Lecture 11: MVT & Global Extrema

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Theorem 11.1 (MVT for functions of two variables) *Let $D \subseteq \mathbb{R}^2$ be a closed disk and suppose $f : D \to \mathbb{R}$ is differentiable. Given any distinct points $A = (x_0, y_0)$ and $B = (x_1, y_1)$ in D, there is $C = (c, d)$ lying on the line segment joining A and B with $C \neq A, C \neq B$ such that*

$$f(B) - f(A) = f'(C) \cdot (B - A).$$

In other words,

$$f(x_1, y_1) - f(x_0, y_0) = (x_1 - x_0, y_1 - y_0) \cdot \nabla f(c, d) = (x_1 - x_0) f_x(c, d) + (y_1 - y_0) f_y(c, d).$$

Proof: Define $\phi : [0, 1] \to \mathbb{R}$ by

$$\phi(t) := f(x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t).$$

Since f is differentiable on D and $x(t) = x_0 + (x_1 - x_0)t, y(t) = y_0 + (y_1 - y_0)t$ are differentiable function on $\mathbb{R}$. Hence ϕ is differentiable on $[0, 1]$ (hence continuous) and by chain rule

$$\phi'(t) = \nabla f(x(t), y(t)) \cdot (x'(t), y'(t))$$

Applying MVT for ϕ, there exist $s \in (0, 1)$ such that

$$\phi(1) - \phi(0) = \phi'(s)(1 - 0).$$

That is

$$f(x_1, y_1) - f(x_0, y_0) = (x_1 - x_0) f_x(c, d) + (y_1 - y_0) f_y(c, d),$$

where $c = x_0 + (x_1 - x_0)s, d = y_0 + (y_1 - y_0)s$. ■

Corollary 11.2 *Let $D \subseteq \mathbb{R}^2$ be a closed disk and let $f : D \to \mathbb{R}$ be a differentiable function. If both f_x and f_y vanish identically on D, then f is constant on D.*

Proof: Let (x_0, y_0) be any point of D. Since $f_x = f_y = 0$ on D, by Theorem 11.1, for any $(x_1, y_1) \in D$ with $(x_1, y_1) \neq (x_0, y_0)$ we have $f(x_1, y_1) - f(x_0, y_0) = 0$, that is, $f(x_1, y_1) = f(x_0, y_0)$. Thus, f is a constant function on D. ■

Review of theory of absolute extrema for functions of one variable

Definition 11.3 *A subset D of $\mathbb{R}$ is said to be bounded if there exists $K > 0$ such that $|x| \leq K$ for all $x \in D$.*

Definition 11.4 *Let $D \subseteq \mathbb{R}$. A point $c \in D$ is called an interior point of D if there exists $r > 0$ such that symmetric neighborhood of c (i.e., $(c-r, c+r)$) is contained in D.*

Definition 11.5 *Let $D \subseteq \mathbb{R}$. We say D is open if all the points of D are interior points.*

Definition 11.6 *Let $D \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a boundary point of D if every symmetric neighborhood of c contains at least one point that lie outside of D and at least one point that lie in D.*

For example $D = (0,1) \cup (2,3)$ has four boundary points $0, 1, 2, 3$.

Remark 11.7 *A boundary point of D itself need not belong to D.*

Definition 11.8 *A subset D of $\mathbb{R}$ is said to be closed if it contains all its boundary points. A subset D of $\mathbb{R}$ is said to be open if all its points are interior points.*

For example $D = (0,1) \cup (2,3)$ is open, $D = [0,1] \cup [2,3]$ is closed and $D = [0,1] \cup (2,3)$ is neither open nor closed.

Exercise 11.9 *Determine whether the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are closed or open ?*

Recall, the following theorem you might have seen while studying single variable calculus.

Theorem 11.10 *Let $-\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$ then f is bounded. Moreover, f attains its absolute maxima and absolute minima on $[a,b]$.*

In fact, this theorem can be generalized.

Theorem 11.11 *Let D closed and bounded subset of $\mathbb{R}$. If $f : D \to \mathbb{R}$ is continuous on D then f is bounded. Moreover, f attains its absolute maxima and absolute minima on D.*

Absolute extrema of functions of two variables

Definition 11.12 *A subset D of $\mathbb{R}^2$ is said to be bounded if there exists $K > 0$ such that $|x| \le K$ for all $x \in D$.*

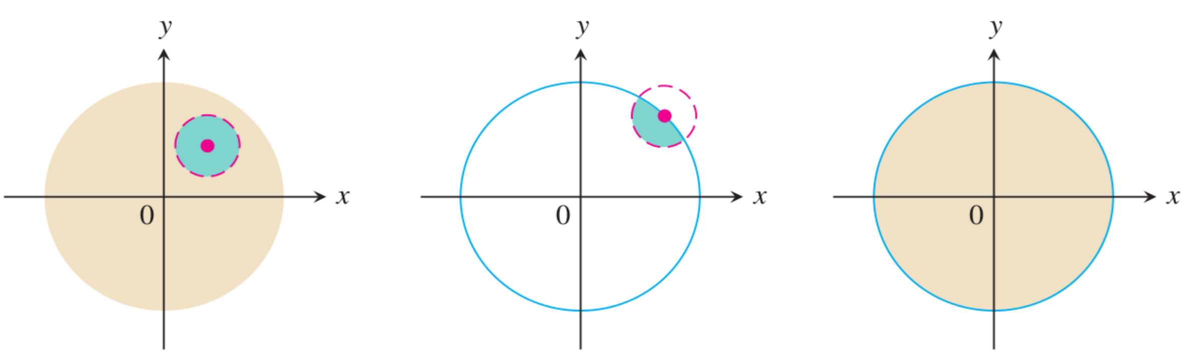
For example, if $D := \{x \in \mathbb{R}^2 : |x| < 1\}$, then D is bounded, whereas its complement $\mathbb{R}^2 \setminus D = \{x \in \mathbb{R}^2 : |x| \ge 1\}$ is not bounded.

Definition 11.13 *Let $R \subseteq \mathbb{R}^2$. We say a point $(x_0, y_0) \in R$ is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.*

A point $(x_0, y_0) \in \mathbb{R}^2$ is a boundary point of R if every disk centered at (x_0, y_0) contains at least one point that lie outside of R and at least one point that lie in R.

Remark 11.14 *A boundary point of D itself need not belong to D.*

Definition 11.15 *Let $D \subseteq \mathbb{R}^2$. We say D is an open subset of $\mathbb{R}^2$ if every point of D is an interior point. We say D is a closed subset of $\mathbb{R}^2$ if every boundary point of D belongs to D.*



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)

$\{(x, y) \mid x^2 + y^2 \le 1\}$
Closed unit disk.
Contains all
boundary points.

As with a half-open interval of real numbers $[a, b)$, some sets in the plane are neither open nor closed. If you start with the open unit disk as in figure above and add to it some of but not all its boundary points, the resulting set is neither open nor closed. The boundary points that are there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

Remark 11.16 *Note that if $(x_0, y_0) \in D$, then (x_0, y_0) is either an interior point of D or a boundary point of D.*

Theorem 11.17 *Let $D \subseteq \mathbb{R}^2$ be closed and bounded, and $f : D \to \mathbb{R}$ is continuous. Then f is bounded absolute extrema for functions of one variable on D and attains its absolute minimum and the absolute maximum on D.*