Lecture 2: Continuity of Functions of several variables

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We were discussing the continuity of

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

And in last lecture we proved the continuity of f at origin. Now let us discuss the continuity at non-zero points. Let $(x_0, y_0) \in \mathbb{R}^2$ be a non-zero point. Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ which converges to (x_0, y_0), that is, $x_n \to x_0$ and $y_n \to y_0$. Since both x_0 and y_0 can not be zero simultaneously, hence without loss of generality we may assume that $x_0 \neq 0$. Convergence of (x_n) to x_0 implies that there exists n_0 such that $x_n \neq 0$ for all $n \geq n_0$, i.e., $x_n^2 + y_n^2 \neq 0$ for all $n \geq n_0$.

Appealing to the limit theorem of sequences, $x_n^2 y_n \to x_0^2 y_0$ and $x_n^2 + y_n^2 \to x_0^2 + y_0^2 \neq 0$,

$$f(x_n, y_n) = \frac{x_n^2 y_n}{x_n^2 + y_n^2} \to \frac{x_0^2 y_0}{x_0^2 + y_0^2}$$

Hence f is continuous at (x_0, y_0). Since (x_0, y_0) was arbitrary a arbitrary non-zero point, therefore f is continuous at every non-zero point.

Combining both the cases we conclude that f is continuous everywhere.

Example 2.1 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

Discuss the continuity of f at $(0,0)$.

Initial Attempts: As in the last example, we start with a sequence $((x_n, y_n))$ in $\mathbb{R}^2$ which converges to $(0,0)$. Then we try to estimate $|f(x_n, y_n) - f(0,0)|$ as follows:

$$|f(x_n, y_n) - f(0,0)| = \left| \frac{x_n y_n}{x_n^2 + y_n^2} - 0 \right| = \frac{|x_n||y_n|}{x_n^2 + y_n^2}$$

You might be tempted to use the A.M.-G.M. inequality, but the bad news is that it just gives an upper bound, i.e.,

$$x_n^2 + y_n^2 \geq 2\sqrt{x_n^2 y_n^2} = 2|x_n y_n| \implies \frac{|x_n||y_n|}{x_n^2 + y_n^2} \leq \frac{1}{2}$$

It is not enough to conclude convergence or divergence $f(x_n, y_n)$. Some other attempts might be

$$\frac{|x_n||y_n|}{x_n^2 + y_n^2} \leq \frac{|x_n||y_n|}{x_n^2} = \frac{|y_n|}{|x_n|}$$

This is also not useful to make any conclusion.

Now one should consider a possibility that f might not be continuous at $(0,0)$. so it is a right time to ask the following question.

Question: When you say $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is not continuous at $(x_0, y_0) \in D$?
Answer: A function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be discontinuous at (x_0, y_0) if there exists a sequence $((x_n, y_n))$ in D such that $(x_n, y_n) \to (x_0, y_0)$, but $f(x_n, y_n) \nrightarrow f(x_0, y_0)$.

Solution: Consider the sequence $\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right)$. Clearly it converge to $(0,0)$. But

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \nrightarrow 0 = f(0,0)$$

Therefore, f is not continuous at origin. ■

A million dollar question, which must be puzzling you guys is that how one would guess the sequence $\left(\left(\frac{1}{n}, \frac{1}{n}\right)\right)$ to conclude the discontinuity of the function f in previous example? Here is the idea. For f to be continuous at (x_0, y_0), the limit of the sequence $f(x_n, y_n)$ must be the same for every sequence $(x_n, y_n) \to (x_0, y_0)$. This means in particular, we can consider sequences along any path (for example if point of interest is origin then $y = mx^n, x = my^n$ etc.) and if f is continuous at $(0,0)$ then limit of $f(x_n, y_n)$ must be same for every sequence $((x_n, y_n))$ along every path. Hence for functions of two or more variables if we want to show that function is not continuous at (x_0, y_0) then there are two ways

1. Find a sequence $((x_n, y_n))$ in domain of f such that $(x_n, y_n) \to (x_0, y_0)$ but $f(x_n, y_n)$ diverges.

2. Find a sequence $((x_n, y_n))$ in domain of f such that $(x_n, y_n) \to (x_0, y_0)$ but $f(x_n, y_n) \nrightarrow f(x_0, y_0)$.

If point of interest is $(0,0)$, then simplest paths along which can consider sequences are $y = mx^n$ or $x = my^n$, here n could any positive real number.

Now using this idea you can get other sequences in solution of Example 2.1, like sequence $\left(\left(\frac{1}{n}, \frac{(-1)^n}{n}\right)\right)$ which converge to $(0,0)$. But

$$f\left(\frac{1}{n}, \frac{(-1)^n}{n}\right) = \frac{(-1)^n}{2}$$

does not converge. This also serves the purpose.

Example 2.2 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is not continuous at $(0,0)$.

Rough Work: We try to identify paths along which limit of the sequence is path dependent. First we try with paths $y = mx^n$.

$$\lim_{x \to 0} f(x, mx^n) = \lim_{x \to 0} \frac{xm^2x^{2n}}{x^2 + m^4x^{4n}} = \lim_{x \to 0} \frac{m^2x^{2n+1}}{x^2 + m^4x^{4n}} = \lim_{x \to 0} \frac{m^2x^{2n-1}}{1 + m^4x^{4n-2}}$$

If $2n - 1 = 0$, i.e., $n = \frac{1}{2}$ then

$$\lim_{x \to 0} \frac{m^2x^{2n-1}}{1 + m^4x^{4n-2}} = \frac{m^2}{1 + m^4}.$$

So limit is going to be path dependent. So we have identify the path as $y = m\sqrt{x}$.

Now we write the solution which will bring full marks in the second quiz and end-term!!

Solution:

Take the sequence $\left(\left(\frac{1}{n}, \frac{1}{\sqrt{n}}\right)\right)$. This sequence converges to $(0,0)$. Now

$$f\left(\frac{1}{n}, \frac{1}{\sqrt{n}}\right) = \frac{\frac{1}{n} \times \frac{1}{n}}{\frac{1}{n^2} + 1\frac{1}{n^2}} = \frac{1}{2}$$

This is a constant sequence which converges to $\frac{1}{2}$, hence f is not continuous at $(0,0)$. ■