# Lecture 1: Functions of several variables

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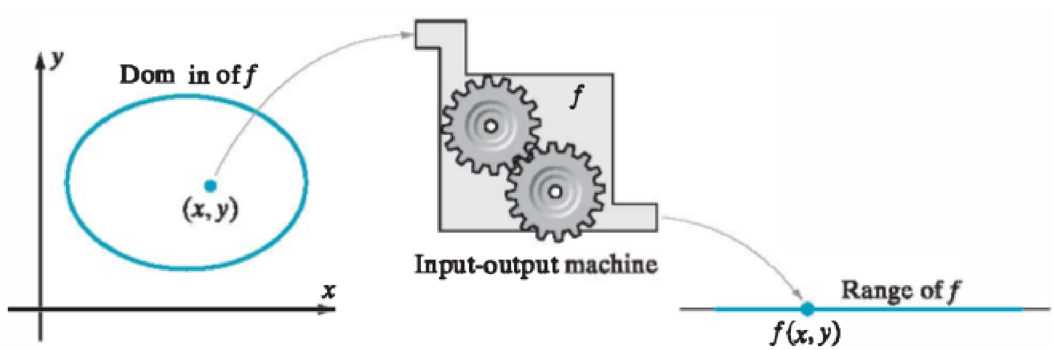
So far we have studied real-valued functions of one real variable, that is, functions $f : D \to \mathbb{R}$, where $D$ is a subset of the set $\mathbb{R}$ of all real numbers. We have seen basic properties of real numbers and functions of one real variable.

Now we begin the study of real-valued functions of two or more real variables. Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and its height, so it is a function $V(r, h)$ of two variables $r$ and $h$. The output of a factory depends on the amount of capital that is allocated to labor and the amount that is allocated to equipment. The air pressure at a point in the atmosphere depends on the altitude of the point and also on the temperature at the point.

**Definition 1.1** *Suppose that $D$ is a set of ordered pairs of real numbers and that $\mathbb{R}$ is the set of real numbers. We say that $f$ is a (real-valued or scalar-valued) function of two variables with domain $D$ and co-domain $\mathbb{R}$ if, for every ordered pair $(x, y)$ in $D$, there is associated a unique real number in $\mathbb{R}$; we denote this number by $f(x, y)$. The number $f(x, y)$ is said to be the is image of the point $(x, y)$ under $f$. We also say that $f$ is evaluated at $(x, y)$, or that $f$ maps $(x, y)$ to the value $f(x, y)$. The notation $(x, y) \mapsto f(x, y)$ is often used; the arrow-like symbol is read as "maps to". A function $F$ of three variables is similarly defined: The only difference is that the domain of $F$ is a set of ordered triples. The notation $(x, y, z) \mapsto F(x, y, z)$ is used for a function $F$ of three variables.*

It is convenient to think of a function as an input-output machine. The domain is thought of as the set of input values. Evaluation is the process of getting a unique output value from an input value that is fed into the machine.



**Example 1.2** *Suppose a rectangular box has length $l$, width $w$, and hight $h$. Then surface area $A(l,h,w) = 2lw+2lh+2wh$ and volume $V(l,w,h) = lwh$ are functions of three variables.*

Sometimes only a defining formula is given for function without mentioning the domain, in this case we take largest possible set as domain of the function.

**Example 1.3** *Discuss the largest possible domain of the functions $f(x,y) = x^2+y^2$, $g(x,y) = \dfrac{1}{x^2+y^2}$ and $h(x,y,z) = \dfrac{z}{x^2+y^2}$.*

**Solution:** The largest possible domain of the function is the set all of points for which the expression makes sense and evaluates to a real number. Hence for $f$ it is $\mathbb{R}^2$. for $g$ it is punctured plane. For $h$, it is the entire space minus the $z$-axis (only on that both $x$ and $y$ are zero). ■

## Sequences in $\mathbb{R}^2$

You may recall that sequences makes life much easier if we want to prove continuity of a real-valued function of a single variable. What was the key? The key is, work hard on proving some beautiful theorems on real sequences and use them to conclude the continuity and limit of a real-valued function of one variable. We adopt the same path in order to define continuity of a real-valued functions of two variable. Sequences in $\mathbb{R}^2$ enjoys many properties similar to real-sequences but here our sole purpose of considering sequences is to define the notion of continuity and limit of a real-valued functions of two variable, hence we defer to discuss the sequences in $\mathbb{R}^2$ in detail as we did for real sequences.

A sequence in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. Typically, a sequence in $\mathbb{R}^2$ is denoted by $((x_n, y_n))$.

**Definition 1.4** *We say $((x_n, y_n))$ converges to $(x_0, y_0)$ if $x_n \to x_0$ and $y_n \to y_0$.*

**Example 1.5**

1. *Sequence $\left(\left(\frac{1}{n}, -\frac{1}{n}\right)\right)$. converge to $(0,0)$.*
2. *Sequence $\left(\left(\frac{1}{n}, (-1)^n\right)\right)$ diverges. Since $((-1)^n)$ is divergent.*

## Continuity

**Definition 1.6** *Let $D$ be a subset of $\mathbb{R}^2$ and let $(x_0, y_0)$ be any point in $D$. A function $f : D \to \mathbb{R}$ is said to be continuous at $(x_0, y_0)$ if for every sequence $(x_n, y_n)$ in $D$ such* *that* $(x_n, y_n) \to (x_0, y_0)$, *we have* $f(x_n, y_n) \to f(x_0, y_0)$. *When* $f$ *is continuous at every* $(x_0, y_0) \in D$, *we say that* $f$ *is continuous on* $D$.

**Example 1.7** *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

*Discuss the continuity of* $f$ *at* $(0,0)$.

**Solution:** Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ which converges to $(0,0)$, that is, $x_n \to 0$ and $y_n \to 0$. Now

$$f(x_n, y_n) = \begin{cases} \dfrac{x_n^2 y_n}{x_n^2 + y_n^2} & \text{if } (x_n, y_n) \neq (0,0) \\ 0 & \text{if } (x_n, y_n) = (0,0). \end{cases}$$

Therefore

$$|f(x_n, y_n)| = \frac{x_n^2 |y_n|}{x_n^2 + y_n^2} \leq |y_n| \text{ if } (x_n, y_n) \neq (0,0)$$

For any $(x_n, y_n) \in \mathbb{R}^2$, we obtain

$$|f(x_n, y_n)| \leq |y_n|$$

$$\begin{aligned} y_n \to 0 &\implies |y_n| \to 0 \\ 0 \leq |f(x_n, y_n)| \leq |y_n| &\implies |f(x_n, y_n)| \to 0 \text{ (by Sandwhich Theorem)} \end{aligned}$$

Hence $f(x_n, y_n) \to 0 = f(0,0)$. As a result, $f$ is continuous at $(0,0)$.

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