Lecture 3: Improper Integral

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Today we will study the improper integrals. You may ask first what is the "proper integral"? We regard the Riemann integral as proper integral. And you may recall that the Riemann integral is defined for a bounded function defined on a closed and bounded interval. So improper integral means either domain of integration is unbounded interval or unbounded functions or both.

We begin by considering bounded functions defined on a semi-infinite interval of the form $[a, \infty)$, where $a \in \mathbb{R}$. Before we jump into technicalities of an improper integral, it would be nice to have some Déjà-vu type of feeling.

Example 3.1 1. *Consider the integral* $\int_0^\infty e^{-x}\, dx$. *How do you evaluate it? This is what you would do.*

$$\int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} + e^{-0} = -0 + 1$$

2. *Now consider the integral* $\int_0^\infty \cos x\, dx$. *Once again we do the same thing to evaluate the integral*

$$\int_0^\infty \cos x\, dx = \sin x\Big|_0^\infty = \sin(\infty) - \sin(0)$$

What is the value of $\sin\infty$*? May be it is good idea to rewrite* $\sin(\infty)$ *as* $\lim_{x\to\infty} \sin x$. *And we know that this limit does not exist. So we can say that the integral* $\int_0^\infty \cos x\, dx$ *is not well defined.*

The common thread in both examples above is, first we are finding the anti-derivative of the integrand and then evaluating the upper limit and lower limit of the integration, at the anti-derivative. If these limits exists we are saying integral exists and if any of the limit fails to exist, as in second example, we declare that integral does not exists.

This observation leads to the following definition.

Definition 3.2 *Let $a \in \mathbb{R}$ and $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, x]$ for every $x \geq a$. We say that an improper integral $\int_a^\infty f(x)dx$ is convergent if the limit*

$$\lim_{x\to\infty} \int_a^x f(t)dt \quad \textit{exists.}$$

An integral of the form $\int_a^\infty f(x)dx$ is called an improper integral of the first kind.

It is clear that if this limit exists, then it is unique, and we may denote it by I. When we write $\int_a^\infty f(x)dx = I$, we mean that I is a real number and the improper integral $\int_a^\infty f(x)dx$ is convergent with I as its value. An improper integral that is not convergent is said to be divergent. In particular, if $\int_a^x f(t)dt \to \infty$ or $\int_a^x f(t)dt \to -\infty$ as $x \to \infty$, then we say that the improper integral diverges to ∞ or to $-\infty$, as the case may be.

Exercise 3.3 *For what values of $p \in \mathbb{R}$ the improper integral $\int_1^\infty \frac{1}{x^p}dx$ converges ?*

Solution: Given any $x \in [1, \infty)$, we have

$$\int_1^x \frac{1}{t^p}dt = \begin{cases} \dfrac{x^{1-p}-1}{1-p} & \text{if} \quad p \neq 1 \\ \ln x & \text{if} \quad p = 1 \end{cases}$$

Since

$$\lim_{x\to\infty} x^{1-p} = \lim_{x\to\infty} \frac{1}{x^{p-1}} = \begin{cases} 0 & \text{if} \quad p > 1 \\ \infty & \text{if} \quad p < 1 \end{cases}$$

It follows that if $p > 1$, then $\int_1^\infty \frac{1}{x^p}dx$ converges to $\frac{1}{1-p}$, while if $p \leq 1$, then it diverges to ∞. ■

Remark 3.4 *It is useful to keep in mind that the convergence of an improper integral $\int_a^\infty f(x)dx$ is not affected by changing the initial point a of the interval $[a, \infty)$, although the value of improper integral may change by doing so. Indeed, if $a' \geq a$, then*

$$\int_a^\infty f(x)dx \textit{ is convergent} \iff \int_{a'}^\infty f(x)dx \textit{ is convergent .}$$

and if this holds, then the values are related by the equation

$$\int_a^\infty f(x)dx = \int_a^{a} f(x)dx + \int_{a'}^\infty f(x)dx$$

Definition 3.5 *An improper integral* $\int_a^\infty f(x)dx$ *is said to be absolutely convergent if the improper integral* $\int_a^\infty |f(x)|dx$ *is convergent.*

Proposition 3.6 *An absolutely convergent improper integral is convergent.*

Convergence Tests for Improper Integrals

If we want to decide the convergence of an improper integral by definition then we need to have anti-derivative of the integrand. In many cases it is not possible not have anti-derivative in term of elementary functions or closed form. For example, consider the improper integral $\int_0^\infty e^{-x^2}\,dx$. Of course by fundamental theorem of calculus, you are 1000 times correct in saying that anti-derivative of e^{-x^2} is the function $F(x) = \int_a^x e^{-t^2}\,dt$ for any $a \in \mathbb{R}$. But what we can do with this F? Can you find limit of $F(x)$ as $x \to \infty$? Of course, answer is no. Is it the end of the world for us? or is it possible to decide the convergence of the improper integral $\int_0^\infty e^{-x^2}\,dx$ by some other way which does not depend on finding anti-derivative explicitly. The answer is yes. Thanks to the following test.

Theorem 3.7 (Comparison Test for Improper Integrals) *Suppose* $a \in \mathbb{R}$ *and* $f, g : [a, \infty) \to \mathbb{R}$ *are such that both* f *and* g *are integrable on* $[a, x]$ *for every* $x \geq a$ *and* $|f| \leq g$. *If* $\int_a^\infty g(x)dx$ *is convergent, then* $\int_a^\infty f(x)dx$ *is absolutely convergent.*

Exercise 3.8 *Discuss the convergence of improper integral* $\int_0^\infty e^{-x^2}\,dx$.

Solution: Note that $0 \leq e^{-x^2} \leq e^{-x}$ for all $x \geq 1$. Also $\int_1^\infty e^{-x}\,dx = e^{-1}$. Hence by comparison test $\int_1^\infty e^{-x^2}\,dx$ converges. Writing

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

Since the integral $\int_0^1 e^{-x^2}\,dx$ is proper, hence we get the convergence of $\int_0^\infty e^{-x^2}\,dx$. ■

Example 3.9 *Determine if the improper integral* $\int_1^\infty \frac{\cos x}{x^2}dx$ *converges.*

Solution: If one tries to go by definition then using integration by parts we obtain

$$\begin{aligned}\int_1^x \frac{\cos t}{t^2}dt &= -\frac{\cos t}{t}\Big|_1^x - \int_1^x \frac{\sin t}{t}dt \\ &= \frac{\cos 1}{1} - \frac{\cos x}{x} - \left[\sin t \ln t\Big|_1^x - \int_1^x \cos t \ln t dt\right]\end{aligned}$$

Now again we will not be able to find anti-derivative if $\cos t \ln t$ in terms of elementary functions, and we are stuck.

Let us try to apply the comparison test. Note that $\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$ for all $x \geq 1$. Since $\int_1^\infty \frac{1}{x^2}dx$ converges, hence improper integral $\int_1^\infty \frac{\cos x}{x^2}dx$ converges.