① You can try to write the pdf for the the $X \sim$ no. of fix points in permutation. as

$$\boxed{f_X(x) = \frac{{}^{n}C_x \, D_x}{n!}} \qquad x \in \{0, 1, 2, \dots, n\}$$

here D_x = dearrangment of $n-x$ elements

and you can see that we are ~~sel~~ ~~se~~ selecting x elements maps them to themselves and Dearrange the rest

But solving the question using the above pdf, won't be an easy task, handling few dearrangement is ~~easy~~ already ~~so~~ hard and here you are dealing with arbitary n. So we will try different approach and breaks X into as sum of X_is

where each X_i is a random variable with following defination.

$$X_i = \begin{cases} 0 & \text{if } f(i) \neq i \\ 1 & \text{if } f(i) = i \end{cases} \quad \text{for any } f$$

belonging to set of Bijective functions

~~then~~

$$X = \sum_{i=1}^{n} X_i$$

$X \sim$ no. of fix points $\Rightarrow \sum_{i=1}^{n} X_i$

for example for a given $f \in \Omega$

[$X(f) = \sum X_i(f)$ = no. of fix points in f]

[~~Rember~~ Remember X is a function from $\Omega \to \mathbb{R}$]

Now $E(X) = \sum_{i=1}^{n} E(X_i)$ [linearity]

now each X_i follow Bernoulli

$$X_i = \begin{cases} 0 & ; (1-p) \text{ probability} \\ 1 & , p \text{ probability} \end{cases}$$

$p = \frac{(n-1)!}{n!} = 1/n$, as $i \mapsto i$, and there are $n-1$ elements available for mapping

$E(X_i) = \sum_{x} x f_X(x) = 0 \times (1-p) + 1 \times (\frac{1}{n})$
$= 1/n$

$$E(X) = \sum_{n=1}^{n} 1/n = 1$$

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Variance

E

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = E((\sum(X_i))^2) - 1$$

$$Var(X) = E\left(\sum_{1}^{n} x_i^2 + \sum_{i=1}^{n}\sum_{i \neq j}(x_i x_j)\right) - 1$$

$$Var(X) = \sum_{i=1}^{n} E(x_i^2) + \sum_{i=1}^{n}\sum_{i \neq j} E(x_i x_j) - 1$$

$$E(x_i^2) = 1/n \quad \left[\begin{matrix} 0^2 = 0 & [1-p] \\ 1^2 = 1 & [p] \end{matrix}\right]$$

$$Var(X) = \sum_{i=1}^{n}\sum_{i \neq j} E(x_i x_j)$$

Now distribution of $x_i x_j$ $[i \neq j]$

$$Y = x_i x_j = \begin{cases} 1, & [\text{when } x_i = 1 \text{ and } x_j = 1] \\ 0 & \text{otherwise} \end{cases}$$

[so $x_i x_j$ follow bernoulli]

You may think $P(X_i X_j = 1) = \underset{=1}{P(X_i)}\, \underset{=1}{P(X_j)}$

$= \frac{1}{n^2}$

but that is not the case as X_i and X_j are not independent so

$P[X_i X_j = 1] = \frac{(n-2)!}{n!}$ $\left[\begin{array}{l} i \longrightarrow i \,,\; j \longrightarrow j \\ (n-2) \text{ elements remaining for mapping} \end{array} \right]$

$P[X_i X_j = 1] = 1/n(n-1)$

$E(X_i X_j) = \frac{1}{n(n-1)}$

$\text{Var}(X) = \sum_{i=1}^{n} \sum_{i \neq j} \frac{1}{n(n-1)} = n(n-1) * \frac{1}{n(n-1)} = 1$

2nd $Z = \min(X, Y)$

Let F_Z be the distribution of Z

then $F_Z(t) = P(Z \leq t)$

$$= 1 - P(Z > t)$$

$$= 1 - P(X, Y > t)$$

$$= 1 - (1 - F_X(t))(1 - F_Y(t))$$

$$\left[\begin{array}{l} F_X(t) = P(X \leq t) \\ F_Y(t) = P(Y \leq t) \end{array} \right]$$

$$\left[\therefore F_Z(t) = F_X(t) + F_Y(t) - F_X(t)F_Y(t) \right]$$

If you try to write pmf ~~in this~~ of X ~~as then~~ problem you will encounter inclusion-exclusion for all $x \in [0, 1, \ldots 99]$ it will difficult to deal with it

(3) Picking of balls can be seen as arrangement of the balls.

$$O_1 \bullet_2 \bullet \; . \; . \; . \; . \; . \; O_{98} O_{99} \bullet$$

now there are 99 places where change can occur.

let Ω = set of all arrangements.

$$X_i : \Omega \longrightarrow \{0, 1\}$$

$$\omega \longrightarrow \begin{cases} 0 & \text{if } i\text{th place doesn't have a color change} \\ 1 & \text{if } i\text{th place have a colour change} \end{cases}$$

so X_i follow bernoulli with probability p

so $$p = \underset{\text{[black - white}}{\frac{50}{100} \times \frac{50}{99}} + \underset{\text{white - black]}}{\frac{50}{100} \times \frac{50}{99}} = \frac{50}{99}$$

Now $X = \sum_{i=1}^{99} X_i$ [similar argument as in case of 1st problem

$X \sim$ no. of changes in any arrangement.

$$E(X) = E(\sum X_i) = \sum E(X_i) = \frac{99 \times 50}{99}$$

$$= 50$$

[X_i is bernoulli with p so $E(X_i) = p$]

(4) for N coupons

~~define same case you cannot directly define pmf of X [no. of coupons required to get all~~

Same case as of previous problem, you cannot directly define pmf of X [no. of tries required to ge all coupon] in a pretty way, so will just break X into X_is.

Now $X_i \sim$ [no. of tries required to get i^{th} coupon given you already have $i-1$ different coupons.]

$$\left[X = \sum_{i=1}^{N} X_i \right]$$

Now each X_i is a geometric distribution

$$X_i \sim \left[f_{X_i} = (1-p)^{K-1} p \right]$$

$K \in \mathbb{N}$

p = probability of success.

geometric distribution has two defination as
1) No. of tries to get first success
2) No. of failures before first success

We are using (1) in this problem

How is this geometric

for X_i, we already have ~~i~~ $i-1$ coupons now if we select any coupon from remaining $N-i+1$ it will be success, we will get the i^{th} different coupon

$$p = \frac{N-i+1}{N}$$ [probability of success]

Now in case of Geometric (1) we have
$$E(X) = \frac{1}{p}$$

so $$E(X_i) = \frac{N}{N-i+1}$$

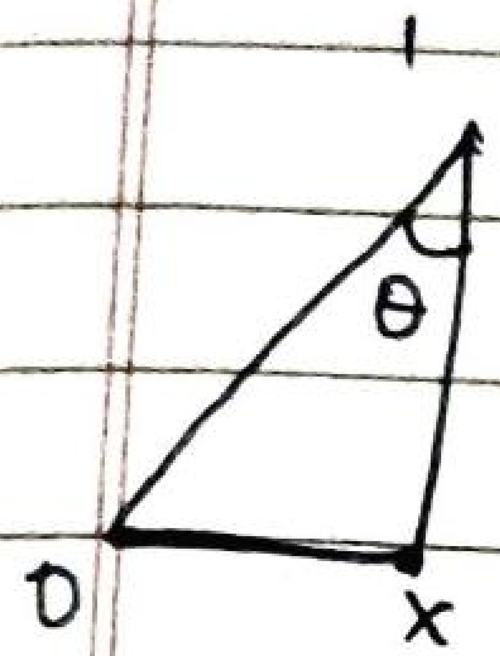
$$E(X) = \sum E(X_i) = N\left[1 + \frac{1}{2} + \cdots \frac{1}{N}\right]$$

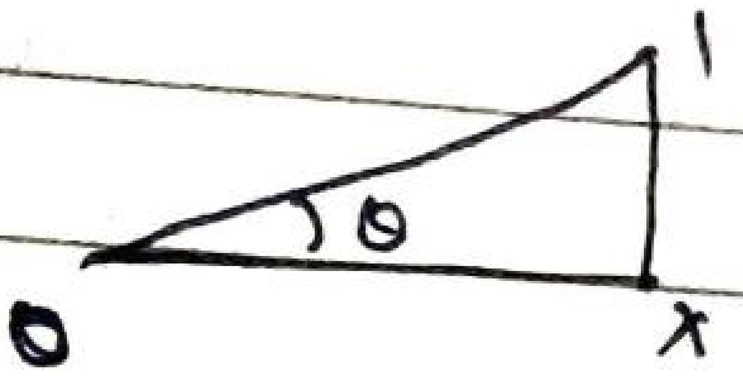
(5) $X \sim$ picking of point

$$\left[f_X(x) = 1 \qquad 0 < x < 1 \right]$$

$$g(x) = \tan\theta = \frac{x}{1-x} \qquad 1 < x < 1/2$$

$$\frac{1-x}{x} \qquad 1/2 < x < 1/2$$





so $E(\tan\theta) = E\,g(x) = \int_0^{1/2} \frac{x}{1-x}\,dx + \int_{1/2}^{1} \frac{1-x}{x}\,dx$

$= 2\ln 2 - 1$

(6) $X \sim U(0,1)$ [uniform]
$Y \sim U(0,1)$

$X =$ time when first person will come in hours [is $x = 1/2$, he will arive as 5:30]

$Y =$ same as above for second person.

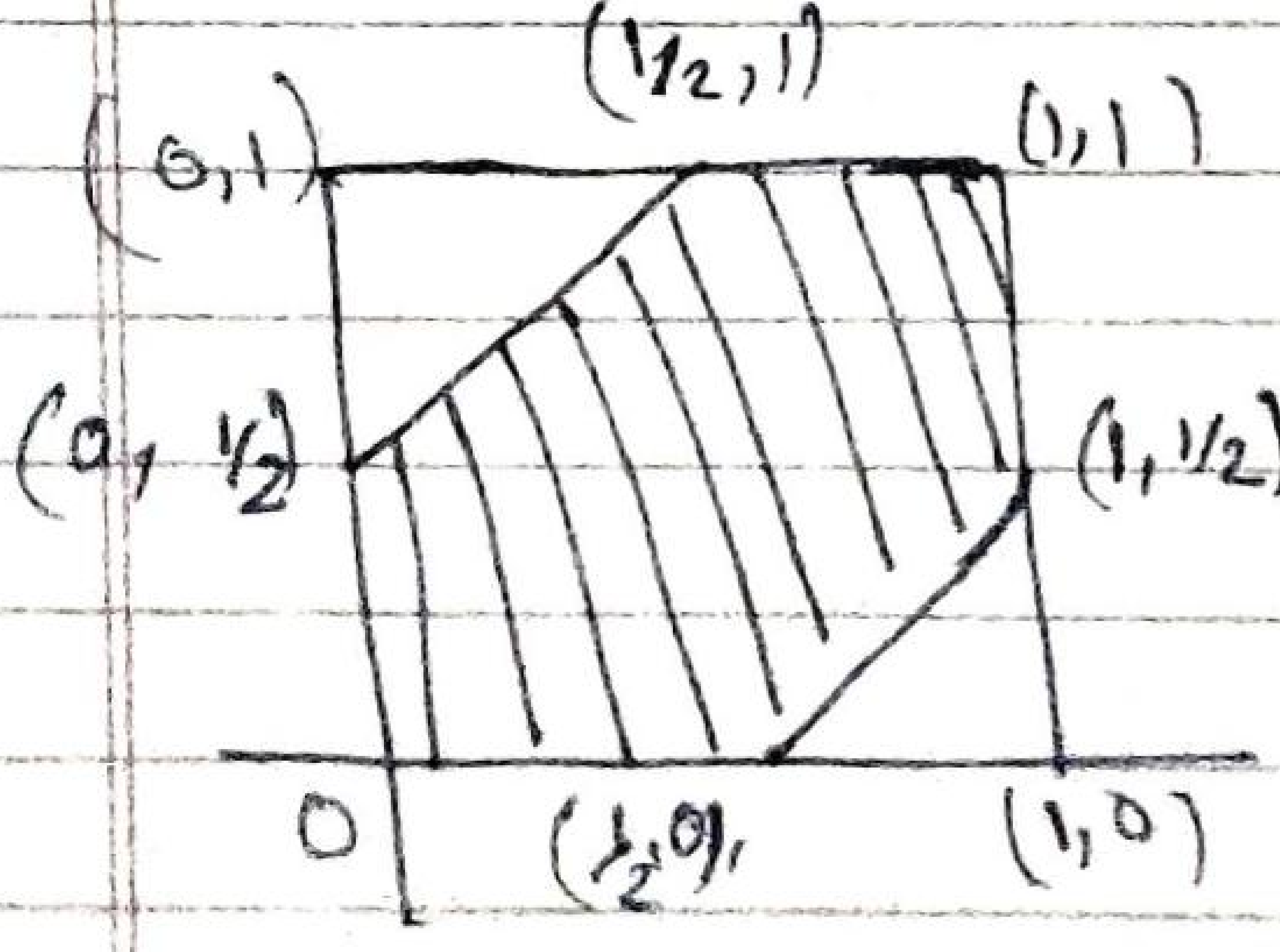
now if $|X-Y| < 1/2$ ~~we~~ the will meet each other,

Now you can define $Z = |X-Y|$

and we will get distribution of

$$f_Z(t) = 2(1-t) \quad 0 < t < 1$$

but right now you cannot find distribution of Z so we will go with graphical approach :-



(X,Y)

(X,Y) can be represented by a point on the square of are ~~a~~ 1

now $|X-Y| < 1/2$ will get us the ~~area~~ shaded area so probability they will meet = $\frac{\text{shaded}}{\text{total}}$

$= 3/4$

Can we use the approach

well in this case every point of square is equally likely so ~~that~~'s as X, Y are uniform. , if it was not the case then this soln. would be wrong.

(8) $E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

Now

$$e^{-\left[\frac{x^2 - 2(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2}{2\sigma^2}\right] - \frac{\mu^2}{2\sigma^2} + \frac{(\mu+\sigma^2 t)^2}{2\sigma^2}}$$

$$\Rightarrow e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2} + \mu t + \frac{\sigma^2 t^2}{2}}$$

$$\Rightarrow e^{\mu t + \frac{\sigma^2 t^2}{2}} \cancel{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}} dx}^{\,1}$$

$$\Rightarrow E(e^{xt}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$