The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-III
Practice Problems Set #3

1. Evaluate $\int_{(0,1)}^{(2,5)}(3x+y)dx+(2y-x)dy$ along:
   (a) the curve $y=x^2+1$;
   (b) the straight line joining (0, 1) to (2, 5);
   (c) the straight line from (0, 1) to (0, 5), and then from (0, 5) to (2, 5).

Ans. (a) $\frac{88}{3}$ (b) 32 (c) 40

2. Evaluate $\int_C(x^2-iy^2)dz$ along
   (a) the parabola $y=2x^2$; from (1, 1) to (2, 8);
   (b) the straight line from (1, 1) to (2, 8);
   (c) the straight line from (1, 1) to (1, 8), and then from (1, 8) to (2, 8).

Ans. (a) $\frac{511}{3}-\frac{49}{5}i$ (b) $\frac{518}{3}-8i$ (c) $\frac{518}{3}-57i$

3. Evaluate $\oint_C \frac{2z+3}{z}dz$ where C is
   (a) upper half of the circle $|z|=2$ in clockwise direction,
   (b) upper half of the circle $|z|=2$ in anti-clockwise direction,
   (c) lower half of the circle $|z|=2$ in clockwise direction,
   (d) lower half of the circle $|z|=2$ in anti-clockwise direction,
   (e) The circle $|z|=2$ in anti-clockwise direction,
   (f) The circle $|z|=2$ in clockwise direction.

Ans. $z=2e^{i\theta}$ and $dz=2ie^{i\theta}d\theta$. Hence,
$I=i\int_C(4e^{i\theta}+3)d\theta=4e^{i\theta}+3i\theta$
(a) $[4e^{i\theta}+3i\theta]_{\pi}^{0}$ = 8 - 3$i\pi$
(b) $[4e^{i\theta}+3i\theta]_{0}^{\pi}$ = -8 + 3$i\pi$
(c) $[4e^{i\theta}+3i\theta]_{-\pi}^{0}$ = -8 - 3$i\pi$
(d) $[4e^{i\theta}+3i\theta]_{-\pi}^{0}$ = 8 + 3$i\pi$
(e) $[4e^{i\theta}+3i\theta]_{0}^{2\pi}=6i\pi$
(f) $[4e^{i\theta}+3i\theta]_{2\pi}^{0}=-6i\pi$

4. Evaluate $\int_C(z^2-z+1)dz$ along
   (a) the parabola $y=2x^2$; from (1, 2) to (2, 8);
   (b) the straight line from (1, 2) to (2, 8);
   (c) the straight line from (1, 2) to (1, 8), and then from (1, 8) to (2, 8).

Ans. In all cases ans will be same $-\frac{-553}{6}-246i$ as (z^2-z+1) is analytic function.

5. Evaluate $\oint_C \bar{z}^2dz$ around the circle (a) $|z|=1$ and (b) $|z-1|=1$.

Ans. (a) 0; (b) $4i\pi$

6. Evaluate $\oint_C(z^4+3)dz$
   (a) around the circle $|z|=1$,
   (b) around the square with vertices at (0, 0), (2, 0), (2, 1), (0, 2).

Ans. 0 in all cases as we are integrating analytic function in closed curve (Using Cauchy's Theorem).

7. Evaluate $\oint_C \frac{(z+5)}{(z-5)(z-4)^3} dz$
   (a) where C is the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$; (b) where C is the circle $|z+4| = 3$.

Ans. 0 in all cases as $f(z)$ is analytic inside the given closed curve (Using Cauchy's Theorem).

8. Find the length of the curve $C : z = (1-i)t^2$, $-1 \le t \le 1$.

Ans $2\sqrt{2}$ using the formula for length of curve $L = \int_a^b |z'(t)|dt$

9. Use the ML-inequality to prove

   (a) $\left| \int_\gamma \frac{1}{1+z^2} dz \right| \le \frac{\pi}{3}$, γ is the arc of $|z| = 2$ from 2 to $2i$.

   (b) $\left| \int_{|z|=R} \frac{\text{Log}z}{z^2} dz \right| \le 2\pi(\frac{\pi + \ln R}{R})$, $\quad R > 1$.

10. Find the upper bound for the absolute value of the integral $I = \int_C e^z dz$, where C is the line segment joining the points (0, 0) and $(1, 2\sqrt{2})$.

Ans $I \le ML = 3e$.

11. Find the upper bound for the absolute value of the integral $I = \int_C e^{\bar{z}^2} dz$, $C : |Z| = 1$, where C is traversed in the anti-clockwise direction.

Ans $I \le ML = 2\pi e$.

12. Determine the nature of all singularities of the following functions f(z).
   (a) $\frac{z}{\sin z}$
   (b) $\cos \frac{1}{z}$
   (c) $\frac{1}{\sin 1/z}$
   (d) $\frac{5}{z^3 \sin z}$
   (e) $\frac{z}{e^z - 1}$

Ans (a) Removable singularity at $z = 0$ as the limit exist.
   (b) z = 0 is the only singularity. It is an essential singularity.
   (c) Has singularities at all the points where $\sin 1/z = 0$, i.e., $z = \frac{1}{n\pi}$, $n = \pm1, \pm2, \ldots$ are isolated singularities and $z = 0$ is is an non-isolated singularity.
   (d) The singularities are $z = 0$ and $z = n\pi, n = \pm1, \pm2, \ldots$. The singularity at $z = 0$ is a pole of order 4.
   (e) Removable singularity at $z = 0$ as the limit exist.

13. Apply the Cauchy-Goursat theorem to evaluate $\oint_C f(z)dz$, where the contour C is the unit circle $|z| = 1$ in either direction and

(a) $f(z) = \frac{z^2}{(z-3)(z-5)}$

(b) $f(z) = \tan z$

(c) $f(z) = Log(z+2)$

(d) $f(z) = z^3 + 2z + 1 + 2i$

(e) $f(z) = \sin(2z+1)$

(f) $f(z) = 2z + e^{(z+i)}$

Ans 0 for all (By Cauchy's Theorem)

14. Let γ denote the positively oriented boundary of the square whose sides lie on the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the following integrals:

$$(a) \int_\gamma \frac{e^{-z}}{z - i\frac{\pi}{2}} dz \quad (b) \int_\gamma \frac{\cos z}{z(z^2+8)} dz \quad (c) \int_\gamma \frac{z}{2z+1} dz \quad (d) \int_\gamma \frac{\cosh z}{z^4} dz$$

Ans: (a) $2\pi i e^{\frac{-\pi i}{2}} = 2\pi$ (b) $2\pi i \frac{\cos 0}{(0+8)} = \frac{\pi i}{4}$ (c) $2\pi i \frac{-\frac{1}{2}}{2} = -\frac{\pi i}{2}$ (d) $\frac{2\pi i}{3!} f'''(0) = 0,\ where\ f(z) = \cosh z$

15. Evaluate the integral $\int_C (z - z_0)^n dz$, $n = 0, \pm 1, \pm 2, \ldots$, where C denote the positively oriented circle $|z - z_0| = R$.

Ans: $\mathrm{I} = \begin{cases} 2\pi i, & \text{if } z - 1 \\ 0, & \text{if } z \neq -1. \end{cases}$

16. Evaluate $\frac{1}{2\pi i} \int_C \frac{ze^z}{(z+1)^3} dz$, where C is a positively oriented simple closed curve enclosing $z = -1$.

Ans: $\frac{1}{2!} f''(-1) = \frac{1}{2e},\ f(z) = ze^z$

17. Evaluate $\int_C \frac{tan\pi z}{z - i} dz$, where C is a positively oriented triangle with vertices 0, $\pm 1 + 2i$.

Ans: $2\pi i \tan \pi i = -2\pi \tanh \pi$ as the othr singular points $\pi z = n\pi + \frac{\pi}{2}$ i.e. $z = n + \frac{1}{2},\ n \in Z$ of $\frac{tan\pi z}{z-i}$ are outside of C

18. Evaluate $\frac{1}{2\pi i} \oint_C \frac{3z-1}{(z^3+2z)} dz$, where C is a positively oriented unit circle enclosing $z = -2$.