

Discrete Mathematical Structures

(DMS)

Unit: 1 - Set theory, Functions and Relations

Unit: 2 - Combinatorics and Matrix Algebra

Unit: 3 - Mathematical Logic

Unit: 4 - Number and Graph theory

* Book: Discrete mathematics with Applications
(Author: Thomas Koshy)

* Assessment Criteria:

(1) Attendance: 6%

- Above 95% - 6 marks
- 90 - 95% - 5 marks
- 85 - 90% - 4 marks
- 80 - 85% - 3 marks
- Less than 70% - 0 marks

(2) Quizzes : 24% (2 + 1 Quizzes)

(3) Mid-term: 30%

(4) End-term: 40%

o Cartesian product:

$$A = \{a_1, a_2\} \Rightarrow |A| = 2$$

$$B = \{b_1, b_2, b_3\} \Rightarrow |B| = 3$$

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}$$

$$|A \times B| = |B \times A| = 6$$

$$\therefore R \subseteq A \times B$$

° Total no. of subsets: $A = \{a_1, a_2, \dots, a_n\}$

$$= 2^n$$

$$= {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$$

OR

each element has 2 choices; either to be part of subset or not to be

$$= (2 \times 2 \times 2 \dots n \text{ times}) = 2^n$$

° How many relations exist on A?

$$\therefore \text{Relation } R \subseteq A \times A$$

$$\therefore \text{no. of } R = |A \times A| = (n \times n) = n^2$$

° All funⁿ are relⁿ but converse is false.

Ex: In pr. ex. of $A \times B$, we take only elements $\{(a_1, b_1), (a_1, b_2)\}$ as a set, then it's a relⁿ but not funⁿ.

* Set theory:
(Georg Cantor)

(1) Null set or Empty set

$$\emptyset = \{\} \neq \{\emptyset\}$$

(2) Singleton set (1 element)

→ Set representation:

(i) Set-BUILDER form:

$$A = \{x \mid P(x)\}$$



predicate/condⁿ on x

(ii) List method / Roaster form:

Listing elements with { }

$$\text{Ex: } A = \{a, e, i, o, u\}$$

In set-builder, $A = \{x \mid x \text{ is a vowel of eng. alphabet}\}$

$$\text{Ex: } A = \{x \mid x^2 - 3x + 2 = 0\}$$

$$\therefore A = \{1, 2\}$$

→ subset:

$$A \subseteq B$$

$$\Leftrightarrow \text{if } x \in A \Rightarrow x \in B \wedge x$$

$A \subseteq B \rightarrow$ improper subset ($A = B$) may be

$A \subset B \rightarrow$ proper subset ($A \neq B$)

$$\text{Ex: } A = \{a, b\}$$

subsets of $A = \emptyset, \{a\}, \{b\}, \{a, b\}$

Power set of given set is collection/set of all the subsets of given set.

$$\therefore P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

NO. of subsets = 2^n ; $n = \text{cardinality of a set.}$
 $= |A| = 2$

$$\text{Ex: } S = \{Y \mid Y \notin Y\}$$

S is a collection of all those elements which do not belong to themselves.

• Does $S \in S$?

Let

(i) $S \in S \Rightarrow$ through def. $S \notin S$ but if

(ii) $S \notin S \Rightarrow S \in S$ or S should be element of S

∴ Answer is Yes. can be neither Yes/No since there's contradiction, being created.

Hence, $S = \{Y | Y \notin Y\}$ is called Russel's paradox.
 \Rightarrow No set contains all sets.

So, acc. to this paradox, U doesn't exist however we chose U in the domain of our question.

Ex: $A = \{a\}$

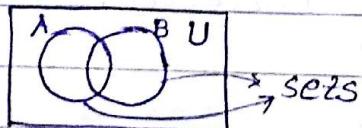
$B = \{a, c, d\}$

$C = \{b, e, f, g, h, i\}$

$D = \{f, i, j\}$

So, $U = \{a, b, c, d, e, f, g, h, i, j\}$ OR $U = \{a, z, \emptyset\}$ & soon.

→ Venn Diagrams:



Finite - Able to count no. of elements.

* Operations on Set:

(i) Union

(v) Cartesian product

(ii) Intersection

(vi) Complement

(iii) Difference

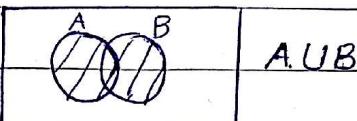
(iv) Symmetric diff.

$y A=B$ and $B \subseteq A \Rightarrow A=B$

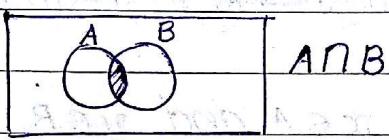
All operations are considering U .

$\{a, a, b, c, c\} = \{a, b, c\} \Rightarrow$ repetition counted only once

(i) $A \cup B \Rightarrow x \in A$ or $x \in B$ or $x \in A \text{ & } B$ both



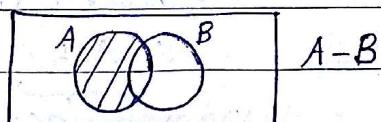
(ii) $A \cap B \Rightarrow x \in A$ and $x \in B$



(iii) A^c or A' \Rightarrow if $x \notin A \Rightarrow x \in A^c = U - A$



(iv) $A - B \Rightarrow$ if $x \in A \not\Rightarrow x \notin B$



(v) Symmetric diff. (\oplus or Δ)

$$A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Proof: Let $x \in (A \cup B) - (A \cap B)$, then
 $\Rightarrow (x \in A \text{ or } x \in B)$ and $x \notin (A \cap B)$

$$\begin{aligned} A \times B &\neq B \times A \\ A - B &\neq B - A \\ A \oplus B &\neq B \oplus A \end{aligned}$$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \notin A \text{ or } x \notin B)$

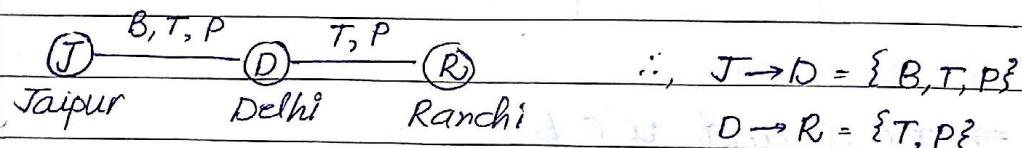
$\Rightarrow (x \in A \text{ or } x \notin B) \text{ and } (x \in B \text{ or } x \notin A)$

$\Rightarrow x \in (A - B) \cup (B - A) \Rightarrow (A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

Similarly $A \cup (A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B) \therefore \text{They are equal}$

* Cartesian product:

Application:



$\therefore \text{No. of ways to go } J \rightarrow R = (J \rightarrow D) \times (D \rightarrow R) = \text{cartes. prod.}$

\rightarrow If $(x, y) \in A \times B \Rightarrow x \in A \text{ and } y \in B.$

$(x_1, x_2, \dots, x_n) \in A_1 \times A_2 \times \dots \times A_n \Rightarrow x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$

* Partition sets:

$$U = \{a, b, c, d, e, f, g, h, i\}$$

$$A = \{a, b, c\}$$

$$B = \{d, e, f\}$$

$$C = \{g, h, i\}$$

• $A, B, C = \text{disjoint sets of } U$

$$\bullet A \cup B \cup C = U$$

$\therefore A, B, C = \underbrace{\text{partition sets of } U}_{\text{each should be non-empty}}$

* Recursively Defined sets:

$$S = \{2, 2^2, 2^{2^2}, 2^{2^2}, 2^{2^2}, \dots\}$$

\downarrow
primitive/basic clause or fund. elec. = which can't be generated by a def. property

Properties:

(i) primitive element existence

(ii) $x \in S \Rightarrow z^x \in S$

(iii) Termination law: (i) and (ii) are only way to define set.

* Fuzzy Sets

$$A = \{x, \mu_x(A)\} \longrightarrow \{0 \leq \mu_x(A) \leq 1\}$$

↓

degree of belongingness of x to A

Ex: $X = \{x_1, x_2, x_3, x_4, x_5\}$

$$S = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.9), (x_4, 1), (x_5, 1)\}; x_5 \notin S \text{ i.e. } (x_5, 0)$$

$$T = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.9), (x_4, 1)\}$$

$$S' = \{(x_1, 0.8), (x_2, 0.5), (x_3, 0.1), (x_4, 1)\}$$

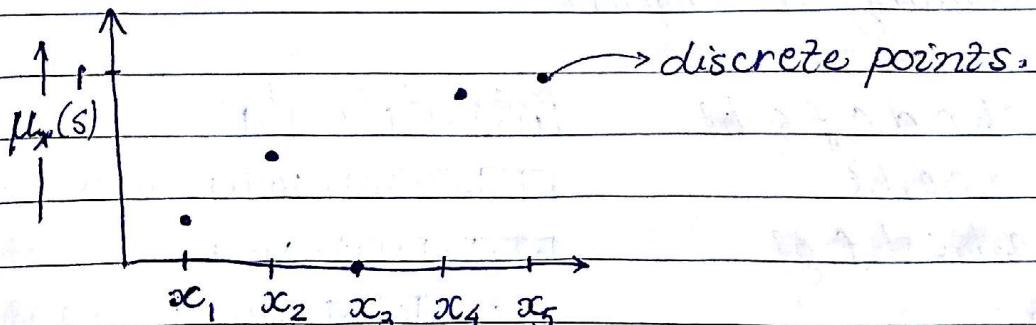
$$T' = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.1), (x_4, 1)\}$$

→ max. membership of x_i from S & T

$$S \cup T = \{(x_1, 0.6), (x_2, 0.8), (x_3, 0.9), (x_4, 1), (x_5, 1)\}$$

$$S \cap T = \{(x_1, 0.2), \dots, (x_5, 0)\}$$

→ min. membership



- Partition set of an ∞ set:

Ex: \mathbb{Z} - set of integers.

Partit' sets: base = remainder left when \div by 5

$$\mathbb{Z}_0 = \{ \dots -10, -5, 0, 5, 10, \dots \}$$

$$\mathbb{Z}_1 = \{ \dots -9, -6, -1, \dots 1, 6, 9, \dots \}$$

$$\mathbb{Z}_2 = \dots$$

$$\mathbb{Z}_3 = \dots$$

$$\mathbb{Z}_4 = \dots$$

$$\Sigma = \{a, b, c\} = \text{alphabets}$$

$$\Sigma^* = \text{words using } \Sigma$$

$$= \{ \lambda, a, b, c, aa, ab, \dots \}$$

$$\|\alpha\| = \text{length of } \alpha = 1$$

$$\|aa\| = 2$$

$$\|\lambda\| = 0$$

subset of Σ^* = language.

Ex: $S = \{a, ba, ca\}$ in set-builder
aa

$$S = \{ x \in \Sigma^* \mid 1 \leq |x| \leq 2 \text{ & } x \text{ ends by } a \}$$

Ex: Representing in computer:

$$U = \{a, b, c, d, e, f, g, h\}$$

$$\begin{matrix} h & g & f & e & d & c & b & a \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$$A = \{a, c, e, h\}$$

$$\begin{matrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{matrix}$$

$$B = \{a, b, c, d, f, g\}$$

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{matrix}$$

$$A \cap B$$

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{matrix}$$

* Proof similar to $A \oplus B$:

$$x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A \text{ and } x \notin B \Rightarrow x \in A' \text{ and } x \in B' \Rightarrow x \in A' \cap B'$$

$$\therefore (A \cup B)' = A' \cap B'$$

$$\text{Similarly, } A' \cap B' = (A \cup B)' \Rightarrow (A \cup B)' = A' \cap B'$$

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$$A \cap B = A \wedge B$$

$$A \cup B = A \vee B \quad \text{OR}$$

$$A \oplus B = A \text{ XOR } B \text{ (single 1)}$$

$$A^c = \sim A$$

$$\boxed{A - B = A \cap B'} *$$

* De-morgan's law:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

* Distributive law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Q Prove using Set laws:

$$(X - Y) - Z = X - (Y \cup Z)$$

$$\xrightarrow{\text{SOLN:}} (X \cap Y') - Z = X \cap (Y' \cup Z)' \quad \text{RHS}$$

$$\begin{aligned} (X \cap Y') \cap Z' &= X \cap (Y' \cap Z') = \text{LHS} \\ &= X \cap (Y \cup Z)' = \text{LHS} = \text{RHS} \\ &= X - (Y \cup Z) \end{aligned}$$

$$\textcircled{P1} |A \cup B| = |A| + |B| \neq |A \cap B|$$

$$\textcircled{P2} |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Proof: Let $X = A$ and $Y = (B \cup C)$, then,

$$(X \cup Y) \text{ from } \textcircled{P1}, = |X| + |Y| - |X \cap Y|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

Now, using $\textcircled{P1}$ on $B \cup C$ and distributive law on $A \cap (B \cup C)$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

Now, using $\textcircled{P1}$ on last term,

$$= |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |A \cap B \cap C|\}$$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Functions

$$f: x \rightarrow y$$

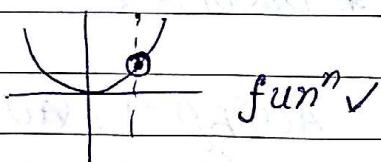
$$f(x) = y \quad \text{output}$$

input/argument
 $x \in \text{domain}(f)$

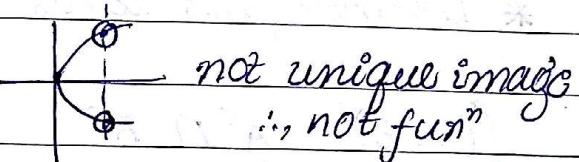
$y = \text{range}$

$f(x) = 2x$

$x = \text{domain}$



$$x^2 = 4ay$$



$$y^2 = 4ax$$

not unique image
, not funⁿ

Ex: $y = 2\sqrt{ax}; a \neq 0$

$$D(y) = R^+ \cup \{0\} = [0, \infty)$$

Ex: $(x-a)(x-b) \leq 0; a < b$

$$x \in [a, b] \quad D(y) = (-\infty, a] \cup [b, \infty)$$

$$\rightarrow D(f+g) = D_1(f) + D_2(g)$$

$$= D_1 \cap D_2$$

$$\rightarrow D(f \cdot g) = D_1 \cap D_2$$

◦ Composition of funⁿ:

$$f: X \rightarrow Y$$

$$g: Y \rightarrow Z$$

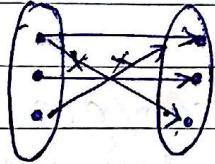
$$g \circ f: X \rightarrow Z \quad \text{Range}(f) \subseteq \text{Domain}(g)$$

$$(g \circ f)(x) = g(f(x))$$

- Types of functions:

- (1) One-One (Injective):

f is one-one if
 $x_1 \neq x_2 \in X \Rightarrow f(x_1) \neq f(x_2)$



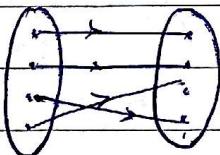
(i.e each element in domain has unique image)

OR

f is one-one if
 $x_1 = x_2 \in X \Rightarrow f(x_1) = f(x_2)$

- (2) Onto (surjective):

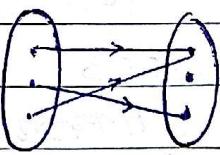
$f: X \rightarrow Y$



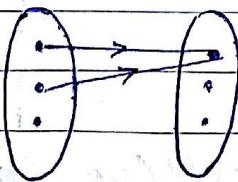
$\text{Range}(f) = Y$

- (3) Into:

$\text{Range}(f) \subset Y$



- (4) Many-One:

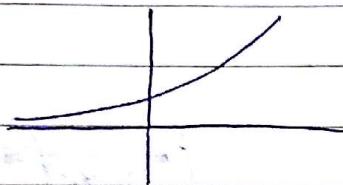


→ one-one correspondence / Bijective:

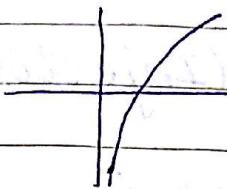
Both one-one and onto.

If f is bijective inverse/inverse mapping exists.

- $e^x, a^x ; a \neq 1$



- $\log_a x ; a \neq 1, x > 0$



- Floor/greatest int. funⁿ: greatest left side int.

$$[3 \cdot 14] = 3$$

$$[-3 \cdot 14] = -4$$

$$[2 \cdot 9] = 2$$

- Ceiling funⁿ:

least int. to right

$$[3 \cdot 14] = 4$$

$$[-3 \cdot 14] = -3$$

- Characteristic funⁿ:

$$S \subseteq U$$

$$f: U \rightarrow \{0, 1\}$$

$$f_S(x) = \begin{cases} 1, & x \in S \\ 0, & \text{otherwise} \end{cases}$$

Q Prove: $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$

$$\rightarrow f_A(x) = 1; x \in A$$

$$\rightarrow f_B(x) = 1; x \in B$$

where, $A, B \subseteq U$

$$\begin{aligned} f_{A \cap B}(x) &= f_A(x) \wedge f_B(x) = 1 \times 1 \\ &\rightarrow = f_A(x) \cdot f_B(x) \end{aligned}$$

- If a and b are 2 positive integers then the no. of positive integers $\leq a$ and divisible by b is given by:

$$\lfloor a/b \rfloor$$

Q Find the no. of +ve integers ≤ 1776 and divisible by 13.

Soln:

$$\left\lfloor \frac{1776}{13} \right\rfloor = 136$$

Q Find the no. of +ve int. ≤ 3000 and not divisible by 7 and 8.

Soln:

$$\left\lfloor \frac{3000}{7} \right\rfloor = 428 \quad \left\lfloor \frac{3000}{8} \right\rfloor = 375 \quad \left\lfloor \frac{3000}{56} \right\rfloor = 53$$

$$= |A| \quad = |B| \quad = |A \cap B|$$

$$A = \{x \in \mathbb{Z}^+ \mid x \text{ is divis. by } 7 \text{ & } x \leq 3000\}$$

$$B = \{x \in \mathbb{Z}^+ \mid x \text{ is divis. by } 8 \text{ & } x \leq 3000\}$$

$$= |A' \cap B'| = |(A \cup B)'| = |U - (A \cup B)| = |U| - |A| - |B| + |A \cap B|$$

$$= 3000 - [(428 + 375) - 53] = (3000 - 750) = 2250.$$

Q The no. of leap year (l) after 1600 and not exceeding a given year $y = ?$ Let $y = 2017$.
(Hint: if century, divis. by 400, otherwise divis. by 4)

Soln: $l = (\text{no. of l.y till } 2017 - \text{no. of l.y till } 1600)$

$$\text{no. of l.y till } 2016 = \frac{1600}{4} - 16 + \frac{1600}{400} = 388$$

$$\text{Similarly till } 2017 = \frac{2017}{400} - 20 + \frac{2017}{400} = 489$$

$$\therefore l = (489 - 388) = 101$$

Q Suppose today is Friday. What day will it be 99 days after from today.

Soln: $99 \% 7 = 1 \quad \therefore (\text{Fri} + 1) = \text{saturday.}$

o Hashing:

$$h(x) = x \bmod m$$

(m = size of array
= prime)

Ex A/c: $207630764 = x$

$$x \% 1209 = 762$$

NOW, if there is another no. which also gives 762, then this is called collision. In such case linearly 763, 764, ... will be checked if empty and if not, then assignment of A/c no. will occur from beginning. This is called linear probing.

- If X and Y are finite sets of same cardinality and f is a mapping from X to Y , which is one-one, if and only if it is onto.

Proof: $f: X \rightarrow Y$, then let

$$X = \{x_1, x_2, \dots, x_n\}$$

$$Y = \{y_1, y_2, \dots, y_m\} \quad \therefore |X| = |Y|$$

Suppose f is one-one, then

$f(x_1), f(x_2), \dots$ are different

Hence, all elements of X are uniquely mapped with all elem. of Y . \therefore its possible

\therefore only if f is onto.

- If X and Y are two finite sets having the same cardinality, then there must be mapping which is one-one and onto.

Proof: For finite sets only,

if $A \subseteq B \Rightarrow |A| \leq |B|$

* Countable & Uncountable Sets:

↓ ↳ which can't be counted.

A set S is called countable if it is either finite or countably infinite.

→ A set S is countably infinite if there exists a bijection between S and \mathbb{N} .

→ Cardinality of every countably infinite set is either \aleph_0 .

(Ex: $|N| = |z^+ \text{ which are even}| = x_0$)

Ex: $\mathbb{N} \times \mathbb{N}$ is countably ∞ \because it can be written in definite pattern

* Pigeon-hole principle:

If S is a mapping from X to Y , where X and Y are finite sets and $|X|=m$, $|Y|=n$, with $m > n$, then there exists at least 2 distinct elements x_1, x_2 such that $f(x_1) = f(x_2)$.

m # pigeons

n. # holes

$m > n$

If m pigeons are assigned to n -pigeon-holes, then there must be a hole containing at least $\lfloor (m-1)/n \rfloor + 1$ pigeons.

Pf: No pigeon-hole contains atmost $\lfloor (m-1)/n \rfloor$ - suppose
max. no. of pigeons = $n \lfloor (m-1)/n \rfloor$
 $\leq n(m-1)$
 $\leq (m-1)$ pigeons but its given m pigeons.

Q If we select 1000 students from campus, then atleast 3 of them must have same birthday.

Soⁿ: $m = 1000$, $\lfloor (m-1)/n \rfloor + 1 = \lfloor 999/366 \rfloor + 1 = 3$
 $n = 366$

* Proof Techniques:

(1) Trivial technique:

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} - \text{induction}$$

Check for $n=1$, If its true (i.e LHS=RHS), its true for other no.

(2) Direct proof:

Product of 2 odd integers is also an odd integer.

$$x = 2m+1$$

$$y = 2n+1$$

then, $x * y = 4mn + 2m + 2n + 1$
 $= 2(mn + m + n) + 1 = \text{odd no.}$

(3) Indirect proof technique:

(i) contrapositive

(ii) contradiction

(i) contrapositive:

If p then $q \equiv p \rightarrow q$
 then $\sim q \rightarrow \sim p$

(ii) Contradiction:

If p then $q \equiv p \rightarrow q$
 then $\sim q$

Q Prove that there exists largest prime no. x
 There's no largest prime no.

Pf by contradiction:

Let there be infinitely many prime no.

\exists a largest prime p_k (say)

p_1, p_2, \dots, p_k are k prime no.

Let $x = (p_1 \cdot p_2 \cdots \cdot p_k) + 1 \Rightarrow x$ is not divisible by any prime.
 $\forall p_i$

x must be either prime or composite

Case 1: x is a prime $> p_k$

Case 2: x must be divisible by a prime no. $\neq p_i$
(say q)

$\therefore p_k$ is = largest prime doesn't exist.

(4) Proof by Cases:

KNIGHT Always speaks the truth

KNAVE Always tells lie.

Suppose, A, B & C are only 3 on an island.

Ex: Statement: A says B is knave & B says that A & C have same type then prove that C is knave.

o Case : 1

If A is knight \Rightarrow B is knave

\because A & C are same \Rightarrow C is knight but C & A are diff.

\Rightarrow C = knave

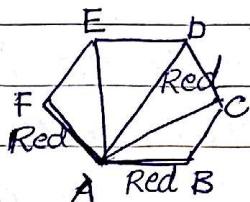
o Case : 2

If A is knave \Rightarrow B is knight and C = knave ✓

Q Suppose we are joining every pair of vertices of hexagon by line segment either red/blue. Then prove each these line segment will form monochromatic Δ .

SOLⁿ:

$m = 5 = \text{no. of line segments to other vertex}$
 $n = \text{no. of colors} = 2$



$$\therefore \lfloor \frac{(m-1)/n}{n} \rfloor + 1 = \lfloor \frac{5-1}{2} \rfloor + 1 = 3$$

Let AF, AD and AB be Red.

Now, considering $\triangle ADF$,

Case: 1: FD is Red $\Rightarrow \triangle ADF$ is monochromatic

Case: 2: FD is blue, then if BD is Red $\Rightarrow \triangle ABD$ = mono.
 BD is blue $\Rightarrow \triangle ABD$ = non-mono.

Now,

If BF is Red, $\triangle AFB$ is mono.

If BF is blue, $\triangle BDF$ is mono.

* Logic:

It specifies the meaning of mathematical statement

* Proposition: Declarative statement.

Ex: Delhi is capital of India ✓

$$2+2=5 \quad \checkmark$$

$$2x1=4 \quad \times$$

* Division Algorithm:

$a \in \mathbb{Z}, b \in \mathbb{Z}^+, q, r \in \mathbb{Z}$ s.t. $0 \leq r < b$ and $a = bq + r$

Q What is the remainder & the quotient when -23 is divided by 5?

Careful: quotient = -5, rem. = 2 \rightarrow its true ✓
 " = -4, rem. = -3 \rightarrow its false x

Q Let b is an integer such that $b \geq 2$ are randomly selected, then prove that the difference of some of the them must be divisible by 3.

$$\text{SOL: } a = bq + r ; \quad 0 \leq r < b$$

b is divisible by a if $r=0$

$$a_i = bq + r_i ; \quad 0 \leq r_i < b \quad \forall i = 1, 2, \dots, (b+1)$$

$$\downarrow \\ a_1, a_2, \dots, a_{b+1}$$

How many $(b+1)$ remainder — 'b'

\therefore applying pigeon-hole principle.

#

① If a & b are integers and
if $(a \text{ divides } b) \& (b \text{ divides } a) \Rightarrow a=b$

② If $a, b, c \in \mathbb{Z}, \quad a/b, b/c \Rightarrow a/c$

③ If $a/b \Rightarrow a/bc$

Euclidean Algorithm (Division method)

④ Relatively prime if $\text{HCF} = 1$

* mathematical Induction:

$$P(n) \quad ; \quad n \geq n_0$$

(1) Basis step: $P(n_0)$ should be true/satisfy

(2) Inductive step: If $P(k)$ is true for any $k \geq n_0$
 $\Rightarrow P(k+1)$ must be true

$\therefore P(n)$ is always true.

Q Prove that: $2n^3 + 3n^2 + n$ is divisible by 6

SOLⁿ: Let $P(n) = 2n^3 + 3n^2 + n$ is divisible by 6 ; $n \geq 1$

(1) $P(1) = 2(1)^3 + 3(1)^2 + 1 = 6$ which is divisible by 6 ✓

(2) $P(k) = 2k^3 + 3k^2 + k$ is if divisible by 6

$\therefore 2k^3 + 3k^2 + k = 6m$; m is any integer

$$P(k+1) = 2(k+1)^3 + 3(k+1)^2 + (k+1)$$

$$= (k+1)[2k^2 + 2 + 4k + 3k + 3 + 1] = (k+1)[2k^2 + 7k + 6]$$

$$= 6[m + (k+1)^2] \checkmark$$

Q A set containing n elements has 2^n subsets. Prove.

SOLⁿ: (1) Basis: If $n=0$, that \emptyset is subset.

$$\therefore 2^0 = 1 \text{ subset} = \{\emptyset\} \checkmark$$

(2) Inductive: Let $|A|=k$ has 2^k subsets, then

$|B|=k+1$ and constructing, $B = A \cup \{x\}$

$\because A \subseteq B \Rightarrow$ every subset of A is a subset of B .

(i) \therefore , say A_1, A_2, \dots, A_k are 2^k subsets of $A \forall i \in B$

(ii) $\therefore A_1 \cup \{x\}, A_2 \cup \{x\}, \dots, A_k \cup \{x\} = 2^{k+1}$

\therefore (i) and (ii) are all subsets of B

$$\therefore \text{Total subsets of } B = (2^k + 2^k) = 2 \cdot 2^k = 2^{k+1} \checkmark$$

* Bernoulli's Inequality:

$$(1 + xc)^n \geq (1 + nxc) \quad ; \quad xc > -1$$

$P(n)$: $n \geq n_0$

- Basis Step: To check the validity of $P(n_0)$
(truthfulness of $P(n_0)$)

- Inductive step: for $k \geq n_0$
Assumptions are $P(n_0)$, $P(n_0+1)$, ..., $P(k)$ all are true
 $\Rightarrow P(k+1)$ is also true.

- Q Every +ve integer ≥ 2 is either a prime or can be represented as a product of primes

Sotn: $P(n)$ = Every +ve $n \geq 2$ is either a prime / product of primes
Basis step: $P(n=n_0=2)$: 2 is prime, $P(n_0)$ is true
 $\because 2$ is prime, $P(n_0)$ is true

Inductive step: any $k \geq 2$

Assumptions: $P(2)$, $P(3)$, ..., $P(k)$ all are true

\therefore We will prove $P(k+1)$: $k+1$ is

Suppose, $k+1$ = prime — case -①

$\Rightarrow P(k+1)$ is true

Suppose $k+1$ = not prime — case -②

$\therefore (k+1) = xc$ $\therefore 2 \leq x, y < (k+1)$

$2 \leq x, y < \frac{(k+1)}{2} < (k+1)$

x, y = prime / prod. of prime

$\therefore P(x), P(y)$ are true

Every nonempty set of non-negative integers has a least element.

Program: start algo

2. product = 0,

3. i = 0,

4. while(i < x)

5. { product = product_k + y;

6. $i_{k+1} = i_k + 1$; }

7. print product

8. terminate

Product_n is the value of product after n^{th} iteration.
 i_n is value of i after n^{th} iteration.

- mathematical Inducⁿ:

$P(n)$: $\text{product}_n = i_n \cdot y$, $0 \leq n < x$

Basic step : $p(0)$

$\text{product}_0 = 0$, $i_0 = 0$

$LHS = RHS \Rightarrow P(0)$ is true

Inductive step: If $p(k)$ is true ; $k \geq 0$

$\Rightarrow p(k)$: $\text{Product}_k = i_k \cdot y$ — (A)

$\text{product}_{k+1} = \text{product}_k + y$ — from line(5)

$= i_k \cdot y + y = (i_k + 1) \cdot y$ — (A)

$= i_{k+1} \cdot y$ — from line(6)

* Congruence:

$a, b \in \mathbb{Z}, m \in \mathbb{N}$, then

$$a \equiv b \pmod{m}$$

$$\text{if } m \mid (a-b)$$

Pf: From division algorithm, $\Rightarrow a = mq_0 + b$

o properties of congruency:

- ① Suppose $a \equiv b \pmod{m}$ & $c \equiv d \pmod{m}$ then
 $(a+c) \equiv (b+d) \pmod{m}$?
 $ac \equiv bd \pmod{m}$?

* Application in computer science:

→ Pseudo-random numbers

Linear congruence method

you input 4 int: $2 \leq a < m$

$$0 \leq c < m; 0 \leq x_0 < m$$

$$x_n = (ax_0 + c) \pmod{m}$$

$$x_1 = (ax_0 + c) \pmod{m}$$

$$x_2 = (ax_1 + c) \pmod{m}$$

Find random no.'s bet" 0 & 1?

* Recursion:

$a \in W$

$$X = \{a, a+1, a+2, \dots\}$$

- (i) Basic clause: Some initial funⁿ values should be defined.
- (ii) Recursive clause: $f(n)$ should be defined for k previous values from $f(n-1)$ to $f(n-k)$.
- (iii) Termination clause:

Ex: No. of handshakes that can be done in a party with n guests (Let this recursive f^n be $h(n)$)?

Soln: If only 1 person is in party: $h(1) = 0$ handshakes similarly for n persons : $h(n) = (n-1)$ " + only pr. "

$$\therefore h(n) = \begin{cases} h(n-1) + (n-1) & ; n \geq 2 \\ 0 & ; n=1 \end{cases}$$

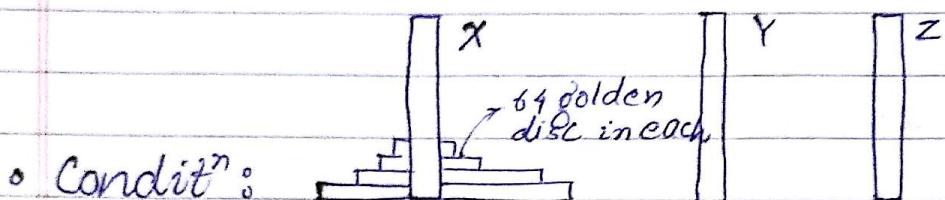
Ex: factorial of no.:

$$\text{Fact}(n) = \begin{cases} 1 & ; n=0 \\ n \text{fact}(n-1) & ; n \geq 0, n \in \mathbb{N} \end{cases}$$

Q Mr. X deposits ₹ 1000 at a bank at an interest rate 8% which is compounded annually. Define recursively the compound amount: $a(n)$, he will have in his account at the end of n years.

* Tower of Brahma

Date / /
Page



Condition:

- (i) Only 1 disc can be moved at a time.
- (ii) No disc can be placed on the top of smaller disc when transferring.

$h(n)$ = no* of moves in transferring disc from 1 peg to another using 1 with the help of n auxiliary peg.

Ex: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$; a_0, a_1, \dots, a_{k-1}
 Its characteristic eqⁿ:

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \dots - c_k = 0$$

- If α is one of the roots of characteristic eqⁿ of multiplicity m

→ Basic solⁿ are:

$$\alpha^n, n\alpha^n, n^2\alpha^n, \dots, n^{m-1}\alpha^n$$

General solⁿ: $A\alpha^n + B \cdot n\alpha^n + \dots + Cn^{m-1}\alpha^n$

Q $a_n = 5a_{n-1} - 6a_{n-2}$; $a_0 = 4, a_1 = 7$

Solⁿ: $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Characteristic eqⁿ: $x^2 - 5x + 6 = 0$

$$= x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$$

All roots have multiplicity $m = 1$

∴ Basic solⁿ: $(2)^n$ & $(3)^n$

General solⁿ: $A(2)^n + B(3)^n = a_n$

∴ $a_n = A \cdot 2^n + B \cdot 3^n$

Now to find A, B . use initial condⁿ:

$$a_0 = 4 = A \cdot 2^0 + B \cdot 3^0 \Rightarrow A + B = 4 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow A = 5$$

$$a_1 = 7 = A \cdot 2^1 + B \cdot 3^1 \Rightarrow 2A + 3B = 7 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow B = -1$$

Hence: $a_n = 5 \cdot 2^n - 3^n$

Q $a_n = 7a_{n-1} - 13a_{n-2} - 3a_{n-3} + 18a_{n-4}$

where, $a_0 = 5, a_1 = 1, a_2 = 6, a_3 = -21$

Solⁿ: It is linear & homogenous of order 4 ✓

∴ Its characteristic eqⁿ:

$$x^4 - 7x^3 + 13x^2 + 3x - 18 = 0 \quad (1)$$

factors: $1, 2, 3, 6, 9, 18$ = possible roots

∴ roots: $x = -1, 2, 3, 3$

$\downarrow \downarrow \quad \downarrow \quad \downarrow$ multiplicity = 2

multiplicity = 1

∴ $(-1)^n, 2^n, 3^n, n3^n$ Basic solⁿ

General solⁿ: $A(-1)^n + B \cdot 2^n + C \cdot 3^n + nD \cdot 3^n = a_n$

Now, using a_0, a_1, a_2, a_3 find A, B, C, D

* Linear Non-Homogeneous Recurrence Relation with constant coefficient of (any order k):

(A) $a_n = c_{k,1}a_{n-1} + c_{k,2}a_{n-2} + \dots + c_{k,k}a_{n-k} + f(n) \rightarrow$ Non-homogeneous

Associated LHRRWCC is $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ (B)

Suppose the solⁿ of B is a_n^R

We will choose another particular solⁿ of given eqⁿ (A)

$$\therefore \text{general sol}^n: a_n = a_n^h + a_n^P$$

Suppose: $f(n) = [b_k n^k + b_{k-1} n^{k-1} + \dots + b_1 n + b_0] \alpha^n$, then

- Rule: 1: If α is not the root of A LHRRWCC
(not characteristic root)

then, $a_n^P = [c_k n^k + c_{k-1} n^{k-1} + \dots + c_1 n + c_0] \alpha^n$

- Rule: 2: If α is the root of A LHRRWCC
[characteristic multiplicity m]

then, $a_n^P = n^m [d_k n^k + d_{k-1} n^{k-1} + \dots + d_1 n + d_0] \alpha^n$