Thesis Monitoring Committee (TMC) meeting

Lax-Wendroff flux reconstruction: unstructured grids and diffusive problems

February 3, 2023

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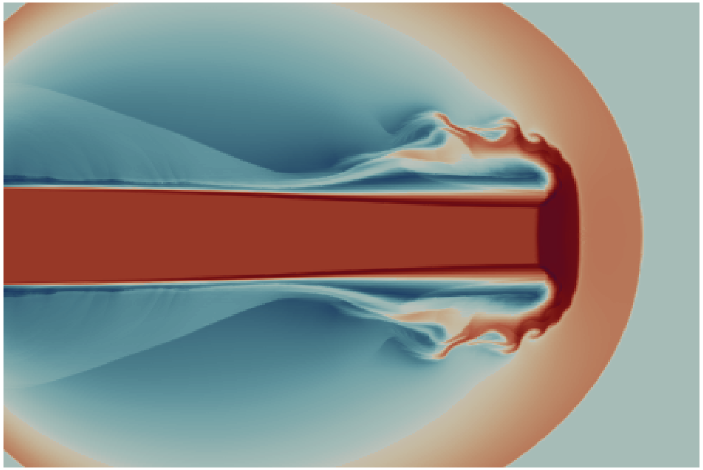
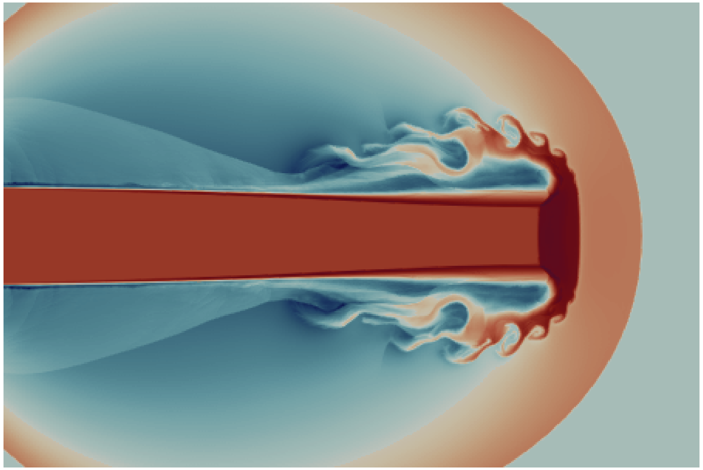
SSFR.jl

High order, admissibility preserving solver with Lax-Wendroff discretization for hyperbolic problems on cartesian grids



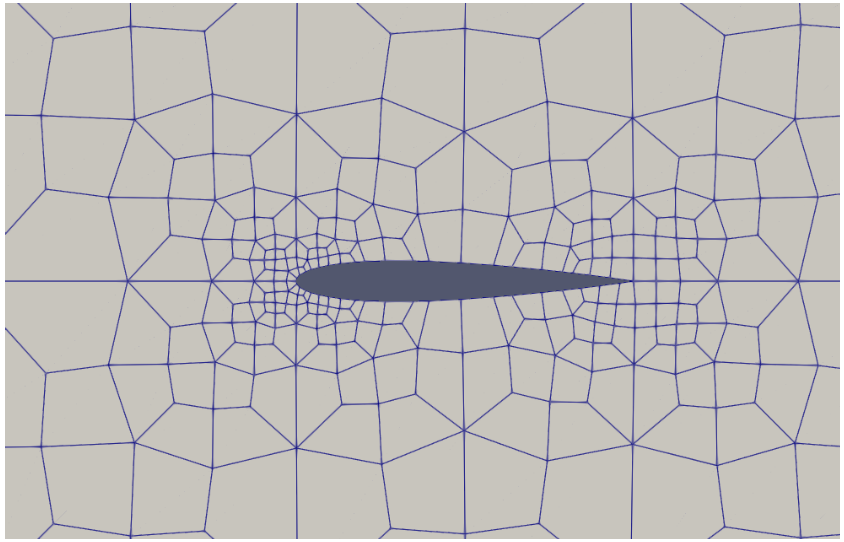
Trixi.jl



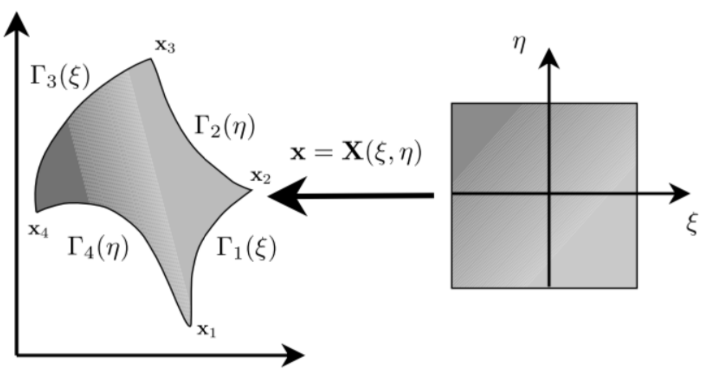
SSFR.jl

```
julia> import Pkg
julia> Pkg.add(url = "https://github.com/path/to/pkg")
julia> using SSFR
julia> include("run_m2000.jl")
```

Unstructured, curvilinear grids



NACA0012 air foil

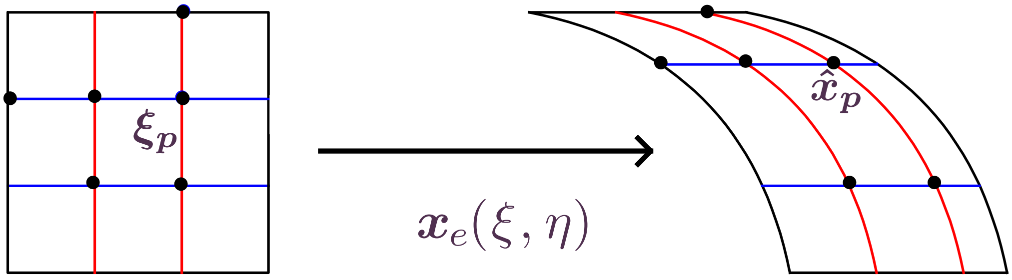


$$\Omega = \bigcup_e \Omega_e$$

$$(\xi, \eta) \mapsto \boldsymbol{x}^e(\xi, \eta)$$

$\boldsymbol{x}^e$ is a **degree** $\boldsymbol{N}$ polynomial in ξ, η

Element definition



$$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{\boldsymbol{p}} \hat{\boldsymbol{x}}_{\boldsymbol{p}} \boldsymbol{\ell}_{\boldsymbol{p}}(\boldsymbol{\xi}),$$

$$\boldsymbol{\ell}_{\boldsymbol{p}} = \ell_{p_1} \ell_{p_2} \ell_{p_3}$$

Conservation Law

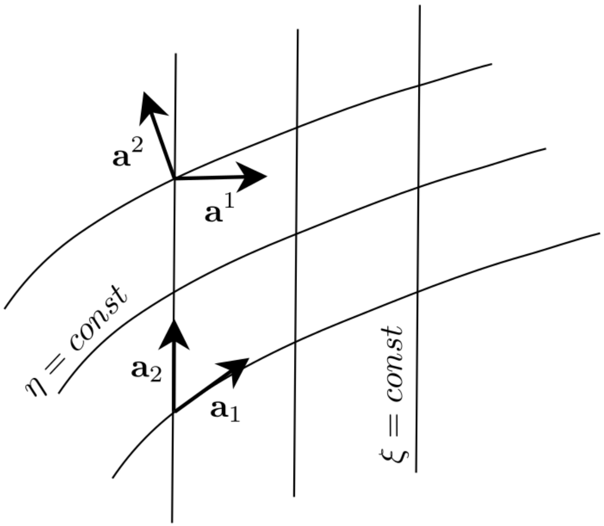
$$\boldsymbol{u}_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f} = 0$$

$$\boldsymbol{u} \in \mathcal{U} \subset \mathbb{R}^p, \qquad \boldsymbol{f} \in \mathbb{R}^d \times \mathbb{R}^p$$

$$\boldsymbol{f} = \left(f_n^i\right)_{\substack{n=1,\dots,d \\ i=1,\dots,p}}$$

Transformation of conservation law

$$\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$$
$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) = (\xi, \eta, \zeta)$$



$$\boldsymbol{a}_i = \frac{\partial}{\partial \xi_i}\boldsymbol{x}$$

$$\boldsymbol{a}^i = \nabla_{\boldsymbol{x}} \xi^i = J^{-1} (\boldsymbol{a}_j \times \boldsymbol{a}_k)$$
i, j, k cyclic

$$J = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)$$

Covariant and contravariant coordinate vectors in relation to the coordinate lines

$$\boldsymbol{u}_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f} = 0 \longrightarrow \boldsymbol{u}_t + \frac{1}{J} \nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{f}} = 0, \qquad \boldsymbol{\xi} \in [0,1]^3$$

where

$$\tilde{\boldsymbol{f}}^i = J \boldsymbol{a}^i \cdot \boldsymbol{f} = \sum_{n=1}^{3} J a_n^i \boldsymbol{f}_n.$$

Free stream

Take $\boldsymbol{u}^n = \underline{\boldsymbol{c}}$ and $\boldsymbol{f}(\underline{\boldsymbol{c}}) = \boldsymbol{c}$, we must have

$$\nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{f}} = -J\,\boldsymbol{u}_t = \boldsymbol{0}.$$

$$\Leftrightarrow \quad \sum_{i=1}^{3} \partial_{\xi_i} \sum_{n=1}^{3} J\,a_n^i\,\boldsymbol{c}_n = \boldsymbol{0} \qquad \forall \boldsymbol{c}$$

$$\Leftrightarrow \quad \sum_{i=1}^{3} \partial_{\xi_i}(J\,\boldsymbol{a}^i) = \boldsymbol{0}$$

$$\text{DG:} \qquad \sum_{i=1}^{3} D_{\xi_i}^{N}\,(J\,a^i) = \boldsymbol{0}$$

$$\sum_{i=1}^{3} D_{\xi_i}^{N}(\partial_{\xi_j}\boldsymbol{x} \times \partial_{\xi_k}\boldsymbol{x}) = \boldsymbol{0}$$

Flux Reconstruction/Discontinuous Galerkin review

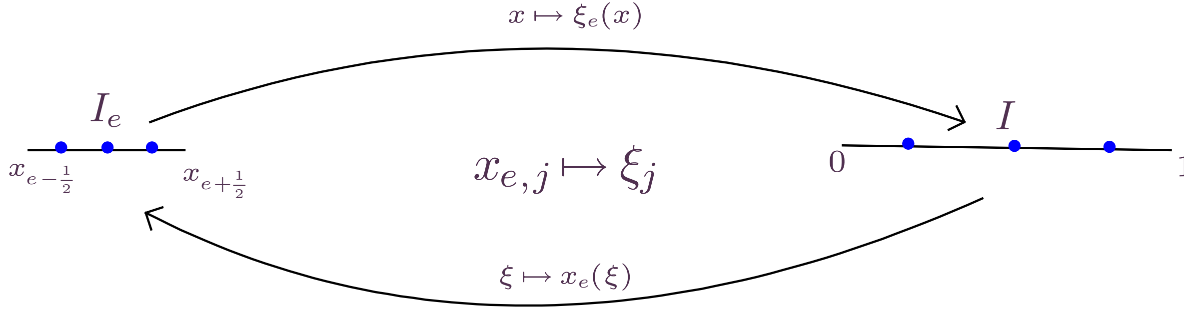
$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$$

Degree N approximation

$$\Omega = \bigcup_e I_e$$

$$u_h = u_h(x)$$

$$r = r(x)$$



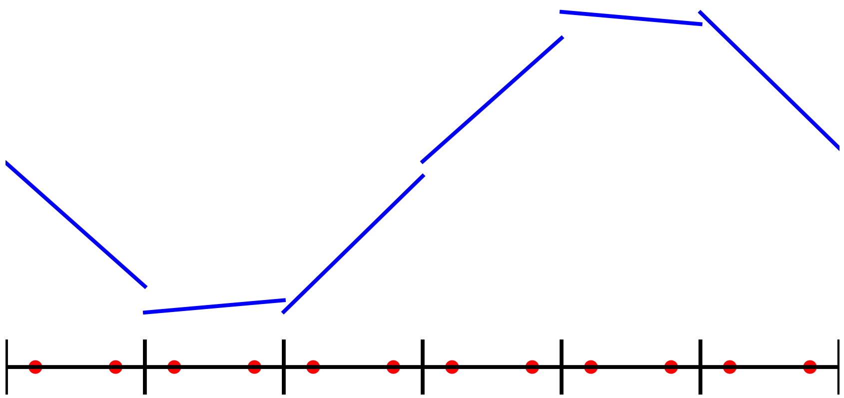
$$\hat{r} = \hat{r}(\xi)$$

$$\frac{\partial r}{\partial x} = \frac{1}{\Delta x_e} \frac{\partial \hat{r}}{\partial \xi}$$

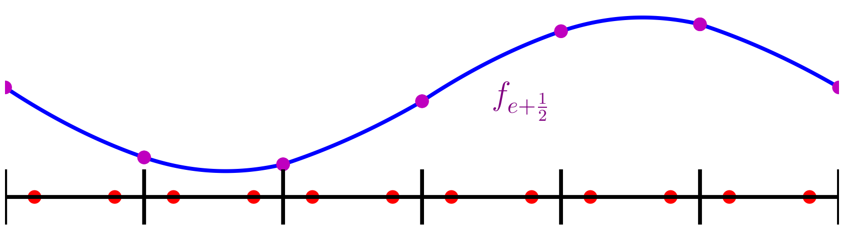
Flux Reconstruction/Discontinuous Galerkin review

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{e,i} + \frac{\partial f_h}{\partial x}(x_{e,i}) = 0, \quad 1 \leq i \leq N+1.$$

Degree N approximate solution u_h



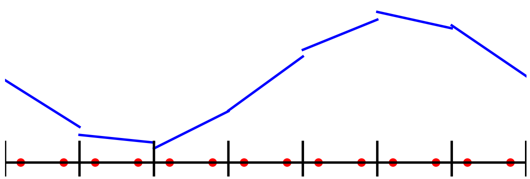
Degree $N+1$ Continuous Flux Approximation f_h



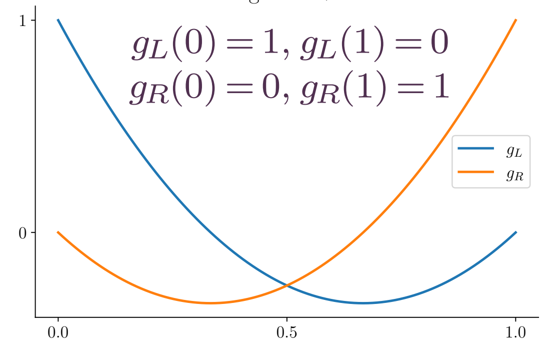
Flux Reconstruction/Discontinuous Galerkin review

Discontinuous Flux Approximation f_h^δ
Degree N

$$\hat{f}_h^\delta(\xi_i) = f(\hat{u}_h(\xi_i))$$

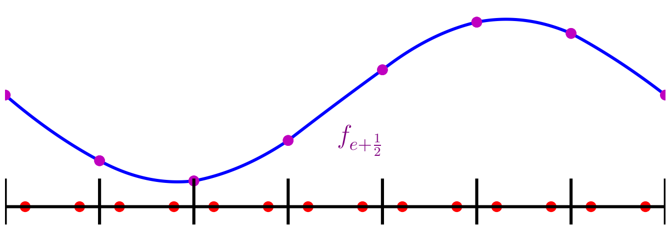


Corrector functions
Degree $N+1$



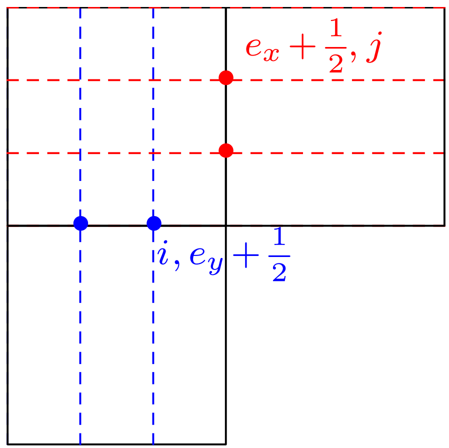
$$\hat{f}_h = \hat{f}_h^\delta + \left(f_{e-\frac{1}{2}} - \hat{f}_h^\delta(0)\right) g_L + \left(f_{e+\frac{1}{2}} - \hat{f}_h^\delta(1)\right) g_R$$

Continuous Flux Approximation f_h
Degree $N+1$



$$
\begin{aligned}
J\,\frac{\mathrm{d}\boldsymbol{u}_{e,i}}{\mathrm{d}t} &+ \partial_\xi\, f_h^\delta(\xi_i) \\
&+ \left(f_{e-\frac{1}{2}} - f_h^\delta(0)\right) g_L'(\xi_i) + \left(f_{e+\frac{1}{2}} - f_h^\delta(1)\right) g_R'(\xi_i) = 0
\end{aligned}
$$

Flux Reconstruction/Discontinuous Galerkin review



$$\boldsymbol{f} = (f, g)$$

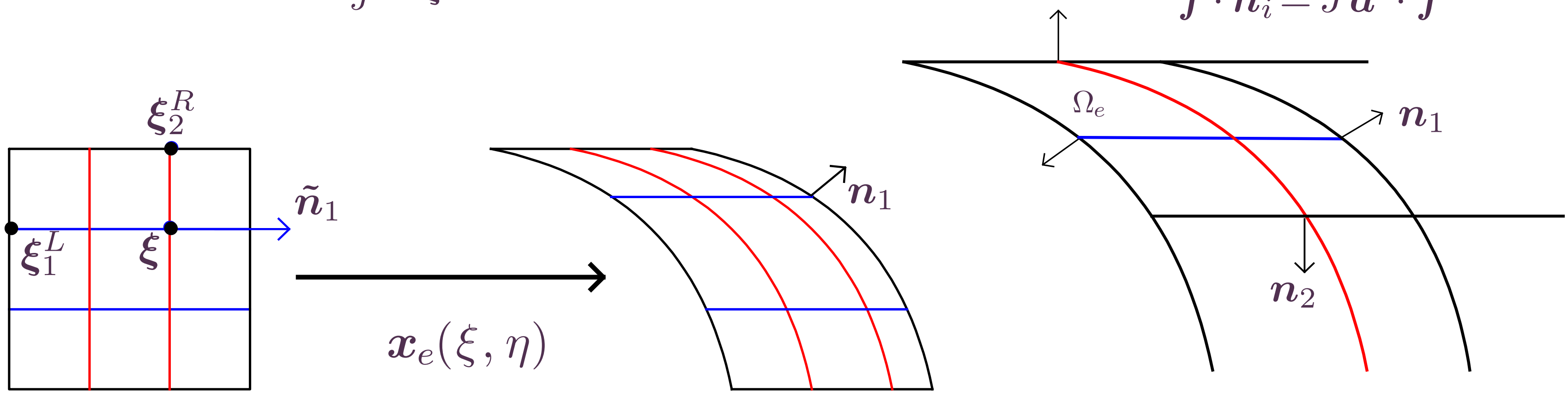
$$
\begin{aligned}
J \frac{\mathrm{d}\boldsymbol{u}_{\boldsymbol{e},i,j}}{\mathrm{d}t} &+ \partial_\xi f_{ij} + \partial_\eta g_{ij} \\
&+ \left(f_{e_x - \frac{1}{2}, j} - f_h^\delta(0, \eta_j)\right) g_L'(\xi_i) + \left(g_{i, e_y - \frac{1}{2}} - g_h^\delta(\xi_i, 0)\right) g_L'(\eta_j) \\
&+ \left(f_{e_x + \frac{1}{2}, j} - f_h^\delta(1, \eta_j)\right) g_R'(\xi_i) + \left(g_{i, e_y + \frac{1}{2}} - g_h^\delta(\xi_i, 1)\right) g_R'(\eta_i) \;=\; 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{u}_{\boldsymbol{e},i,j}}{\mathrm{d}t} &+ \frac{1}{J} \nabla_{\boldsymbol{\xi}}^N \cdot \boldsymbol{f}_{ij} \\
&+ \frac{1}{J} \left((\boldsymbol{f}_e \cdot \boldsymbol{n}_1)^* - \boldsymbol{f}_e^\delta \cdot \boldsymbol{n}_1 \right)(0, \eta_j) g_L'(\xi_i) + \left((\boldsymbol{f}_e \cdot \boldsymbol{n}_2)^* - \boldsymbol{f}_e^\delta \cdot \boldsymbol{n}_2 \right)(\xi_i, 0)\, g_L'(\eta_j) \\
&+ \frac{1}{J} \left((\boldsymbol{f}_e \cdot \boldsymbol{n}_1)^* - \boldsymbol{f}_e^\delta \cdot \boldsymbol{n}_1 \right)(1, \eta_j)\, g_R'(\xi_i) + \left((\boldsymbol{f}_e \cdot \boldsymbol{n}_2)^* - \boldsymbol{f}_e^\delta \cdot \boldsymbol{n}_2 \right)(\xi_i, 1)\, g_R'(\eta_i) \;=\; 0
\end{aligned}
$$

Flux Reconstruction for transformed conservation law

$$\boldsymbol{u}_t + \frac{1}{J}\,\nabla_{\boldsymbol{\xi}}^N \cdot \tilde{\boldsymbol{f}}_e = \boldsymbol{0}$$

$$\boldsymbol{n}_i = J\,\nabla_{\boldsymbol{x}}\,\xi_i = \boldsymbol{a}^i$$
$$\tilde{\boldsymbol{f}} \cdot \tilde{\boldsymbol{n}}_i = J\,\boldsymbol{a}^i \cdot \boldsymbol{f}$$



$$\frac{\mathrm{d}\boldsymbol{u}_{e,\boldsymbol{p}}^{\delta}}{\mathrm{d}t} + \frac{1}{J}\,\nabla_{\boldsymbol{\xi}}^N \cdot \tilde{\boldsymbol{f}}_e^{\delta}(\boldsymbol{\xi})$$
$$+\frac{1}{J}\sum_{d=1}^{3}\left((\tilde{\boldsymbol{f}}_e \cdot \tilde{\boldsymbol{n}}_d)^* - \tilde{\boldsymbol{f}}_e^{\delta} \cdot \tilde{\boldsymbol{n}}_d\right)(\boldsymbol{\xi}_d^R)\; g_R'(\xi_{p_d})$$
$$+\frac{1}{J}\sum_{d=1}^{3}\left((\tilde{\boldsymbol{f}}_e \cdot \tilde{\boldsymbol{n}}_d)^* - \tilde{\boldsymbol{f}}_e^{\delta} \cdot \tilde{\boldsymbol{n}}_d\right)(\boldsymbol{\xi}_d^L)\; g_L'(\xi_{p_d}) = \boldsymbol{0},$$

$$\boldsymbol{p} = (p_1, p_2, p_3),\, p_i = 1, \ldots, N+1$$
$$\boldsymbol{\xi} = \boldsymbol{\xi}_p = (\xi_{p_1}, \xi_{p_2}, \xi_{p_3})$$

$$(\boldsymbol{\xi}_i^S)_k = \begin{cases} \xi_k, & k \neq i \\ 0, & k = i,\, S = L \\ 1, & k = i,\, S = R \end{cases}$$

Lax-Wendroff Flux Reconstruction

$$\boldsymbol{u}_t + \frac{1}{J}\nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{f}} = 0,$$

$$\tilde{\boldsymbol{f}}^i \;=\; J\boldsymbol{a}^i \cdot \boldsymbol{f} = \sum_{n=1}^{3} J\, a_n^i\, \boldsymbol{f}_n.$$

Taylor's expansion

$$\boldsymbol{u}^{n+1}(\boldsymbol{\xi}) = \boldsymbol{u}^n(\boldsymbol{\xi}) - \frac{1}{J}\Delta t\, \nabla_{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{F}}$$

$$\tilde{\boldsymbol{F}} = \sum_{k=0}^{N} \frac{\Delta t^k}{(k+1)!}\, \partial_t^k\, \tilde{\boldsymbol{f}}$$

Local time averaged flux

$$N = 1: \qquad \partial_t\, \tilde{\boldsymbol{f}}^\delta = \frac{\tilde{\boldsymbol{f}}(\boldsymbol{u} + \Delta t\, \boldsymbol{u}_t) - \tilde{\boldsymbol{f}}(\boldsymbol{u} - \Delta t\, \boldsymbol{u}_t)}{2\,\Delta t} = \boldsymbol{0}$$

$$\boldsymbol{u}_t = -\frac{1}{J}\nabla_{\boldsymbol{\xi}}^N \cdot \tilde{\boldsymbol{f}}^\delta$$

$$\tilde{F} = \tilde{\boldsymbol{f}} + \frac{\Delta t}{2}\, \tilde{\boldsymbol{f}}_t$$

Lax-Wendroff Flux Reconstruction

Corrected update

$$
\begin{aligned}
\boldsymbol{u}_{e,\boldsymbol{p}}^{n+1} - \boldsymbol{u}_{e,\boldsymbol{p}}^{n} &+ \frac{1}{J}\,\Delta t\,\nabla_{\boldsymbol{\xi}}\cdot\tilde{\boldsymbol{F}}_e^{\delta}(\boldsymbol{\xi}_{\boldsymbol{p}}) \\
&+ \frac{1}{J}\,\Delta t \sum_{i=1}^{3}\int_{\partial\Omega_{s,i}} \left((\tilde{\boldsymbol{F}}_e^{\delta}\cdot\boldsymbol{n}_{s,i})^{*} - \tilde{\boldsymbol{F}}_e^{\delta}\cdot\boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^{R})\, g_R'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} \\
&+ \frac{1}{J}\,\Delta t \sum_{i=1}^{3}\int_{\partial\Omega_{s,i}} \left((\tilde{\boldsymbol{F}}_e^{\delta}\cdot\boldsymbol{n}_{s,i})^{*} - \tilde{\boldsymbol{F}}_e^{\delta}\cdot\boldsymbol{n}_{s,i}\right)(\boldsymbol{\xi}_i^{L})\, g_L'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} = \boldsymbol{0}.
\end{aligned}
$$

Free stream preservation of Lax-Wendroff

Free stream conditions

$$\sum_{i=1}^{3} D_{\xi^i}^N (J\boldsymbol{a}^i) = \boldsymbol{0}$$

Assume $\boldsymbol{u}^n = \underline{\boldsymbol{c}}$ and $\boldsymbol{f}(\underline{\boldsymbol{c}}) = \boldsymbol{c}$

$$\boldsymbol{u_t} \;=\; -\frac{1}{J}\nabla_{\boldsymbol{\xi}}^N \cdot \tilde{\boldsymbol{f}}_e^{\delta} \;=\; -\sum_{n=1}^{3}\left(\sum_{i=1}^{3} D_{\xi_i}^N (J a_n^i)\right)\boldsymbol{c}_n \;=\; \boldsymbol{0}.$$

$$N = 1: \qquad \partial_t \tilde{\boldsymbol{f}}^{\delta} = \frac{\tilde{\boldsymbol{f}}(\boldsymbol{u} + \Delta t\, \boldsymbol{u}_t) - \tilde{\boldsymbol{f}}(\boldsymbol{u} - \Delta t\, \boldsymbol{u}_t)}{2\,\Delta t} = \boldsymbol{0}$$

$$\Rightarrow \quad \tilde{\boldsymbol{F}}^{\delta} = \tilde{\boldsymbol{f}}^{\delta} + \frac{\Delta t}{2}\,\partial_t\, \tilde{\boldsymbol{f}}^{\delta} = \tilde{\boldsymbol{f}}^{\delta},$$

where

$$\tilde{\boldsymbol{f}}_i^{\delta} = J\,\boldsymbol{a}^i \cdot \boldsymbol{c} = J\,\boldsymbol{a}_n^i \cdot \boldsymbol{c}_n.$$

Free stream preservation of Lax-Wendroff

$$
\begin{aligned}
&\boldsymbol{u}^{n+1} - \boldsymbol{u}^n + \frac{1}{J}\Delta t \left(\sum_{i=1}^{3} D_{\xi^i}^N (J\boldsymbol{a}^i) \right) \cdot \boldsymbol{c} \\
&\quad + \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c})(\boldsymbol{\xi}_i^R)\, g_R'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} \\
&\quad + \frac{1}{J}\Delta t \sum_{i=1}^{3} \int_{\partial\Omega_{s,i}} ((J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c})(\boldsymbol{\xi}_i^L)\, g_L'(\xi_{p_i})\, dS_{\boldsymbol{\xi}} = \boldsymbol{0}.
\end{aligned}
$$

Metric identities

$$
\sum_{i=1}^{3} D_{\xi^i}^N (J\boldsymbol{a}^i) = \boldsymbol{0}
$$

Conformality

$$
(J\boldsymbol{a}^i \cdot \boldsymbol{c})^* - J\boldsymbol{a}^i \cdot \boldsymbol{c} = \boldsymbol{0}
$$

Metric identities in practise (Kopriva [3])

$$J\boldsymbol{a}^1 = (y_\eta, -x_\eta), \qquad J\boldsymbol{a}^2 = (-y_\xi, x_\xi).$$

$$D_\xi^N J\boldsymbol{a}^1 + D_\eta^N J\boldsymbol{a}^2 = \boldsymbol{0}$$

2-D

$$\begin{aligned} D_\xi^N(y_\eta) - D_\eta^N(y_\xi) &= 0 \\ -D_\xi^N(x_\eta) + D_\eta^N(x_\xi) &= 0 \end{aligned}$$

Condition: Degree of mesh $\leq N$

3-D

$$J\boldsymbol{a}^1 = \boldsymbol{a}^2 \times \boldsymbol{a}^3, \quad J\boldsymbol{a}^2 = \boldsymbol{a}^3 \times \boldsymbol{a}^1, \quad J\boldsymbol{a}^3 = \boldsymbol{a}^1 \times \boldsymbol{a}^2$$

Condition: Degree of mesh $\leq N/2$.

Metric identities in practise (Kopriva 2006)

Conservative form of metric terms

$$J a_n^i = -\boldsymbol{e}_i \cdot \nabla_\xi \times (x_m \nabla_\xi x_l), \quad i = 1, 2, 3, \quad n = 1, 2, 3, \quad (n, m, l) \text{ cyclic.}$$

Compute metric terms with **conservative curl form**

$$J a_n^i = -\boldsymbol{e}_i \cdot \nabla_\xi^N \times (x_l \nabla_\xi x_m)$$

to always get

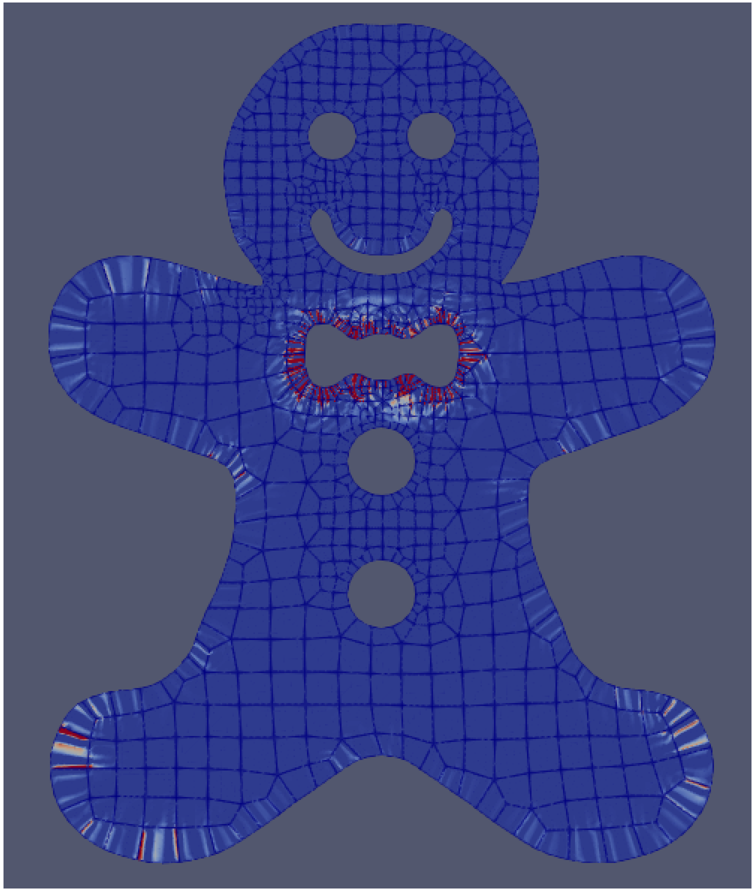
$$\sum_{n=1}^{3} D_{\xi_i}^N (J \boldsymbol{a}_n^i) = \boldsymbol{0}.$$

In 2-D, this can be simplified as

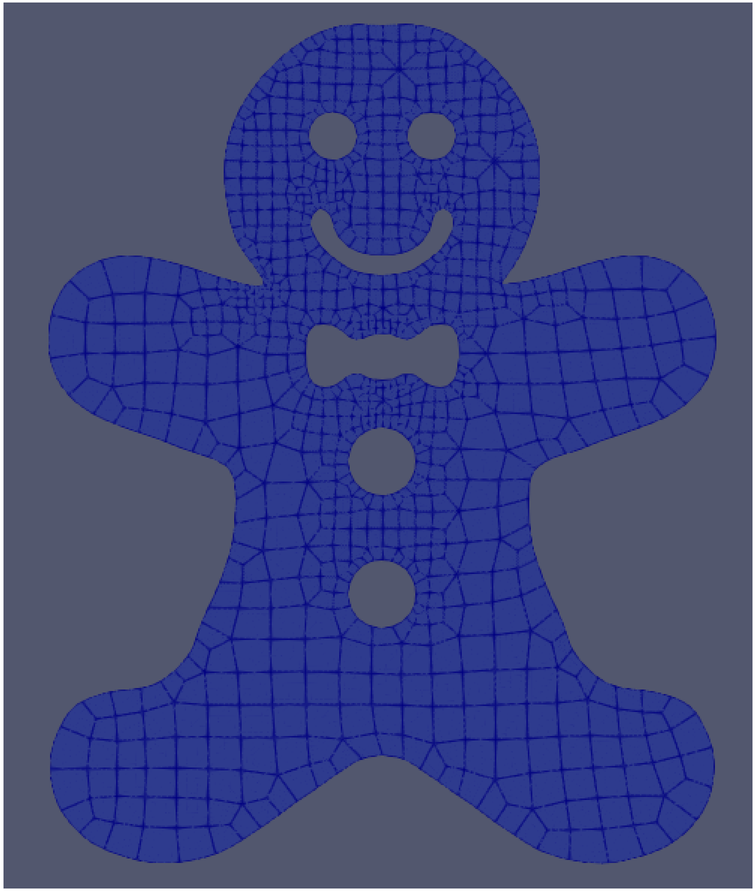
$$J \boldsymbol{a}^1 = (D_\eta^N y, -D_\eta^N x), \qquad J \boldsymbol{a}^2 = (-D_\xi^N y, D_\xi^N x).$$

Lax-Wendroff free stream condition verified

Mesh degree $= 6$



Solution polynomial degree 5



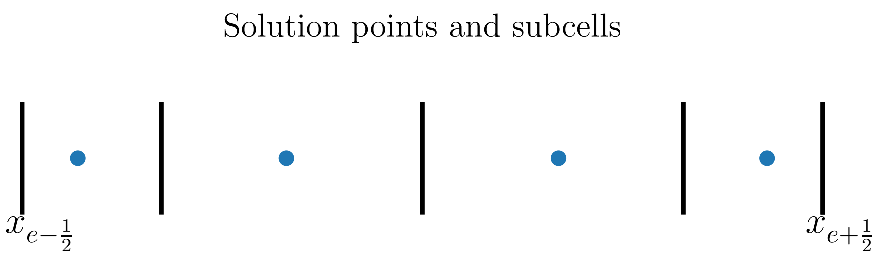
Solution polynomial degree 6

Blending limiter [2]

High order LWFR update

$$\boldsymbol{u}_e^{H,n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e}\boldsymbol{R}_e^H.$$

Lower order subcell update



$$\boldsymbol{u}_e^{L,n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e}\boldsymbol{R}_e^L.$$

Blend residual with $\alpha_e \in [0,1]$

$$\boldsymbol{R}_e = (1-\alpha_e)\,\boldsymbol{R}_e^H + \alpha_e\,\boldsymbol{R}_e^L,$$

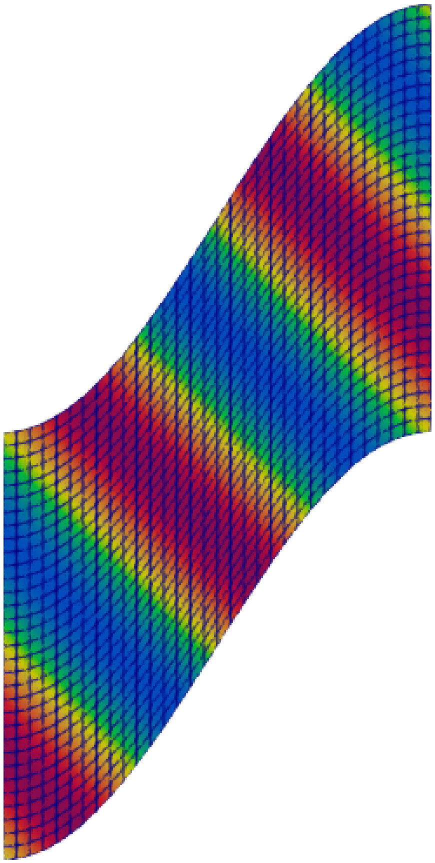
Limited update

$$\boldsymbol{u}_e^{n+1} = \boldsymbol{u}_e^n - \frac{\Delta t}{\Delta x_e}\boldsymbol{R}_e.$$

Numerical Results

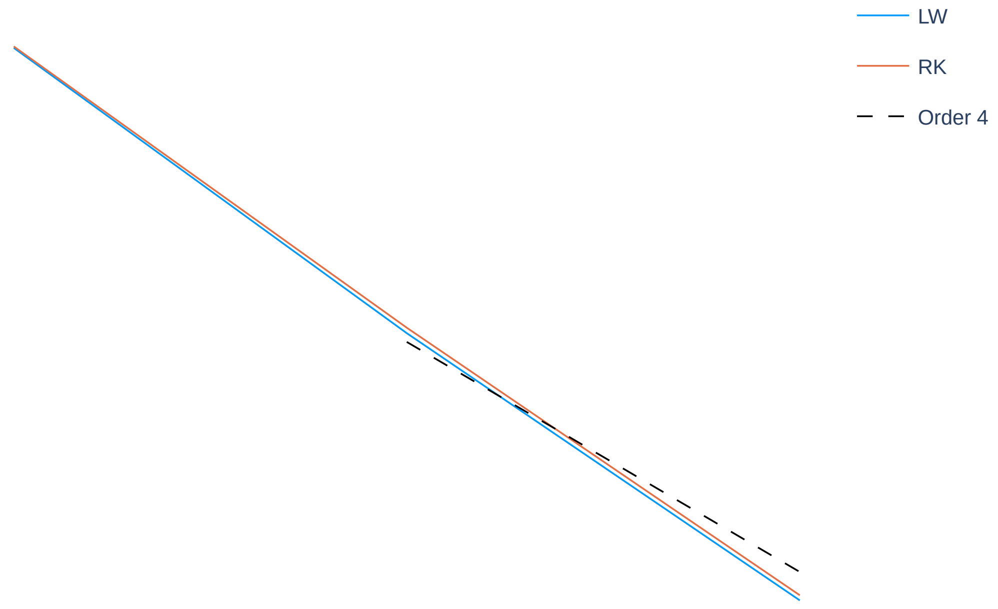
2-D Euler's equations

Density wave convergence test



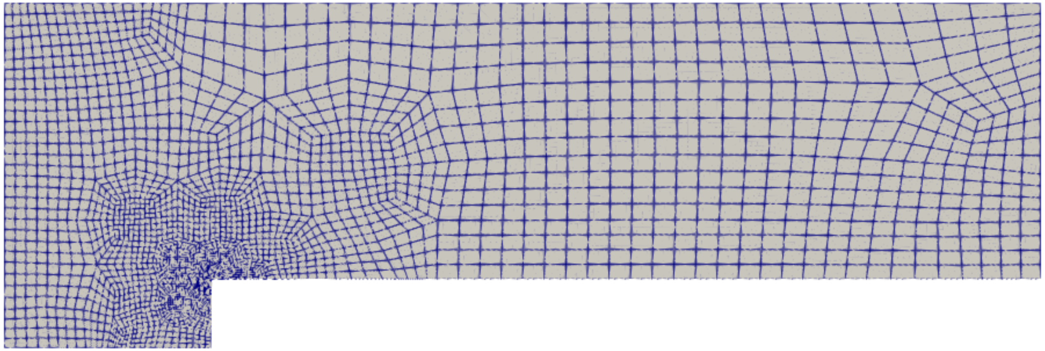
Sine wave manufactured as a solution for **Euler's equations** with **source term**

Density wave convergence test



Forward step

Mach 3 flow with solid walls at bottom (step) and top

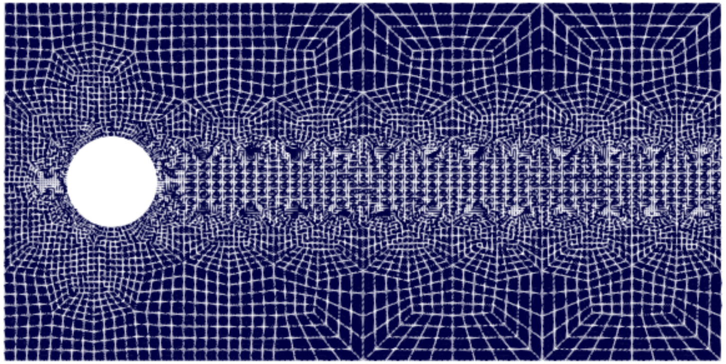


Mesh

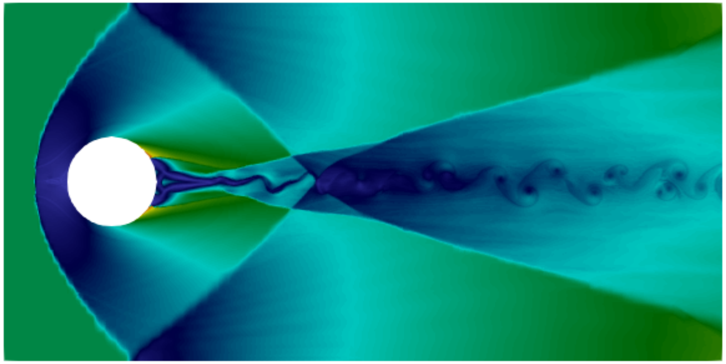
Mach number profile

Supersonic flow through cylinder

Mach 3 flow with solid walls at sphere, bottom and top



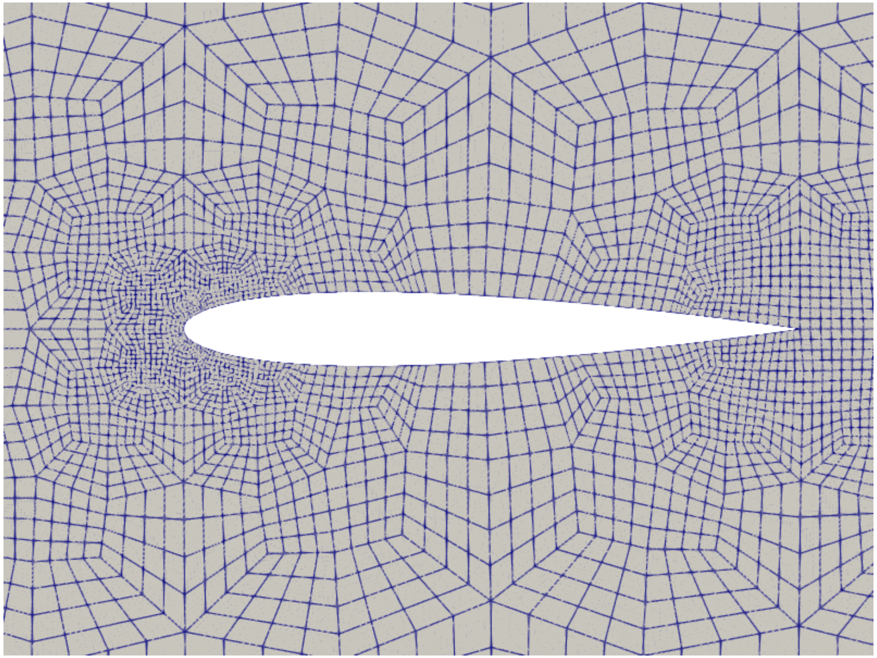
Mesh



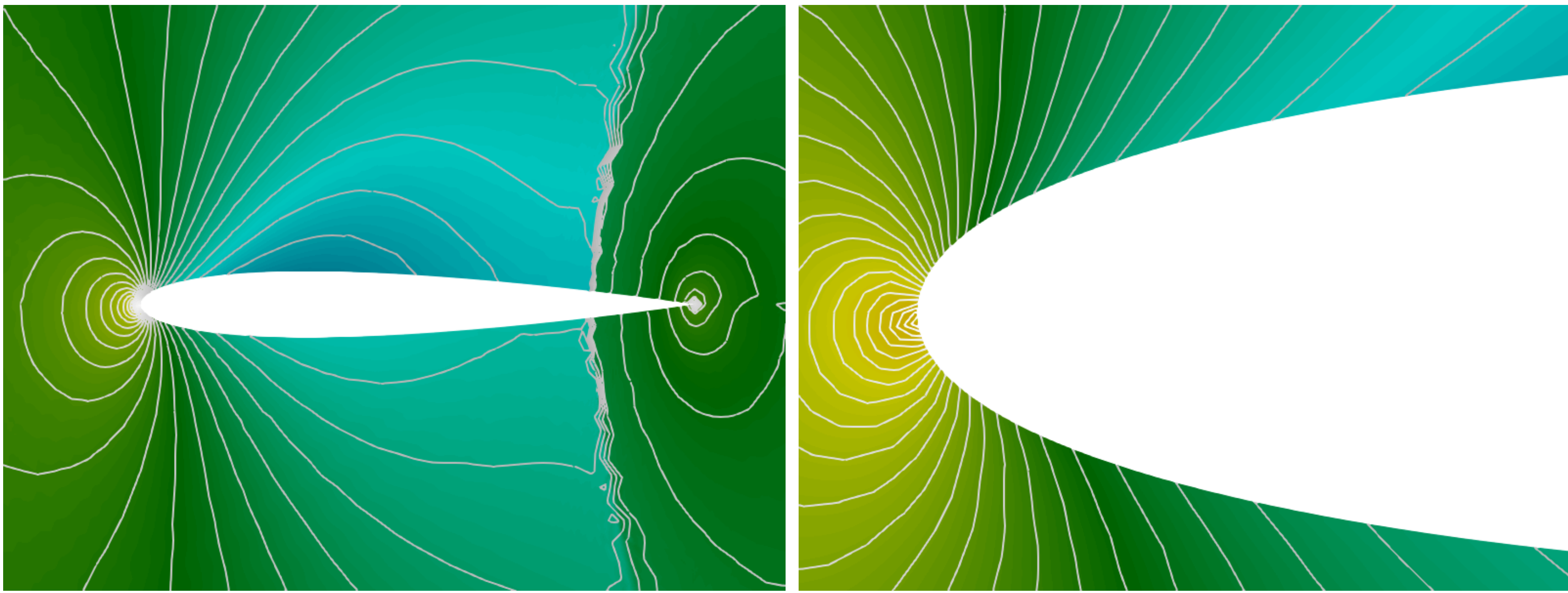
Mach number profile

NACA0012 airfoil

Mach 0.85 (transonic) flow with an angle of attack of 2 degrees



NACA0012 airfoil



Full airfoil Zoomed at nose

Density profile with **Mach number** contour lines

Second order equations

$$\partial_t \boldsymbol{u} + \nabla \cdot \boldsymbol{f}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{0}$$

$$\boldsymbol{f} = \boldsymbol{f}_a(\boldsymbol{u}) - \boldsymbol{f}_{\boldsymbol{v}}(\boldsymbol{u}, \nabla \boldsymbol{u})$$

Reduce to first order (Bassi, Rebay [1])

$$\begin{aligned} \boldsymbol{S} - \nabla \boldsymbol{u} &= 0 \\ \partial_t \boldsymbol{u} + \nabla \cdot \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{S}) &= 0 \end{aligned}$$

Solve first equation by FR

$$\boldsymbol{u}_e^h = \boldsymbol{u}_e^\delta + \left(\hat{\boldsymbol{u}}_{e-\frac{1}{2}} - \boldsymbol{u}^\delta(x_{e-\frac{1}{2}+}) \right) g_L + \left(\hat{\boldsymbol{u}}_{e+\frac{1}{2}} - \boldsymbol{u}^\delta(x_{e+\frac{1}{2}-}) \right) g_R$$

$$\hat{\boldsymbol{u}}_{e+\frac{1}{2}} = \frac{\boldsymbol{u}^\delta(x_{e+\frac{1}{2}-}) + \boldsymbol{u}^\delta(x_{e+\frac{1}{2}+})}{2}$$

$$\boldsymbol{S} := \nabla \boldsymbol{u}_e^h$$

Second order equations ($N = 1$)

$$
\begin{aligned}
\boldsymbol{u}_e^{n+1} &= \boldsymbol{u}_e^n - \Delta t\, \nabla \cdot \left(\boldsymbol{f}(\boldsymbol{u}_e^n, \nabla \boldsymbol{u}_e^n) + \frac{\Delta t}{2}\, \partial_t\, \boldsymbol{f}(\boldsymbol{u}_e^n, \nabla \boldsymbol{u}_e^n) \right) \\
&= \boldsymbol{u}_e^n - \Delta t\, \nabla \cdot \boldsymbol{F}, \\
& \qquad \qquad \boldsymbol{F} = \boldsymbol{F}_a - \boldsymbol{F}_v
\end{aligned}
$$

Local time averaged fluxes

$$
\begin{gathered}
\partial_t\, \boldsymbol{u}_e = -\nabla \cdot \boldsymbol{f}(\boldsymbol{u}_e^n, \nabla \boldsymbol{u}_e^n) = -\nabla \cdot \boldsymbol{f}(\boldsymbol{u}_e^n, \boldsymbol{S}_e) \\
(\boldsymbol{f}_a)_t = \frac{\boldsymbol{f}_a(\boldsymbol{u}_e + \Delta t\, \partial_t\, \boldsymbol{u}_e^n) - \boldsymbol{f}_a(\boldsymbol{u}_e - \Delta t\, \partial_t\, \boldsymbol{u}_e^n)}{2\, \Delta t}
\end{gathered}
$$

$$
(\boldsymbol{f}_v)_t = \frac{\boldsymbol{f}_v(\boldsymbol{u}_e + \Delta t\, \partial_t\, \boldsymbol{u}_e, \nabla \boldsymbol{u}_e + \Delta t\, \partial_t\, \nabla \boldsymbol{u}_e^n) - \boldsymbol{f}_v(\boldsymbol{u}_e - \Delta t\, \partial_t\, \boldsymbol{u}_e, \nabla \boldsymbol{u}_e + \Delta t\, \partial_t\, \nabla \boldsymbol{u}_e^n)}{2\, \Delta t}
$$

$$
\partial_t\, \nabla \boldsymbol{u}_e^n = \nabla (\partial_t\, \boldsymbol{u}_e^n)
$$

$$
\boldsymbol{F}^\delta = \boldsymbol{f} + \frac{\Delta t}{2}\, \boldsymbol{f}_t
$$

Second order equations

$$\boldsymbol{F}_{e+\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{F}^{-}_{e+\frac{1}{2}} + \boldsymbol{F}^{+}_{e+\frac{1}{2}}\right) - \frac{\lambda_a}{2}\left(\boldsymbol{U}^{-}_{e+\frac{1}{2}} - \boldsymbol{U}^{+}_{e+\frac{1}{2}}\right),$$

$$\lambda_a \approx \sigma\left(\boldsymbol{f}_a'(\boldsymbol{u})\right)_{e+\frac{1}{2}}$$

The above is equivalent to

$$\begin{aligned}
\boldsymbol{F}^{a}_{e+\frac{1}{2}} &= \frac{1}{2}\left(\boldsymbol{F}^{-}_{a,e+\frac{1}{2}} + \boldsymbol{F}^{+}_{a,e+\frac{1}{2}}\right) - \frac{\lambda_a}{2}\left(\boldsymbol{U}^{-}_{e+\frac{1}{2}} - \boldsymbol{U}^{+}_{e+\frac{1}{2}}\right) \\
\boldsymbol{F}^{\nu}_{e+\frac{1}{2}} &= \frac{1}{2}\left(\boldsymbol{F}^{-}_{\nu,e+\frac{1}{2}} + \boldsymbol{F}^{+}_{\nu,e+\frac{1}{2}}\right)
\end{aligned}$$

Numerical Results

$$\Delta t = \frac{C_{\mathrm{CFL}}}{\mathrm{dim}} \left[\frac{\lambda_a^{\max}\,(2\,N+1)}{h_{\min}} + \lambda_\nu^{\max}\,\frac{2\,(2\,N+1)^2}{h_{\min}^2} \right]^{-1}$$

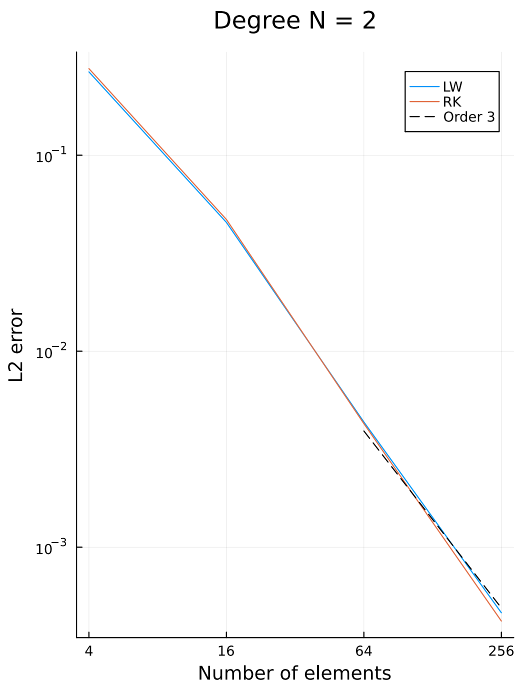
$$C_{\mathrm{CFL}} = 0.98$$

Convergence tests

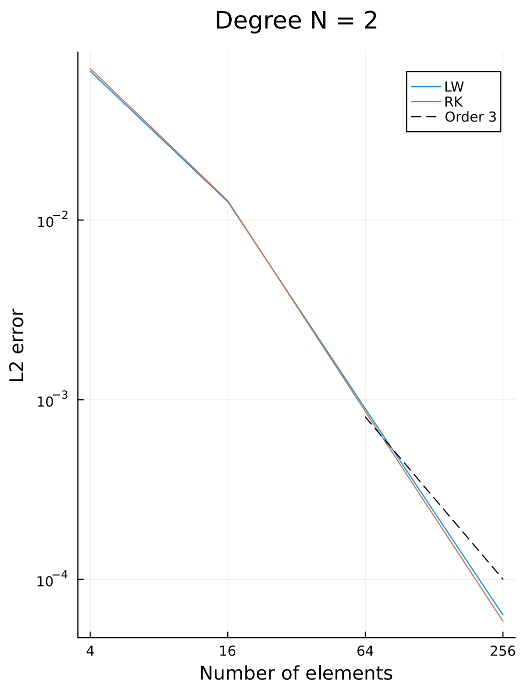
Periodic case for

$$u_t + \boldsymbol{b} \cdot \nabla u = \nu \, \Delta u, \qquad (x, y) \in [-1, 1]^2$$

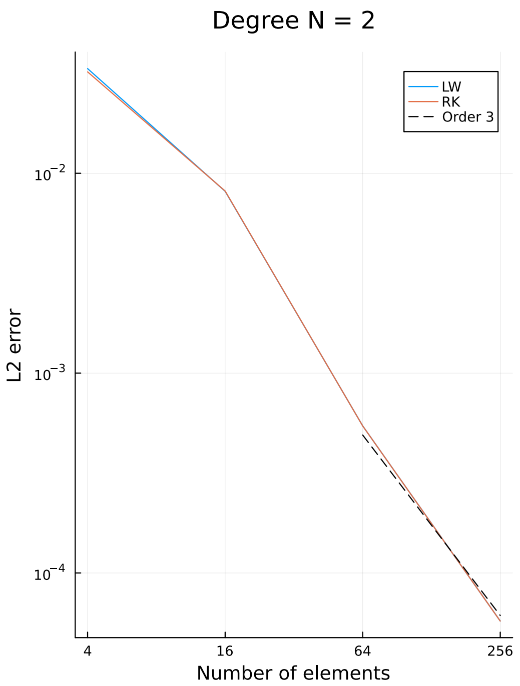
with sinusoidal initial data.



$$\boldsymbol{b} = (1, 1.5)$$
$$\nu = 10^{-6}$$



$$\boldsymbol{b} = (1, 1.5)$$
$$\nu = 5 \times 10^{-2}$$



$$\boldsymbol{b} = (0, 0)$$
$$\nu = 5 \times 10^{-2}$$

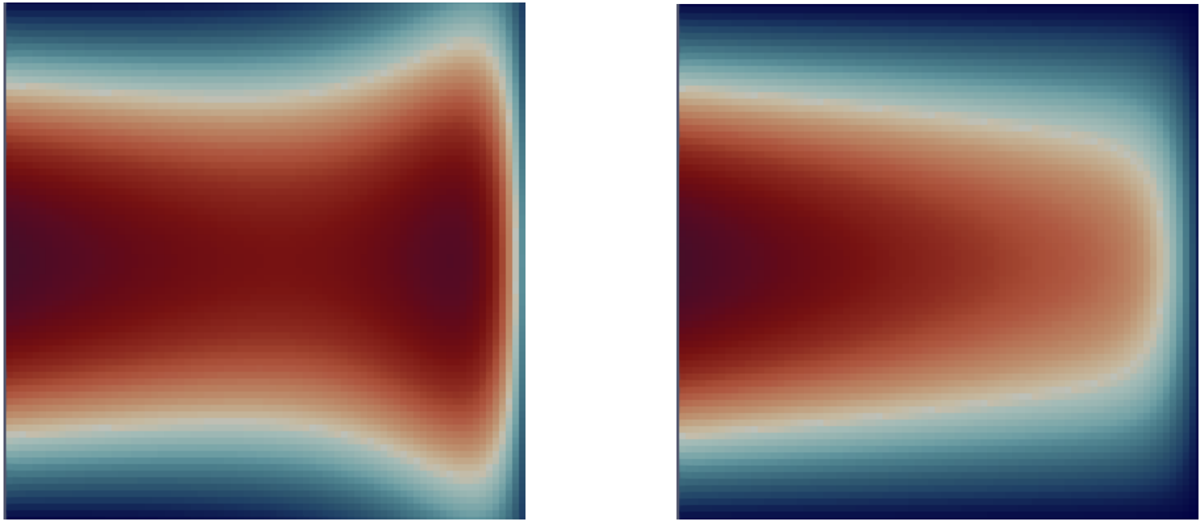
Non-periodic problem: Eriksson-Johnson

Non-periodic case for

$$u_t + u_x = \epsilon\, \Delta_{x,y}\, u, \qquad (x,y) \in [-1,0] \times [-0.5, 0.5]$$

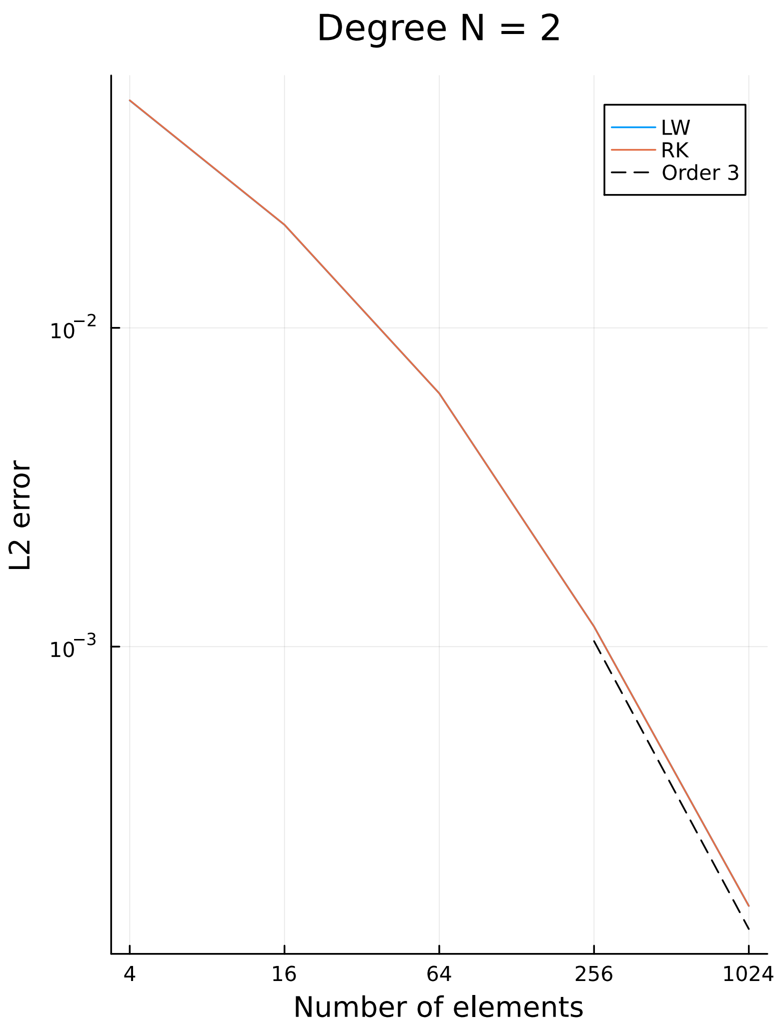
with $\epsilon = 0.05$ and the exact solution given by

$$u(x,t) = e^{-lt}\,(e^{\lambda_1 x} - e^{\lambda_2 x}) + \cos(\pi\, y)\,\frac{e^{s_1 x} - e^{r_1 x}}{e^{-s_1} - e^{-r_1}}$$



$t = 0$ $\qquad$ $t = 1.5$

Non-periodic problem: Eriksson-Johnson



Navier Stokes equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho\,\boldsymbol{v} \\ \rho\,e\end{pmatrix} + \nabla\cdot\begin{pmatrix}\rho\,\boldsymbol{v} \\ \rho\,\boldsymbol{v}\otimes\boldsymbol{v} + p\,\underline{I} \\ (\rho\,e + p)\,\boldsymbol{v}\end{pmatrix} = \nabla\cdot\begin{pmatrix}0 \\ \underline{\tau} \\ \underline{\tau}\,\boldsymbol{v} - \nabla q\end{pmatrix}$$

$$p = (\gamma - 1)\left(\rho\,e - \frac{1}{2}\,\rho\,(v_1^2 + v_2^2)\right)$$ Ideal gas assumption

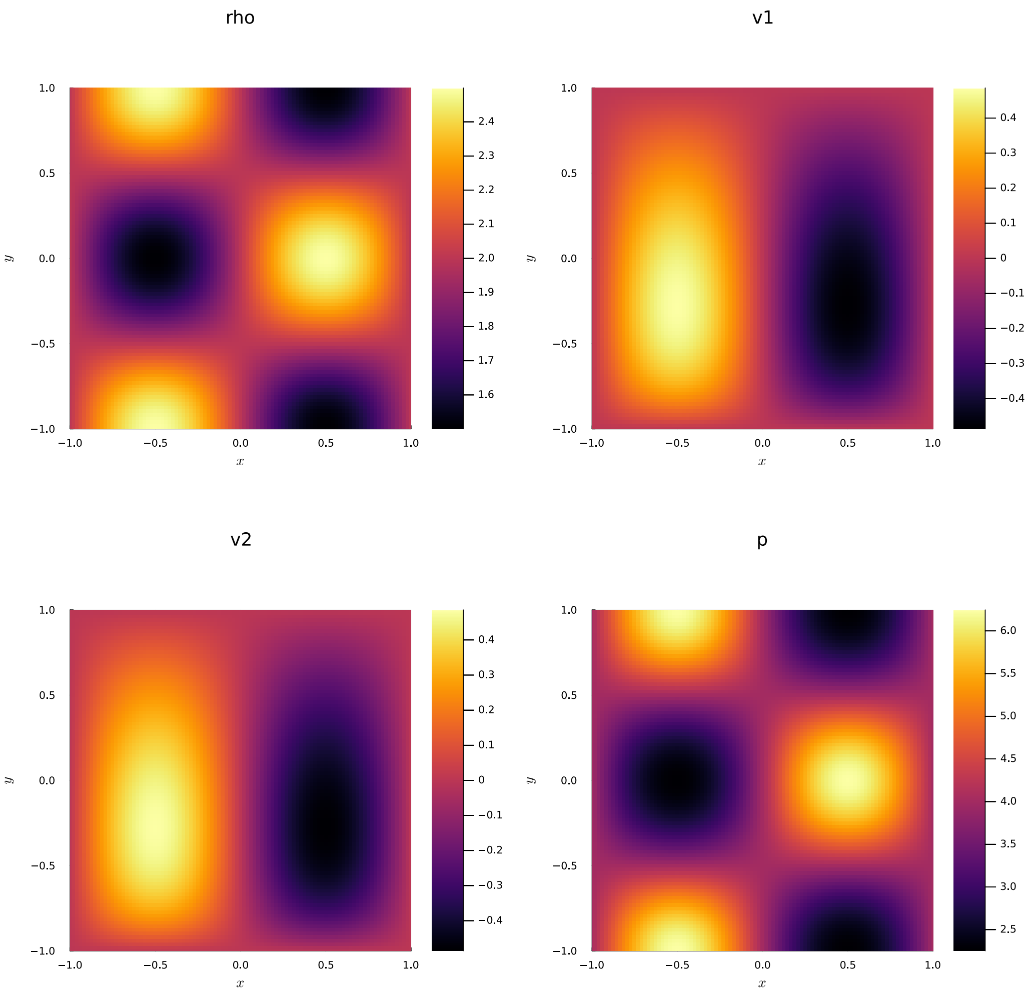
$$\underline{\tau} = \mu\,(\nabla\,\boldsymbol{v} + (\nabla\,\boldsymbol{v})^T) - \frac{2}{3}\,\mu\,(\nabla\cdot\boldsymbol{v})\,\underline{I}$$ Viscous stress tensor

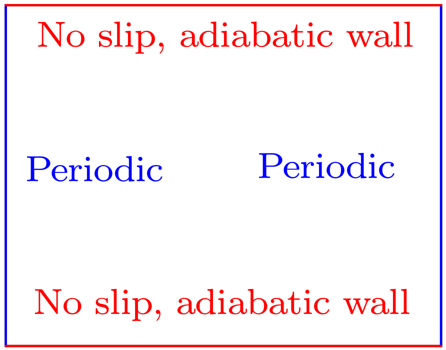
$$\nabla q = -\kappa\,\nabla(T), \quad T = \frac{p}{R\,p}$$ Fick's law

$$\kappa = \frac{\gamma\,\mu\,R}{(\gamma - 1)\,\mathrm{Pr}}$$ Thermal conductivity under constant Pr

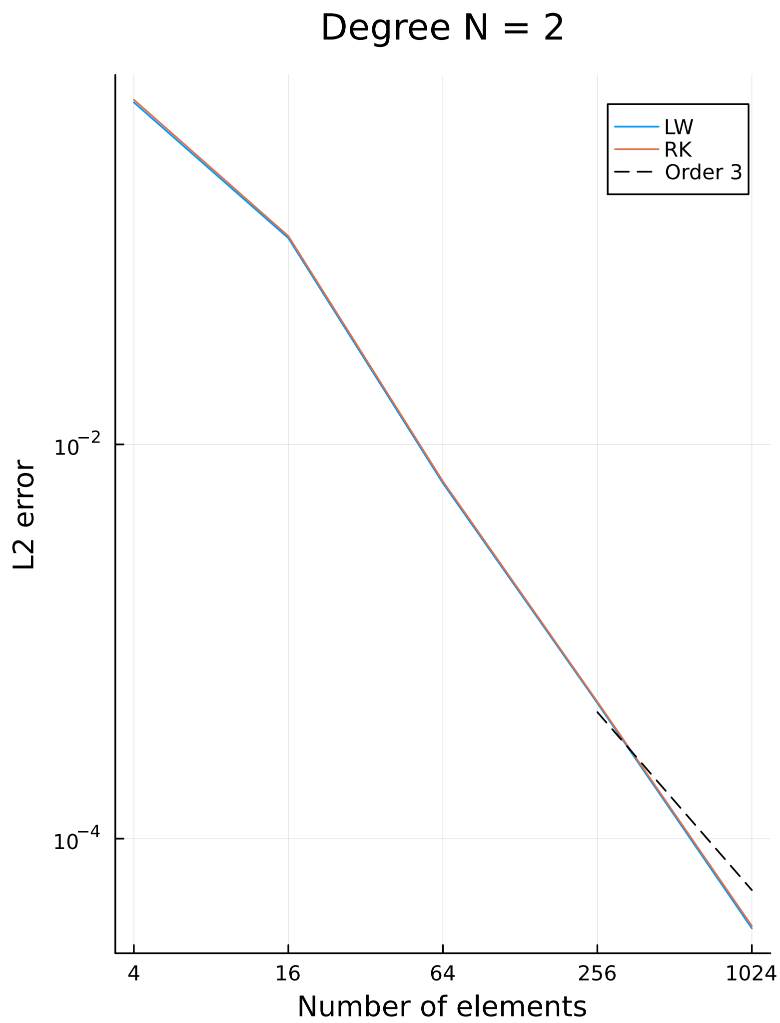
$$\mathrm{Pr} = 0.72, \quad \mu = 0.01, \quad \gamma = 1.4$$

Navier Stokes convergence test - manufactured solution



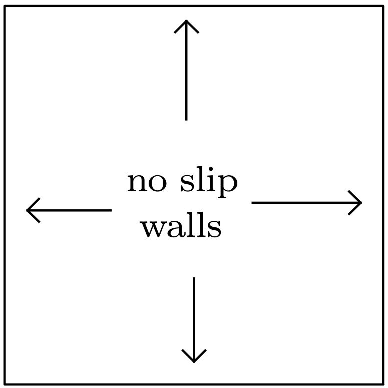


Navier Stokes convergence test - manufactured solution

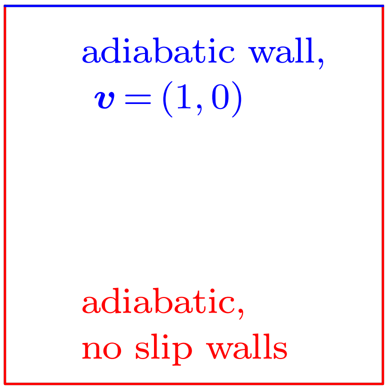


Lid Driven cavity

$$\rho = 1.0, \quad v_1 = v_2 = 0, \quad p = \frac{1}{\gamma M^2},$$
$$M = 0.1.$$

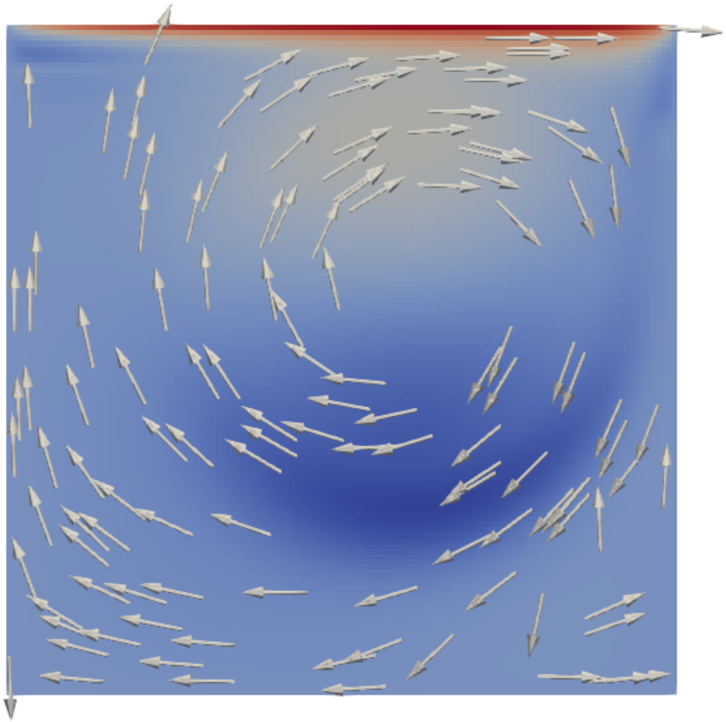


Advective boundary conditions



Parabolic Boundary conditions

Lid driven cavity



x velocity profile with velocity vectors for $N = 3$

	LW	**RK**
WCT	621 s	930 s

Conclusions

- The Lax-Wendroff scheme was extended to unstructured grids and the free stream condition was proven to be equivalent to that of Flux Reconstruction schemes.
- Lax-Wendroff scheme was extended to second order equations using the scheme of Bassi, Rebay [1].
- The second order scheme shows optimal convergence rates for advection and diffusion dominated problems
- Initial results for second order scheme indicate speed benefit over Runge-Kutta schemes.

Plans and WIP

- Adaptive mesh refinement
- Time step computation based on temporal derivatives of $\boldsymbol{u}_t$
- Non-conservative systems
- Blending limiter for second order equations
- Second order equations on curved grids
- Implicit handling of source terms
- Fourier analysis of viscous Lax-Wendroff
- Other choices of solution points
- MUSCL-Hancock blending on unstructured grids
- Code optimization, composability and modularity

Bibliography

[1] F. Bassi and S. Rebay. A high-order accurate discontinuous finite element method for the numerical solution of the compressible navier–stokes equations. *Journal of Computational Physics*, 131(2):267–279, 1997.

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[3] David A. Kopriva. Metric identities and the discontinuous spectral element method on curvilinear meshes. *Journal of Scientific Computing*, 26(3):301–327, 2006.

Thank you

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