13 Markov Chains: Classification of States

We say that a state j is *accessible from state* i, $i \to j$, if $P_{ij}^n > 0$ for some $n \ge 0$. This means that there is a possibility of reaching j from i in some number of steps. If j is not accessible from i, $P_{ij}^n = 0$ for all $n \ge 0$, and thus the chain started form i never visits j:

$$\begin{aligned} P(\text{ever visit } j|X_0 = i) &= P(\bigcup_{n=0}^{\infty}\{X_n = j\}\,|X_0 = i) \\ &\le \sum_{n=0}^{\infty} P(X_n = j|X_0 = i) = 0. \end{aligned}$$

Also, note that for accessibility the size of entries of P does not matter, all that matters is which are positive and which 0. For computational purposes, one should also observe that, if the chain has m states, then j is accessible from i if and only if $(P + P^2 + \ldots + P^m)_{ij} > 0$.

If i is accessible from j, and j is accessible from i, then we say that i and j *coummunicate*, $i \leftrightarrow j$. It is easy to check that this is an *equivalence relation*:

1. $i \leftrightarrow i$;

2. $i \leftrightarrow j$ implies $j \leftrightarrow i$; and

3. $i \leftrightarrow j$ and $j \leftrightarrow k$ together imply $i \leftrightarrow k$.

The only nontrivial part is (3), and to prove it, let's assume $i \to j$ and $j \to k$. This means that there exists an $n \ge 0$ so that $P_{ij}^n > 0$ and an $m \ge 0$ so that $P_{jk}^m > 0$. Now, one can get from i to j in $m + n$ steps by going first to j in n steps, and then from j to k in m steps, so that that

$$P_{ik}^{n+m} \ge P_{ij}^n P_{jk}^m > 0.$$

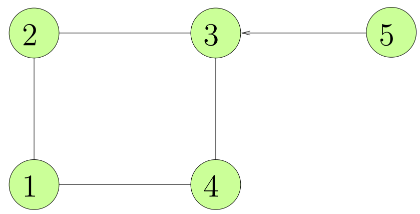
(Alternatively, one can use that $P^{m+n} = P^n \cdot P^m$ and then

$$P_{ik}^{n+m} = \sum_{\ell} P_{i\ell}^n P_{\ell k}^m \ge P_{ij}^n P_{jk}^m,$$

as the sum of nonnegative numbers is at least as large as one of its terms.)

The accessibility relation divides states into *classes*. Within each class, all states communicate to each other, but no pair of states in different classes communicates. The chain is *irreducible* if there is only one class. If the chain has m states, irreducibility means that all entries of $I + P + \ldots + P^m$ are nonzero.

Example 13.1. To determine the classes we may give the Markov chain as a graph, in which we only need to depict edges which signify nonzero transition probabilities (their precise value is irrelevant for this purpose); by convention, we draw an undirected edge when probabilities in both directions are nonzero. Here is an example:



Any state $1, 2, 3, 4$ is accessible from any of the five states, but 5 is not accessible from $1, 2, 3, 4$. So we have two classes: $\{1, 2, 3, 4\}$, and $\{5\}$. The chain is not irreducible.

Example 13.2. Consider the chain on states 1, 2, 3, and

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

As $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$, this is an irreducible chain.

Example 13.3. Consider the chain on states 1, 2, 3, 4, and

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This chain has three classes $\{1, 2\}$, $\{3\}$ and $\{4\}$ and is hence not irreducible.

For any state i, denote

$$f_i = P(\text{ever reenter } i | X_0 = i)$$

We call a state i *recurrent* if $f_i = 1$, and *transient* if $f_i < 1$.

Example 13.4. Back to the previous example. Obviously, 4 is recurrent, as it is an absorbing state. The only possibility to return to 3 is to do so in one step, so we have $f_3 = \frac{1}{4}$, and 3 is transient. Moreover, $f_1 = 1$ because in order never to return to 1 we need to go to to state 2 and stay there forever. We stay at 2 for n steps with probability

$$\left(\frac{1}{2}\right)^n \to 0,$$

as $n \to \infty$, so so the probability of staying at 1 forever is 0 and consequently $f_1 = 1$. By a similar logic, $f_2 = 1$.

Starting from an any state, a Markov Chain visits a recurrent state infinitely many times, or not at all. Let us now compute, in two different ways, the expected number of visits to i (i.e., the times, including time 0, when the chain is at i). First we observe that at every visit to i, the probability of never visiting i again is $1 - f_i$, therefore,

$$P(\text{exactly } n \text{ visits to } i | X_0 = i) = f_i^{n-1}(1 - f_i).$$

This formula says that the number of visits to i is a Geometric$(1-f_i)$ random variable so its expectation equals

$$E(\text{number of visits to } i|X_0=i) = \frac{1}{1-f_i}.$$

The second way to compute this expectation is by using the indicator trick:

$$E(\text{number of visits to } i|X_0=i) = E(\sum_{n=0}^{\infty} I_n|X_0=i),$$

where $I_n = I_{\{X_n=i\}}$, $n = 0, 1, 2, \ldots$ Then

$$\begin{aligned} E(\sum_{n=0}^{\infty} I_n|X_0=i) &= \sum_{n=0}^{\infty} P(X_n=i|X_0=i) \\ &= \sum_{n=0}^{\infty} P_{ii}^n. \end{aligned}$$

We have thus proved the following theorem.

Theorem 13.1. *Characterization of recurrence via n step return probabilities:*

$$\boxed{\text{A state } i \text{ is recurrent if and only if } \sum_{n=1}^{\infty} P_{ii}^n = \infty.}$$

We call a subset $S_0 \subset S$ of states *closed* if $p_{ij} = 0$ for each $i \in S_0$ and $j \notin S_0$. In plain language, once entered, a closed set cannot be exited.

Proposition 13.2. *If a closed subset S_0 only has finitely many states, then there must be at least one recurrent state. In particular any finite Markov chain must contain at least one recurrent state.*

Proof. Start from any state from S_0. By definition, the chain stays in S_0 forever. If all states in S_0 are transient, then each of them is visited not at all or only finitely many times. This is impossible. □

Proposition 13.3. *If i is recurrent, and $i \to j$, then also $j \to i$.*

Proof. There is some n_0 so that $P_{ij}^{n_0} > 0$, i.e., starting from i, the chain can reach from j in n_0 steps. Thus every time it is at i, there is a fixed positive probability that it will be at j n_0 steps later. Starting from i, the chain returns to i infinitely many times, and every time it does so it has an independent chance to reach j n_0 steps later; thus, eventually the chain does reach j. Now assume that it is not true that $j \to i$. Then, once the chain reaches j, it never returns to i, but then i is not recurrent. This contradiction ends the proof. □

Proposition 13.4. *If i is recurrent, and $i \to j$, then j is also recurrent. Therefore, in any class, either all states are recurrent or all are transient. In particular, if the chain is irreducible, then either all states are recurrent or all are transient.*

In light of this proposition, we can classify each class, and an irreducible Markov chain, as recurrent or transient.

Proof. By the previous proposition, we know that also $j \to i$. We will now give two arguments for the recurrence of j.

We could use the same logic as before: starting from j, the chain must visit i with probability 1 (or else the chain starting at i has a positive probability of no return to i, by visiting j), then it returns to i infinitely many times, and each of those times it has an independent chance of getting to j some time later — so it must do so infinitely often.

For another argument, we know that there exist $k, m \geq 0$ so that $P_{ij}^k > 0, P_{ji}^m > 0$. Furthermore, for any $n \geq 0$, one way to get from j to j in $m + n + k$ steps is by going from j to i in m steps, then from i to i in n steps, and then from i to j in k steps, thus,

$$P_{ij}^{m+n+k} \geq P_{ji}^m P_{ii}^n P_{ij}^k.$$

If $\sum_{n=0}^{\infty} P_{ii} = \infty$, then $\sum_{n=0}^{\infty} P_{jj}^{m+n+k} = \infty$, and finally $\sum_{\ell=0}^{\infty} P_{jj}^{\ell} = \infty$. In short, if i is recurrent, then so is j. □

Proposition 13.5. *Any recurrent class is a closed subset of states.*

Proof. Let S_0 be a recurrent class, $i \in S_0$ and $j \notin S_0$. We need to show that $p_{ij} = 0$. Assume the converse, $p_{ij} > 0$. As j does not communicate with i, the chain never reaches i from j, i.e., i is not accessible from j. But this is a contradiction with Proposition 13.3. □

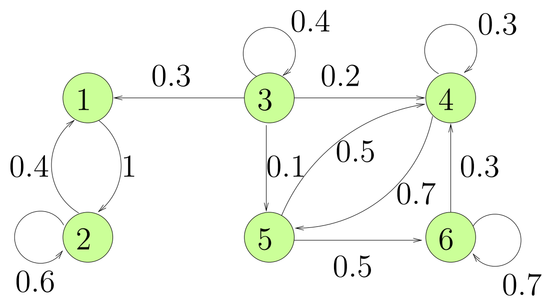
Example 13.5. Assume the states 1, 2, 3, 4 and transition matrix

$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

By inspection, every state is accessible from any other state, so this chain is irreducible. Therefore, every state is recurrent.

Example 13.6. Assume now that the states are $1, \ldots, 6$ and

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$



We observe that 3 can only be reached from 3, therefore 3 is in a class of its own. States 1 and 2 can reach each other but no other state, so the form a class together. Furthermore, 4, 5, 6 all communicate with each other. The division into classes thus is $\{1,2\}$, $\{3\}$, and $\{4,5,6\}$. Clearly, $f_3 = 0.4$, so $\{3\}$ is a transient class. On the other hand, $\{1,2\}$ and $\{4,5,6\}$ are both closed and therefore recurrent.

Example 13.7. *Recurrence of simple random walk on* $\mathbb{Z}$*.* Recall that such a walker moves from x to $x+1$ with probability p and to $x-1$ with probability $1-p$. We will assume that $p \in (0,1)$, and denote the chain $S_n = S_n^{(1)}$. (The superscript indicates the dimension. We will make use of this in subsequent examples, in which the walker will move in higher dimensions.) As such walk is irreducible, we only need to check whether state 0 is recurrent or transient, so we assume that the walker begins at 0. First, we observe that the walker will be at 0 at a later time only if she makes an equal number of left and right moves. Thus, for $n = 1, 2, \ldots$,

$$P_{00}^{2n-1} = 0,$$

and

$$P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n.$$

Now you need to recall Stirling's formula:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

(the symbol "$\sim$" means that the quotient of two quantities converges to 1 as $n \to \infty$). Therefore,

$$\begin{aligned}
\binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \\
&\sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi 2n}}{n^{2n} e^{-2n} 2\pi n} \\
&= \frac{2^{2n}}{\sqrt{n\pi}},
\end{aligned}$$

and therefore

$$\begin{aligned}
P_{00}^{2n} &= \frac{2^{2n}}{\sqrt{n\pi}} p^n (1-p)^n \\
&\sim \frac{1}{\sqrt{n\pi}} (4p(1-p))^n
\end{aligned}$$

In the symmetric case, when $p = \frac{1}{2}$,

$$P_{00}^{2n} \sim \frac{1}{\sqrt{n\pi}},$$

therefore

$$\sum_{n=0}^{\infty} P_{00}^{2n} = \infty,$$

and the random walk is recurrent.

When $p \neq \frac{1}{2}$, $4p(1-p) < 1$, so that P_{00}^{2n} goes to 0 faster than terms of a convergent geometric series,

$$\sum_{n=0}^{\infty} P_{00}^{2n} < \infty,$$

and the random walk is transient. What is in this case the probability f_0 that the chain ever reenters 0? We need to recall the Gambler's ruin probabilities,

$$P(S_n \text{ reaches } N \text{ before } 0|S_0 = 1) = \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^N}.$$

As $N \to \infty$, the probability

$$P(S_n \text{ reaches } 0 \text{ before } N|S_0 = 1) = 1 - P(S_n \text{ reaches } N \text{ before } 0|S_0 = 1)$$

converges to

$$P(P(S_n \text{ ever reaches } 0|S_0 = 1) = \begin{cases} 1 & \text{if } p < \frac{1}{2}, \\ \frac{1-p}{p} & \text{if } p > \frac{1}{2}. \end{cases}$$

Assume that $p > \frac{1}{2}$. Then,

$$\begin{aligned} f_0 &= P(S_1 = 1, S_n \text{ returns to 0 eventually}) + P(S_n = -1, S_n \text{ returns to 0 eventually}) \\ &= p \cdot \frac{1-p}{p} + (1-p) \cdot 1 \\ &= 2(1-p). \end{aligned}$$

If $p < 1/2$, we may use the fact replacing the walker with its mirror image replaces p by $1 - p$, to obtain that $f_0 = 2p$ when $p < \frac{1}{2}$.

Example 13.8. *Is simple symmetric random walk on $\mathbb{Z}^2$ recurrent?* A walker now moves on integer points in two dimensions: each step is a distance 1 jump in one of the four directions (N, S, E, or W). We denote this Markov chain by $S_n^{(2)}$, and imagine a drunk wondering at random through rectangular grid of streets of a large city (Chicago would be a good approximation). The question is whether the drunk will eventually return to her home at $(0, 0)$. All starting positions in this and in the next example will be the appropriate origins. Note again that the walker can only return in even number of steps, and in fact both the number of steps in the x direction (E or W) and in the y direction (N or S) must be even (otherwise the respective coordinate cannot be 0).

We condition on the number N of times walk moves in the x-direction:

$$\begin{aligned}
P(S_{2n}^{(2)} = (0,0)) &= \sum_{k=0}^{n} P(N = 2k) P(S_{2n}^{(2)} = (0,0) | N = 2k) \\
&= \sum_{k=0}^{n} P(N = 2k) P(S_{2k}^{(1)} = 0) P(S_{2(n-k)}^{(1)} = 0).
\end{aligned}$$

In order not to obscure the computation, we will go into full detail from now on; filling in the missing pieces is an excellent computational exercise.

First, as the walker chooses to go horizontally or vertically with equal probability, $N \sim \frac{2n}{2} = n$ with overwhelming probability, and so we can assume $k \sim \frac{n}{2}$. Taking this into account

$$\begin{aligned}
P(S_{2k}^{(1)} = 0) &\sim \frac{\sqrt{2}}{\sqrt{n\pi}}, \\
P(S_{2(n-k)}^{(1)} = 0) &\sim \frac{\sqrt{2}}{\sqrt{n\pi}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(S_{2n}^{(2)} = (0,0)) &\sim \frac{2}{n\pi} \sum_{k=0}^{n} P(N = 2k) \\
&\sim \frac{2}{n\pi} P(N \text{ is even}) \\
&\sim \frac{1}{n\pi},
\end{aligned}$$

as we know that (see Problem 1 in Chapter 11)

$$P(N \text{ is even}) = \frac{1}{2}.$$

Therefore,

$$\sum_{n=0}^{\infty} P(S_{2n}^{(2)} = (0,0)) = \infty.$$

and we have demonstrated that this chain is still recurrent, albeit barely. In fact, there is an easier proof, a very slick one that does not generate to higher dimensions, which demonstrates that

$$P(S_{2n}^{(2)} = 0) = P(S_{2n}^{(1)} = 0)^2.$$

Here is how it goes. If you let each coordinate of a two-dimensional random walk move *independently*, then the above is certainly true. Such a walk makes *diagonal* moves, from (x, y) to $(x+1, y+1)$, $(x-1, y+1)$, $(x+1, y-1)$, or $(x-1, y-1)$ with equal probability. At first this appears to be a different walk, but if you rotate the lattice by 45 degrees, scale by $\frac{1}{\sqrt{2}}$, and ignore half of the points that are never visited, this becomes the same walk as $S_n^{(2)}$. In particular, it is at the origin exactly when $S_n^{(2)}$ is.

Example 13.9. *Is simple symmetric random walk on* $\mathbb{Z}^3$ *recurrent?* Now imagine a squirrel running around in a 3 dimensional maze. The process $S_n^{(3)}$ moves from a point (x, y, z) to one of the six neighbors $(x \pm 1, y, z)$, $(x, y \pm 1, z)$, $(x, y, z \pm 1)$ with equal probability. To return to $(0, 0, 0)$, it has to make an even number number of steps in each of the three directions. We will condition on the number N of steps in z direction. This time $N \sim \frac{2n}{3}$, and thus

$$
\begin{aligned}
P(S_{2n}^{(3)} = (0,0,0)) &= \sum_{k=0}^{n} P(N = 2k) P(S_{2k}^{(1)} = 0) P(S_{2(n-k)}^{(2)} = (0,0)) \\
&\sim \sum_{k=0}^{n} P(N = 2k) \frac{\sqrt{3}}{\sqrt{\pi n}} \frac{3}{2\pi n} \\
&= \frac{3\sqrt{3}}{2\pi^{3/2} n^{3/2}} P(N \text{ is even}) \\
&\sim \frac{3\sqrt{3}}{4\pi^{3/2} n^{3/2}}.
\end{aligned}
$$

Therefore,

$$\sum_n P(S_{2n}^{(3)} = (0,0,0)) < \infty,$$

and the three-dimensional random walk is transient, so the squirrel may never return home. The probability $f_0 = P(\text{return to } 0)$ is thus not 1, but can we compute it? One approximation is obtained using

$$\frac{1}{1 - f_0} = \sum_{n=0}^{\infty} P(S_{2n}^{(3)} = (0,0,0)) = 1 + \frac{1}{6} + \ldots,$$

but this series converges slowly and its terms are difficult to compute. Instead, one can use this remarkable formula, derived by Fourier analysis,

$$\frac{1}{1 - f_0} = \frac{1}{(2\pi)^3} \iiint_{(-\pi,\pi)^3} \frac{dx\, dy\, dz}{1 - \frac{1}{3}(\cos(x) + \cos(y) + \cos(z))},$$

which gives, to four decimal places,

$$f_0 \approx 0.3405.$$

Problems

1. For the following transition matrices, determine the classes and specify which are recurrent and which transient.

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad P_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Assume a Markov chain X_n has states $0, 1, 2 \ldots$ and transitions from each $i > 0$ to $i+1$ with probability $1 - \frac{1}{2 \cdot i^\alpha}$ and to 0 with probability $\frac{1}{2 \cdot i^\alpha}$. Moreover, from 0 it transitions to 1 with probability 1. (a) Is this chain irreducible? (b) Assume $X_0 = 0$ and let R be the first return time to 0 (i.e., the first time after the initial time the chain is back at the origin). Determine for which α is

$$1 - f_0 = P(\text{no return}) = \lim_{n \to \infty} P(R > n) = 0.$$

(c) Depending on α, determine which classes are recurrent.

3. Consider the one-dimensional simple symmetric random walk, $S_n = S_n^{(1)}$ with $p = \frac{1}{2}$. As in the Gambler's ruin problem, fix an N, and start at some $0 \le i \le N$. Let E_i be the expected time at which the walk first hits either 0 or N. (a) By conditioning on the first step, determine the recursive equation for E_i. Also determine the boundary conditions E_0 and E_N. (b) Solve the recursion. (c) Assume that the chain starts at 0 and let R be the first time (after time 0) that it revisits 0. By recurrence, we know that $P(R < \infty) = 1$; use (b) to show that $ER = \infty$. The walk will eventually return 0, but the expected waiting time is infinite!

Solutions to problems

1. Assume states are $1, \ldots, 5$. For P_1: $\{1, 2, 3\}$ recurrent, $\{4\}$ transient, $\{5\}$ transient. For P_2: irreducible, so all states recurrent. For P_3: $\{1, 2, 3\}$ recurrent, $\{4, 5\}$ recurrent. For P_4: $\{1, 2\}$ recurrent, $\{3\}$ recurrent (absorbing), $\{4\}$ transient, $\{5\}$ transient.

2. (a) The chain is irreducible. (b) If $R > n$, then the chain, after moving to 1, makes $n - 1$

consecutive steps to the right, so

$$P(R > n) = \prod_{i=1}^{n-1} \left(1 - \frac{1}{2 \cdot i^{\alpha}}\right).$$

The product converges to 0 if and only if its logarithm converges to $-\infty$ and that is if and only if the series

$$\sum_{i=1}^{\infty} \frac{1}{2 \cdot i^{\alpha}}$$

diverges, which is when $\alpha \leq 1$. (c) For $\alpha \leq 1$, the chain is recurrent, and transient otherwise.

3. For (a), the walk uses one step, and then proceeds from $i+1$ or $i-1$ with equal probability, so that

$$E_i = 1 + \frac{1}{2}(E_{i+1} + E_i),$$

with $E_0 = E_N = 0$. For (b), the homogeneous equation is the same as the one in Gambler's ruin, so its general solution is linear: $Ci + D$. We look for a particular solution of the form Bi^2, and we get $Bi^2 = 1 + \frac{1}{2}(B(i^2 + 2i + 1) + B(i^2 - 2i + 1)) = Bi^2 + B$, so $B = -1$. By plugging in boundary conditions we can solve for C and D to get $D = 0$, $C = N$. Therefore

$$E_i = i(N - i).$$

For (c), after a step the walk proceeds either from 1 or -1, and by symmetry the expected time to get to 0 is the same for both. So, for every N,

$$ER \geq 1 + E_1 = 1 + 1 \cdot (N - 1) = N,$$

and so $ER = \infty$.