Lecture 19: Triple integral & Cylindrical Coordinate System

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Example 19.1 *Let W be the region bounded by the planes $x = 0, y = 0$ and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0, y \geq 0$. Compute $\iiint_W x \; dx \; dy \; dz$.*

Solution: The shadow of the region is part of disk $x^2 + y^2 = 2$. Hence region can be described by

$$0 \leq \sqrt{2}, 0 \leq y \leq \sqrt{2 - x^2}, x^2 + y^2 \leq z \leq 2$$

$$\begin{aligned}
\iiint_W x \; dx \; dy \; dz &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \; dx \; dy \; dz \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2 - x^2 - y^2) \; dx \; dy \; dz \\
&= \int_0^{\sqrt{2}} x \left[(2 - x^2)^{\frac{3}{2}} - \frac{(2 - x^2)^{\frac{3}{2}}}{3} \right] \; dx \; dy \; dz \\
&= \frac{8\sqrt{2}}{15}
\end{aligned}$$

19.1 Change of variable in triple integrals

Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$$

Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G. If $g, h,$ and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, y, w)$ over G by the equation

$$\iiint_R F(x, y, z) dxydydz = \iiint_G H(u, v, w)|J(u, v, w)|dudvdw$$

where

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}.$$

Many times our calculation of triple integral involves a cylinder, cone, or sphere. Then we can often simplify our work by using cylindrical or spherical coordinates.

Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

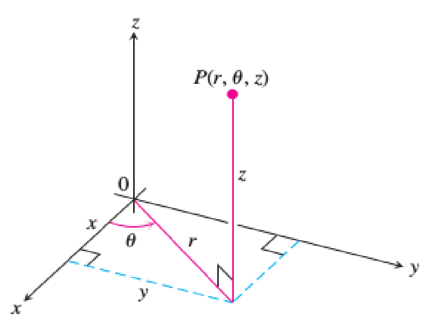


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

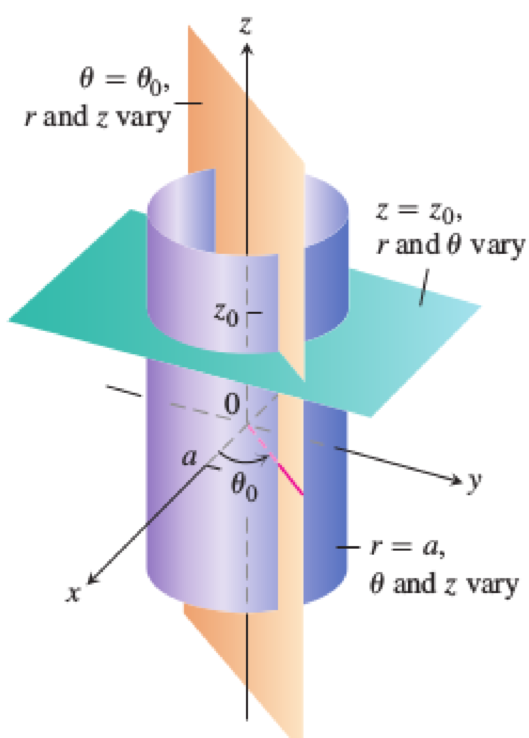
Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Figure 15.43). The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$r = 4$	Cylinder, radius 4, axis the z-axis
$\theta = \dfrac{\pi}{3}$	Plane containing the z-axis
$z = 2.$	Plane perpendicular to the z-axis



Example 19.2 *For cylindrical coordinates, The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations*

$$x = r\cos\theta, y = r\sin\theta, z = z$$

The Jacobian of the transformation is

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Hence

$$\iiint_R F(x,y,z)dxydydz = \iiint_G H(r,\theta,z)|r|drd\theta dz$$

We can drop the absolute value signs since $r \geq 0$ in polar coordinates.

Example 19.3 *Evaluate $\iiint_W (z^2x^2 + z^2y^2)dxdydz$, where W is the cylinderical region determined by $x^2 + y^2 \leq 1$ and $-1 \leq z \leq 1$*

Solution: The region W is described in cylindrical coordinates as $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1$, so $\iiint_W (z^2x^2 + z^2y^2)dxdydz = \int_{-1}^{1}\int_{0}^{2\pi}\int_{1}^{1} z^2r^2 rdrd\theta dz = \frac{\pi}{3}$. ■