Lecture 7: Differentiability

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If we look at the definition of continuity and limit of a real-valued function of two variables, it's natural extension of the notion of continuity and limit of real-valued function of one variable. So it is natural to ask that, can we extend similarly the notion of derivative of one variable function, to the real-valued function of two variables?

First recall the differentiability of a single variable function.

Definition 7.1 *Let $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. We say that f is differentiable at point c if the limit*

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists.

Let us try extend this definition to define differentiability of a real-valued function of two variables.

Let $f : B_r(x_0, y_0) \to \mathbb{R}$ be any function. It might seem natural to consider a limit such as

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{(h, k)}$$

But this doesn't make sense for the simple reason that division of a real number by a point in $\mathbb{R}^2$ has not been defined. Now how to overcome this difficulty?

First we recast the definition of derivative for function of one variable as below and take a clue from here to get a correct notion of differentiability for function of two variables.

Exercise 7.2 *Let $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. Show that f is differentiable at point c if and only if there is $\alpha \in \mathbb{R}$ such that*

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} = 0.$$

Solution: First we assume that f is differentiable at c. That is

$$\lim_{h \to 0} F(h) \text{ exists}, \quad \text{where } F(h) := \frac{f(c+h) - f(c)}{h}.$$

We denote this limit by $f'(c)$, i.e., $\lim\limits_{h\to 0} F(h) = f'(c)$. Now we consider a constant function $G(h) \equiv f'(c)$. Trivially $\lim\limits_{h\to 0} G(h) = f'(c)$. Therefore, by properties of limits of functions of one variable

$$\begin{aligned}
\lim_{h\to 0}[F(h) - G(h)] &= 0\\
\text{i.e., } \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - f'(c)\right] &= 0\\
\text{i.e., } \lim_{h\to 0}\left[\frac{f(c+h)-f(c)-f'(c)h}{h}\right] &= 0\\
\iff \lim_{h\to 0}\left|\frac{f(c+h)-f(c)-f'(c)h}{h}\right| &= 0,
\end{aligned}$$

where the last equivalence is follows from the fact that " $\lim\limits_{x\to c} g(x) = 0 \iff \lim\limits_{x\to c} |g(x)| = 0$"

Hence we choose $\alpha = f'(c)$.

Now we assume that there is $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0}\frac{|f(c+h)-f(c)-\alpha h|}{|h|} = 0.$$

Then by previous discussion, we have

$$\lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - \alpha\right] = 0$$

Hence

$$\lim_{h\to 0}\left(\left[\frac{f(c+h)-f(c)}{h} - \alpha\right] + G(h)\right) = 0 + \alpha$$

Above statement is same as saying that

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \alpha$$

That is f' at c exists and is equal to α. ■

Now the last exercise and a realization that the derivative of a real-valued function of two variables may not be a single number but possibly a pair of real numbers suggests the way to define the differentiability of function of two variables.

Definition 7.3 *Let $D \subseteq \mathbb{R}^2$ and Let $(x_0, y_0) \in D$ be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be differentiable at (x_0, y_0) if there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that*

$$\lim_{(h,k)\to(0,0)}\frac{|f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha_1 h - \alpha_2 k|}{\sqrt{h^2+k^2}} = 0$$

In this case, we call the pair (α_1, α_2) the derivative of f at (x_0, y_0).

Let us note that if f is differentiable at (x_0, y_0) and if (α_1, α_2) is the derivative of f at (x_0, y_0), then letting (h, k) approach $(0, 0)$ along the x-axis we see that

$$\lim_{h\to 0} \frac{|f(x_0 + h, y_0) - f(x_0, y_0) - \alpha_1 h|}{\sqrt{h^2}} = \lim_{h\to 0} \frac{|f(x_0 + h, y_0) - f(x_0, y_0) - \alpha_1 h|}{|h|} = 0$$

that is $\alpha_1 = f_x(x_0, y_0)$. Similarly, letting (h, k) approach $(0, 0)$ along the y-axis we see that

$$\lim_{k\to 0} \frac{|f(x_0, y_0 + k) - f(x_0, y_0) - \alpha_2 k|}{\sqrt{k^2}} = \lim_{k\to 0} \frac{|f(x_0, y_0 + k) - f(x_0, y_0) - \alpha_2 k|}{|k|} = 0$$

that is $\alpha_2 = f_y(x_0, y_0)$. Hence if f is differentiable, then the gradient of f at (x_0, y_0) exists and the derivative of f at $(x_0, y_0) = \nabla f(x_0, y_0)$. Thus in checking the differentiability of f at (x_0, y_0), First check wether partial derivatives exists and then check whether the corresponding two-variable limit exists and is equal to zero. Also, if either of the partial derivatives does not exist at a point, then we can be sure that f is not differentiable at that point.

Example 7.4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |xy|$.*

Differentiability at $(0, 0)$: *Since $f(x, 0) = 0$ for all $x \in \mathbb{R}$ and $f(0, y) = 0$ for all $y \in \mathbb{R}$ hence $f_x(x, 0) = 0, \forall x \in \mathbb{R}$ and $f_y(0, y) = 0$ for all $y \in \mathbb{R}$. In particular, $f_x(0, 0) = 0 = f_y(0, 0)$. Now we show that*

$$\lim_{(h,k)\to(0,0)} \frac{|f(0 + h, 0 + k) - f(0, 0) - 0.h - 0k|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2 + k^2}} = 0$$

Let $((h_n, k_n))$ be sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(h_n, k_n) \to (0, 0)$.

Note that $|h_n| \leq \sqrt{h_n^2 + k_n^2}$ Hence

$$0 \leq \frac{|h_n k_n|}{\sqrt{h_n^2 + k_n^2}} \leq |k_n|$$

Since $k_n \to 0$ hence $|k_n| \to 0$. Hence $\dfrac{|h_n k_n|}{\sqrt{h_n^2 + k_n^2}} \to 0$. This completes the proof of the claim.

Hence, f is differentiable at $(0, 0)$.

Differentiability at $(x_0, 0), x_0 \neq 0$: *Let*

$$g(y) := f(x_0, y) = |x_0||y|.$$

Since function g is not differentiable at $y = 0$, therefore $f_y(x_0, 0)$ does not exists. Hence f is not differentiable at $(x_0, 0), x_0 \neq 0$.

Differentiability at $(0, y_0), y_0 \neq 0$: *Now*

$$h(x) := f(x, y_0) = |y_0||x|$$

Since function h is not differentiable at $x = 0$, therefore $f_x(0, y_0)$ does not exists. Hence f is not differentiable at $(0, y_0), y_0 \neq 0$.

Differentiability at $(x_0, y_0), x_0 \neq 0, y_0 \neq 0$: *In this case the point (x_0, y_0) is an interior point of one of quadrants, i.e., $\exists r > 0$ such that $B_r(x_0, y_0)$ is contained in the same quadrant.*

1. *If $x_0 > 0, y_0 > 0$ then*

$$f(x, y) = xy \quad \forall (x, y) \in B_r(x_0, y_0).$$

Then

$$f_x(x_0, y_0) = y_0, f_y(x_0, y_0) = x_0$$

It remains to check if double limit goes to zero or not? Note that in the double limit as $(h, k) \to (0, 0)$, what matters is values of f in an open disk centered at (x_0, y_0). Hence

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) - y_0 h - k x_0|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|(x_0 + h)(y_0 + k) - x_0 y_0 - y_0 h - k x_0|}{\sqrt{h^2 + k^2}}$$
$$\lim_{(h,k)\to(0,0)} \frac{|x_0 y_0 + x_0 k + y_0 h + hk - x_0 y_0 - y_0 h - k x_0|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2 + k^2}} = 0$$

2. *Similarly if $x_0 > 0, y_0 < 0$, we can find an open disk centered at (x_0, y_0) which is complete contained in fourth quadrant. then*

$$f(x, y) = -xy \quad \forall (x, y) \in B_r(x_0, y_0).$$

Then

$$f_x(x_0, y_0) = -y_0, f_y(x_0, y_0) = -x_0$$

It remains to check if double limit goes to zero or not? Note that in the double limit as $(h, k) \to (0, 0)$, what matters is values of f in an open disk centered at (x_0, y_0). Hence

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) + y_0 h + k x_0|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|(x_0 + h)(-y_0 - k) + x_0 y_0 + y_0 h + k x_0}{\sqrt{h^2 + k^2}}$$
$$\lim_{(h,k)\to(0,0)} \frac{|x_0 y_0 + x_0 k + y_0 h + hk - x_0 y_0 - y_0 h - k x_0|}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2 + k^2}} = 0$$