Random Variables and Probability Distributions

Reference: https://web.njit.edu/~dhar/math279/ch03.ppt

CHAPTER OUTLINE

3-1 Introduction

- Experiment
- Random
- Random experiment

3-1 Introduction

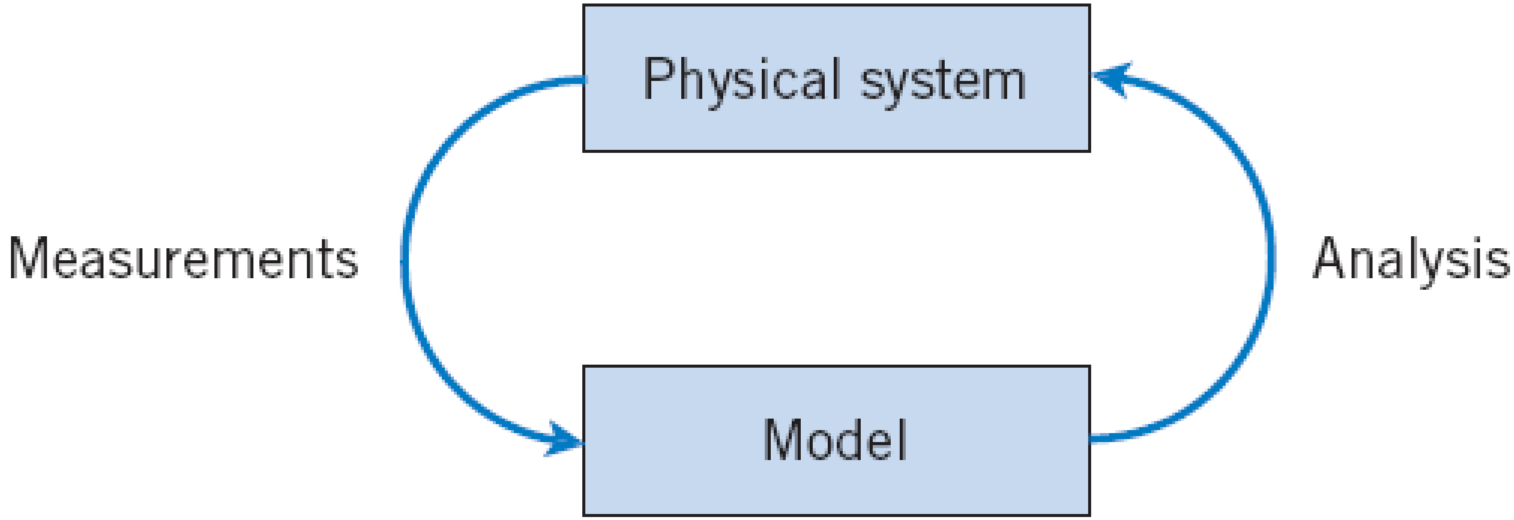


Figure 3-1 Continuous iteration between model and physical system.

3-1 Introduction

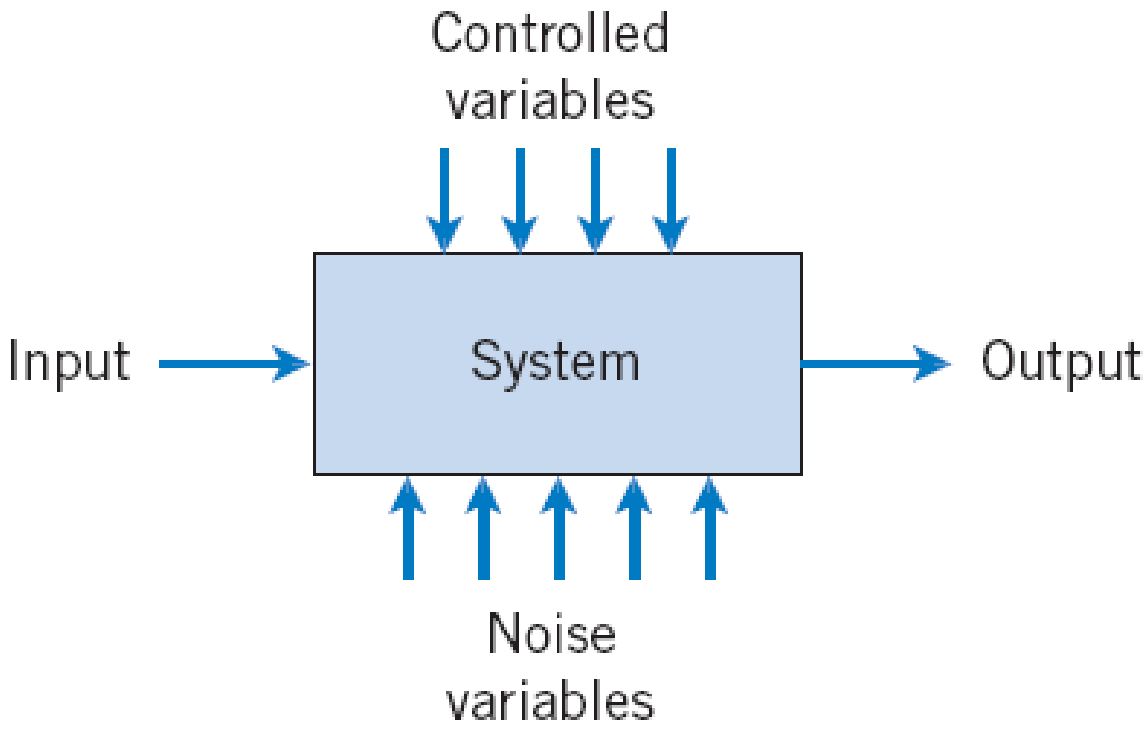


Figure 3-2 Noise variables affect the transformation of inputs to outputs.

3-1 Introduction

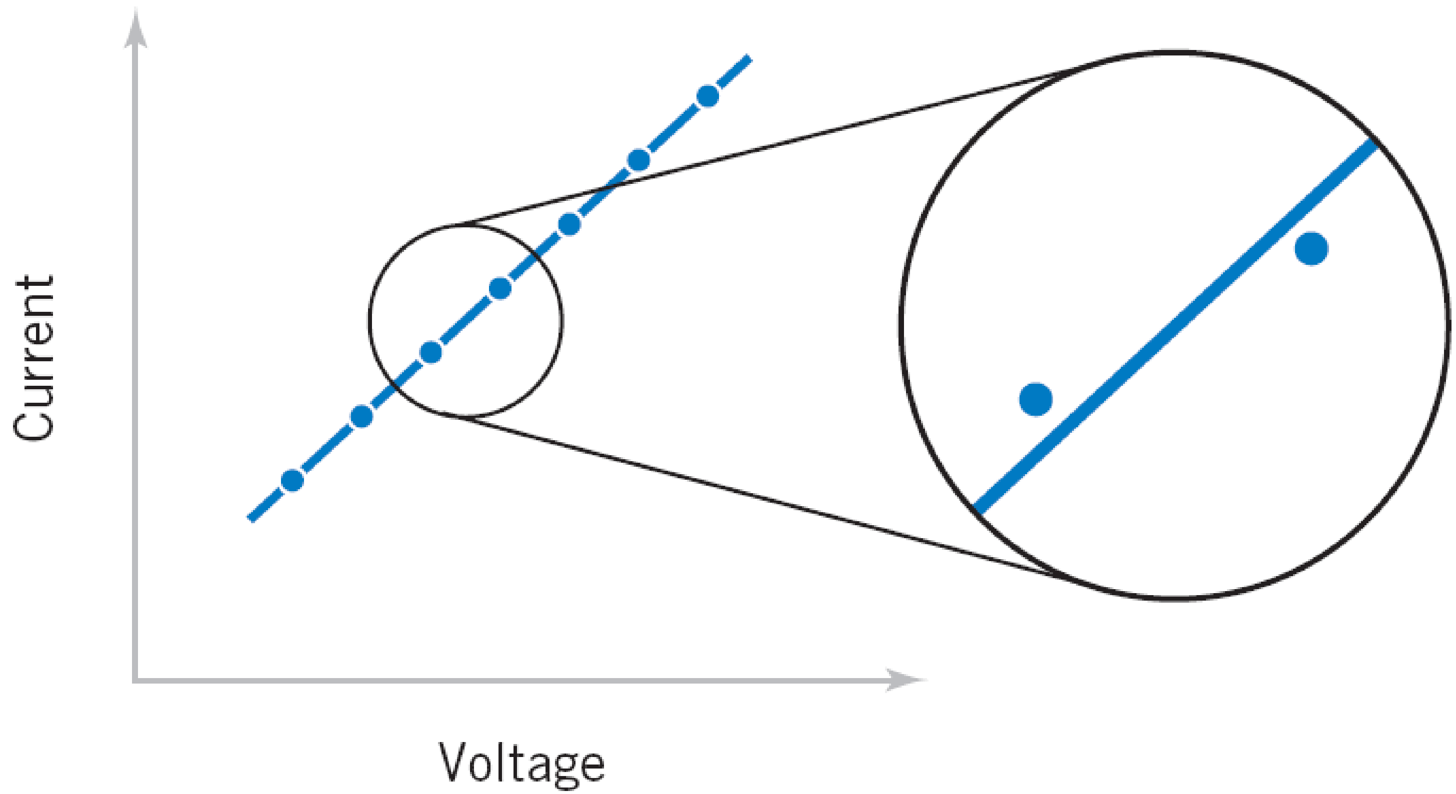


Figure 3-3 A closer examination of the system identifies deviations from the model.

3-1 Introduction

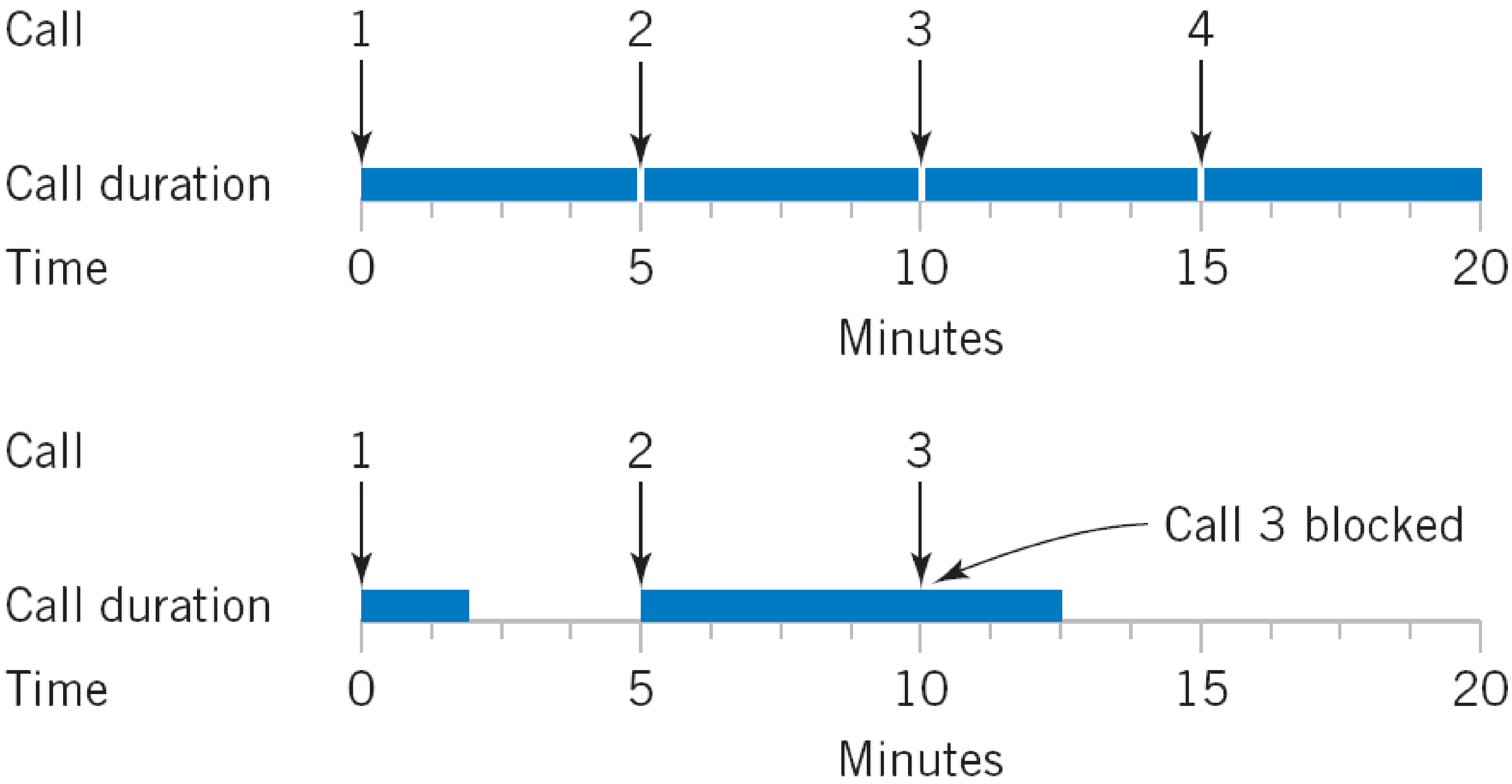


Figure 3-4 Variation causes disruptions in the system.

3-2 Random Variables

- In an experiment, a measurement is usually denoted by a variable such as X.

- In a random experiment, a variable whose measured value can change (from one replicate of the experiment to another) is referred to as a **random variable**.

3-2 Random Variables

A **random variable** is a numerical variable whose measured value can change from one replicate of the experiment to another.

A **discrete** random variable is a random variable with a finite (or countably infinite) set of real numbers for its range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples of **continuous** random variables:
electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:
number of scratches on a surface, proportion of defective parts among 1,000 tested, number of transmitted bits received in error

3-3 Probability

- Used to quantify likelihood or chance

- Used to represent risk or uncertainty in engineering applications

3-3 Probability

- Probability statements describe the likelihood that particular values occur.

- The likelihood is quantified by assigning a number from the interval [0, 1] to the set of values (or a percentage from 0 to 100%).

- Higher numbers indicate that the set of values is more likely.

3-3 Probability

- A probability is usually expressed in terms of a random variable.
- For the part length example, X denotes the part length and the probability statement can be written in either of the following forms

$$P(X \in [10.8, 11.2]) = 0.25 \qquad \text{or} \qquad P(10.8 \leq X \leq 11.2) = 0.25$$

- Both equations state that the probability that the random variable X assumes a value in [10.8, 11.2] is 0.25.

3-3 Probability

Complement of an Event

- Given a set E, the complement of E is the set of elements that are not in E. The **complement** is denoted as E'.

Mutually Exclusive Events

- The sets E_1 , E_2 ,...,E_k are **mutually exclusive** if the intersection of any pair is empty. That is, each element is in one and only one of the sets E_1 , E_2 ,...,E_k .

3-3 Probability

Probability Properties

1. $P(X \in R) = 1$, where R is the set of real numbers.
2. $0 \leq P(X \in E) \leq 1$ for any set E. (3-1)
3. If $E_1, E_2, \ldots, E_k$ are mutually exclusive sets,
   $P(X \in E_1 \cup E_2 \cup \ldots \cup E_k) = P(X \in E_1) + \cdots + P(X \in E_k)$.

3-3 Probability

Events

Illustrations

- The current measurement might only be recorded as *low*, *medium*, or *high*; a manufactured electronic component might be classified only as defective or not; and either a message is sent through a network or not.

3-4 Continuous Random Variables

3-4.1 Probability Density Function

- The **probability distribution** or simply **distribution** of a random variable X is a description of the set of the probabilities associated with the possible values for X.

The **probability density function** (or pdf) $f(x)$ of a continuous random variable is used to determine probabilities from areas as follows:

$$P(a < X < b) = \int_a^b f(x)\,dx \qquad (3\text{-}2)$$

The properties of the pdf are

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x) = 1$

3-4 Continuous Random Variables

3-4.1 Probability Density Function

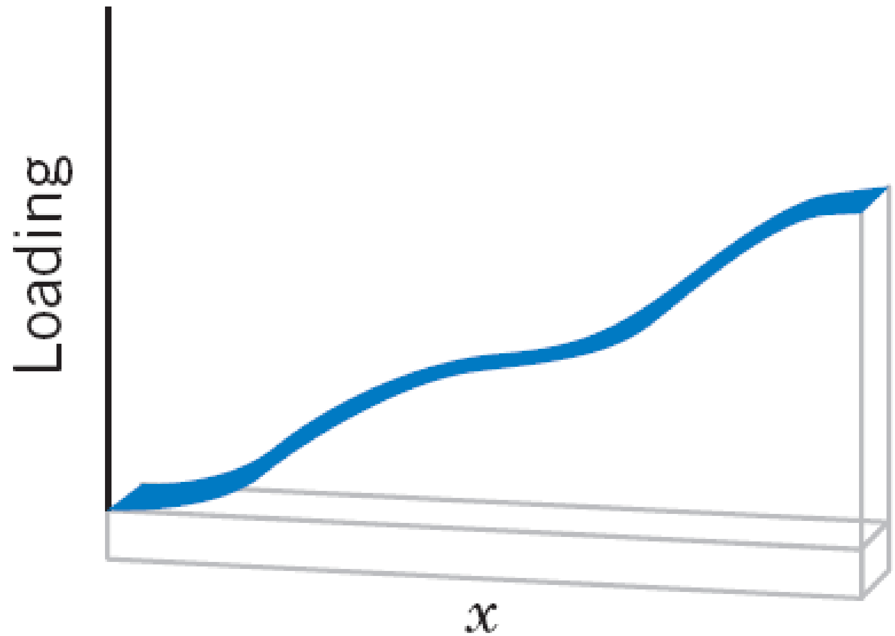


Figure 3-5 Density function of a loading on a long, thin beam.

3-4 Continuous Random Variables

3-4.1 Probability Density Function

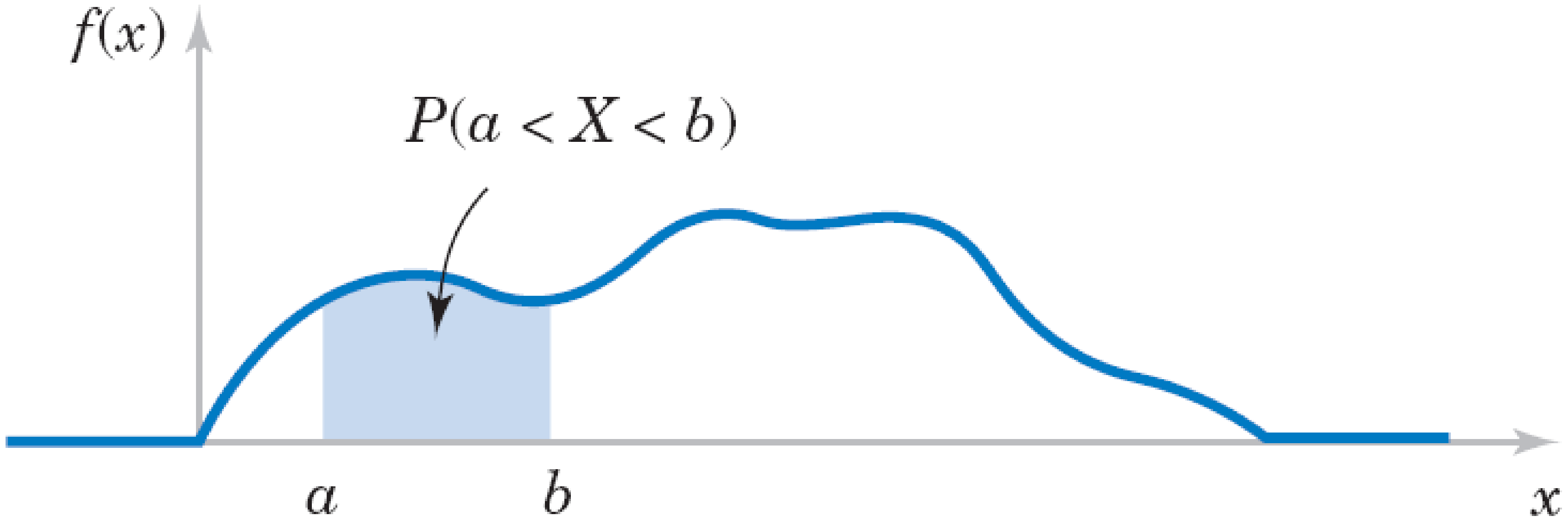


Figure 3-6 Probability determined from the area under $f(x)$.

3-4 Continuous Random Variables

3-4.1 Probability Density Function

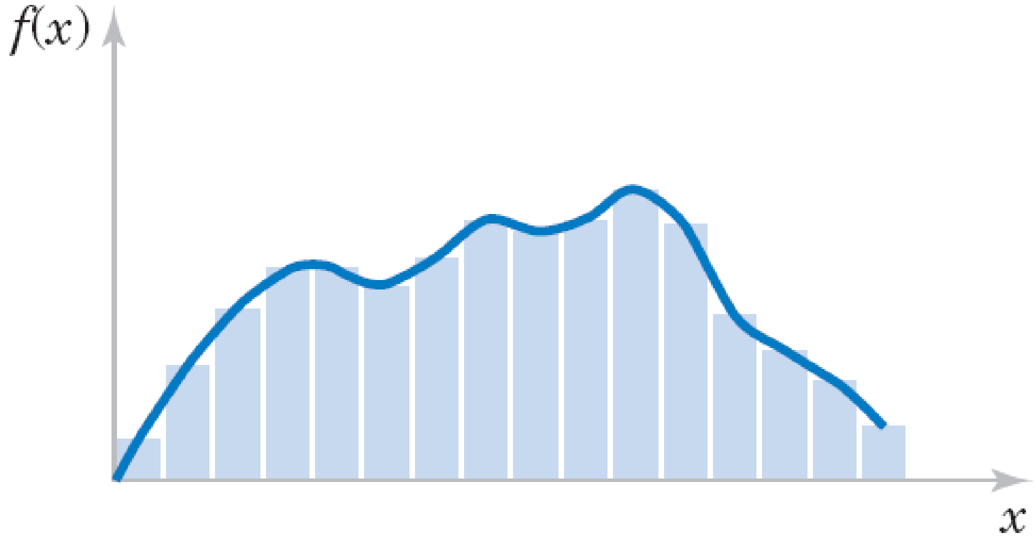


Figure 3-7 A histogram approximates a probability density function. The area of each bar equals the relative frequency of the interval. The area under $f(x)$ over any interval equals the probability of the interval.

EXAMPLE 3-2

Flaw on a Magnetic Disk

Let the continuous random variable X denote the distance in micrometers from the start of a track on a magnetic disk until the first flaw. Historical data show that the distribution of X can be modeled by a pdf $f(x) = \frac{1}{2000} e^{-x/2000}$, $x \geq 0$. For what proportion of disks is the distance to the first flaw greater than 1000 micrometers?

Solution. The density function and the requested probability are shown in Fig. 3-9.
Now,

$$P(X > 1000) = \int_{1000}^{\infty} f(x)\, dx = \int_{1000}^{\infty} \frac{e^{-x/2000}}{2000}\, dx = -e^{-x/2000} \Big|_{1000}^{\infty} = e^{-1/2} = 0.607$$

What proportion of parts is between 1000 and 2000 micrometers?

Solution. Now,

$$P(1000 < X < 2000) = \int_{1000}^{2000} f(x)\, dx = -e^{-x/2000} \Big|_{1000}^{2000} = e^{-1/2} - e^{-1} = 0.239$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(X < 1000) = 1 - P(X > 1000) = 1 - 0.607 = 0.393$.

3-4 Continuous Random Variables

EXAMPLE 3-2

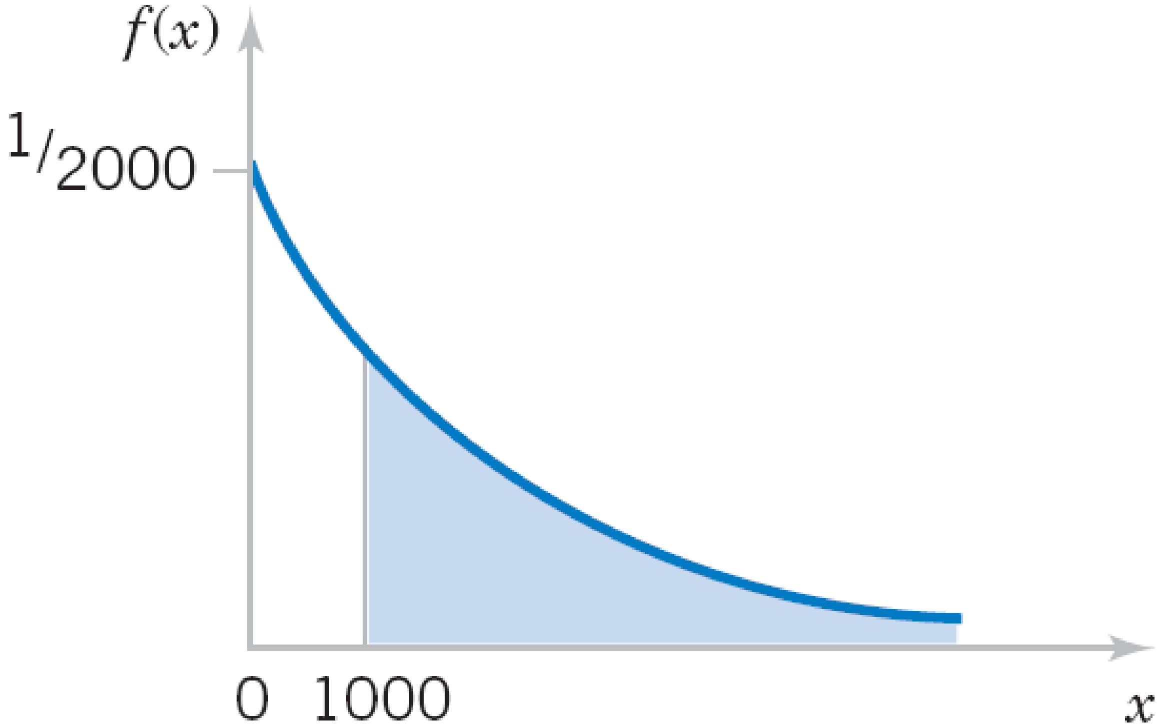


Figure 3-9 Probability density function for Example 3-2.

3-4 Continuous Random Variables

3-4.2 Cumulative Distribution Function

The **cumulative distribution function** (or cdf) of a continuous random variable X with probability density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du$$

for $-\infty < x < \infty$.

3-4 Continuous Random Variables

EXAMPLE 3-3

Flaw on a Magnetic Disk Distribution Function

The distance in micrometers from the start of a track on a magnetic disk until the first surface flaw is a random variable with the cdf

$$F(x) = 1 - \exp\left(-\frac{x}{2000}\right) \text{ for } x > 0$$

A graph of $F(x)$ is shown in Fig. 3-10. Note that $F(x) = 0$ for $x \leq 0$. Also, $F(x)$ increases to 1 as mentioned. The following probabilities should be compared to the results in Example 3-2. Determine the probability that the distance until the first surface flaw is less than 1000 micrometers.

3-4 Continuous Random Variables

EXAMPLE 3-3

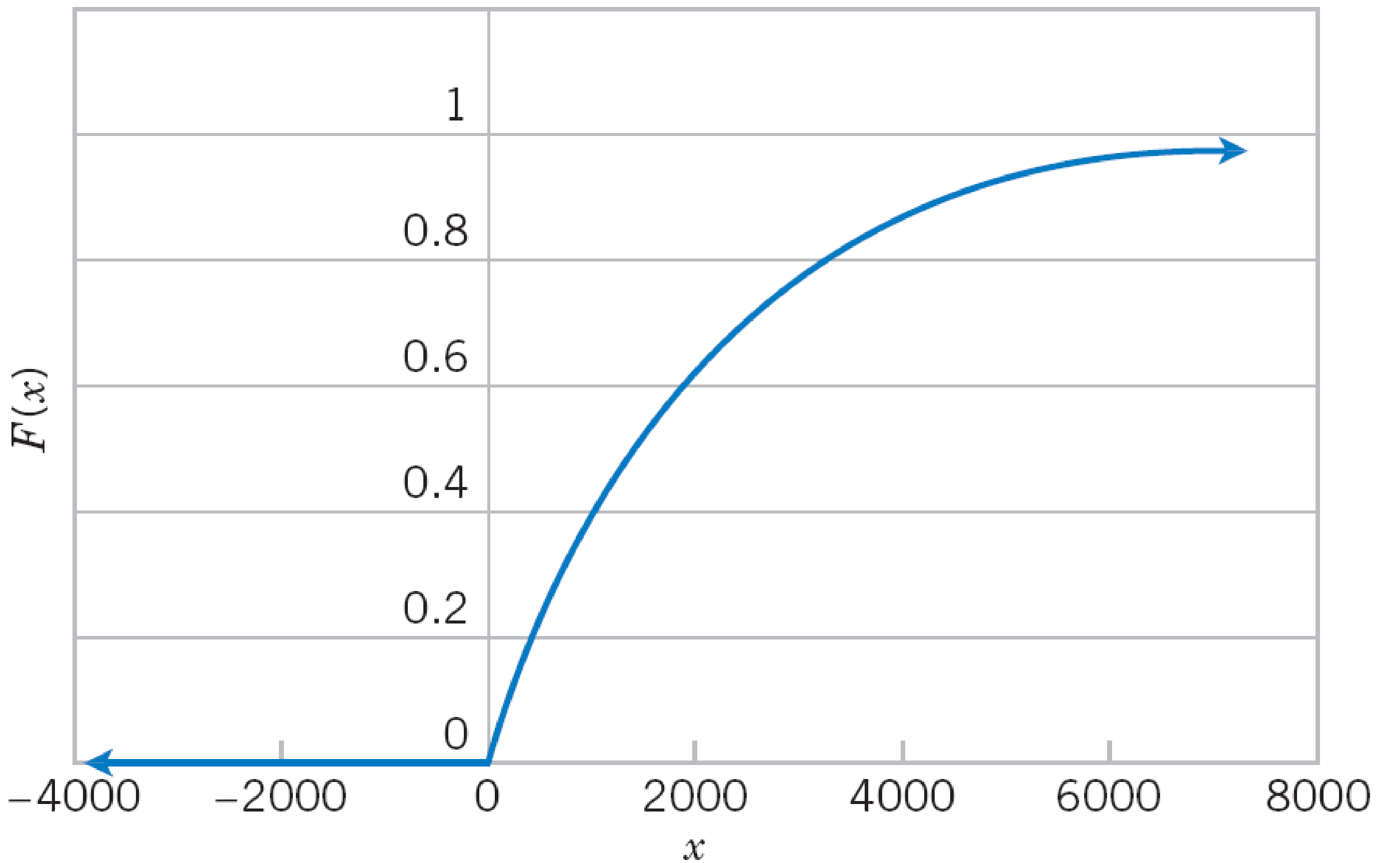


Figure 3-10 Cumulative distribution function for Example 3-3.

3-4 Continuous Random Variables

EXAMPLE 3-3

Solution. The random variable is the distance until the first surface flaw with distribution given by $F(x)$. The requested probability is

$$P(X < 1000) = F(1000) = 1 - \exp\left(-\frac{1}{2}\right) = 0.393$$

Determine the probability that the distance until the first surface flaw exceeds 2000 micrometers.

Solution. Now we use

$$\begin{aligned} P(2000 < X) &= 1 - P(X \leq 2000) = 1 - F(2000) = 1 - [1 - \exp(-1)] \\ &= \exp(-1) = 0.368 \end{aligned}$$

Determine the probability that the distance is between 1000 and 2000 micrometers.

Solution. The requested probability is

$$\begin{aligned} P(1000 < X < 2000) &= F(2000) - F(1000) = 1 - \exp(-1) - [1 - \exp(-0.5)] \\ &= \exp(-0.5) - \exp(-1) = 0.239 \end{aligned}$$

3-4 Continuous Random Variables

3-4.3 Mean and Variance

Suppose X is a continuous random variable with pdf $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{3-3}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is σ.

3-4 Continuous Random Variables

For the distance to a flaw in Example 3-2, the mean of X is

$$E(x) = \int_0^\infty x f(x)\, dx = \int_0^\infty x \frac{e^{-x/2000}}{2000}\, dx$$

A table of integrals or integration by parts can be used to show that

$$E(x) = -xe^{-x/2000} \Big|_0^\infty + \int_0^\infty e^{-x/2000}\, dx = 0 - 2000\, e^{-x/2000} \Big|_0^\infty = 2000$$

The variance of X is

$$V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_0^\infty (x - 2000)^2 \frac{e^{-x/2000}}{2000}\, dx$$

A table of integrals or integration by parts can be applied twice to show that

$$V(X) = 2000^2 = 4{,}000{,}000$$

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

Undoubtedly, the most widely used model for the distribution of a random variable is a **normal distribution**.

- Central limit theorem
- Gaussian distribution

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

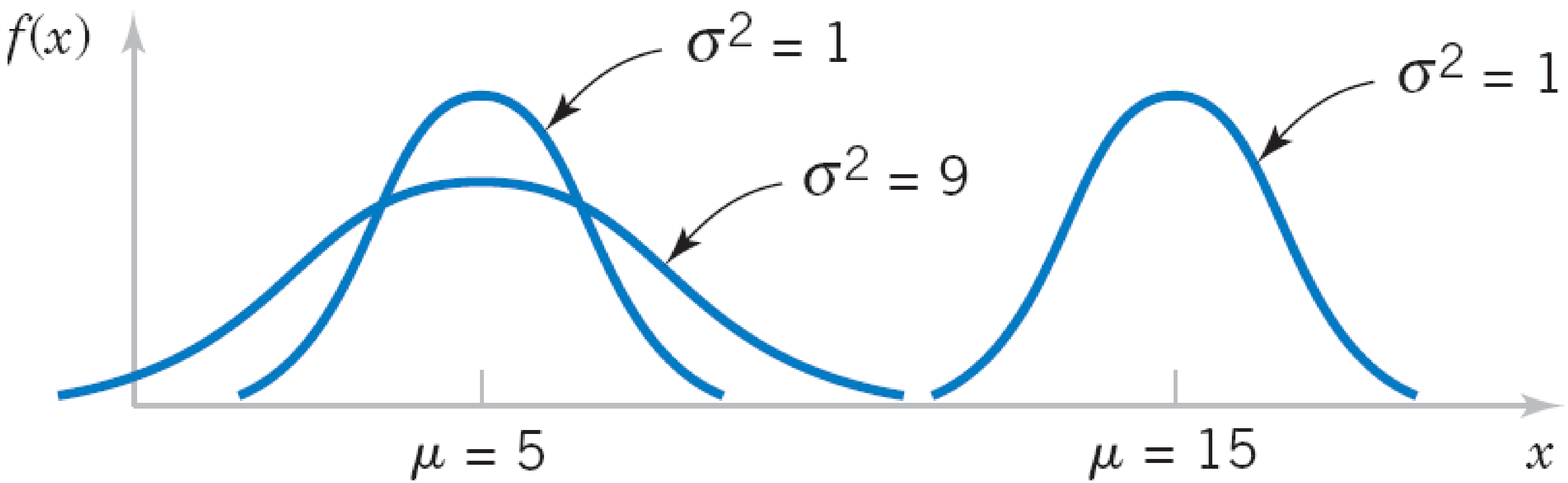


Figure 3-11 Normal probability density functions for selected values of the parameters μ and σ^2.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \text{for} \quad -\infty < x < \infty \tag{3-4}$$

has a **normal distribution** (and is called a **normal random variable**) with parameters μ and σ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \qquad \text{and} \qquad V(X) = \sigma^2$$

The mean and variance of the normal distribution are derived at the end of this section.

3-5 Important Continuous Distributions

EXAMPLE 3-6

Current in a Wire: Normal Distribution

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 milliamperes^2. What is the probability that a measurement exceeds 13 milliamperes?

Solution. Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. This probability is shown as the shaded area under the normal probability density function in Fig. 3-12. Unfortunately, there is no closed-form expression for the integral of a normal pdf, and probabilities based on the normal distribution are typically found numerically or from a table (which we will introduce later).

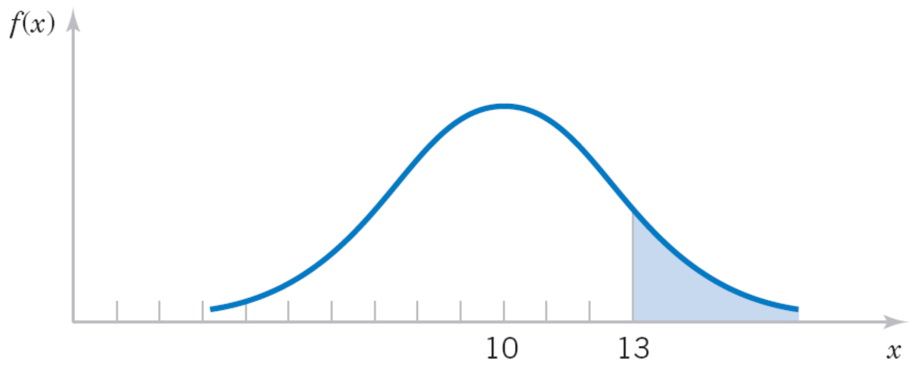


Figure 3-12 Probability that $X > 13$ for a normal random variable with $\mu = 10$ and $\sigma^2 = 4$ in Example 3-6.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

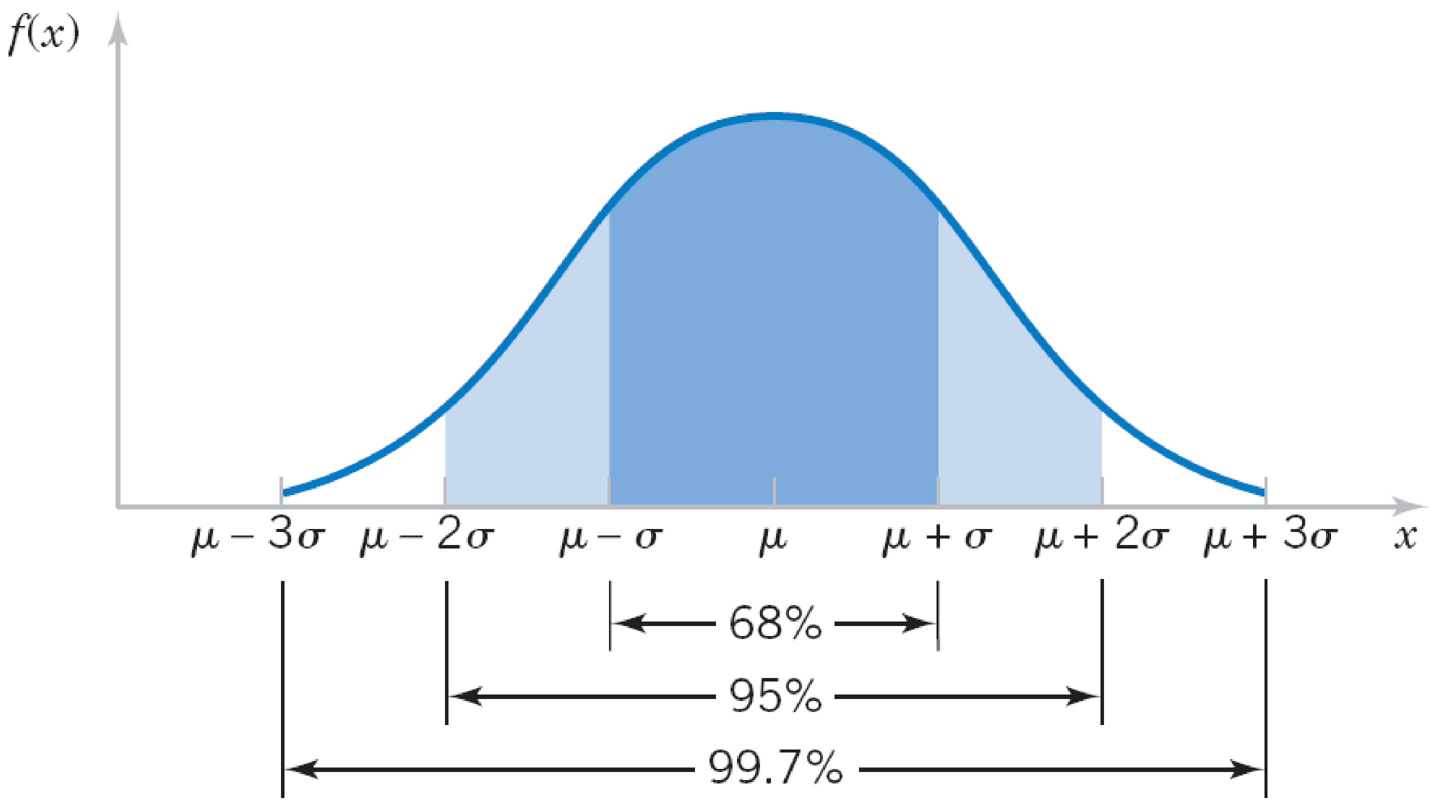


Figure 3-13 Probabilities associated with a normal distribution.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **standard normal** random variable. A standard normal random variable is denoted as Z.

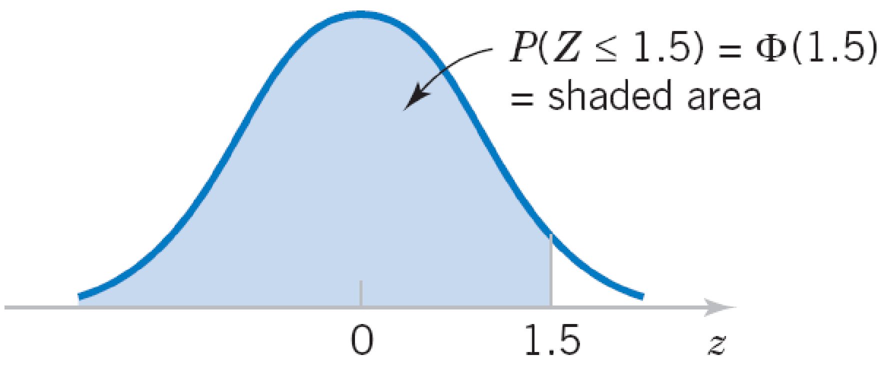
The function

$$\Phi(z) = P(Z \leq z)$$

is used to denote a probability from Appendix A Table I. It is the **cumulative distribution function** of a standard normal random variable. A table (or computer software) is required because the probability can't be determined by elementary methods.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 3-14 Standard normal probability density function.

3-5 Important Continuous Distributions

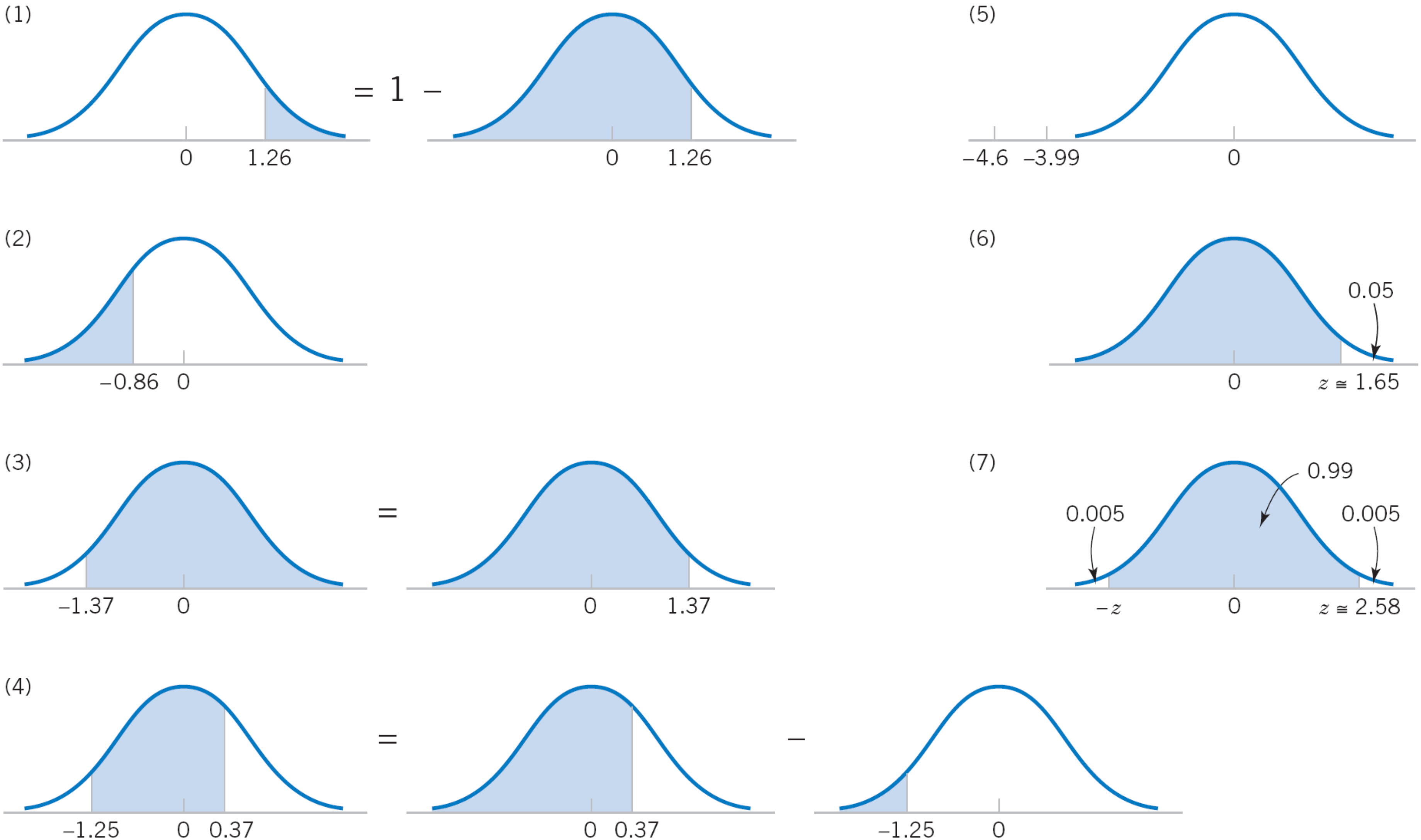


Figure 3-15 Graphical displays for Example 3-8.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a **standard normal** random variable.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z) \tag{3-5}$$

where

Z is a **standard normal** random variable, and

$z = (x - \mu)/\sigma$ is the ***z*-value** obtained by **standardizing** x.

The probability is obtained by entering **Appendix A Table I** with $z = (x - \mu)/\sigma$.

EXAMPLE 3-12

Diameter of a Shaft

The diameter of a shaft in an optical storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are 0.2500 ± 0.0015 inch. What proportion of shafts conforms to specifications?

Solution. Let X denote the shaft diameter in inches. The requested probability is shown in Fig. 3-19 and

$$\begin{aligned} P(0.2485 < X < 0.2515) &= P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right) \\ &= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6) \\ &= 0.91924 - 0.0000 = 0.91924 \end{aligned}$$

EXAMPLE 3-12

Most of the nonconforming shafts are too large, because the process mean is located very near to the upper specification limit. If the process is centered so that the process mean is equal to the target value of 0.2500,

$$P(0.2485 < X < 0.2515) = P\left(\frac{0.2485 - 0.2500}{0.0005} < Z < \frac{0.2515 - 0.2500}{0.0005}\right)$$
$$= P(-3 < Z < 3) = P(Z < 3) - P(Z < -3)$$
$$= 0.99865 - 0.00135 = 0.9973$$

By recentering the process, the yield is increased to approximately 99.73%.

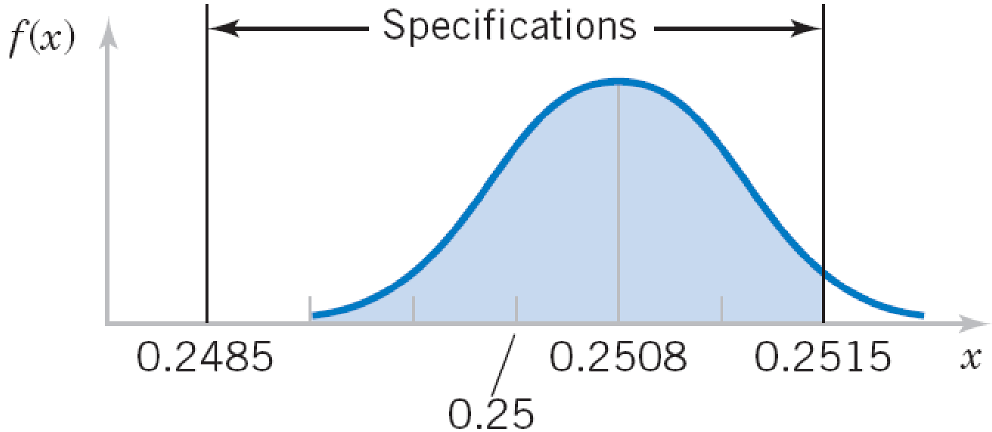


Figure 3-19 Distribution for Example 3-12.

3-5 Important Continuous Distributions

3-5.2 Lognormal Distribution

Let W have a normal distribution with mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with probability density function

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln(x) - \theta)^2}{2\omega^2}\right] \qquad 0 < x < \infty \tag{3-6}$$

The mean and variance of X are

$$E(X) = e^{\theta+\omega^2/2} \qquad \text{and} \qquad V(X) = e^{2\theta+\omega^2}(e^{\omega^2} - 1) \tag{3-7}$$

3-5 Important Continuous Distributions

3-5.2 Lognormal Distribution

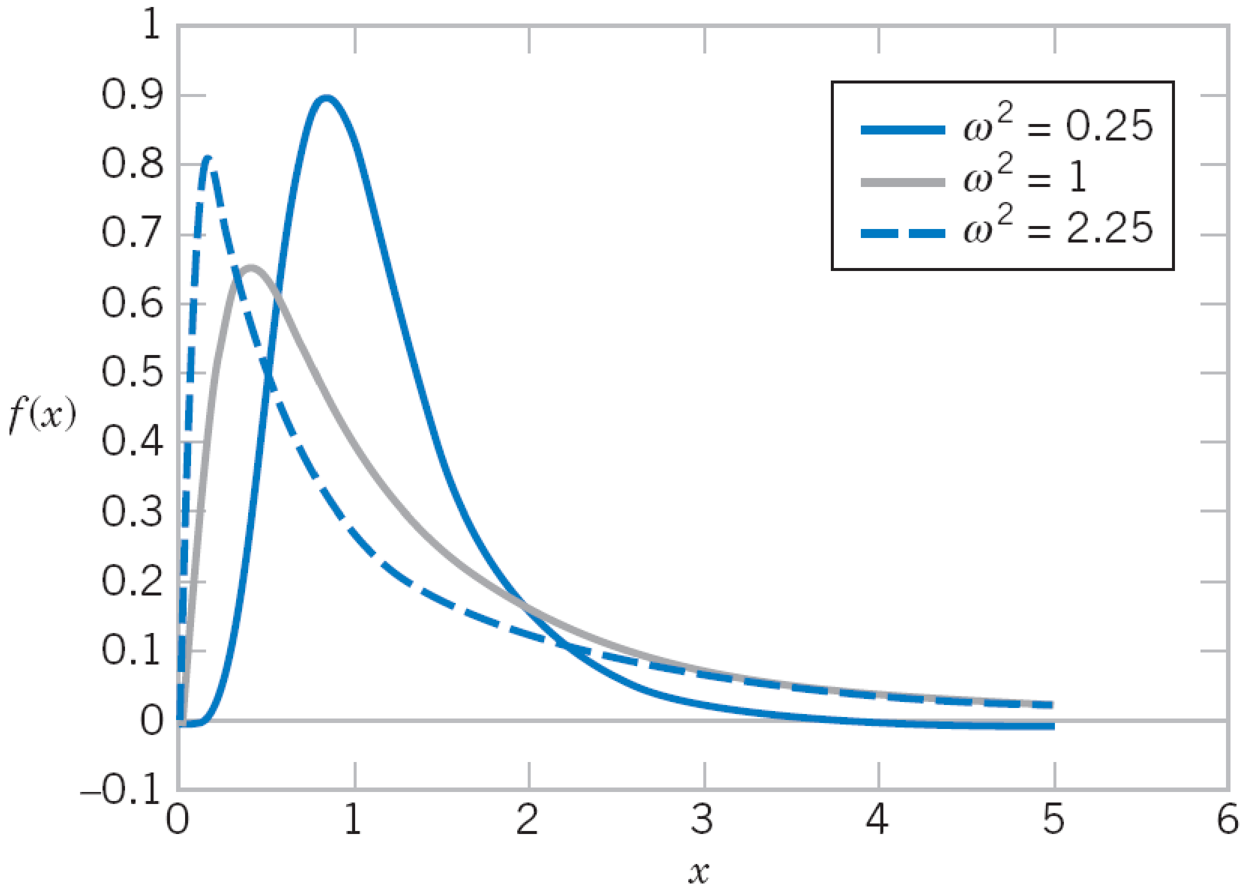


Figure 3-20 Lognormal probability density functions with $\theta = 0$ for selected values of ω^2.

3-5 Important Continuous Distributions

3-5.3 Gamma Distribution

The **gamma function** is

$$\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x} \, dx, \quad \text{for } r > 0 \tag{3-8}$$

3-5 Important Continuous Distributions

3-5.3 Gamma Distribution

The random variable X with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0 \tag{3-9}$$

is a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$. The mean and variance are

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \tag{3-10}$$

3-5 Important Continuous Distributions

3-5.3 Gamma Distribution

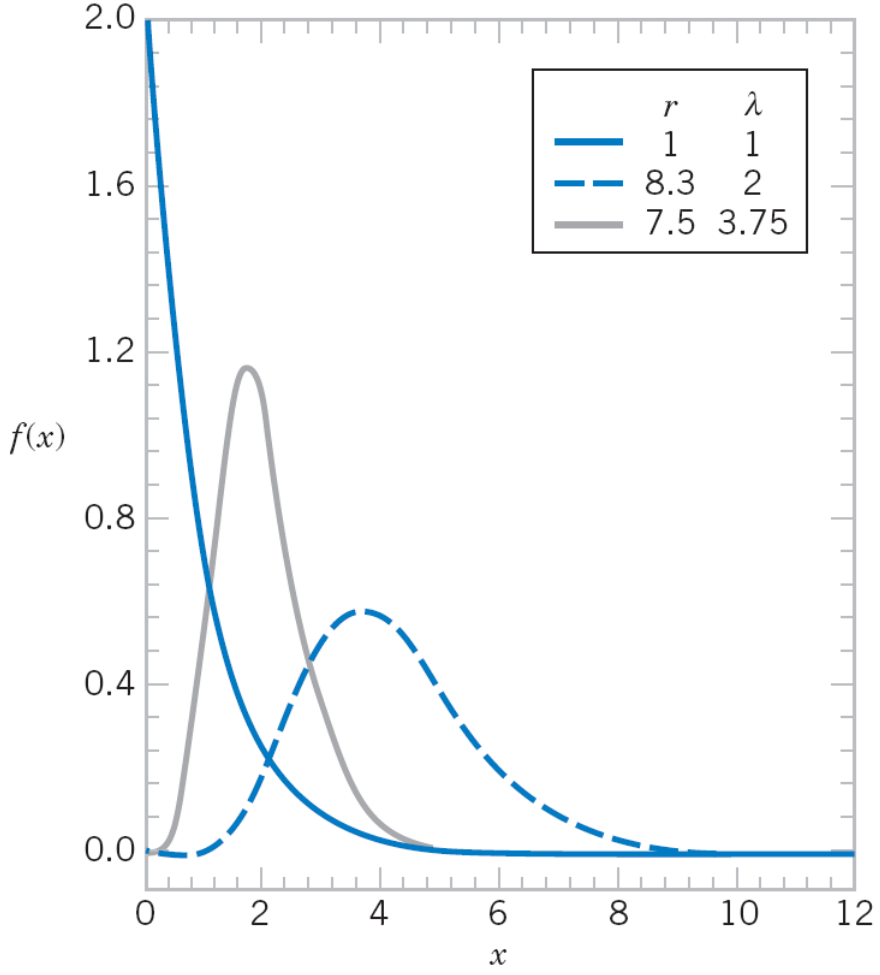


Figure 3-21 Gamma probability density functions for selected values of λ and r.

3-5 Important Continuous Distributions

3-5.4 Weibull Distribution

The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0 \tag{3-11}$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

3-5 Important Continuous Distributions

3-5.4 Weibull Distribution

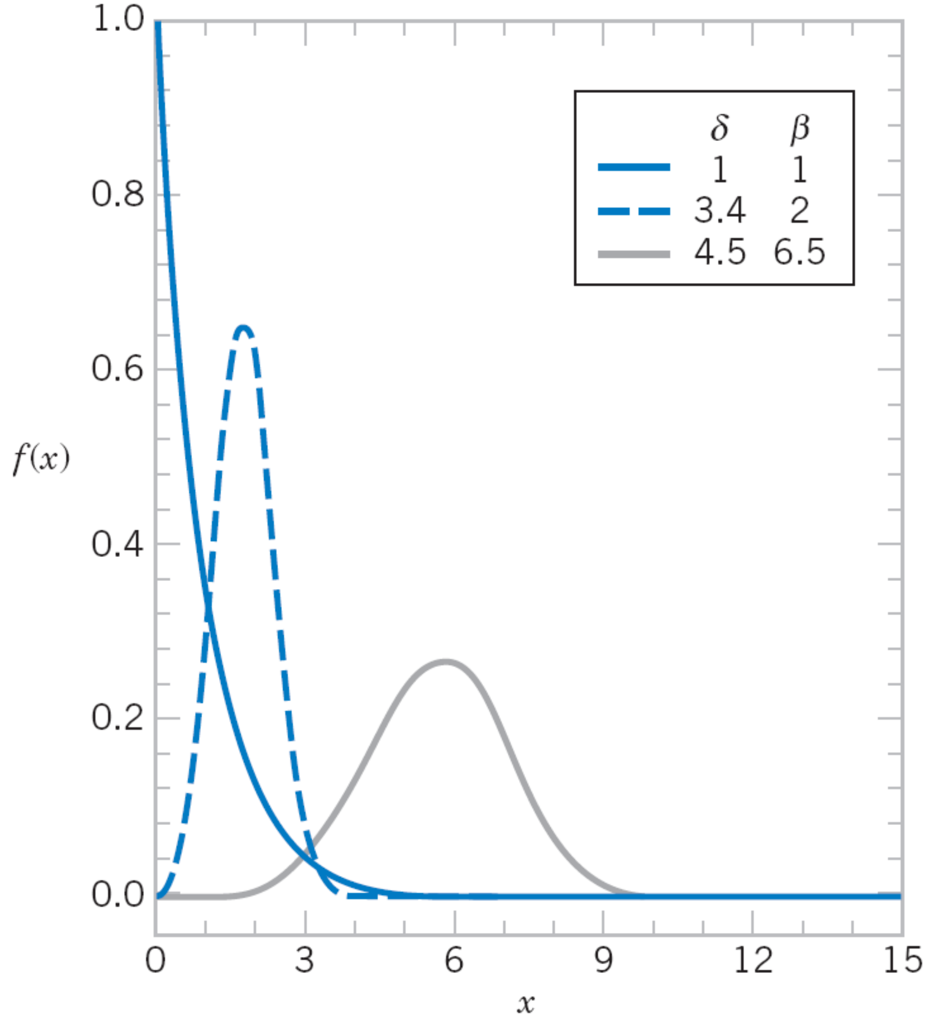


Figure 3-22 Weibull probability density functions for selected values of δ and β.

3-5 Important Continuous Distributions

3-5.4 Weibull Distribution

If X has a Weibull distribution with parameters δ and β, the cumulative distribution function of X is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

The mean and variance of the Weibull distribution are as follows.

If X has a Weibull distribution with parameters δ and β,

$$\mu = \delta\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad \sigma^2 = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \qquad (3\text{-}12)$$

3-6 Probability Plots

3-6.1 Normal Probability Plots

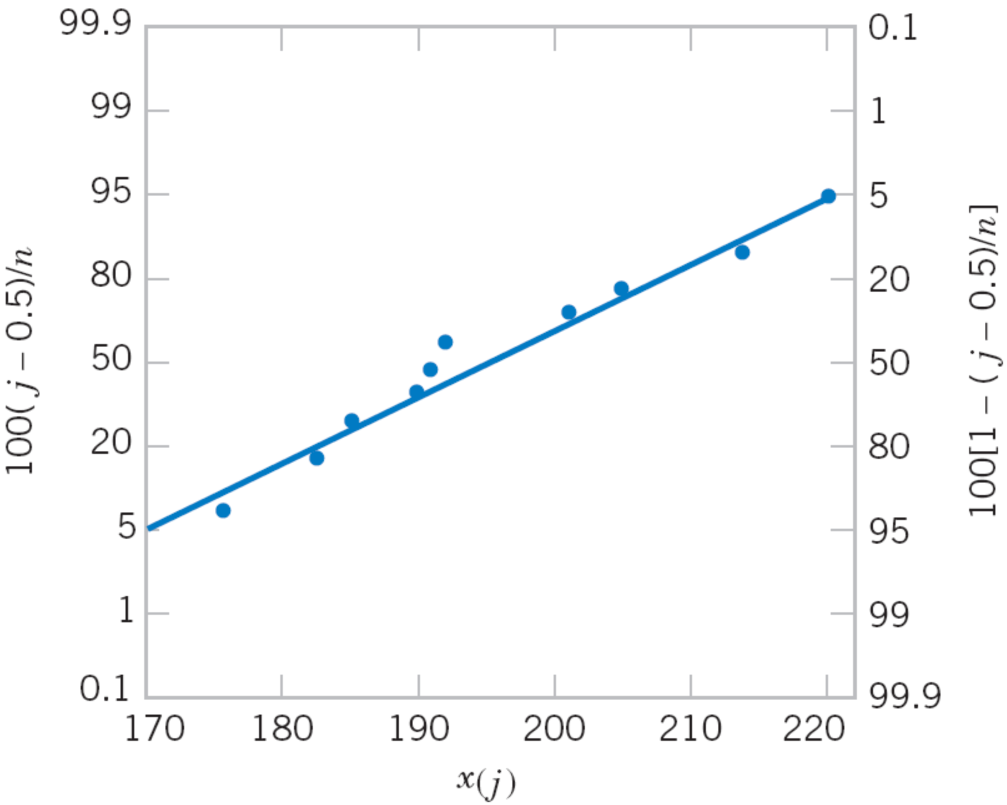
- How do we know if a normal distribution is a reasonable model for data?

- **Probability plotting** is a graphical method for determining whether sample data conform to a hypothesized distribution based on a subjective visual examination of the data.

- Probability plotting typically uses special graph paper, known as **probability paper,** that has been designed for the hypothesized distribution. Probability paper is widely available for the normal, lognormal, Weibull, and various chi-square and gamma distributions.

3-6 Probability Plots

3-6.1 Normal Probability Plots



j	$x_{(j)}$	$(j - 0.5)/10$
1	176	0.05
2	183	0.15
3	185	0.25
4	190	0.35
5	191	0.45
6	192	0.55
7	201	0.65
8	205	0.75
9	214	0.85
10	220	0.95

Figure 3-23 Normal probability plot for the battery life.

3-6 Probability Plots

3-6.1 Normal Probability Plots

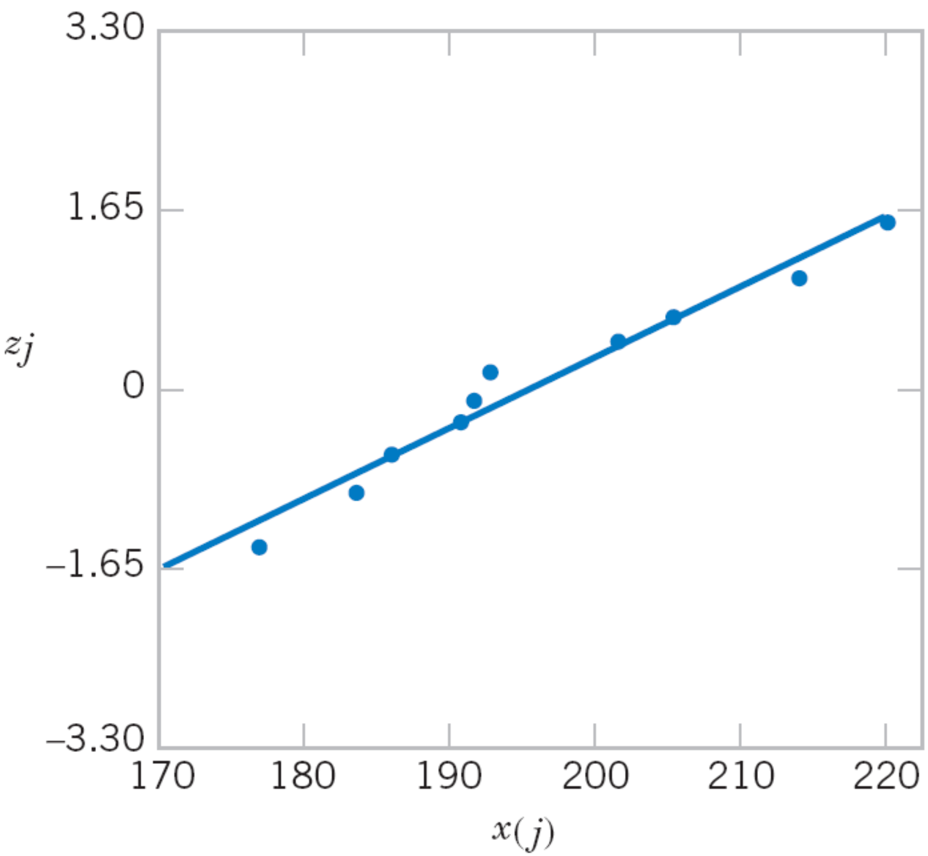


Figure 3-24 Normal probability plot obtained from standardized normal scores.

j	$x_{(j)}$	$(j - 0.5)/10$	z_j
1	176	0.05	−1.64
2	183	0.15	−1.04
3	185	0.25	−0.67
4	190	0.35	−0.39
5	191	0.45	−0.13
6	192	0.55	0.13
7	201	0.65	0.39
8	205	0.75	0.67
9	214	0.85	1.04
10	220	0.95	1.64

3-6 Probability Plots

3-6.2 Other Probability Plots

Table 3-1 Crack Length (mm) for an Aluminum Alloy

81	98	291	101	98	118	158	197	139	249
249	135	223	205	80	177	82	64	137	149
117	149	127	115	198	342	83	34	342	185
227	225	185	240	161	197	98	65	144	151
134	59	181	151	240	146	104	100	215	200

3-6 Probability Plots

3-6.2 Other Probability Plots

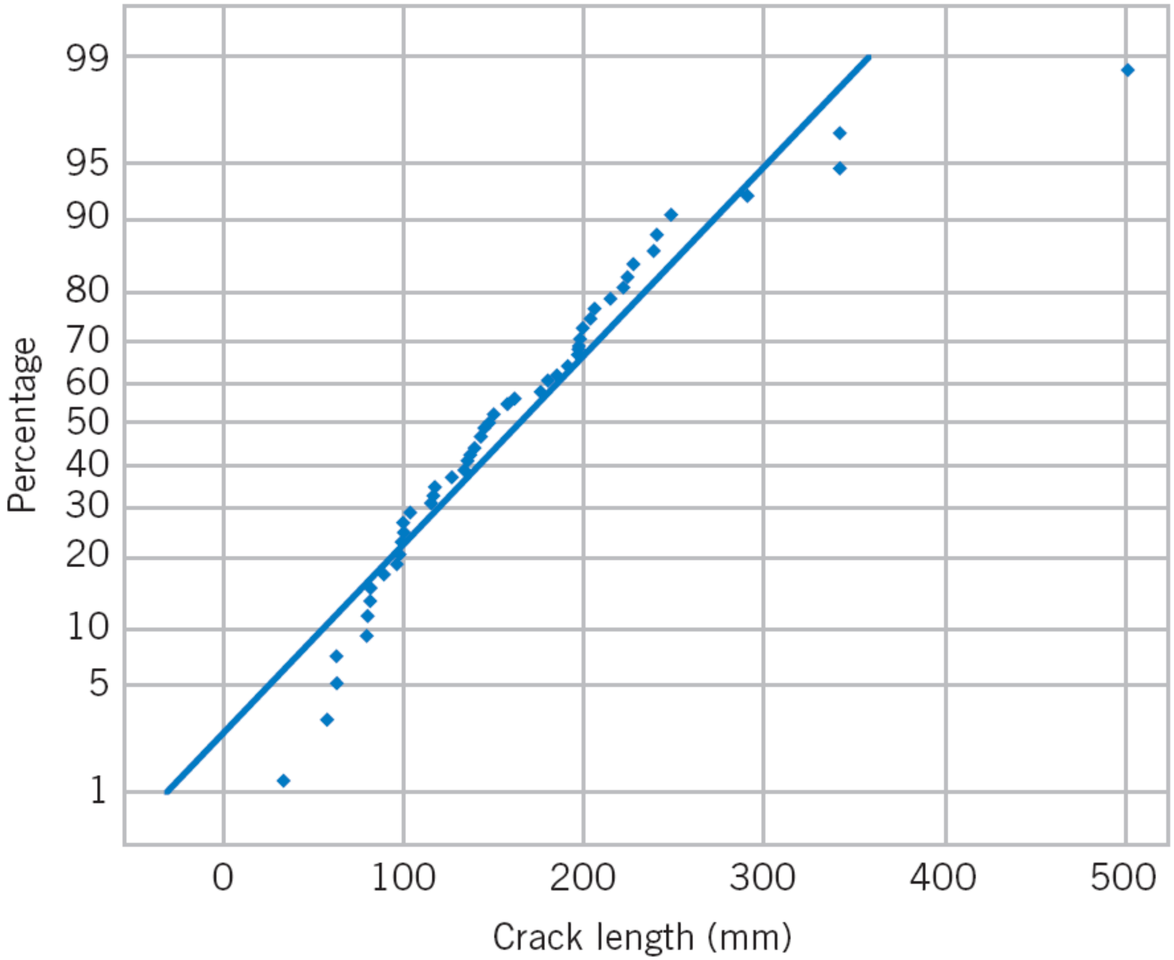


Figure 3-25 Normal probability plot for the crack-length data in Table 3-1.

3-6 Probability Plots

3-6.2 Other Probability Plots

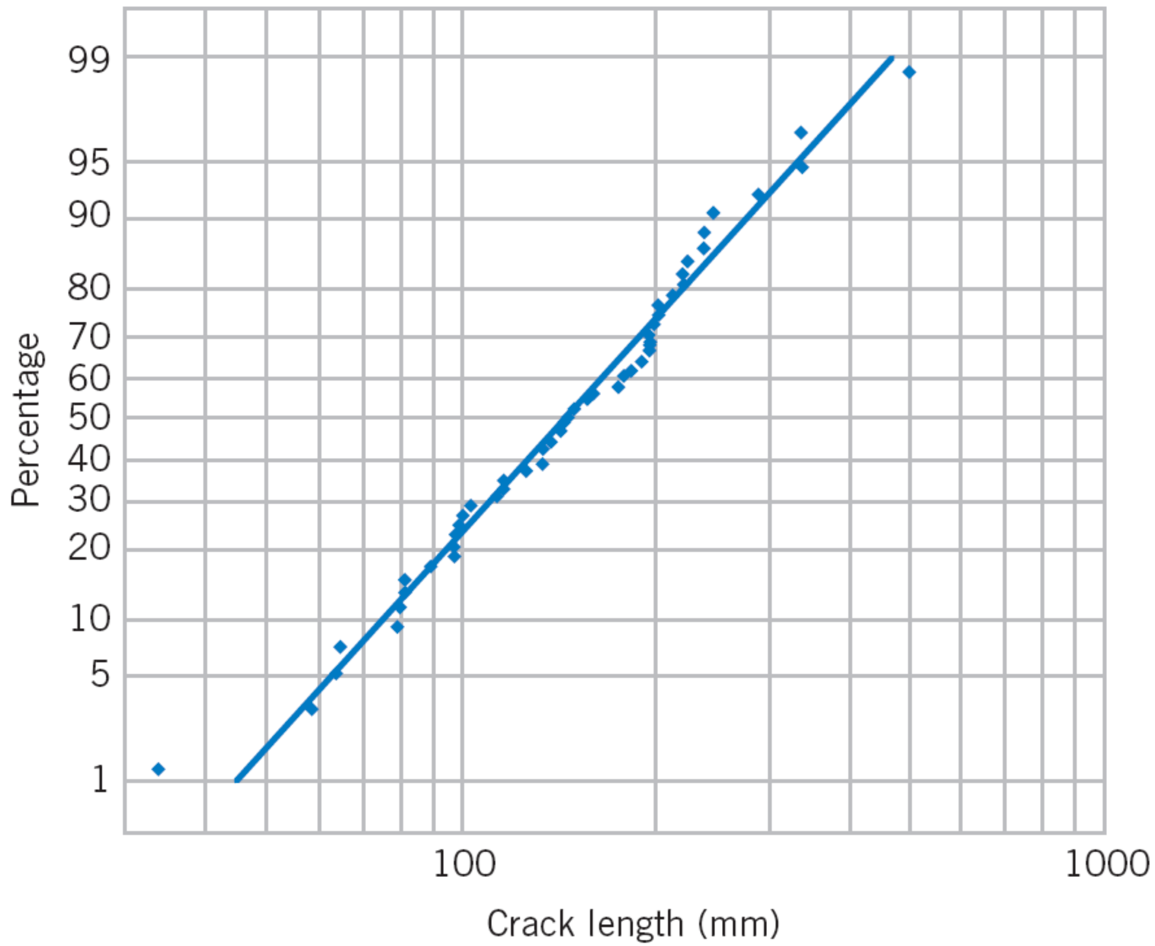


Figure 3-26 Lognormal probability plot for the crack-length data in Table 3-1.

3-6 Probability Plots

3-6.2 Other Probability Plots

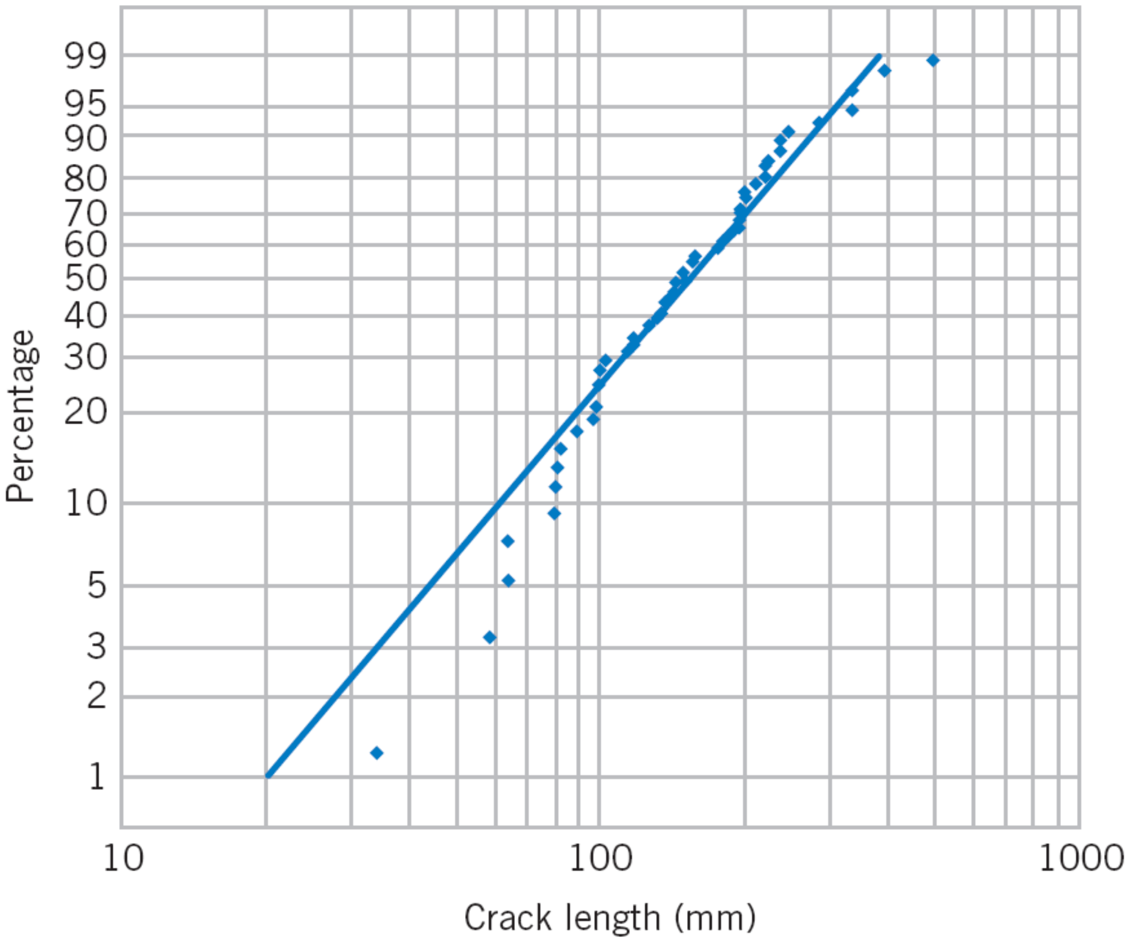


Figure 3-27 Weibull probability plot for the crack-length data in Table 3-1.

3-7 Discrete Random Variables

- Only measurements at **discrete** points are possible

EXAMPLE 3-17

Semiconductor Wafer Contamination

The analysis of the surface of a semiconductor wafer records the number of particles of contamination that exceed a certain size. Define the random variable X to equal the number of particles of contamination.

The possible values of X are integers from 0 up to some large value that represents the maximum number of these particles that can be found on one of the wafers. If this maximum number is very large, it might be convenient to assume that any integer from zero to ∞ is possible. ■

3-7 Discrete Random Variables

3-7.1 Probability Mass Function

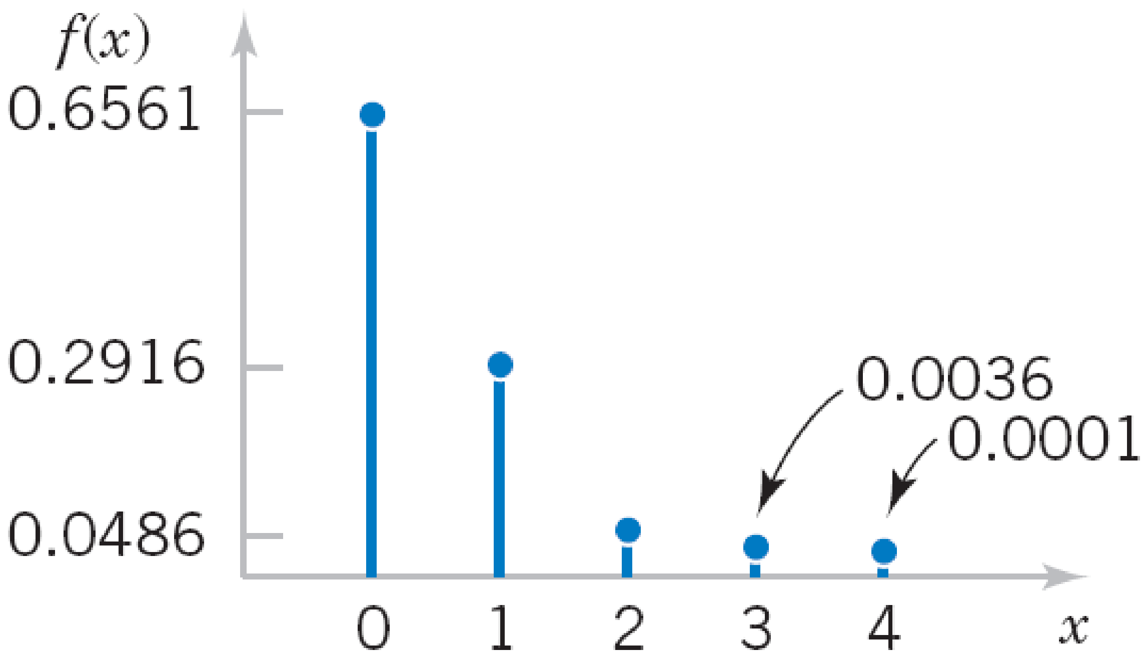


Figure 3-28 Probability distribution for X in Example 3-18.

3-7 Discrete Random Variables

3-7.1 Probability Mass Function

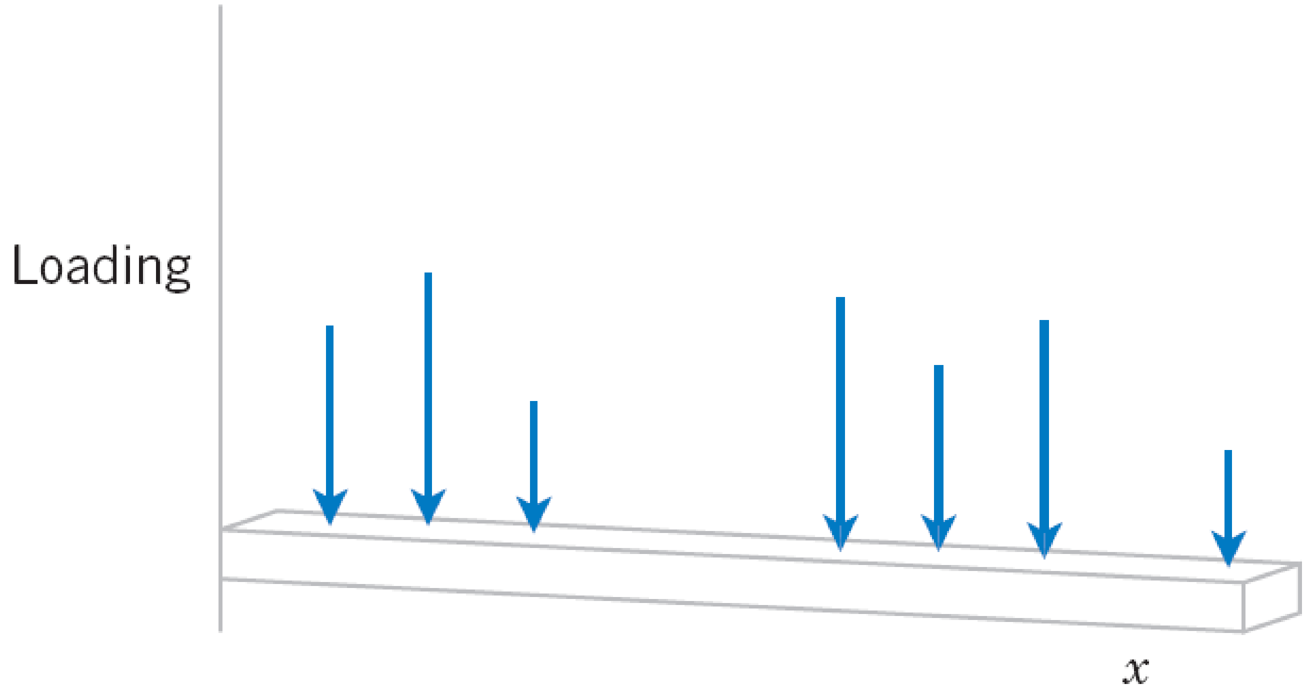


Figure 3-29 Loadings at discrete points on a long, thin beam.

3-7 Discrete Random Variables

3-7.1 Probability Mass Function

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** (or pmf) is

$$f(x_i) = P(X = x_i) \tag{3-13}$$

3-7 Discrete Random Variables

3-7.2 Cumulative Distribution Function

The **cumulative distribution function** of a discrete random variable X is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

3-7 Discrete Random Variables

3-7.2 Cumulative Distribution Function

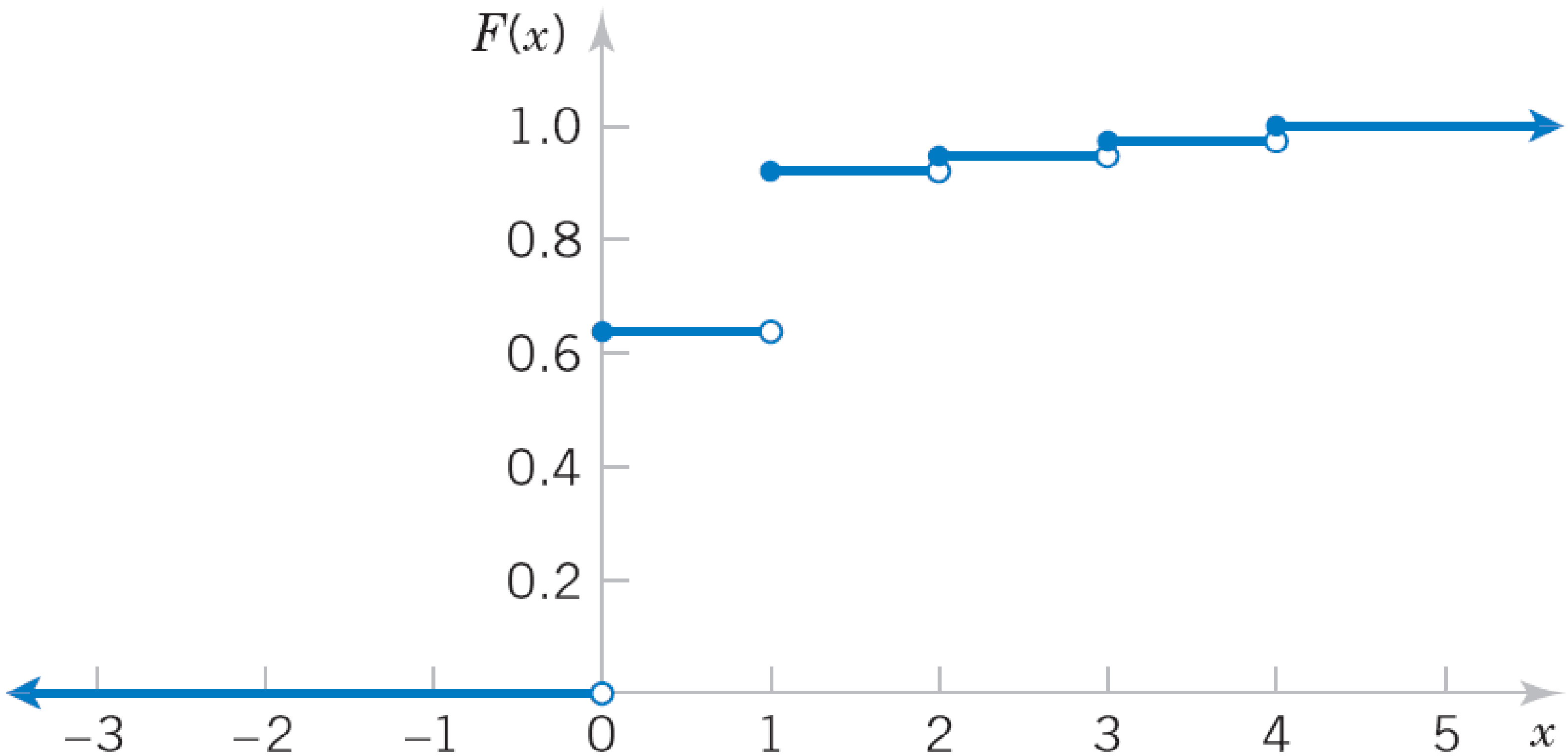


Figure 3-30 Cumulative distribution function for x in Example 3-19.

3-7 Discrete Random Variables

3-7.3 Mean and Variance

Let the possible values of the random variable X be denoted as $x_1, x_2, \ldots, x_n$. The pmf of X is $f(x)$, so $f(x_i) = P(X = x_i)$.

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \qquad (3\text{-}14)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$$

The **standard deviation** of X is σ.

3-7 Discrete Random Variables

3-7.3 Mean and Variance

EXAMPLE 3-21

Product Revenue

Two new product designs are to be compared on the basis of revenue potential. Marketing feels that the revenue from design A can be predicted quite accurately to be \$3 million. The revenue potential of design B is more difficult to assess. Marketing concludes that there is a probability of 0.3 that the revenue from design B will be \$7 million, but there is a 0.7 probability that the revenue will be only \$2 million. Which design would you choose?

Solution. Let X denote the revenue from design A. Because there is no uncertainty in the revenue from design A, we can model the distribution of the random variable X as \$3 million with probability one. Therefore, $E(X) = \$3$ million.

3-7 Discrete Random Variables

3-7.3 Mean and Variance

EXAMPLE 3-21

As an extension of this example, let Y denote the revenue from design B. The expected value of Y in millions of dollars is

$$E(Y) = \$7(0.3) + \$2(0.7) = \$3.5$$

Because $E(Y)$ exceeds $E(X)$, we might choose design B. However, the variability of the result from design B is larger. That is,

$$\sigma^2 = (7 - 3.5)^2(0.3) + (2 - 3.5)^2(0.7) = 5.25 \text{ millions of dollars squared}$$

and

$$\sigma = \sqrt{5.25} = 2.29 \text{ millions of dollars}$$

3-8 Binomial Distribution

- A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli trial**.

- It is usually assumed that the trials that constitute the random experiment are **independent.** This implies that the outcome from one trial has no effect on the outcome to be obtained from any other trial.

- Furthermore, it is often reasonable to assume that the **probability of a success on each trial is constant**.

3-8 Binomial Distribution

- Consider the following random experiments and random variables.
  - Flip a coin 10 times. Let X = the number of heads obtained.
  - Of all bits transmitted through a digital transmission channel, 10% are received in error. Let X = the number of bits in error in the next 4 bits transmitted.

Do they meet the following criteria:

1. Does the experiment consist of **Bernoulli trials**?
2. **Are the trials that constitute the random experiment are independent?**
3. **Is probability of a success on each trial is constant?**

3-8 Binomial Distribution

A random experiment consisting of n repeated trials such that

1. the trials are independent,
2. each trial results in only two possible outcomes, labeled as *success* and *failure,* and
3. the probability of a success on each trial, denoted as p, remains constant

is called a *binomial experiment.*

The random variable X that equals the number of trials that result in a *success* has a **binomial distribution** with parameters p and n where $0 < p < 1$ and $n = \{1, 2, 3, \ldots\}$. The pmf of X is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n \tag{3-15}$$

3-8 Binomial Distribution

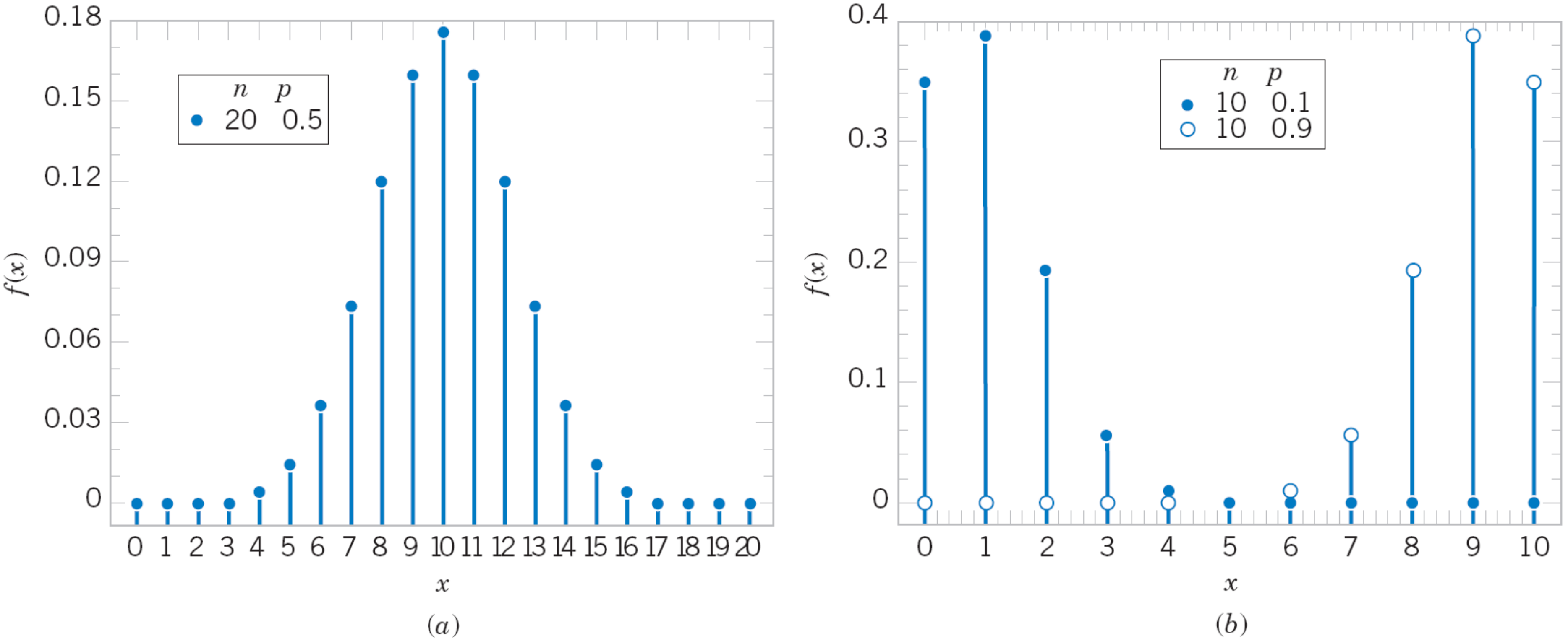


Figure 3-31 Binomial distribution for selected values of n and p.

3-8 Binomial Distribution

If X is a binomial random variable with parameters p and n,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p) \tag{3-16}$$

EXAMPLE 3-25

Bit Transmission Errors: Binomial Mean and Variance

For the number of transmitted bits received in error in Example 3-18, $n = 4$ and $p = 0.1$ so

$$E(X) = 4(0.1) = 0.4$$

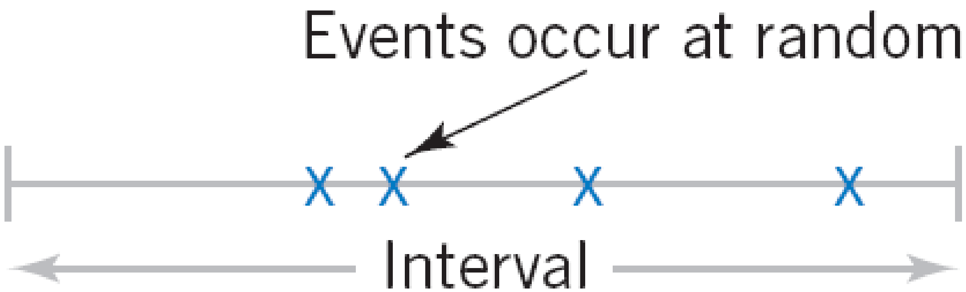
The variance of the number of defective bits is

$$V(X) = 4(0.1)(0.9) = 0.36$$

These results match those that were calculated directly from the probabilities in Example 3-20.

3-9 Poisson Process

Figure 3-32 In a Poission process, events occur at random in an interval.



3-9 Poisson Process

3-9.1 Poisson Distribution

EXAMPLE 3-26

Limit of Bit Errors

Consider the transmission of n bits over a digital communication channel. Let the random variable X equal the number of bits in error. When the probability that a bit is in error is constant and the transmissions are independent, X has a binomial distribution. Let p denote the probability that a bit is in error. Then, $E(X) = pn$. Now, suppose that the number of bits transmitted increases and the probability of an error decreases exactly enough that pn remains equal to a constant—say, λ. That is, n increases and p decreases accordingly, such that $E(X)$ remains constant. Then,

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x\, x!}(np)^x(1-p)^n(1-p)^{-x}$$

With some work, it can be shown that

$$\lim_{n\to\infty} P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

Also, because the number of bits transmitted tends to infinity, the number of errors can equal any nonnegative integer. Therefore, the possible values for X are the integers from zero to infinity.

3-9 Poisson Process

3-9.1 Poisson Distribution

Assume that events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

1. The probability of more than one event in a subinterval is zero,
2. The probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
3. The event occurrence in each subinterval is independent of other subintervals, the random experiment is called a *Poisson process.*

If the mean number of events in the interval is $\lambda > 0$, the random variable X that equals the number of events in the interval has a **Poisson distribution** with parameter λ, and the pmf of X is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots \tag{3-17}$$

The mean and variance of X are

$$E(X) = \lambda \qquad \text{and} \qquad V(X) = \lambda \tag{3-18}$$

3-9 Poisson Process

3-9.1 Poisson Distribution

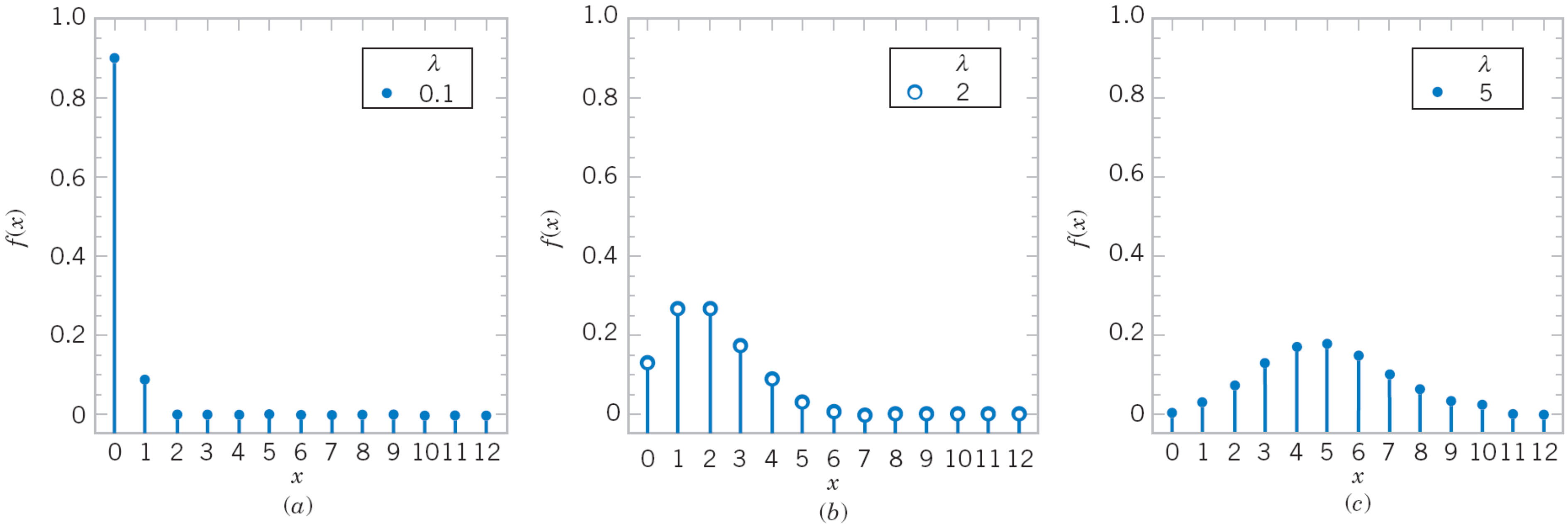


Figure 3-33 Poisson distribution for selected values of the parameter λ.

3-9 Poisson Process

3-9.1 Poisson Distribution

EXAMPLE 3-29

Contamination on Optical Disks

Contamination is a problem in the manufacture of optical storage disks. The number of particles of contamination that occur on an optical disk has a Poisson distribution, and the average number of particles per centimeter squared of media surface is 0.1. The area of a disk under study is 100 squared centimeters. Determine the probability that 12 particles occur in the area of a disk under study.

Solution. Let X denote the number of particles in the area of a disk under study. Because the mean number of particles is 0.1 particles per cm^2,

$$\begin{aligned} E(X) &= 100 \text{ cm}^2 \times 0.1 \text{ particles/cm}^2 \\ &= 10 \text{ particles} \end{aligned}$$

Therefore,

$$P(X = 12) = \frac{e^{-10}10^{12}}{12!} = 0.095$$

3-9 Poisson Process

3-9.1 Poisson Distribution

EXAMPLE 3-29

Determine the probability that zero particles occur in the area of the disk under study.

Solution. Now, $P(X = 0) = e^{-10} = 4.54 \times 10^{-5}$.

Determine the probability that 12 or fewer particles occur in the area of a disk under study.

Solution. This probability is

$$P(X \le 12) = P(X = 0) + P(X = 1) + \cdots + P(X = 12)$$
$$= \sum_{i=0}^{12} \frac{e^{-10} 10^i}{i!}$$

Because this sum is tedious to compute, many computer programs calculate cumulative Poisson probabilities. From Minitab, we obtain $P(X \le 12) = 0.7916$. ■

3-9 Poisson Process

3-9.2 Exponential Distribution

• The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire. The distance between flaws is another random variable that is often of interest.

• Let the random variable X denote the *length* from any starting point on the wire until a flaw is detected.

• As you might expect, the distribution of X can be obtained from knowledge of the distribution of the number of flaws. The key to the relationship is the following concept:

> The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters—simple, but sufficient for an analysis of the distribution of X.

3-9 Poisson Process

3-9.2 Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ has an **exponential distribution** with parameter λ. The pdf of X is

$$f(x) = \lambda e^{-\lambda x}, \qquad \text{for } 0 \le x < \infty \tag{3-19}$$

The mean and variance of X are

$$E(X) = \frac{1}{\lambda} \quad \text{and} \quad V(X) = \frac{1}{\lambda^2} \tag{3-20}$$

3-9 Poisson Process

3-9.2 Exponential Distribution

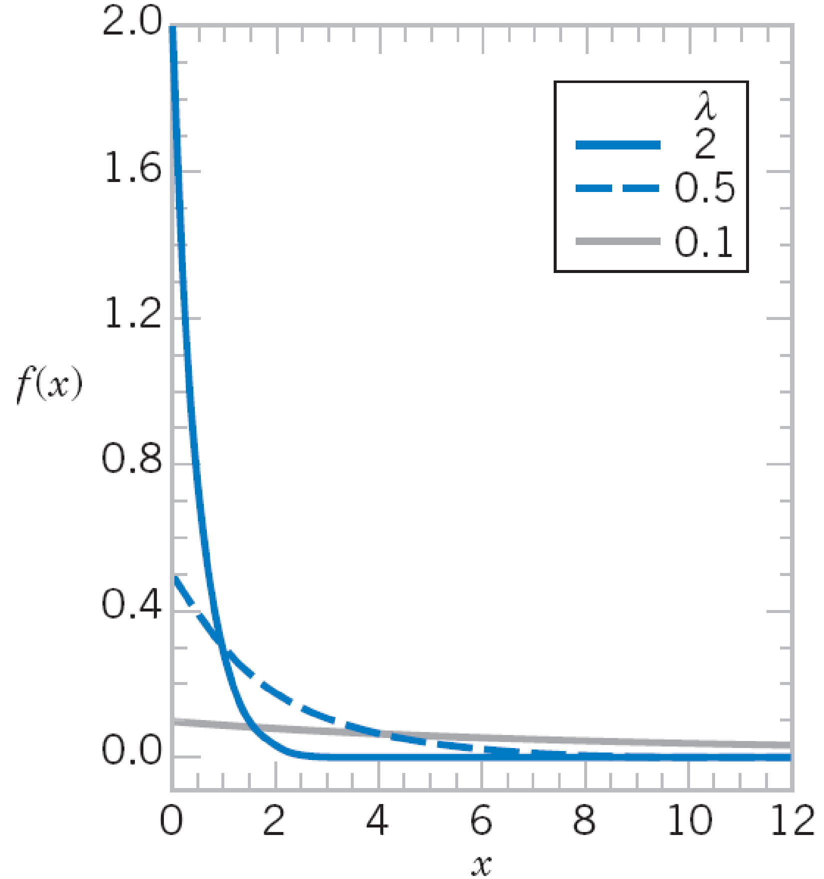


Figure 3-34 Probability density function of an exponential random variable for selected values of λ.

3-9 Poisson Process

3-9.2 Exponential Distribution

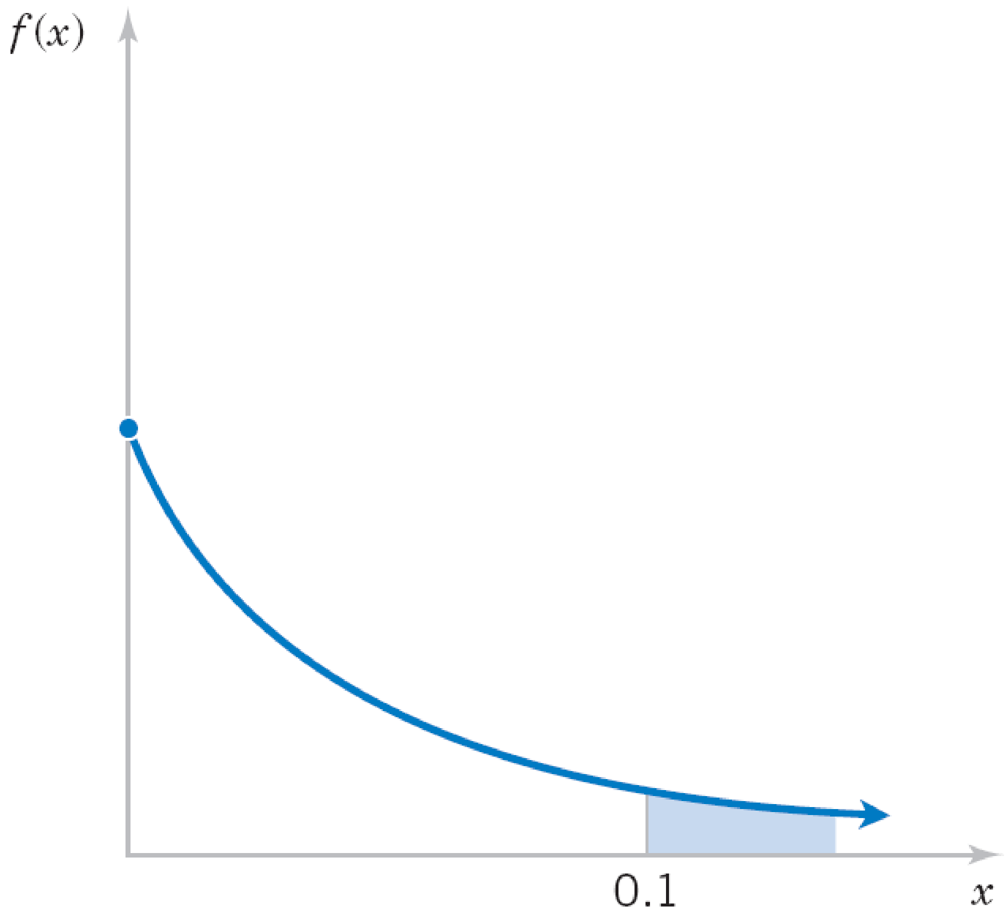


Figure 3-35 Probability for the exponential distribution in Example 3-30.

3-9 Poisson Process

3-9.2 Exponential Distribution

- The exponential distribution is often used in reliability studies as the model for the time until failure of a device.

- For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours. The **lack of memory property** of the exponential distribution implies that the *device does not wear out*. The lifetime of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable.

- However, the lifetime of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution that does not lack memory.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{3-21}$$

is approximately a standard normal random variable. Consequently, probabilities computed from Z can be used to approximate probabilities for X.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial

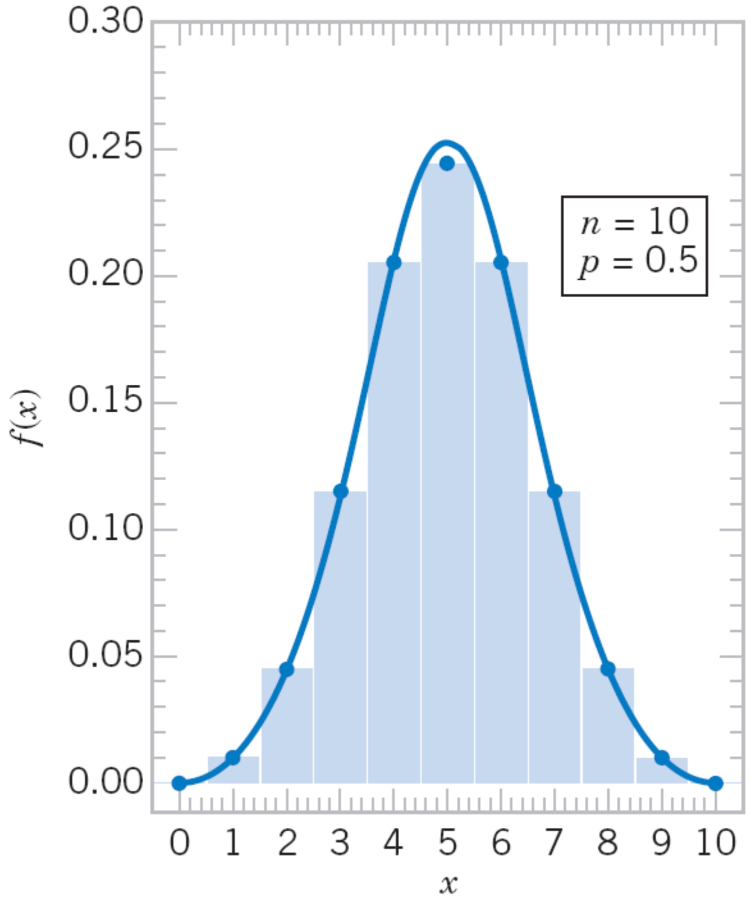


Figure 3-36 Normal approximation to the binomial distribution.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial

EXAMPLE 3-32

Again consider the transmission of bits in the previous example. To judge how well the normal approximation works, assume that only $n = 50$ bits are to be transmitted and that the probability of an error is $p = 0.1$. The exact probability that 2 or fewer errors occur is

$$P(X \le 2) = \binom{50}{0} 0.9^{50} + \binom{50}{1} 0.1(0.9^{49}) + \binom{50}{2} 0.1^2(0.9^{48}) = 0.11$$

Based on the normal approximation,

$$P(X \le 2) = P\left(\frac{X - 5}{\sqrt{50(0.1)(0.9)}} < \frac{2.5 - 5}{\sqrt{50(0.1)(0.9)}}\right) = P(Z < -1.18) = 0.12$$

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Poisson

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{3-22}$$

is approximately a standard normal random variable.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

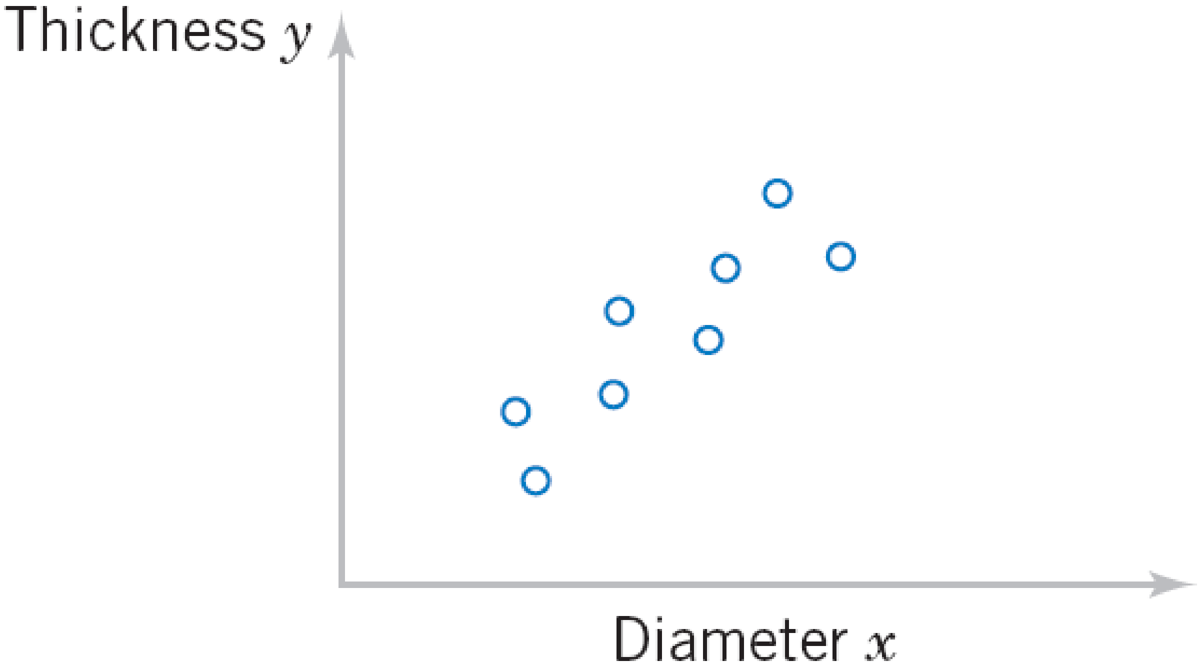


Figure 3-38 Scatter diagram of diameter and thickness measurements.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

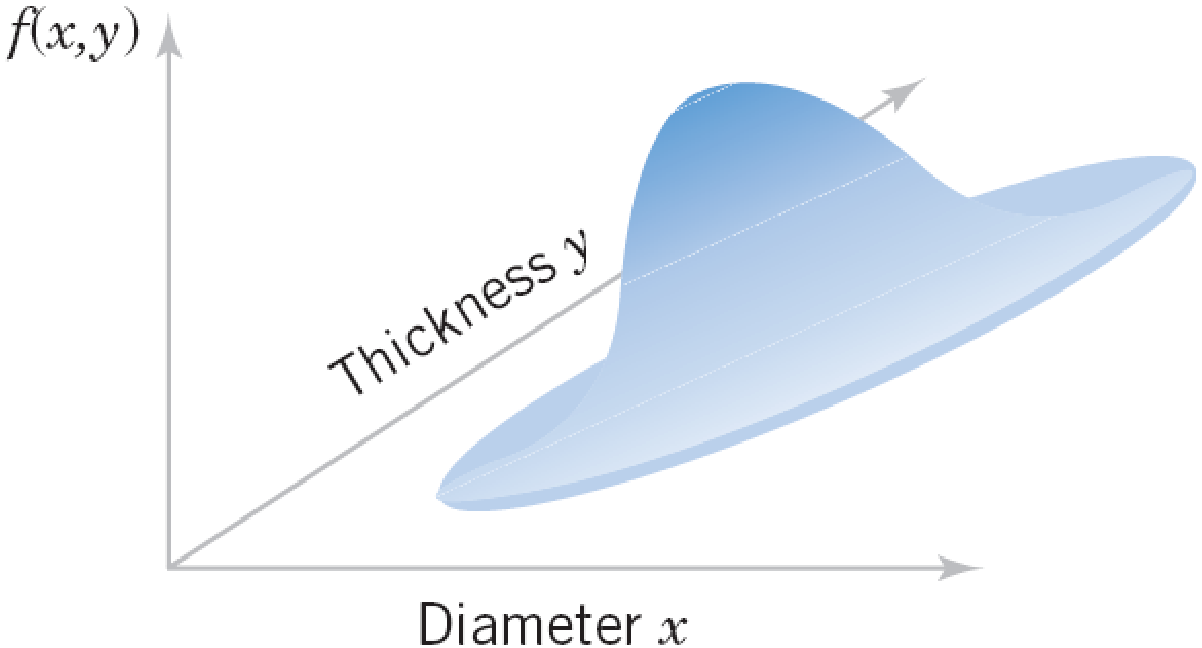


Figure 3-39 Joint probability density function of x and y.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

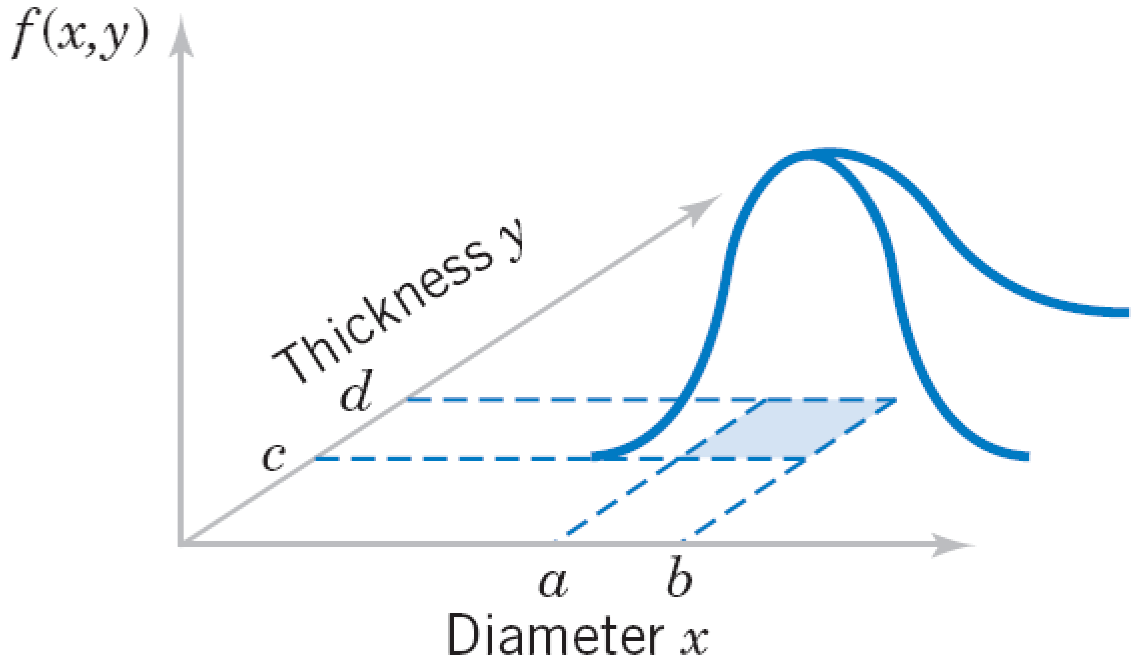


Figure 3-40 Probability of a region is the volume enclosed by $f(x, y)$ over the region.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

$$P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

3-11 More Than One Random Variable and Independence

3-11.2 Independence

The random variables $X_1, X_2, \ldots, X_n$ are **independent** if

$$P(X_1 \in E_1, X_2 \in E_2, \ldots, X_n \in E_n) = P(X_1 \in E_1)P(X_2 \in E_2)\cdots P(X_n \in E_n)$$

for *any* sets $E_1, E_2, \ldots, E_n$.

3-11 More Than One Random Variable and Independence

3-11.2 Independence

EXAMPLE 3-34

Optical Drive Diameters

In Example 3-12, the probability that a diameter meets specifications was determined to be 0.919. What is the probability that 10 diameters all meet specifications, assuming that the diameters are independent?

Solution. Denote the diameter of the first shaft as X_1, the diameter of the second shaft as X_2, and so forth, so that the diameter of the tenth shaft is denoted as X_{10}. The probability that all shafts meet specifications can be written as

$$P(0.2485 < X_1 < 0.2515, 0.2485 < X_2 < 0.2515, \ldots, 0.2485 < X_{10} < 0.2515)$$

In this example, the only set of interest is

$$E_1 = (0.2485, 0.2515)$$

With respect to the notation used in the definition of independence,

$$E_1 = E_2 = \cdots = E_{10}$$

3-11 More Than One Random Variable and Independence

3-11.2 Independence

EXAMPLE 3-34

Recall the relative frequency interpretation of probability. The proportion of times that shaft 1 is expected to meet the specifications is 0.919, the proportion of times that shaft 2 is expected to meet the specifications is 0.919, and so forth. If the random variables are independent, the proportion of times in which we measure 10 shafts that we expect all to meet the specifications is

$$P(0.2485 < X_1 < 0.2515, 0.2485 < X_2 < 0.2515, \ldots, 0.2485 < X_{10} < 0.2515)$$
$$= P(0.2485 < X_1 < 0.2515) \times P(0.2485 < X_2 < 0.2515) \times \cdots \times P(0.2485 < X_{10} < 0.2515) = 0.919^{10} = 0.430$$

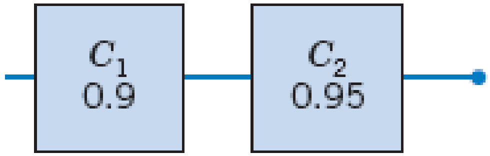
3-11 More Than One Random Variable and Independence

3-11.2 Independence

EXAMPLE 3-36

Series System

The system shown here operates only if there is a path of functional components from left to right. The probability that each component functions is shown in the diagram. Assume that the components function or fail independently. What is the probability that the system operates?



Solution. Let C_1 and C_2 denote the events that components 1 and 2 are functional, respectively. For the system to operate, both components must be functional. The probaility that the system operates is

$$P(C_1, C_2) = P(C_1)P(C_2) = (0.9)(0.95) = 0.855$$

Note that the probability that the system operates is smaller than the probability that any component operates. This system fails whenever *any* component fails. A system of this type is called a **series system.**

3-12 Functions of Random Variables

$$Y = X + c$$

$$E(Y) = E(X) + c = \mu + c \tag{3-23}$$

$$V(Y) = V(X) + 0 = \sigma^2 \tag{3-24}$$

$$Y = cX$$

$$E(Y) = E(cX) = cE(X) = c\mu \tag{3-25}$$

$$V(Y) = V(cX) = c^2V(X) = c^2\sigma^2 \tag{3-26}$$

3-12 Functions of Random Variables

3-12.1 Linear Combinations of Independent Random Variables

The mean and variance of the linear function of **independent** random variables are

$$Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

$$E(Y) = c_0 + c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n \tag{3-27}$$

and

$$V(Y) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_n^2 \sigma_n^2 \tag{3-28}$$

3-12 Functions of Random Variables

3-12.1 Linear Combinations of Independent Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent, normally distributed random variables with means $E(X_i) = \mu_i$, $i = 1, 2, \ldots, n$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$. Then the linear function

$$Y = c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

is normally distributed with mean

$$E(Y) = c_0 + c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n$$

and variance

$$V(Y) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_n^2 \sigma_n^2$$

3-12 Functions of Random Variables

3-12.1 Linear Combinations of Independent Random Variables

EXAMPLE 3-40

Perimeter of a Molded Part: Normal Distribution

Once again, consider the manufactured part described previously. Now suppose that the length X_1 and the width X_2 are normally and independently distributed with $\mu_1 = 2$ centimeters, $\sigma_1 = 0.1$ centimeters, $\mu_5 = 5$ centimeters, and $\sigma_2 = 0.2$ centimeters. In the previous example we determined that the mean and variance of the perimeter of the part $Y = 2X_1 + 2X_2$ were $E(Y) = 14$ centimeters and $V(Y) = 0.2$ square centimeters, respectively. Determine the probability that the perimeter of the part exceeds 14.5 centimeters.

Solution. From the above result, Y is also a normally distributed random variable, so we may calculate the desired probability as follows:

$$P(Y > 14.5) = P\left(\frac{Y - \mu_Y}{\sigma_Y} > \frac{14.5 - 14}{0.447}\right) = P(Z > 1.12) = 0.13$$

Therefore, the probability is 0.13 that the perimeter of the part exceeds 14.5 centimeters. ■

3-12 Functions of Random Variables

3-12.2 What If the Random Variables Are Not Independent?

The correlation between two random variables X_1 and X_2 is

$$\rho_{X_1X_2} = \frac{E(X_1X_2) - \mu_1\mu_2}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\sigma_1^2\sigma_2^2}} \tag{3-29}$$

with $-1 \leq \rho_{X_1X_2} \leq +1$, and $\rho_{X_1X_2}$ is usually called the **correlation coefficient.**

3-12.2 What If the Random Variables Are Not Independent?

Let $X_1, X_2, \ldots, X_n$ be random variables with means $E(X_i) = \mu_i$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$, and covariances $\mathrm{Cov}(X_1, X_2)$, $i, j = 1, 2, \ldots, n$ with $i < j$. Then the mean of the linear combination

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

is

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \tag{3-30}$$

and the variance is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 + 2\sum_{i<j}\sum c_ic_j\,\mathrm{Cov}(X_i, X_j) \tag{3-31}$$

3-13 Random Samples, Statistics, and The Central Limit Theorem

Independent random variables $X_1, X_2, \ldots, X_n$ with the same distribution are called a **random sample.**

A **statistic** is a function of the random variables in a random sample.

The probability distribution of a statistic is called its **sampling distribution.**

3-13 Random Samples, Statistics, and The Central Limit Theorem

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{3-39}$$

as $n \rightarrow \infty$, is the standard normal distribution.

3-13 Random Samples, Statistics, and The Central Limit Theorem

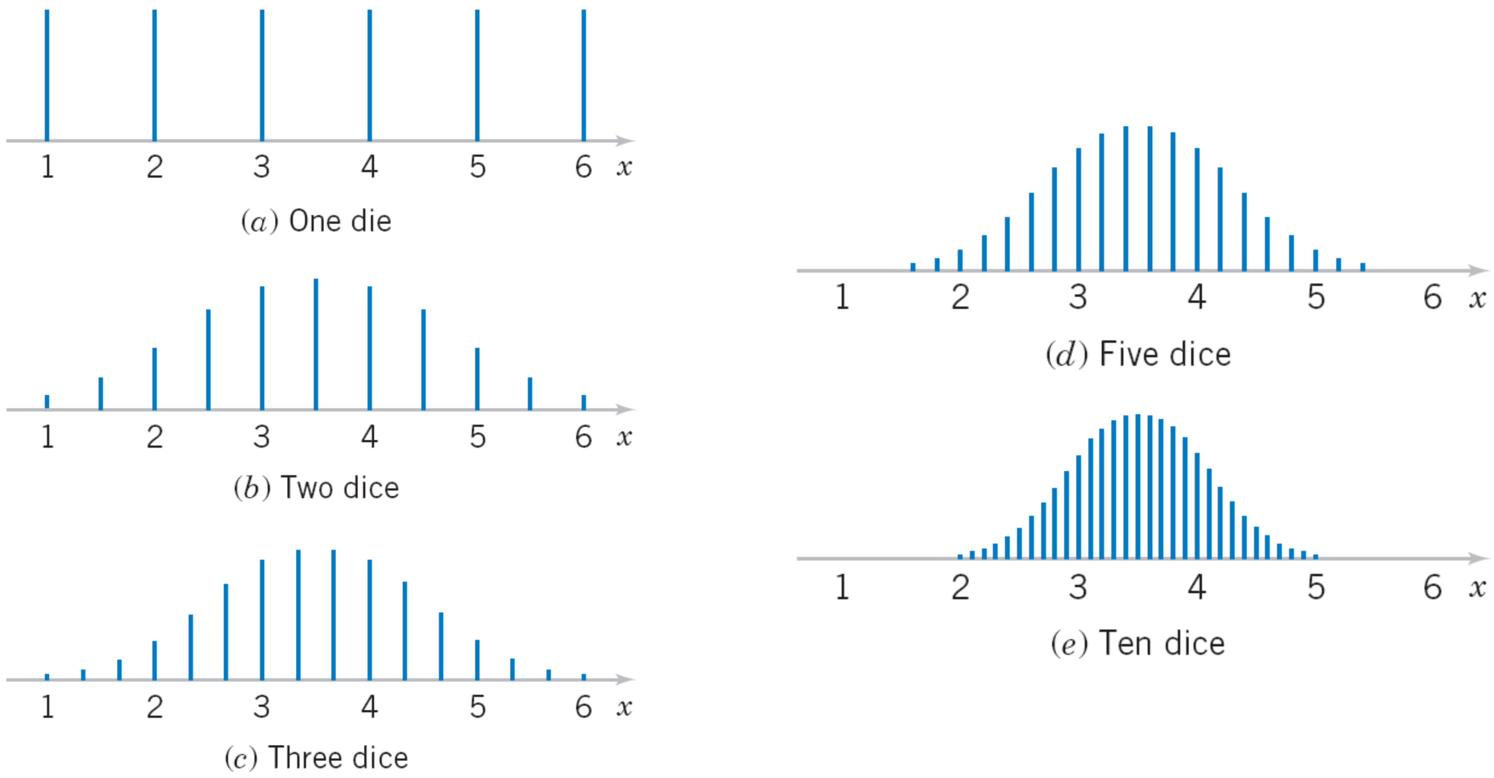


Figure 3-42 Distributions of average scores from throwing dice. [Adapted with permission from Box, Hunter, and Hunter (1978).]

3-13 Random Samples, Statistics, and The Central Limit Theorem

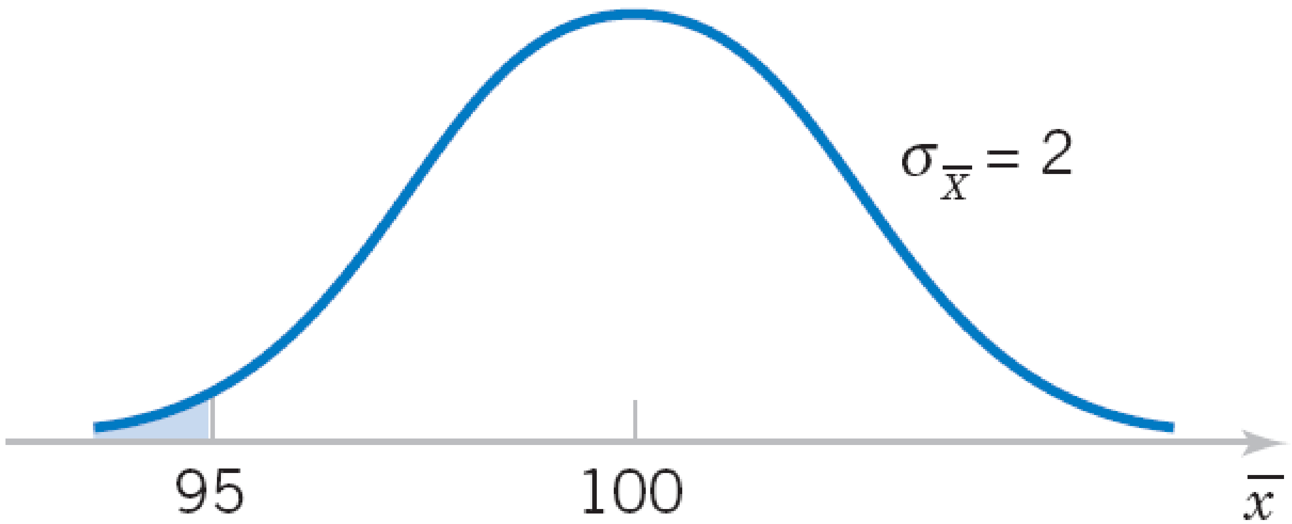


Figure 3-43 Probability density function of average resistance.

3-13 Random Samples, Statistics, and The Central Limit Theorem

EXAMPLE 3-45

Average Resistance

An electronics company manufactures resistors that have a mean resistance of 100 Ω and a standard deviation of 10 Ω. Find the probability that a random sample of $n = 25$ resistors will have an average resistance less than 95 Ω.

Note that the sampling distribution of $\overline{X}$ is approximately normal, with mean $\mu_{\overline{X}} = 100\ \Omega$ and a standard deviation of

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$$

Therefore, the desired probability corresponds to the shaded area in Fig. 3-43. Standardizing the point $\overline{X} = 95$ in Fig. 3-43, we find that

$$z = \frac{95 - 100}{2} = -2.5$$

and therefore,

$$P(\overline{X} < 95) = P(Z < -2.5) = 0.0062$$ ■

IMPORTANT TERMS AND CONCEPTS

- Binomial distribution
- Central limit theorem
- Continuity correction
- Continuous random variable
- Cumulative distribution function
- Delta method
- Discrete random variable
- Events
- Exponential distribution
- Gamma distribution
- Independence
- Joint probability distribution
- Lognormal distribution
- Mean of a random variable
- Normal approximations to binomial and Poisson distributions
- Normal distribution
- Normal probability plot
- Poisson distribution
- Poisson process
- Probability
- Probability density function
- Probability distribution
- Probability mass function
- Probability plots
- Propagation of error
- Random experiment
- Random sample
- Random variable
- Sampling distribution
- Standard deviation of a random variable
- Standard normal distribution
- Statistic
- Variance of a random variable
- Weibull distribution