Q 1

$$Mgf_{X+Y} = Mgf_X * Mgf_Y$$

now $M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX} \times e^{tY})$

Now X and Y are independent so e^{tX} and e^{tY} are also independent then

$$M_{X+Y}(t) = E(e^{tX} e^{tY}) = E(e^{tX}) E(e^{tY}) = M_X(t) M_Y(t)$$

Finding the distrib^n of $Y = \Sigma x_i/n$ for $x_i \sim N(\mu, \sigma^2)$

Ans:

$Y = \Sigma x_i/n$ x_i: is a random independent variable observing pdf: N

$x_i \sim N(\mu, \sigma^2)$

$$x_i' = \frac{x_i}{n} \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n^2}\right)$$

$$M_{x_i'}(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} \frac{n}{\sigma} e^{-\frac{(x-\mu/n)^2}{2(\sigma/n)^2}} dx.$$

$$= e^{\mu/n\, t + \frac{\sigma^2 t^2}{2n^2}}$$

Now WKT for $Z = X + Y$, X, Y independent

$$M_Z(t) = M_X(t) \cdot M_Y(t).$$

$\therefore$ We can say that,

$$M_Y = M_{x_1'} M_{x_2'} M_{x_3'} M_{x_4'} \cdot - \cdot M_{x_n'}$$

$$= \prod_{n=1}^{n} M_{x_i'}(t).$$

$$M_Y = \left(e^{\mu/n\, t + \frac{\sigma^2 t^2}{2n^2}}\right)^n = e^{\mu t + \sigma^2/n \, t^2/2}$$

By observation we can see that M_Y is comparable to the MGF of pdf: $N(\mu, \sigma^2/n)$.

$\therefore$ pdf: $Y \sim N(\mu, \sigma^2/n)$.

Q3 $X_1 \sim Exp(1)$

$$f_{X_1}(x) = e^{-x} \qquad x > 0$$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} e^{-x} \quad x \geq$$

$$M_X(t) = \int_0^\infty e^{-(1-t)x} \, dx$$

thus integration is only well defined when $t < 1$

$$M_X(t) = \frac{1}{1-t}$$

$Y = \sum X_i$

$$M_Y(t) = \prod M_{X_i}(t) = \left(\frac{1}{1-t}\right)^n$$

Now $Z \sim \text{Gamma}(n, 1)$

$$f_Z(x) = \frac{1}{\Gamma n} x^{n-1} e^{-x} \quad x > 0$$

$$M_Z(t) = \int_0^\infty \frac{1}{\Gamma n} x^{n-1} e^{-(1-t)x} dx$$

now we know $\Rightarrow$ $$\int_0^\infty x^{a-1} e^{-bx} dx = \frac{\Gamma a}{(b)^a}$$

so $$M_Z(t) = \frac{1}{(1-t)^n} = M_Y(t)$$

so $Z \sim Y$