Marginal PDF

Theorem-I: If $f_{XY}(x,y)$ is the joint pdf of random variable X and Y, then

$$f_X(a) = \int_{-\infty}^{\infty} f_{XY}(a,y)\,dy$$

$$f_Y(a) = \int_{-\infty}^{\infty} f_{XY}(x,a)\,dx$$

Proof: We have

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\} =$$
$$= \int_{-\infty}^{a}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$F_X(a) = P\{X \le a\} = \int_{-\infty}^{a} g(x)\,dx, \text{ where } g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_X(a) = \frac{dF_X(a)}{d(a)} = g(a) = \int_{-\infty}^{\infty} f(a,y)\,dy$$

$$\Rightarrow \boxed{f_X(a) = \int_{-\infty}^{\infty} f(a,y)\,dy}$$ This is called marginal pdf f_X of X w.r.t f_{XY}

Simillarly, $$\boxed{f_Y(a) = \int_{-\infty}^{\infty} f_{XY}(x,a)\,dx}$$ This is called marginal pdf f_Y of Y w.r.t f_{XY}

Example-I: The joint pdf of (X,Y) is given as

$$f_{XY}(x,y) = \begin{cases} 6(1-x), & 0<y<x,\ 0<x<1 \\ 0, & \text{else.} \end{cases}$$

Determine marginal density of r.v X and Y.

Solu-: Density of Y: we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx, \quad \forall\, y \in R$$

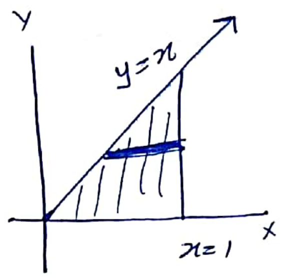
Since $f_{XY}(x,y) = 0$ for $y \geq 1$ and $y \leq 0$.

So $f_Y(y) = 0$ if $y \geq 1$ or $y \leq 0$

Nor For $y \in (0,1)$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^1 f_{XY}(x,y)\,dx$$

$$f_Y(y) = \int_y^1 6(1-x)\,dx = 6\left[x - \frac{x^2}{2}\right]_y^1$$



$$f_Y(y) = 3(y-1)^2$$

So

$$f_Y(y) = \begin{cases} 3(y-1)^2, & 0 < y < 1 \\ 0, & \text{else.} \end{cases}$$

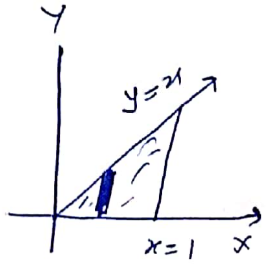
Density of X:

Again $f_{XY}(x,y) = 0$, if $x \geq 1$ and $x \leq 0$.

So $f_X(x) = 0$ if $x \geq 1$ or $x \leq 0$.

For $x \in (0,1)$, we have

$$f_X(x) = \int_0^x 6(1-x)\,dy = \left[6(y - xy)\right]_0^x$$



$$f_X(x) = 6(x - x^2)$$

$$f_X(x) = \begin{cases} 6(x - x^2), & 0 < x < 1 \\ 0, & \text{else.} \end{cases}$$

JOINT DISTRIBUTION FUNCTION

Defination: Let (X, Y) be a random vector on (Ω, F, P). Then the function $F: R^2 \rightarrow R$ given by

$$F(a, b) = P\{X \le a, Y \le b\}, \quad -\infty < a, b < \infty$$

is called joint distribution of (X, Y).

NOTE: 1 Distribution of X can be obtained from joint distribution of X and Y as

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\}$$

$$= P\{\lim_{b \to \infty} \{X \le a, Y \le b\}\}$$

$$= \lim_{b \to \infty} P\{X \le a, Y \le b\}$$

$$F_X(a) = \lim_{b \to \infty} F_{XY}(a, b) = F_{XY}(a, \infty)$$

if continous

$$\int_{-\infty}^{a}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \boxed{F_X(a) = F_{XY}(a, \infty)}$$

This is called marginal distribution.

If X and Y are discrete

$$F_X(a) = \sum_{y \in R(Y)} P\{X \le a, Y = y\}$$

Similarly, $F_Y(b) = F_{XY}(\infty, b)$

$$= \int_{-\infty}^{y}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \quad \text{if continous.}$$

$$F_Y(b) = \sum_{x \in R(X)} P\{X = x, Y \le b\}$$

NOTE-2

$$P\{X > a, Y > b\} = 1 - P(\{X > a, Y > b\}^c)$$

$$= 1 - P(\{X > a\}^c \cup \{Y > b\}^c)$$

$$= 1 - P(\{X \le a\} \cup \{Y \le b\})$$

$$P\{X > a, Y > b\} = 1 - P\{X \le a\} - P\{Y \le b\} + P\{X \le a, Y \le b\}$$

$$\boxed{P\{X > a, Y > b\} = 1 - F_X(a) - F_X(b) + F_{XY}(a, b)}$$

NOTE-3

$$\boxed{P\{a_1 \le X \le a_2,\ b_1 \le Y \le b_2\} = F(a_2, b_2) + F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1)}$$

≥ 0 since it is probability

NOTE-4 (I) $\lim_{x \to -\infty} F_{XY}(x,y) = 0, \ \forall y \in \mathbb{R}$

(II) $\lim_{y \to -\infty} F_{XY}(x,y) = 0, \ \forall x \in \mathbb{R}$

(III) $\lim_{(x,y) \to (-\infty,-\infty)} F_{XY}(x,y) = 0,$

Any function $F(x,y)$ on R^2 satisfy NOTE-3, 4, 5, 6 can be identified as joint CDF of some 2-dimensional random vector.

NOTE-5 $\lim_{(x,y) \to (\infty,\infty)} F_{XY}(x,y) = 1.$

NOTE-6 F is right continuous.

NOTE-6: Distribution function F_X and F_Y are called marginal distribution of X and Y w.r.t $F_{XY}(x,y)$.

NOTE-7: F is non-decreasing

Joint Density Function From Distribution Function

Let $F_{XY}(x,y)$ be joint distribution function of continuous random vector (X,Y). ~~we ge~~ The joint density function $f_{XY}(x,y)$ of (X,Y) can be obtained by partially differentiation as follows:

$$f_{XY}(a,b) = \frac{\partial^2 F(a,b)}{\partial(a,b)}$$

$\forall (x,y) \in R^2$ at which the joint pdf is continuous.

Example-2: Suppose the joint pmf of X and Y is given as $f(0,0) = f(0,1) = \frac{1}{6}$, $f(1,0) = f(1,1) = \frac{1}{3}$

or

X \ Y	0	1
0	1/6	1/6
1	1/3	1/3

Determine the joint CDF of X and Y.

Soln

$x \backslash y$	0	1
0	1/6	1/6
1	1/3	1/3

$$F_{xy}(x,y) = \sum_{(i,j):\, i\le x,\ j \le y} P(X=i,\, Y=j)$$

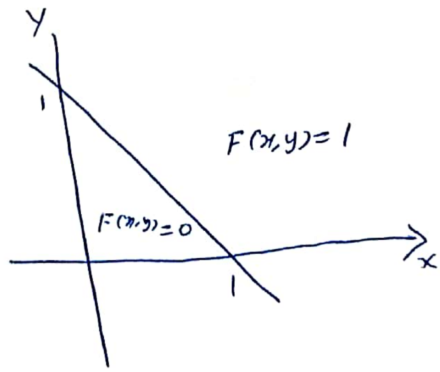
$$F_{xy}(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ f(0,0) = 1/6, & 0\le x<1,\ 0\le y<1 \\ f(0,0)+f(0,1) = \frac{2}{6}, & 0\le x<1,\ y\ge 1 \\ f(0,0)+f(1,0) = \frac{1}{2}, & x\ge 1,\ 0\le y<1 \\ f(0,0)+f(0,1)+f(1,0)+f(1,1) = 1, & x\ge 1,\ y\ge 1 \end{cases}$$

Example-3: Let $F(x,y)$ be a function of two variables defined by

$$F(x,y) = \begin{cases} 0, & x<0, \text{ or } x+y<1 \text{ or } y<0 \\ 1, & \text{else.} \end{cases}$$

Determine whether $F(x,y)$ is a joint CDF?

Soln: we can check clearly



- $\lim_{x\to -\infty} F(x,y) = 0, \ \forall y \in R$
- $\lim_{y\to -\infty} F(x,y) = 0$
- $\lim_{(x,y)\to(-\infty,-\infty)} F(x,y) = 0$
- F is non-decreasing
- Take $(\frac{1}{3},\frac{1}{3}) = (a_1, b_1)$, $(a_2, b_2) = (1,1)$

$$P\{\tfrac{1}{3}\le X\le 1,\ \tfrac{1}{3}\le Y\le 1\} = F(1,1) + F(\tfrac{1}{3},\tfrac{1}{3}) - F(\tfrac{1}{3},1) - F(1,\tfrac{1}{3})$$
$$= 1 + 0 - 1 - 1 = -1 \ne 0$$

Can not be negative. So not CDF

Example-4 In example-2, we have joint CDF

$$F_{XY}(x,y) = \begin{cases} 0 & x<0 \text{ or } y<0 \\ \frac{1}{6}, & 0\le x<1, \quad 0\le y<1 \\ \frac{2}{6}, & 0\le x<1, \quad y\ge 1 \\ \frac{1}{2}, & x\ge 1, \quad 0\le y<1 \\ 1, & x\ge 1, \quad y\ge 1 \end{cases}$$

X \ Y	0	1
0	1/6	1/6
1	1/3	1/3

Find Marginal distribution $F_X(x)$ and $F_Y(y)$

Soln

$$F_X(x) = F_{XY}(x,\infty) = \lim_{y\to\infty} F_{XY}(x,y)$$

$$= \begin{cases} 0 & x<0 \\ \frac{1}{3} & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$$

$F_X(x)$ is not continuous at $x=0,1$ So corresponding pmf

$$f_X(0) = P(X=0) = F_X(0) - F_X(0^-) = \frac{1}{3}$$

$$f_X(1) = P(X=1) = F_X(1) - F_X(1^-) = 1-\frac{1}{3} = 2/3$$

Similarly

$$F_Y(y) = F_{XY}(\infty,y) = \lim_{x\to\infty} F_{XY}(x,y)$$

$$F_Y(y) = \begin{cases} 0 & y<0 \\ \frac{1}{2} & 0\le y<1 \\ 1 & y\ge 1 \end{cases}$$

Corresponding pmf $\quad P(Y=0) = f_Y(0) = \frac{1}{2}$,

$$P(Y=1) = f_Y(1) = 1-\frac{1}{2} = \frac{1}{2}$$

Example-5 The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0<x<\infty, \ 0<y<\infty \\ 0 & \text{; else} \end{cases}$$

Find CDF

Soln. if either $x\le 0$ or $y\le 0 \Rightarrow F_{XY}(x,y) = 0$

if $x>0$, $y>0$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} e^{-(s+t)}\,ds\,dt = \left[\int_0^x e^{-s}ds\right]\left[\int_0^y e^{-t}dt\right]$$

$$F_{XY}(x,y) = (1-e^{-x})(1-e^{-y})$$

So $$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & 0<x<\infty, \ 0<y<\infty \\ 0 & \text{. else} \end{cases}$$

Independent Random Variables

Defination. We say $X_1, X_2, \ldots X_n$ are independent if events $\{X_1 \in A_1\}, \{X_2 \in A_2\}, \ldots \{X_n \in A_n\}$ are independent for all $A_1, A_2, \ldots A_n$ Borel subset of R.

In terms of Two random variables X and Y

Random variables X and Y are said to be independent if for ~~any~~ every two Borel selbsel A and B of R

$$P\{X \in A, Y \in B\} = P\{X \in A\} \cdot P\{Y \in B\} \quad \text{— } (1)$$

or X and Y are independent if for all Borel subset A and B of R, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

In terms of Joint Distribution

Theorm-1 let $F_{XY}(x, y)$ be CDF of X and Y. X and Y are said to be independent if

$$F_{XY}(a, b) = F_X(a)\, F_Y(b) \qquad \forall\, a, b$$

NOTE-I when X and Y are discrete random variables then X and Y are independent if

$$P\{X = a, Y = b\} = P\{X = a\} \cdot P\{X = b\} \qquad \forall\, a \in R(X),\ b \in R(Y)$$

NOTE-2 When X and Y are continuous random variable with yoint pdf $f_{XY}(x, y)$. X and Y are independent if

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

$\forall\, (x, y) \in R^2$ where both $f_{XY}(x, y)$ and $g(x, y) = f_X(x)\, f_Y(y)$ are continuous.

Countably Infinite Collection of random variables

Defination: We say a sequence of random variables $(X_n)_{n \in N}$ is independent if for every $n = 2, 3, \ldots$ the random variables $X_1, X_2 \ldots X_n$ are independent.

For three variables:

(I) $P\{X = x, Y = y, Z = z\} = P[X = x] \cdot P\{Y = y\} \; P[Z = z]$ $\quad \forall x \in R(X),\; y \in R(Y),\; z \in R(Z)$

Discrete Case

(II) $f_{XYZ}(x, y, z) = f_X(x) \, f_Y(y) \, f_Z(z) \quad \forall (x, y, z) \in R^3$

where f_{XYZ} and $g(x, y, z) = f_X(x) f_Y(y) f_Z(z)$ are continuous

Random Variables that are not independent are called dependent

Remark-I Let us recall that if we are given only marginal distributions of random variables X and Y, in general it is impossible to define the joint distribution of X and Y. But in a very special situation, knowledge about marginal distributions is enough to construct the joint distribution, namely when random variables X and Y are independent. So Thanks to theorem-I.

Example-6 Let the random vector (X, Y) has joint probability as follows

X \ Y	-1	0	1
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

. Whether X and Y are independent

Soln $P\{X = -1\} = P\{X = 1\} = \frac{1}{4}$, $\quad P\{X = 0\} = \frac{1}{2}$

$P\{X = -1, Y = -1\} = 0 \neq P\{X = -1\} \; P\{X = 1\} = \frac{1}{4} \times \frac{1}{4} = 1/16$

So Not Independent.

Example-7 The joint pdf of (X,Y) is

$$f(x,y) = \begin{cases} 6(1-x), & 0<y<x, \quad 0<x<1 \\ 0, & \text{else.} \end{cases}$$

Determine whether X and Y are independent

Soln. Recall we computed (In example - I)

$$f_Y(y) = \begin{cases} 3(y-1)^2, & 0<y<1 \\ 0, & \text{else} \end{cases}$$

$$f_X(x) = \begin{cases} 6(x-x^2) & 0<x<1 \\ 0 & \text{else} \end{cases}$$

Now At $(\frac{1}{2}, \frac{1}{4})$

$$f_{XY}(x,y) = 6(1-x) = 3$$

$$f_X(x) f_Y(y) = 6(x-x^2) \times 3(y-1)^2 = 6\{\tfrac{1}{2}-\tfrac{1}{4}\} \times 3(\tfrac{1}{4}-1)^2$$

$$= \frac{\cancel{6}^{3}}{\cancel{4}_{2}} \times 3 \times \frac{9}{16} = \frac{81}{32}$$

So $f_{XY}(x,y) \neq f_X(x) f_Y(y)$ at $(\frac{1}{2}, \frac{1}{4})$

Clearly at $(\frac{1}{2}, \frac{1}{4})$ $f_{XY}(x,y)$ and $f_X(x) f_Y(y)$ are continuous