Lecture 18: Triple Integral

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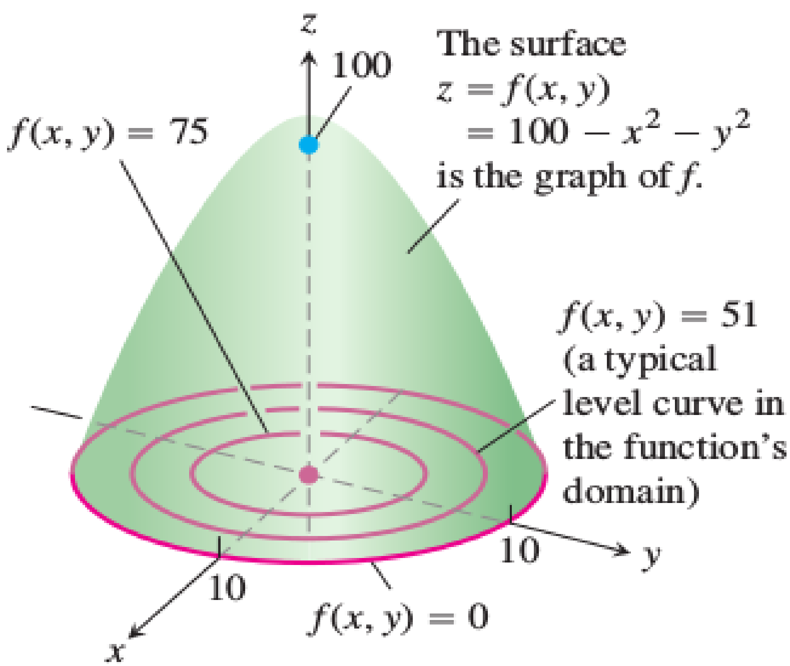
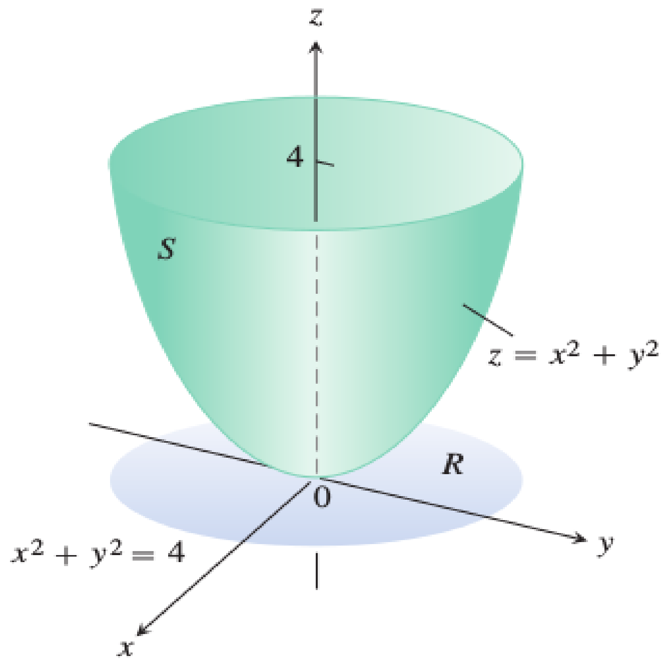
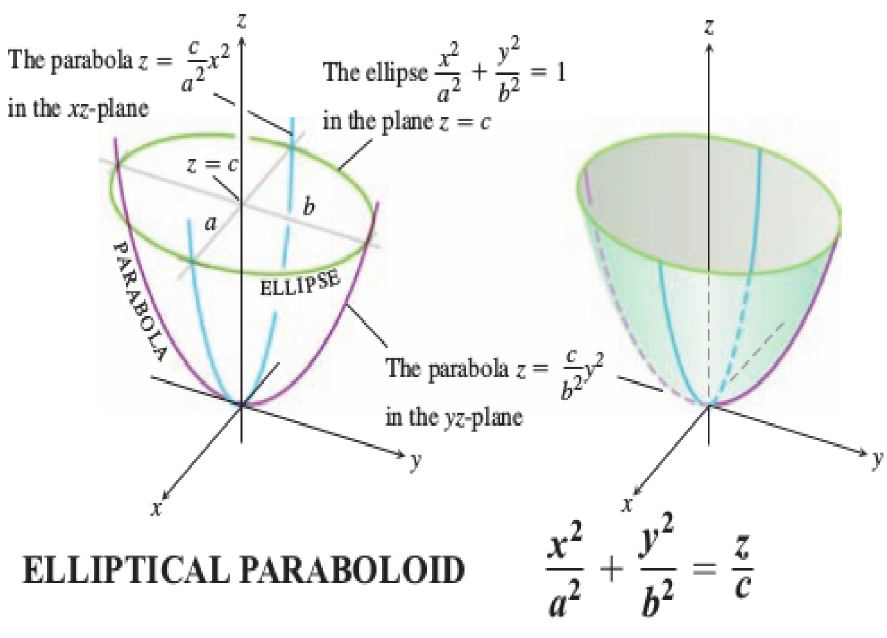
Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

In evaluating triple integrals, we need to identify limits of integration and for that the we need have to some idea about how geometrically surface looks like. We will not go in detail how to plot a surface but discuss the idea with some standard surfaces.

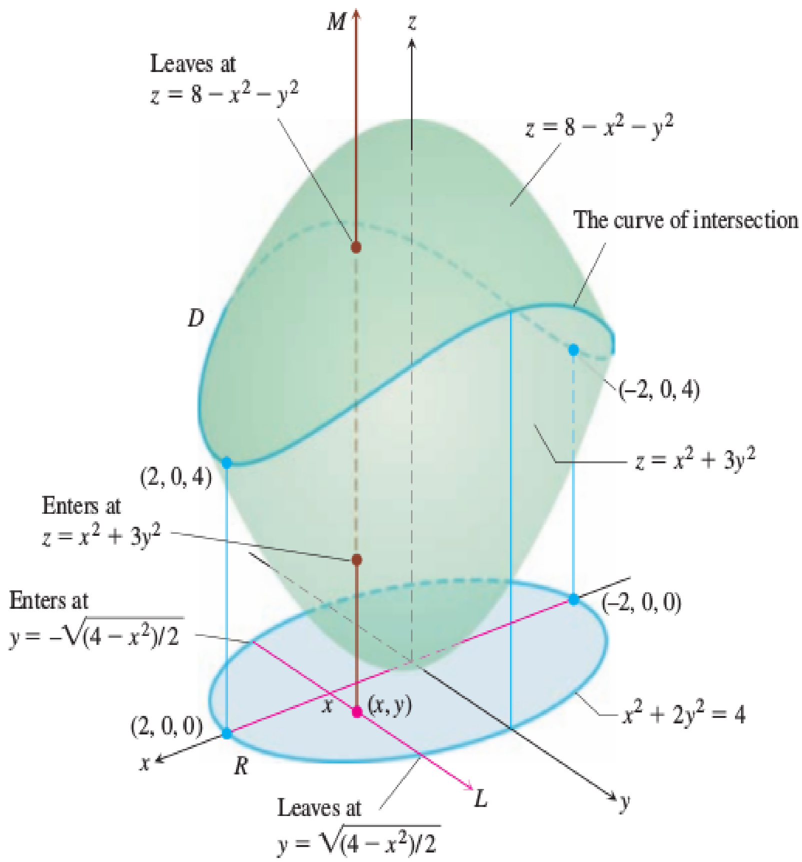






Example 18.1 *Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.*

Solution:



We find intersection curve of both the surfaces. Equate both the equations:

$$x^2 + 3y^2 = 8 - x^2 - y^2 \implies 2x^2 + 4y^2 = 8 \implies \frac{x^2}{4} + \frac{y^2}{2} = 1$$

The surfaces intersect on the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{2} = 1$, $z > 0$.

Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4-x^2)/2}$ and leaves at $y = \sqrt{(4-x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$
\begin{aligned}
V &= \iiint_D dz\, dy\, dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \\
&= \int_{-2}^{2}\left[(8-2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right) dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\, dx \\
&= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u
\end{aligned}
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Example 18.2 *Let W be the region bounded by the planes $x = 0, y = 0$ and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0, y \geq 0$. Compute $\iiint_W x\, dx\, dy\, dz$.*

Solution: The shadow of the region is part of disk $x^2 + y^2 = 2$. Hence region can be described by

$$0 \leq \sqrt{2}, 0 \leq y \leq \sqrt{2 - x^2}, x^2 + y^2 \leq z \leq 2$$

$$
\begin{aligned}
\iiiint_W x \; dx \; dy \; dz &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \; dx \; dy \; dz \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2 - x^2 - y^2) \; dx \; dy \; dz \\
&= \int_0^{\sqrt{2}} x \left[(2-x^2)^{\frac{3}{2}} - \frac{(2-x^2)^{\frac{3}{2}}}{3} \right] \; dx \; dy \; dz \\
&= \frac{8\sqrt{2}}{15}
\end{aligned}
$$