Lecture 10: Absolute Extrema of functions of two variables

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Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

10.1 Open and Closed Subsets of $\mathbb{R}$

Definition 10.1 *Let $D \subseteq \mathbb{R}$. A point $c \in D$ is called an interior point of D if there exists $r > 0$ such that $(c - r, c + r)$ (called symmetric neighborhood of c) is contained in D.*

Definition 10.2 *Let $D \subseteq \mathbb{R}$. We say D is open if all the points of D are interior points.*

Definition 10.3 *Let $D \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a boundary point of D if every symmetric neighborhood of c contains at least one point that lie outside of D and at least one point that lie in D.*

For example $D = (0, 1) \cup (2, 3)$ has four boundary points $0, 1, 2, 3$.

Remark 10.4 *A boundary point of D itself need not belong to D.*

Definition 10.5 *A subset D of $\mathbb{R}$ is said to be closed if it contains all its boundary points.*

For example $D = (0, 1) \cup (2, 3)$ is open, $D = [0, 1] \cup [2, 3]$ is closed and $D = [0, 1] \cup (2, 3)$ is neither open nor closed.

Exercise 10.6 *Determine whether the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are closed or open subsets of $\mathbb{R}$?*

10.2 Absolute extrema for real-valued functions of one variable

Definition 10.7 *A subset D of $\mathbb{R}$ is said to be bounded if there exists $K > 0$ such that $|x| \leq K$ for all $x \in D$.*

Recall, the following theorem you might have seen while studying single variable calculus.

Theorem 10.8 *Let $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$. If f is continuous on $[a, b]$ then f is bounded. Moreover, f attains its absolute maxima and absolute minima on $[a, b]$.*

In fact, the following generalized theorem holds.

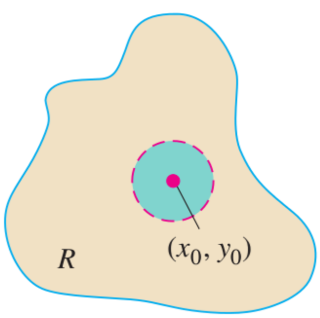
Theorem 10.9 *Let D be a closed and bounded subset of $\mathbb{R}$. If $f : D \to \mathbb{R}$ is continuous on D then f is bounded. Moreover, f attains its absolute maxima and absolute minima on D.*

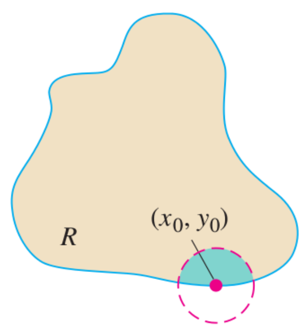
10.3 Open, Closed subsets of $\mathbb{R}^2$

Definition 10.10 *A subset D of $\mathbb{R}^2$ is said to be bounded if there exists $K > 0$ such that $\sqrt{x^2 + y^2} \leq K$ for all $(x, y) \in D$.*

For example, if $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, then D is bounded, whereas its complement $\mathbb{R}^2 \setminus D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$ is not bounded.

Definition 10.11 *Let $R \subseteq \mathbb{R}^2$. We say a point $(x_0, y_0) \in R$ is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.*

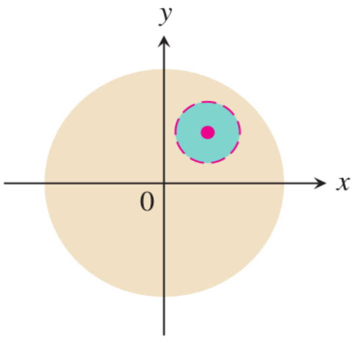




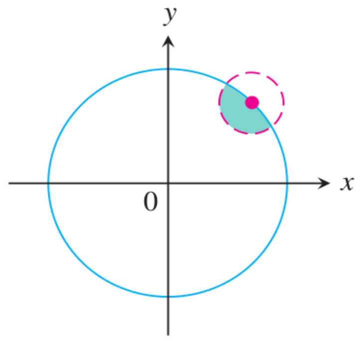
A point $(x_0, y_0) \in \mathbb{R}^2$ is a boundary point of R if every disk centered at (x_0, y_0) contains at least one point that lie outside of R and at least one point that lie in R.

Remark 10.12 *A boundary point of D itself need not belong to D.*

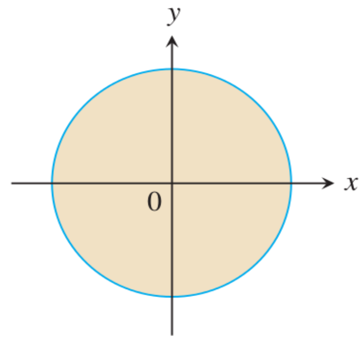
Definition 10.13 *Let $D \subseteq \mathbb{R}^2$. We say D is an open subset of $\mathbb{R}^2$ if every point of D is an interior point. We say D is a closed subset of $\mathbb{R}^2$ if every boundary point of D belongs to D.*



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an
interior point.



$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit
disk. (The unit
circle.)



$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all
boundary points.

As with a half-open interval of real numbers $[a, b)$, some sets in the plane are neither open nor closed. If you start with the open unit disk as in figure above and add to it some of but not all its boundary points, the resulting set is neither open nor closed. The boundary points that are there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

Remark 10.14 *Note that if $(x_0, y_0) \in D$, then (x_0, y_0) is either an interior point of D or a boundary point of D.*

10.4 Absolute extrema for real-valued functions of two variables

Theorem 10.15 *Let $D \subset \mathbb{R}^2$ be closed and bounded, and $f : D \to \mathbb{R}$ is continuous. Then f is bounded on D, and attains its absolute minimum and the absolute maximum on D.*

10.4.1 Local Extrema

Next question arises naturally. Knowing the function f, how does one find the absolute extrema and points where they are attained? As in one-variable calculus, it helps to consider the interior points of D at which the partial derivatives vanish or fail to exist, and also the boundary points of D.

Definition 10.16 *Given $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$, a point $(x_0, y_0) \in D$ is called a critical point of f if*

1. *(x_0, y_0) is an interior point of D.*

2. *either $\nabla f(x_0, y_0)$ does not exist, or if $\nabla f(x_0, y_0)$ exists, then $\nabla f(x_0, y_0) = (0, 0)$.*

Definition 10.17 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. A point $(x_0, y_0) \in D$ is said to be a point of local maximum of f if*

1. *(x_0, y_0) is an interior point of D*

2. *there exists $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and $f(x, y) \leq f(x_0, y_0)$ for all $(x, y) \in B_\delta(x_0, y_0)$.*

Definition 10.18 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. A point $(x_0, y_0) \in D$ is said to be a point of local minimum of f if*

1. *(x_0, y_0) is an interior point of D*

2. *there exists $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and $f(x, y) \geq f(x_0, y_0)$ for all $(x, y) \in B_\delta(x_0, y_0)$.*

Theorem 10.19 (Necessary condition for local extrema) *Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) is an interior point of D. Suppose $f : D \to \mathbb{R}$ has a local extremum at (x_0, y_0). If $\nabla f(x_0, y_0)$ exists, then $\nabla f(x_0, y_0) = (0, 0)$.*

Example 10.20 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := -(x^2 + y^2)$. Find the local extreme values (if any) of f.*

Solution: Being a polynomial f is differentiable everywhere and $\nabla f(x, y) = (-2x, -2y)$ for $(x, y) \in \mathbb{R}^2$. Thus the only point where f can possibly have a local extremum is $(0, 0)$. Since $f(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$, we see that f does have a local maximum at $(0, 0)$ and this value is equal to 0. ■

Example 10.21 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy$. Find the points of local extrema (if any) of f.*

Solution: Note that f is differentiable everywhere and $\nabla f(x, y) = (y, x)$ for $(x, y) \in \mathbb{R}^2$. Thus the only point where f can possibly have a local extremum is $(0, 0)$. But $f(0, 0) = 0$ and for any $\delta > 0$, there are $(x_1, y_1), (x_2, y_2) \in B_\delta(0, 0)$ such that $f(x_1, y_1) < 0$ and $f(x_2, y_2) > 0$. For example, one can choose any $t \in (0, \delta)$ and let $(x_1, y_1) := (t, -t)$ and $(x_2, y_2) := (t, t)$. It follows that f has neither a local maximum nor a local minimum at $(0, 0)$. ■