Discrete Mathematics Assignment-3

1. Let a_n denote the number of n-bit words containing no two consecutive 1'**s.** Define a_n recursively.
2. The nth **Lucas number** *Ln*, named after the French mathematician Franois-Edouard-Anatole Lucas, is defined recursively as follows"

$$L_1 = 1, L_2 = 3$$
$$L_n = L_{n-1} + L_{n-2}, n \geq 3$$

(The Lucas sequence and the Fibonacci sequence satisfy the same recurrence relation, but have different initial conditions.) Compute the first six Lucas numbers.

The gcd of two integers x (> 0) and y (_> 0) can be defined recursively as follows:

$$\text{gcd}\{x, y\} = \begin{cases} \text{gcd}\{y, x\} & if y > x \\ \text{x} & \text{if y} \leq \text{x and y} = 0 \\ \text{gcd}\{y, xmody\} & if\ y \leq x\ and\ y > 0 \end{cases}$$

3. Using this definition, compute the gcd of pair of integers gcd{28,18}.
4. Define recursively each sequence of numbers. *(Hint:* Look for a pattern and define the *n*th term a_n recursively.)
   a) 1,4,7,10,13..
   b) 1,2,5,26,677..
5. The 91-function f, invented by John McCarthy, is defined recursively on W as follows.

$$\text{f}\{x\} = \begin{cases} x - 10 & if\ x > 100 \\ f\big(f(x + 11)\big) & if\ 0 \leq x \leq 100 \end{cases}$$

   Compute each
   a) f(99)
   b) f(f(99))
6. Let a_n denote the number of times the assignment statement x <- x + 1 is executed by each nested for loop. Define a_n recursively.
   a) $for\ i = 1\ to\ n\ do$
      $for\ j = 1\ to\ i\ do$
      $x \leftarrow x + 1$
   b) $for\ i = 1\ to\ n\ do$
      $for\ j = 1\ to\ i\ do$
      $for\ k = 1\ to\ i\ do$
      $x \leftarrow x + 1$
   c) $for\ i = 1\ to\ n\ do$
      $for\ j = 1\ to\ \lfloor i/2 \rfloor\ i\ do$
      $x \leftarrow x + 1$

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7. **Stirling numbers of the second kind,** denoted by *S(n, r)* and used in combinatorics, are defined recursively as follows, where n, r ∈ N:

$$S\{n,r\} = \begin{cases} 1 & if r = 1\ or\ r = n \\ S(n-1,r-1) + rS(n-1,r) & if\ 1 < r < n \\ 0 & if\ r > n \end{cases}$$

They are named after the English mathematician James Stirling (1692-1770). Compute S(2,2) Stirling number.

8. A function of theoretical importance in the study of algorithms is the **Ackermann's function,** named after the German mathematician and logician Wilhelm Ackermann (1896-1962). It is defined recursively as follows, where *m*,*n* ∈ **W:**

$$\mathrm{A(m,n)} = \begin{cases} n+1 & if\ m = 0 \\ A(m-1,1) & if\ n = 0 \\ A(m-1, A(m,n-1)) & otherwise \end{cases}$$

Compute each.
   a) A(0, 7)
   b) A(4, 0)

9. Using the iterative method, predict a solution to each recurrence relation satisfying the given initial condition.
   a) $a_0 = 0$
      $a_n = a_{n-1} + 4n, \quad n \geq 1$
   b) $S_1 = 1$
      $S_n = S_{n-1} + n^3, \quad n \geq 2$
   c) $a_1 = 1$
      $a_n = 2a_{n-1} + (2^n - 1), \quad n \geq 2$