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Ans 1: $\lim_{z \to z_0} f(z)$ exist.

To Prove : it is unique

$\because$ using method of contradiction,

let us assume there exist two limit l_1 and l_2.

then $|l_1 - f(z) + f(z) - l_2| \leq |l_1 - f(z)| + |f(z) - l_2|$

$|l_1 - f(z)| < \epsilon$

$|l_2 - f(z)| < \epsilon$

$\therefore$ $|l_1 - f(z)| + |f(z) - l_2| < 2\epsilon$

$\therefore$ $l_1 = l_2$.

hence only one limit exist

Ans 2: a) given: $f(z)$ is continous.

$g(z)$ is continous.

Then, $|f(z) - f(z_0)| < \epsilon$ where $|z - z_0| < \delta_1$.

$|g(z) - g(z_0)| < \epsilon$ where $|z - z_0| < \delta_2$.

Then for all z satisfying $|z - z_0| < \delta$

$|f(z) + g(z) - f(z_0) - g(z_0)| < |f(z) - f(z_0)| + |g(z) - g(z_0)| < \epsilon + \epsilon$

$= 2\epsilon$.

b) $|f(z)g(z) - f(z_0)g(z_0)|$ $f(z)g(z)$.

= We know $|f(z) - f(z_0)| < \epsilon$.

$|g(z) - g(z_0)| < \epsilon$

then, $|f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)| <$

$|f(z)g(z) - f(z)g(z_0)| + |f(z)g(z_0) - f(z_0)g(z_0)|$

taking $|f(z)|$ and $|g(z_0)|$ common

$$|f(z)|\,|\underbrace{g(z)-g(z_0)}_{<\epsilon}| + |g(z_0)|\,|\underbrace{f(z)-f(z_0)}_{<\epsilon}| <$$

$\therefore$

$$|f(z)|\epsilon + |g(z_0)|\epsilon \to 0.$$

$$\leq \epsilon + |f(z_0)|$$

$$\therefore \quad \underbrace{\epsilon^2 + |f(z_0)|\epsilon}_{\epsilon'} = 0$$

$\therefore$ proved.

c) $\dfrac{f(z)}{g(z)}$

we know $|f(z)-f(z_0)| < \epsilon$ for $|z-z_0| < \delta_1$

$|g(z)-g(z_0)| < \epsilon$ for $|z-z_0| < \delta_2$

Then, $\left|\dfrac{f(z)}{g(z)} - \dfrac{f(z_0)}{g(z_0)}\right|$

$$= \frac{|f(z)g(z_0) - f(z_0)g(z)|}{|g(z)g(z_0)|} <$$

$$= \frac{|f(z)g(z_0) - f(z)g(z) + f(z)g(z) - f(z_0)g(z)|}{|g(z)g(z_0)|}$$

$$< \frac{|f(z)|\,|g(z)-g(z_0)|}{|g(z)g(z_0)|} + \frac{|g(z)|\,|f(z)-f(z_0)|}{|g(z)|\,|g(z_0)|}$$

$$< \underbrace{\frac{(\epsilon + |f(z_0)|)\epsilon}{(\epsilon + |g(z_0)|)|g(z_0)|}}_{\epsilon' \to 0.} + \frac{\epsilon}{|g(z_0)|} \to 0$$

ϵ

$\therefore$ proved.

3. a) $f(z) = \frac{1}{z}$ continuous except at $z = 0$.

b) $g(z) = \frac{z}{z^2+1}$ everywhere except $z = \pm i$.

4. a) $\lim_{z \to 1} (z^2 - 3z + 3) = 1$.

$|f(z) - f(z_0)| < \epsilon$. for all $|z - z_0| < \delta$. i.e. $0 < |z-1| < \delta$

then $|z^2 - 3z + 3 - 1| = |z^2 - 3z + 2| = |(z-1)(z-2)|$

$|z-2| - 1 < |z - 2 + 1|$ $\cancel{< |z-2| + 1}$

$|z-2| < |z-1| + 1 < \delta + 1$.

$\therefore$ $|z-1||z-2| < (\delta+1)\delta < \delta^2 + \delta < \delta' = \epsilon$

$\therefore$

$0 < |(z^2 - 3z + 3) - 1| < \epsilon$ for $|z-1| < \delta$.

b) $\lim_{z \to -i} \frac{iz}{3} = \frac{1}{3}$.

$\left| \frac{iz}{3} - \frac{(+1)}{3} \right| < \epsilon$ for all $|z - (-i)| < \delta$.

i.e, $|z + i| < \delta$.

$\cancel{\left| \frac{iz}{3} + i \right|} = \cancel{|i| \left| \frac{z}{3} + i \right|}$ $\left| \frac{iz}{3} + \frac{i^2}{3} - \frac{i^2}{3} + i \right|$

$\cancel{\left| \frac{iz}{3} + i \right|} > \left| \frac{i}{3} \right| \cancel{|z + i|} - \cancel{\left| i - \frac{i^2}{3} \right|}$

$\cancel{\left| \frac{iz}{3} + i \right| + \left| i + \frac{1}{3} \right| > \frac{1}{3} |z+i|}$.

Now, $\left| \frac{iz}{3} + \frac{i^2}{3} \right| = \frac{1}{3} |z + i| < |z + i| < \delta$.

$\therefore$ there exist ϵ

such that $\left| \frac{zi}{3} - \frac{1}{3} \right| < \epsilon$

for $|z + i| < \delta$.

5. a) $\lim_{z \to z_0} \frac{\bar{z}}{z} = \lim_{x \to x_0, iy \to iy_0} \frac{x - iy}{x + iy}$

put $y = mx$ we get -

$\lim_{x \to x_0} \frac{x - imx}{x + im} = \frac{1 - im}{1 + im}$ $\therefore$ limit does not exist $= 0$

b) $\lim_{z \to z_0} \frac{z^2}{|z|^2} = \lim_{x \to x_0} \frac{(x + imx)^2}{|(x + imx)|^2} = \frac{x^2 + 2imx^2 - m^2x^2}{x^2 + m^2x^2}$

$= \left(\frac{1 + 2im - m^2}{1 + m^2}\right) = \frac{(1+m)^2}{1+m^2}$

$\therefore$ limit does not exist.

c) $\lim_{z \to z_0} \frac{[\operatorname{Re} z - \operatorname{Im} z]^2}{|z|^2} = \lim_{x \to x_0} \frac{[x - ixm]^2}{x^2 + m^2x^2} = \frac{x^2 - m^2x^2 - 2imx^2}{1 + m^2}$

(putting $y = mx$)

$= \frac{(1 - im)^2}{(1 + m^2)}$

limit does not exist

6. a) $\begin{cases} \frac{\operatorname{Im}(z)}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$

let $y = mx$.

then $\lim_{z \to 0} \frac{mx}{\sqrt{x^2 + m^2x^2}} = \frac{m}{\sqrt{1 + m^2}}$ $\therefore$ limit does not exist.

b) $\begin{cases} \frac{\operatorname{Re}(z^2)}{|z|^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$

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$\lim_{z\to 0} \frac{x^2}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$ limit does not exist

5. a)

c) $\lim \begin{cases} |z|^2 & z \neq 0 \\ 1 & z = 0 \end{cases}$

$\lim_{z\to 0} \sqrt{x^2+m^2x^2} = x\sqrt{1+m^2} = 0$

$\therefore 1 \neq 0$ hence limit does not exist.

Ans7: a) $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \neq 0)$, $f(0) = 0$.

$\lim_{z\to 0} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3(1+i)}{x^2} =$

putting $y=0$

$x(1+i) = 0$.

Now putting $x = 0$.

$0 - \frac{y^3(1-i)}{y^2} = y(1-i) = 0$.

Now putting $y = mx$.

$\lim_{z\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2+m^2x^2} = \lim_{x\to 0} x\left[\frac{((1+i) - m^3(1-i))}{1+m^2}\right]$.

$= 0$

hence continous

Now for differentiable:

we can check through CR equation:

$f(z) = \frac{(x^3 - y^3) + i(x^3+y^3)}{x^2+y^2}$

u, v

$f(z) = u + iv$.

$u = \frac{x^3 - y^3}{x^2+y^2}$ $v = \frac{x^3+y^3}{x^2+y^2}$

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Now, $\because u_x = \dfrac{3x^2(x^2+y^2)-(x^3-y^3)2x}{(x^2+y^2)^2}$ $u_y = \dfrac{-3y^2(x^2+y^2)-(x^3-y^3)(2y)}{(x^2+y^2)^2}$

$v_y = \dfrac{3y^2(x^2+y^2)-(x^3+y^3)(2y)}{(x^2+y^2)^2}$ $v_x = \dfrac{3x^2(x^2+y^2)-(x^3+y^3)(2x)}{(x^2+y^2)^2}$

at $x=0,\ y=0$

$u_x = v_y$ and $u_y = -v_x$

$\therefore$ follows CR equation.

Now, $f'(z) = \lim_{z\to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$

$= \lim_{z\to z_0} \dfrac{\dfrac{x^3(1+i)-y^3(1-i)-0}{x^2+y^2}}{(x+iy)-0}$

Now putting $y=mx$ we get –

$= \lim_{z\to z_0} \dfrac{x^3(1+i)-x^3m^3(1-i)}{(x^2+m^2x^2)(x+im)} = \dfrac{(1+i)-m^3(1-i)}{(1+m^2)(1+m)}$

$\because$ it depends on m $\therefore$ not differentiable.

b) $f(z) = \sqrt{|xy|}$ $\forall z \in \mathbb{C}$.

$u = \sqrt{|xy|}$ $v=0$

$\cancel{u_x = \dfrac{y}{2\sqrt{|xy|}} = \dfrac{1}{2}\sqrt{\dfrac{y}{x}}}$ / $\cancel{u_y = \dfrac{x}{2\sqrt{|xy|}} = \sqrt{\dfrac{x}{y}}}$

$u_x(0,0) = \lim_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x-0} = 0$

$u_y(0,0) = \lim_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y-0} = 0$

$\therefore$ satisfies CR equation

$v_x = 0$

$v_y = 0$

8. $|f(z)| = c$.

$u = x^2+y^2$.

$f(z) = u+iv$

$|f(z)| = \sqrt{u^2+v^2}$

$\Rightarrow uu_x + vv_x = 0$.

$uu_y + vv_y = 0$.

$u\frac{du}{dx} + v\frac{dv}{dx} = 0$

$uu_x + vv_x = 0$

$(uu_x - vu_y = 0)u$

$(uu_y + vu_x = 0)v$

$u_x(u^2+v^2) = 0$.

constant //

$u_x = v_y$,

$v_x = -u_y$.

$f(z) = u+iv$.

$u_y = 0$.

$c_1 + ic_2$

Contd 7.b $f'(z) = \lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0} = \frac{\sqrt{|xy|} - 0}{x+iy-0}$

$y = mx$ $\frac{\sqrt{m}}{1+im}$

$(x-iy)^2$

$= x^2 + (iy)^2$

c) $f(z) = \frac{\bar{z}^2}{z}$ $z \neq 0$

0 $z = 0$.

$\frac{(x-iy)^2}{x+iy} = \frac{x^2-y^2-2xyi}{x+iy}$

$\frac{(x^2-y^2-2xyi)(x-iy)}{x^2+y^2}$

$u_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h-0} = \frac{h^3}{h \cdot h} = 1$.

$u = \frac{(x^2-y^2)x - 2xy^2}{x^2+y^2}$

$u_y(0,0) = \lim_{k\to 0} \frac{-k^2}{ik \cdot k} \neq 0$.

$V_x(0,0) = 0$

$v = \frac{-2x^2yi - (x^2-y^2)iy}{x^2+y^2}$

$V_y(0,0) = \frac{+y^3}{y^3} = 1$.

Real part $\rightarrow \frac{(x^2-y^2)x - 2xy^2}{x^2+y^2}$

Img part $\rightarrow \frac{-2x^2y - (x^2-y^2)y}{x^2+y^2}$.

9. $|z|^4 = (x^2+y^2)^2$

Now, using CR equation $u = (x^2+y^2)^2$ $v = 0$

$u_x = 2(x^2+y^2)(2x)$

$u_y = 2(x^2+y^2)(2y)$

at $z=0$ it follows CR equation.

Now, $f'(z) = \lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0} = \frac{(x^2+y^2)^2 - 0}{x+iy}$

putting $y = mx$

$\frac{(x^2+m^2x^2)^2}{x+imx} = x^3\left(\frac{(1+m^2)^2}{1+im}\right) = 0.$

$\therefore$ differentiable

Now, but it is not analytic $\because$ it is not diff at nbd $z=0$ and it does not follow CR equation.

10. $f(z) = u+iv$

a) $v=0$, then using CR eqn $v_x = 0 = -u_y$ and $v_y = 0 = u_x$. $\therefore$ u is constant c.

Hence $f(z) = u+iv = c$.

b) $V = c$ then using CR eqn $v_x = 0 = -u_y$, $v_y = 0 = u_x$

$\therefore$ u is constant K Hence $f(z) = u+iv = K+ic$

c) $u = c$ then using CR equation $v_x = 0 = -u_y$ and $v_y = 0 = u_x$. $\therefore u$ is constant.

Hence $f(z) = c+iK$.

d) $\arg(f(z)) = K$. $\tan^{-1}\frac{v}{u} = K$.

$\cancel{\frac{(uu_x + vu_x)}{1+\left(\frac{v}{u}\right)^2} = 0}$.

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diff w.r.t x.

$$\frac{1}{1+\left(\frac{v^2}{u^2}\right)} \cdot \left(\frac{1}{u} \frac{dv}{dx} + v\left(\frac{-1}{u^2}\right)\frac{du}{dx} \right)$$

$$= \frac{u^2}{u^2+v^2}\left(\frac{v_x}{u} - \frac{v}{u^2} \cdot u_x \right) = 0$$

$$= \frac{u v_x - v u_x}{u^2+v^2} = 0.$$

similarily w.r.t y -

$$\frac{u v_y - v u_y}{u^2+v^2} = 0$$

$u v_x = v u_x$ using CR eqn $u_x = v_y$.

$u v_y = v u_y$. and $u_y = -v_x$.

$(-u_y u = v u_x.) u$

$(u u_x = v u_y). v$

$v^2 u_y = -u^2 u_y$

$\Rightarrow (v^2+u^2) u_y = 0$ $\therefore$ $\underline{u_y = 0}$

$\therefore u_x = 0$ hence $v_x = v_y = 0$.

$\therefore$ $f(z) = c_1 + i c_2$.

11. $f(z) = \sin x \cosh y - i \cos x \sinh y$

$u = \sin x \cosh y$

$v = -\cos x \sinh y$.

Now using CR eqn:- $u_x = \cos x \cosh y$

$u_y = \sin x(-\sinh y)$

$v_x = \sin x \sinh y$

$v_y = -\cos x \cosh y$.

$v_x = -u_y$ but $u_x \neq v_y$

$\therefore$ not analytic.

12. a) $u = 2x(1-y)$

$u_x = 2(1-y) = v_y$

$u_y = +2x = +v_x$

$V_y = \int 2(1-y)\,dy = 2y - y^2 + \phi(x)$.

$V_x = \cancel{x^2 x + \phi_1(x)}$ $\int 2(1-y)\,dx$ $x^2 + \phi(x)$.

$= \cancel{2x - 2yx} + \phi_2(x)$.

$V_y = 2xy + \phi_3(x)$

$\cancel{V_x = \frac{2x^2}{x}} + \phi_4(x)$ $\quad v_x + v_y = x^2 - y^2 + 2y$.

b) $u = e^{-x}(x\sin y - y\cos y)$

$u_x = -e^{-x} x\sin y + e^{-x}\sin y$.

$u_y = e^{-x}(x\cos y - \cos y + y\sin y)$.

$V_x = -e^{-x}\sin y + e^{-x}\cdot x\sin y - \int e^{-x}\,dx \sin y$

$= -e^{-x}\cancel{\sin y} + e^{-x}\cdot x\sin y + e^{-x}\cancel{\sin y}$

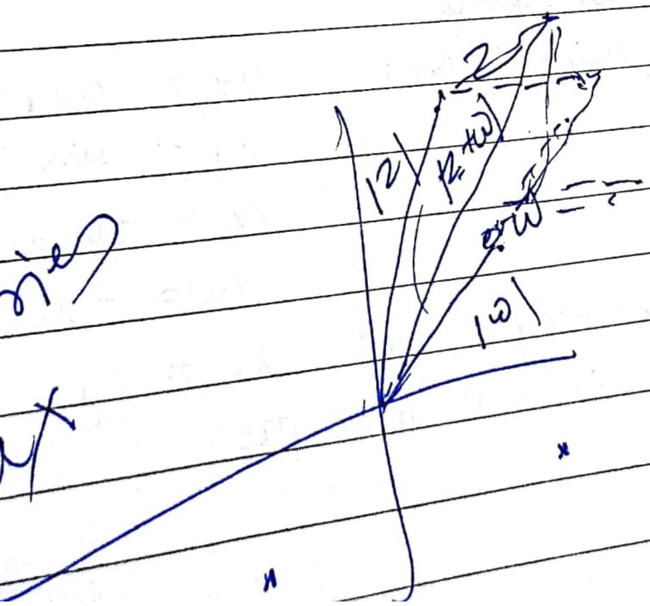
$= e^{-x}\cdot x\sin y + \phi(x)$.

$V_y = +e^{-x} x\cos y - e^{-x}\cos y$.

$Vx = -e^{-x} x\cos y + e^{-x}\cos y - e^{-x}\cos y(x)$ $te^{-x} y\sin y x$.

$= e^{-x}$

13. $f(z) = u + iv \qquad z = x + iy \qquad u - v = e^{x}(\cos y - \sin y)$.



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12. a) $u = 2$

$u = 2x(1-y)$

$u_x = (1-y) = v_y$

$u_y = -2x = -v_x \Rightarrow v_x = 2x$ — (1)

$\frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$v = y - \frac{y^2}{2} + \phi(x)$

$v_x = \phi'(x)$ — (2)

c) $u = \cos x \cosh y$

$u_x = -\sin x \cosh y = v_y$

$u_y = \cos x(+\sinh y) = -v_x$

$\phi'(x) = 2x$

$\phi(x) = x^2 + c$

$v_x = 2x$

$f(z) = x - iy$

$e^{-x}(xa$

$v = y - \frac{y^2}{2} + \phi(x)$

$v_x = \phi'(x)$

(x^2)

$6(x+iy)$

$u_x = 1$

$u_y = -1$

$f(z)$

b) $\phi(x) + -\sin x \sinh y = v$

$\phi'(x) - \cos x \sinh y = v_x$

$\phi'(x) = 2\cos x \sinh y$.

$\sqrt{x+iy}$

u_y

v_x

$u^2 + v^2 = c$

$u^2 + v^2 = c^2$

$$\left.\begin{aligned} 2uu_x + 2vv_x &= 0 \\ uu_y + vv_y &= 0 \end{aligned}\right\}$$

$(1) \leftarrow R$

b)