Lecture 13: Test for Local Extrema and Saddle Point

October 27, 2016

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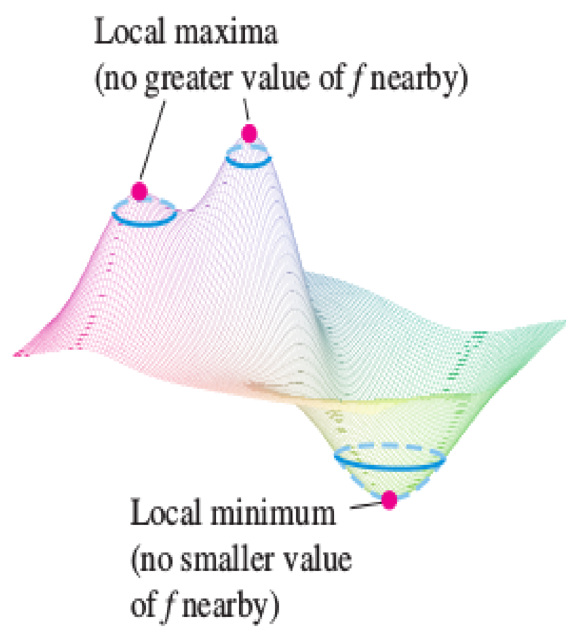
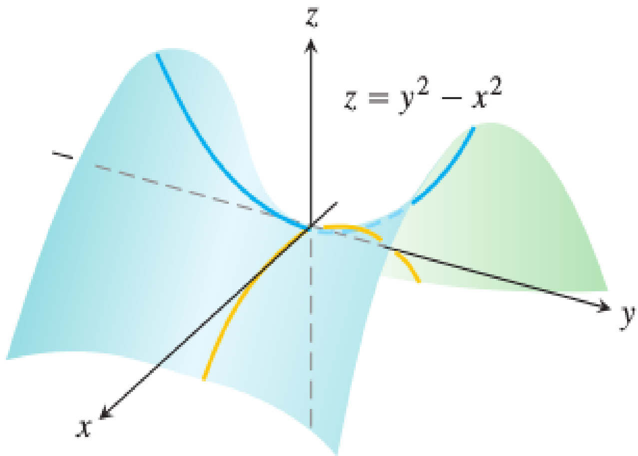
Geometrically, local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms.



Saddle Point We have seen in the previous lecture that the only points where a function $f(x, y)$ can assume extreme values are critical points and boundary points. As functions of a single variable, not every critical point gives rise to a local extremum. A function of a single variable might have a point of inflection. In the same spirit, a function of two variables might have a saddle point.

Definition 13.1 *A function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.*

The name "saddle point" is motivated by the shape of surface near such a point. Look at the following picture.



Surface $f(x, y) = y^2 - x^2$ look like a saddle near origin. Note that $f(r, 0) = -r^2 < 0 = f(0, 0)$ and $f(0, r) = r^2 > 0 = f(0, 0)$ for any $r \in \mathbb{R}$. Therefore, however small open disk centered at $(0, 0)$ we consider, there will always be some points from x-axis and y-axis. Hence, $(0, 0)$ is a saddle point as per Definition 13.1.

Theorem 13.2 (Second Derivative Test or Discriminant Test) *Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then*

1. *f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
2. *f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
3. *f has a saddle point at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).*

The test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the Hessian or discriminant of f. It is denoted by Δf

Example 13.3 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := 4xy - x^4 - y^4$. Find the points of local extrema (if any) of f.*

Solution: Then f has continuous partial derivatives of all orders. Also, $f_x = 4y - 4x^3$ and $f_y = 4x - 4y^3$, and so $\nabla f(x, y) = (0, 0) \iff y = x^3, x = y^3 \implies x = (x^3)^3 \implies x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \implies x = 0, \pm 1 \implies (x, y) = (x, x^3) = (0, 0), (1, 1), (1, -1)$.

Further, $f_{xx} = -12x^2, f_{xy} = 4$, and $f_{yy} = -12y^2$, and so the discriminant is given by $\Delta f = f_{xx}f_{yy} - f_{xy} = 16(9x^2y^2 - 1)$. In particular, $\Delta f(0,0) = -16 < 0$ and $\Delta f(1,1) = \Delta f(-1,-1) = 128 > 0$. Also $f_{xx}(1,1) = f_{xx}(-1,-1) = -12 < 0$. By the Discriminant Test, f has a saddle point at $(0,0)$ and a local maximum at $(1,1)$ as well as at $(-1,-1)$. ■

Example 13.4 *Find all saddle points (if any) for the function* $f(x,y) = x^4 + y^3$.

Solution: Then $f_x = 4x^3, f_y = 3y^2$. So $(0,0)$ is the only critical point. $f_{xx} = 12x^2, f_{yy} = 6y, f_{xy} = 0$. Hence $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$. So test fails. We claim that f neither has local maximum nor a local minimum at $(0, 0)$. To see this, note that $f(0,0) = 0$ and f takes both positive as well as negative values in any open disk centered at the origin. For example, $f(r,0) = r^4 > 0$ and $f(0,-r) = -r^3 < 0$ for any $r > 0$. It turns out that f does have a saddle point at $(0,0)$. ■