Duhamel's Principle for 1-D Wave Equation

Consider

$$u_{tt} = c^2 u_{xx} + F(x,t), \qquad x \in \mathcal{R},\ t > 0 \tag{1}$$

$$u(x,0) = u_t(x,0) = 0, \qquad x \in \mathcal{R} \tag{2}$$

Let $v(x,t,\tau)$, for $t > \tau$, satisfy

$$v_{tt} = c^2 v_{xx}, \qquad x \in \mathcal{R},\ t > 0 \tag{3}$$

with the conditions $(t \geq \tau > 0)$

$$v(x,\tau,\tau) = 0, \quad v_t(x,\tau,\tau) = F(x,\tau), \quad x \in \mathcal{R} \tag{4}$$

The solution of (3) and (4) is

$$v(x,t,\tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds \tag{5}$$

Define:

$$u(x,t) = \int_0^t v(x,t,\tau)\, d\tau \tag{6}$$

Now

$$u_t = v(x,t,t) + \int_0^t v_t(x,t,\tau)\, d\tau = \int_0^t v_t(x,t,\tau)\, d\tau \qquad [\text{since } v(x,t,t) = 0]$$

Similarly,

$$u_{tt} = v_t(x,t,t) + \int_0^t v_{tt}(x,t,\tau)\, d\tau$$

or

$$u_{tt} = F(x,t) + \int_0^t v_{tt}(x,t,\tau)\, d\tau \tag{7}$$

We also have

$$u_{xx} = \int_0^t v_{xx}(x,t,\tau)\, d\tau \tag{8}$$

From (7) and (8), we get

$$u_{tt} - c^2 u_{xx} = F(x,t) + \int_0^t [v_{tt} - c^2 v_{xx}]\, d\tau = F(x,t) \tag{9}$$

Now let us consider

$$u_{tt} = c^2 u_{xx} + F(x,t), \qquad x \in \mathcal{R},\ t > 0 \tag{10}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad x \in \mathcal{R} \tag{11}$$

Let u_1 be the solution of (10) with

$$u(x,0) = u_t(x,0) = 0, \qquad x \in \mathcal{R} \tag{12}$$

and u_2 be the solution of

$$u_{tt} = c^2 u_{xx}, \qquad x \in \mathcal{R},\, t > 0 \tag{13}$$

with initial conditions (11).

Then $u = u_1 + u_2$ is the solution of (10) and (11) (easy verification) and hence

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)\, ds\, d\tau \tag{14}$$

Thus u given by (14) solves (10) and (11).