MATH-221: Probability and Statistics

Tutorial # 5 (Function of Random Variable, Expectation, Variance, Variance and Expectation of function of Random Variable)

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1. Let X be a random variable with distribution function F. Find the distribution function of the following random variables in terms of F. (i) $\max\{X, a\}$, where $a \in \mathbb{R}$ (ii) $|X|^{\frac{1}{3}}$ (iii) $|X|$ (iv) e^X (v) $-\ln|X|$.

2. Let X be the uniform random variable on $[0,1]$. Then Determine pdf of (i) $\sqrt{X}$ (ii) $aX+b$, where $a \neq 0, b \in \mathbb{R}$ (iii) $X^{\frac{1}{4}}$.

3. Let X be a random variable with PMF

$$f_X(x) = \begin{cases} \dfrac{x^2}{a} & \text{if} \quad x = -3,-2,-1,0,1,2,3 \\ 0 & \quad \text{otherwise} \end{cases}$$

   Find a. What is the PMF of the random variable $Z = (X-a)^2$.

4. Consider $X \sim U[-1,1]$. Another random variable Y is formed by using the transformation $Y = X^2 + X$. Find the distribution function $F_Y(y)$ and density function $f_Y(y)$ of the new transformed random variable Y.

5. (a) Let a random variable X has a continuous distribution function $F_X(x)$ which is strictly increasing on $\mathbb{R}$. Then show that $Y \sim U(0,1)$, where $Y = F_X(X)$. (Actually the result is true even for continuous distribution function which are constant over some intervals but proof of this is more involved).

   (b) Use the general version of result in part (a), to show that if X is exponentially distributed with parameter λ, then $-\dfrac{1}{\lambda}\ln(1-Y)$ is also exponentially distributed, where $Y = F_X(X)$.

6. Find the mean and variance of (i) Bernoulli(p) (ii)binomial(n,p) (iii) geometric(p) (iv) Poisson(λ) (v) continuous uniform on $[a,b]$ (vi)normal(μ, σ^2) (vii) exponential(λ) distributions.

7. Let X be exponentially distributed with parameter λ. Find the 4th moment.

8. Let X be standard normal random variable. Find $E|X|$.

9. Let X be a Binomial random variable with parameters $n = 4, p = \dfrac{1}{2}$. Find $E\left[\sin\left(\dfrac{\pi x}{2}\right)\right]$.

10. Let X be a geometrically distributed random variable with parameter p and M be a positive integer. Find $E[\min\{X, M\}]$.

11. Assume that every time you attend your lecture there is a probability of 0.1 that your Professor will not show up. Assume his arrival to any given lecture is independent of his arrival (or non-arrival) to any other lecture. What is the expected number of classes you must attend until you arrive to find your Professor absent?

12. If X is the number of points rolled with a fair die, find the expected value of $g(X) = 2X^2+1$.

13. Let X be a random variable with $E(X) = 6$ and $E(X^2) = 45$, and let $Y = 20 - 2X$. Find $E(Y)$ and $Var\ (Y)$.

14. The thickness of a conductive coating in micrometers has a density function of $f_X(x) = 600x^{-2}$ for $100\mu m < x < 120\mu m$. Let X denote the coating thickness.

   (a) Determine the mean and variance of X.

   (b) If the coating costs 5 Rs. per micrometer of thickness on each part, what is the average cost of the coating per part?

   (c) Find the probability that the coating thickness exceeds $110\mu m$.

15. An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, X, follows a distribution with density function:

$$f_X(x) = \begin{cases} \dfrac{2}{x^3}, & x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

   What is the expected value of the benefit paid under the insurance policy? Also find the variance of the benefit paid under the insurance policy.

16. Let

$$P_X(x) = \begin{cases} \dfrac{1}{2^x}, & x = 1, 2, 3, \cdots, \\ 0, & otherwise. \end{cases}$$

   be the pmf of the random variable X, then find all the possible values of t such that $E[e^{tX}]$ exist.

17. Let a random variable X of the continuous type have a pdf $f(x)$ whose graph is symmetric with respect to $x = c$. If the mean value of X exists, then prove that $E[X] = c$.

18. Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 1 & x > 1, \\ \frac{x}{2} + \frac{x^2}{2}, & 0 \le x \le 1 \\ 0, & x < 0. \end{cases}$$

   then find $E(e^X)$.

19. The number of messages sent per hour over a computer network has the following distribution:

x = number of messages	10	11	12	13	14	15
$P_X(x)$	0.08	0.15	0.30	0.20	0.20	0.07

   Determine the mean and standard deviation of the number of messages sent per hour.