Assignment 11: Laplace Transformation

1. (a) Taking LaplaceTransform,we get

$$Y(s) = \frac{e^{-as}}{s+1} + \frac{e^{-bs}}{s+1} \implies y(t) = u(t-a)e^{-(t-a)} + u(t-b)e^{-(t-b)}$$

(b) Taking Laplace Transform, we get

$$Y(s) = \frac{s^2+1}{(s+1)^3} = \frac{1}{1+s} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

Thus,

$$y(t) = e^{-t}(t-1)^2$$

(c) Taking Laplace Transform, we get

$$Y(s) = -\frac{2}{s^2+1} + \frac{24}{3(s^2+4)} \implies y(t) = -2\sin t + 4\sin 2t$$

2. We can use the following result: For a function $f(x)$ with period a,

$$F(s) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

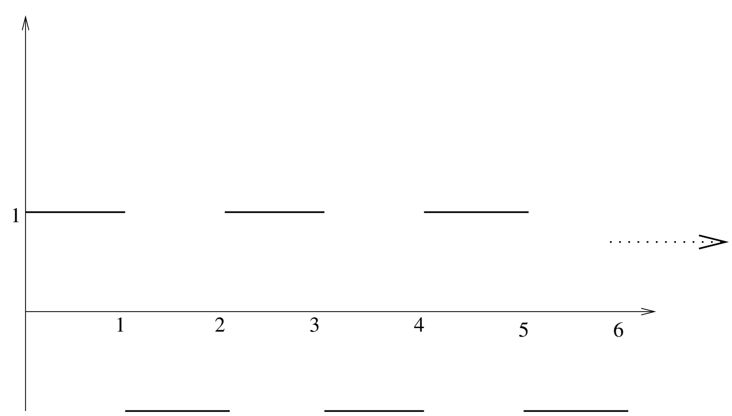

Figure 1: Q 8(a)]

(a) Use the above formula to get

$$F(s) = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st}\, dt - \int_1^2 e^{-st}\, dt \right] = \frac{1-e^{-s}}{s(1+e^{-s})} = \frac{1}{s}\tanh(s/2)$$

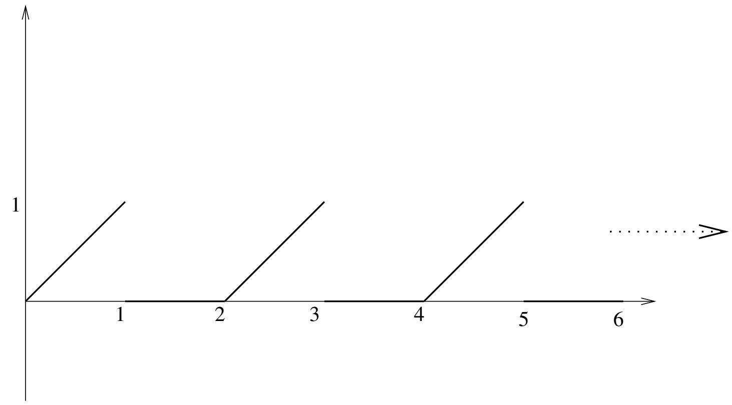

Figure 2: Q 8(b)]

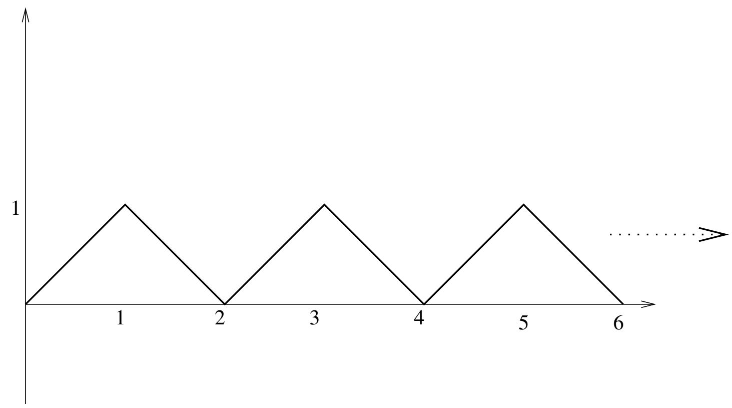

Figure 3: Q 8(c)]

(b) Using the above formula,

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^1 te^{-st}\,dt = \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-2s})}$$

(c) Using the above formula,

$$F(s) = \frac{1}{1-e^{-2s}} \left[\int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt\right] = \frac{1}{s^2}\tanh(s/2)$$

Assignment-11 : Fourier Series

3. (a) $a_n = 0$ and $\pi\, b_n = 2\int_0^\pi \sin nx\, dx = 2[1-(-1)^n]/n$. Thus, $b_{2m} = 0$ and $b_{2m+1} = 4/[\pi(2m+1)]$. Thus,

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right), \qquad f(101\pi/2) = f(\pi/2) = 1$$

(b) $a_n = 0$ and

$$\pi b_n = 2\int_0^{\pi/2} x \sin nx\, dx \implies b_n = -\frac{1}{n}\cos(n\pi/2) + \frac{2}{n^2\pi}\sin(n\pi/2)$$

Thus,

$$f(x) = \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \cdots$$

$$f(101\pi/2) = f(\pi/2) = \frac{f(\pi/2+) + f(\pi/2-)}{2} = \frac{\pi}{4}$$

(c) $b_n = 0$, $a_0 = 2\pi^2/3$ and

$$a_n = (-1)^n \frac{4}{n^2}$$

Thus,

$$f(x) = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \frac{\cos 4x}{16} + \cdots\right), \qquad f(101\pi/2) = f(\pi/2) = \frac{\pi^2}{16}$$

(d) $a_0 = 2\pi$, $a_n = 0$, $n \geq 1$ and $b_n = -2/n$ and

$$f(x) = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right), \qquad f(101\pi/2) = f(\pi/2) = \frac{\pi}{2}$$

4. (i) Put $x = \pi/2$ in 3(a) (ii) Put $x = 0$ in 3(c)

5. We have

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos(n\pi x/2)\,dx \implies a_0 = 1, a_n = 0, n \geq 1$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(n\pi x/2)\,dx \implies b_n = \frac{1 + (-1)^{n+1}}{n\pi}$$

Thus,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin(\pi x/2) + \frac{\sin(3\pi x/2)}{3} + \frac{\sin(5\pi x/2)}{5} + \cdots\right)$$

6. (i) Here $p = 2L = 2$. For cosine series

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx \implies a_0 = 2 + 4/\pi, a_1 = 0,\ a_n = -\frac{2}{\pi}\frac{1 + (-1)^n}{n^2 - 1},\ n \geq 2$$

Thus, cosine series is

$$f(x) = 1 + \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2\pi x}{3} + \frac{\cos 4\pi x}{15} + \frac{\cos 6\pi x}{35} + \cdots\right)$$

(ii) For sine series

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^x \sin(nx)\,dx = \frac{2}{\pi}\left[-n\left\{(-1)^n e^{\pi} - 1\right\} - \frac{n^2\pi}{2} b_n\right] \Rightarrow b_n = \frac{2n}{\pi}\left[\frac{1 - (-1)^n e^{\pi}}{1 + n^2}\right].$$

Thus, sine series is

$$\frac{2}{\pi}\sum_{1}^{\infty} \frac{n\left[1 - (-1)^n e^{\pi}\right]}{1 + n^2}\sin nx.$$

7. The Fourier integral can be written as

$$f(x) = \int_0^\infty [A(\omega)\cos x\omega + B(\omega)\sin x\omega]d\omega$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos x\omega\,dx \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin x\omega\,dx$$

(a)

$$A(\omega) = \int_0^\infty e^{-x}\cos x\omega\,dx = \frac{1}{1+\omega^2}, \qquad B(\omega) = \int_0^\infty e^{-x}\sin x\omega\,dx = \frac{\omega}{1+\omega^2}$$

(b) $f(x) = \pi e^{-x}/2$ for $x > 0$. For Fourier cosine integral, we make even extension of the function. Thus, $B(\omega) = 0$ and

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos x\omega\,dx = \frac{1}{1+\omega^2}$$

8. Here $f(x) = 1/(1+x^2)$ is even function. Thus, $B(\omega) = 0$ and

$$A(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\cos x\omega}{1+x^2}\,dx = e^{-\omega}, \ (\omega > 0)$$

(See the Kreyszig Book: Laplace Intergral)