**Exercise 4.1:** We consider the pure transport problem

$$\begin{aligned} \beta \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \Gamma_-, \end{aligned}$$

where $\Omega = (0,1)^2$ and $\Gamma_- = \{\, x \in \partial\Omega \mid \beta(x) \cdot n(x) < 0 \,\}$. Here $n(x)$ is the outer unit normal vector in the coordinates $x \in \partial\Omega$. $\beta \in \mathbb{R}^2$ is the direction of transport.

In Exercise 3.3 we learned that the Galerkin discretization of the convection diffusion problem has instabilities for small values of $\varepsilon$. One possibility to overcome this problem was the so called stremline diffusion method. That means artificial diffusion is added in direction of $\beta$. To keep the variational formulation consistent we add the following terms for stabilization:

$$\sum_{K \in \mathcal{T}_h} \delta_K(-\varepsilon\Delta u + \beta \cdot \nabla u - f, \beta \cdot \nabla \varphi)_K \quad \text{with} \quad \delta_K = \delta_0 \min\left\{ \frac{h_K^2}{\varepsilon}, \frac{h_K}{\|\beta\|} \right\}.$$

Here $\delta_0$ is constant. Usually one takes $\delta_0 = 0.1$ and $h_K$ the diameter of the cell $K$.

In the case of a pure transport problem the stabilization term reduces to

$$\sum_{K \in \mathcal{T}_h} \delta_K(\beta \cdot \nabla u - f, \beta \cdot \nabla \varphi)_K \quad \text{with} \quad \delta_K = \delta_0 \frac{h_K}{\|\beta\|}.$$

Solve the transport problem on a sequence of globally refined meshes with parameters $\beta = (\cos 13°, \sin 13°)^T$, $f \equiv 0$ and

$$u_0(x) = \begin{cases} 1 & \text{for } x_1 = 0,\ x_2 \geq \frac{1}{2} \\ 0 & \text{else} \end{cases} .$$

How does $\Gamma_-$ look like in this case?

**Exercise 4.2:** Another possibility to overcome the problems with instabilities that occur in convection dominated problems is to use discontinuous Galerkin methods ("DG methods"). In the case of a pure transport problem

$$\begin{aligned} \beta \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{on } \Gamma_- \end{aligned}$$

(nomenclature like in Exercise 4.1) the variational formulation reads: Find $u_h \in V_h$ such that for all $\varphi_h \in V_h$ we have:

$$\sum_{K \in \mathcal{T}_h} \left\{ -(u_h, \beta \cdot \nabla \varphi_h)_K + (\beta \cdot n u_h, \varphi_h)_{\partial K_+} + (\beta \cdot n u_h^-, \varphi_h)_{\partial K_- \setminus \Gamma_-} \right\} = (f, \varphi_h) - (\beta \cdot n u_0, \varphi_h)_{\Gamma_-}.$$

Here $\partial K_- = \{ x \in \partial K \mid \beta(x) \cdot n(x) < 0 \}$ denotes the inflow boundary and $\partial K_+ = \partial K \setminus \partial K_-$ denotes the outflow boundary of a cell $K$. Futhermore let $u_h^-$ be the value of the function $u_h$ on the neighbouring cell (recall that the function $u_h$ does not necessarily have to be continuous across cell edges).

Solve the transport problem on a sequence of globally refined grids with a DG method where

$$V_h := \left\{ v \in L^2(\Omega) \,\middle|\, v\big|_K \in Q_1(K) \right\},$$

with parameters $\beta = (\cos 13°, \sin 13°)^T$, $f \equiv 0$ and

$$u_0(x) = \begin{cases} 1 & \text{for } x_1 = 0,\ x_2 \geq \frac{1}{2} \\ 0 & \text{else} \end{cases} .$$