Lecture 6: Know your Limits

October 17, 2016

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Definition 6.1 *Let $f : B_r(x_0, y_0) \setminus \{(x_0, y_0)\} \to \mathbb{R}$ for some $r > 0$. We say that $f(x, y)$ has a limit as (x, y) approaches (x_0, y_0) if there exists $L \in \mathbb{R}$ such that for every sequence $((x_n, y_n))$ in $D \setminus \{x_0, y_0\}$ such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to L$. We then write*

$$f(x, y) \to L \text{ as } (x, y) \to (x_0, y_0) \text{ or } \lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

Remark 6.2

1. *Note that (x_0, y_0) need not be in the domain of f. So in order to talk about limit of a function at point, it is irrelevant what is the value of the function at that point, what matters is how the values of f behaves in the neighborhood of the point (x_0, y_0).*

2. *Even if (x_0, y_0) lies in the domain of f, the limit L need not be $f(x_0, y_0)$. See Example 6.3 below.*

3. *One may wonder that why the definition of limit is given on punctured disk rather than any arbitrary subset D of $\mathbb{R}^2$ as was the case with definition of continuity. The reason is, for continuity of f at point (x_0, y_0), the function f has to be defined at (x_0, y_0) but for limit it is not necessary. So if (x_0, y_0) is an "isolated point", where we are interested in limits then we loose the uniqueness of the limit. In order to avoid this absurdity we need to put some restriction on the domain of f, and this achieved by saying that f is defined on some punctured disk centered at (x_0, y_0). Actually one can weaken this also further.*

   If you do not completely understand this remark, you may ignore this one (I promise, nothing will come in exam based on this remark!!). But if you are curious to understand it then discuss with me. This remark justifies, why we teach continuity of a function before the concept of the limit, in M-I.

Example 6.3 *Consider the function*

$$f(x, y) = \begin{cases} \sin(xy) & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

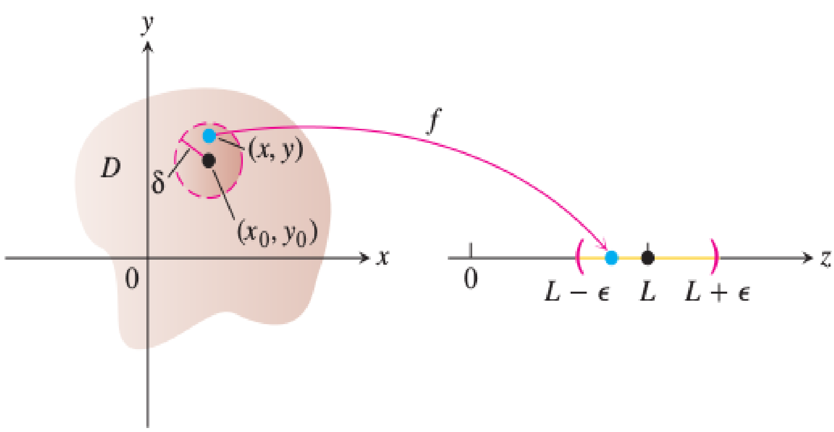
Find $\lim_{(x,y)\to(0,0)} f(x, y)$ *if it exists.*

Solution: Let $((x_n, y_n))$ is a sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$, i.e., $x_n \to 0$ and $y_n \to 0$. Therefore, $x_n y_n \to 0$ and by the continuity of the $g(t) = \sin t, t \in \mathbb{R}$, $\sin(x_n y_n) \to \sin 0 = 0$, that is, $f(x_n, y_n) \to 0$. Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ ■

Exercise 6.4 *In example above, what you can say about* $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ *if* $(x_0, y_0) \neq (0,0)$*?*

Definition 6.5 *Let* $f : B_r(x_0, y_0) \setminus \{(x_0, y_0)\} \to \mathbb{R}$ *for some* $r > 0$*. We say that* $f(x,y)$ *has a limit as* (x,y) *approaches* (x_0, y_0) *if there exists* $L \in \mathbb{R}$ *such that for every* $\epsilon > 0$*, there is* $\delta > 0$ *with* $\delta < r$*, we have* $|f(x,y) - L| < \epsilon$ *for all* $(x,y) \in B_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$*. We then write*

$$f(x,y) \to L \text{ as } (x,y) \to (x_0,y_0) \text{ or } \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$



Example 6.6 *Consider the function*

$$f(x,y) = \frac{x^2 y}{x^4 + y^2}, \text{ for all } (x,y) \neq (0,0).$$

Determine the limit of f *at* $(0,0)$ *if it exists.*

Solution: Let us find limit of $f(x,y)$ as $(x,y) \to (0,0)$ along parabolas $y = mx^2$, that is we calculate $\lim\limits_{x\to 0} f(x, mx^2)$

$$\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{m^2 x^4}{x^4 + m^2 x^4} = \frac{m^2}{1+m^2}$$

Hence limit depends on the path, therefore f does not have limit at $(0,0)$. ■

Exercise 6.7 *In example above, what you can say about* $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ *if* $(x_0,y_0) \neq (0,0)$*?*

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \qquad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   n a positive integer, and if n is even, we assume that $L > 0$.

Limits and Continuity

Theorem 6.8 *A function* $f(x,y)$ *is continuous at the point* (x_0,y_0) *if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Composition of Continuous Functions We now observe that the composition of continuous functions is continuous. It may be noted that for functions of two variables, three types of composites are possible.

Theorem 6.9 *Let* $D \subset \mathbb{R}^2$*, and let* $f : D \to \mathbb{R}$ *be continuous on* D*.*

1. *Suppose* $E \subseteq \mathbb{R}$*. If* $g : E \to \mathbb{R}$ *is continuous on* E*, then* $g \circ f : D \to \mathbb{R}$ *is continuous* D*.*

2. *Suppose* $E \subseteq \mathbb{R}$, *and* $x, y : E \to \mathbb{R}$ *are continuous on* E, *then* $F : E \to \mathbb{R}$ *defined by* $F(t) := f(x(t), y(t))$ *is continuous* E.

3. *Suppose* $E \subseteq \mathbb{R}^2$, *and* $x, y : E \to \mathbb{R}$ *continuous on* E, *then* $F : E \to \mathbb{R}$ *defined by* $F(u, v) := f(x(u, v), y(u, v))$ *is continuous at* E.

Example 6.10 1. *Function* $\cos(x + y)$ *is continuous by part 1 of Theorem 6.9.*

2. *By part 2 of Theorem 6.9, if* $f(x, y)$ *is any polynomial in two variables, then* $F : \mathbb{R} \to \mathbb{R}$ *defined by* $F(t) := f(e^t, \sin t)$ *for* $t \in \mathbb{R}$ *is continuous at everywhere.*

3. *By part 3 of Theorem 6.9, if* $f(x, y)$ *is any polynomial in two variables, then* $F : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $F(u, v) := f(\sin(uv), \cos(u+v))$ *for* $(u, v) \in \mathbb{R}^2$ *is continuous at everywhere.*