Discrete Mathematics Assignment-4

1. Using the relation R= {(a, b), (b, b), (c, a), (c, c)} on {a,b,c}, find $R^2 and\ R^3$.
2. Let A, B, and C be finite sets. Let R be a relation from A to B, and S a relation from B to C. Then $M_{R\circ S} = M_R \Theta M_S$.
3. Give an example of a relation on {a, b, c} that is:
   a) Reflexive, symmetric, and transitive.
   b) Reflexive, symmetric, but not transitive.
   c) Reflexive, transitive, but not symmetric.
   d) Symmetric, transitive, but not reflexive.
   e) Reflexive, but neither symmetric nor transitive.
   f) Symmetric, but neither transitive nor reflexive.
   g) Transitive, but neither reflexive nor symmetric.
4. Determine if the relation R on {a, b, c} defined by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

   is antisymmetric.?
5. Determine if each relation from *{a,b,c,d}* to {0, 1,2, 3, 4} is a function.
   a) {(a, 0), (b, 1), (c, 0), (d, 3)}
   b) {(a, 3), (b, 3), (b, 4), (c, 1), (d, 0)}
   c) {(a, 3), (b, 3), (c, 3), (d, 3)}
   d) {(a, 1), (b, 2), (c, 3)}
6. Find the number of primes < 100.
7. Find the total number of sub-matrices of an m x n matrix. Find the number of ways 10 quarters can be distributed among three people--Aaron, Beena, and Cathy--so that both Aaron and Beena get at least one quarter, Beena gets no more than three, and Cathy gets at least two.

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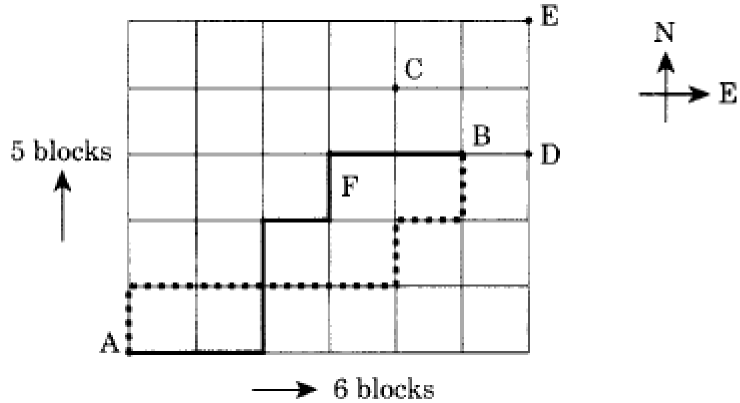


8. Use above Figure to find the number of possible routes from A to the given point, traveling easterly or northerly for the given number of blocks.
   a) Point F and 5 blocks.
   b) Point E and 11 blocks.

9. Find the number of solutions to each equation, where $x_i \geq 1$

$$x_1 + x_2 + x_3 + x_4 = 11$$

10. Find the number of ways a committee of five can be formed from a group of five boys and four girls, if each committee must contain:
    a) At least one boy and at least one girl.
    b) At most one girl.
11. Find the number of lines that can be drawn using 10 distinct points, no three being collinear.
12. A botanist would like to plant three coleus, four zinnias, and five dahlias in a row in her front garden. How many ways can she plant them if:
    a) They can be planted in any order.
    b) Plants of the same family must be next to each other.
13. A salesperson at a computer store would like to display six models of personal computers, five models of computer monitors, and four models of keyboards. In how many different ways can he arrange them in a row if items of the same family are to be next to each other?
14. An old zip code in the United States consists of five digits. Find the total number of possible zip codes that"
    a) Have no repetitions.

b) Begin with 0.

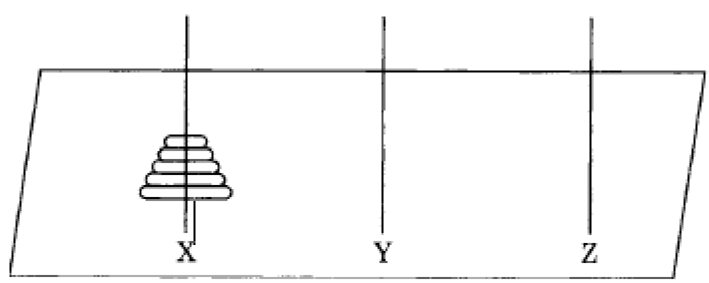
c) End in K.

15. A word over the alphabet {0, 1, 2} is called a ternary word. Find the number of ternary words of length n that can be formed.
16. Find the number of positive integers < 1976 and divisible by 2 or 3.
17. Write a recursive algorithm to compute the nth Fibonacci number F_n.
18. Write a recursive algorithm to compute the gcd of two positive integers x and y.
19. (Binary Search Algorithm**)** Write a recursive algorithm to search an ordered list X of n items and determine if a certain item *(key)* occurs in the list. Return the location of *key* if the search is successful.
20. Write a recursive algorithm to print the moves and the total number of moves needed to transfer the n disks from peg X to peg Z in the Tower of Brahma puzzle.

    [(Tower of Brahma) According to a legend of India, at the beginning of creation, God stacked 64 golden disks on one of three diamond pegs on a brass platform in the temple of Brahma at Benares. The priests on duty were asked to move the disks from peg X to peg Z using Y as an auxiliary peg under the following conditions"

    - Only one disk can be moved at a time.
    - No disk can be placed on the top of a smaller disk.

    The priests were told that the world would end when the job was completed.]



21. Using generating functions, solve each LHRRWCC.

    a) $a_n = a_{n-1} + 2a_{n-2}, a_0 = 3, a_1 = 0$

    b) $L_n = L_{n-1} + L_{n-2}, L_1 = 1, L_2 = 3$

    c) $a_n = 6a_{n-1} + 9a_{n-2}, a_0 = 2, a_1 = 3$

    d) $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}, a_0 = 3, a_1 = -7 , a_2 = 7$

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22. Solve each LHRRWCC.

    a) $L_n = L_{n-1} + L_{n-2}, L_1 = 1, L_2 = 3$

    b) $a_n = -a_{n-1} + 16a_{n-2} + 4a_{n-3} - 48a_{n-4}, a_0 = 0, a_1 = 16\ , a_2 = -2, a_3 = 142$

    c) $a_n = 7a_{n-1} - 12a_{n-2} + 3n4^n, a_0 = 0, a_1 = 2$

    d) $a_n = a_{n-1} + n, a_0 = 1$

23. Let $a, b \in \mathbb{R}$ and $b \neq 0$. Let α be a real or complex solution of the equation $x^2 - ax - b = 0$ with degree of multiplicity two. Then $a_n = Aa^n + Bna^n$ is the general solution of the LHRRWCC $a_n = aa_{n-1} + ba_{n-2}$.

24. Let a_n denote the number of subsets of the set S**{** 1, 2 , . . . n} that do not contain consecutive integers, where $n \geq 0$. When $n = 0, S = \emptyset$. Find an explicit formula for a_n.

25. Prove that the given predicate P(n) in each algorithm is a loop invariant.

    a) Algorithm Euclid(x, y, divisor)

       This algorithm returns gcd{x,y} in *divisor*, where $x \geq y > 0$

       Begin (* algorithm *)
       dividend ⟵x
       divisor ⟵ y
       remainder ⟵ dividend mod divisor
       while remainder > 0 do (* update dividend,divisor, and remainder *)
       begin (* while *)
           dividend ⟵ divisor
           divisor ⟵ remainder
           remainder ⟵ dividend mod divisor
       endWhile
       End (* algorithm *)

       $$P(n): \gcd\{x_n, y_n\} = \gcd\{x, y\}$$

       where x_n and y_n denote the values of x =dividend and y=divisor after n iterations.

    b) **Algorithm square** (x) (* This algorithm prints the square of x ∈W. *)

       Begin (* algorithm *)

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```
answer ⟵ 0
i ⟵ 0 (* counter *)
While i < x do
begin (* while *)
    answer ⟵answer + (2i + 1):
i ⟵i +i
endwhile
End (* algorithm *)
```

P(n): $a_n = a^2$, where a_n denotes the value of answer at the end of n iterations.

26. Prove that the binary search algorithm works correctly for every ordered list of size n > 0.

```
Algorithm binary search(X,1,n,key,mid)
(* This algorithm searches an ordered list X of n elements for a special
   item (key). It returns the location of key if the search is
   successful and zero otherwise. The variable mid returns such a value.
   The variables low and high denote the lower and upper indices of the
   list being searched. *)
Begin (* algorithm *)
low ← 1
high ← n
while low ≤ high do      (* list is nonempty *)
begin (* while *)
  mid ← ⌊(low + high)/2⌋
  if key = x_mid then      (* key exists in the list*)
    exit the loop
  else if key < x_mid then      (* search lower half*)
        high ← mid − 1
      else  (* search the upper half *)
        low ← mid + 1
endwhile
  if low > high then      (* search is unsuccessful *)
    mid ← 0
End (* algorithm *)
```

27. Below algorithm is an iterative algorithm for computing $n!$ where $n \geq 0$. Let fact(n) be the value *of factorial* at the end of n iterations of the loop. Prove that *P(n): fact(n)- n!* is a loop invariant.

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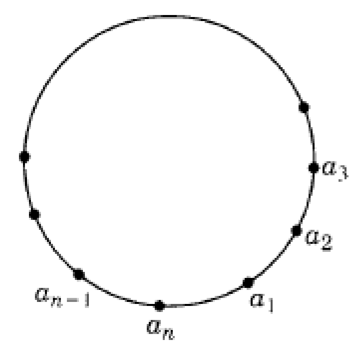
```
Algorithm factorial (n)
(* This algorithm computes and prints the value of
   n! for every n ≥ 0. *)
  Begin (* algorithm *)
    factorial ← 1          (* initialize *)
    i ← 1                  (* counter *)
    while i < n do
    begin (* while *)
      i ← i + 1
      factorial ← factorial * i
    endwhile
  End (* algorithm *)
```

28. Find the number of trailing zeros in 123!

29. Let $a_1, a_2 \dots a_n$ be the first n positive integers in some order. Suppose they are arranged around a circle (see Figure). Let k be any positive integer ≤ n. Prove that there exists a set of k consecutive elements in the arrangement with a sum$\lfloor kn(n+1) - 2]/2n\rfloor$, where $\lfloor x \rfloor$denotes the floor of x.



30. Disprove each statement.
    a) If gcd{a, b}=1 and gcd{b,c}=1, then gcd{a,c}=1, where a, b, and c are positive integers.
    b) n! + 1 is a prime for every n ≥ 0.

31. Euler's phi-function φ is another important number-theoretic function on ℕ, defined by $\varphi(n) =$ number of positive integers ≤ n and relatively prime to n. For example, $\varphi(1)$ =1= $\varphi(2), \varphi(3) = 2 = \varphi(4)$, and$\varphi(5) = 4$. Evaluate φ(n) for value of n=10.

32. Let n be an integer ≥ 2 and let $a_1, a_2 \dots a_n \in$ Z. Prove that there exist integers k and l such that $a_{k+1}, a_{k+2} \dots a_l$ is divisible by n, where $1 \le k < l \le n$; that is, there exist consecutive elements $a_{k+1}, a_{k+2} \dots a_l$ whose sum is divisible by n.