Lecture 17: Double & Triple Integrals

November 7, 2016

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Recall that while evaluating a double integral $\iint_{[a,b]\times[c,d]} f(x,y)dA$ using Fubini's theorem, what we are actually doing: Suppose we first integrate with respect to y, that is, for each fixed $x \in [a,b]$, we compute Riemann integral $A(x) = \int_c^d f(x,y)dy$. Then we compute the Riemann integral $\int_a^b A(x)dx$.

If we first integrate with respect to x, that is, for each fixed $y \in [c,d]$, we compute Riemann integral $B(y) = \int_a^b f(x,y)dx$. Then we compute the Riemann integral $\int_c^d B(y)dy$.

Example 17.1 *Consider the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{1}{x^2} & \text{if } \; 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } \; 0 < x < y < 1 \\ 0 & \qquad \textit{otherwise} \end{cases}$$

Show that both the iterated integral exists but they are not equal.

Solution:

Claim 17.2 $\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = 1$

If $x = 0$ or $x = 1$, then

$$A(x) := \int_0^1 f(x,y)dy = \int_0^1 0dy = 0$$

whereas if $0 < x < 1$, then

$$\begin{aligned} A(x) &:= \int_0^1 f(x,y)dy = \int_0^x f(x,y)dy + \int_x^1 f(x,y)dy = \int_0^x \frac{1}{x^2}dy + \int_x^1 -\frac{1}{y^2}dy \\ &= \frac{1}{x^2}[y]_{y=0}^{y=x} + \frac{1}{y}\Big|_{y=x}^{y=1} = \frac{1}{x} + 1 - \frac{1}{x} = 1 \end{aligned}$$

Thus except at the two endpoints of $[0,1]$, A is the constant function 1 on $[0,1]$. So it follows that the function $A:[0,1]\to\mathbb{R}$ is Riemann integrable and

$$\int_0^1 A(x)dx = \int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = 1$$

Claim 17.3 $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = -1$

If $y=0$ or if $y=1$, then

$$B(y) := \int_0^1 f(x,y)dx = \int_0^1 0dx = 0$$

whereas if $0<y<1$, then

$$\begin{aligned} B(y) := \int_0^1 f(x,y)dx &= \int_0^y f(x,y)dy + \int_y^1 f(x,y)dx = \int_0^y -\frac{1}{y^2}dx + \int_y^1 \frac{1}{x^2}dx \\ &= -\frac{1}{y^2}[x]_{x=0}^{x=y} + -\frac{1}{x}\Big|_{x=y}^{x=1} = -\frac{1}{y} - 1 + \frac{1}{y} = -1 \end{aligned}$$

Thus except at the two endpoints of $[0,1]$, B is the constant function -1 on $[0,1]$. So it follows that the function $B:[0,1]\to\mathbb{R}$ is Riemann integrable and

$$\int_0^1 B(y)dx = \int_0^1 \left(\int_0^1 f(x,y)ds\right) dy = -1$$

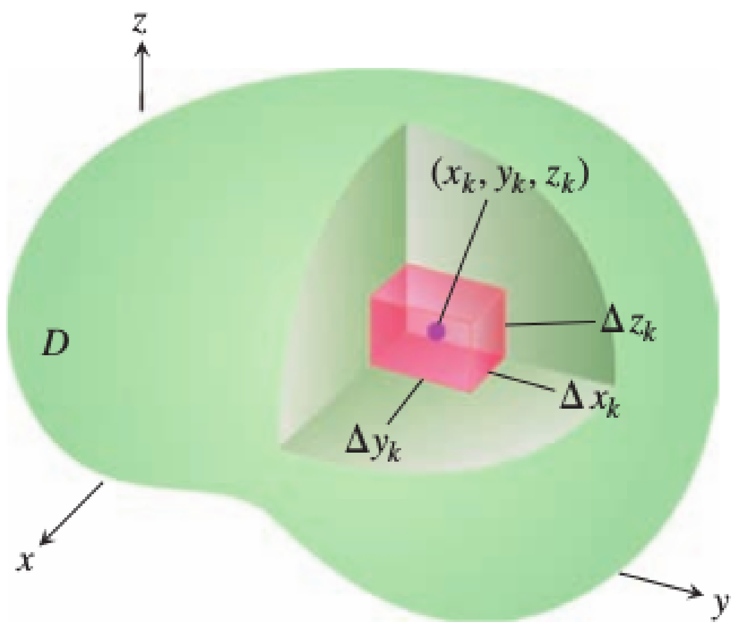
■

Question: Does this example contradicts the Fubini's Theorem?
Answer: No. The reason Fubini's Theorem does not apply here is that f is not continuous on $[0,1]\times[0,1]$.

Remark 17.4 *We said that even if function $f(x,y)$ is not continuous along a smooth curve then also double integral is defined. But here in Example 17.1, function f is not even bounded over the $[0,1]\times[0,1]$. 1 Note that if $x_n := \frac{1}{n}$ and $y_n := \frac{1}{\sqrt{n}}$ for $n\in\mathbb{N}$, then $f(x_n,y_n) = -n \to -\infty$, whereas if $x_n := \frac{1}{\sqrt{n}}$ and $y_n := \frac{1}{n}$ for $n\in\mathbb{N}$, then $f(x_n,y_n) = n\to\infty$. Thus f is neither bounded below nor bounded above. Hence the double integral of f is not defined.*

Triple Integral

If $F(x,y,z)$ is a function defined on a closed, bounded region D in space, such as the region occupied by a solid ball, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes.



We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is integrable over D. The limit is called the triple integral of f over D, written as

$$\iiint_D F(x, y, z)dV, \quad \text{or} \quad \iiint_D F(x, y, z)dxdydz$$

Volume of a Region in Space If F is the constant function whose value is 1, then the sums reduce to

$$S_n = \sum_{k=1}^{n} \Delta V_k$$

As Δx_k, Δy_k and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$V = \iiint_D dV.$$

Evaluation of the triple integral

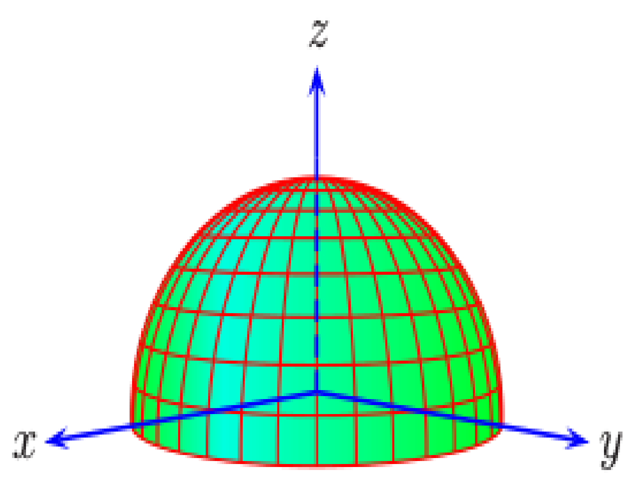
If $D = [a, b] \times [c, d] \times [p, q]$ then

$$\begin{aligned}\iiint_D F(x, y, z)dV &= \int_p^q \int_c^d \int_a^b F(x, y, z)dxdydz \\ &= \int_a^b \int_c^d \int_p^q F(x, y, z)dzdydx\end{aligned}$$

There are six possible orders altogether. Now we focus on elementary region in the space.

Example 17.5 *Find the volume of the region* $D = \{(x, y, z)|z \geq 0, x^2 + y^2 + z^2 \leq 1\}$.

Solution: It is clear that D is the region bounded by hemisphere.



So now we need to find the limit of integration. We integrate first with respect to z, then with respect to y, and finally with respect to x

1. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. In this case shadow is unit circle $x^2 + y^2 = 1$.
2. Find the z-limits of integration. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = 0$ and leaves at $z = \sqrt{1 - x^2 - y^2}$. These are the z-limits of integration.
3. Find the y-limits of integration. $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$

Hence desired volume is given by

$$\begin{aligned}
\iiint_D dV &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dy dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\ dy dx \\
&= 2 \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\ dy dx \quad \left[\int \sqrt{a^2 - t^2} dt = \frac{t}{2}\sqrt{a^2 - t^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{t}{a}\right) \right] \\
&= \int_{-1}^{1} \frac{1 - x^2}{2} \pi dx \\
&= \pi \int_0^1 (1 - x^2) dx \\
&= \frac{2}{3}\pi
\end{aligned}$$