The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-III
Practice Problems Set #5

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1. Find all $z \in \mathbb{C}$ such that the following series converges absolutely.
$$(a) \sum \frac{z^n}{n^2} \quad (b) \sum \frac{z^n}{n!} \quad (c) \sum \frac{z^n}{2^n} \quad (d) \sum \frac{1}{2^n}\left(\frac{1}{z}\right)^n.$$

2. Let $a_n = \dfrac{(-1)^n}{\sqrt{n}} + i\dfrac{1}{n^2}, n \in \mathbb{N}$. Show that $\sum a_n$ converges but does not converge absolutely.

3. Determine the radius of convergence of the power series $\sum a_n z^n$ where a_n is
$$(a)\,(\ln n)^2 \quad (b)\, n! \quad (c)\, \frac{n^2}{4^n + 3n} \quad (d)\, \frac{(n!)^3}{(3n)!} \quad (e)\, \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}}.$$

4. Show that $\displaystyle\sum_{n=0}^{\infty} (n+1)^2 z^n = \frac{1+z}{(1-z)^3}$, where $|z| < 1$.

5. Find the Taylor series expansion of
$$(a)\, f(z) = \frac{1}{z^2} \text{ at } z = a \neq 0, \quad (b)\, f(z) = \frac{6z+8}{(2z+3)(4z+5)} \text{ at } z = 1, \quad (c)\, f(z) = \frac{e^z}{z+1} \text{ at } z = 1.$$

6. Find the Laurent series expansions for the following functions around $z = 0$:
$(a)\, f(z) = (z-3)^{-1}$ for $|z| > 3$, $(b)\, f(z) = (z(z-1))^{-1}$ for $0 < |z| < 1$, $(c)\, f(z) = z^3 e^{\frac{1}{z}}$ for $|z| > 0$.

7. Write all possible Laurent series in powers of z that represent the function $f(z) = (z(1+z^2))^{-1}$ in certain domains and specify those domains.

8. Let f and g be entire functions which satisfy $|f(z)| < |g(z)|\,\forall z \in \mathbb{C}$. Show that there exist a constant $\lambda \in \mathbb{C}$ such that $f(z) = \lambda g(z)\,\forall z \in \mathbb{C}$. Determine all the entire functions f such that $|f'(z)| < |f(z)|$.

9. For each function given below determine its isolated singular point and weather that point is a pole, a removable singular point, or an essential singular point:
$$(a)\, z\exp\frac{1}{z} \quad (b)\, \frac{\sin z}{z}, \quad (c)\, \frac{1-\cos z}{z^2}, \quad (d)\, \frac{\pi\cot\pi z}{z^2}, \quad (e)\, \frac{z - \sin(z-1)}{z-1}, \quad (f)\, \frac{z^2 + \sin z}{\cos z - 1}.$$

10. Which of the following singularities are removable/pole?
$$(a)\, \frac{\sin z}{z^2 - \pi^2} \text{ at } z = \pi, \quad (b)\, \frac{\sin z}{(z-\pi)^2} \text{ at } z = \pi, \quad (c)\, \frac{z\cos z}{1 - \sin z} \text{ at } z = \pi/2.$$

11. Find the residue at $z = 0$ of the following functions and indicate the type of singularity they have at $z = 0$:
$$(a)\, \frac{1}{z+z^2}, \quad (b)\, z\cos\frac{1}{z}, \quad (c)\, \frac{z - \sin z}{z}, \quad (d)\, \frac{\cot z}{z^4}.$$

12. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z| = \pi$.
$$(a)\, \frac{e^{-z}}{z^2}, \quad (b)\, \frac{e^{-z}}{(z-1)^2}, \quad (c)\, z^2 e^{\frac{1}{z}}, \quad (d)\, \frac{z+1}{z^2 - 2z}, \quad (e)\, \frac{e^z}{z^2-1}, \quad (f)\, \frac{\pi\cot\pi z}{(z+\frac{1}{2})^2}.$$

13. Show that
$$(a)\, \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4}dx = \frac{\pi}{4}, \quad (b)\, \int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

14. Show that
$$(a)\, \int_0^\infty \frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a},\, (a \geq 0) \quad (b)\, \int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right),\, (a > b > 0).$$

15. Show that
$$(a)\, \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}},\, (a > |b|), \quad (b)\, \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \pi.$$