(Quiz-1: Solution)

1. (a) Determine and sketch the sets in the complex plane $Re(\frac{4}{z}) < 1$ [3 Marks]

Solution: $Re(\frac{4}{z}) < 1$ or, $Re(\frac{4z}{z\bar{Z}}) < 1$ or, $Re(\frac{4(x+iy)}{x^2+y^2 z}) < 1.$ or, $\frac{4x)}{x^2+y^2} < 1.$ or, $x^2+y^2-4x > 0.$ Or, $(x-2)^2+y^2 > 4 = 2^2$. Therefore the set of ponts are outside the disk with centre at (2,0) and radius is 2.

(b) Find all possible solutions of $z^5+1-i=0$. [3 Marks]

Solution: $z^5+1-i=0$ Or, $z^5 = -1+i = \sqrt{2}(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}) = \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}e^{\frac{3\pi i}{4}+2k\pi i}$, $k \in Z$. Or, $z = 2^{\frac{1}{10}}e^{\frac{(3+8k)\pi i}{20}}$; $k = 0, \pm1, \pm2$ or, $k = 0,1,2,3,4$. Then $z = 2^{\frac{1}{10}}e^{\frac{3\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{11\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{-5\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{19\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{-13\pi i}{20}}$ or $z = 2^{\frac{1}{10}}e^{\frac{3\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{11\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{19\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{27\pi i}{20}}, 2^{\frac{1}{10}}e^{\frac{35\pi i}{20}}$

2. Find out the region of analyticity of the function $f(z) = Log(z+4-i\sqrt{2})$, where $Logz$ denotes the the principal value of the logarithm. Justify your claim. [6 Marks]

Solution We have to find the all possible points where $f(z) = Log(z+4-i\sqrt{2}) = Log(x+4)+i(y-\sqrt{2}) = ln|(x+4)+i(y-\sqrt{2})| + iArg(x+4+i(y-\sqrt{2})$ is analytic.

The given function is analytic except the points $y=\sqrt{2}$ and $x \le -4$.

Thus, the region of analyticity is $C - \{z = x+iy : y = \sqrt{2}\ and\ x \le -4\}$.

Justification: as argument $Arg(x+4+i(y-\sqrt{2})$ is not continuous for the points $y=\sqrt{2}$ and $x \le -4$.

For any point $x \le -4$ on line $y=\sqrt{2}$, we get limit of $Arg(x+4+i(y-\sqrt{2})) = \pi$ and $-\pi$. This shows that given function is not continuous on $y=\sqrt{2}$ and $x \le -4$. Thus not analytic for all the points $y=\sqrt{2}$ and $x \le -4$.

Now investigate in $C - \{z = x+iy : y = \sqrt{2}\ and\ x \le -4\}$

$Log(z+4-i\sqrt{2}) = Log(x+4)+i(y-\sqrt{2}) = ln|(x+4)+i(y-\sqrt{2})| + iArg(x+4+i(y-\sqrt{2}) = \frac{1}{2}ln[(x+4)^2+(y-\sqrt{2})^2] + itan^{-1}\frac{y-\sqrt{2}}{x+4}$ gives

$u = \frac{1}{2}ln[(x+4)^2+(y-\sqrt{2})^2]$ and $v = tan^{-1}\frac{y-\sqrt{2}}{x+4}$

$u_x = \frac{x+4}{(x+4)^2+(y-\sqrt{2})^2} = v_y$ and $u_y = \frac{y-\sqrt{2}}{(x+4)^2+(y-\sqrt{2})^2} = -v_x$

Partial derivatives exist and C-R equations are satisfies, and partial derivatives are continuous except the point $y=\sqrt{2}, x \le -4$. Thus, $f(z)$ is analytic in $C - \{z = x+iy : y = \sqrt{2}\ and\ x \le -4\}$.

3. For what value of the integer $n > 1$, $u(x,y) = x^n - y^n$ is harmonic? Then, for the value of $n > 1$ for which $u(x,y)$ is harmonic, find the conjugate harmonic. Construct $f(z) = u(x,y) + iv(x,y)$. Finally, find the function $f(z)$ in terms of z. [8 Marks]

Solution: For harmonic: $u_{xx} + u_{yy} = 0$
For given $u(x, y) = x^n - y^n$, we get $n(n-1)(x^{n-2} - y^{n-2}) = 0 - - - - - - (A)$
It is clear that equation (A) will satisfies for $n = 0, 1, 2$

the value of $n > 1$ for which $u(x, y)$ is harmonic is $n = 2$. So, we will try to get the conjugate harmonic of $u(x, y) = x^2 - y^2$.

$u_x = 2x$ and $u_y = -2y$
Using C-R equations $u_x = v_y$ and $u_y = -v_x$, we get

$$
\begin{aligned}
v_y &= 2x &(1)\\
v_x &= 2y &(2)
\end{aligned}
$$

Now integrating $v_y = 2x$, we get $v = 2xy + \phi(x)$.

Differentiating partially $v = 2xy + \phi(x)$ with respect to x, we get

$$v_x = 2y + \phi'(x) \tag{3}$$

From (2) and (3), we get $\phi'(x) = 0$. This gives $\phi(x) = c$

Finally, we get

$$
\begin{aligned}
v &= 2xy + c &(4)\\
f(z) &= x^2 - y^2 + i2xy + ic &(5)\\
f(z) &= z^2 + ic \qquad (in\ terms\ of\ z) &(6)
\end{aligned}
$$