Lecture 15: More Constraints & Double Integral

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The Lagrange Multiplier Method can also be adapted to a situation in which there is more than one constraint. For example, suppose we want to find the absolute extremum of a function f of three variables x, y, z subject to the constraints given by $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Step I We seek simultaneous solutions $\lambda, \mu \in \mathbb{R}$, and $(x, y, z) \in \mathbb{R}^3$ of

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \text{ and } g(x, y, z) = 0 = h(x, y, z).$$

Step II If it can be ensured that f does have an absolute extremum on the intersection of the zero sets of g and h (which will certainly be the case if this intersection is closed and bounded, and f is continuous), then an absolute extremum is necessarily attained either at a simultaneous solution $P_0 := (x_0, y_0, z_0)$ of the above three equations for which $\nabla g(P_0)$ and $\nabla h(P_0)$ are nonzero and are not multiples of each other or at exceptional points such as those where $\nabla f, \nabla g$, or ∇h does not exist or where ∇g or ∇h vanishes or where they are multiples of each other.

Example 15.1 *Minimize the function $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.*

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2, g(x, y, z) = x + 2y + 3z - 6$ and $h(x, y, z) = x + 3y + 9z - 9$. Consider the equation

$$\begin{aligned}
\nabla f &= \lambda \nabla f + \mu \nabla g \\
\implies (2x, 2y, 2z) &= \lambda(1, 2, 3) + \mu(1, 3, 9) \\
\implies x &= \frac{\lambda + \mu}{2}, y = \frac{2\lambda + 3\mu}{2}, z = \frac{3\lambda + 9\mu}{2}
\end{aligned}$$

Substituting these in the equations $g(x, y, z) = 0$ and $h(x, y, z) = 0$, we obtain

$$\begin{aligned}
\frac{\lambda + \mu}{2} + 2\frac{2\lambda + 3\mu}{2} + 3\frac{3\lambda + 9\mu}{2} = 6 \implies 14\lambda + 34\mu = 12 \implies 7\lambda + 17\mu = 6 \\
\frac{\lambda + \mu}{2} + 3\frac{2\lambda + 3\mu}{2} + 9\frac{3\lambda + 9\mu}{2} = 9 \implies 34\lambda + 91\mu = 18
\end{aligned}$$

This gives $\lambda = \frac{240}{59}, \mu = -\frac{78}{59}$. Therefore,

$$x = \frac{81}{59}, y = \frac{123}{59}, z = \frac{9}{59}$$

is the unique simultaneous solution of $\nabla f = \lambda \nabla g + \mu \nabla h$ and $g = h = 0$. Now f is continuous, and although the set $\{(x, y, z) \in \mathbb{R}^3 | x + 3y + 9z - 9 = 0, x + 2y + 3z - 6 = 0\}$ is not bounded (it is a straight line in the space) But geometrically, we know that there will be some point on this straight line nearest to the origin, hence f attains its absolute minimum on the intersection of zero sets of g and h. Now f, g and h are polynomials so gradient exists everywhere. Also ∇g and ∇h are constant non-zero vectors and are not multiple of each other. Therefore, absolute minimum will be attained at the point $P_0 = \left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right)$ and the minimum value is

$$f(P_0) = \frac{21771}{3481} = \frac{369}{59}$$

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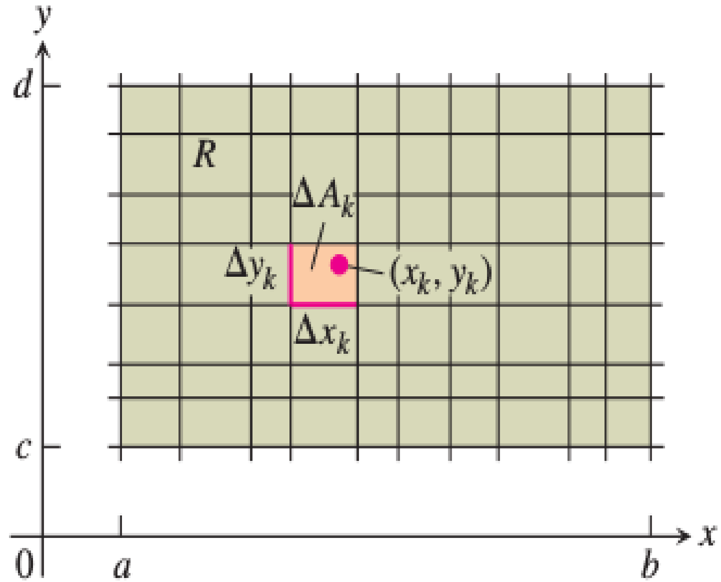
Double Integral

We consider a function $f(x, y)$ defined on a rectangular region $R = [a, b] \times [c, d]$. We subdivide R into small rectangles using lines parallel to the x- and y-axes. The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a partition of R. A sub-rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the sub-pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$, where ΔA_k is the area of the kth small rectangle. To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth sub-rectangle, multiply the value of f at that point by the area A_k, and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$



Depending on how we pick (x_k, y_k) in the kth sub-rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The norm of a partition P, written $\|P\|$, is the largest width or height of any rectangle in the partition. If $\|P\| = 0.1$ then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of P goes to zero, written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be integrable and the limit is called the double integral of f over R, written as

$$\iint_R f(x,y)dA, \quad \text{or} \quad \iint_R f(x,y)dxdy$$

It can be shown that if $f(x,y)$ is a continuous function throughout R, then f is integrable. Many discontinuous functions are also integrable, including functions that are discontinuous only on a finite number of points or smooth curves.

Calculating Double Integrals

Calculating double integral from definition is very tedious. Thanks to the following theorem which says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration.

Theorem 15.2 (Fubini's Theorem (First Form)) *If $f(x, y)$ is continuous throughout the rectangular region $R : a \leq x \leq b, c \leq y \leq d$, then*

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example 15.3 *Calculate* $\displaystyle\iint_{[0,2]\times[-1,1]} (100 - 6x^2 y)dA$

Solution: By Fubini's Theorem,

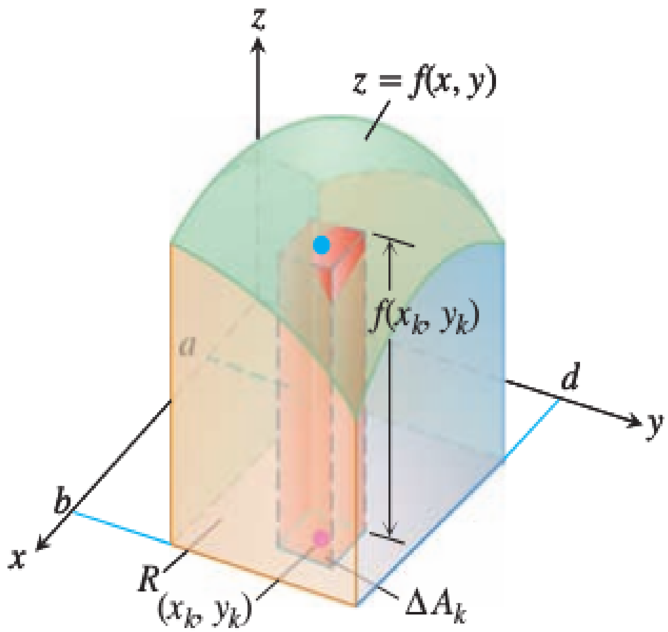
$$\begin{aligned}\iint_{[0,2]\times[-1,1]} (100 - 6x^2 y)dA &= \int_{-1}^1 \int_0^2 (100 - 6x^2 y)dxdy = \int_{-1}^1 [100x - 2x^3 y]_{x=0}^{x=2} \ dy \\ &= \int_{-1}^1 [200 - 16y]dy = [200y - 8y^2]_{-1}^1 = 400\end{aligned}$$

Reversing the order of integration gives the same answer:

$$\begin{aligned}\iint_{[0,2]\times[-1,1]} (100 - 6x^2 y)dA &= \int_0^2 \int_{-1}^1 (100 - 6x^2 y)dydx = \int_0^2 [100y - 3x^2 y^2]_{y=-1}^{y=1} \ dx \\ &= \int_{-1}^1 [100 - 3x^2 - (-100 - 3x^2)]dx \\ &= \int_{-1}^1 200 \ dx = 400\end{aligned}$$

Double Integrals as Volumes

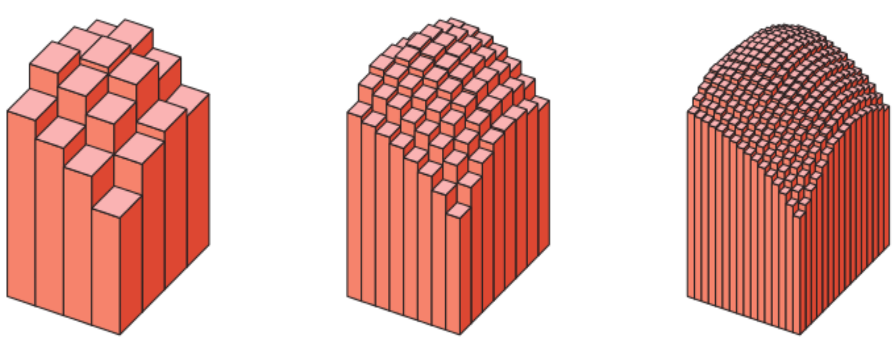
When $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$.



Each term $f(x_k, y_k)\Delta A_k$ in the sum $S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base A_k. The sum S_n thus approximates what we want to call the total volume of the solid. We define this volume to be

$$Volume = \lim_{n\to} S_n = \iint_R f(x, y)dA$$

where $\Delta A_k \to 0$ as $n \to \infty$.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Example 15.4 *Find the volume of the region bounded above by the ellipitical paraboloid* $z = 10 + x^2 + 3y^2$ *and below by the rectangle* $R : 0 \leq x \leq 1, 0 \leq y \leq 2$.

Solution: The volume is given by the double integral

$$\begin{aligned}
V &= \iint_R (10 + x^2 + 3y^2)dA \\
&= \int_0^1 \int_0^2 (10 + x^2 + 3y^2)dydx = \int_0^1 [10y + x^2y + y^3]_{y=0}^{y=2} \ dx \\
&= \int_{-1}^1 [20 + 2x^2 + 8]dx \\
&= 28x + \frac{2}{3}x^3 \Big|_{x=0}^{x=1} \\
&= \frac{86}{3}
\end{aligned}$$