma\Omega_D\theta_P}{\alpha_D}\right) \tag{9}$$

Therefore:

$$\cos(\phi) = \frac{\theta_P}{\alpha_D}(\omega_0^2 - \Omega_D^2) \tag{10}$$

$$\sin(\phi) = \frac{2\gamma\Omega_D\theta_P}{\alpha_D} \tag{11}$$

Finding, $\theta_P(\Omega_D)$ and $\phi(\Omega_D)$ from $\sin(\phi)$ and $\cos(\phi)$

$$\theta_P(\Omega_D) = \frac{\alpha_D}{\sqrt{(\omega_0^2 - \Omega_D^2)^2 + 4\gamma^2\Omega_D^2}} \tag{12}$$

$$\phi(\Omega_D) = \arctan\left(\frac{2\gamma\Omega_D}{\omega_0^2 - \Omega_D^2}\right) \tag{13}$$

To find Ω_{Res} analytically, we can equate the first derivative of to zero:

$$\frac{d\theta_P}{d\Omega} = 0 \tag{13}$$

On solving we will get

$$\Omega_{\text{Res}} = \sqrt{\omega_0^2 - 2\gamma^2} \tag{14}$$

On taking,

$$\omega_0 = 1sec^{-1} \tag{15}$$

$$\gamma = 0.25sec^{-1} \tag{16}$$

We get,

$$\Omega_{\text{Res}} = 0.935 \text{ rad/sec} \tag{17}$$