Lecture 4: Limits of Functions of several variables

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Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Why one should study continuity first and limits thereafter?

Example 4.1 *Suppose $D = \{(k, 0) : k \in \mathbb{Z}\}$. Then show that any function from D to $\mathbb{R}$ is continuous at every point of D.*

Solution: Fix some $(k_0, 0) \in D$. Let $((x_n, y_n))$ be a sequence in D such that $(x_n, y_n) \to (k_0, 0)$, i.e., $x_n \to k_0$ and $y_n \to 0$. Since $x_n \in \mathbb{Z}$ and $x_n \to k_0$, hence there exists n_0 such that $x_n = k_0$ for all $n \geq n_0$. By definition $y_n = 0$ for all n. Hence $(x_n, y_n) = (k_0, 0)$ for all $n \geq n_0$. Hence for any function f, we have

$$f(x_n, y_n) = f(k_0, 0), \forall n \geq n_0.$$

Hence $f(x_n, y_n) \to f(k_0, 0)$. Therefore f is continuous at $(k_0, 0)$. Since $(k_0, 0)$ is arbitrary, f is continuous at every point of D. ■

Remark 4.2 *Let (x_0, y_0) be any point in the plane. Then the domain D in Example 4.1, does not have nonempty intersection with every punctured disk centered (x_0, y_0), hence we can not talk about limit of any function f at (x_0, y_0).*

Question: One may wonder that why the definition of limit requires constraints on the domain of f, whereas definition of continuity does not require any condition on the domain of the function. What happens if we allow any domain in definition of limit?

Answer: The reason is, for continuity of f at point (x_0, y_0), the function f has to be defined at (x_0, y_0) but for limit it is not necessary. So if (x_0, y_0) is an "isolated point", where we are interested in limits then we loose the uniqueness of the limit. In order to avoid this absurdity we need to put some restriction on the domain of f, and this achieved by saying that the domain f has nonempty intersection with every punctured disk centered at (x_0, y_0).

Suppose we make the following definition of limit:

Suppose that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function. Let (x_0, y_0) be a point in the plane (not necessarily in D). We say that the f has a limit as (x, y) approaches (x_0, y_0) if there exists

$L \in \mathbb{R}$ such that for every sequence $((x_n, y_n))$ in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to L$.

Now apply this definition in Example 4.1. Then any number is a limit at every point. To see this let (x_0, y_0) be any point in the plane. Now take a sequence $((x_n, y_n))$ in $D \setminus \{(x_0, y_0)\}$ which converges to (x_0, y_0). Basically there is no such sequence in $D \setminus \{(x_0, y_0)\}$. Hence the statement $f(x_n, y_n) \to l$ is vacuously true for every real number l.

Theorem 4.3 *Suppose that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function. Let (x_0, y_0) be a point in D such that every punctured disk centered at (x_0, y_0) contains at least one point from the set D. If f has a limit as (x, y) approaches (x_0, y_0), and if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0),$$

then the function f is continuous at the point (x_0, y_0).

Example 4.4 *Suppose that $f(x, y) = \dfrac{(x^2 - y^2)^2}{x^2 + y^2}$. Discuss the limiting behavior of f as $(x, y) \to (0, 0)$.*

Solution: The largest possible domain of f is the punctured plane. So we can talk about limit at $(0, 0)$. So let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Hence

$$0 \le f(x_n, y_n) = \frac{(x_n^2 - y_n^2)^2}{x_n^2 + y_n^2} \le \frac{(x_n^2 + y_n^2)^2}{x_n^2 + y_n^2} = x_n^2 + y_n^2$$

Hence by sandwich theorem $f(x_n, y_n) \to 0$. Therefore as as $(x, y) \to (0, 0)$, f has limit 0. ■

If λ is a real constant and if f and g real-valued functions with a common domain $D \subset \mathbb{R}^2$, then we define the functions $\lambda f, f + g, f - g, fg$ and $\dfrac{f}{g}$ on D by

$$\begin{aligned}
(\lambda f)(x, y) &= \lambda f(x, y) \\
(f + g)(x, y) &= f(x, y) + g(x, y) \\
(f - g)(x, y) &= f(x, y) - g(x, y) \\
(fg)(x, y) &= f(x, y)g(x, y) \\
\left(\frac{f}{g}\right)(x, y) &= \frac{f(x, y)}{g(x, y)}
\end{aligned}$$

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

Polynomials in two variable: A polynomial in two variables x and y (with real coefficients) is a finite sum of terms of the form cx^iy^j, where i, j are nonnegative integers and $c \in \mathbb{R}$; For instance, $p(x,y) = x^5y + 2x^4 + y^2 + 1$ and $q(x,y) = x^3 + x^2y + 6xy^2$ are polynomials. Thus, if $p(x,y)$ is a polynomial in the variables x and y, then for any $(x_0, y_0) \in \mathbb{R}^2$, by substituting x_0 for x and y_0 for y in $p(x,y)$, we obtain a real number, denoted by $p(x_0, y_0)$. So each polynomial defines a function from $\mathbb{R}^2$ to $\mathbb{R}$.

It is easy to see that a polynomial in two variable (with real coefficients) is continuous everywhere.

Rational Functions A function f of the from $f(x,y) = \dfrac{p(x,y)}{q(x,y)}$ where p and q are polynomials, is called a rational function. By quotient rule f is continuous at all points in the plane where $q(x,y) \neq 0$.

Example 4.5 *Find* $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x+y+1}{x^2-y^2}$ *(if it exists).*

Solution: It is a rational function and $x^2 - y^2$ is not zero at $(1,2)$ hence limit exists and it is equal to $-4/3$ just by evaluation. ∎