Lecture 14: With Constraints

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Now we discuss a method of finding the absolute extrema of $f(x, y)$ subject to the constraint $g(x, y) = 0$.

Theorem 14.1 (Lagrange Multiplier Theorem) *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f, g : D \to \mathbb{R}$ are such that the partial derivatives of f and g exist and are continuous in $B_r(x_0, y_0)$ for some $r > 0$ with $B_r(x_0, y_0) \subseteq D$. Let $C := \{(x, y) \in D : g(x, y) = 0\}$. Suppose the following three conditions are satisfied.*

1. *$(x_0, y_0) \in C$, that is, $g(x_0, y_0) = 0$,*
2. *$\nabla g(x_0, y_0) \neq (0, 0)$, and*
3. *the function f, when restricted to C, has a local extremum at (x_0, y_0).*

Then $\nabla f(x_0, y_0) = \lambda_0 \nabla g(x_0, y_0)$ for some $\lambda_0 \in \mathbb{R}$

The Lagrange Multiplier Theorem, gives us the following recipe to determine constrained extrema.

To determine the absolute extremum of a real-valued function f of two variables, subject to the constraint $g(x, y) = 0$.

Step I Solve simultaneous equations for λ and points (x, y)

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and } g(x, y) = 0.$$

Step II If it can be ensured that f does have an absolute extremum on the zero set of g (which will certainly be the case if the zero set of g is closed and bounded, and f is continuous), then check the values of f at such simultaneous solutions (x_0, y_0) of the above two equations for which $\nabla g(x_0, y_0) \neq (0, 0)$, and also at exceptional points such as points in the zero set of g at which ∇f or ∇g does not exist, or at which ∇g vanishes.

Example 14.2 *Find the maximum and the minimum of the function f given by $f(x, y) := xy$ on the unit circle.*

Solution: Following the Lagrange Multiplier Method, we let $g(x,y) := x^2 + y^2 - 1$ for $(x,y) \in \mathbb{R}^2$ and consider the equations $\nabla f = \lambda \nabla g$ and $g(x,y) = 0$, that is,

$$y = 2\lambda x, x = 2\lambda y, \text{ and } x^2 + y^2 - 1 = 0.$$

Hence $y = 2\lambda(2\lambda y) \implies 4\lambda^2 = 1$. Thus $\lambda = \pm\frac{1}{2}$, Substituting $y = \pm x$ in $x^2 + y^2 - 1 = 0$ gives us $x = \pm\frac{1}{\sqrt{2}}$. So the simultaneous solutions of the above equations are given by $(x,y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Note that $\nabla g = (2x, 2y)$ is nonzero at every solution of $g(x,y) = 0$. Also, the zero set of g, that is, the unit circle, is closed and bounded and f is continuous. Thus by the Lagrange Multiplier Theorem, the maximum of f on the unit circle is $\frac{1}{2}$, which is attained at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ while the minimum is $-\frac{1}{2}$, which is attained at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. ■

Remark 14.3 *The condition $\nabla f = \lambda \nabla g$ is not sufficient. Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x,y)$ subject to the conditions $g(x,y) = 0$ and $\nabla g \neq 0$, it does not in itself guarantee that one exists.*

Example 14.4 *Use the method of Lagrange multipliers to find a maximum value of $f(x,y) = x + y$ subject to the constraint that $xy = 16$. $\nabla f = \lambda g$ gives us $(1,1) = \lambda(y,x)$ and $g(x,y) = xy - 16 = 0$. Since $x = 0$ and $y = 0$ both are not solutions of $xy = 16$ hence $1 = \lambda x, 1 = \lambda y$ implies $\lambda \neq 0$. So put $x = y = \frac{1}{\lambda}$ in $xy = 16$ to get $\lambda = \pm\frac{1}{4}$ Hence points $(4,4)$ and $(-4,-4)$ as candidates for the location of extreme values. Also ∇g is not zero at both points. Yet the sum $x + y$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x,y) = x + y$ becomes.*

Exercise 14.5 *Find the point on the curve $(x-1)^3 = y^2$ which is closet to the origin.*

Solution: This amounts to finding the minimum of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) := x^2 + y^2$ subject to the constraint given by $g(x,y) := (x-1)^3 - y^2 = 0$.

1. $\nabla f = \lambda \nabla g \implies (2x, 2y) = \lambda(3(x-1)^2, -2y) \iff 2x = \lambda(x-1)^2, 2y = -2\lambda y$
2. $g(x,y) = 0 \implies (x-1)^3 = y^2$.
3. To find $\lambda, x, y \in \mathbb{R}$ such that

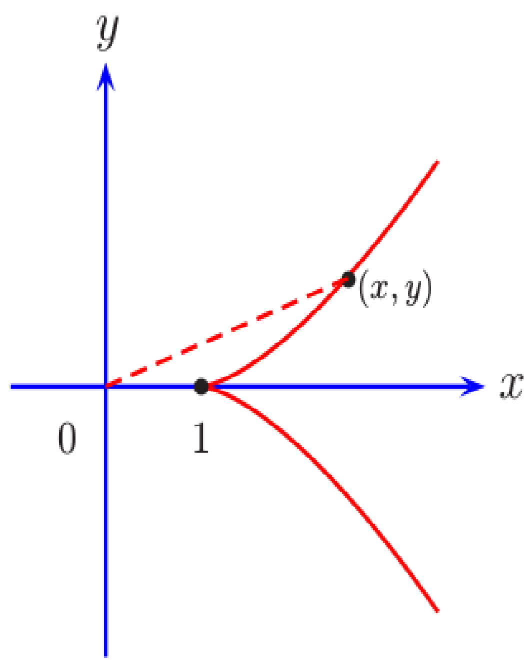
$$2x = 3\lambda(x-1)^2 \tag{14.1}$$
$$2y = -2\lambda y \tag{14.2}$$
$$y^2 = (x-1)^3 \tag{14.3}$$

If $y \neq 0$, then from (14.2), $\lambda = -1$. Using $\lambda = -1$ in (14.1), we get $2x + 3(x-1)^2 = 0$. But $2x + 3(x-1)^2 = 3x^2 + 3 - 4x = 0$ does not have any real roots since $b^2 - 4ac = (-4)^2 - 36 < 0$.

Now if $y = 0$ then from (14.3), we get $x = 1$. Substituting $x = 1$ in (14.1) we get $2 = 0$ which is absurd.

Hence there is no solution to the simultaneous equations (14.1)-(14.3).

Question: Can we say that there is no point on the curve $(x-1)^3 = y^2$ which is nearest to the origin ?
Geometrically thinking, answer would be no. In fact if you draw this curve, this is how it looks like.



it is obvious that the minimum is 1 and it is attained at $(1, 0)$.
So now we are ensured (geometrically) that f attains it's absolute minimum on zero set of g. Also f and g are everywhere differentiable hence only possibility left where f attains its absolute minimum is the point where ∇g is zero and indeed we note that $\nabla g(1, 0) = (0, 0)$.