Lecture 15,16: Double Integral

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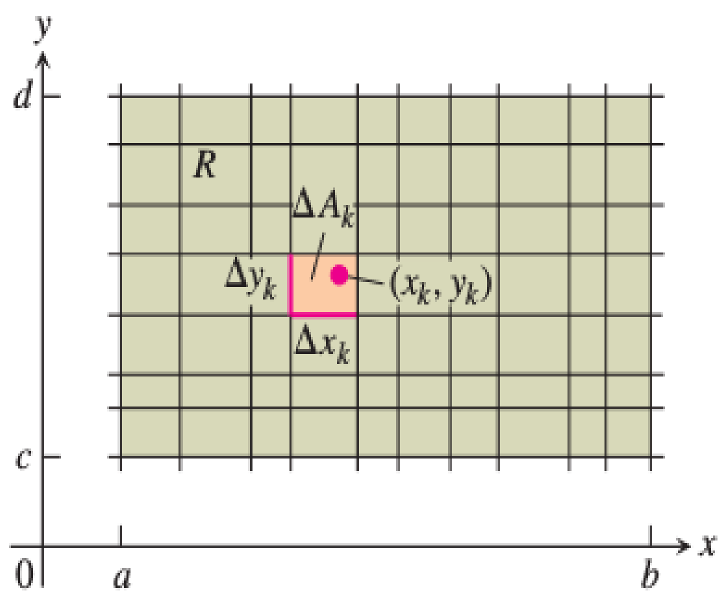
We consider a function $f(x, y)$ defined on a rectangular region $R = [a, b] \times [c, d]$. We subdivide R into small rectangles using lines parallel to the x- and y-axes. The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a partition of R. A sub-rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the sub-pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$, where ΔA_k is the area of the kth small rectangle. To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth sub-rectangle, multiply the value of f at that point by the area A_k, and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$



Depending on how we pick (x_k, y_k) in the kth sub-rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The norm of a partition P, written $\|P\|$, is the largest width or height of any rectangle in the partition. If $\|P\| = 0.1$ then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of P goes to zero, written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be integrable and the limit is called the double integral of f over R, written as

$$\iint_R f(x,y)dA, \quad \text{or} \quad \iint_R f(x,y)dxdy$$

It can be shown that if $f(x,y)$ is a continuous function throughout R, then f is integrable. Many discontinuous functions are also integrable, including functions that are discontinuous only on a finite number of points or smooth curves.

Calculating Double Integrals

Calculating double integral from definition is very tedious. Thanks to the following theorem which says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration.

Theorem 15.1 (Fubini's Theorem (First Form)) *If $f(x,y)$ is continuous throughout the rectangular region $R : a \le x \le b, c \le y \le d$, then*

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Example 15.2 *Calculate* $\displaystyle\iint_{[0,2]\times[-1,1]} (100 - 6x^2y)dA$

Solution: By Fubini's Theorem,

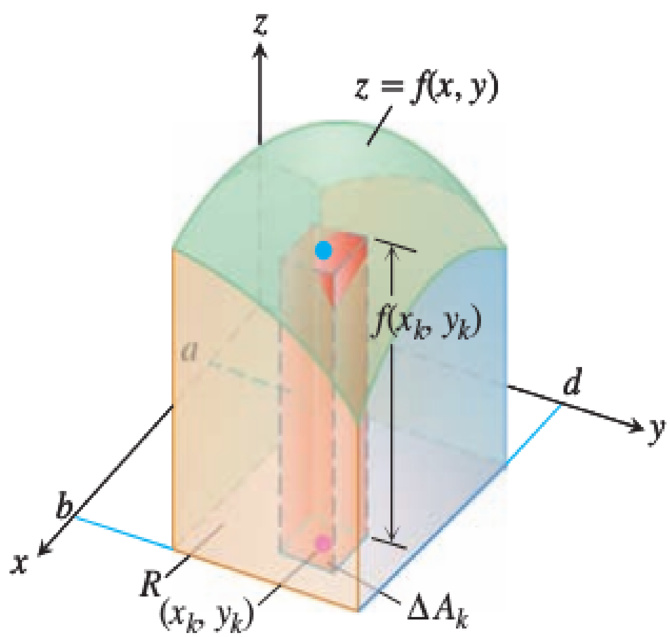
$$\begin{aligned}\iint_{[0,2]\times[-1,1]} (100 - 6x^2y)dA &= \int_{-1}^1 \int_0^2 (100 - 6x^2y)dxdy = \int_{-1}^1 [100x - 2x^3y]_{x=0}^{x=2} \;\; dy \\ &= \int_{-1}^1 [200 - 16y]dy = [200y - 8y^2]_{-1}^1 = 400\end{aligned}$$

Reversing the order of integration gives the same answer:

$$\iint_{[0,2]\times[-1,1]} (100 - 6x^2y)dA = \int_0^2 \int_{-1}^1 (100 - 6x^2y)dydx = \int_0^2 [100y - 3x^2y^2]_{y=-1}^{y=1} \; dx$$
$$= \int_{-1}^1 [100 - 3x^2 - (-100 - 3x^2)]dx$$
$$= \int_{-1}^1 200 \; dx = 400$$

Double Integrals as Volumes

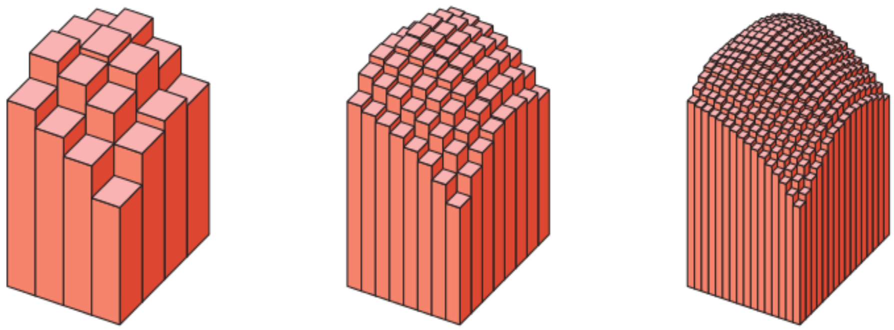
When $f(x,y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x,y)$.



Each term $f(x_k,y_k)\Delta A_k$ in the sum $S_n = \sum_{k=1}^{n} f(x_k,y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base A_k. The sum S_n thus approximates what we want to call the total volume of the solid. We define this volume to be

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)dA$$

where $\Delta A_k \to 0$ as $n \to \infty$.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

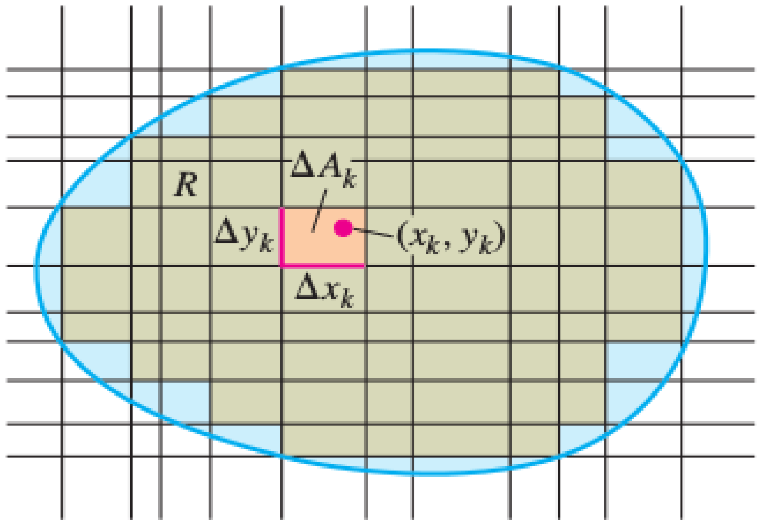
Example 15.3 *Find the volume of the region bounded above by the ellipitical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R : 0 \leq x \leq 1, 0 \leq y \leq 2$.*

Solution: The volume is given by the double integral

$$\begin{aligned}
V &= \iint_R (10 + x^2 + 3y^2) dA \\
&= \int_0^1 \int_0^2 (10 + x^2 + 3y^2) dy dx = \int_0^1 [10y + x^2 y + y^3]_{y=0}^{y=2} \; dx \\
&= \int_{-1}^1 [20 + 2x^2 + 8] dx \\
&= 28x + \frac{2}{3} x^3 \Big|_{x=0}^{x=1} \\
&= \frac{86}{3}
\end{aligned}$$

15.1 Double Integrals over General Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.



How to evaluate them ?

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

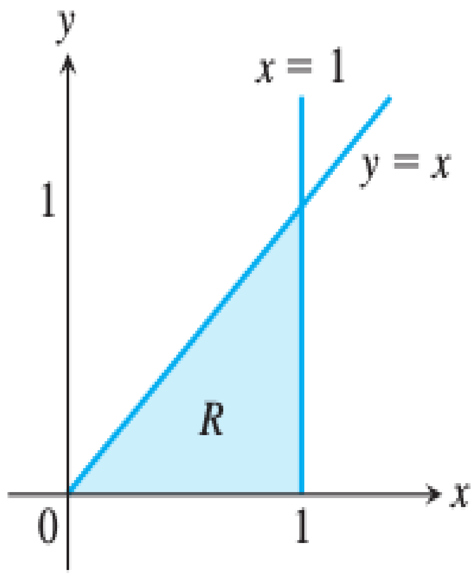
1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.



Solution: The Region R can be identified as: $0 \leq y \leq 1$, $y \leq x \leq 1$. Draw a line parallel to x-axis. Then by Fubini's theorem

$$\iint_A \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

Now we run into a problem because $\dfrac{\sin x}{x}$ has no closed from of antiderivative. So let us change the order of integration. The same region R can be identified as: $0 \leq x \leq 1$, $0 \leq y \leq x$. Draw a line parallel to y-axis. Then by Fubini's theorem

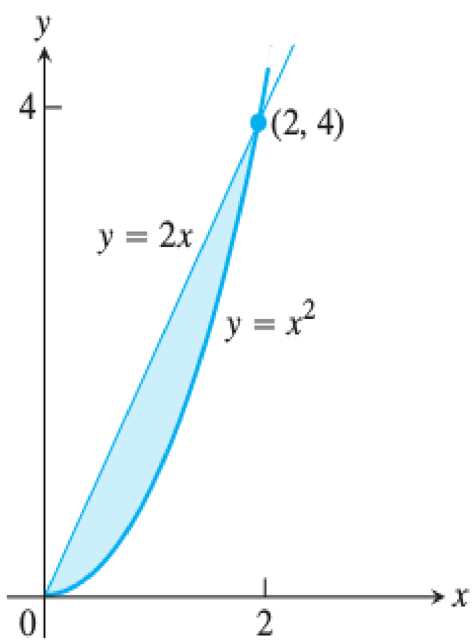
$$\begin{aligned}\iint_A \frac{\sin x}{x} dA &= \int_0^1 \int_0^x \frac{\sin x}{x} dy dx \\ &= \int_0^1 \frac{\sin x}{x} [y]_{y=0}^{y=x} dx \\ &= \int_0^1 \sin x dx \\ &= -\cos x|_{x=0}^{x=1} = 1 - \cos 1\end{aligned}$$

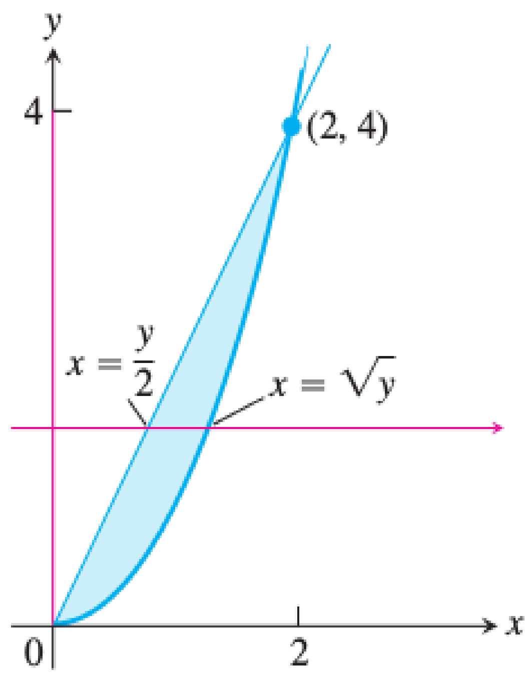
Example 15.4 *Sketch the region of integration for the integral*

$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx.$$

and write an equivalent integral with the order of integration reversed.

Solution: It is clear that the region R is : $0 \leq x \leq 2, x^2 \leq y \leq 2x$.





To find limits for integrating in the reverse order, draw a line parallel to the x-axis.

$$\int_0^2 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2)dxdy.$$

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

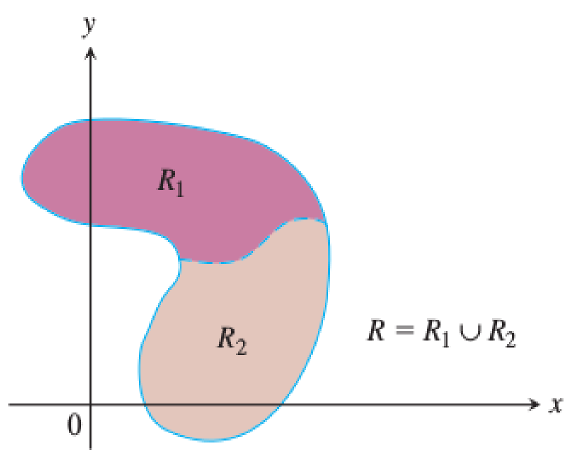
3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

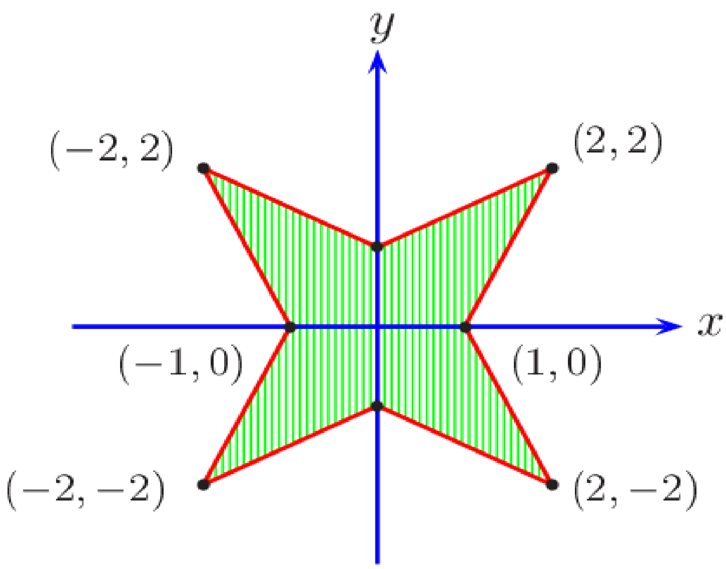
4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

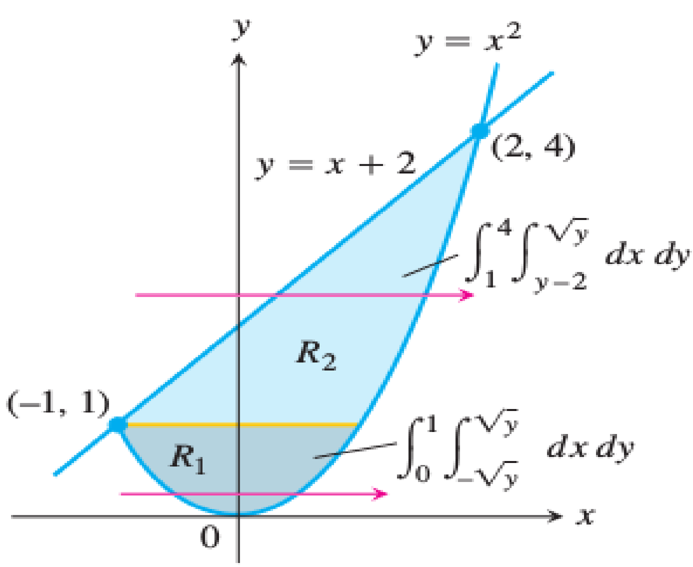
if R is the union of two nonoverlapping regions R_1 and R_2



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

This additivity property is very useful for evaluating double integral over the following regions.





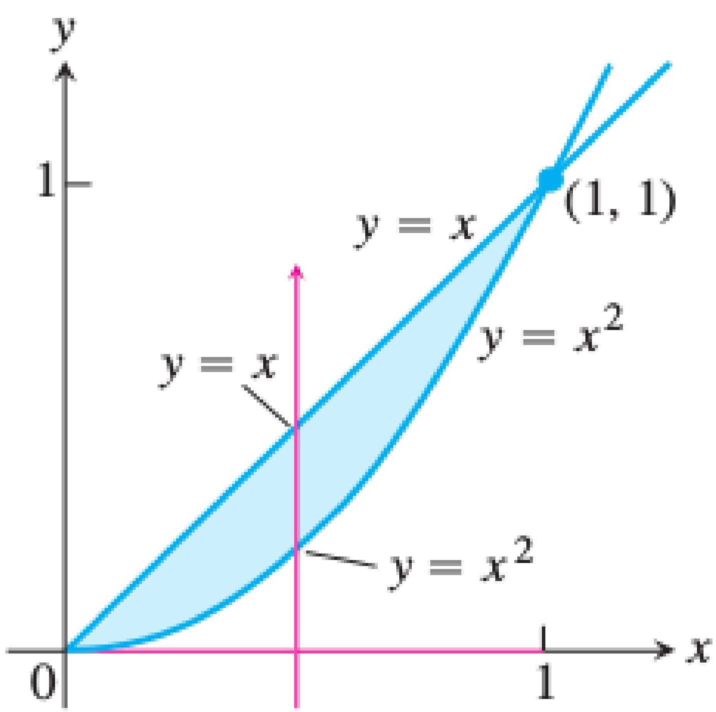
Areas of Bounded Regions in the Plane as a double integral

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, the Riemann sums reduce to $S_n = \sum_{k=1}^{n} \Delta A_k$ This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R. The area of a closed, bounded plane region R is the double integral

$$\iint_R dA$$

Example 15.5 *Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.*

Solutions:



Hence

$$Area = \int_0^1 \int_{x^2}^{x} dydx = \frac{1}{6}$$