Lecture 14: Lagrange Multiplier Method: II

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14.1 Case of Three Variables

The Lagrange Multiplier Method for finding absolute extrema of a real-valued function of three variables $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ is similar to two variable situation.

Example 14.1 *Find the maximum and the minimum of the function f given by $f(x, y, z) := x^2y^2z^2$ subject to the constraint that $x^2 + y^2 + z^2 = 1$.*

Solution: Following the Lagrange Multiplier Method,

Step I Note that zero set of g is a closed and bounded subset of $\mathbb{R}^2$. Also f being a polynomial function is continuous everywhere, therefore f attains its absolute maximum and absolute minimum on the unit sphere.

Step II We let $g(x, y, z) := x^2+y^2+z^2-1$ and consider the equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = 0$, that is,

$$(2xy^2z^2, 2yx^2z^2, 2zx^2y^2) = \lambda(2x, 2y, 2z), \quad x^2 + y^2 + z^2 - 1 = 0.$$

Therefore we get the system

$$\begin{aligned} xy^2z^2 &= \lambda x && (14.1)\\ yx^2z^2 &= \lambda y && (14.2)\\ zx^2y^2 &= \lambda z && (14.3)\\ x^2 + y^2 + z^2 &= 1 && (14.4) \end{aligned}$$

If $x \neq 0$ then from (14.1) we get $\lambda = y^2z^2$. Using this in equations (14.2) and (14.3), we get

$$yx^2z^2 = y^2z^2y, \quad zx^2y^2 = y^2z^2z$$

Assuming $y \neq 0$ and $z \neq 0$, we get $x^2 = y^2 = z^2$. Using this in (14.4) we get $x^2 = \dfrac{1}{3}$ which implies $x = \pm\dfrac{1}{\sqrt{3}}$. Therefore points $\left(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{1}{\sqrt{3}}\right)$ are solutions. If

$x = 0$ then from (14.2), $\lambda y = 0$ and from (14.3), $\lambda z = 0$ and from (14.4) $y^2 + z^2 = 1$. Therefore both y and z cannot be zero simultaneously. Assuming $y \neq 0$, we get $\lambda = 0$ and $z = \pm\sqrt{1-y^2}$. So other solutions are $\left(0, y, \pm\sqrt{1-y^2}\right)$ where $y \neq 0$. If $y = 0$ then $z = \pm 1$. So other solutions are $(0, 0, \pm 1)$. By symmetry we may conclude that other solutions are $\left(t, 0, \pm\sqrt{1-t^2}\right)$ and $\left(\pm\sqrt{1-t^2}, t, 0\right)$ where $t \in [-1, 1]$.

Step III (a) Note that $\nabla g = (2x, 2y, 2z)$ is zero only at $(0, 0, 0)$, in particular ∇g is non-zero at all the solutions of simultaneous equations. Also

$$f\left(\frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}\right) = \frac{1}{27}, f\left(0, t, \sqrt{1-t^2}\right) = f\left(t, 0, \sqrt{1-t^2}\right) = f\left(t, \sqrt{1-t^2}, 0\right) = 0 \; \forall\, t \in [-1, 1]$$

(b) Since f is a polynomial function so ∇f is defined everywhere.

(c) Since g is a polynomial function so ∇g is defined everywhere.

(d) There are no points in the zero set of g at which $\nabla g = (0, 0, 0)$.

Step IV Therefore f attains it's maximum value at $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and while the minimum is 0, which is attained at infinitely many points $\left(0, t, \sqrt{1-t^2}\right), \left(t, 0, \sqrt{1-t^2}\right), \left(t, \sqrt{1-t^2}, 0\right)$ $\forall\, t \in [-1, 1]$

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14.2 More Than One constraints

The Lagrange Multiplier Method can also be adapted to a situation in which there is more than one constraint. For example, suppose we want to find the absolute extremum of a function f of three variables x, y, z subject to the constraints given by $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Step I Ensured that f does have an absolute extremum on the intersection of the zero sets of g and h (which will certainly be the case if this intersection is closed and bounded, and f is continuous).

Step II We seek simultaneous solutions $\lambda, \mu \in \mathbb{R}$, and $(x, y, z) \in \mathbb{R}^3$ of

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \text{ and } g(x, y, z) = 0 = h(x, y, z). \tag{14.5}$$

Step III Make a table of f values at the following points:

(a) At simultaneous solutions $P_0 = (x_0, y_0, z_0)$ of the system (14.5) for which $\nabla g(P_0), \nabla h(P_0)$, are nonzero and are not multiples of each other.

(b) At points in the intersection of zero sets of g and h at which ∇f does not exist.

(c) At points in the intersection of zero sets of g and h which ∇g does not exist.

(d) At points in the intersection of zero sets of g and h which ∇h does not exist.

(e) At points in the intersection of zero sets of g and h at which ∇g exists but $\nabla g = (0, 0, 0)$.

(f) At points in the intersection of zero sets of g and h at which ∇g exists but $\nabla h = (0, 0, 0)$.

(g) At points in the intersection of zero sets of g and h at which ∇g and ∇h are multiple of each other.

Step IV Largest value in the table corresponds to the absolute maximum and minimum value in the table corresponds to the absolute minimum of f subject to $g = 0$ and $h = 0$.

Example 14.2 *Minimize the function $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.*

Solution:

Step I Note that f is continuous, and but the set $\{(x, y, z) \in \mathbb{R}^3 | x + 3y + 9z - 9 = 0, x + 2y + 3z - 6 = 0\}$ is not bounded (it is a straight line in the space) But geometrically, we know that there will be some point on this straight line nearest to the origin, hence f attains its absolute minimum on the intersection of zero sets of g and h.

Step II Let $f(x, y, z) = x^2 + y^2 + z^2, g(x, y, z) = x + 2y + 3z - 6$ and $h(x, y, z) = x + 3y + 9z - 9$. Consider the equation

$$\begin{aligned}
\nabla f &= \lambda \nabla f + \mu \nabla g \\
\implies (2x, 2y, 2z) &= \lambda(1, 2, 3) + \mu(1, 3, 9) \\
\implies x &= \frac{\lambda + \mu}{2}, y = \frac{2\lambda + 3\mu}{2}, z = \frac{3\lambda + 9\mu}{2}
\end{aligned}$$

Substituting these in the equations $g(x, y, z) = 0$ and $h(x, y, z) = 0$, we obtain

$$\begin{aligned}
\frac{\lambda + \mu}{2} + 2\frac{2\lambda + 3\mu}{2} + 3\frac{3\lambda + 9\mu}{2} = 6 \implies 14\lambda + 34\mu = 12 &\implies 7\lambda + 17\mu = 6 \\
\frac{\lambda + \mu}{2} + 3\frac{2\lambda + 3\mu}{2} + 9\frac{3\lambda + 9\mu}{2} = 9 &\implies 34\lambda + 91\mu = 18
\end{aligned}$$

This gives $\lambda = \dfrac{240}{59}, \mu = -\dfrac{78}{59}$. Therefore,

$$x = \frac{81}{59}, y = \frac{123}{59}, z = \frac{9}{59}$$

is the unique simultaneous solution of $\nabla f = \lambda \nabla g + \mu \nabla h$ and $g = h = 0$.

Step III Now f, g and h are polynomials so gradient exists everywhere. Also ∇g and ∇h are constant non-zero vectors and are not multiple of each other.

Step IV Therefore, absolute minimum will be attained at the point $P_0 = \left(\dfrac{81}{59}, \dfrac{123}{59}, \dfrac{9}{59}\right)$ and the minimum value is

$$f(P_0) = \frac{21771}{3481} = \frac{369}{59}$$

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