Lecture 13: Lagrange Multiplier Method

November 2, 2018

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To determine the absolute extremum of a real-valued function f of two variables, subject to the constraint $g(x, y) = 0$.

Step I Ensure (by giving some argument) that f does have an absolute extremum on the zero set of g.

Step II Solve simultaneous equations for $\lambda \in \mathbb{R}$ and points $(x, y) \in \mathbb{R}^2$

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and } g(x, y) = 0. \tag{13.1}$$

Step III Make a table of f values at the following points:

(a) At simultaneous solutions (x_0, y_0) of the system (13.1) for which $\nabla g(x_0, y_0) \neq (0, 0)$,

(b) At points in the zero set of g at which ∇f does not exist.

(c) At points in the zero set of g at which ∇g does not exist.

(d) At points in the zero set of g at which ∇g exists but $\nabla g = (0, 0)$.

Step IV Largest value in the table corresponds to the absolute maximum and minimum value in the table corresponds to the absolute minimum of f subject to $g = 0$

Example 13.1 *Find the maximum and the minimum of the function f given by $f(x, y) := xy$ on the unit circle.*

Solution: Following the Lagrange Multiplier Method,

Step I We let $g(x, y) := x^2 + y^2 - 1$ and consider the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, that is,

$$(y, x) = \lambda(2x, 2y), \quad x^2 + y^2 - 1 = 0.$$

Therefore we get the system

$$y = 2\lambda x \tag{13.2}$$

$$x = 2\lambda y \tag{13.3}$$

$$x^2 + y^2 = 1 \tag{13.4}$$

Using (13.3) in (13.2) we get $y = 2\lambda(2\lambda y) \implies 4\lambda^2 = 1$ provided $y \neq 0$. With condition $y \neq 0$ we get $\lambda = \pm\frac{1}{2}$. Hence we get $y = \pm x$ from (13.2). We substitute this in (13.4), to get $x = \pm\frac{1}{\sqrt{2}}$. So we get four simultaneous solutions of the above equations, which are given by $(x, y) = \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Now we consider the case $y = 0$, then $x = 0$ by (13.3). But $(0, 0)$ is not a solution of (13.4). Hence there are no other solutions.

Step II Note that zero set of g is a closed and bounded subset of $\mathbb{R}^2$. Also f being a polynomial function is continuous everywhere, therefore f attains its absolute maximum and absolute minimum on the unit circle.

Step III (a) Note that $\nabla g = (2x, 2y)$ is zero only at $(0, 0)$, in particular ∇g is non-zero at all the four solutions of simultaneous equations. Also

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2},\ f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

(b) Since f is a polynomial function so ∇f is defined everywhere.

(c) Since g is a polynomial function so ∇g is defined everywhere.

(d) There are no points in the zero set of g at which $\nabla g = (0, 0)$.

Step IV Therefore f attains it's maximum value at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ while the minimum is $-\frac{1}{2}$, which is attained at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

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The following example illustrates the importance of Step I while applying the Lagrange's multiplier method.

Example 13.2 *Find the maximum value of $f(x, y) = x+y$ subject to the constraint $xy = 16$.*

Solution: Following the Lagrange Multiplier Method,

Step I Note that zero set of g is a closed a subset of $\mathbb{R}^2$ but it is not a bounded subset of $\mathbb{R}^2$. Is there any other way to ensure that f attains it's absolute maximum on the zero set of g? If one tries to ignore the step I and proceed to next step then

Step II We let $g(x, y) := xy - 16$ and consider the equations $\nabla f(x, y) = \lambda\nabla g(x, y)$ and $g(x, y) = 0$, that is,

$$(1, 1) = \lambda(y, x), \quad xy - 16 = 0.$$

Therefore we get the system

$$\begin{aligned} 1 &= \lambda y &(13.5)\\ 1 &= \lambda x &(13.6)\\ xy &= 16 &(13.7)\end{aligned}$$

From (13.7), it follows that neither $x = 0$ nor $y = 0$. Therefore from (13.5) and (13.6) it follows that $y = x$. Using this in (13.7) we get points $(4, 4)$ and $(-4, -4)$ as solutions of the system. the location of extreme values. Also ∇g is not zero at both points.

Step III (a) Note that $\nabla g = (y, x)$ is zero only at $(0, 0)$, in particular ∇g is non-zero at both the solutions of simultaneous equations. Also

$$f(4, 4) = 8, f(-4, -4) = -8$$

(b) Since f is a polynomial function so ∇f is defined everywhere.

(c) Since g is a polynomial function so ∇g is defined everywhere.

(d) There are no points in the zero set of g at which $\nabla g = (0, 0)$.

Step IV Therefore we may wish to declare that f attains it's maximum value at $(4, 4)$. Yet the sum $x + y$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

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Example 13.3 *Find the point on the curve $(x - 1)^3 = y^2$ which is closet to the origin.*

Solution: This amounts to finding the minimum of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := x^2 + y^2$ subject to the constraint given by $g(x, y) := (x - 1)^3 - y^2 = 0$.

Following the Lagrange Multiplier Method,

Step I Consider the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$, that is,

$$(2x, 2y) = \lambda(3(x - 1)^2, -2y), \quad (x - 1)^3 = y^2.$$

Therefore we get the system

$$\begin{aligned} 2x &= 3\lambda(x - 1)^2 &(13.8)\\ 2y &= -2\lambda y &(13.9)\\ y^2 &= (x - 1)^3 &(13.10)\end{aligned}$$

If $y \neq 0$, then from (13.9), $\lambda = -1$. Using $\lambda = -1$ in (13.8), we get $2x + 3(x-1)^2 = 0$. But $2x + 3(x-1)^2 = 3x^2 + 3 - 4x = 0$ does not have any real roots since $b^2 - 4ac = (-4)^2 - 36 < 0$.

Now if $y = 0$ then from (13.10), we get $x = 1$. Substituting $x = 1$ in (13.8) we get $2 = 0$ which is absurd.

Hence there is no solution to the simultaneous equations (13.8)-(13.10).

Step II Note that zero set of g is a closed a subset of $\mathbb{R}^2$ but it is not a bounded subset of $\mathbb{R}^2$. Is there any other way to ensure that f attains it's absolute maximum on the zero set of g? Yes, geometrically given any curve in the plane there has to be some point which will be closet to the origin. Therefore, we can say that f attains it's absolute minimum on zero set of g.

Step III (a) There are no points to evaluate.

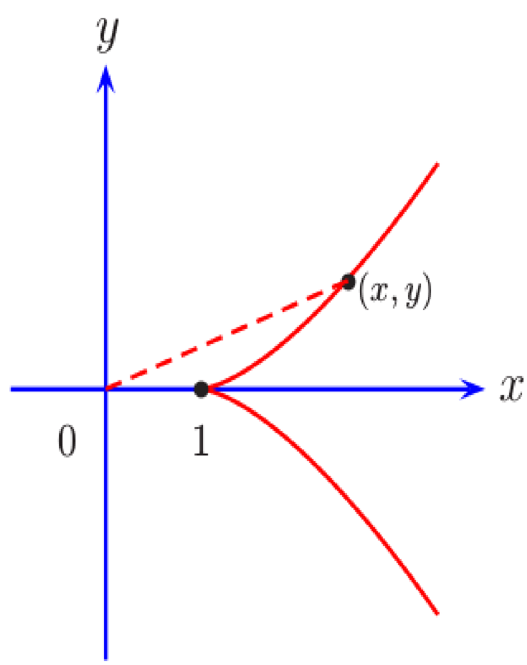
(b) Since f is a polynomial function so ∇f is defined everywhere.

(c) Since g is a polynomial function so ∇g is defined everywhere.

(d) $\nabla g = (0, 0)$ only at $(1, 0)$ which is in the zero set of g.

Step IV Therefore f attains it's absolute minimum value 1 at point $(1, 0)$ on the curve.

In fact if you draw this curve, this is how it looks like.



it is obvious that the minimum is 1 and it is attained at $(1, 0)$. ■