

Lecture 17: Change of variable in double integral

November 17, 2018

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In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u , we can write the Substitution Rule (4.5.5) as

$$\boxed{1} \quad \int_a^b f(x) \, dx = \int_c^d f(g(u)) g'(u) \, du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

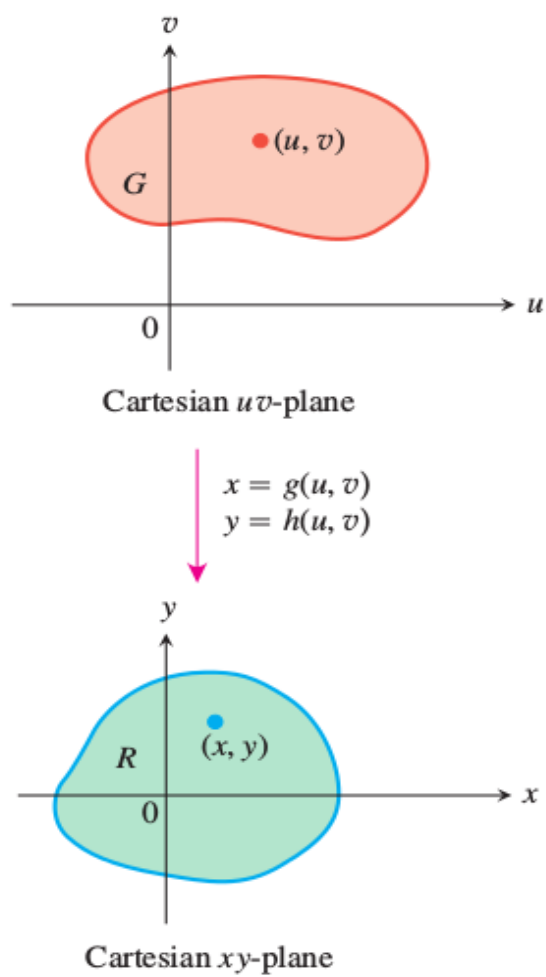
$$\boxed{2} \quad \int_a^b f(x) \, dx = \int_c^d f(x(u)) \frac{dx}{du} \, du$$

A change of variables can also be useful in double integrals.

Suppose that a region G in the uv -plane is transformed one-to-one into the region R in the xy -plane by equations of the form

$$x = g(u, v), y = h(u, v),$$

Region R is the image of G under the transformation, and G the preimage of R .



V EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \quad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S . The first side, S_1 , is given by $v = 0$ ($0 \leq u \leq 1$). (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leq x \leq 1$. Thus S_1 is mapped into the line segment from $(0, 0)$ to $(1, 0)$ in the xy -plane. The second side, S_2 , is $u = 1$ ($0 \leq v \leq 1$) and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \quad y = 2v$$

Eliminating v , we obtain

$$\boxed{4} \quad x = 1 - \frac{y^2}{4} \quad 0 \leq x \leq 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ ($0 \leq u \leq 1$), whose image is the parabolic arc

$$\boxed{5} \quad x = \frac{y^2}{4} - 1 \quad -1 \leq x \leq 0$$

Finally, S_4 is given by $u = 0$ ($0 \leq v \leq 1$) whose image is $x = -v^2$, $y = 0$, that is, $-1 \leq x \leq 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x -axis and the parabolas given by Equations 4 and 5.

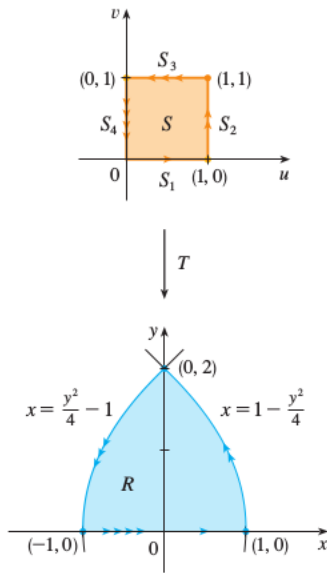


FIGURE 2

Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well.

Question: How the double integral $\iint_R f(x, y) dx dy$ gets change when we apply the change of variables $x = g(u, v)$, $y = h(u, v)$?

Before we give answer to this question we have one definition.

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}. \quad (2)$$

Theorem 17.1 (Change of variable formula for double integrals) If g , h , and f have

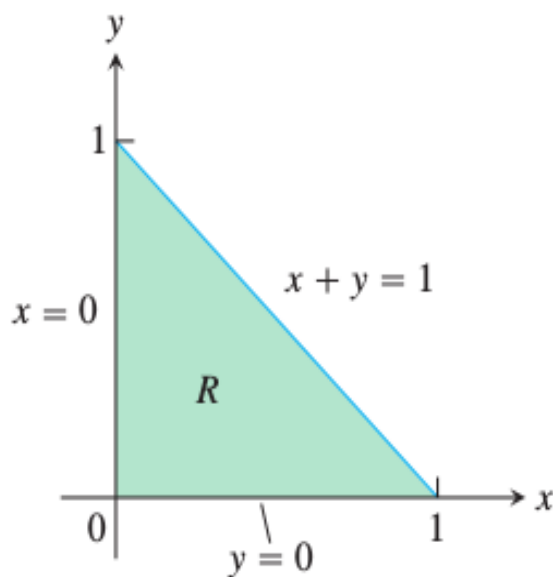
continuous partial derivatives and J is zero only at isolated points, if at all, then

$$\iint_R f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| du dv$$

EXAMPLE 3 Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx.$$

Solution: We first sketch the region R of integration in the xy -plane and identify its boundaries.



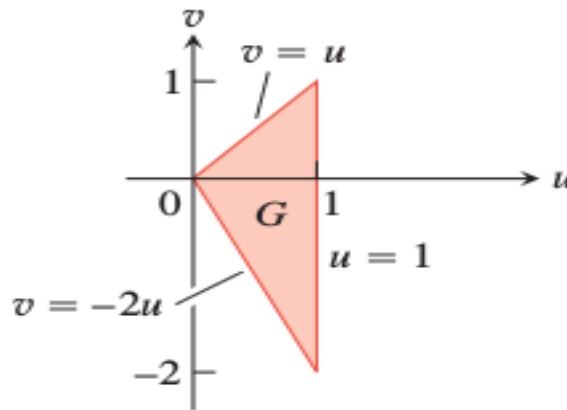
The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Solving for x and y we get

$$x = \frac{u}{3} - \frac{v}{3}, y = \frac{2u}{3} + \frac{v}{3}$$

In order to identify the image of R under the transformation we look at boundary of R .

<i>xy</i>-equations for the boundary of <i>R</i>	Corresponding <i>uv</i>-equations for the boundary of <i>G</i>	Simplified <i>uv</i>-equations
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

Hence the region G in the uv -plane is



The Jacobian of the transformation in Equations (6) is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Equation (1), we evaluate the integral:

$$\begin{aligned}
 \int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx &= \int_{u=0}^1 \int_{v=-2u}^{v=u} u^{1/2} v^2 |J(u, v)| dv du \\
 &= \int_0^1 \int_{-2u}^u u^{1/2} v^2 \left(\frac{1}{3}\right) dv du = \frac{1}{3} \int_0^1 u^{1/2} \left[\frac{1}{3} v^3 \right]_{v=-2u}^{v=u} du \\
 &= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3) du = \int_0^1 u^{7/2} du = \left[\frac{2}{9} u^{9/2} \right]_0^1 = \frac{2}{9}.
 \end{aligned}$$

Suppose that we want to evaluate a double integral $\iint_R f(x, y) dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

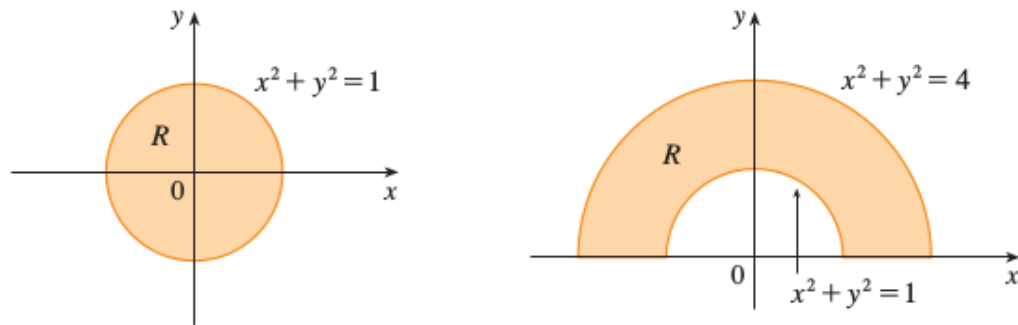


FIGURE 1

$$(a) R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

$$(b) R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

Example 17.2 For the polar coordinate transformation $x = r \cos \theta, y = r \sin \theta$, write the Cartesian integral $\iint_R f(x, y) dx dy$ as a polar integral.

Solution:

For polar coordinates, we have r and θ in place of u and v . With $x = r \cos \theta$ and $y = r \sin \theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r.$$

Since we assume $r \geq 0$ when integrating in polar coordinates, $|J(r, \theta)| = |r| = r$, so that Equation (1) gives

$$\iint_R f(x, y) dx dy = \iint_G f(r \cos \theta, r \sin \theta) r dr d\theta. \quad (3)$$

↳ (a) We define the improper integral (over the entire plane \mathbb{R}^2)

$$\begin{aligned} I &= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx \\ &= \lim_{a \rightarrow \infty} \iint_{D_a} e^{-(x^2+y^2)} dA \end{aligned}$$

where D_a is the disk with radius a and center the origin.
Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dA = \pi$$

Solution: Using polar coordinates

$$\iint_{D_a} e^{-x^2-y^2} dx dy = \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta = \pi[1 - e^{-a^2}]$$

Therefore

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \pi.$$

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