Lecture 19: Line Integral

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In previous lecture you have seen integral of a real-valued function defined on an interval, bounded subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$. Now we extend the theory of integration to curves in space. Now we learn how to integrate vector-valued functions over curves. These integrals are called line integral. Line integrals are used to find the work done by a force in moving an object along a path. In the end we learn how to integrate real-valued function over curves, these are used to find the mass of a curved wire with variable density.

Parametric Curves in Space

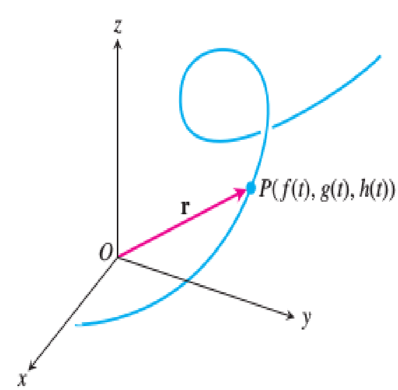
When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), y = g(t), z = h(t)$$

The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$, make up the curve in space that we call the particle's path. A curve in space can also be represented in vector form. The vector

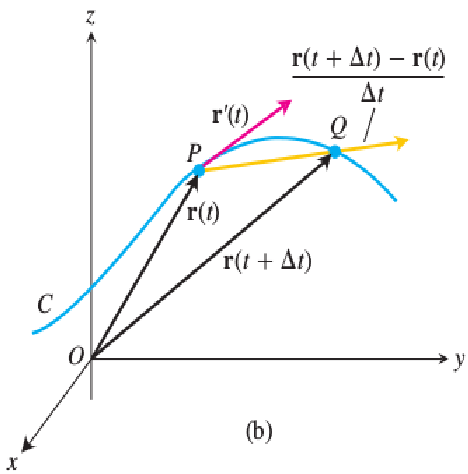
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t) + \mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's position vector.



Note that the curve $\mathbf{r}$ is determined by its parametrization, that is, by the three functions $f, g, h : I \to \mathbb{R}$, and not by the subset $\{f(t), g(t), h(t) : t \in I\}$ of $\mathbb{R}^3$ traced by $\mathbf{r}$. For example, the curve $\mathbf{r_1}$ in $\mathbb{R}^2$ given by $(\cos t, \sin t), t \in [0, 2\pi]$, and the curve $\mathbf{r_2}$ in $\mathbb{R}^2$ given by $(\cos 2t, \sin 2t), t \in [0, 2\pi]$, have the same parameter domain and they trace the same subset of $\mathbb{R}^2$ the unit circle, but they are obviously different curves, since $\mathbf{r_1}$ goes around the unit circle once, while $\mathbf{r_2}$ goes around the unit circle twice. Similarly the curve $\mathbf{r_3}$ in $\mathbb{R}^2$ given by $(-\cos t, -\sin t), t \in [0, 2\pi]$ is different from both $\mathbf{r_1}, \mathbf{r_2}$ since the direction of traveling is reversed.

The vector $\mathbf{r}'(t) = (f'(t), g'(t), h'(t))$, is the vector tangent to the curve.



(b)

Reparametrization of a curve Let $\mathbf{r}(\mathbf{t})$ be a given curve in $\mathbb{R}^n$, $t \in [a, b]$. Let $s = \alpha(t)$ be a new variable, where α is a strictly increasing differentiable function on $[a, b]$. Then for each $s \in [\alpha(a), \alpha(b)]$ there is a unique $t \in [a, b]$ with $\alpha(t) = s$. Define the function $\mathbf{c} : [\alpha(a), \alpha(b)] \to \mathbb{R}^n$ by $\mathbf{c}(s) = \mathbf{r}(t)$. The path $\mathbf{c}$ is called a reparametrization of $\mathbf{r}$.

Example 19.1

1. *Consider a curve in the plane, line segment joining* $(0, 0)$ *and* $(1, 0)$. *Then* $(t, 0)$ *where* $t \in [0, 1]$. *Clearly* $\alpha(t) = t^n$ *for any* $n \in \mathbb{N}$ *is a strictly increasing* ($\because \alpha'(t) > 0$ *over* $(0, 1]$) *differentiable function on* $[0, 1]$. *Hence* $(t^n, 0)$ *is a parameterization for all* n.

2. *Consider unit circle* $(\cos t, \sin t)$ *where* $t \in [0, 2\pi]$. *Then* $(\cos 2t, \sin 2t)$ *for* $t \in [0, \pi]$ *is an another parameterization of the same curve.*

Definition 19.2 *Let* $\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$ *be a vector field in the plane or space and* C *be continuously differentiable path (with parameterization* $\mathbf{r}(t) =$

$x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$) defined on the interval $[a,b]$. The line integral of $\mathbf{F}$ over C is defined by

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &:= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt \\ &= \int_a^b [f_1(x(t),y(t),z(t))\mathbf{i} + f_2(x(t),y(t),z(t))\mathbf{j} + f_3(x(t),y(t),z(t))\mathbf{k}] \\ &\qquad \cdot [x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}]dt\end{aligned}$$

Example 19.3 *Find the line integral of the vector field $F(x,y,z) = y\hat{i} - x\hat{j} + \hat{k}$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \le t \le 2\pi$ joining $(1,0,0)$ to $(1,0,1)$.*

Solution:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} [\sin t\mathbf{i} - \cos t\mathbf{j} + \mathbf{k}]\cdot[-\sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{2\pi}\mathbf{k}]dt \\ &= \int_0^{2\pi}\left(-\sin^2 t - \cos^2 t + \frac{1}{2\pi}\right)dt \\ &= (\frac{1}{2\pi} - 1)\int_0^{2\pi} dt = 1 - 2\pi\end{aligned}$$

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Suppose $\mathbf{F}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$ be a vector field and C be continuously differentiable path (with parameterization $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$) Then line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is also written as

$$\int_C f_1 dx + f_2 dy + f_3 dz = \int_a^b [f_1(x(t),y(t),z(t))\frac{dx}{dt} + f_2(x(t),y(t),z(t))\frac{dy}{dt} + f_3(x(t),y(t),z(t))\frac{dz}{dt}]dt.$$

Example 19.4 *Evaluate the line integral $\int_C \frac{xdy - ydx}{x^2+y^2}$ where C is a circle of radius a with center at origin.*

Solution: Curve C has a parameterization $r(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$. Then line integral

$$\begin{aligned}\int_C \frac{xdy - ydx}{x^2+y^2} &= \int_C \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx \\ &= \int_0^{2\pi}\left[\frac{a\cos t}{a^2}\frac{dy}{dt} - \frac{a\sin t}{a^2}\frac{dx}{dt}\right]dt \\ &= \int_0^{2\pi}\left[\cos^2 + \sin^2 t\right]dt = 2\pi\end{aligned}$$

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