Lax-Wendroff Flux Reconstruction for hyperbolic conservation laws

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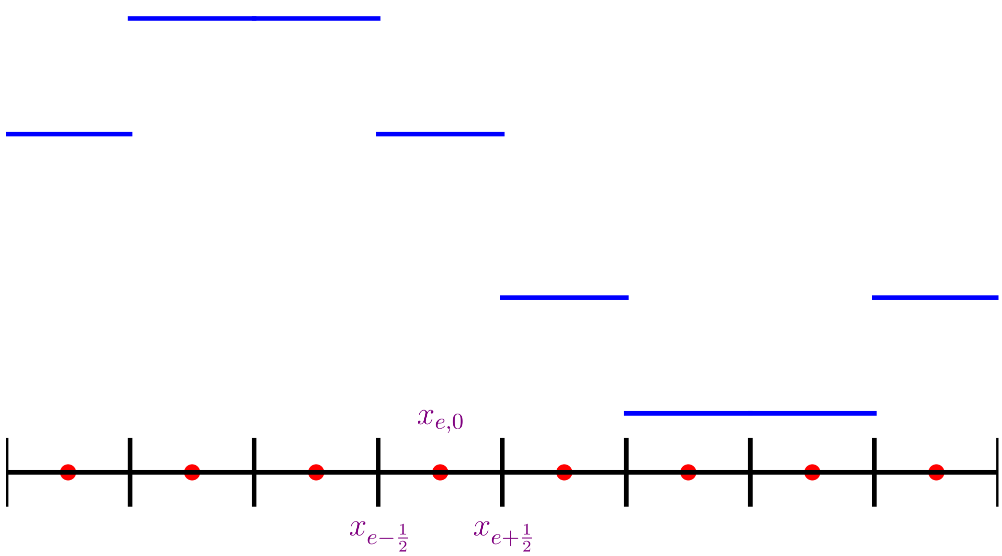
Outline

- Review of Finite Volume Method (FVM) and Flux Reconstruction (FR)
- Introduce Lax-Wendroff Flux Reconstruction (LWFR) where numerical fluxes are carefully constructed to improve accuracy and stability
- Introduce a second order variant of blending limiter of Henneman Et Al [8] in context of LW schemes which will be used to create a provably admissibility preserving LW scheme
- Extend LWFR to unstructured, curved meshes with free stream preservation

Finite Volume Method

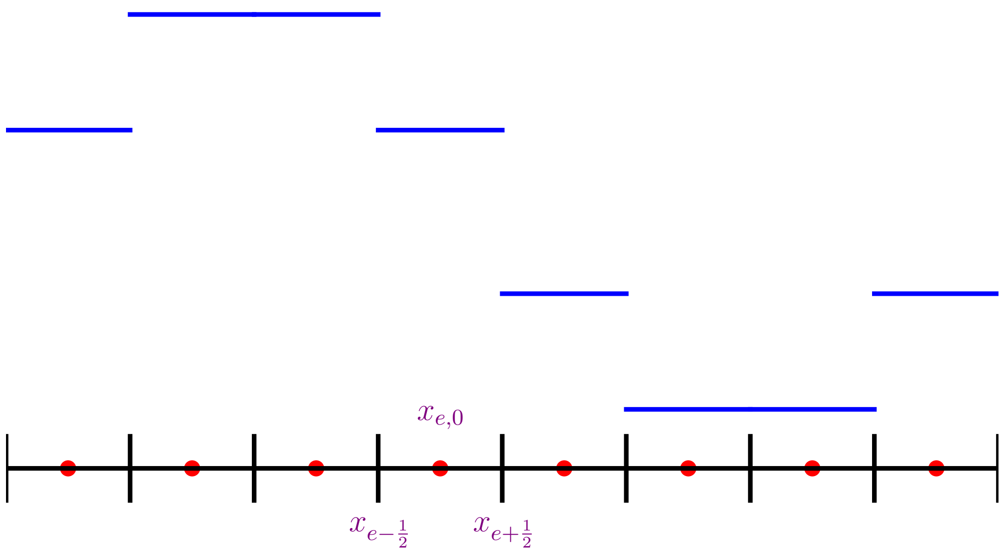
$$\begin{aligned} u_t + f(u)_x &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$



Piecewise constant states

Finite Volume Method

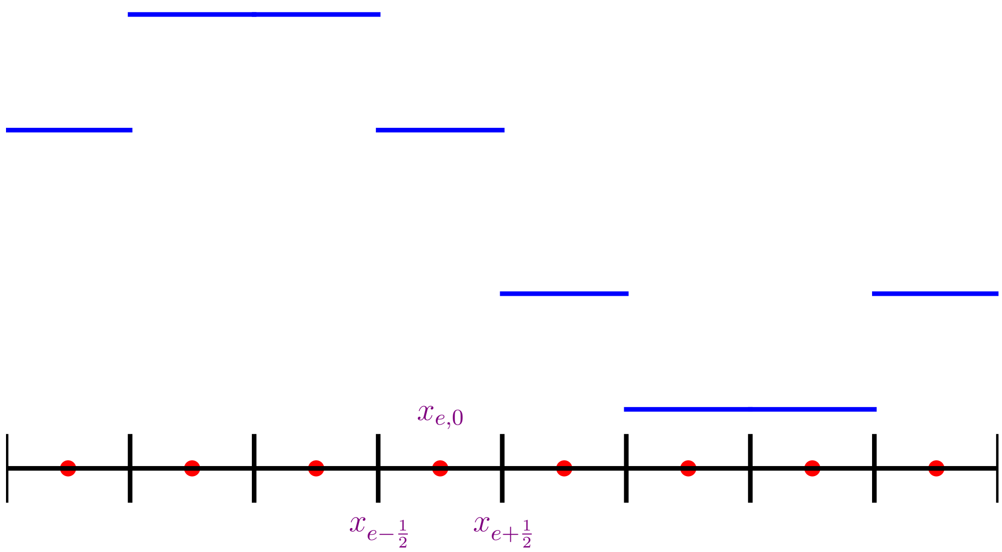
$$\int_{I_e} u_t + \int_{I_e} f(u)_x = 0$$



Piecewise constant states

Finite Volume Method

$$\frac{\mathrm{d}\overline{u}_e^n}{\mathrm{d}t} + \frac{f_{e+\frac{1}{2}} - f_{e-\frac{1}{2}}}{\Delta x_e} = 0$$

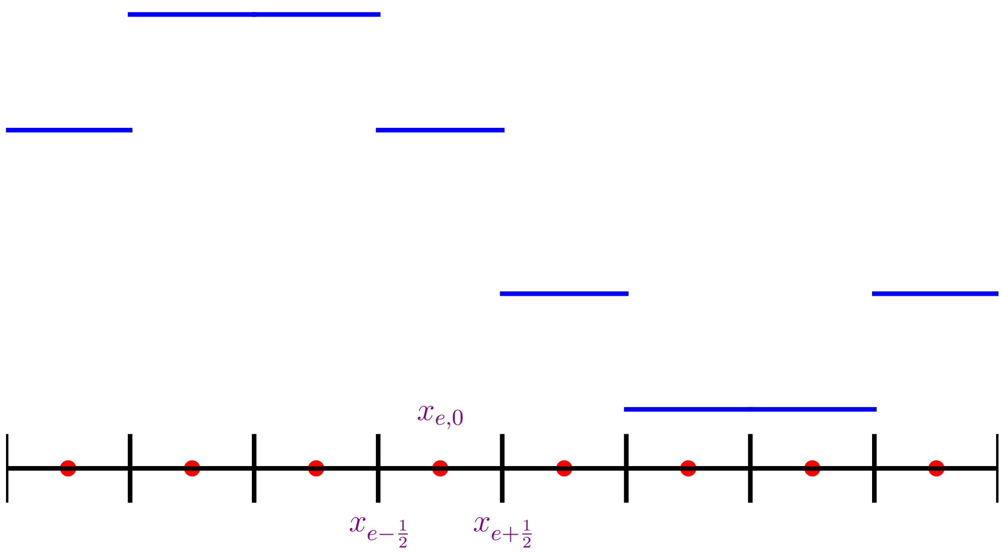


Piecewise constant states

Finite Volume Method

$$\frac{\mathrm{d}\overline{u}_e^n}{\mathrm{d}t} + \frac{f_{e+\frac{1}{2}} - f_{e-\frac{1}{2}}}{\Delta x_e} = 0$$

$$f_{e+\frac{1}{2}} = \frac{f(\overline{u}_e) + f(\overline{u}_{e+1})}{2} + \max_{u \in I[\overline{u}_e, \overline{u}_{e+1}]} |f'(u)| \frac{\overline{u}_e - \overline{u}_{e+1}}{2}$$



Piecewise constant states

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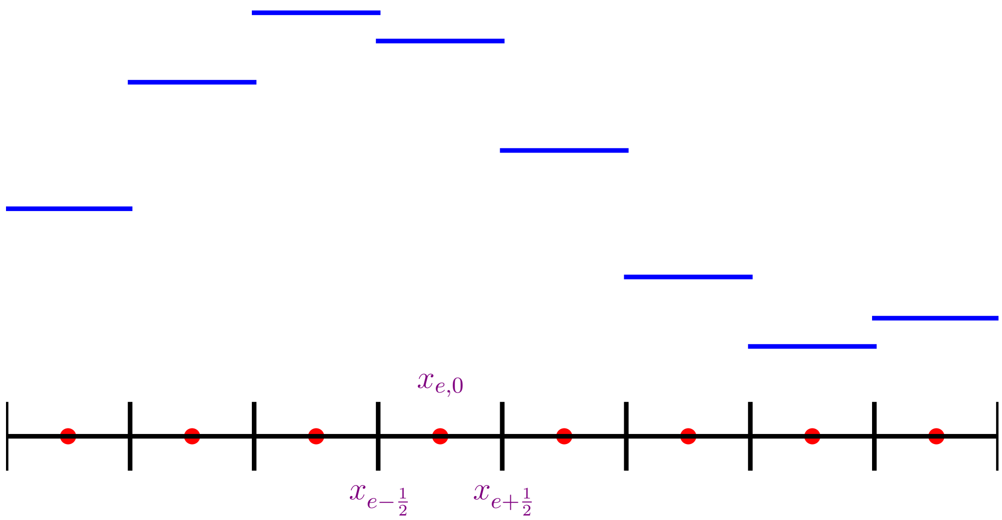


Piecewise constant states

Flux Reconstruction (FR) by Huynh [9]

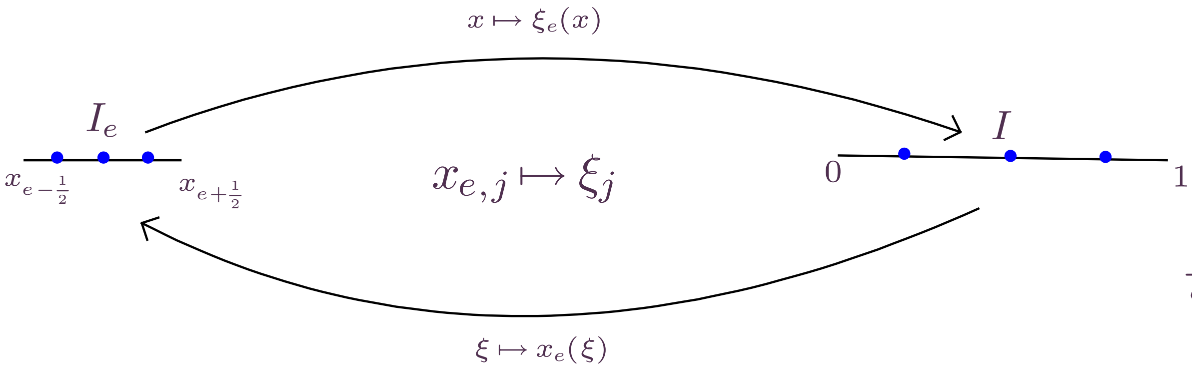
$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$$

Degree N approximation

$$\Omega = \bigcup_e I_e$$

$$u_h = u_h(x)$$

$$r = r(x)$$



$$\hat{r} = \hat{r}(\xi)$$

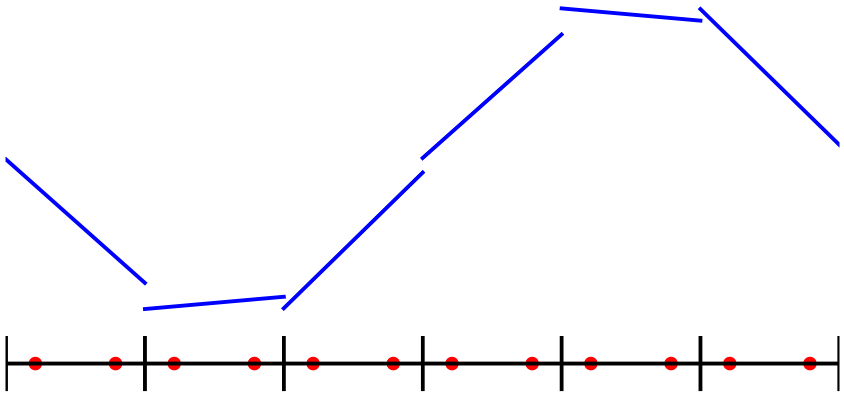
$$\frac{\partial r}{\partial x} = \frac{1}{\Delta x_e} \frac{\partial \hat{r}}{\partial \xi}$$

We use Gauss-Legendre solution points.

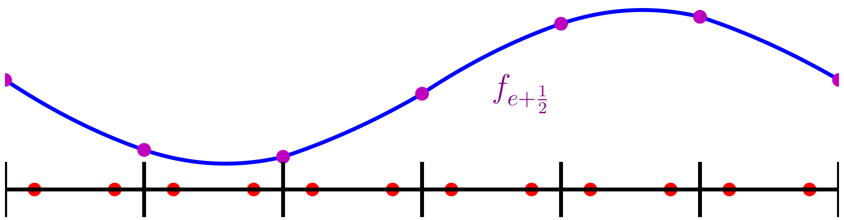
Flux Reconstruction (FR) by Huynh [9]

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{e,i} = -\frac{\partial f_h}{\partial x}(x_{e,i}), \quad 1 \le i \le N+1.$$

Degree N approximate solution u_h



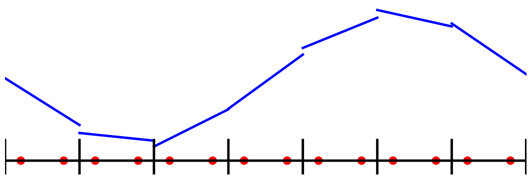
Degree $N+1$ Continuous Flux Approximation f_h



Flux Reconstruction (FR) by Huynh [9]

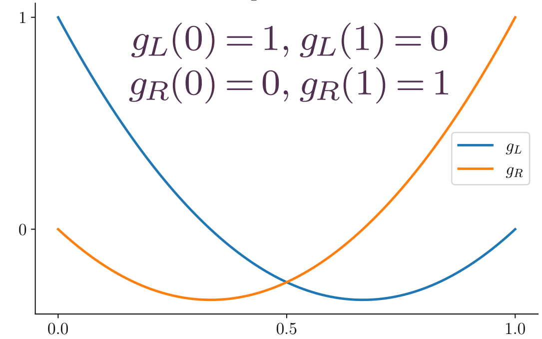
Discontinuous Flux Approximation f_h^δ
Degree N

$$\hat{f}_h^\delta(\xi_i) = f(\hat{u}_h(\xi_i))$$



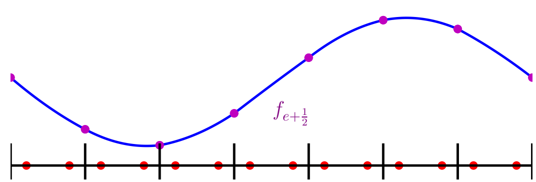
Corrector functions
Degree $N+1$

$$g_L(0) = 1,\; g_L(1) = 0$$
$$g_R(0) = 0,\; g_R(1) = 1$$



$$\hat{f}_h = \hat{f}_h^\delta + \left(f_{e-\frac{1}{2}} - \hat{f}_h^\delta(0) \right) g_L + \left(f_{e+\frac{1}{2}} - \hat{f}_h^\delta(1) \right) g_R$$

Continuous Flux Approximation f_h
Degree $N+1$



$$f_{e+\frac{1}{2}} = \frac{f_h^\delta(u^-_{e+\frac{1}{2}}) + f(u^+_{e+\frac{1}{2}})}{2} + \max_{u \in I[u^-_{e-1/2}, u^+_{e+1/2}]} |f'(u)| \frac{u^-_{e+\frac{1}{2}} - u^+_{e+\frac{1}{2}}}{2}$$

Lax-Wendroff Flux Reconstruction (LWFR)

$$u^{n+1} = u^n - \Delta t\, F_x^n,$$

$$\text{where } F = f(u) + \frac{\Delta t}{2}(f(u))_t + \frac{\Delta t^2}{3!} f(u)_{tt} + \cdots + \frac{\Delta t^N}{(N+1)!}\frac{\partial^N}{\partial t^N} f(u) \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u)\, dt$$

Taylor series method

$$f(u)_t = f'(u)\, u_t, \qquad u_t = -f(u)_x.$$

Approximate Lax-Wendroff procedure (Zorio Et Al. [21])

$$\begin{aligned} f(u)_t &\approx \frac{f(u(x,t+\Delta t)) - f(u(x,t-\Delta t))}{2\Delta t} + O(\Delta t^2) \\ &\approx \frac{f(u+\Delta t\, u_t) - f(u-\Delta t\, u_t)}{2\Delta t} + O(\Delta t^2), \end{aligned}$$

and $u_t = -f(u)_x$.

F_h^δ corrected using $F_{e+\frac{1}{2}}$.

Numerical flux: D2

Past works : **Dissipation 1** (D1)

$$F_{e+\frac{1}{2}} = \frac{1}{2}\,[F^-_{e+\frac{1}{2}} + F^+_{e+\frac{1}{2}}] - \frac{1}{2}\,\lambda_{e+\frac{1}{2}}\,[u^+_{e+\frac{1}{2}} - u^-_{e+\frac{1}{2}}].$$

Dissipation 2 (D2) flux

$$F_{e+\frac{1}{2}} = \frac{1}{2}\,[F^-_{e+\frac{1}{2}} + F^+_{e+\frac{1}{2}}] - \frac{1}{2}\,\lambda_{e+\frac{1}{2}}\,[U^+_{e+\frac{1}{2}} - U^-_{e+\frac{1}{2}}].$$

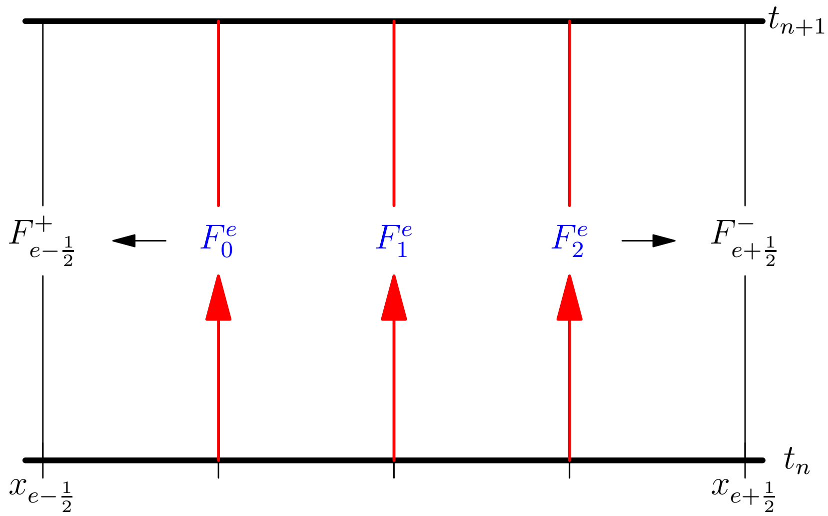
where

$$U = u + \frac{\Delta t}{2}u_t + \frac{\Delta t^2}{3!}u_{tt} + \cdots + \frac{\Delta t^N}{(N+1)!}\frac{\partial^N u}{\partial t^N} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u_h\,dt.$$

Upwind flux for $u_t + a\,u_x = 0$

$$F_{e+\frac{1}{2}} = \begin{cases} F^-_{e+\frac{1}{2}} & a > 0, \\ F^-_{e+\frac{1}{2}} & a \le 0. \end{cases}$$

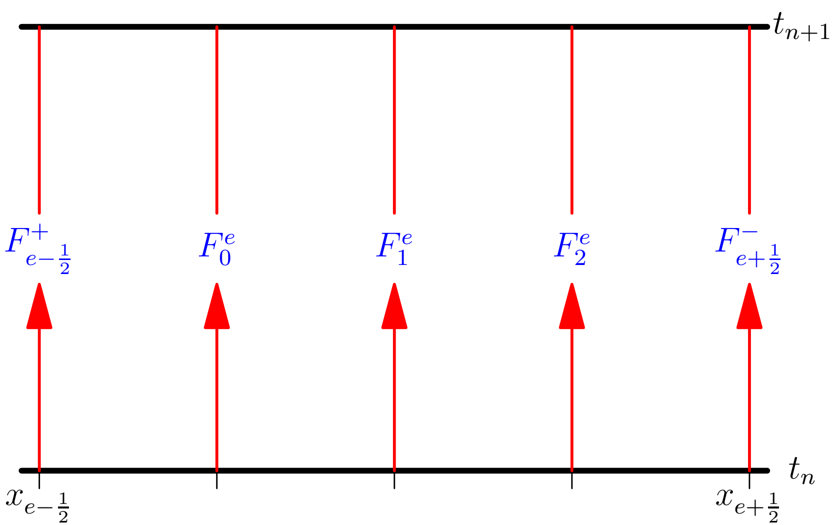
Equivalent to ADER-DG for linear problems.



Average and Extrapolate (AE)

$$F^{\pm}_{e+\frac{1}{2}} = F^{\delta}_{h}(x^{\pm}_{e+\frac{1}{2}})$$

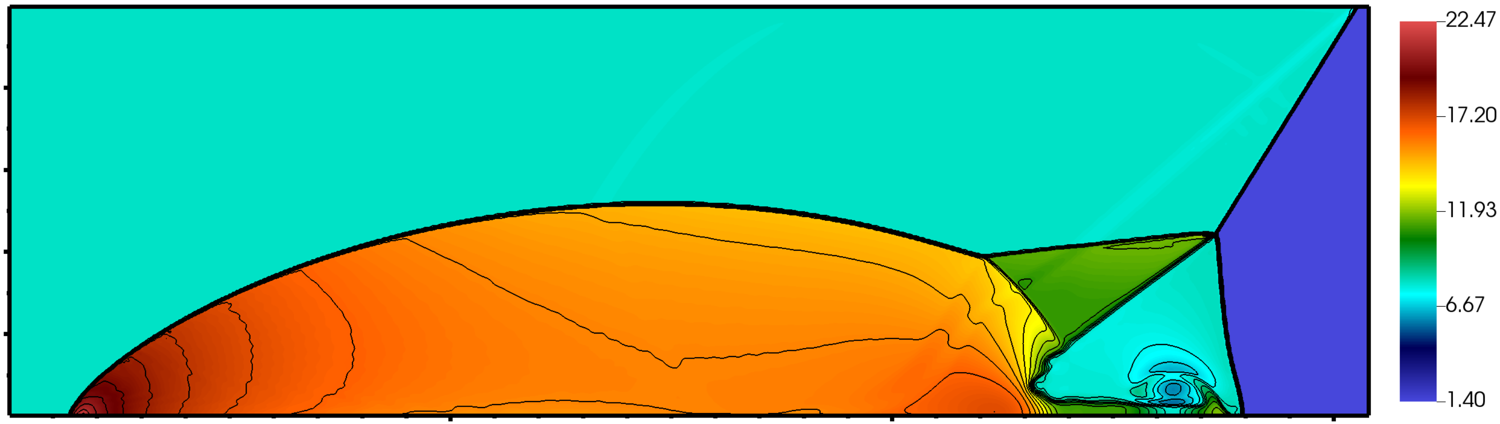
Sub-optimal accuracy for some non-linear problems, robustness issues



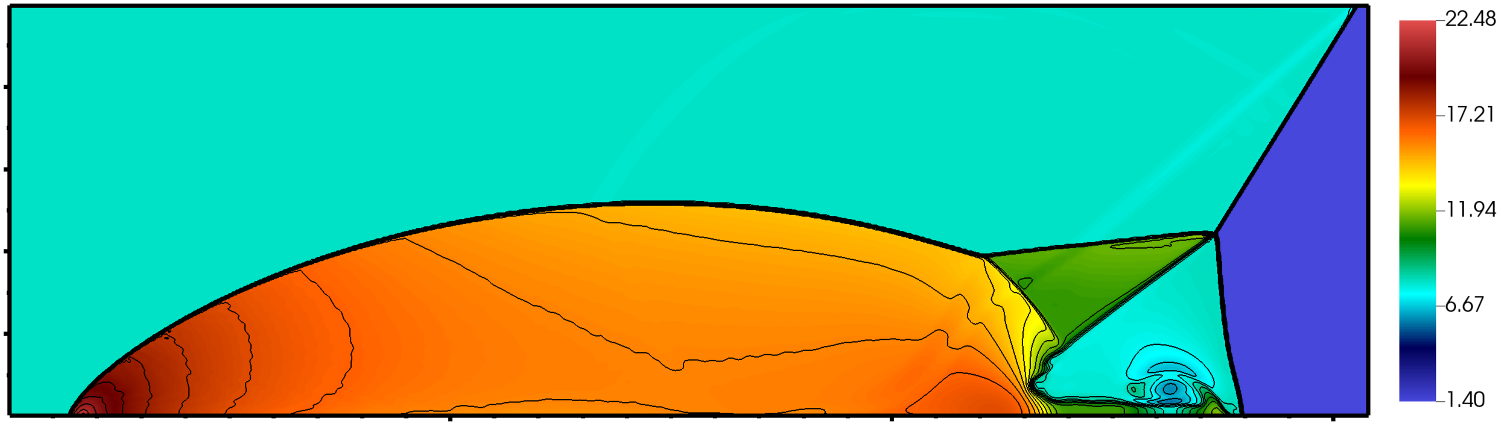
Extrapolate and Average (EA)

Lax-Wendroff time averaging performed at faces, only $\partial_t^{(k)}\boldsymbol{u}$ terms extrapolated

Accuracy compared

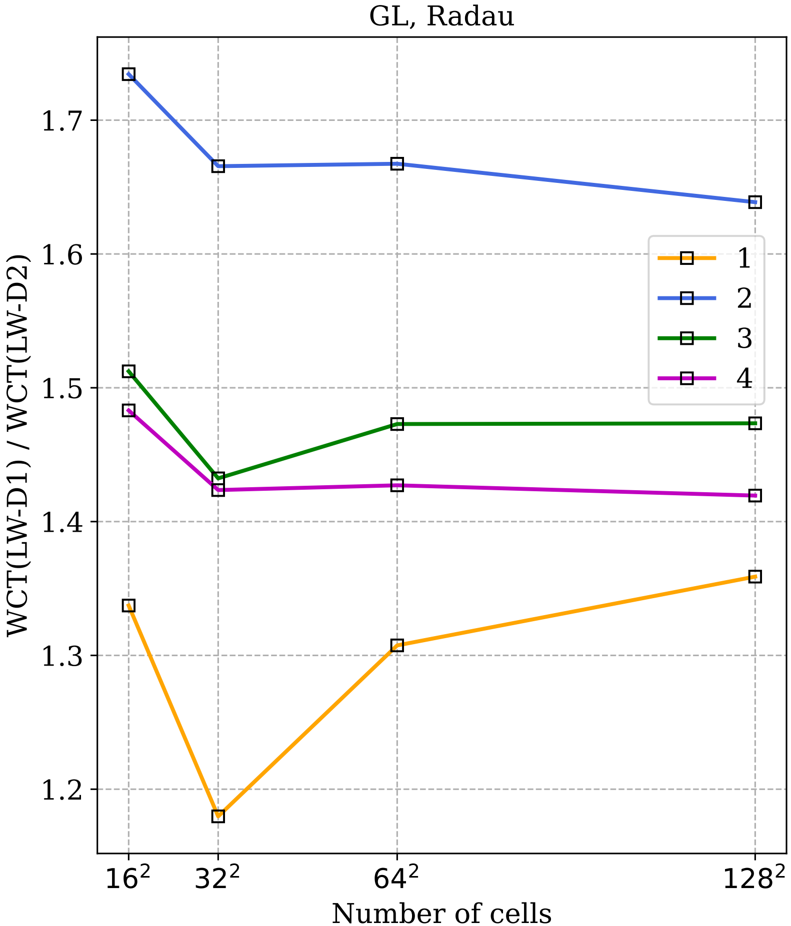


LWFR-D2 with **EA** scheme

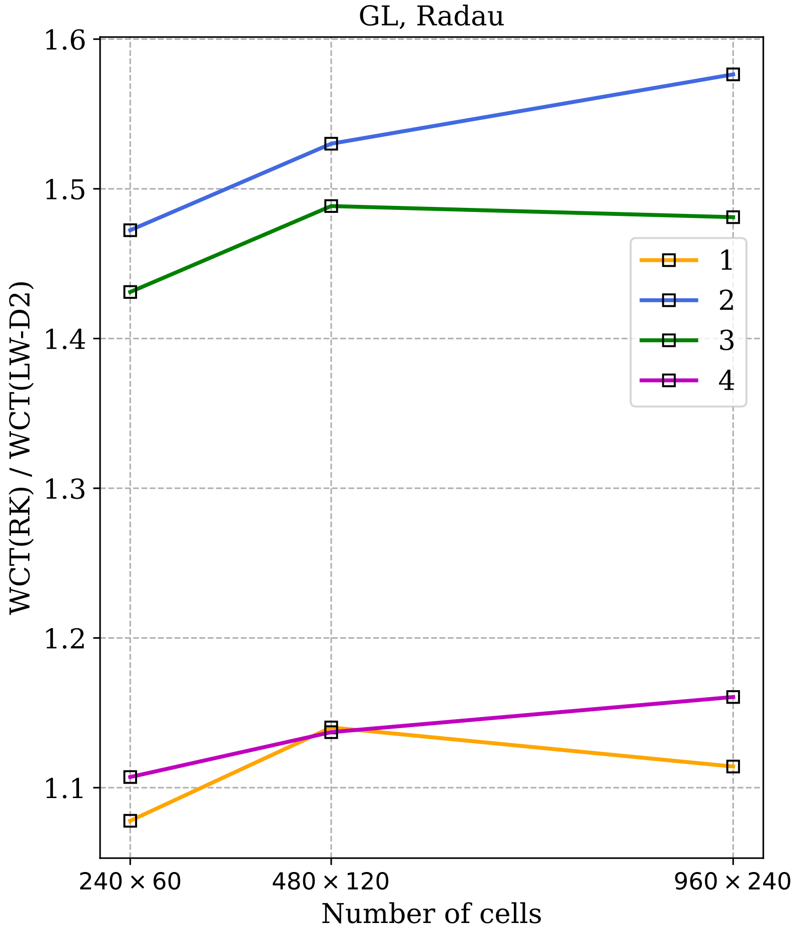


RKFR

Performance gains measured



Performance gain of D2 over D1



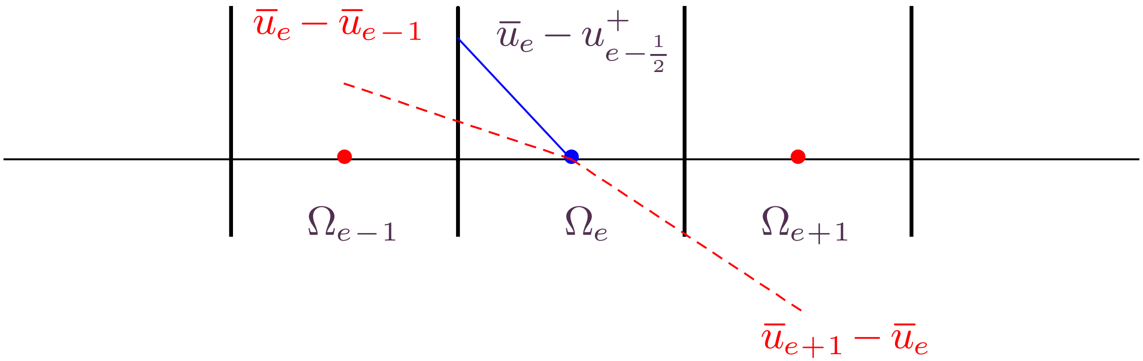
Performance gain of LW over RK

For more results comparing LW-D2/LW-D1 and LW/RK, see [2].

Godunov's order barrier theorem

Linear schemes which do not add oscillations can be at most first order accurate.

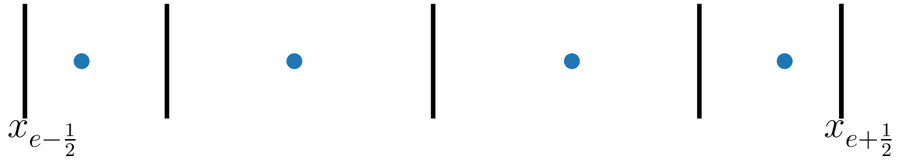
TVD/TVB Limiter



$$\alpha_e = \alpha(\boldsymbol{u}_e)$$

Blending limiter
(Hennemann Et Al. [8])

Solution points and subcells



Blending limiter (Hennemann Et Al. [8])

High order LWFR update

$$\boldsymbol{u}_e^{H,n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e} \boldsymbol{R}_e^H .$$

Lower order subcell update (FO FVM or MUSCL-Hancock)

$$\boldsymbol{u}_e^{L,n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e} \boldsymbol{R}_e^L .$$

Blend residual with $\alpha_e \in [0, 1]$

$$\boldsymbol{R}_e = (1 - \alpha_e)\, \boldsymbol{R}_e^H + \alpha_e\, \boldsymbol{R}_e^L ,$$

Limited update

$$\boldsymbol{u}_e^{n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e} \boldsymbol{R}_e .$$

Choice of α_e : Smoothness indicator [8]

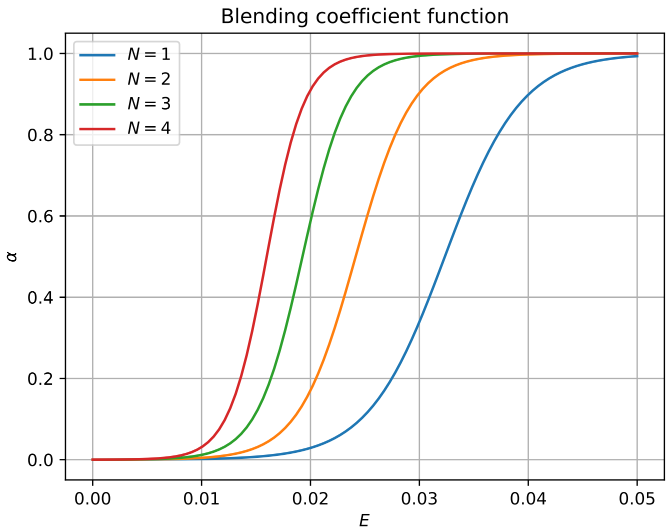
Legendre expansion of degree N polynomial $\epsilon = \epsilon(\xi)$

$$\epsilon = \sum_{j=1}^{N+1} m_j L_j, \qquad m_j = \langle \epsilon, L_j \rangle_{L^2},$$

Energy content (Persson and Peraire [12])

$$\mathbb{E} := \max\left(\frac{m_{N+1}^2}{\beta_1 m_1^2 + \sum_{j=2}^{N+1} m_j^2}, \frac{m_N^2}{\beta_2 m_1^2 + \sum_{j=2}^{N} m_j^2} \right), \qquad 0 \le \beta_i \le 1.$$

$$\epsilon = \rho\, p$$



$$\alpha(\mathbb{E}) = \frac{1}{1 + \exp\left(-\frac{s}{\mathbb{T}}(\mathbb{E} - \mathbb{T})\right)}$$

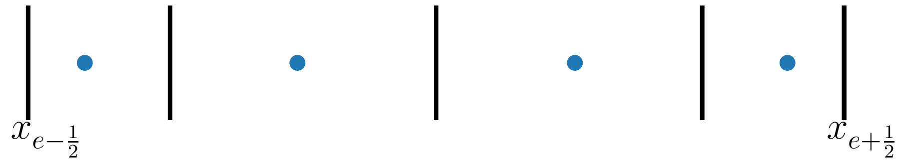
$$\mathbb{T}(N) = 0.5 \cdot 10^{-1.8\,(N+1)^{1/4}}, \qquad \alpha(\mathbb{E} = 0) = 0.0001$$

$$\tilde{\alpha} = \begin{cases} 0, & \text{if } \alpha < \alpha_{\min} \\ \alpha, & \text{if } \alpha_{\min} \le \alpha \le 1 - \alpha_{\min} \\ 1, & \text{if } 1 - \alpha_{\min} < \alpha \end{cases}$$

$$\alpha^{\text{final}} = \max_{e \in V_e} \{\alpha, 0.5\, \alpha_e\}$$

Lower order update

Solution points and subcells

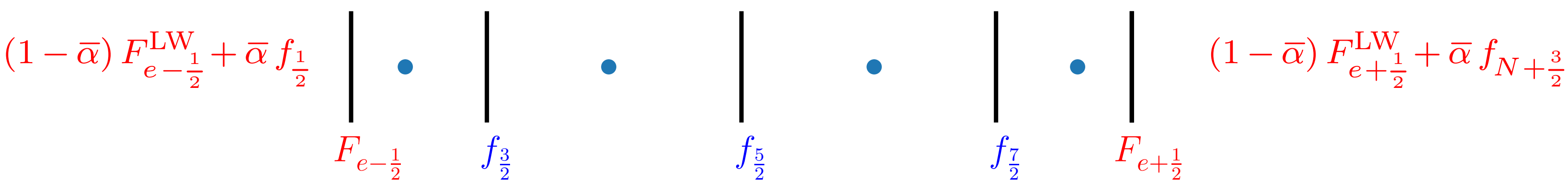


Subcell $[x^e_{j-\frac{1}{2}}, x^e_{j+\frac{1}{2}}]$

$$x^e_{j+\frac{1}{2}} - x^e_{j-\frac{1}{2}} = \Delta x_e \, w_j, \qquad 1 \le j \le N+1,$$

where $\{w_j\}_{j:1}^{N+1}$ are the Gauss-Legendre quadrature weights.

FVM



$$\overline{u}_e^{n+1} = \overline{u}_e^n - \frac{\Delta t}{\Delta x_e}(F_{e+\frac{1}{2}} - F_{e-\frac{1}{2}}).$$

Interface numerical flux

Initial candidate

$$\tilde{\boldsymbol{F}}_{e+\frac{1}{2}} = \left(1 - \alpha_{e+\frac{1}{2}}\right) \boldsymbol{F}^{\text{LW}}_{e+\frac{1}{2}} + \alpha_{e+\frac{1}{2}} \boldsymbol{f}_{e,N+3/2}, \qquad \alpha_{e+\frac{1}{2}} = \frac{1}{2}(\alpha_e + \alpha_{e+1}).$$

Lower order update of last solution point of cell e

$$\tilde{\boldsymbol{u}}^{n+1} = \boldsymbol{u}^n_{e,N+1} - \frac{\Delta t}{\Delta x_e w_{N+1}} \left(\tilde{\boldsymbol{F}}_{e+\frac{1}{2}} - \boldsymbol{f}_{e,N+1/2}\right).$$

Assume concave p such that

$$\mathcal{U}_{\text{ad}} = \{\boldsymbol{u} \in \Omega \colon p(\boldsymbol{u}) > 0\}.$$

For purely low order

$$\tilde{\boldsymbol{u}}^{n+1}_{\text{low}} = \boldsymbol{u}^n_{e,N+1} - \frac{\Delta t}{\Delta x_e w_{N+1}} (\boldsymbol{f}_{e,N+3/2} - \boldsymbol{f}_{e,N+1/2}) \in \mathcal{U}_{\text{ad}}.$$

Thus, for

$$\theta = \min\left(\left|\frac{\epsilon - p(\tilde{\boldsymbol{u}}^{n+1}_{\text{low}})}{p(\tilde{\boldsymbol{u}}^{n+1}) - p(\tilde{\boldsymbol{u}}^{n+1}_{\text{low}})}\right|, 1\right),$$

we will have

$$p(\theta\, \tilde{\boldsymbol{u}}^{n+1} + (1-\theta)\, \boldsymbol{u}^{n+1}_{\text{low}}) \geq \theta\, p(\tilde{\boldsymbol{u}}^{n+1}) + (1-\theta)\, p(\boldsymbol{u}^{n+1}_{\text{low}}) > \epsilon.$$

$$\text{Final choice:} \qquad \boldsymbol{F}_{e+\frac{1}{2}} = \theta\, \tilde{\boldsymbol{F}}_{e+\frac{1}{2}} + (1-\theta)\, \boldsymbol{f}_{e,N+3/2}.$$

Extension of Zhang-Shu's limiter to Lax-Wendroff schemes

Low order residual

$$
\begin{aligned}
(\tilde{\boldsymbol{u}}_1^e)^{n+1} &= (\boldsymbol{u}_1^e)^n - \frac{\Delta t}{w_1\,\Delta x_e}\Big[f_{\frac{3}{2}} - F_{e-\frac{1}{2}}\Big],\\
(\tilde{\boldsymbol{u}}_j^e)^{n+1} &= (\boldsymbol{u}_j^e)^n - \frac{\Delta t}{w_j\,\Delta x_e}\Big[f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}\Big], \qquad 2\le j\le N,\\
(\tilde{\boldsymbol{u}}_N^e)^{n+1} &= (\boldsymbol{u}_N^e)^n - \frac{\Delta t}{w_{N+1}\,\Delta x_e}\Big[F_{e+\frac{1}{2}} - f_{N-\frac{1}{2}}\Big].
\end{aligned}
$$

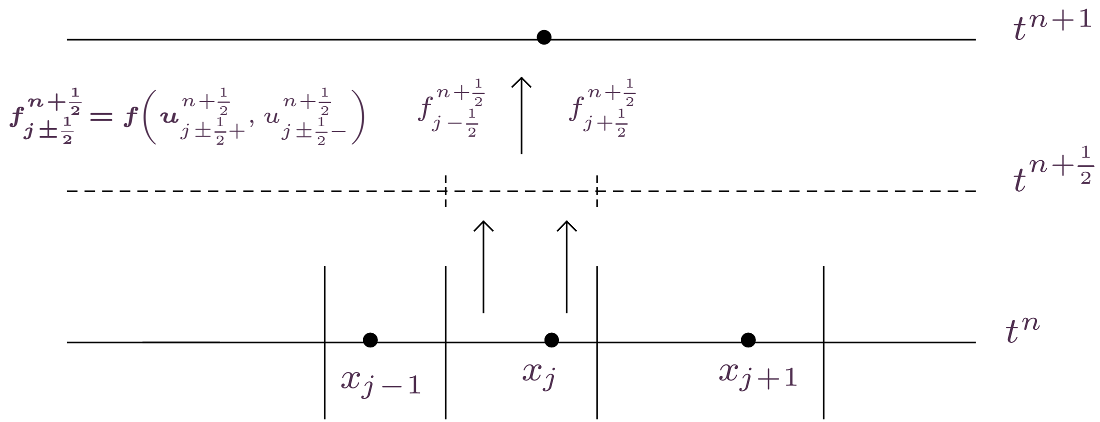
By appropriate choice of $F_{e\pm\frac{1}{2}}$,

$$(\tilde{\boldsymbol{u}}_j^e)^{n+1} \in \mathcal{U}_{\mathrm{ad}}, \qquad 1\le j\le N+1.$$

$$\Rightarrow \overline{\boldsymbol{u}}_e^{n+1} = \overline{\boldsymbol{u}}_e^n - \frac{\Delta t}{\Delta x_e}\Big(F_{e+\frac{1}{2}} - F_{e-\frac{1}{2}}\Big) = \sum_{j=1}^{N+1} w_j\,\tilde{\boldsymbol{u}}_j^{n+1} \in \mathcal{U}_{\mathrm{ad}}.$$

Zhang-Shu's scaling limiter applies.

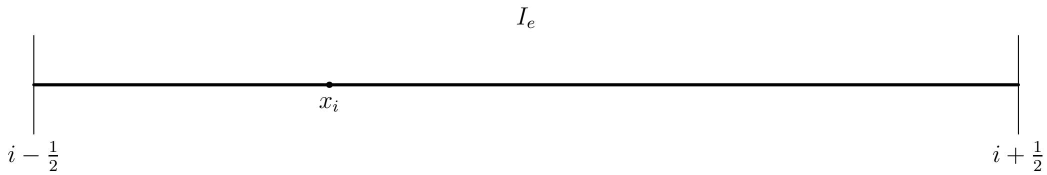
$$\boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\Delta t}{\Delta x_j}\left(\boldsymbol{f}_{j+1/2}^{n+1/2} - \boldsymbol{f}_{j-1/2}^{n+1/2}\right)$$



$$\boldsymbol{u}_{j-\frac{1}{2}+} = \boldsymbol{u}_j(x_{j-\frac{1}{2}}), \quad \boldsymbol{u}_{j+\frac{1}{2}-} = \boldsymbol{u}_j(x_{j+\frac{1}{2}})$$
$$\boldsymbol{u}_j(x) = \boldsymbol{u}_j^n + \boldsymbol{\sigma}_j\,(x - x_j)$$

$$\boldsymbol{\sigma}_j = \operatorname{minmod}\left(\beta_e \frac{\boldsymbol{u}_{j+1} - \boldsymbol{u}_j}{x_{j+1} - x_j}, D_{\text{cent}}(\boldsymbol{u})_j, \beta_e \frac{\boldsymbol{u}_j^n - \boldsymbol{u}_{j-1}^n}{x_j - x_{j-1}}\right)$$
$$\beta_e = 2 - \alpha_e$$

Admissibility of low order method



Theorem. (*Extension of Berthon* [3]) *Consider the hyperbolic conservation law* $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_{\boldsymbol{x}} = \boldsymbol{0}$ *which preserves the convex set* Ω*. Let* $\{\boldsymbol{u}_i^n\}_{i\in\mathbb{Z}}$ *be the approximate solution at time level* n *and assume that* $\boldsymbol{u}_i^n \in \Omega$ *for all* $i \in \mathbb{Z}$*. Consider conservative reconstructions*

$$\boldsymbol{u}_i^{n,+} = \boldsymbol{u}_i + (x_{i+1/2} - x_i)\boldsymbol{\sigma}_i, \quad \boldsymbol{u}_i^{n,-} = \boldsymbol{u}_i + (x_{i-1/2} - x_i)\boldsymbol{\sigma}_i.$$

Define $\boldsymbol{u}_i^{*,\pm}$ *satisfying*

$$\mu^- \boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,\pm} + \mu^+ \boldsymbol{u}_i^{n,+} = 2\boldsymbol{u}_i^{n,\pm},$$

where

$$\mu^- = \frac{x_{i+1/2} - x_i}{x_{i+1/2} - x_{i-1/2}}, \quad \mu^+ = \frac{x_i - x_{i-1/2}}{x_{i+1/2} - x_{i-1/2}}.$$

Assume that the slope $\boldsymbol{\sigma}_i$ *is chosen so that*

$$\boldsymbol{u}_i^{*,\pm} \in \mathcal{U}_{\text{ad}}.$$

Then, under appropriate time step restrictions, the updated solution $\boldsymbol{u}_i^{n+1}$*, defined by the MUSCL-Hancock procedure is in* $\mathcal{U}_{\text{ad}}$*.*

Berthon defined $\boldsymbol{u}_i^{*,\pm}$

$$\frac{1}{2}\boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,\pm} + \frac{1}{2}\boldsymbol{u}_i^{n,+} = 2\boldsymbol{u}_i^{n,\pm}.$$

Our generalization

$$\mu^- \boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,\pm} + \mu^+ \boldsymbol{u}_i^{n,+} = 2\boldsymbol{u}_i^{n,\pm},$$

where

$$\mu^- = \frac{x_{i+1/2} - x_i}{x_{i+1/2} - x_{i-1/2}}, \quad \mu^+ = \frac{x_i - x_{i-1/2}}{x_{i+1/2} - x_{i-1/2}}.$$

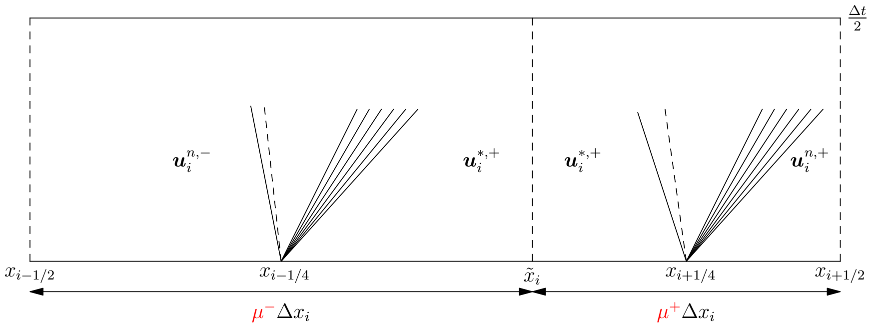
For **conservative reconstruction**

$$\boldsymbol{u}_i^{*,\pm} = \boldsymbol{u}_i^n + 2(x_{i\pm1/2} - x_i)\boldsymbol{\sigma}_i,$$

noting that

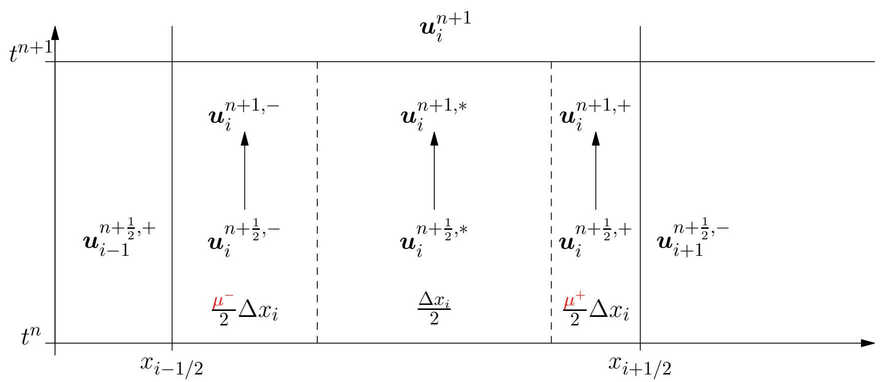
$$\boldsymbol{u}_i^{n,+} = \boldsymbol{u}_i + (x_{i+1/2} - x_i)\boldsymbol{\sigma}_i, \quad \boldsymbol{u}_i^{n,-} = \boldsymbol{u}_i + (x_{i-1/2} - x_i)\boldsymbol{\sigma}_i.$$

Idea of proof



$$\frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}^h(x,\Delta t/2)\,dx = \boldsymbol{u}_i^{n+\frac{1}{2},+}$$

$$\frac{\mu^-}{2}\boldsymbol{u}_i^{n+\frac{1}{2},-} + \frac{1}{2}\boldsymbol{u}_i^{n+\frac{1}{2},*} + \frac{\mu^+}{2}\boldsymbol{u}_i^{n+\frac{1}{2},+} = \boldsymbol{u}_i^n$$



Enforcing slope restriction

Given candidate slope $\boldsymbol{\sigma}_i$,

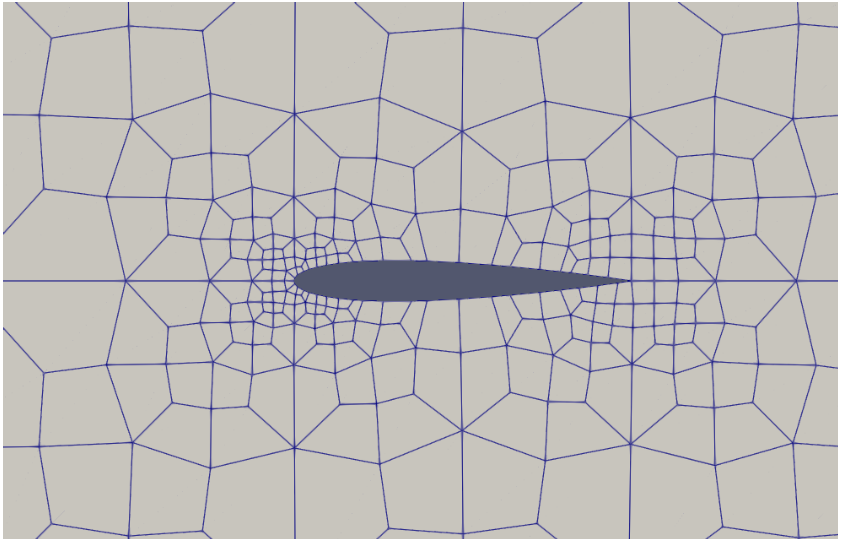
$$\boldsymbol{u}_i^{*,\pm} := \boldsymbol{u}_i^n + 2\,(x_{i\pm 1/2} - x_i)\,\boldsymbol{\sigma}_i.$$

Find $\theta \in [0,1]$

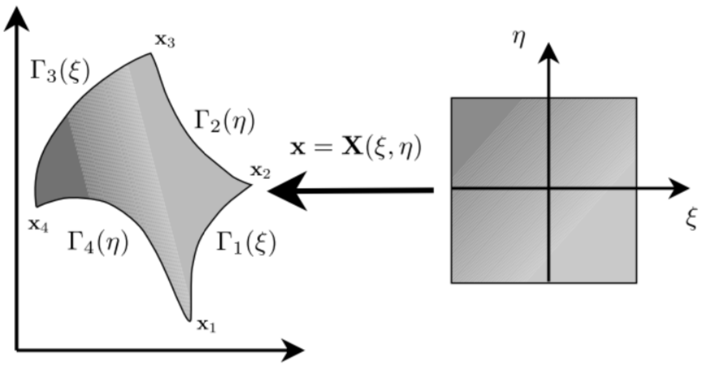
$$\boldsymbol{u}_i^n + 2\,(x_{i\pm 1/2} - x_i)\,\theta\,\boldsymbol{\sigma}_i \in \mathcal{U}_{\mathrm{ad}}$$

by Zhang-Shu type procedure.

Unstructured, curvilinear grids



NACA0012 air foil

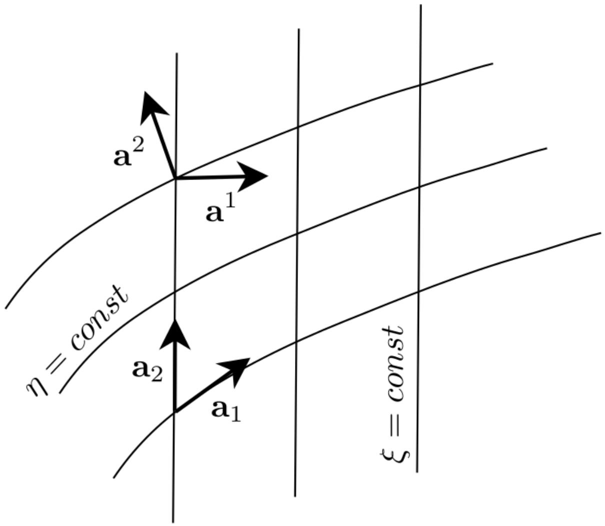


$$(\xi, \eta) \mapsto \boldsymbol{x}^e(\xi, \eta)$$

$\boldsymbol{x}^e$ is a **degree** $\boldsymbol{k}$ polynomial in ξ, η

Transformation of conservation law

$$\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$$
$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) = (\xi, \eta, \zeta)$$



$$\boldsymbol{a}_i = \frac{\partial}{\partial \xi_i}\boldsymbol{x}$$
$$\boldsymbol{a}^i = \nabla_{\boldsymbol{x}}\, \xi^i = J^{-1}\,(\boldsymbol{a}_j \times \boldsymbol{a}_k)$$
$$J = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)$$

Covariant and contravariant coordinate vectors in relation to the coordinate lines

$$\boldsymbol{u}_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f} = 0 \longrightarrow \tilde{\boldsymbol{u}}_t + \nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{f}} = 0,$$

where

$$\begin{aligned} \tilde{\boldsymbol{u}} &= J\vec{Q}, \\ \tilde{\boldsymbol{f}}^i &= J\boldsymbol{a}^i \cdot \boldsymbol{f} = \sum_{n=1}^{3} J a_n^i\, f_n. \end{aligned}$$

Flux Reconstruction for the transformed PDE

$$
\begin{aligned}
&\frac{\mathrm{d}\boldsymbol{u}_{e,\boldsymbol{p}}^{\delta}}{\mathrm{d}t} + \frac{1}{J}\,\nabla_{\boldsymbol{\xi}}^{N} \cdot \tilde{\boldsymbol{f}}_e(\boldsymbol{\xi}_{\boldsymbol{p}}) \\
&\quad + \frac{1}{J}\sum_{i=1}^{3} \left((\tilde{\boldsymbol{f}}_e \cdot \boldsymbol{n}_{s,i})^* - \tilde{\boldsymbol{f}}_e \cdot \boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^R)\, g_R'(\xi_{p_i}) \\
&\quad + \frac{1}{J}\sum_{i=1}^{3} \left((\tilde{\boldsymbol{f}}_e \cdot \boldsymbol{n}_{s,i})^* - \tilde{\boldsymbol{f}}_e \cdot \boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^L)\, g_L'(\xi_{p_i}) = \boldsymbol{0},
\end{aligned}
$$

Lax-Wendroff Flux Reconstruction for the transformed PDE

$$\boldsymbol{u}_t + \frac{1}{J}\nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{f}} = 0,$$

$$\tilde{\boldsymbol{f}}^i \;=\; J\boldsymbol{a}^i \cdot \boldsymbol{f} = \sum_{n=1}^{3} J a_n^i \, \boldsymbol{f}_n.$$

$$\boldsymbol{u}^{n+1}(\boldsymbol{\xi}) = \boldsymbol{u}^n(\boldsymbol{\xi}) - \frac{1}{J}\Delta t \, \nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{F}}$$

$$\tilde{\boldsymbol{F}} = \sum_{k=0}^{N} \frac{\Delta t^k}{(k+1)!} \, \partial_t^k \, \tilde{\boldsymbol{f}}$$

$$\begin{aligned}
\boldsymbol{u}_{e,\boldsymbol{p}}^{n+1} - \boldsymbol{u}_{e,\boldsymbol{p}}^{n} + \frac{1}{J}\Delta t \, \nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{F}}_e^\delta(\boldsymbol{\xi}_{\boldsymbol{p}}) \\
+ \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((\tilde{\boldsymbol{F}}_e^\delta \cdot \boldsymbol{n}_{s,i})^* - \tilde{\boldsymbol{F}}_e^\delta \cdot \boldsymbol{n}_{s,i})(\boldsymbol{\xi}_i^R) \, g_R'(\xi_{p_i}) \, dS_{\boldsymbol{\xi}} \\
+ \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((\tilde{\boldsymbol{F}}_e^\delta \cdot \boldsymbol{n}_{s,i})^* - \tilde{\boldsymbol{F}}_e^\delta \cdot \boldsymbol{n}_{s,i})(\boldsymbol{\xi}_i^L) \, g_L'(\xi_{p_i}) \, dS_{\boldsymbol{\xi}} = \boldsymbol{0}.
\end{aligned}$$

Free stream preservation of Lax-Wendroff

Claimed free stream preservation conditions

$$\sum_{i=1}^{3} \partial_{\xi^i}^N (J\boldsymbol{a}^i) = \mathbf{0}$$

Assume a free stream

$$\boldsymbol{u}^n = \underline{\boldsymbol{c}} \text{ and } \boldsymbol{f}(\underline{\boldsymbol{c}}) = \boldsymbol{c}$$

$$\begin{aligned} \boldsymbol{u_t} &= -\frac{1}{J} \nabla_{\boldsymbol{\xi}}^N \cdot \tilde{\boldsymbol{f}}_e^\delta \\ &= -\sum_{n=1}^{3} \left(\sum_{i=1}^{3} \partial_{\xi_i}^N (J a_n^i) \right) \boldsymbol{c}_n = 0. \end{aligned}$$

$$N = 1: \qquad \partial_t \tilde{\boldsymbol{f}}^\delta = \frac{\tilde{\boldsymbol{f}}(\boldsymbol{u} + \Delta t\, \boldsymbol{u}_t) - \tilde{\boldsymbol{f}}(\boldsymbol{u} - \Delta t\, \boldsymbol{u}_t)}{2\,\Delta t} = \mathbf{0}$$

$$\Rightarrow \tilde{\boldsymbol{F}}^\delta = \tilde{\boldsymbol{f}}^\delta + \frac{\Delta t}{2} \partial_t \tilde{\boldsymbol{f}}^\delta = \tilde{\boldsymbol{f}}^\delta,$$

where

$$\tilde{\boldsymbol{f}}_i^\delta = J\boldsymbol{a}^i \cdot \boldsymbol{c} = J\boldsymbol{a}^i \cdot \boldsymbol{c}.$$

Free stream preservation of Lax-Wendroff

$$\begin{aligned}
&\boldsymbol{u}^{n+1} - \boldsymbol{u}^n + \frac{1}{J}\Delta t \left(\sum_{i=1}^{3} \partial_{\xi^i}^{N} (J\boldsymbol{a}^i) \right) \cdot \boldsymbol{c} \\
&\quad + \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c})(\boldsymbol{\xi}_i^R)\, g_R'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} \\
&\quad + \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c})(\boldsymbol{\xi}_i^L)\, g_L'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} = \boldsymbol{0}.
\end{aligned}$$

Metric identities

$$\sum_{i=1}^{3} \partial_{\xi^i}^{N} (J\boldsymbol{a}^i) = \boldsymbol{0}$$

Conformality

$$(J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c} = \boldsymbol{0}$$

Metric identities in practise (Kopriva 2006)

$$J\boldsymbol{a}^1 = (y_\eta, -x_\eta), \qquad J\boldsymbol{a}^2 = (-y_\xi, x_\xi).$$

$$\partial_\xi^N J\boldsymbol{a}^1 + \partial_\eta^N J\boldsymbol{a}^2 = \boldsymbol{0}$$

2-D

$$\begin{aligned} \partial_\xi^N (y_\eta) - \partial_\eta^N (x_\xi) &= 0 \\ -\partial_\xi^N (x_\eta) + \partial_\eta^N (x_\xi) &= 0 \end{aligned}$$

$$\text{Condition:} \quad \text{Degree of mesh} \le N$$

$$J\boldsymbol{a}^1 = \boldsymbol{a}^2 \times \boldsymbol{a}^3, \quad J\boldsymbol{a}^2 = \boldsymbol{a}^3 \times \boldsymbol{a}^1, \quad J\boldsymbol{a}^3 = \boldsymbol{a}^1 \times \boldsymbol{a}^2$$

3-D

$$\text{Condition:} \qquad \text{Degree of mesh} \le N/2.$$

Conservative form of metric terms

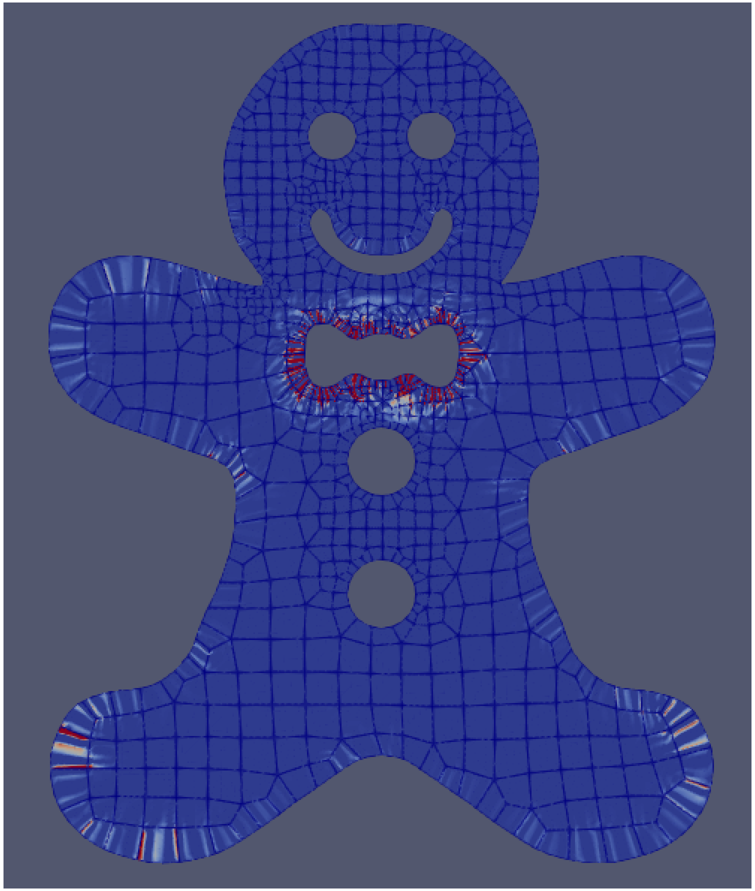
$$J a_n^i = -\boldsymbol{e}_i \cdot \nabla_\xi \times (x_m \nabla_\xi x_l), \quad i = 1,2,3, \quad n = 1,2,3, \quad (n,m,l) \text{ cyclic.}$$

Free stream always satisfied with **conservative curl form**

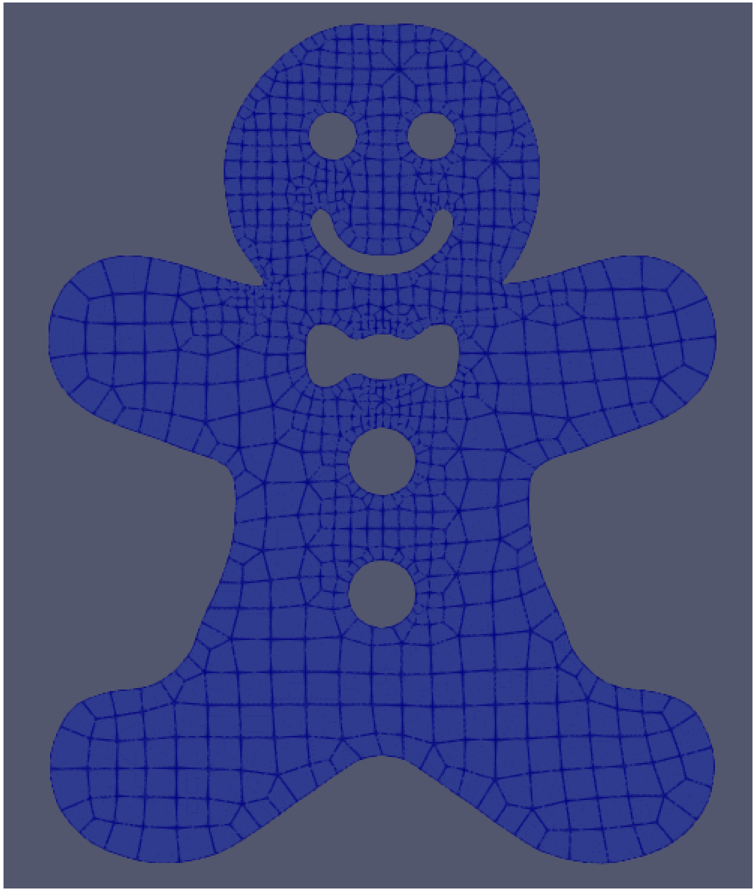
$$J a_n^i = -\boldsymbol{e}_i \cdot \nabla_\xi^N \times (x_l \nabla_\xi x_m)$$

Free stream condition verified

Mesh degree = 6



Solution polynomial degree 5

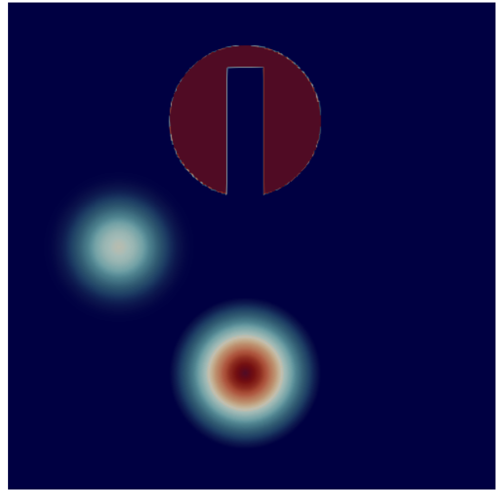


Solution polynomial degree 6

Numerical Results

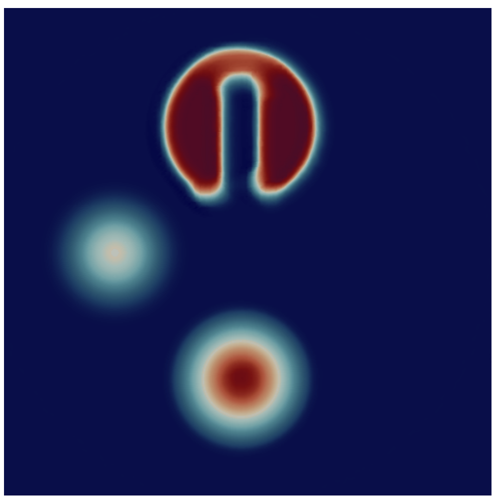
2-D Composite signal [11]

Min = 0, Max = 1



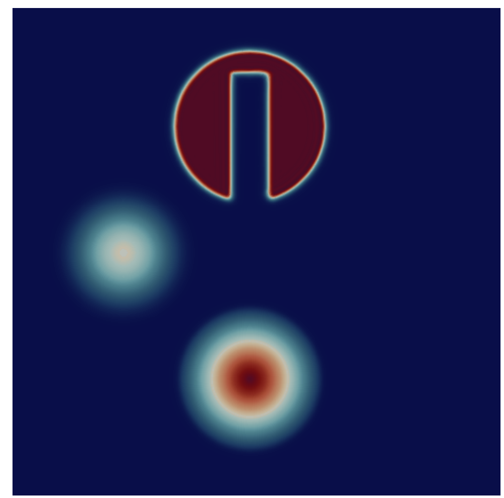
Initial State

Min= -0.01572, Max = 1.01422



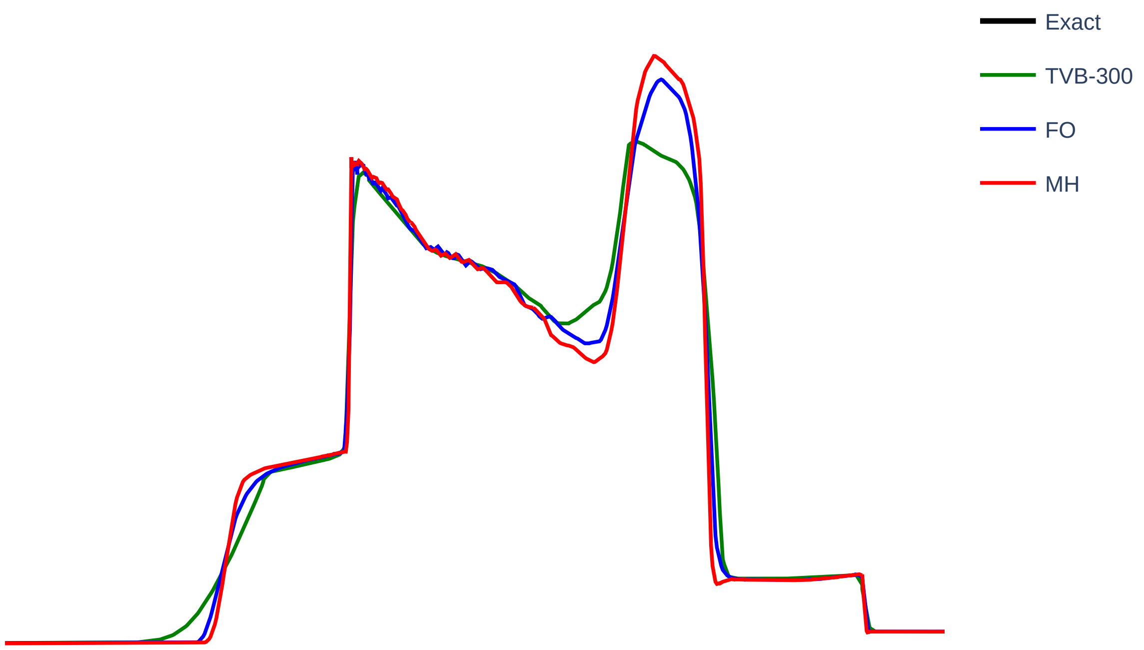
$\mathrm{TVB}-100$

Min = -0.0016, Max = 1.00684



MUSCL-Hancock

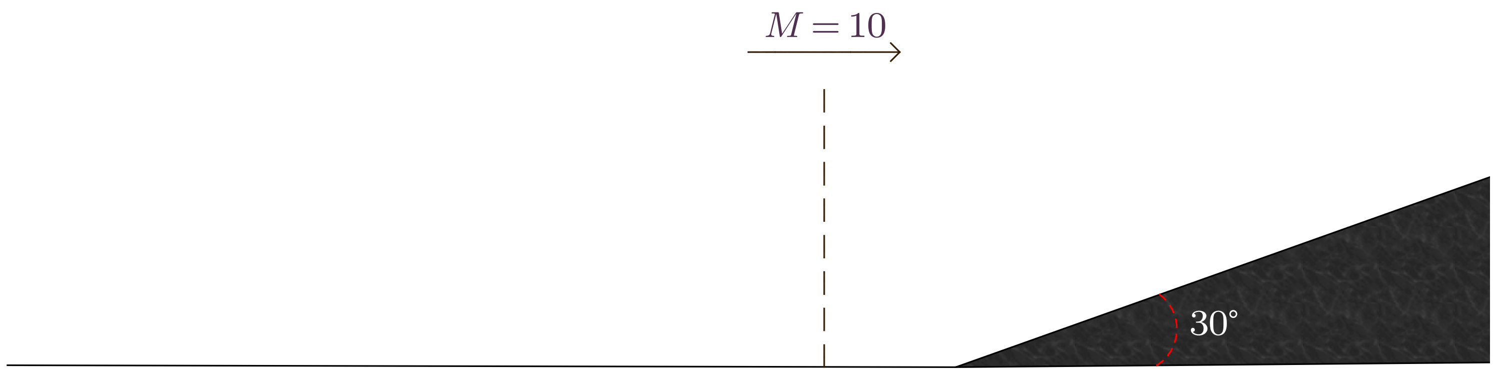
Blast wave [19]



Solid wall boundary conditions on $[0,1]$ with intial condition

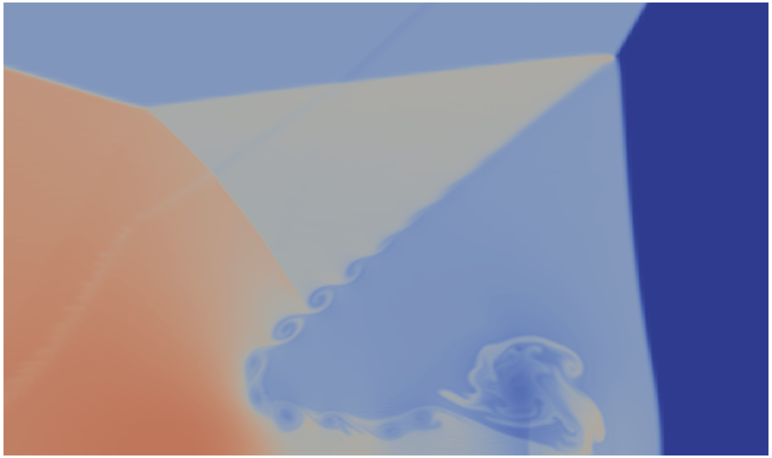
$$(\rho, v, p) = \begin{cases} (1, 0, 1000), & \text{if } x < 0.1, \\ (1, 0, 0.01), & 0.1 < x < 0.9, \\ (1, 0, 100), & x > 0.9. \end{cases}$$

Double Mach reflection [19]

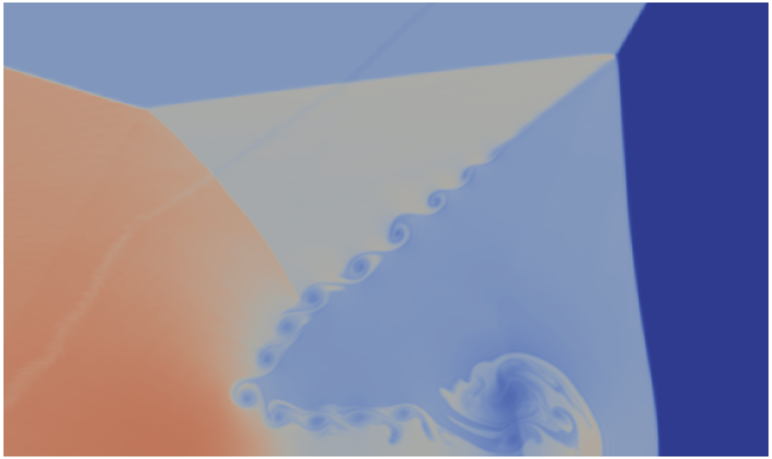


Animation link

$t = 0.2$, NC $= 568 \times 142$, Rusanov, Degree $N = 4$



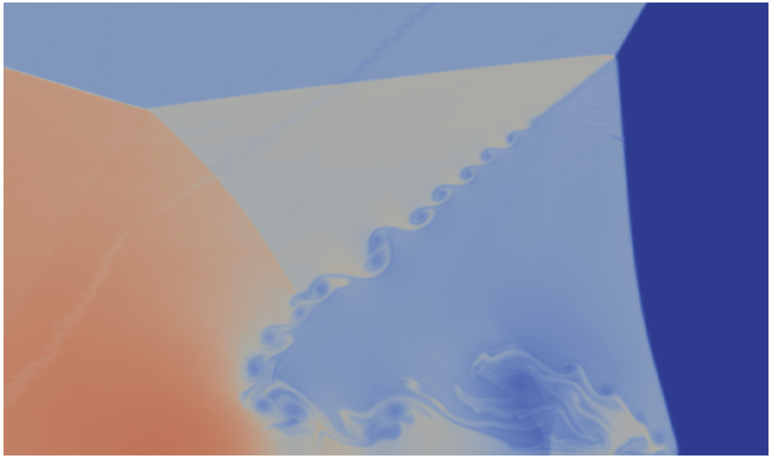
`Trixi.jl`



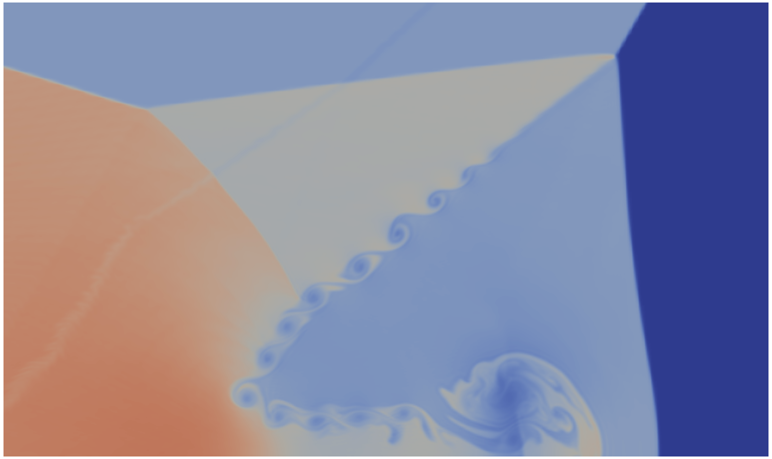
LWFR

Double Mach Reflection

$t = 0.2$, NC$=568 \times 142$, LWFR, Degree $N = 4$

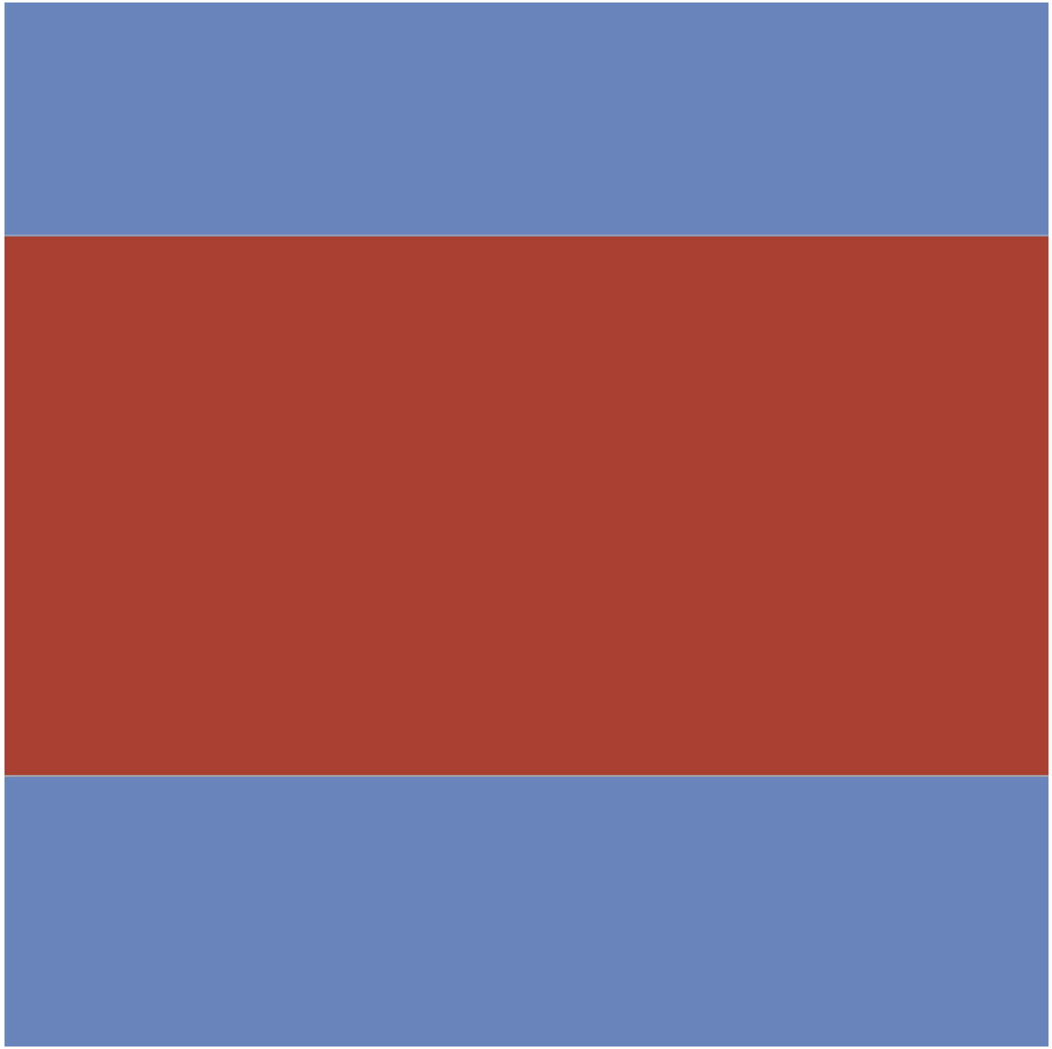


HLLC, $\beta_1 = 0.1$, $\beta_2 = 1.0$



Rusanov. $\beta_1 = \beta_2 = 1.0$

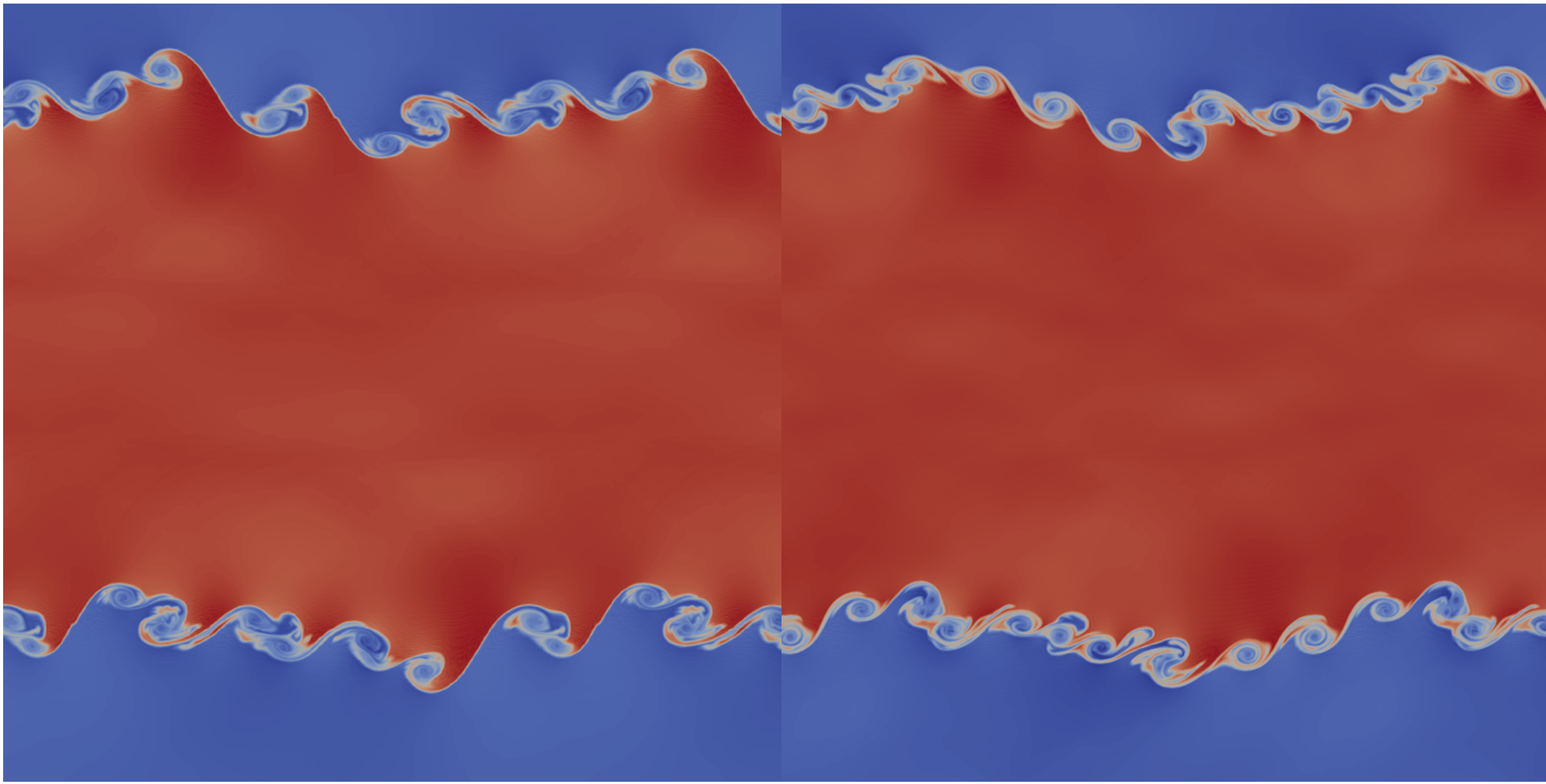
Kelvin-Helmholtz Instability [18, 15]



Animation link

Kelvin-Helmholtz Instability [18, 15]

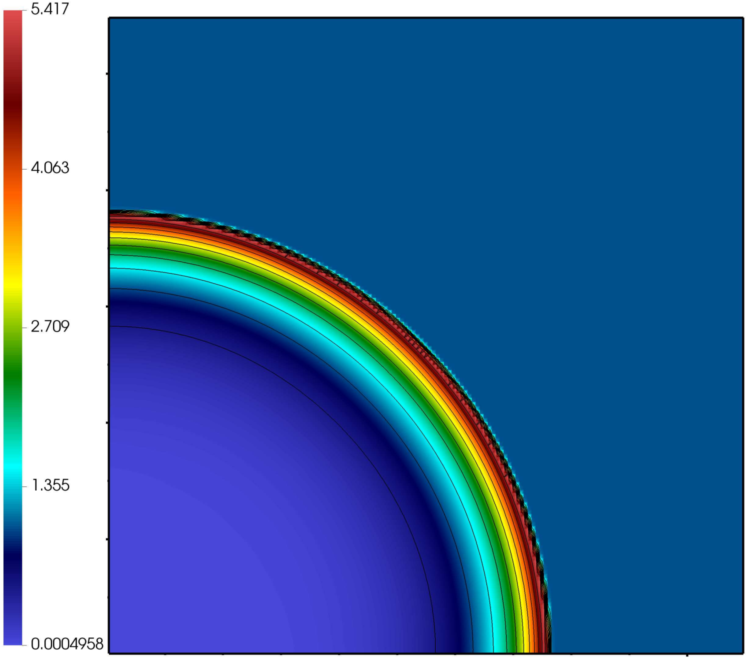
Density profile $t = 0.4$, $\mathrm{NC} = 256^2$, Degree $N = 4$



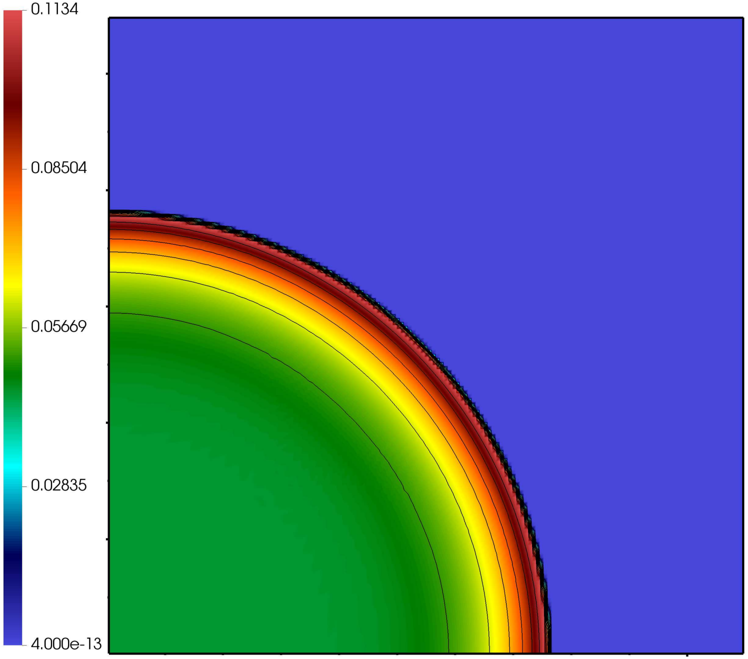
`Trixi.jl` LWFR

Sedov blast [16]

$t = 0.001, \quad \mathrm{NC} = 160^2, \quad \text{Degree } N = 4$



Density



Pressure

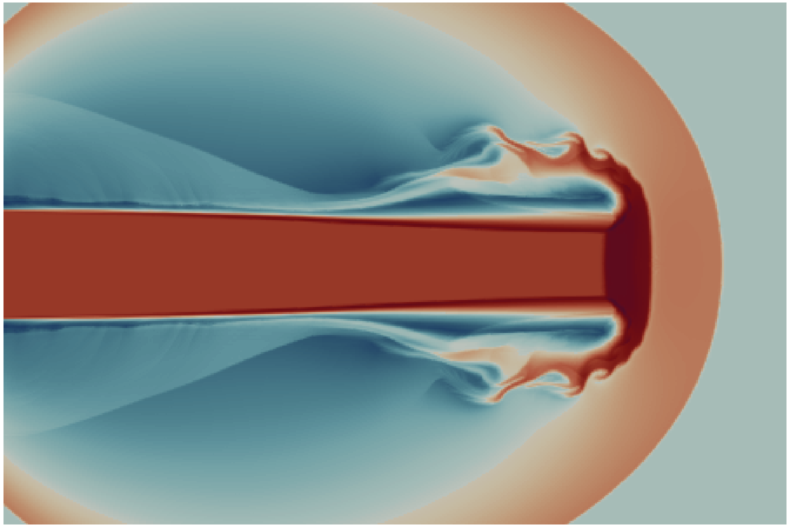
The initial condition is given by

$$\rho = 1.0, \quad v_1 = v_2 = 0.0, \quad E(x,y) = \begin{cases} \dfrac{0.244816}{\Delta x\, \Delta y} & x < \Delta x,\, y < \Delta y, \\ 10^{-12} & \text{otherwise.} \end{cases}$$

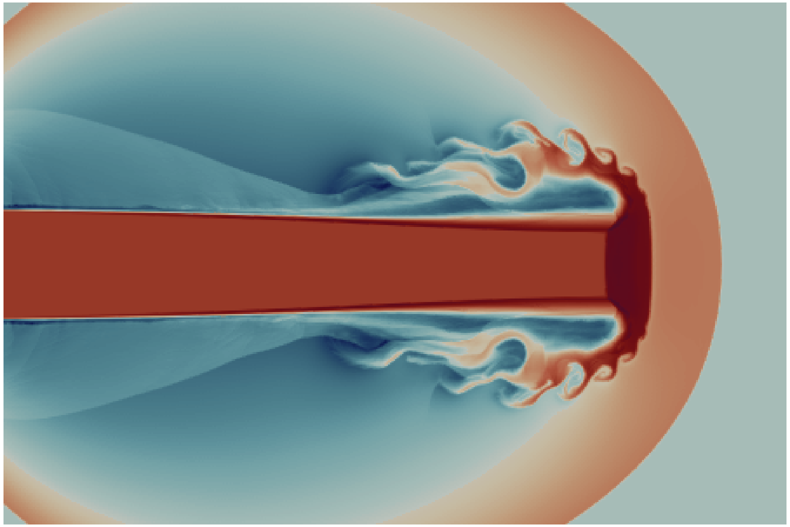
Astrophysial jet [7]

Animation link

Density profile, $\mathrm{NC} = 450 \times 225$, $t = 0.001$, Degree $N = 4$



`Trixi.jl`



LWFR

Conclusions

- A Jacobian free Lax-Wendroff scheme presented in the collocation based Flux Reconstruction (FR) framework.
- Dissipative part of numerical flux computed with time averaged solution, leading to an increase in CFL number. The obtained scheme is equivalent to ADER-DG for linear problems.
- Central part of numerical flux computed by performing the Lax-Wendroff procedure at the face, leading to improvement in accuracy.
- Sub-cell based blending limiter [8] with MUSCL-Hancock reconstruction introduced for Lax-Wendroff schemes and is found to be more accurate than the first order blending initially proposed in [8].
- Problem independent slope limiting procedure proposed for MUSCL-Hancock schemes on general grids which leads to provable admissibility preservation.
- Using the admissibility preserving MUSCL-Hancock and a flux correction, an admissibility preserving Lax-Wendroff scheme was constructed.
- The scheme was extended to unstructured grids and the free stream condition was proven to be equivalent to that of Flux Reconstruction schemes.

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Thank you

Joint Work With

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Sudarshan Kumar Kenettinkara, IISER-Trivandrum

Conservation property of LWFR

$$(u_j^e)^{n+1} = (u_j^e)^n - \frac{\Delta t}{\Delta x_e} \frac{\partial F_h}{\partial \xi}(\xi_j), \qquad 1 \le j \le N+1.$$

For $\{w_j\}_{j=1}^{N+1}$ being the quadrature weights associated with solution points:

$$\sum_{j=1}^{N+1} w_j (u_j^e)^{n+1} = \sum_{j=1}^{N+1} (u_j^e)^n - \frac{\Delta t}{\Delta x_e} \sum_{j=1}^{N+1} w_j \frac{\partial F_h}{\partial \xi}(\xi_j),$$

$$\Rightarrow \overline{u}_e^{n+1} = \overline{u}_e^n - \frac{\Delta t}{\Delta x_e}\Big[F_{e+\frac{1}{2}} - F_{e-\frac{1}{2}}\Big].$$

Interface numerical flux

High order

$$\overline{u}_e^{H,n+1} = \overline{u}_e^n - \frac{\Delta t}{\Delta x_e}\left(F^H_{e+\frac{1}{2}} - F^H_{e-\frac{1}{2}}\right)$$

Low-order

$$\overline{u}_e^{L,n+1} = \overline{u}_e^n - \frac{\Delta t}{\Delta x_e}\left(F^L_{e+\frac{1}{2}} - F^L_{e-\frac{1}{2}}\right).$$

For blended update

$$\overline{u}_e^{n+1} = \overline{u}_e^n - \Delta t\left(F_{e+\frac{1}{2}} - F_{e-\frac{1}{2}}\right),$$

$$\text{where} \quad F_{e+\frac{1}{2}} = \alpha_e\, F^L_{e+\frac{1}{2}} + (1-\alpha_e)\, F^H_{e+\frac{1}{2}}.$$

Conservation requires

$$\alpha_e\, F^L_{e+\frac{1}{2}} + (1-\alpha_e)\, F^H_{e+\frac{1}{2}} = \alpha_{e+1}\, F^L_{e+\frac{1}{2}} + (1-\alpha_{e+1})\, F^H_{e+\frac{1}{2}}$$

$$\Rightarrow F^L_{e+\frac{1}{2}} = F^H_{e+\frac{1}{2}}$$

Admissibility preservation in 2-D

Initial candidate:
$$\begin{aligned}\tilde{F}_{e_x+\frac{1}{2},e_y,j} &= (1-\alpha_{e_x+\frac{1}{2},e_y})\,F^{\text{LW}}_{e_x+\frac{1}{2},e_y,j} + \alpha_{e_x+\frac{1}{2},e_y}\,f_{\boldsymbol{e},N+\frac{3}{2},j}, \qquad 1\le j\le N+1,\\ \tilde{F}_{e_x,e_y+\frac{1}{2},i} &= (1-\alpha_{e_x,e_y+\frac{1}{2}})\,F^{\text{LW}}_{e_x,e_y+\frac{1}{2},i} + \alpha_{e_x,e_y+\frac{1}{2}}\,f_{\boldsymbol{e},i,N+\frac{3}{2}}, \qquad 1\le i\le N+1.\end{aligned}$$

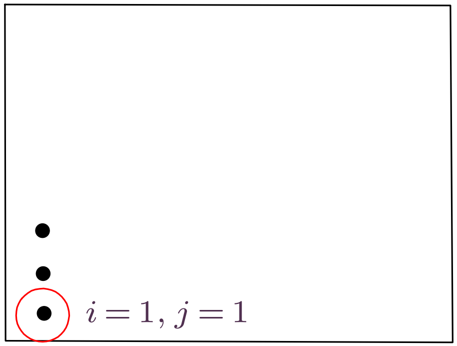
$$\begin{aligned}\tilde{\boldsymbol{u}}^{n+1} &= \boldsymbol{u}^n_{\boldsymbol{e},1,j} - \frac{\Delta t}{\Delta x_{\boldsymbol{e}} w_1}\left(\tilde{F}_{e_x+\frac{1}{2},e_y,j} - f_{\boldsymbol{e},\frac{1}{2},j}\right) - \frac{\Delta t}{\Delta y_e w_j}\left(f_{\boldsymbol{e},1,j+\frac{1}{2}} - f_{\boldsymbol{e},1,j-\frac{1}{2}}\right), \quad 1<j<N+1,\\ \tilde{\boldsymbol{u}}^{n+1} &= \boldsymbol{u}^n_{\boldsymbol{e},1,1} - \frac{\Delta t}{\Delta x_{\boldsymbol{e}} w_1}\left(\tilde{F}_{e_x+\frac{1}{2},e_y,1} - f_{\boldsymbol{e},\frac{1}{2},1}\right) - \frac{\Delta t}{\Delta y_e w_1}\left(\tilde{F}_{e_x,e_y+\frac{1}{2},1} - f_{\boldsymbol{e},1,\frac{1}{2}}\right), \quad j=1.\end{aligned}$$

In the 2-D code, there's two separate face loops for vertical and horizontal faces. This poses a challenge because to ensure $\tilde{u}^{n+1}$ is admissible, we need to correct both $\tilde{F}_{e_x+\frac{1}{2},e_y,1}$ and $\tilde{F}_{e_x,e_y+\frac{1}{2},1}$ and these values are never available together.

To avoid having to store values and doing aposteriori correction, we find appropriate λ_x, λ_y such that

$$\lambda_x + \lambda_y = 1,$$

and then, following the 1-D procedure, construct corrected $F_{e_x+\frac{1}{2},e_y,1}$ and $F_{e_x,e_y+\frac{1}{2},1}$ such that

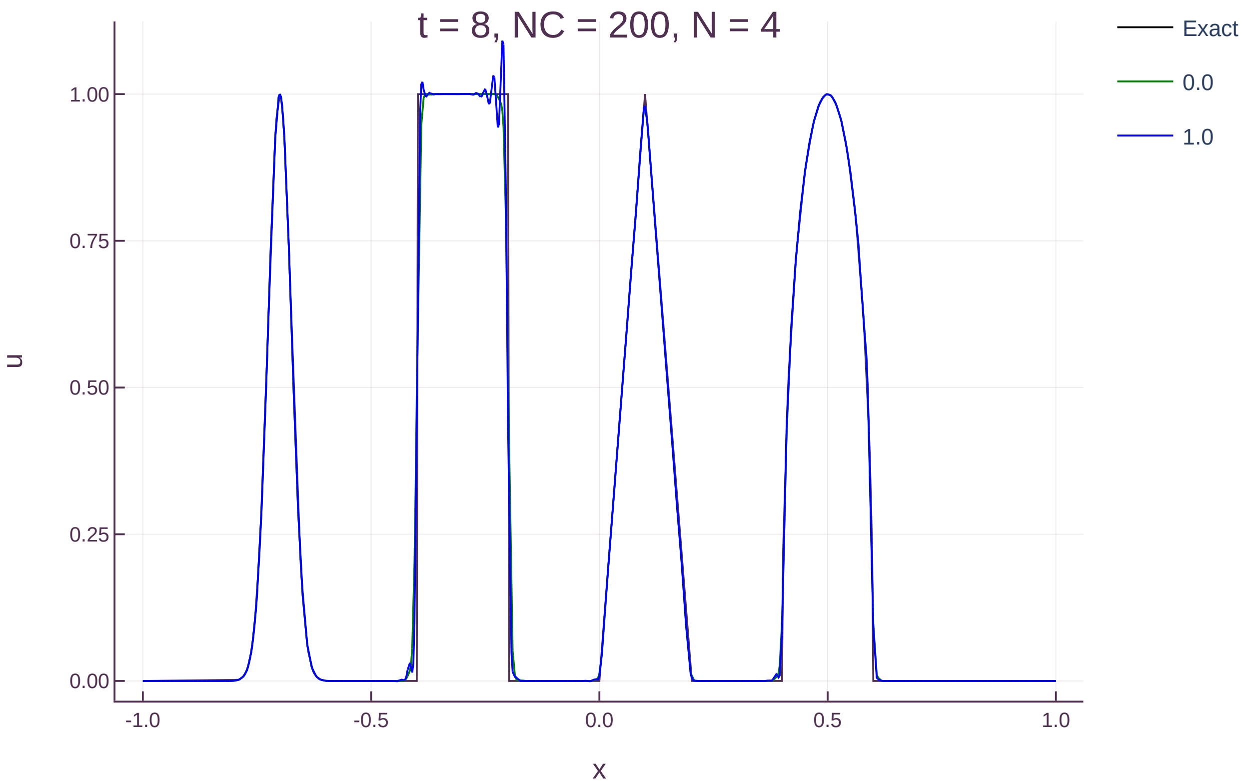


$$\boldsymbol{u}^n_{\boldsymbol{e},1,1} - \frac{\Delta t}{\Delta x_{\boldsymbol{e}} \lambda_x w_1}\left(\tilde{F}_{e_x+\frac{1}{2},e_y,1} - f_{\boldsymbol{e},\frac{1}{2},1}\right) \in \Omega,$$
$$\boldsymbol{u}^n_{\boldsymbol{e},1,1} - \frac{\Delta t}{\Delta y_e \lambda_y w_1}\left(\tilde{F}_{e_x,e_y+\frac{1}{2},1} - f_{\boldsymbol{e},1,\frac{1}{2}}\right) \in \Omega.$$

$$\lambda_x = \frac{|s_x^{\boldsymbol{e}}| \,/\, \Delta x_{\boldsymbol{e}}}{|s_x^{\boldsymbol{e}}| \,/\, \Delta x_{\boldsymbol{e}} + |s_y^{\boldsymbol{e}}| \,/\, \Delta y_{\boldsymbol{e}}}, \qquad \lambda_y = \frac{|s_y^{\boldsymbol{e}}| \,/\, \Delta y_{\boldsymbol{e}}}{|s_x^{\boldsymbol{e}}| \,/\, \Delta x_{\boldsymbol{e}} + |s_y^{\boldsymbol{e}}| \,/\, \Delta y_{\boldsymbol{e}}}$$

Choosing β_1, β_2

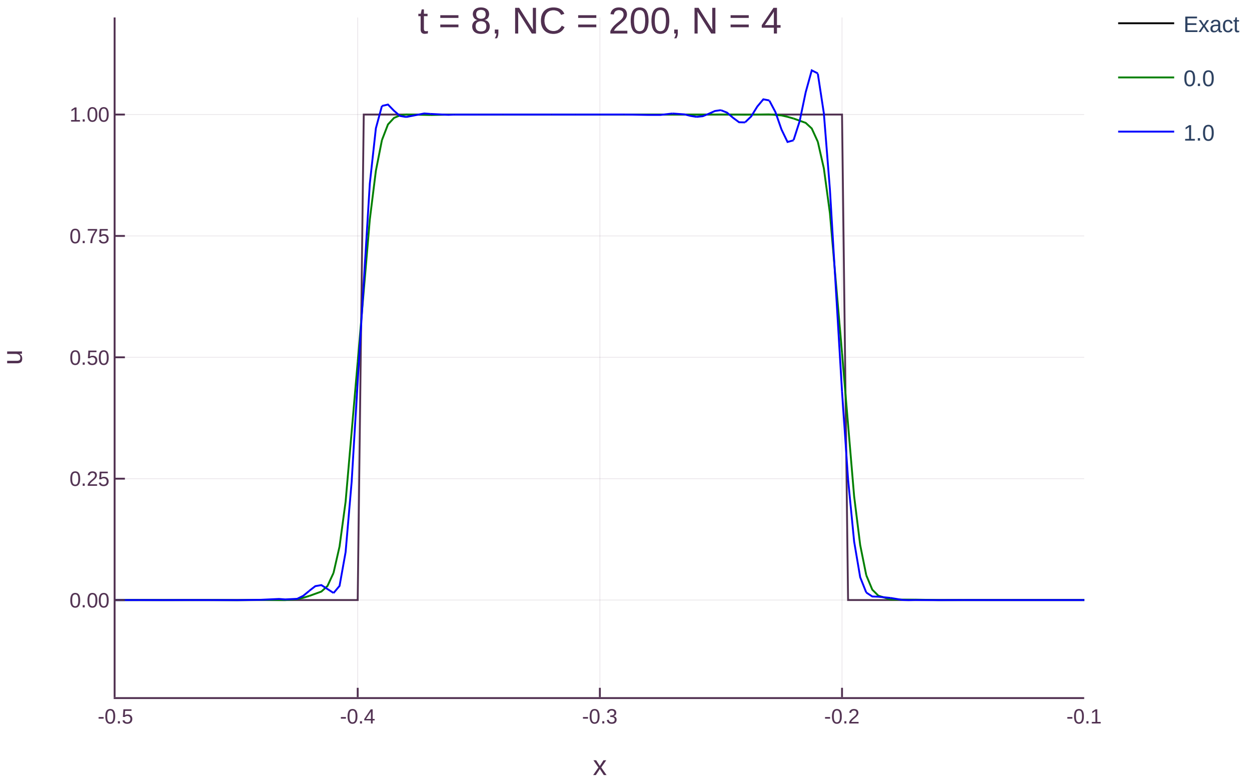
$$\mathbb{E} = \max\left(\frac{m_{N+1}^2}{\beta_1 m_1 + \sum_{j=2}^{N+1} m_j^2}, \frac{m_N^2}{\beta_2 m_1 + \sum_{j=2}^{N} m_j^2}\right)$$



$\beta_2 = 1.0$

Choosing β_1, β_2

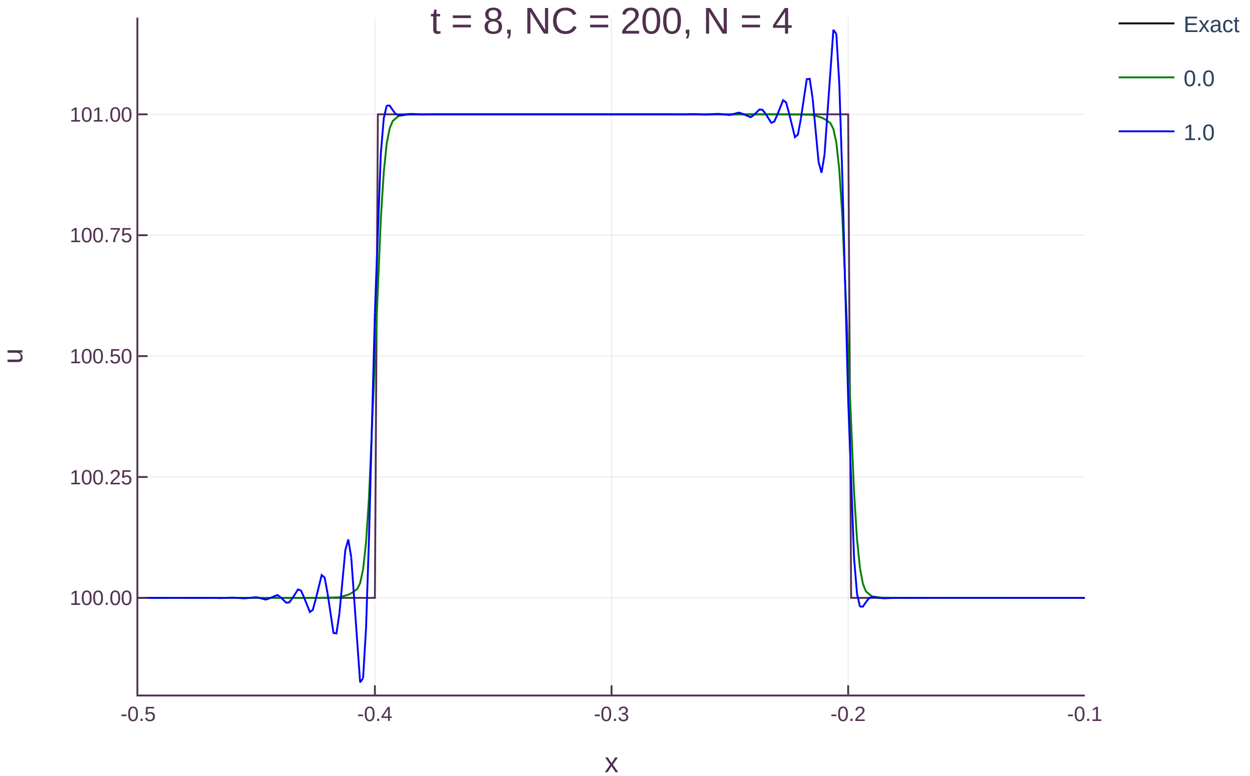
$$\mathbb{E} = \max\left(\frac{m_{N+1}^2}{\beta_1 m_1 + \sum_{j=2}^{N+1} m_j^2}, \frac{m_N^2}{\beta_2 m_1 + \sum_{j=2}^{N} m_j^2} \right)$$



$\beta_2 = 1.0$

Choosing β_1, β_2

$$\mathbb{E} = \max\left(\frac{m_{N+1}^2}{\beta_1 m_1 + \sum_{j=2}^{N+1} m_j^2}, \frac{m_N^2}{\beta_2 m_1 + \sum_{j=2}^{N} m_j^2}\right)$$



$\beta_2 = 1.0$

Lemma. *Consider the 1-D Riemann problem*

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x &= 0, \\ \boldsymbol{u}(x,0) &= \begin{cases} \boldsymbol{u}_l, & x<0, \\ \boldsymbol{u}_r, & x>0, \end{cases} \end{aligned}$$

in $[-h,h]\times[0,\Delta t]$ *where*

$$\frac{\Delta t}{h}\left|\sigma_e(\boldsymbol{u}_l,\boldsymbol{u}_r)\right| \le \frac{1}{2},$$

where $\sigma_e(\boldsymbol{u}_1,\boldsymbol{u}_2)$ *denotes all eigenvalues of all Jacobian matrices at the states between* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$. *Then, for all* $t \le \Delta t$, *we have*

$$\int_{-h}^{h} \boldsymbol{u}(x,t)\,dx = h\left(\boldsymbol{u}_l + \boldsymbol{u}_r\right) - t(\boldsymbol{f}(\boldsymbol{u}_r) - \boldsymbol{f}(\boldsymbol{u}_l)).$$

Step 1 : Evolution to $n+1/2$

Lemma 1. ***(Evolution)*** *Pick*

$$\mu^{-}=\frac{x_{i+1/2}-x_i}{x_{i+1/2}-x_{i-1/2}}, \qquad \mu^{+}=\frac{x_i-x_{i-1/2}}{x_{i+1/2}-x_{i-1/2}},$$

so that

$$\frac{\mu^{-}}{2}\boldsymbol{u}_i^{n,-}+\frac{1}{2}\boldsymbol{u}_i^{*,\pm}+\frac{\mu^{+}}{2}\boldsymbol{u}_i^{n,+}=\boldsymbol{u}_i^{n,\pm}.$$

Then, assume that

$$\boldsymbol{u}_i^{n,\pm}\in\mathcal{U}_{\mathrm{ad}} \quad \textit{and} \quad \boldsymbol{u}_i^{*,\pm}\in\mathcal{U}_{\mathrm{ad}},$$

and the CFL restrictions

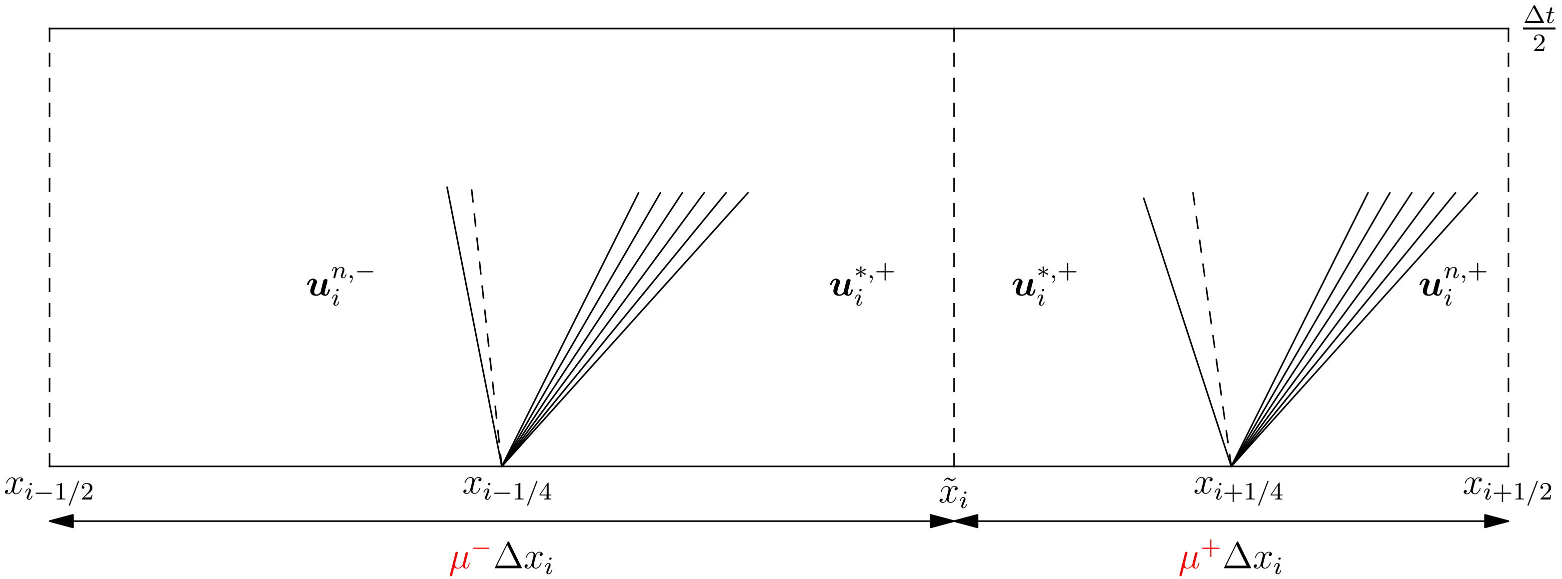
$$\begin{aligned}\frac{\Delta t/2}{\mu^{-}\Delta x/2}\max_{i\in\mathbb{Z}}\left(|\sigma_e(\boldsymbol{u}_i^{n,-},\boldsymbol{u}_i^{*,\pm})|\right)\leq\frac{1}{2},\\ \frac{\Delta t/2}{\mu^{+}\Delta x/2}\max_{i\in\mathbb{Z}}\left(|\sigma_e(\boldsymbol{u}_i^{*,\pm},\boldsymbol{u}_i^{n,+})|\right)\leq\frac{1}{2},\end{aligned} \tag{1}$$

where $\sigma_{\mathrm{e}}(\boldsymbol{u}_1,\boldsymbol{u}_2)$ denotes the maximum spectral radius among all Jacobian matrices at states between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

Then, we have invariance of $\mathcal{U}_{\mathrm{ad}}$ under the first step of MUSCL-Hancock scheme, i.e.,

$$\boldsymbol{u}_i^{n+1/2,\pm}\in\mathcal{U}_{\mathrm{ad}}.$$

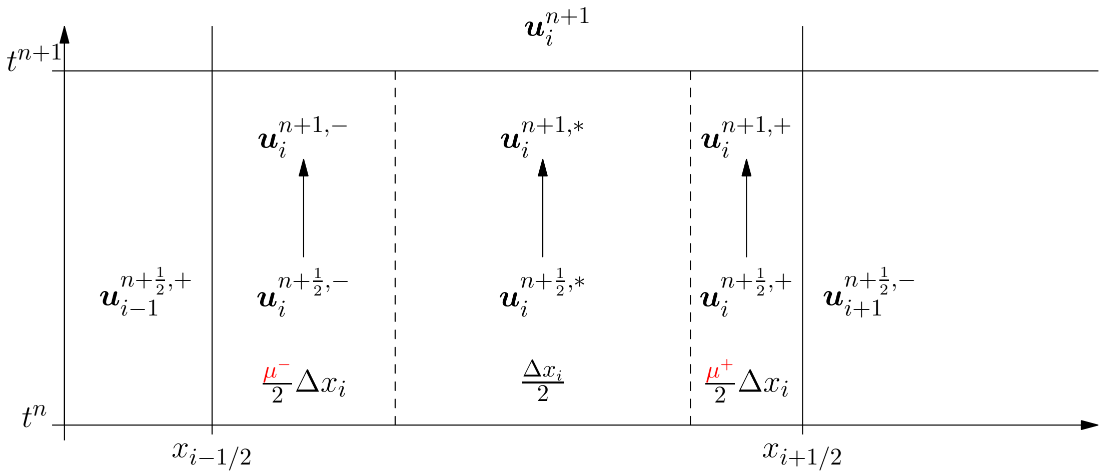
Proof



$$
\begin{aligned}
\frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}^h(x,\Delta t/2)\,dx &= \frac{1}{2}(\mu^-\boldsymbol{w}_i^{n,-}+\boldsymbol{u}_i^{*,+}+\mu^+\boldsymbol{u}_i^{n,+}) - \frac{\Delta t/2}{\Delta x_i}(f(\boldsymbol{u}_i^{n,+})-f(\boldsymbol{u}_i^{n,-}))\\
&= \boldsymbol{u}_i^{n,+} - \frac{\Delta t/2}{\Delta x_i}(f(\boldsymbol{u}_i^{n,+})-f(\boldsymbol{u}_i^{n,-})) = \boldsymbol{u}_i^{n+\frac{1}{2},+} \qquad \square
\end{aligned}
$$

Define $\boldsymbol{u}_i^{n+\frac{1}{2},*}$

$$\frac{\mu^-}{2}\boldsymbol{u}_i^{n+\frac{1}{2},-}+\frac{1}{2}\boldsymbol{u}_i^{n+\frac{1}{2},*}+\frac{\mu^+}{2}\boldsymbol{u}_i^{n+\frac{1}{2},+}=\boldsymbol{u}_i^n.$$



$$\begin{aligned}
\boldsymbol{u}_i^{n+1,-} &:= \boldsymbol{u}_i^{n+\frac{1}{2},-}-\frac{\Delta t}{\mu^-\Delta x_i/2}\left(f\left(\boldsymbol{u}_i^{n+\frac{1}{2},-},\boldsymbol{u}_i^{n+\frac{1}{2},*}\right)-f\left(\boldsymbol{u}_{i-1}^{n+\frac{1}{2},+},\boldsymbol{u}_i^{n+\frac{1}{2},-}\right)\right)\\
\boldsymbol{u}_i^{n+1,*} &:= \boldsymbol{u}_i^{n+\frac{1}{2},*}-\frac{\Delta t}{\Delta x_i/2}\left(f\left(\boldsymbol{u}_i^{n+\frac{1}{2},*},\boldsymbol{u}_i^{n+\frac{1}{2},+}\right)-f\left(\boldsymbol{u}_i^{n+\frac{1}{2},-},\boldsymbol{u}_i^{n+\frac{1}{2},*}\right)\right)\\
\boldsymbol{u}_i^{n+1,+} &:= \boldsymbol{u}_i^{n+\frac{1}{2},+}-\frac{\Delta t}{\mu^+\Delta x_i/2}\left(f\left(\boldsymbol{u}_i^{n+\frac{1}{2},+},\boldsymbol{u}_{i+1}^{n+\frac{1}{2},-}\right)-f\left(\boldsymbol{u}_i^{n+\frac{1}{2},*},\boldsymbol{u}_i^{n+\frac{1}{2},+}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{u}_i^{n+1} &= \boldsymbol{u}_i^n-\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}\left(\boldsymbol{u}_i^{n+\frac{1}{2},+},\boldsymbol{u}_{i+1}^{n+\frac{1}{2},-}\right)-\boldsymbol{f}\left(\boldsymbol{u}_{i-1}^{n+\frac{1}{2},+},\boldsymbol{u}_i^{n+\frac{1}{2},-}\right)\right)\\
&= \frac{\mu^-}{2}\boldsymbol{u}_i^{n+1,-}+\frac{1}{2}\boldsymbol{u}_i^{n+1,*}+\frac{\mu^+}{2}\boldsymbol{u}_i^{n+1,+}
\end{aligned}$$

Lemma 2. (***Riemann solver***) *Assume that the states* $\left(\boldsymbol{u}_i^{n+\frac{1}{2},\pm}\right)_{i\in\mathbb{Z}}, \left(\boldsymbol{u}_i^{n+\frac{1}{2},*}\right)_{i\in\mathbb{Z}}$ *belong to* $\mathcal{U}_{\mathrm{ad}}$, *where* $\boldsymbol{u}_i^{n+\frac{1}{2},*}$ *was defined above as*

$$\mu^- \boldsymbol{u}_i^{n+\frac{1}{2},-} + \boldsymbol{u}_i^{n+\frac{1}{2},*} + \mu^+ \boldsymbol{u}_i^{n+\frac{1}{2},+} = 2\boldsymbol{u}_i^n.$$

Then, the updated solution of MUSCL-Hancock scheme is in Ω *under the CFL conditions*

$$\begin{aligned}
&\frac{\Delta t}{\mu^- \Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_i^{n+\frac{1}{2},-}, \boldsymbol{u}_i^{n+\frac{1}{2},*}\right)\right|\right) \leq \frac{1}{2},\\
&\frac{\Delta t}{\mu^- \Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_{i-1}^{n+\frac{1}{2},+}, \boldsymbol{u}_i^{n+\frac{1}{2},-}\right)\right|\right) \leq \frac{1}{2},\\
&\frac{\Delta t}{\Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_i^{n+\frac{1}{2},*}, \boldsymbol{u}_i^{n+\frac{1}{2},+}\right)\right|\right) \leq \frac{1}{2},\\
&\frac{\Delta t}{\Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_i^{n+\frac{1}{2},-}, \boldsymbol{u}_i^{n+\frac{1}{2},*}\right)\right|\right) \leq \frac{1}{2},\\
&\frac{\Delta t}{\mu^+ \Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_i^{n+\frac{1}{2},+}, \boldsymbol{u}_{i+1}^{n+\frac{1}{2},-}\right)\right|\right) \leq \frac{1}{2},\\
&\frac{\Delta t}{\mu^+ \Delta x_i/2} \max_{i\in\mathbb{Z}} \left(\left|\sigma_e\left(\boldsymbol{u}_i^{n+\frac{1}{2},*}, \boldsymbol{u}_i^{n+\frac{1}{2},+}\right)\right|\right) \leq \frac{1}{2}.
\end{aligned} \tag{2}$$

$$\mu^- \boldsymbol{u}_i^{n+\frac{1}{2},-} + \boldsymbol{u}_i^{n+\frac{1}{2},*} + \mu^+ \boldsymbol{u}_i^{n+\frac{1}{2},+} = 2\boldsymbol{u}_i^n$$

Lemma 3. (***Link previous lemmas***) *Define* $\boldsymbol{u}_i^{*,*}$ *to satisfy*

$$\mu^- \boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,*} + \mu^+ \boldsymbol{u}_i^{n,+} = 2(2\boldsymbol{u}_i^n - (\mu^- \boldsymbol{u}_i^{n,-} + \mu^+ \boldsymbol{u}_i^{n,+})), \tag{3}$$

where, as defined before,

$$\mu^- = \frac{x_{i+1/2} - x_i}{x_{i+1/2} - x_{i-1/2}}, \quad \mu^+ = \frac{x_i - x_{i-1/2}}{x_{i+1/2} - x_{i-1/2}}.$$

Assume that $\boldsymbol{u}_i^{n,\pm}$ *and* $\boldsymbol{u}_i^{*,*}$ *are in* $\mathcal{U}_{\mathrm{ad}}$. *Consider the CFL conditions*

$$\begin{aligned} \frac{\Delta t/2}{\mu^- \Delta x/2} \max_{i \in \mathbb{Z}} (|\sigma_e(\boldsymbol{u}_i^{*,*}, \boldsymbol{u}_i^{n,-})|) \leq \frac{1}{2}, \\ \frac{\Delta t/2}{\mu^+ \Delta x/2} \max_{i \in \mathbb{Z}} (|\sigma_e(\boldsymbol{u}_i^{n,+}, \boldsymbol{u}_i^{*,*})|) \leq \frac{1}{2}, \end{aligned} \tag{4}$$

then $\boldsymbol{u}_i^{n+\frac{1}{2},*} \in \mathcal{U}_{\mathrm{ad}}$.

Remark 4. For conservative reconstruction, we actually have $\boldsymbol{u}_i^{*,*} = \boldsymbol{u}_i$. So, this lemma isn't placing new restrictions.

Final admissibility condition

$$\boldsymbol{u}_i^{n+\frac{1}{2},*} = (2\boldsymbol{u}_i^n - (\mu^-\boldsymbol{u}_i^{n,-} + \mu^+\boldsymbol{u}_i^{n,+})) - \frac{\Delta t}{2\Delta x_i}[(f(\boldsymbol{u}_i^{n,-}) - f(\boldsymbol{u}_i^{n,+}))]$$

Lemma 5. (***Link previous lemmas***) *Define $\boldsymbol{u}_i^{*,*}$ to satisfy*

$$\mu^-\boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,*} + \mu^+\boldsymbol{u}_i^{n,+} = 2(2\boldsymbol{u}_i^n - (\mu^-\boldsymbol{u}_i^{n,-} + \mu^+\boldsymbol{u}_i^{n,+})), \tag{5}$$

where, as defined before,

$$\mu^- = \frac{x_{i+1/2} - x_i}{x_{i+1/2} - x_{i-1/2}}, \qquad \mu^+ = \frac{x_i - x_{i-1/2}}{x_{i+1/2} - x_{i-1/2}}.$$

Assume that $\boldsymbol{u}_i^{n,\pm}$ and $\boldsymbol{u}_i^{*,*}$ are in $\mathcal{U}_{\text{ad}}$. Consider the CFL conditions

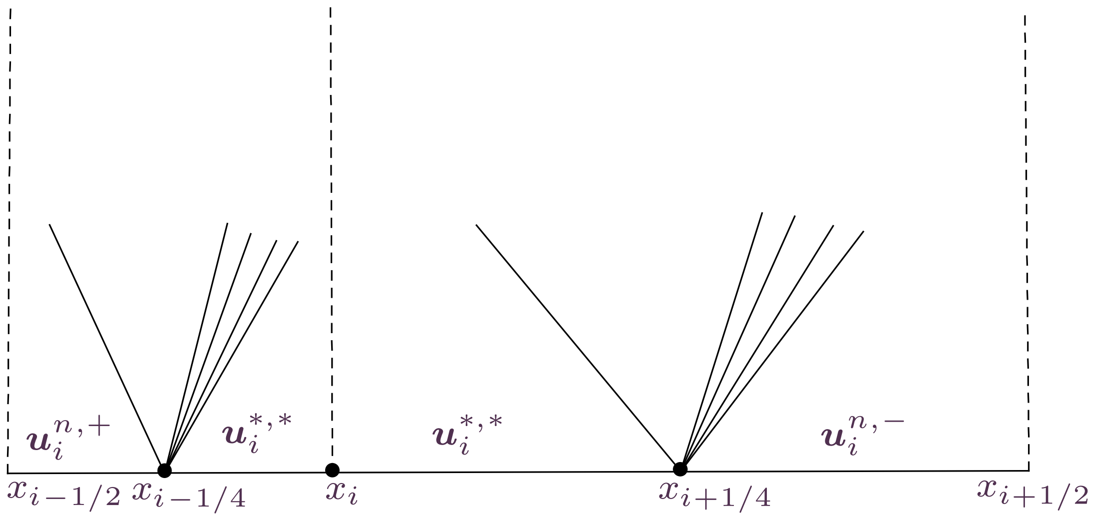
$$\begin{aligned} \frac{\Delta t/2}{\mu^-\Delta x/2}\max_{i\in\mathbb{Z}}(|\sigma_e(\boldsymbol{u}_i^{*,*}, \boldsymbol{u}_i^{n,-})|) \le \frac{1}{2}, \\ \frac{\Delta t/2}{\mu^+\Delta x/2}\max_{i\in\mathbb{Z}}(|\sigma_e(\boldsymbol{u}_i^{n,+}, \boldsymbol{u}_i^{*,*})|) \le \frac{1}{2}, \end{aligned} \tag{6}$$

then $\boldsymbol{u}_i^{n+\frac{1}{2},*} \in \mathcal{U}_{\text{ad}}$.

Remark 6. For conservative reconstruction, we actually have $\boldsymbol{u}_i^{*,*} = \boldsymbol{u}_i$. So, this lemma isn't placing new restrictions.

Final admissibility condition

Proof.



$$
\begin{aligned}
\frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}^h\left(x, \frac{\Delta t}{2}\right) dx &= \frac{1}{2}(\mu^+ \boldsymbol{u}_i^{n,+} + \boldsymbol{u}_i^{*,*} + \mu^- \boldsymbol{u}_i^{n,-}) - \frac{\Delta t}{2\,\Delta x_i}(f(\boldsymbol{u}_i^{n,-}) - f(\boldsymbol{u}_i^{n,+})) \\
&= \boldsymbol{u}_i^{n+\frac{1}{2},*}
\end{aligned}
$$

Final admissibility condition

Theorem 7. *Consider the conservation law*

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$$

which preserves a convex set $\mathcal{U}_{\mathrm{ad}}$. Let $\{\boldsymbol{u}_i^n\}_{i\in\mathbb{Z}}$ be the approximate solution at time level n and assume that $\boldsymbol{u}_i^n \in \mathcal{U}_{\mathrm{ad}}$ for all $i \in \mathbb{Z}$. Consider conservative reconstructions

$$\boldsymbol{u}_i^{n,+} = \boldsymbol{u}_i + (x_{i+1/2} - x_i)\boldsymbol{\sigma}_i, \quad \boldsymbol{u}_i^{n,-} = \boldsymbol{u}_i + (x_{i-1/2} - x_i)\boldsymbol{\sigma}_i.$$

Define $\boldsymbol{u}_i^{*,\pm}$ to be

$$\boldsymbol{u}_i^{*,\pm} = \boldsymbol{u}_i^n + 2\Big(x_{i\pm\frac{1}{2}} - x_i\Big)\,\boldsymbol{\sigma}_i$$

and assume that the slope σ_i is chosen so that

$$\boldsymbol{u}_i^{*,\pm} \in \mathcal{U}_{\mathrm{ad}}.$$

Then, under time step restrictions (1), (2), (6), the updated solution $\boldsymbol{u}_i^{n+1}$, defined by the MUSCL-Hancock procedure is in $\mathcal{U}_{\mathrm{ad}}$.

Proof. We only need $\boldsymbol{u}_i^{n,\pm} \in \mathcal{U}_{\mathrm{ad}}$ to apply the Lemmas 1, 2, 5. To that end, notice

$$\boldsymbol{u}_i^{n,\pm} = \frac{1}{2}\boldsymbol{u}_i^{*,\pm} + \frac{1}{2}\boldsymbol{u}_i^n. \qquad \square$$

Enforcing slope restriction

Given candidate slope $\boldsymbol{\sigma}_i$,

$$\boldsymbol{u}_i^{*,\pm} := \boldsymbol{u}_i^n + 2\,(x_{i\pm 1/2} - x_i)\,\boldsymbol{\sigma}_i.$$

Find $\theta \in [0,1]$

$$\boldsymbol{u}_i^n + 2\,(x_{i\pm 1/2} - x_i)\,\theta\,\boldsymbol{\sigma}_i \in \mathcal{U}_{\mathrm{ad}}. \tag{7}$$

For concave $p = p(\boldsymbol{u})$, assume

$$\mathcal{U}_{\mathrm{ad}} = \{\boldsymbol{u} \in \Omega : p(\boldsymbol{u}) > 0\}$$

We pick

$$\theta_\pm = \min\left(\left|\frac{\epsilon - p(\boldsymbol{u}_i^n)}{p(\boldsymbol{u}_i^{*,\pm}) - p(\boldsymbol{u}_i^n)}\right|, 1\right)$$

and

$$\theta = \min\,(\theta_+, \theta_-).$$

By concavity,

$$p(\theta\,\boldsymbol{u}_i^{*,\pm} + (1-\theta)\,\boldsymbol{u}_i^n) > \theta\,p(\boldsymbol{u}_i^{*,\pm}) + (1-\theta)\,p(\boldsymbol{u}_i^n) > \epsilon > 0.$$

Thus, this θ will give us (7).

Consider non-conservative variables

$$\boldsymbol{U}_i^n = \kappa(\boldsymbol{u}_i^n),$$

so that reconstruction is given by

$$\boldsymbol{U}^n(x) = \boldsymbol{U}_i^n + \sigma_i(x - x_i)$$

$$\boldsymbol{u}_i^{n,\pm} \; := \; \kappa^{-1}(\boldsymbol{U}_i^{n,\pm}) \tag{8}$$

Theorem 8. *Assume that $\boldsymbol{u}_i^n \in \Omega$ for all $i \in \mathbb{Z}$. Consider $\boldsymbol{u}_i^{n,\pm}$ defined in (8), $\boldsymbol{u}_i^{*,\pm}$ defined so that*

$$\mu^- \boldsymbol{u}_i^{n+\frac{1}{2},-} + \boldsymbol{u}_i^{n+\frac{1}{2},*} + \mu^+ \boldsymbol{u}_i^{n+\frac{1}{2},+} = 2\boldsymbol{u}_i^n,$$

and $\boldsymbol{u}_i^{*,*}$ defined explicitly as

$$\boldsymbol{u}_i^{*,*} = 4\boldsymbol{u}_i^n - 3(\mu^- \boldsymbol{u}_i^{n,-} + \mu^+ \boldsymbol{u}_i^{n,+}).$$

Assume that the slope is chosen so that

$$\boldsymbol{u}_i^{n,\pm} \in \Omega, \quad \boldsymbol{u}_i^{*,\pm} \in \Omega \quad \text{and} \quad \boldsymbol{u}_i^{*,*} \in \Omega.$$

Consider the same CFL conditions (1),(2),(6). Then the updated solution $\boldsymbol{u}_i^{n+1}$ of MUSCL-Hancock procedure is in Ω.

Remark 9. The definition of $\boldsymbol{u}_i^{*,*}$ comes from

$$\mu^- \boldsymbol{u}_i^{n,-} + \boldsymbol{u}_i^{*,*} + \mu^+ \boldsymbol{u}_i^{n,+} = 2(2\boldsymbol{u}_i^n - (\mu^- \boldsymbol{u}_i^{n,-} + \mu^+ \boldsymbol{u}_i^{n,+}))$$

ADER-DG : Predictor step

Current solution

$$x \in I_e: \qquad u_h^n(\xi) = \sum_{j=1}^{N+1} u_j^n \ell_j(\xi)$$

Cell local space-time solution and flux: $\tau = (t - t_n)\,/\,\Delta t$

$$\tilde{u}_h(\xi,\tau) = \sum_{r,s=1}^{N+1} \tilde{u}_{rs}\ell_r(\xi)\ell_s(\tau), \quad \tilde{f}_h(\xi,\tau) = \sum_{r,s=1}^{N+1} f(\tilde{u}_{rs})\ell_r(\xi)\ell_s(\tau).$$

Find $\tilde{u}_h$ by cell local Galerkin method

$$\int_{t_n}^{t_{n+1}} \int_{I_e} (\partial_t \tilde{u}_h + \partial_x \tilde{f}_h)\, \ell_r(\xi)\ell_s(\tau)\, dx\, dt, \qquad 1 \le r,s \le N+1.$$

Integrate by parts in time

$$-\int_{t_n}^{t_{n+1}} \int_{I_e} \tilde{u}_h \ell_r(\xi)\partial_t \ell_s(\tau)\, dx\, dt + \int_{I_e} \tilde{u}_h(\xi,1)\ell_r(\xi)\ell_s(1)\, dx - \int_{I_e} u_h^n(\xi)\ell_r(\xi)\ell_s(0)\, d\xi$$
$$+\int_{t_n}^{t_{n+1}} \int_{I_e} \partial_x \tilde{f}_h\, \ell_r(\xi)\ell_s(\tau) dx\, dt = 0.$$

ADER correction step

$$\int_{t_n}^{t_{n+1}} \int_{I_e} (\partial_t u_h + \partial_x f(u_h))\ell_i(\xi)\, dx\, dt = 0$$

Integrate by parts in space

$$\begin{aligned}\int_{I_e} u_h^{n+1} \ell_i(\xi)\, dx &= \int_{I_e} u_h^n \, \ell_i(\xi)\, dx + \int_{t^n}^{t^{n+1}} \int_{I_e} \tilde{f}_h \partial_x \ell_i(\xi)\, dx\, dt \\ &\quad + \int_{t^n}^{t^{n+1}} f(\tilde{u}_{e-\frac{1}{2}}^{-}(t), \tilde{u}_{e-\frac{1}{2}}^{+}(t))\ell_i(0) dt - \int_{t^n}^{t^{n+1}} f(\tilde{u}_{e+\frac{1}{2}}^{-}(t), \tilde{u}_{e+\frac{1}{2}}^{+}(t))\ell_i(1) dt.\end{aligned}$$

FR form

$$\begin{aligned} & u_i^n - \int_{t^n}^{t^{n+1}} \partial_x \tilde{f}_h(\xi_i, \tau)\, dt \\ \Rightarrow \; u_i^{n+1} = & \quad - g_L'(\xi_i) \int_{t^n}^{t^{n+1}} f(\tilde{u}_{e-\frac{1}{2}}^{-}(t), \tilde{u}_{e-\frac{1}{2}}^{+}(t)) - \tilde{f}_h(0, \tau) dt \\ & \quad - g_R'(\xi_i) \int_{t^n}^{t^{n+1}} f(\tilde{u}_{e+\frac{1}{2}}^{-}(t), \tilde{u}_{e+\frac{1}{2}}^{+}(t)) - \tilde{f}_h(1, \tau) dt \end{aligned}$$

ADER correction step - writing as FR

$$\int_{t_n}^{t_{n+1}} \int_{I_e} (\partial_t u_h + \partial_x f(u_h)) \ell_i(\xi)\, dx\, dt = 0$$

Integrate by parts in space

$$\begin{aligned} \int_{I_e} u_h^{n+1} \ell_i(\xi)\, dx \;&=\; \int_{I_e} u_h^n\, \ell_i(\xi)\, dx + \int_{t^n}^{t^{n+1}} \int_{I_e} \tilde{f}_h \partial_x \ell_i(\xi)\, dx\, dt \\ &+ \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) \ell_i(0) dt - \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) \ell_i(1) dt. \end{aligned}$$

Another integration by parts

$$\begin{aligned} \int_{I_e} u_h^{n+1} \ell_i(\xi) dx \;&=\; \int_{I_e} u_h^n \ell_i(\xi) dx - \int_{t^n}^{t^{n+1}} \int_{I_e} \partial_x \tilde{f}_h \ell_i(\xi)\, dx\, dt \\ &+ \int_{t^n}^{t^{n+1}} \Big(f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) - \tilde{f}_h(0, \tau) \Big) \ell_i(0) dt \\ &- \int_{t^n}^{t^{n+1}} \Big(f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) - \tilde{f}_h(1, \tau) \Big) \ell_i(1) dt, \end{aligned}$$

Quadrature on solution points

$$
\Rightarrow \quad u_i^{n+1} \; = \; \begin{aligned}
& u_i^n - \int_{t^n}^{t^{n+1}} \partial_x \tilde{f}_h(\xi_i, \tau)\, dt \\
& + \frac{\ell_i(0)}{w_i} \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) - \tilde{f}_h(0, \tau)\, dt \\
& - \frac{\ell_i(1)}{w_i} \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) - \tilde{f}_h(1, \tau)\, dt
\end{aligned}
$$

ADER correction step - writing as FR

$$\int_{t_n}^{t_{n+1}} \int_{I_e} (\partial_t u_h + \partial_x f(u_h)) \ell_i(\xi)\, dx\, dt = 0$$

Integrate by parts in space

$$\begin{aligned}
\int_{I_e} u_h^{n+1} \ell_i(\xi)\, dx \;&=\; \int_{I_e} u_h^n\, \ell_i(\xi)\, dx + \int_{t^n}^{t^{n+1}} \int_{I_e} \tilde{f}_h \partial_x \ell_i(\xi)\, dx\, dt \\
&\quad + \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) \ell_i(0) dt - \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) \ell_i(1) dt.
\end{aligned}$$

Another integration by parts

$$\begin{aligned}
\int_{I_e} u_h^{n+1} \ell_i(\xi) dx \;&=\; \int_{I_e} u_h^n \ell_i(\xi) dx - \int_{t^n}^{t^{n+1}} \int_{I_e} \partial_x \tilde{f}_h \ell_i(\xi)\, dx\, dt \\
&\quad + \int_{t^n}^{t^{n+1}} \Big(f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) - \tilde{f}_h(0, \tau) \Big) \ell_i(0) dt \\
&\quad - \int_{t^n}^{t^{n+1}} \Big(f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) - \tilde{f}_h(1, \tau) \Big) \ell_i(1) dt,
\end{aligned}$$

Quadrature on solution points

$$
\begin{aligned}
\Rightarrow \quad u_i^{n+1} \quad = \quad & u_i^n - \int_{t^n}^{t^{n+1}} \partial_x \tilde{f}_h(\xi_i, \tau)\, dt \\
& - g_L'(\xi_i) \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e-\frac{1}{2}}(t), \tilde{u}^+_{e-\frac{1}{2}}(t)) - \tilde{f}_h(0,\tau) dt \\
& - g_R'(\xi_i) \int_{t^n}^{t^{n+1}} f(\tilde{u}^-_{e+\frac{1}{2}}(t), \tilde{u}^+_{e+\frac{1}{2}}(t)) - \tilde{f}_h(1,\tau) dt
\end{aligned}
$$

ADER and LWFR-D2 for constant linear advection

For

$$u_t + u_x = 0,$$

ADER update

$$\begin{aligned} u_i^{n+1} \;=\; & u_i^n - \partial_x \int_{t^n}^{t^{n+1}} \tilde{u}_h(\xi_i,\tau)\,dt \\ & - g_L'(\xi_i)\Big[f\Big(\int_{t^n}^{t^{n+1}} \tilde{u}_{e-\frac{1}{2}}^-(t)\,dt, \int_{t^n}^{t^{n+1}} \tilde{u}_{e-\frac{1}{2}}^+(t)\,dt \Big) - \int_{t^n}^{t^{n+1}} \tilde{u}_h(0,\tau) dt \Big] \\ & - g_R'(\xi_i)\Big[f\Big(\int_{t^n}^{t^{n+1}} \tilde{u}_{e+\frac{1}{2}}^-(t)\,dt, \int_{t^n}^{t^{n+1}} \tilde{u}_{e+\frac{1}{2}}^+(t)\,dt \Big) - \int_{t^n}^{t^{n+1}} \tilde{u}_h(1,\tau)\,dt \Big] \end{aligned} \tag{9}$$

LWFR-D2 update

$$u_i^{n+1} = u_i^n - \Delta t \Big[\partial_x U_h(\xi_i) - g_L'(\xi_i)\Big[f(U_{e-\frac{1}{2}}^-, U_{e-\frac{1}{2}}^+) - U_h(0) \Big] - g_R'(\xi_i)\Big[f(U_{e+\frac{1}{2}}^-, U_{e+\frac{1}{2}}^+) - U_h(1) \Big]\Big], \tag{10}$$

where

$$\begin{aligned} U_h^n &= u + \frac{\Delta t}{2} u_t + \frac{\Delta t^2}{3!} u_{tt} + \cdots + \frac{\Delta t^N}{(N+1)!}\frac{\partial^N u}{\partial t^N} \\ &= u - \frac{\Delta t}{2} u_x + \frac{\Delta t^2}{3!} u_{xx} + \cdots + (-1)^N \frac{\Delta t^N}{(N+1)!}\frac{\partial^N u}{\partial x^N}. \end{aligned}$$

Equivalence of ADER-FR and LWFR-D2 for linear case

Theorem 10. *For the linear advection equation*

$$u_t + u_x = 0,$$

the Lax-Wendroff Flux Reconstruction scheme with D2 dissipation and ADER-FR scheme are equivalent.

Proof. Let $u_e^n = u_e^n(x)$ denote the solution polynomial at time level n in element e.

Then, $\tilde{u}_h(x,t) := u_e^n(x-(t-t^n))$ is a weak solution of the equation

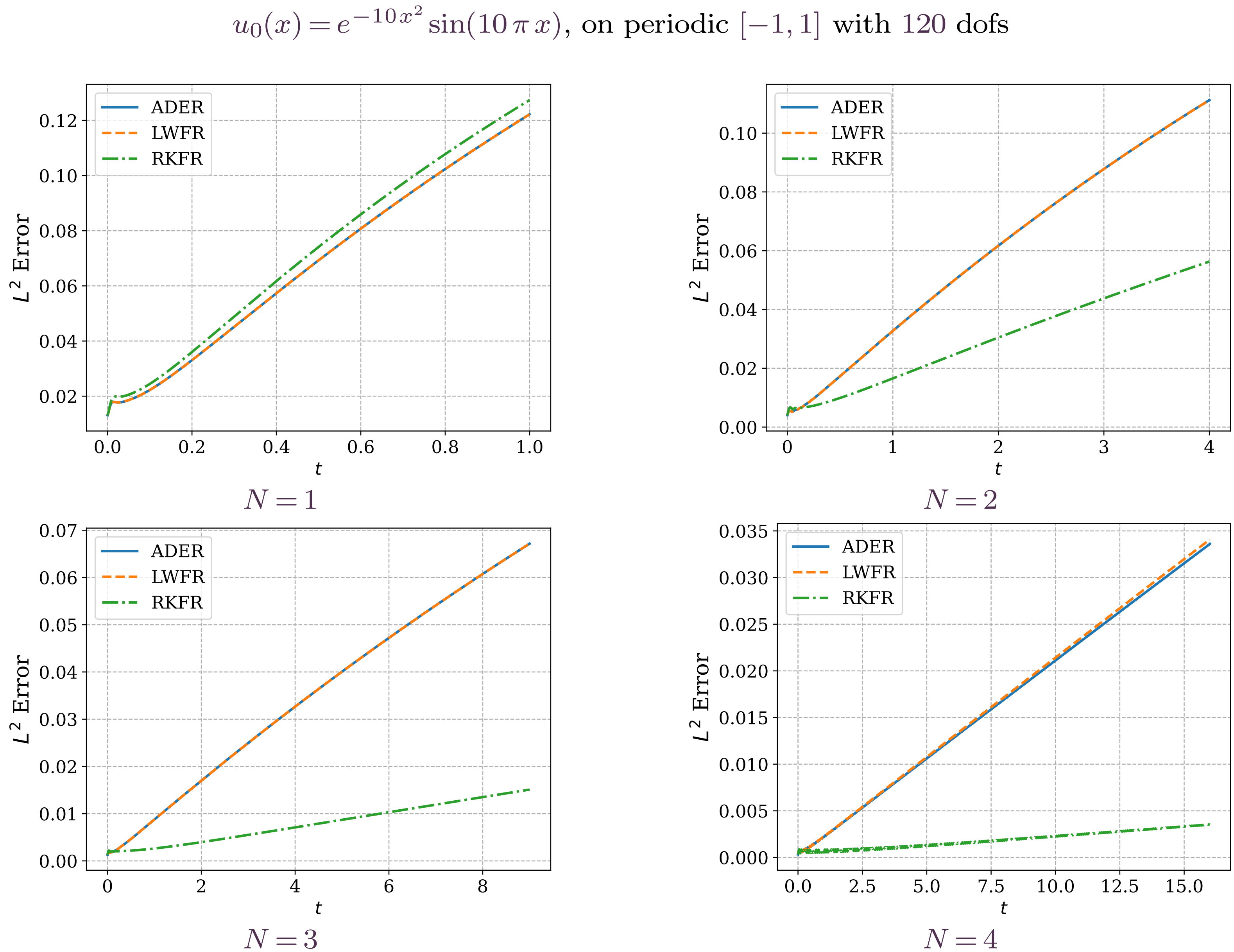
$$\begin{aligned} \tilde{u}_t + \tilde{u}_x &= 0, \qquad x \in [x_{e-1/2}, x_{e+1/2}], t \in (t_n, t_{n+1}], \\ \tilde{u}(x,t^n) &= u_e^n(x), \qquad t = t_n. \end{aligned}$$

Since the predictor equation has a unique solution of degree N [10, 5], the specified $\tilde{u}_h$ must be the predictor solution.

$$\begin{aligned} \tilde{u}_h(x,t) &= \tilde{u}_h(x,t^n) + (t-t^n)\frac{\partial}{\partial t}\tilde{u}_h(x,t^n) + \ldots + \frac{(t-t^n)^N}{N!}\frac{\partial^N}{\partial t^N}\tilde{u}_h(x,t^n) \\ &= u^n(x) - (t-t^n)\frac{\partial}{\partial x}u^n(x) + \ldots + (-1)^N\frac{(t-t^n)^N}{N!}\frac{\partial^N}{\partial x^N}u^n(x) \\ \Rightarrow \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \tilde{u}_h(x,t)\,dt &= u^n(x) - \frac{\Delta t}{2}\frac{\partial}{\partial x}u^n(x) + \ldots + (-1)^N\frac{\Delta t^N}{(N+1)!}\frac{\partial^N}{\partial x^N}u^n dt, \\ \Rightarrow \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \tilde{u}_h(x,t)\,dt &= U_h^n(x) \end{aligned}$$

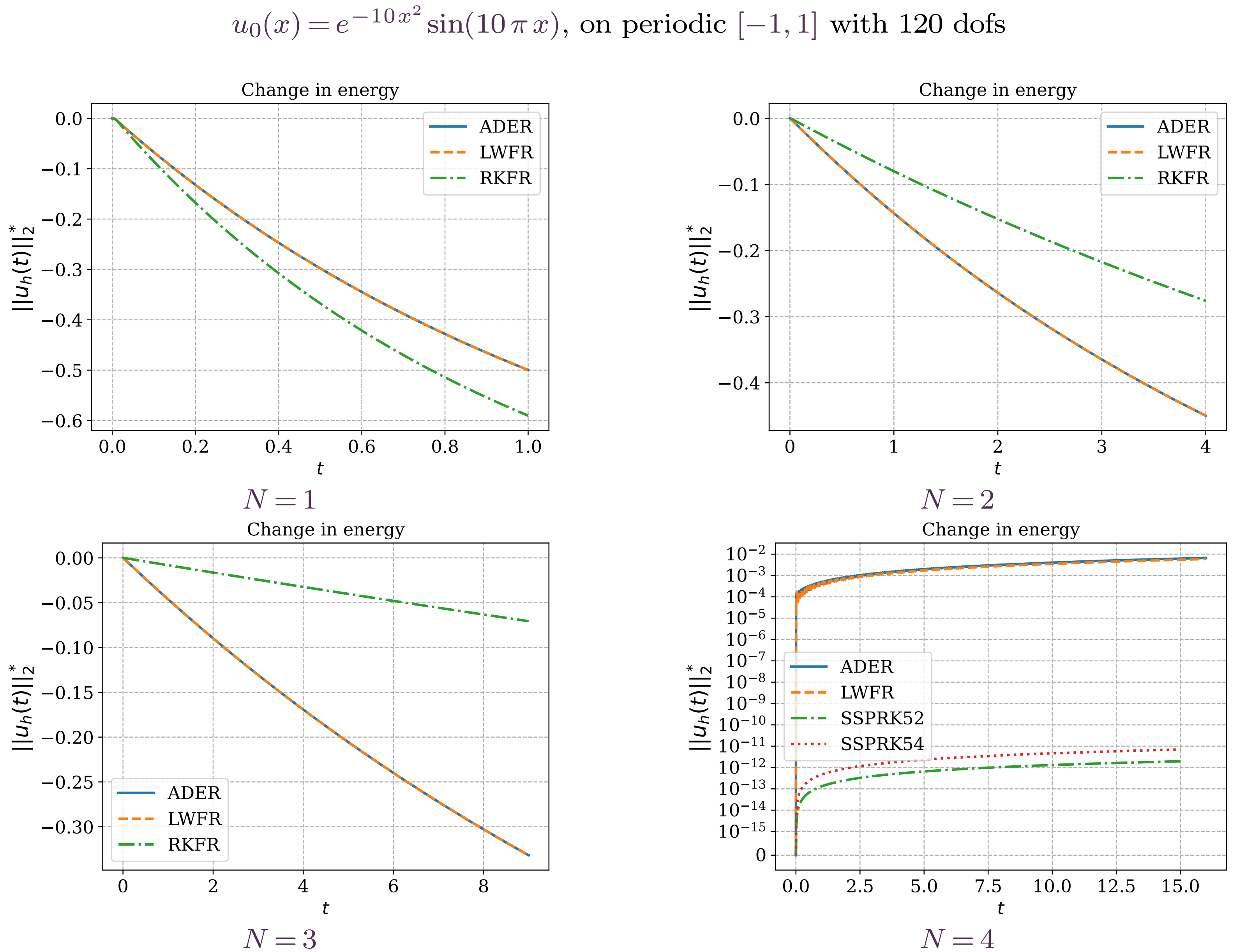
Thus, the ADER update (9) and the LWFR-D2 update (10) are the same. $\square$

Equivalence of ADER-FR and LWFR for linear case



$u_0(x) = e^{-10x^2}\sin(10\pi x)$, on periodic $[-1, 1]$ with 120 dofs
ADER
LWFR
RKFR
L^2 Error
t
$N = 1$
$N = 2$
$N = 3$
$N = 4$

Equivalence of ADER-FR and LWFR for linear case



$u_0(x) = e^{-10x^2}\sin(10\pi x)$, on periodic $[-1, 1]$ with 120 dofs
Change in energy
$||u_h(t)||_2^*$
t
ADER
LWFR
RKFR
$N = 1$
Change in energy
$||u_h(t)||_2^*$
t
ADER
LWFR
RKFR
$N = 2$
Change in energy
$||u_h(t)||_2^*$
t
ADER
LWFR
RKFR
$N = 3$
Change in energy
$||u_h(t)||_2^*$
t
ADER
LWFR
SSPRK52
SSPRK54
$N = 4$

Theorem 11. *If the predictor solution of ADER satisfies*

$$\|U^n\|_{C^{N+1}} \leq C$$

where C *is a constant independent of* $n, \Delta x, \Delta t$, *then the ADER and LW solution will satisfy*

$$\|u_{\text{ADER}}^{n+1} - u_{\text{LW}}^{n+1}\|_\infty = O(\Delta t^{N+1}).$$

Idea of proof. The weak formulation will give us

$$\tilde{u}_t + (\tilde{f}_h)_x = 0, \tag{11}$$

for all (x_r, t_s) where $s > 0$, i.e., $t_s > t^n$. Then, we can extrapolate to $t = t^n$ as

$$\tilde{u}_t + (\tilde{f}_h)_x = O(\Delta t^N) \qquad \text{at} \quad t = t^n,$$

so that we will have

$$\tilde{u}_h(x, t^n) \;=\; \tilde{u}_h(x, t^n) + \Delta t\, (\tilde{f}_h)_x + \ldots + \frac{\Delta t^N}{N!} \frac{\partial^{N-1}}{\partial t^{N-1}} (\tilde{f}_h)_x + O(\Delta t^{N+1}).$$

- **Modularity** - new conservation law can be added without modifying base code. User need only supply physical flux, numerical flux and wave speed estimates.
- **Portability** - Dependencies are handled by Julia's package manager
- **Parallelization** - Shared-memory via multithreading
- **Efficiency** - noticeably faster than some C++ implementations
- **Visualization** - Postprocessing to `vtr` format

Type instabilities

```julia
container = Dict( "u" => u, ... )
...
u = container["u"]
for cell in eachelement(grid)
    ! heavy computation with u
end
```

Bad version

```julia
container = (; u, ...)
...
u = container.u
for cell in eachelement(grid)
    ! heavy computation with u
end
```

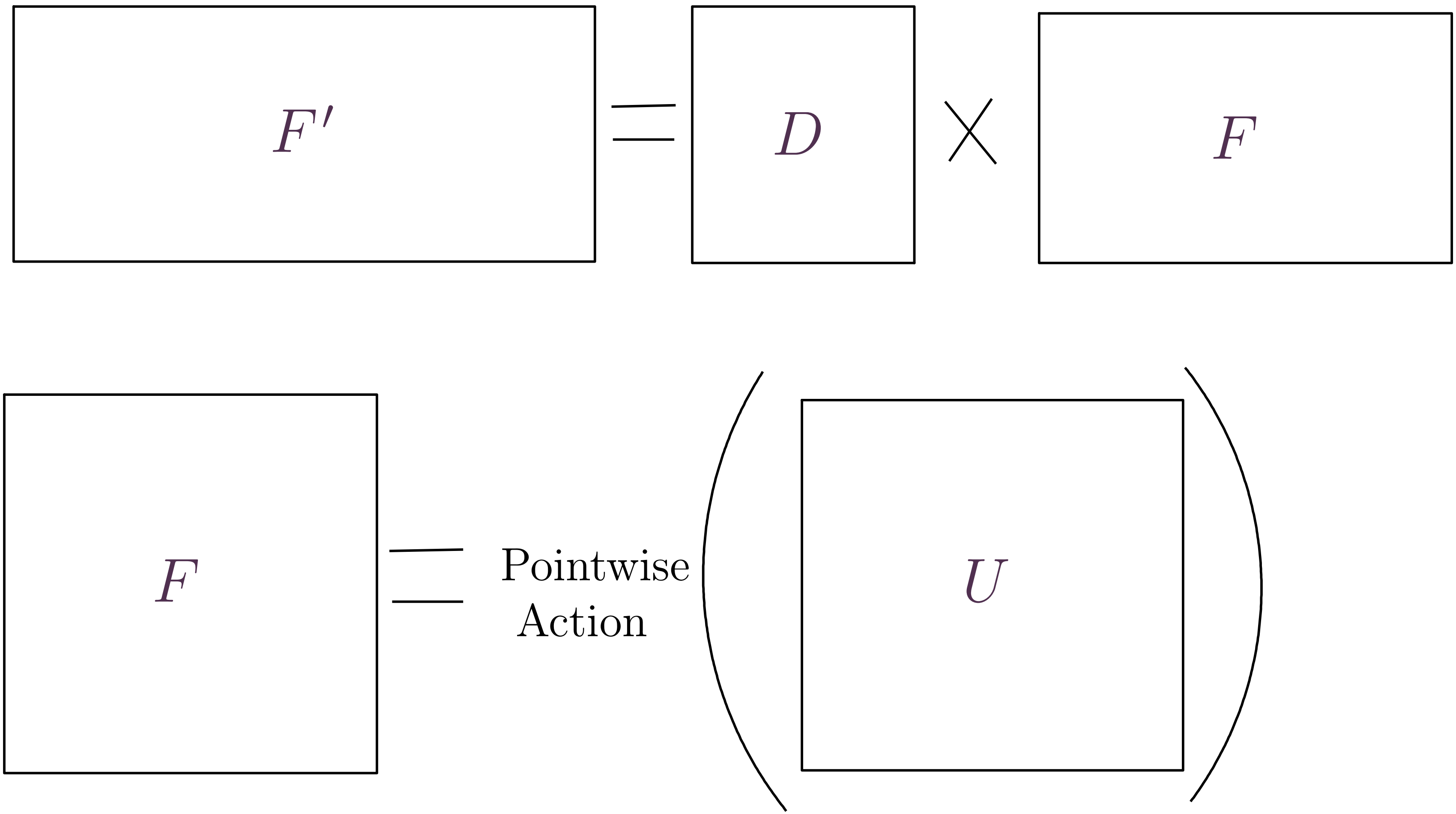
Good version

Tool to find type instabilities - `ProfileView.jl`

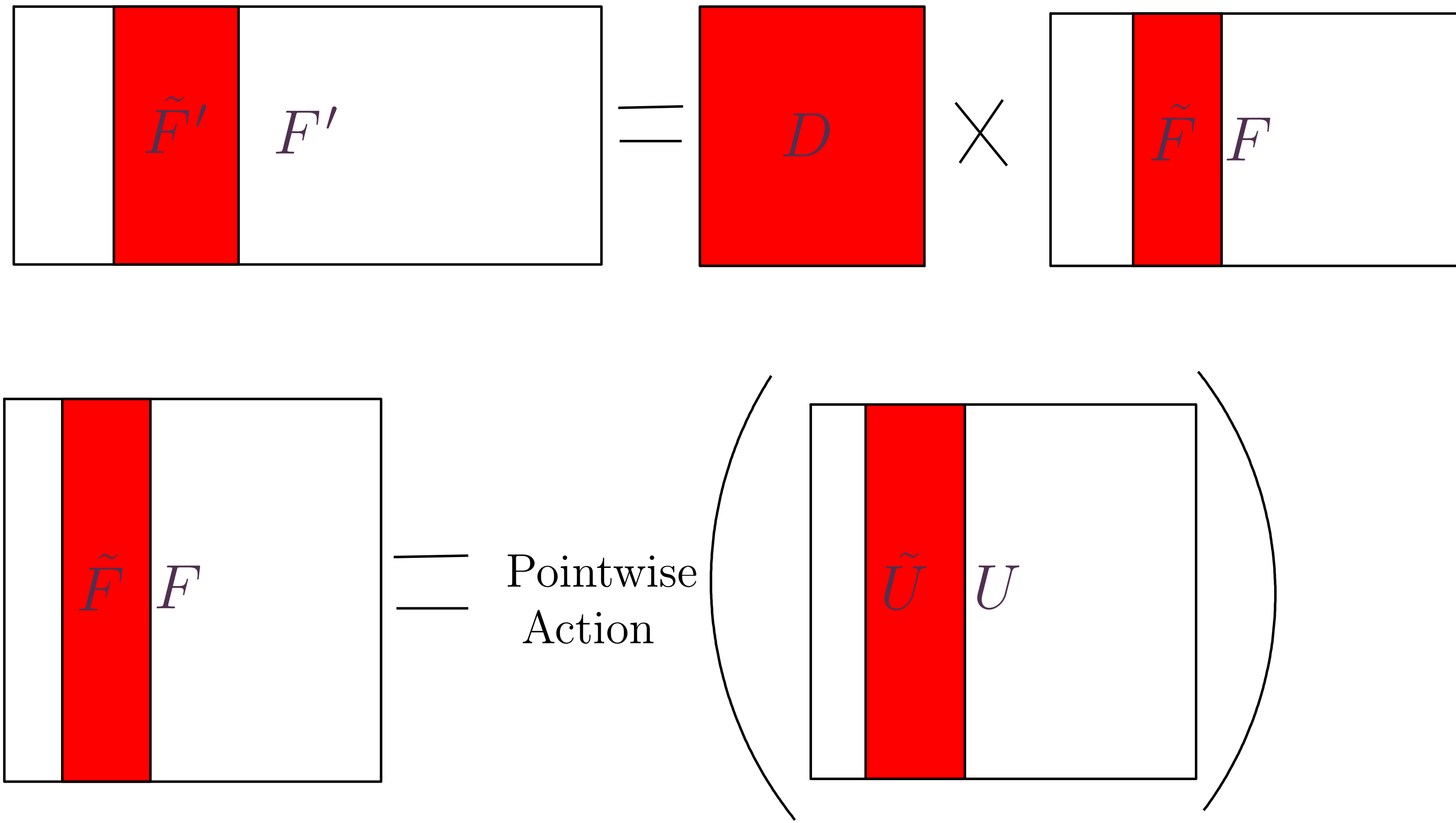
Tool to measure allocations - `BenchmarkTools.jl`, `TimerOutputs.jl`

Fixing the problem - JuliaLang - forum, Zulip, Slack.

Cache blocking (`Trixi.jl` [13], Akkurt Et Al [1])



Cache blocking (`Trixi.jl` [13], Akkurt Et Al [1])



Cache blocking (Trixi.jl [13], Akkurt Et Al [1])

```
for cell in eachelement(grid) ! Cell loop
  for i in eachnode(basis)    ! DoF loop
    f[:,i,cell] = flux(u[:,i,cell])
  end
  BLAS.mul(D, f, res)
end
```

Bad version

```
for cell in eachelement(grid) ! Cell loop
  for i in eachnode(basis)    ! DoF loop
    u_node = get_node_vars(eq, u, i, cell)
    f_node = flux(u_node)
    for ix in eachnode(basis)
      ! res[:,ix,i,cell] += D[ix,i] * f_node
      multiply_add_to_node_vars(eq, D[ix,i],
                                f_node, res,
                                iix, cell)

    end
  end
end
```

Good version

Low order residual : MUSCL-Hancock

Solution points and subcells

Integrate conservation law $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_{\boldsymbol{x}} = \boldsymbol{0}$ over $I_j^e \times [t^n, t^{n+1}]$

$$\Delta x_e\, w_j\, (\boldsymbol{u}_j^{n+1} - \boldsymbol{u}_j^n) + \int_{t^n}^{t^{n+1}} (\boldsymbol{f}_{j+\frac{1}{2}} - \boldsymbol{f}_{j-\frac{1}{2}})\, dt = 0.$$

$$\text{Midpoint rule: } \boldsymbol{u}_j^{n+1} = \boldsymbol{u}_j^n - \frac{\Delta t}{\Delta x_e w_j} \left(\boldsymbol{f}_{j+1/2}^{n+1/2} - \boldsymbol{f}_{j-1/2}^{n+1/2} \right)$$
$$\boldsymbol{f}_{j+1/2}^{n+1/2} = \boldsymbol{f}\left(\boldsymbol{u}_{j+1/2-}^{n+1/2}, \boldsymbol{u}_{j+1/2+}^{n+1/2} \right)$$

$$\boldsymbol{u}_{j-\frac{1}{2}+}^{n+1/2} = \boldsymbol{u}_{j-\frac{1}{2}+}^{n} - \frac{\Delta t}{2} \frac{f(\boldsymbol{u}_{j+\frac{1}{2}+}) - f(\boldsymbol{u}_{j-\frac{1}{2}-})}{x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}}, \quad \boldsymbol{u}_{j+\frac{1}{2}-}^{n+1/2} = \boldsymbol{u}_{j+\frac{1}{2}-}^{n} - \frac{\Delta t}{2} \frac{f(\boldsymbol{u}_{j+\frac{1}{2}-}) - f(\boldsymbol{u}_{j-\frac{1}{2}+})}{x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}}.$$

$$\boldsymbol{u}_{j-\frac{1}{2}+} = \boldsymbol{u}_j(x_{j-\frac{1}{2}}), \quad \boldsymbol{u}_{j+\frac{1}{2}-} = \boldsymbol{u}_j(x_{j+\frac{1}{2}})$$
$$\boldsymbol{u}_j(x) = \boldsymbol{u}_j^n + \boldsymbol{\sigma}_j\, (x - x_j)$$

$$\boldsymbol{\sigma}_j = \operatorname{minmod}\left(\beta_e \frac{\boldsymbol{u}_{j+1} - \boldsymbol{u}_j}{x_{j+1} - x_j}, D_{\text{cent}}(\boldsymbol{u})_j, \beta_e \frac{\boldsymbol{u}_j^n - \boldsymbol{u}_{j-1}^n}{x_j - x_{j-1}} \right)$$
$$\beta_e = 2 - \alpha_e$$

Flux Reconstruction for the transformed PDE

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{u}_{e,\boldsymbol{p}}^{\delta}}{\mathrm{d}t} &+ \frac{1}{J}\nabla_{\boldsymbol{\xi}}^{N}\cdot \tilde{\boldsymbol{f}}_e(\boldsymbol{\xi}_{\boldsymbol{p}}) \\
&+ \frac{1}{J}\sum_{i=1}^{3}\left((\tilde{\boldsymbol{f}}_e\cdot\boldsymbol{n}_{s,i})^{*} - \tilde{\boldsymbol{f}}_e\cdot\boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^{R})\, g_R'(\xi_{p_i}) \\
&+ \frac{1}{J}\sum_{i=1}^{3}\left((\tilde{\boldsymbol{f}}_e\cdot\boldsymbol{n}_{s,i})^{*} - \tilde{\boldsymbol{f}}_e\cdot\boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^{L})\, g_L'(\xi_{p_i}) = \boldsymbol{0},
\end{aligned}
$$

where

$\boldsymbol{p} = (p_1, p_2, p_3)$ for $1 \le p_i \le N+1$

$\boldsymbol{\xi}_{\boldsymbol{p}} = \boldsymbol{\xi}_{p_1p_2p_3} = (\xi_{p_1}, \xi_{p_2}, \xi_{p_3})$ is the solution point of collocation, $\boldsymbol{e}$ is the element index

$\nabla_{\boldsymbol{\xi}}^{N}$ is the degree N gradient

$(\tilde{\boldsymbol{f}}_e\cdot\boldsymbol{n})^{*}$ is the numerical flux along normal $\boldsymbol{n}$

$\boldsymbol{n}_{s,i}$ is the normal vector in reference cell along i^{th} direction

$\boldsymbol{\xi}_i^{L/R}$ is the point that agrees with $\boldsymbol{\xi}$ at j, k and equals 0 or 1 for L or R at i.