Lecture 4: Functions of several variables

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So far we have studied real-valued functions of one real variable, that is, functions $f : D \to \mathbb{R}$, where D is a subset of the set $\mathbb{R}$ of all real numbers. We have seen basic properties of real numbers and functions of one real variable.

Now basic object of our study will be the n-dimensional (Euclidean) space $\mathbb{R}^n$ consisting of n-tuples of real numbers, namely,

$$\mathbb{R}^n := \{(x_1, \cdots, x_n) : x_1, \cdots, x_n \in \mathbb{R}\}$$

and real-valued functions on subsets of $\mathbb{R}^n$. Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and its height, so it is a function $V(r, h)$ of two variables r and h. We extend the basic ideas of single variable calculus to functions of several variables. Their derivatives are more varied and interesting because of the different ways the variables can interact. The applications of these derivatives are also more varied than for single-variable calculus, and we will see that the same is true for integrals involving several variables.

Whenever we write $\mathbb{R}^n$, it will be tacitly assumed that $n \in \mathbb{N}$, that is, n is a positive integer. Elements of $\mathbb{R}^n$ are sometimes referred to as vectors in n-space when $n > 1$. In contrast, the elements of $\mathbb{R}$ are referred to as scalars. Given a vector $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ and $1 \le i \le n$, the scalar x_i is called the ith coordinate of x.

The algebraic operations on $\mathbb{R}$ can be easily extended to $\mathbb{R}^n$ in a component-wise manner. Thus, we define the sum of $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ to be $x + y := (x_1 + y_1, \cdots, x_n + y_n)$. It is easily seen that addition defined in this way satisfies properties analogous to those in $\mathbb{R}$. In particular, the zero vector $0 := (0, ..., 0)$ plays a role similar to the number 0 in $\mathbb{R}$. We might wish to define the product of $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ to be $(x_1 y_1, \cdots, x_n y_n)$. However, this kind of component-wise multiplication is not well behaved. For example, the component-wise product of the nonzero vectors $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2$ is the zero vector $(0, 0)$, and consequently, the reciprocals of these nonzero vectors cannot be defined.

The theory of functions of several variables differs significantly from that of functions of one-variable. However, once $n > 1$, there is not a great deal of difference between the smaller values of n and the larger values of n. This is particularly true with the basic aspects of the theory of functions of several variables that we are going to discuss here. With this in view and for the sake of simplicity, we shall almost exclusively restrict ourselves to the case $n = 2$. In this case, the space $\mathbb{R}^n$ can be effectively visualized as the plane.

Also, graphs of real-valued functions of two variables may be viewed as surfaces in 3-space.

As indicated earlier, we do not have a good notion of multiplication of points in $\mathbb{R}^n$ when $n > 2$. But we have useful notions of scalar multiplication and dot product that are defined as follows. Given any $c \in \mathbb{R}$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define

$$cx := (cx_1, \cdots, cx_n).$$

This is referred to as the scalar multiplication of the vector x by the scalar c.

Given any $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$, the dot product (also known as the inner product or the scalar product) of x and y is the real number denoted by $x \cdot y$ and defined by

$$x \cdot y := x_1 y_1 + \cdots + x_n y_n.$$

We also have an analogue of the notion of the absolute value of a real number, which is defined as follows. Given any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the norm (also known as the magnitude or the length) of x is the nonnegative real number denoted by $|x|$ and defined by

$$|x| := \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Geometrically speaking, the norm $|x|$ represents the distance between x and the origin $0 := (0, \cdots, 0)$. More generally, for any $x, y \in \mathbb{R}^n$, the norm of their difference, that is, $|x - y|$, represents the distance between x and y.

A vector u in $\mathbb{R}^n$ for which $|u| = 1$ is called a unit vector in $\mathbb{R}^n$. For example, in $\mathbb{R}^2$ the vectors $\mathbf{i} := (1, 0)$ and $\mathbf{j} := (0, 1)$ are unit vector.

Sequences in $\mathbb{R}^2$

You may recall that sequences makes life much easier if we want to prove continuity of a real-valued function of a single variable. What was the key? The key is, work hard on proving some beautiful theorems on real sequences and use them to conclude the continuity and limit of a real-valued function of one variable. We adopt the same path in order to define continuity of a real-valued functions of two variable. Sequences in $\mathbb{R}^2$ enjoys many properties similar to real-sequences but here our sole purpose of considering sequences is to define the notion of continuity and limit of a real-valued functions of two variable, hence we defer to discuss the sequences in $\mathbb{R}^2$ in detail as we did for real sequences.

A sequence in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. Typically, a sequence in $\mathbb{R}^2$ is denoted by $((x_n, y_n))$.

Definition 4.1 *We say $((x_n, y_n))$ converges to (x_0, y_0) if $x_n \to x_0$ and $y_n \to y_0$.*

Example 4.2 *1. Sequence* $\left(\left(\frac{1}{n}, -\frac{1}{n}\right)\right)$*. converge to* $(0,0)$*.*

2. Sequence $\left(\left(\frac{1}{n}, (-1)^n\right)\right)$ *diverges. Since* $((-1)^n)$ *is divergent.*

Continuity

Definition 4.3 *Let* D *be a subset of* $\mathbb{R}^2$ *and let* (x_0, y_0) *be any point in* D*. A function* $f : D \to \mathbb{R}$ *is said to be continuous at* (x_0, y_0) *if for every sequence* (x_n, y_n) *in* D *such that* $(x_n, y_n) \to (x_0, y_0)$*, we have* $f(x_n, y_n) \to f(x_0, y_0)$*. When* f *is continuous at every* $(x_0, y_0) \in D$*, we say that* f *is continuous on* D*.*

Example 4.4 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Discuss the continuity of f *at* $(0,0)$*.*

Solution: Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ which converges to $(0,0)$, that is, $x_n \to 0$ and $y_n \to 0$. Now

$$f(x_n, y_n) = \begin{cases} \dfrac{x_n^2 y_n}{x_n^2 + y_n^2} & \text{if } (x_n, y_n) \neq (0,0) \\ 0 & \text{if } (x_n, y_n) = (0,0). \end{cases}$$

Therefore

$$|f(x_n, y_n)| = \frac{x_n^2 |y_n|}{x_n^2 + y_n^2} \leq |y_n| \text{ if } (x_n, y_n) \neq (0,0)$$

For any $(x_n, y_n) \in \mathbb{R}^2$, we obtain

$$|f(x_n, y_n)| \leq |y_n|$$

$$\begin{aligned} y_n \to 0 &\implies |y_n| \to 0 \\ 0 \leq |f(x_n, y_n)| \leq |y_n| &\implies |f(x_n, y_n)| \to 0 \text{ (by Sandwhich Theorem)} \end{aligned}$$

Hence $f(x_n, y_n) \to 0 = f(0,0)$. As a result, f is continuous at $(0,0)$. ■