Lecture 5: Let's Continue...

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We were discussing the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

And in last lecture we proved the continuity of f at origin. Now let us discuss the continuity at non-zero points. Let $(x_0, y_0) \in \mathbb{R}^2$ be a non-zero point. Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ which converges to (x_0, y_0), that is, $x_n \to x_0$ and $y_n \to y_0$. Since both x_0 and y_0 can not be zero simultaneously, hence without loss of generality we may assume that $x_0 \neq 0$. Convergence of (x_n) to x_0 implies that there exists n_0 such that $x_n \neq 0$ for all $n \geq n_0$, i.e., $x_n^2 + y_n^2 \neq 0$ for all $n \geq n_0$.

Appealing to the limit theorem of sequences, $x_n^2 y_n \to x_0^2 y_0$ and $x_n^2 + y_n^2 \to x_0^2 + y_0^2 \neq 0$,

$$f(x_n, y_n) = \frac{x_n^2 y_n}{x_n^2 + y_n^2} \to \frac{x_0^2 y_0}{x_0^2 + y_0^2}$$

Hence f is continuous at (x_0, y_0). Since (x_0, y_0) was arbitrary, therefore f is continuous at every non-zero point.

Combining both the cases we conclude that f is continuous everywhere.

Remark 5.1 *A polynomial in two variables x and y (with real coefficients) is a finite sum of terms of the form cx^iy^j, where i, j are nonnegative integers and $c \in \mathbb{R}$; here c is called the coefficient of the term and $i + j$ is called its total degree, provided it is a nonzero term, that is, $c \neq 0$. For instance, $p(x,y) = x^5y + 2x^4 + y^2 + 1$ and $q(x,y) = x^3 + x^2y + 6xy^2$ are polynomials of total degree 6 and 3 respectively. Thus, if $p(x,y)$ is a polynomial in the variables x and y, then for any $(x_0, y_0) \in \mathbb{R}^2$, by substituting x_0 for x and y_0 for y in $p(x,y)$, we obtain a real number, denoted by $p(x_0, y_0)$. So each polynomial defines a function from $\mathbb{R}^2$ to $\mathbb{R}$. As argued in example above, one can show that a polynomial in two variable (with real coefficients) is continuous everywhere.*

By the same line of argument, one can show that a rational function $r(x,y) = \dfrac{p(x,y)}{q(x,y)}$ (where p and q are polynomials) is continuous all points where $q(x,y) \neq 0$.

Example 5.2 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Discuss the continuity of f at $(0,0)$.

Initial Attempts: As in the last example, we start with a sequence $((x_n, y_n))$ in $\mathbb{R}^2$ which converges to $(0,0)$. Then we try to estimate $|f(x_n, y_n) - f(0,0)|$ as follows:

$$|f(x_n, y_n) - f(0,0)| = \left| \frac{x_n y_n}{x_n^2 + y_n^2} - 0 \right| = \frac{|x_n||y_n|}{x_n^2 + y_n^2}$$

You might be tempted to use the A.M.-G.M. inequality, but the bad news is that it just gives an upper bound, i.e.,

$$x_n^2 + y_n^2 \geq 2\sqrt{x_n^2 y_n^2} = 2|x_n y_n| \implies \frac{|x_n||y_n|}{x_n^2 + y_n^2} \leq \frac{1}{2}$$

It is not enough to conclude convergence or divergence $f(x_n, y_n)$. Some other attempts might be

$$\frac{|x_n||y_n|}{x_n^2 + y_n^2} \leq \frac{|x_n||y_n|}{x_n^2} = \frac{|y_n|}{|x_n|}$$

This is also not useful to make any conclusion.

Now one should consider a possibility that f might not be continuous at $(0,0)$. so it is a right time to ask the following question.

Question: When you say $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is not continuous at $(x_0, y_0) \in D$?
Answer: A function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be discontinuous at (x_0, y_0) if there exists a sequence $((x_n, y_n))$ in D such that $(x_n, y_n) \to (x_0, y_0)$, but $f(x_n, y_n) \nrightarrow f(x_0, y_0)$.

Solution: Consider the sequence $\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right)$. Clearly it converge to $(0,0)$. But

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \nrightarrow 0 = f(0,0)$$

Therefore, f is not continuous at origin. ■

A million dollar question, which must be puzzling you guys is that how one would guess the sequence $\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right)$ to conclude the discontinuity of the function f in previous example?

Here is the idea. For f to be continuous at (x_0, y_0), the limit of the sequence $f(x_n, y_n)$ must be the same for every sequence $(x_n, y_n) \to (x_0, y_0)$. This means in particular, we can consider sequences along any path (for example if point of continuity is origin then $y = mx, y = mx^2$ etc.) and if f is continuous at $(0, 0)$ then limit must be same for every sequence along every path. Hence For functions of two or more variables if we want to show that function is not continuous then there are two ways

1. Find a path, along which $f(x_n, y_n)$ diverges.
2. Find two different paths, along which $f(x_n, y_n)$ converges to two different limits.

The simplest path are $y = mx^n$ for $n = 1, 2, \cdots$.

Now using this idea you can get other sequences also like sequence $\left(\left(\frac{1}{n}, \frac{(-1)^n}{n}\right)\right)$. Clearly converge to $(0, 0)$. But

$$f\left(\frac{1}{n}, \frac{(-1)^n}{n}\right) = \frac{(-1)^n}{2}$$

does not converge. This also serves the purpose.

Recall the $\epsilon - \delta$ definition of continuity for function of one variable

Definition 5.3 *Let $D \subseteq \mathbb{R}$. Consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that f is continuous at c if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

We want to have a similar definition for functions of two-variables. Let us see what are the changes we need. Let $D \subset \mathbb{R}^2, f : D \to \mathbb{R}, c = (x_0, y_0) \in D$. We say that f is continuous at c if for every $\epsilon > 0$, there is $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

where $|x - c| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

So moral of the story is, in one dimension we consider open symmetric neighborhood of the point c, in n dimension it is replaced with the following

$$B_r(c) := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x - c| < r\}.$$

Definition 5.4 *Let D be a subset of $\mathbb{R}^2$ and let (x_0, y_0) be any point in D. A function $f : D \to \mathbb{R}$ is said to be continuous at (x_0, y_0) if for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(x, y) - f(x_0, y_0)| < \epsilon$ for all $(x, y) \in B_\delta(x_0, y_0) \cap D$.*

Theorem 5.5 *Both definitions are equivalent.*