# Volume Integral

$$\int_{\mathcal{V}} T \, d\tau \qquad\qquad d\tau = dx \, dy \, dz$$

Calculate the volume integral of $T = xyz^2$
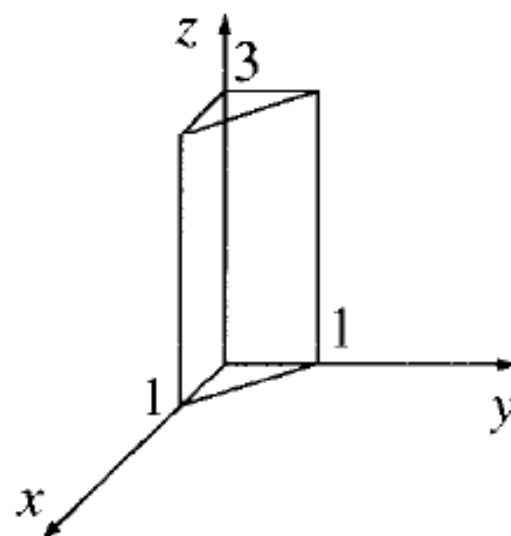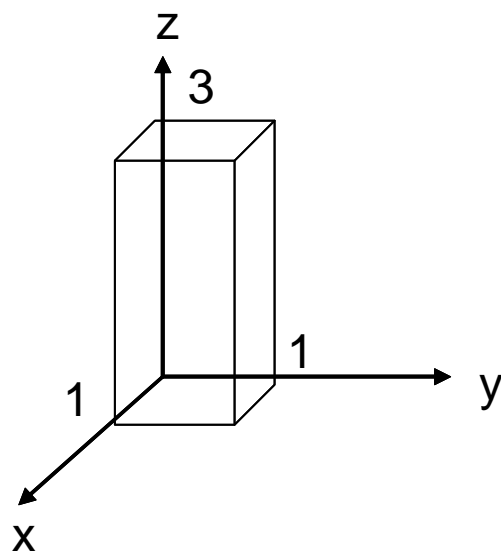


$$\int T \, d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[ \int_0^{1-y} x \, dx \right] dy \right\} dz =$$

$$\frac{1}{2} \int_0^3 z^2 \, dz \int_0^1 (1-y)^2 y \, dy = \frac{1}{2}(9)(\tfrac{1}{12}) = \frac{3}{8}.$$

# The Fundamental Theorem for Gradients

The integral of a derivative over a region is equal to the value of the function at the boundary

$$\int_{\substack{a \\ \mathcal{P}}}^{b} (\nabla T) \cdot dl = T(\mathbf{b}) - T(\mathbf{a}).$$

$$dT = (\nabla T) \cdot dl_1$$

Difference of function's value at b and a

**Corollary 1:** $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot dl$ is independent of path taken from **a** to **b**.
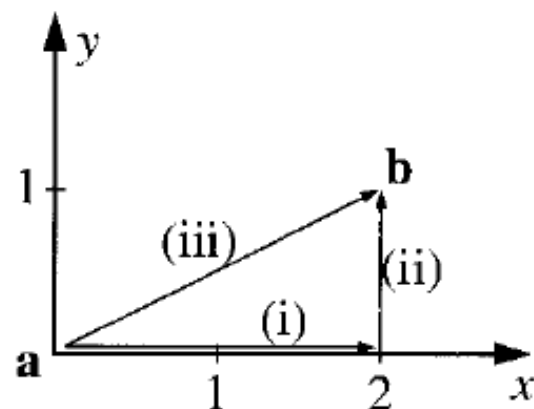
**Corollary 2:** $\oint (\nabla T) \cdot dl = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

$$\vec{\nabla} \times \vec{\nabla} \mathbf{T} = 0 \qquad\qquad \vec{\nabla} \times \vec{F} = 0 \qquad\qquad \text{F conservative field}$$

Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.
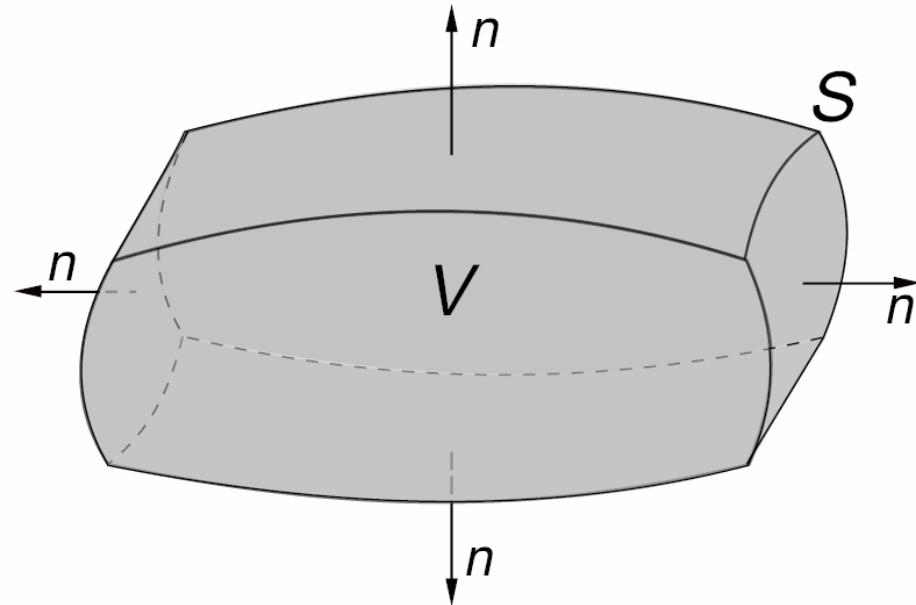
**Divergence**

$$\text{div } \vec{F} = \underset{\Delta v \to 0}{Lim} \frac{\oiint_S \vec{F} \cdot d\vec{s}}{\Delta v}$$

**Curl**

$$\underset{\Delta S \to 0}{Lim} \frac{\oint \vec{F} \cdot d\vec{l}}{\Delta S} = (curl\, F) \cdot \hat{n}$$

# The Fundamental Theorem for Divergences

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v}) \, d\tau = \oint_{S} \mathbf{v} \cdot d\mathbf{a}.$$
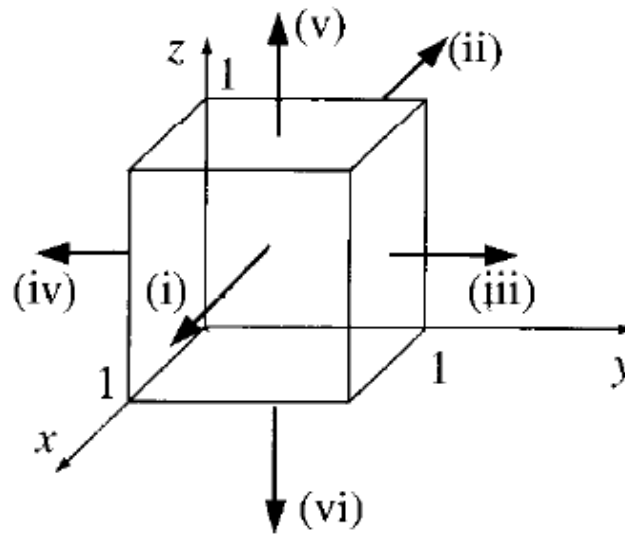
The integral of a derivative over a region is equal to the value of the function at the boundary

$$\nabla \cdot \vec{V} = \text{outflow} - \text{inflow} \longrightarrow \begin{array}{l} \text{+ve} \quad \text{(source)} \\ \text{-ve} \quad \text{(sink)} \end{array}$$

$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface})$$

Check the divergence theorem using the function $\quad \mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$



(iii) $\qquad \displaystyle\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 (2x + z^2)\, dx\, dz = \tfrac{4}{3}.$
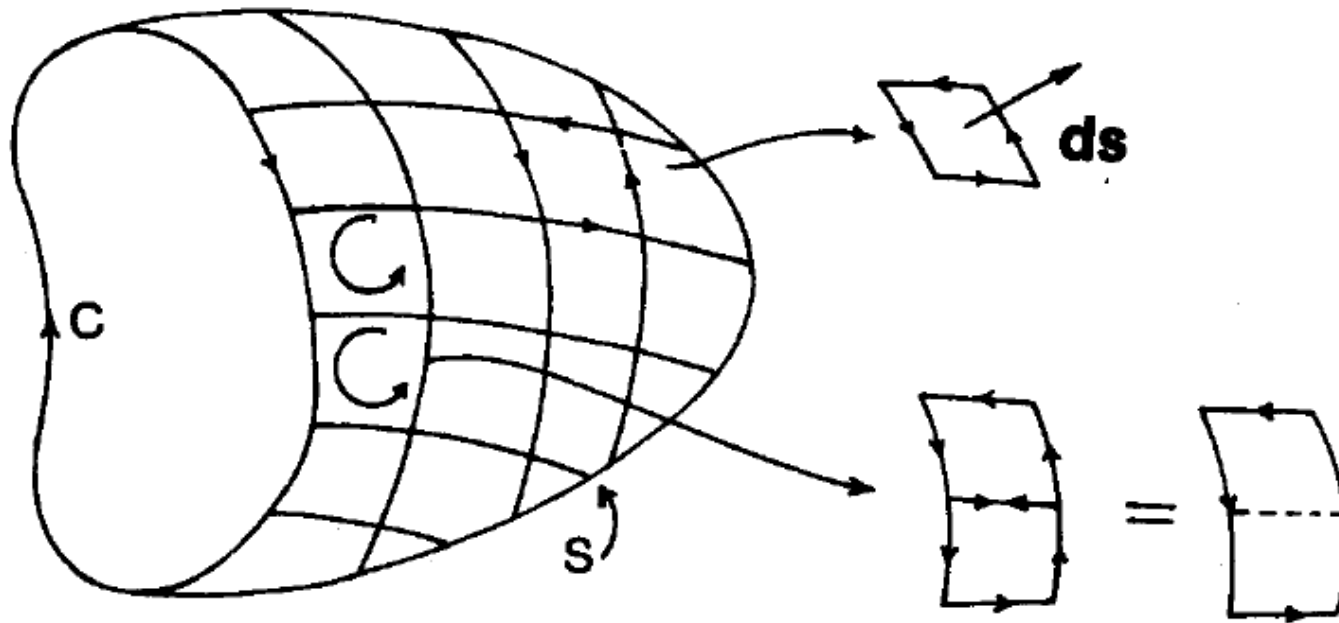
(iv) $\qquad \displaystyle\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 z^2\, dx\, dz = -\tfrac{1}{3}.$

(v) $\qquad \displaystyle\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 2y\, dx\, dy = 1.$

# The Fundamental Theorem for Curls   Stokes' theorem

$$\int_{S} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{P} \mathbf{v} \cdot d\mathbf{l}.$$

The integral of a derivative over a region is equal to the value of the function at the boundary
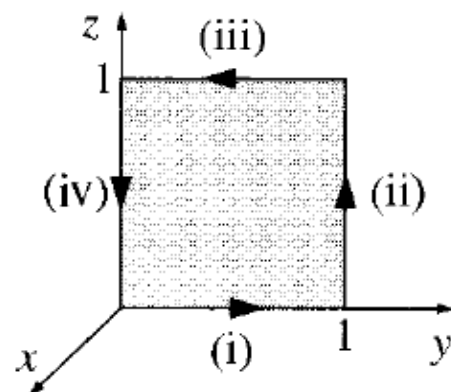


ds

C

S

rotational force field / non-conservative force field

**Corollary 1:** $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

**Corollary 2:** $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface.

Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface



$$\nabla \times \mathbf{v} = (4z^2 - 2x)\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}.$$

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\,dy\,dz = \tfrac{4}{3}.$$

(i)    $x = 0,\quad z = 0,\quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy,\quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2\,dy = 1,$

(ii)    $x = 0,\quad y = 1,\quad \mathbf{v} \cdot d\mathbf{l} = 4z^2\,dz,\quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2\,dz = \tfrac{4}{3},$

(iii)   $x = 0,\quad z = 1,\quad \mathbf{v} \cdot d\mathbf{l} = 3y^2\,dy,\quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2\,dy = -1,$

(iv)   $x = 0,\quad y = 0,\quad \mathbf{v} \cdot d\mathbf{l} = 0,\quad\quad\quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0\,dz = 0.$

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \tfrac{4}{3} - 1 + 0 = \tfrac{4}{3}$$

- **Concepts of Vector Field**

- **Gradient [converts scalar field to a vector field]**

- **Divergence [A measure of change of flux per unit volume]**

- **Curl [measure of rotational nature of a vector field]**

- **Line integration**

- **Surface integration**

- **Volume integration**

- **Fundamental theorem for Gradient**

- **Fundamental theorem for Divergence**

- **Fundamental theorem for Curl**