Notes for First Order Partial Differential Equations (Quasilinear PDE only)

1 Preliminaries

1.1 Curves

Let $I = [0, 1]$ and $f : I \to \mathcal{R}^3$, be a "smooth" function. For $t \in I$, $f(t) = (f_1, f_2, f_3) \in \mathcal{R}^3$ and

$$x = f_1(t), y = f_2(t), z = f_3(t), \quad t \in I \tag{1}$$

represents a curve in $\mathcal{R}^3$. Here t is called a parameter and (1) is called the parametric representation of the curve. We shall use the following notation:

$$C : \ (f_1(t), f_2(t), f_3(t)), \quad t \in I$$

or $\qquad C : \ (f_1, f_2, f_3)$, if no confusion arises.

1.2 Surfaces

Let us start with something with which we are familiar. Let

$$f(x, y, z) = x^2 + y^2 + z^2 - 1, \quad (x, y, z) \in \mathcal{R}^3.$$

We know that

$$f(x, y, z) = 0 \tag{2}$$

is the surface of a sphere (in $\mathcal{R}^3$) of radius 1 with centre at $(0, 0, 0)$. Here we see that the function f represents a surface via (2). Generalizing this observation, in genreal, a "smooth" function $f : D \to \mathcal{R}$ (where $D \subset \mathcal{R}^3$ is suitable domain) defines a surface S through the equation

$$f(x, y, z) = 0, \quad (x, y, z) \in D \tag{3}$$

Notation:

$$S : \ f(x, y, z) = 0, \quad (x, y, z) \in D$$

or $\qquad S : \ f = 0$, if no confusion arises.

Let $f \in C^1$ (i.e. partial derivatives are continuous). Then ∇f at $P(x, y, z)$ is normal to the surface at P. Often, it is convenient to describe a surface through a parametric form:

$$S : \ (x, y, z) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in D$$

for a “suitable” domain $D \subset \mathcal{R}^2$.

Example 1. Consider the surface

$$S:\ x^2 + y^2 + z^2 = a^2, \quad a > 0$$

S has the following two parametric representations:

$$(i)\ x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$$

$$(ii)\ x = a\frac{(1-v^2)}{1+v^2}\cos u,\ y = a\frac{(1-v^2)}{1+v^2}\sin u,\ z = \frac{2av}{1+v^2}$$

The example also shows that the parametric representation may not be unique

A few remarks/overservations

(a) Let $S:\ F(x,y,z) = 0$ be a surface. Let a curve

$$C:\ (x(t), y(t), z(t))\,,\ t \in I$$

lies on S. For some value of $t \in I$, let $P = P(x(t), y(t), z(t))$. The normal to S at P is in the direction

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

Also the tangent direction at P to C is in the direction

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \qquad (\text{at } t \in I)$$

Since the normal is orthogonal to (any) tangent direction at P, we have

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \tag{4}$$

(b) Parametric vs. explicit representation:

Suppose a surface

$$S:\ x = F_1(u,v), y = F_2(u,v), z = F_3(u,v), \quad (u,v) \in D \tag{5}$$

is a parametric representation of S. Consider the Jacobian J defined by

$$J = \frac{\partial(F_1, F_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial F_2}{\partial u}$$

Let $P(x_0, y_0, z_0) \in S$ and $J \neq 0$ in a nbd of P. Then first two equations in (5) can be locally solved for u, v, i.e.

$$u = \lambda(x,y), v = \mu(x,y) \qquad \text{locally around } P$$

(one uses “inverse function theorem”). This implies

$$z = F_3(\lambda(x,y), \mu(x,y)) = G(x,y) \tag{6}$$

which is an “explicit” representation of S.

Example 2. Let a surface S have the parametric representation:

$$S: \; x = u, y = v, z = 1 - u - v, \quad (u, v) \in \mathcal{R}^2$$

Thus

$$x = F_1(u, v) = u, \; y = F_2(u, v) = v, \; z = F_3(u, v) = 1 - u - v$$

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = 1 \neq 0, \quad \forall u, v$$

which shows that we can solve for u and v in terms of x, y “locally” at each point. In fact we have

$$u = x = \lambda(x, y), \; v = y = \mu(x, y) \qquad \text{locally around } P$$

Thus

$$z = F_3 = 1 - u - v = 1 - x - y \quad \text{or} \quad S: \, x + y + z = 1$$

which is a plane. [Note: Things may not be as easy as shown above].

Example 3. Let us consider the surface $S : \mathcal{R}^2$, namely the xy-plane. In S, consider a family of curves

$$\Gamma_1 : L, \; L \text{ is a line passing through } (0, 0).$$

It is easy to see that

$$S = \bigcup_{L \in \Gamma_1} L.$$

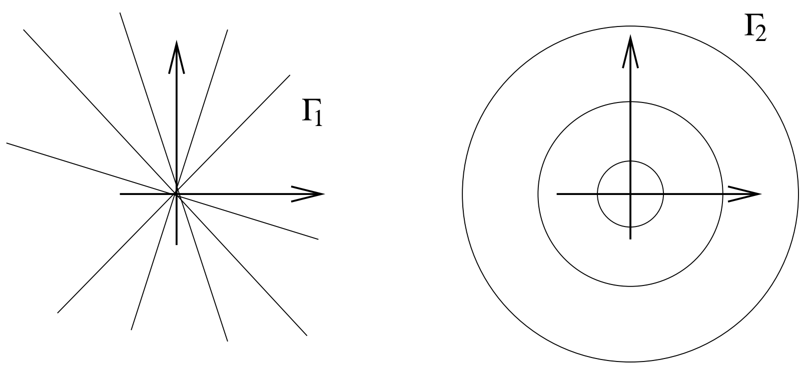
Let us consider another family Γ_2 of all circles with centres at $(0, 0)$, i.e.

$$\Gamma_2 : C, \; C \text{ is a circle with centre at} (0, 0).$$

Again

$$S = \bigcup_{C \in \Gamma_2} C,$$

i.e. S can be generated by the union of elements of Γ_2.



The elements of Γ_1 satisfy a differential equation $xy' = y$, while each member of Γ_2 satisfies $x + yy' = 0$.

2 Introduction to First Order Partial Differential Equation (in two independent variables)

2.1 Preliminaries

When number of independent variables are more than one, e.g., $z = z(x, y)$ be an unknown function, then a relation of the type

$$f(x, y, z, z_x, z_y) = 0 \tag{7}$$

is called a Partial Differential Equation of order one.

[*In general, let* $z(x, y, t, \ldots)$ *be an unknown function s.t.*

$$f(x, y, t, \ldots, z, z_x, z_y, z_t, \ldots, z_{xy}, z_{tt}, \ldots) = 0$$

defines a Partial Differential Equation.

The highest order of the partial derivative present in the relation gives the order of Partial Differential Equation.

Example:

(1) $z_x + z_y = x + y$: 1^{st} *order, (2)* $z_{xx} + z = y$: 2^{nd} *order, (3)* $z_{xy} + z_y = z$: 2^{nd} *order*]

The Partial Differential Equation arise in many problems related to Physics, Engineering, Mathematics, etc. For example:

• Recall integrating factor $\mu(x, y)$ for first order ODE

$$M(x, y) + N(x, y)y' = 0 \equiv M dx + N dy = 0$$

is given by

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

which gives a first order Partial Differential Equation in μ

$$N\mu_x - M\mu_y = \mu(M_y - N_x).$$

• If $f(z) = u(x, y) + iv(x, y)$ is analytic function then

$$\nabla^2 u = 0, \;\; \nabla^2 u = 0$$

i.e. u and v satisfy Laplace Equation ($\nabla^2 \phi = 0.$)

• If we have a family of surfaces:

$$z = F(x, y, a, b)$$

then $z_x = p$ and $z_y = q$.

From these three equations 'a' and 'b' can be eliminated to give a first order Partial Differential Equation provided

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$$

has rank 2.
e.g. For the family of surfaces

$$z = x + ax^2y^2 + b^2,$$

we have $p = 1 + 2axy^2$ and $q = 2ax^2y$, which gives

$$xp - yq = x.$$

• Consider a surface $F(u, v) = 0$, where $u = u(x, y, z)$ and $v = v(x, y, z)$ are defined on $D \subset \mathcal{R}^3$, then

$$\frac{\partial F}{\partial x} = F_u(u_x + pu_z) + F_v(v_x + pv_z) = 0$$
$$\frac{\partial F}{\partial y} = F_u(u_y + qu_z) + F_v(v_y + qv_z) = 0$$

(note $z = z(x, y)$) elimination of F_u and F_v gives a first order pde

$$p\frac{\partial(u, v)}{\partial(y, z)} + q\frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)} \tag{8}$$

• **Heat equation:** If $\theta(x, t)$ is temperature in a thin rod then the equation governing θ is given by

$$\theta_t = k\theta_{xx}.$$

• **Wave equation** Let $u(x, t)$ be displacement in a vibrating string then

$$u_{tt} = c^2u_{xx}.$$

2.2 Classification of First order PDE of two variables:

a. Equation (7) is called a quasilinear first order PDE (in two variables) if

$$f(x, y, z, p, q) = A(x, y, z)p + B(x, y, z)q - C(x, y, z)$$

i.e. (7) is linear in p and q; hence can be written as

$$A(x, y, z)p + B(x, y, z)q = C(x, y, z). \tag{9}$$

b. In case $A(x, y, z)$ and $B(x, y, z)$ in equation (9) are functions of the independent variables x and y only, then the equation is called semilinear. *i.e.*

$$A(x, y)p + B(x, y)q = C(x, y, z). \tag{10}$$

c. Equation (7) is called a linear first order PDE (in x, y) if (7) is linear in z, p and q, i.e. it can be written as

$$A(x, y)p + B(x, y)q = C(x, y)z + D(x, y), \tag{11}$$

where A, B, C and D are known “smooth” functions. If $D(x, y) \equiv 0$, then (11) is called *linear homogeneous*, otherwise it is called linear non-homogeneous.

Equation (7) is called fully nonlinear if it is not any one of the above types.

Note: The linear equations is a subclass of semilinear equations which in turn is a subclass of quasilinear equations.

Examples:

(1) $2xp - 2yq = u + \sin x$ is a linear non-homogeneous equation,

(2) $p + q = -2z$ is a linear homogeneous equation,

(3) $yp + (x + y)q = z^2$ is a semilinear equation,

(4) $zp + q = 0$ is a quasilinear equation but non a linear one,

(5) $p^2 + q^2 = z$ is a nonlinear equation.

2.3 Solution of PDE

A continuously differentiable function $z = z(x, y)$, $(x, y) \in D \subset \mathcal{R} \times \mathcal{R}$ is said to be solution of Equation (7) if z and its partial derivative w.r.t. x, y (*i.e.* p and q,) when substituted in equation reduce it to an identity.

For example (i) $z = ax + by$, $(a, b \in \mathcal{R})$ is a solution of $z = px + qy$.
(ii) $z = yg(x/y)$ is also a solution of above equation *i.e.* $z = px + qy$.
(iii) $z^2 = y^2 + (x + 1)^2$ is a solution of $(p^2 + q^2) = 1$.

<u>Remark:</u> A solution $z = z(x, y)$ in three dimensional space can be interpreted as surface and hence is called **integral surface** of the pde.

Complete Integral: A two parameter family of solutions $z = F(x, y, a, b)$ is called complete integral of Equation (7) if the rank of the matrix

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix},$$

is two in D.

General Integral or General Solution: $F(u, v) = 0$, where, $u = u(x, y, z)$ and $v = v(x, y, z)$ and F is an arbitrary smooth function, is called a (implicit or explicit) general solution of (7) if z, $p = z_x$, $q = z_y$ as determined by the relation $F(u, v) = 0$ satisfy the pde (7).

<u>Remark:</u> Since F is arbitrary, we may choose F in the form

$$F(u, v) = u - g(v),$$

where g is arbitrary smooth function and where u and v are known functions of x, y and z. Depending on the requirement, we can have different choices of F, especially the arbitrariness of F is exploited to suit the needs.

Singular Solution: Singular solution is an envelope of complete integral (*i.e.* two parameter family of solution surfaces $z = F(x, y, a, b)$ of equation (7)) and is obtained by eliminating a and b from

$$z = F(x, y, a, b), \quad \frac{\partial F}{\partial a} = 0, \quad \frac{\partial F}{\partial b} = 0.$$

Cauchy's Problem (IVP): To find the integral surface of equation (7), which contains an initial curve

$$\Gamma:\ x = x_0(s),\ y = y_0(s),\ z = z_0(s), \quad s \in I. \tag{12}$$

Note that the initial data of the Cauchy's Problem removes the arbitrary parameters/ arbitrary function in complete integral/general solution, hence we get the integral surface of the pde.

Example: $F(x+y, x-\sqrt{z}) = 0$ is a general solution of $p - q = 2\sqrt{z}$.

From the given relation we have, by differentiating partially w.r.t. x, y [let $u = x+y, v = x-\sqrt{z}$]

$$F_u + \left(1 - \frac{p}{2\sqrt{z}}\right) F_v = 0$$

$$F_u + \left(-\frac{q}{2\sqrt{z}}\right) F_v = 0$$

Eliminating F_u and F_v, we have

$$p - q = 2\sqrt{z}$$

Example: The relation

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

is a complete solution of

$$z^2(1+p^2+q^2) = 1$$

Example: Another simpler example is

$$z = ax + y/a + b,$$

which is a complete solution of $pq = 1$

3 Lagrange's Method

Consider a quasilinear equation

$$A(x,y,z)p + B(x,y,z)q = C(x,y,z), \tag{13}$$

where $A, B, C : D \to \mathcal{R}$ are given (known) "smooth" functions. $D \in \mathcal{R}^3$ is a suitable domain. We usually assume that A, B and C are C^1 functions which do not vanish simultaneously.

Let $S:\ z = z(x,y),\ (x,y) \in D$ be "integral" surface of (13) and $P \equiv P(x,y,z)$ be a point on S, then the normal at P to S is in the direction $(p,q,-1)$. Hence equation (13) is equivalent to saying that $(p,q,-1)$ and (A,B,C) are orthogonal at each $P \in S$. *i.e.* $A\hat{i} + B\hat{j} + C\hat{k}$ lies on the tangent plane at P. Hence for a curve $C:\ x = x(t),\ y = y(t),\ z = z(t)$ on the surface S we have $A\hat{i} + B\hat{j} + C\hat{k}$ is parallel to $\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$, which gives

$$\frac{\dot{x}}{A} = \frac{\dot{y}}{B} = \frac{\dot{z}}{C}. \tag{14}$$

This is equivalent to saying that

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}. \tag{15}$$

This motivates us to the following result,
Theorem: Let A, B, C be C^1 functions (of x, y and z). The general solution of

$$A(x,y,z)p + B(x,y,z)q = C(x,y,z) \tag{16}$$

is

$$F(u,v) = 0, \tag{17}$$

where F is an arbitrary smooth function of u and v and $u(x,y,z) = C_1$ and $v(x,y,z) = C_2$ are two independent solutions of Auxiliary equations (A.E.)

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C} \tag{18}$$

Proof: Let $u = c_1$ and $v = c_2$ be solution of A.E. then

$$du = 0, \text{ and } dv = 0$$

which gives

$$u_x dx + u_y dy + u_z dz = 0$$

But

$$\frac{\dot{x}}{A} = \frac{\dot{y}}{B} = \frac{\dot{z}}{C}$$

thus we get

$$Au_x + Bu_y + Cu_z = 0 \tag{19}$$

similarly

$$Av_x + Bv_y + Cv_z = 0. \tag{20}$$

Solving for A, B and C we get,

$$\frac{A}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{B}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{C}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{21}$$

Also recall that

$$F(u,v) = 0$$

leads to Partial Differential Equation(see equation (8))

$$\frac{\partial(u,v)}{\partial(y,z)}p + \frac{\partial(u,v)}{\partial(z,x)}q = \frac{\partial(u,v)}{\partial(x,y)}. \tag{22}$$

Comparing equations (21) and (22) we get,

$$Ap + Bq = C$$

Hence $F(u,v) = 0$ is general solution of quasilinear pde (16).

Thus, to find general solution of (16) we write AE (18). Solve this system of ODE to get $u(x,y) = c_1$ and $u(x,y) = c_2$. Then general solution is given by equation (17).
Example: $x^2p + y^2q = (x+y)z$

A.E. is $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$

(i) $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} \;\Rightarrow\; \dfrac{1}{x} - \dfrac{1}{y} = c_1$

(ii) $\dfrac{dx - dy}{x^2 - y^2} = \dfrac{dz}{(x+y)z}$

$\dfrac{d(x-y)}{x-y} = \dfrac{dz}{z}$

$x - y = c_2 z \;\Rightarrow\; F\left(\dfrac{1}{x} - \dfrac{1}{y}, \dfrac{z}{x-y}\right) = 0$

or $\dfrac{z}{x-y} = C_1\left(\dfrac{1}{x} - \dfrac{1}{y}\right)$

Example: Find the integral surface of

$$xp + yq = z \tag{23}$$

which contains

$$\Gamma : x_0 = s^2,\ y_0 = s+1,\ z_0 = s.$$

Recall that $u = \dfrac{y}{z} = c_1$ and $v = \dfrac{y}{x} = c_2$ are two solutions of A.E.

Now if Γ is the curve on this integral surface then it must lie on both of them. Thus,

$$\frac{s+1}{s} = c_1. \ \& \ \frac{s+1}{s^2} = c_2$$

which gives $(c_1 - 1)c_1 = c_2$ OR $(u-1)u = v$ i.e. $(y-z)x = z^2$.

Example: Discuss the existence of integral surface of

$$2p + 3q + 8z = 0$$

which contains the curve:

(i) $\Gamma :\ z = 1 - 3x$ in x-y plane,

(ii) $\Gamma :\ z = x^2$ on the line $2y = 1 + 3x$,

(iii) $\Gamma :\ z = e^{-4x}$ on the line $2y = 3x$.

Solution: It is easy to see that general solution in this case is given by:

$$z = e^{-4x}G(2y - 3x)$$

(i) If Γ lies on integral surface, then

$$1 - 3x = e^{-4x}G(0 - 3x),\ \text{so}\ G(s) = (1+s)e^{-\frac{4s}{3}}.$$

So integral surface is:

$$z = e^{-4x}(1 + 2y - 3x)e^{-4(2y-3x)/3}$$

(ii) If Γ lies on integral surface, then

$$x^2 = e^{-4x} G(1)$$

which is not possible, so no solution of cauchy's problem here.

(iii) The initial condition gives:

$$e^{-4x} = e^{-4x} G(0) \Rightarrow G(0) = 1$$

Hence there are infinitely many choices for $G(t)$, e.g. $G(t) = \cos t,\ e^t,\ 1+t$ *etc.* (*No unique solution in this case*)

Remark: Thus note that Cauchy's problem can have unique solution, no solution or infinitely many situations depending on the initial data. So for unique solution the choice of initial data is not arbitrary.

4 Method of Characteristics

4.1 Semilinear Case

For semilinear equation

$$A(x,y)p + B(x,y)q = C(x,y,z) \tag{24}$$

Consider one parametric family of curves in the xy plane given by

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)} \tag{25}$$

[Or equivalently $\dot{x} = A(x(t), y(t)),\ \dot{y} = B(x(t), y(t))$]

Along these curves, $z = z(x, y(x))$ will satisfy the relation

$$\begin{aligned} \frac{dz}{dx} &= z_x + y' z_y \\ &= \frac{A(x,y)p + B(x,y)q}{A(x,y)} \\ &= \frac{C(x,y,z)}{A(x,y)} \end{aligned}$$

(Or equivalently $\dot{z} = C(x(t), y(t), z(t))$.)

The one parameter family of curves $y_\lambda = y(x)$ given by (25) are called 'Characteristic Curves of PDE (24)'.

Let (x_0, y_0) be a point in xy plane, then

$$y' = \frac{B}{A}; \quad y(x_0) = y_0$$

has unique solution (*assuming* $\frac{B}{A}$ *is continuous function of* x, y *having derivative w.r.t.* y).

Now if we assign a value to z at (x_0, y_0) then

$$\frac{dz}{dx} = \frac{C(x, y(x), z(x))}{A(x, y(x))}; \quad z(x_0) = z_0 \tag{26}$$

will determine a unique $z = (x, y, x_0, y_0)$ (*assuming* $\frac{C}{A}$ *is continuous function of* x *and* z *having derivative w.r.t.* z).

Thus $z(x, y)$ is uniquely determined along the whole characteristic curve passing through (x_0, y_0) if $z(x_0, y_0) = z_0$.

Consider a curve Γ_0 in xy plane intersecting the characteristic curves of one parameter family y_λ. Assign z at each point of Γ_0 i.e. $z(x_0, y_0) = z_0$ on Γ_0 then (26) will determine a unique curve. Now if we consider the region covered by C_λ in xy plane then $z = z(x, y)$ will determine the integral surface containing $(x_0, y_0, z(x_0, y_0))$; $(x_0, y_0) \in \Gamma_0$

Note that the initial curve Γ consists of (x_0, y_0, z_0) where $x_0, y_0 \in \Gamma_0$.

Example: $p - q = 2yz$

Characteristic curve $y' - 1 \quad \Rightarrow \quad x + y = c$.

Now $\frac{dy}{dx} = 2yz$ which on the curve $x + y = c$ gives

$$\frac{dz}{dx} = 2(c - x)z \quad \Rightarrow z = k(c)e^{-(c-x)^2}$$

Hence $z = k(x + y)e^{-y^2}$. Let $z = x$ on $x = y$ then

$$k(2x) = xe^{x^2} \quad \Rightarrow z = \frac{(x + y)}{2} e^{\frac{(x+y)}{2}} e^{-y^2}.$$

4.2 Quasilinear Case

Consider the quasilinear equation

$$A(x, y, z)p + B(x, y, z)q = C(x, y, z). \tag{27}$$

Recall that if $z = z(x, y)$ is an integral solution of (27) then for a curve $\overline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ through a point P on this integral surface, the tangential direction $(\dot{x}, \dot{y}, \dot{z})$ at P is perpendicular to the normal to the integral surface, i.e. (A, B, C) is parallel to $(\dot{x}, \dot{y}, \dot{z})$.

<u>Characteristic curve:</u> A curve $\mathcal{C} : (x(t), y(t), z(t)), t \in I$ is called a characteristic curve for (27) if x, y and z satisfy the following:

$$\left.\begin{aligned} \frac{dx}{dt} &= A(x(t), y(t), z(t)) = A(x, y, z) \\ \frac{dy}{dt} &= B(x(t), y(t), z(t)) = B(x, y, z) \\ \frac{dz}{dt} &= C(x(t), y(t), z(t)) = C(x, y, z) \end{aligned}\right\} \tag{28}$$

Let $P = P(x_0, y_0, z_0) \in S$. Let A, B, C be C^1 functions. Consider the initial conditions

$$x(t_0) = x_0, \; y(t_0) = y_0, \; z(t_0) = z_0 \tag{29}$$

The IVP (28) and (29) will have a unique solution in an interval I containing t_0(this follows as a consequences of the Picard's theorem or existence and uniqueness theorem for a system of equations with initial conditions). Consider the surface generated by this family of characteristic curves, obtained by various choices of initial conditions, then this is integral surface of (27). If we consider a point P on this surface, then the tangent plane will contain the tangent direction (A, B, C). Hence it is perpendicular to the normal $(z_x, z_y, -1) \equiv (p, q, -1)$. So the surface is an integral surface.

Therefore, integral surface is generated by characteristic curves as it contains characteristic curves through each point. Converse of this result is also true i.e. integral surface $z = z(x, y)$ of (27) is generated by the characteristic curves of (27). This is the essence of the following result.

Theorem: let A, B, C in (27) be C^1 functions. Let $S : \; z = u(x, y)$ be an integral surface for (27) and $P = P(x_0, y_0, z_0)$ be a point on S. Let $\mathcal{C}$ be a characteristic curve through P, then $\mathcal{C}$ lies on S.

Proof: Let x, y, z be solutions of IVP (28) and (29). Then, by the Picard's theorem for system with initial conditions, x, y, z exists on interval I with $t_0 \in I$. define

$$\mathcal{C} \;\; (x(t), y(t), z(t)) \, , \quad t \in I$$

By definition $\mathcal{C}$ is the (by uniqueness) characteristic passing through $P \equiv (x_0, y_0, z_0)$. Define

$$U(t) := z(t) - u\left(x(t), y(t)\right), \quad t \in I$$

Since $P \in S \implies U(t_0) = 0$. Also

$$\frac{dU(t)}{dt} = C(x, y, z) - A(x, y, z)u_x - B(x, y, z)u_y$$

or in other words, $U(t)$ satisfies the IVP

$$\left.\begin{aligned} &\frac{dU(t)}{dt} = C\left(x, y, U + u(x, y)\right) - A\left(x, y, U + u(x, y)\right)u_x - B\left(x, y, U + u(x, y)\right)u_y, \\ &U(t_0) = 0 \end{aligned}\right\} \tag{30}$$

We note that $U \equiv 0$ satisfies (30). By uniqueness theorem

$$U(t) \equiv 0, \quad t \in I,$$

$$i.e. z(t) = u(x(t), y(t))$$

hence, $\mathcal{C}$ lies on S.

Remark: In short, to find an integral surface it is sufficient to find the characteristic curves [defined by (28)] and take its union. Thus, Cauchey's problem reduces to to find "the" integral surface of (27) containing a given curve. In other words, we wish to select an integral surface S so that a given curve

$$\Gamma : \; x_0 = f(s), y_0 = g(s), z_0 = h(s), \quad s \in I \tag{31}$$

lies on $S : \; z = u(x, y)$. Equivalently, we are seeking a solution u of (27) so that

$$h(s) = u\left(f(s), g(s)\right), \quad s \in I$$

Remark:

1. On many occasions, we relate one of the independent variables to time. Then the initial condition
$$u(x,0) = h(x) \tag{32}$$
is called an initial condition stated at $y = 0$. Solving for u given (27) and (32) is also referred to as an initial value problem. Here h is called initial data.

2. Note that for unique solution, we need that the initial data curve is not one of the characteristic curve, Hence the initial data curve should be such that
$$\frac{dx_0}{ds}B(x_0(s), y_0(s), z_0(s)) - \frac{dy_0}{ds}A(x_0(s), y_0(s), z_0(s)) \neq 0.$$

Below, we outline a procedure (Method of Characteristics) to solve a Cauchy problem (27) and (31). Of course, we are satisfied with "local solution" u defined for (x, y) near $x_0 = f(s_0), y_0 = f(s_0)$ for some $s_0 \in I$.

From the above argument/analysis, it is clear that the integral surface
$$S: \; z = u(x, y)$$
passing through Γ consists of characteristics curves passing through each point of Γ. Consider
$$x = X(s,t), \; y = Y(s,t), \; z = Z(s,t) \tag{33}$$
with
$$x(s,0) = f(s), \; y(s,0) = g(s), \; z(s,0) = h(s) \tag{34}$$
where the (34) is equivalent to
$$X(s,0) = f(s), \; Y(s,0) = g(s), \; Z(s,0) = h(s)$$
From the general theory of continuous dependence on parameters
$$X(s,t), \; Y(s,t), \; Z(s,t)$$
are C^1 functions in (s,t) if a, b, c, f, g and h are C^1 functions lies in the respective domain of definitions. Now
$$x_0 = X(s_0, 0), \; y_0 = Y(s_0, 0)$$
By the "Implicit function" theorem we have
$$s = S(x,y), \; t = T(x,y)$$
to be the solutions of
$$x = X\left(S(x,y), T(x,y)\right), \; y = Y\left(S(x,y), T(x,y)\right) \tag{35}$$
of class C^1 near $(s_0, 0)$ with $s_0 = S(x_0, y_0)$; $0 = T(x_0, y_0)$ provided
$$J = \begin{vmatrix} X_s(s_0,0) & Y_s(s_0,0) \\ X_t(s_0,0) & Y_t(s_0,0) \end{vmatrix} \neq 0 \tag{36}$$

Now (36) is equivalent to

$$J = \begin{vmatrix} f'(s_0) & g'(s_0) \\ a(x_0, y_0, z_0) & b(x_0, y_0, z_0) \end{vmatrix} \neq 0 \tag{37}$$

From (35), we have

$$z = u(x, y) := Z\left(S(x, y), T(x, y)\right)$$

is the explicit representation of the integral surface in a neighbourhood of (x_0, y_0).

Example 6: Let us find the solution u of the Cauchy problem

$$u_x = 1$$

so that the corresponding integral surface $S:\ z = u(x, y)$ conatins the curve

$$\Gamma:\ f(s) = s,\ g(s) = -s \quad \text{and} \quad h(s) = 0, \quad s \in \mathcal{R}$$

Comparing with (27), we have $a \equiv 1 \equiv c$ and $b = 0$. The characteristic equations are:

$$\frac{dx}{dt} = 1,\ \frac{dy}{dt} = 0,\ \frac{dz}{dt} = 1$$

with

$$x(0) = s,\ y(0) = -s,\ z(0) = 0$$

Solving we get

$$x(t) = t + s,\ y(t) = -s \quad \text{and} \quad z(t) = t$$

$$[X(t, s) = t + s,\ Y(t, s) = -s \quad \text{and} \quad Z(t, s) = t]$$

which shows that

$$x = t - y \implies t = x + y = T(x, y)$$

and

$$z(t) = t \implies z = x + y = u(x, y)$$

Thus, $u(x, y) = x + y$ is the desired solution and the required integral surface S is

$$S:\ z = x + y$$

Example 7: We wish to find an integral surface for

$$xp + yq = u \tag{38}$$

which passes through the curve

$$\Gamma:\ f(s) = s^2,\ g(s) = s + 1,\ h(s) = s \tag{39}$$

Equivalently we wish to find a solution u of (38) so that

$$S:\ z = u(x, y)$$

is an integral surface with $\Gamma \subseteq S$. The characteristic equations are

$$\frac{dx}{dt} = x, \; \frac{dy}{dt} = y, \; \frac{dz}{dt} = z \tag{40}$$

with conditions

$$x(0) = s^2, \; y(0) = s + 1, \; z(0) = s \tag{41}$$

The solutions of the IVP (40) and (41) is

$$x = X(t, s) = s^2 e^t, \; y = Y(t, s) = (1 + s)e^t, \; z = Z(t, s) = se^t$$

Eliminating t and s between the three relations, we have

$$s = x/z, \; y/z = \frac{1 + s}{s} = \frac{x + z}{x}$$

Thus

$$xy = zx + z^2 \tag{42}$$

is the implicit solution.

$\longleftarrow \;\; \pi \;\; \longrightarrow$