The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-II
Assignment 10

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1. Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:

$$(i)\ f(x) = x^3 + x + 1 \quad (ii)\ f(x) = \begin{cases} 0 & \text{if } \ -1 \le x < 0, \\ x & \text{if } \ 0 \le x \le 1 \end{cases} \quad \text{(first three non-zero terms)}$$

2. Locate and classify the singular points in the following:

$$(i)\ x^3(x-1)y'' - 2(x-1)y' + 3xy = 0 \quad (ii)\ (3x+1)xy'' - xy' + 2y = 0.$$

3. Find the eigenvalues and eigen functions of the following Strum-Liouville problems:
$(i)\ y'' + \lambda y = 0, \quad y(0) = y'(1) + y(1) = 0$
$(ii)\ (xy')' + \frac{\lambda}{x} y = 0, \quad y(1) = y'(e) = 0.$

4. If $p(x)$, $q(x)$, $r(x)$ are all greater than zero on (a, b), then prove that the eigenvalues of the Strum-Liouville problem, $(p(x)y')' - q(x)y + \lambda r(x)y = 0$, are positive with any of the boundary conditions: $(i)\ p(a) = 0,\ p(b) = 0,$ $(ii)\ p(a) = p(b)$ with $y(b) = y(a),\ y'(b) = y'(a)$ $(iii)\ y(a) - ky'(a) = 0,$ $y(b) + my'(b) = 0,\ k, m > 0.$

5. Consider the Strum-Liouville problem

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$

with $p(x) > 0$ on $[a, b]$ and $y(a) \ne y(b)$, $y'(a) \ne y'(b)$. Show that every eigen function is unique except for a constant factor.

6. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at)$ $(a > 0)$.

7. Find the Laplace transforms:

$(a)\ [t]$, (greatest integer function) $(b)\ t^m \cos bt\ (m \in$ non-negative integers), $(c)\ e^t \sin at,$

$$(d)\ \frac{e^t \sin at}{t}, \qquad (e)\ \frac{\sin t \cos t}{t}, \qquad (f)\ f(t) = \begin{cases} \sin 3t, & \text{if } \ 0 < t < \pi, \\ 0, & \text{if } \ t > \pi. \end{cases}$$

8. Find the Laplace transforms (Hint: second shifting theorem):

$$(a)\ f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t < 2\pi, \\ \cos t, & t > 2\pi. \end{cases} \qquad (b)\ f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos \pi t, & 1 < t < 2, \\ 0, & t > 2. \end{cases}$$

9. Find the inverse Laplace transforms of

$$(a)\ \tan^{-1}(a/s), \quad (b)\ \ln \frac{s^2+1}{(s+1)^2}, \quad (c)\ \frac{s+2}{(s2+4s-5)^2}, \quad (d)\ \frac{se^{-\pi s}}{s^2+4}, \quad (e)\ \frac{(1-e^{-2s})(1-3e^{-2s})}{s^2}.$$

10. Use Laplace transform to solve the initial value problems:

$(a)\ y'' + 4y = \cos 2t,\ y(0) = 0,\ y'(0) = 1.$

$$(b)\ y'' + 3y' + 2y = \begin{cases} 4t, & \text{if } \ 0 < t < 1, \\ 8, & \text{if } \ t > 1, \end{cases} \qquad y(0) = y'(0) = 0.$$

$$(c)\ y'' + 9y = \begin{cases} 8 \sin t, & \text{if } \ 0 < t < \pi, \\ 0, & \text{if } \ t > \pi, \end{cases} \qquad y(0) = 0,\ y'(0) = 4.$$