The LNM Institute of Information Technology
Department of Mathematics
Probability and Statistics (MATH-221)
Mid Term

Maximum Time: 90 minutes   Date: 26/02/2019   Maximum Marks: 25

Instruction: 1. You should attempt all the questions.
2. Calculator is not allowed.
3. There are total eight questions.
4. Please make an index showing the question number and page number on the front page of your answer sheet in the following format, otherwise you may be penalized by the deduction of 2 Marks.

Question No.				
Page No.				

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1. Let X be exponential random variable with parameter $\lambda > 0$. Show that **[3 Marks]**

$$P(X > x + y | X > x) = P(X > y), \forall\, x, y \in [0, \infty).$$

Ans. The random variable X has the pdf

$$f_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \lambda e^{-\lambda x} & \text{if} \quad x \geq 0. \end{cases}$$

Hence if $t \geq 0$, then

$$P(X > t) = \int_t^\infty f_X(x)dx = \int_t^\infty \lambda e^{-\lambda x} dx = -\left[e^{-\lambda x}\right]_t^\infty = e^{-\lambda t}.$$

[1 Marks]

Note that for $x, y \geq 0$, $\{X > x + y\} \subseteq \{X > x\}$. Hence

$$\begin{aligned} P(X > x + y | X > x) &= \frac{P(\{X > x + y\} \cap \{X > x\})}{P(X > x)} && \textbf{[0.5 Marks]} \\ &= \frac{P(X > x + y)}{P(X > x)} && \textbf{[0.5 Marks]} \\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\ &= e^{-\lambda y} = P(X > y) && \textbf{[1 Marks]} \end{aligned}$$

2. Let X have a normal distribution with parameters μ and $\sigma^2 = 0.25$. Find a real number c such that $P(|X-\mu| \le c) = 0.9$. **[3 Marks]**

Ans. Let $Y = \frac{X-\mu}{\sigma}$. Then Y is a standard normal variable. **[1 Marks]**
Now $0.9 = P(|X-\mu| \le c) = P(\frac{|X-\mu|}{\sigma} \le \frac{c}{\sigma})$, since $\sigma = 0.5 > 0$.
$= P(|Y| \le \frac{c}{\sigma}) = P(-\frac{c}{\sigma} \le Y \le \frac{c}{\sigma})$
$= N(\frac{c}{\sigma}) - N(-\frac{c}{\sigma}) = N(\frac{c}{\sigma}) - (1 - N(\frac{c}{\sigma}))$
$= 2N(\frac{c}{\sigma}) - 1$. Or, $N(\frac{c}{\sigma}) = 0.95$. **[1.5 Marks]**
From standard normal table, we get $\frac{c}{\sigma} = 1.65$. Or, $c = 0.5 \times 1.65 = 0.825$. **[0.5 Marks]**

3. If events A, B, and C are independent, then show that $A \cup B^c$ and C are independent. **[3 Marks]**

Ans.

$$
\begin{aligned}
P(A \cap B^c \cap C) &= P(A \cap C) - P(B \cap C \cap A) \\
&= P(A)P(C) - P(A)P(B)P(C) \\
&= P(A)P(C)\,[1 - P(B)] = P(A)P(C)P(B^c) \qquad (1) \\
P(B^c \cap C) &= P(C \setminus (B \cap C)) = P(C) - P(B \cap C) \quad (\because B \cap C \subset C) \\
&= P(C) - P(C)P(B) = P(C)[1 - P(B)] = P(C)P(B^c) \qquad (2)
\end{aligned}
$$

Now

$$
\begin{aligned}
P((A \cup B^c) \cap C) &= P\{(A \cap C) \cup (B^c \cap C)\} \\
&= P(A \cap C) + P(B^c \cap C) - P\{(A \cap C) \cap (B^c \cap C)\} \qquad [\textbf{1Marks}] \\
&= P(A)P(C) + P(B^c)P(C) - P(A \cap B^c \cap C) \\
&= P(A)P(C) + P(B^c)P(C) - P(A)P(B^c)P(C)\text{justification!}[\textbf{1Marks}] \\
&= P(C)\,[P(A) + P(B^c) - P(A)P(B^c)] \\
&= P(C)\,[P(A) + P(B) - P(A \cap B^c)] = P(C)P(A \cup B^c).[\textbf{1Marks}]
\end{aligned}
$$

Alt. Sol.

$$
\begin{aligned}
P((A \cup B^c) \cap C) &= P(A \cup B^c) + P(C) - P(A \cup B^c \cup C) \\
&= P(A \cup B^c) + P(C) - 1 + P(A^c \cap B \cap C^c) \qquad [\mathbf{1Marks}] \\
[\mathbf{1Marks}] &= P(A \cup B^c) + P(C) - 1 + P(A^c \cap B)P(C^c)(\text{as } A,\ B,\ C \text{ are independent} \\
&= P(A \cup B^c) - P(C^c)(1 - P(A^c \cap B)) \\
&= P(A \cup B^c) - P(C^c)P(A \cup B^c) \\
&= P(A \cup B^c)(1 - P(C^c)) \\
&= P(A \cup B^c)P(C). \qquad [\mathbf{1Marks}]
\end{aligned}
$$

4. Consider the function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \le x < 1 \\ x - \frac{1}{2} & \text{if } 1 \le x < \frac{3}{2} \\ 1 & \text{if } x \ge \frac{3}{2} \end{cases}$$

Show that F is a distribution function. Find pdf or pmf (if exists). Also compute $P(1 \le X < 3)$, where X has distribution function F. **[4 Marks]**

Ans. Note that F satisfies three properties

(a) As $F(x) = 0$ for all $x < 0$, hence $\lim_{x \to -\infty} F(x) = 0$. Similarly, $F(x) = 1$ for all $x \ge 1.5$, hence $\lim_{x \to +\infty} F(x) = 1$.

(b) Also F is continuous everywhere on $\mathbb{R}$.

(c) Note that

$$F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1 \\ 1 & \text{if } 1 < x < \frac{3}{2} \\ 0 & \text{if } x > \frac{3}{2} \end{cases}$$

Which is non-negative, Hence F is non-decreasing.

Hence F is a distribution function of some random variable X. **[1 Marks]**

Since F is continuous everywhere and differentiable everywhere except at $x = 0, \frac{1}{2}, 1, \frac{3}{2}$ and $F'(x)$ is continuous everywhere except at $x = 0, \frac{1}{2}, 1, \frac{3}{2}$, Hence pdf exists. **[1 Marks]** and it is given as follows:

$$f_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \le x \le 1 \\ 1 & \text{if } 1 < x < \frac{3}{2} \\ 0 & \text{if } x \ge \frac{3}{2} \end{cases}$$

[1 Marks].

$$\begin{aligned} P(1 \le X < 3) &= P(X < 3) - P(X < 1) = F(3-) - F(1-) = F(3) - F(1) \\ &= 1 - 1/2 = 1/2. \end{aligned}$$

[1 Marks].

5. Let X be a uniform random variable taking values in the interval $(-2, 1)$. Find the pdf of X^2 (if it exists). **[3 marks]**

Ans. Set $Y = X^2$. When $0 \leq X < 1$, then $0 \leq Y < 1$ and if $-2 < X \leq 0$ then $0 \leq Y < 4$. Therefore range of Y is $[0, 4)$. Let $y \in \mathbb{R}$ be given. Then CDF of random variable Y is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= \begin{cases} P(\emptyset) = 0 & \text{if } \; y < 0 \\ P\left\{-\sqrt{y} \leq X \leq \sqrt{y}\right\} & \text{if } \; y \geq 0 \end{cases} \end{aligned}$$

[0.5 marks]

Hence

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)dx, \quad \text{for } y \geq 0,$$

Since

$$f_X(x) = \begin{cases} 0 & \text{if } \; x \leq -2 \\ 1/3 & \text{if } \; -2 < x < 1 \\ 0 & \text{if } \; x \geq 1 \end{cases}$$

[0.5 marks]

we have

(a) If $y \in [0, 1]$, then $-1 \leq -\sqrt{y} \leq \sqrt{y} \leq 1$, Hence

$$F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3} dx = \frac{2}{3}\sqrt{y}$$

[0.5 marks]

(b) If $y \in [1, 4]$ then $-2 \leq -\sqrt{y} \leq -1 < 1 \leq \sqrt{y} \leq 2$, Hence

$$F_Y(y) = \int_{-\sqrt{y}}^{1} \frac{1}{3} dx = \frac{1 + \sqrt{y}}{3}$$

[0.5 marks]

(c) If $y \in [4, \infty)$, then $-\sqrt{y} \leq -2$ and $\sqrt{y} \geq 2$, Hence

$$F_Y(y) = \int_{-2}^{1} \frac{1}{3} dx = 1$$

[0.5 marks]

Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } \; y < 0 \\ \frac{2}{3}\sqrt{y} & \text{if } \; 0 \leq y \leq 1 \\ \frac{\sqrt{y}+1}{3} & \text{if } \; 1 \leq y \leq 4 \\ 1 & \text{if } \; y \geq 4 \end{cases}$$

Which is continuous everywhere and differentiable everywhere except at $y = 0, 1, 4$ and $F'(x)$ is continuous everywhere except at $y = 0, 1, 4$, Hence pdf exists. Hence we may differentiate it to get the pdf of Y.

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & \text{if } \; 0 < y < 1 \\ \dfrac{1}{6\sqrt{y}} & \text{if } \; 1 < y < 4 \\ 0 & \quad\text{otherwsie} \end{cases}$$

[0.5 marks]

Alt. Sol.: Let $Y = X^2$. When $0 \leq X < 1$, then $0 \leq Y < 1$ and if $-2 < X \leq 0$ then $0 \leq Y < 4$. Therefore range of Y is $[0, 4)$. Let $y \in \mathbb{R}$ be given. Then CDF of random variable Y is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= \begin{cases} P(\emptyset) = 0 & \text{if } \; y < 0 \\ P\left\{-\sqrt{y} \leq X \leq \sqrt{y}\right\} & \text{if } \; 0 \leq y < 4 \\ 1 & \text{if } \; y \geq 4 \end{cases} \\ &= \begin{cases} P(\emptyset) = 0 & \text{if } \; y < 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{if } \; 0 \leq y < 4 \\ 1 & \text{if } \; y \geq 4 \end{cases} \end{aligned}$$

[0.5 marks] Since

$$f_X(x) = \begin{cases} 0 & \text{if } \; x \leq -2 \\ 1/3 & \text{if } \; -2 < x < 1 \\ 0 & \text{if } \; x \geq 1 \end{cases},$$

$$F_X(x) = \begin{cases} 0 & \text{if } \; x \leq 0 \\ \int_{-\infty}^{x} f_X(t)dt = \int_{-2}^{x} \frac{1}{3}dt = \frac{x+2}{3} & \text{if } \; -2 < x < 1 \\ 1 & \text{if } \; x \geq 1 \end{cases}$$

, **[0.5 marks]**

(a) If $y \in [0,1)$, then $0 \leq \sqrt{y} < 1, \quad -1 < -\sqrt{y} \leq 0$, Hence

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{\sqrt{y}+2}{3} - \frac{-\sqrt{y}+2}{3}$$

[0.5 marks]

(b) If $y \in [1,4)$ then $1 \leq \sqrt{y} < 2, \quad -2 < -\sqrt{y} \leq -1$, Hence

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = 1 - \frac{-\sqrt{y}+2}{3}$$

[0.5 marks]

Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } \; y < 0 \\ \frac{2\sqrt{y}}{3} & \text{if } \; 0 \leq y < 1 \\ \dfrac{\sqrt{y}+1}{3} & \text{if } \; 1 \leq y < 4 \\ 1, & \quad \text{otherwsie} \end{cases},$$

which is continuous everywhere and differentiable everywhere except at $y = 0, 1, 4$ and $F_Y'(y)$ is continuous everywhere except at $y = 0, 1, 4$. Hence pdf exists. **[0.5 marks]**
Hence we may differentiate it to get the pdf of Y.

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & \text{if } \; 0 < y < 1 \\ \dfrac{1}{6\sqrt{y}} & \text{if } \; 1 < y < 4 \\ 0, & \quad \text{otherwsie} \end{cases}$$

[0.5 marks]

Alter. One may use the Fundamental Theorem to determine density of $Y = g(X)$ (given in Page 130 Of book by Papoulis & Pillai). Set $Y = X^2$. The range of Y is $[0, 4)$. **[0.5 marks]**
For $y \in [0, 4)$, we solve $y = g(x) = x^2$ for $x \in \mathbb{R}$. So we get $x = \pm\sqrt{y}$.

For $y = 0$, we get one solution $x = 0$ but $g'(0) = 0$ Hence we can not apply the conclusion of the fundamental theorem that

$$f_Y(0) = \frac{f_X(0)}{|g'(0)|}.$$

Since this is only one point for which the density of Y does not get defined via formula, it does not matter or we may set $f_Y(0) = 0$.

Hence for $0 < y < 4$.

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|} = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right]$$

[1.5 marks]

(a) If $y \in (0, 1]$, then $-1 \le -\sqrt{y} < \sqrt{y} \le 1$, Hence $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = 1/3$. Therefore

$$f_Y(y) = \frac{1}{2\sqrt{y}}[1/3 + 1/3]$$

[0.5 marks]

(b) If $y \in (1, 4)$ then $-2 < -\sqrt{y} < -1 < 1 < \sqrt{y} < 2$, Hence $f_X(\sqrt{y}) = 0, f_X(-\sqrt{y}) = 1/3$. Therefore

$$f_Y(y) = \frac{1}{2\sqrt{y}}[1/3 + 0]$$

[0.5 marks]

Therefore

$$f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & \text{if } \ 0 < y < 1 \\ \dfrac{1}{6\sqrt{y}} & \text{if } \ 1 < y < 4 \\ 0 & \quad \text{otherwsie} \end{cases}$$

[0.5 marks]

6. Let N be a positive integer and let X be a random variable with the following pmf:

$$P(X=k)=\frac{2k}{N(N+1)},\ \text{for } k=1,2\cdots,N.$$

Find the mean of X. Find cumulative distribution function of X. **[3 marks]**

Hint: $\sum_{i=1}^{N} i=\frac{N(N+1)}{2}, \quad \sum_{i=1}^{N} i^2=\frac{N(N+1)(2N+1)}{6}$

Ans.

$$E[X]=\sum_{k=1}^{N} kP(X=k)=\sum_{k=1}^{N} k\frac{2k}{N(N+1)}=\frac{2}{N(N+1)}\sum_{k=1}^{N} k^2=\frac{2N+1}{3}.$$

[1 marks] CDF of X is

$$F_X(x)=\begin{cases} 0 & \text{if } x<1[\mathbf{0.25 marks}] \\ \frac{2}{N(N+1)} & \text{if } 1\le x<2 \\ \frac{2}{N(N+1)}[1+2] & \text{if } 2\le x<3 \\ \vdots & \vdots \\ \frac{2}{N(N+1)}[1+2+\cdots+k]=\frac{k(k+1)}{N(N+1)} & \text{if } k\le x<k+1[\mathbf{1.5 marks}] \\ \vdots & \vdots \\ 1 & \text{if } x\ge N[\mathbf{0.25 marks}] \end{cases}$$

7. Let $(\Omega,\mathcal{F},P)$ be a probability space. For events $A_1,A_2,\cdots,A_n\in\mathcal{F}$, show that

$$P(A_1\cap A_2\cap\cdots\cap A_n)\ge\sum_{i=1}^{n} P(A_i)-n+1.$$ **[3 marks]**

Ans. Proof is by induction. Recall $P(A_1\cup A_2)=P(A_1)+P(A_2)-P(A_1\cap A_2)$. Hence for $n=2$

$$P(A_1\cap A_2)=P(A_1)+P(A_2)-P(A_1\cup A_2)\ge P(A_1)+P(A_2)-1(\because P(A_1\cup A_2)\le 1)$$

. **[1 marks]**

Now let us assume the inequality for $n=m$. Consider

$$\begin{aligned} P(A_1\cap A_2\cap\cdots\cap A_{m+1}) &\ge P(A_1\cap A_2\cap\cdots\cap A_m)+P(A_{m+1})-1 \qquad [\mathbf{1 marks}] \\ &\ge \sum_{k=1}^{m} P(A_k)-m+1+P(A_{m+1})-1 \\ &= \sum_{k=1}^{m+1} P(A_k)-(m+1)+1 \end{aligned}$$

Therefore result is true for $n=m+1$. **[1 marks]**

8. Show that all the outcomes of a countably infinite sample space can not be equally likely. **[3 marks]**

Ans. Since Ω is countably infinite, hence we may write $\Omega = \{\omega_n | n \in \mathbb{N}\}$. Suppose the contrary, i.e., there exists $p > 0$ such that $P(\omega_n) = p$ for all $n = 1, 2, \cdots$. **[1.5 marks]**

Now by countable additivity

$$P(\Omega) = \sum_{n=1}^{\infty} P(\omega_n) = \sum_{n=1}^{\infty} p = \infty.$$

which is a contradiction. **[1.5 marks]**

TABLE 1 Values of the standard normal distribution function

x	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5363	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9278	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9430	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9648	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9762	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9874	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000

End of The Paper