Exercise 3.1: Solve the Poisson problem

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_\mathrm{D}, \\ \partial_n u &= -1 && \text{on } \Gamma_\mathrm{N} \end{aligned}$$

on the unit square $\Omega = (0,1)^2$. Let $\Gamma_\mathrm{N} := \{\, x \in \partial\Omega \mid x_2 = 1 \,\}$ and let $\Gamma_\mathrm{D} = \partial\Omega \setminus \Gamma_\mathrm{N}$.

Exercise 3.2: Change the program in Exercise 3.1 such that the following convection diffusion problem is solved:

$$\begin{aligned} -\varepsilon\Delta u + \beta \cdot \nabla u &= 0 && \text{in } \Omega, \\ u &= u_0 && \text{on } \Gamma_\mathrm{D}, \\ \varepsilon\partial_n u &= 0 && \text{on } \Gamma_\mathrm{N}. \end{aligned}$$

Again let $\Omega = (0,1)^2$, $\Gamma_\mathrm{N} := \{\, x \in \partial\Omega \mid x_2 = 1 \,\}$ and let $\Gamma_\mathrm{D} = \partial\Omega \setminus \Gamma_\mathrm{N}$. The Dirichlet boundary conditions are given by

$$u_0(x) = \begin{cases} 1 & \text{for } x_1 = 0,\ x_2 \geq \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad .$$

where $\beta \in \mathbb{R}^2$ is the direction of transport.

Solve the problem with the following parameters: $\varepsilon = 1$ and $\beta = (\cos 13°, \sin 13°)^T$. The corresponding variational problem reads: Find $u \in u_0 + H_0^1(\Omega;\Gamma_\mathrm{D})$ such that

$$\varepsilon(\nabla u, \nabla\varphi) + (\beta \cdot \nabla u, \varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega;\Gamma_\mathrm{D}),$$

where $H_0^1(\Omega;\Gamma_\mathrm{D}) := \left\{\, v \in H^1(\Omega) \;\middle|\; v|_{\Gamma_\mathrm{D}} = 0 \,\right\}$. How do the entries of the system matrix look like? Keep in mind that the resulting system matrix is not symmetric. This has impact on the iterative solver we choose for the algebraic linear systems.

Exercise 3.3: The same problem as in Exercise 3.2 has to be solved but for smaller values for ε. Compute solutions for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$. How does the solver behave? How do the solutions (on various grids) look like?