Gradient Operator (∇)

In Cartesian Coordinate

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}$$

Find out gradient of the following function

$T(x,y) = -(\cos^2 x + \cos^2 y)^2$

T=T(x,y,z)

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz.$$

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Geometrical Interpretation of the Gradient

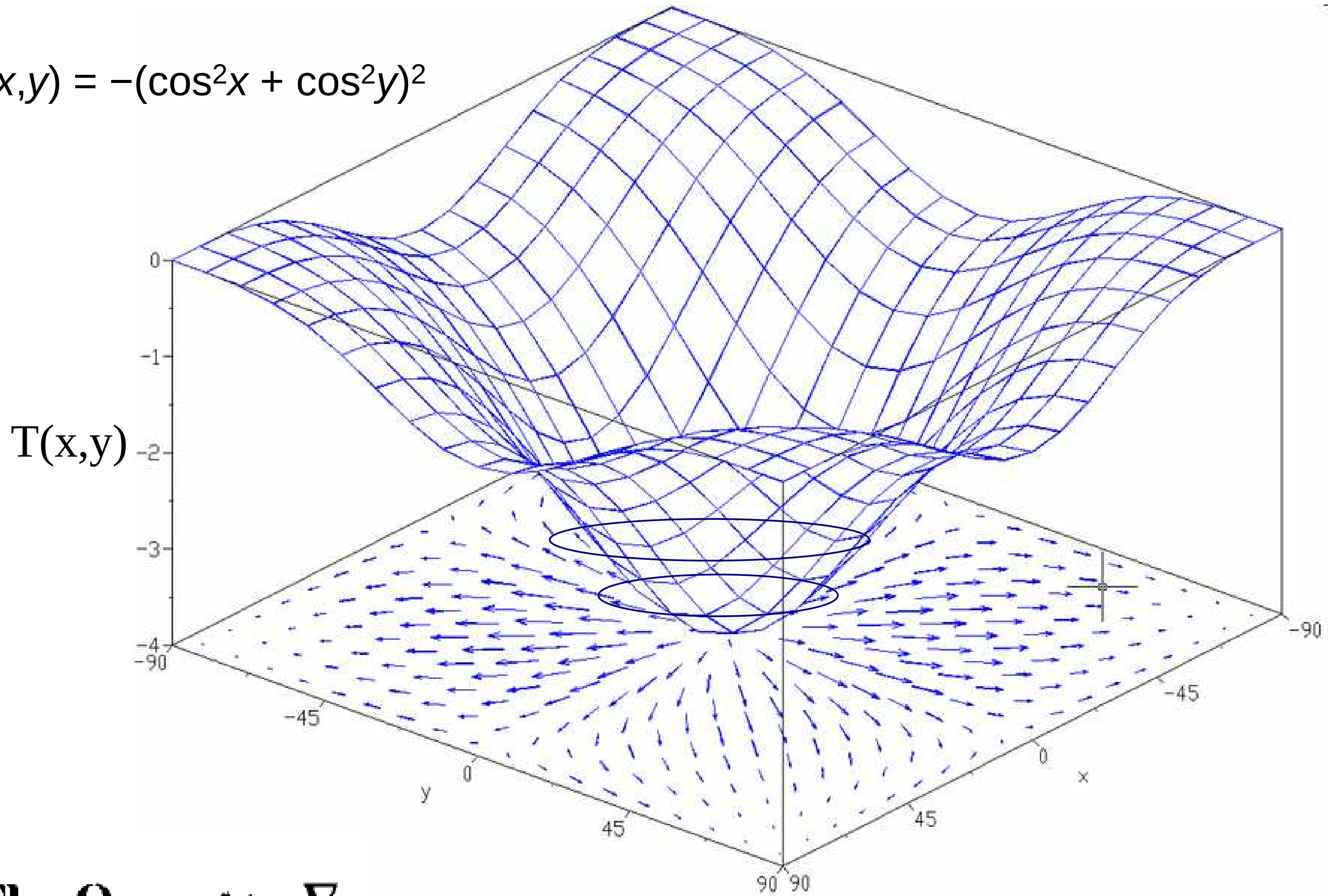
$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}| \cos\theta$$

when $\theta = 0$ (for then $\cos\theta = 1$)

The gradient ∇T points in the direction of maximum increase of the function T.

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

$T(x,y) = -(\cos^2 x + \cos^2 y)^2$



The Operator ∇

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.}$$

http://en.wikipedia.org/wiki/Gradient

$$\nabla T = \underbrace{\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)}_{\text{Vector Field}} T$$

Scalar Field

The Divergence

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}})$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

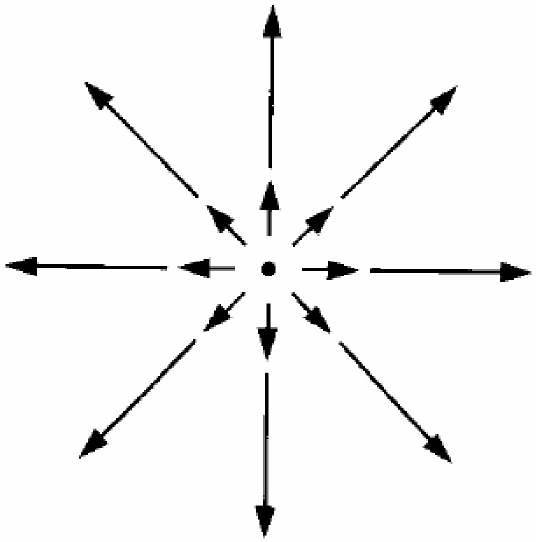
$$\nabla \cdot \vec{V} \neq \vec{V} \cdot \nabla$$

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

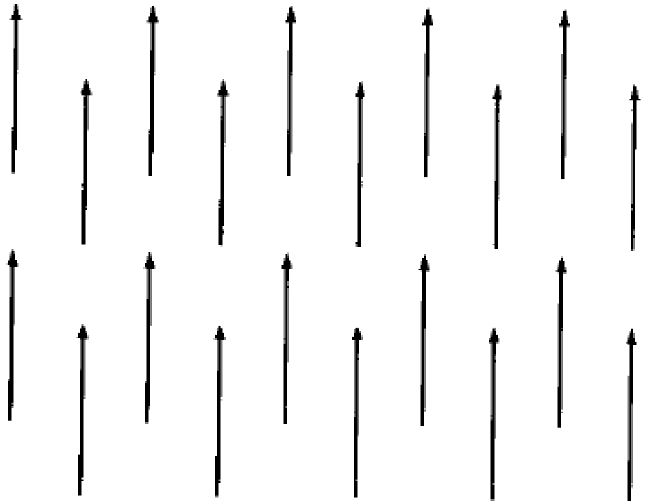
(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

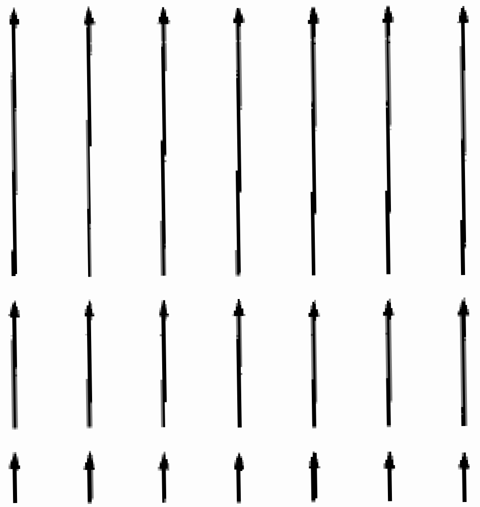
$$\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$



$$\mathbf{v}_b = \hat{\mathbf{z}}$$



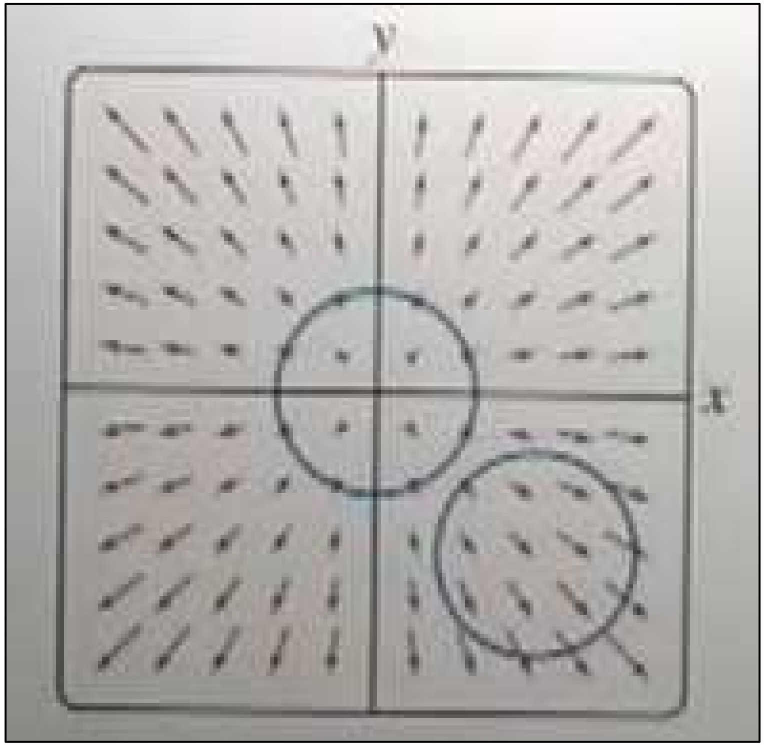
$$\mathbf{v}_c = z\,\hat{\mathbf{z}}$$



Introduction to Electrodynamics by David. J. Griffiths

In many cases, the divergence of a vector function at point P may be predicted by considering a closed surface surrounding P and analyzing the flow over the boundary, keeping in mind that at P:

$$\nabla \cdot \vec{F} = \text{outflow} - \text{inflow}$$

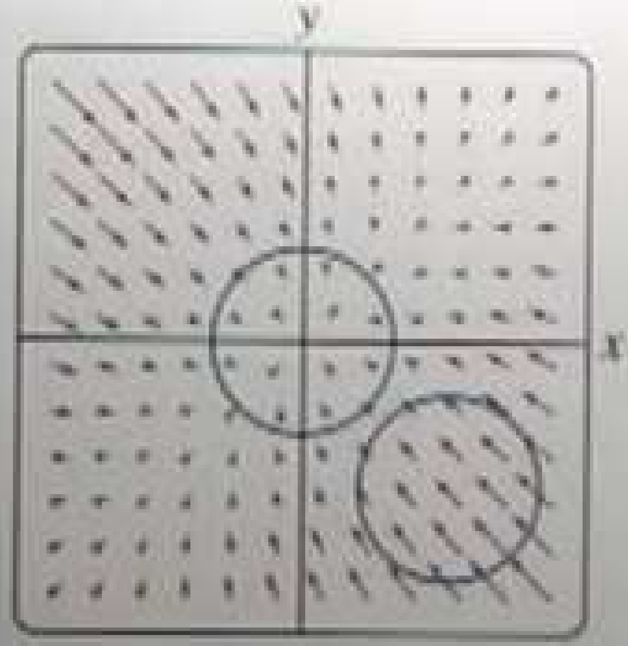


$$\vec{V} = x\hat{i} + y\hat{j}$$

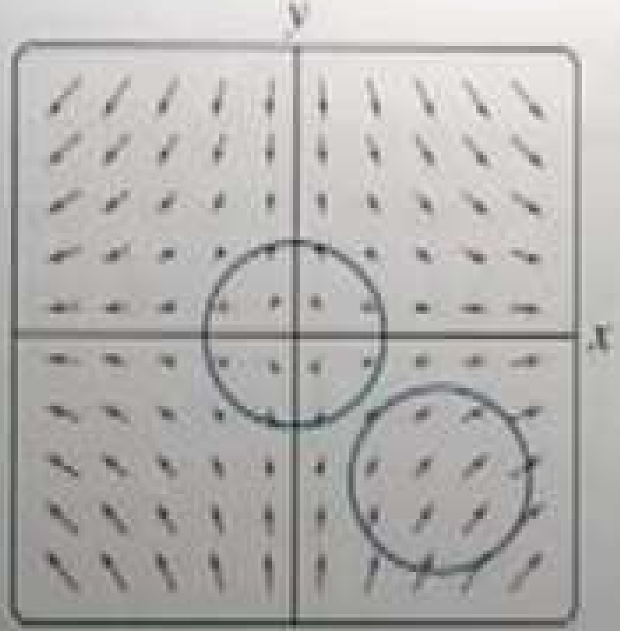
$$\vec{\nabla} \cdot \vec{V} = 2$$

In 3-D, divergence is a measure of Change of flux per unit volume

Note: All the figures related of vector field has been taken from a lecture series given by Dr. Chris Tisdell, UNSW



(B) The force field
$\mathbf{F} = \langle y - 2x, x - 2y \rangle$
with $\operatorname{div}(\mathbf{F}) = -4$.
There is a net inflow
into every circle.

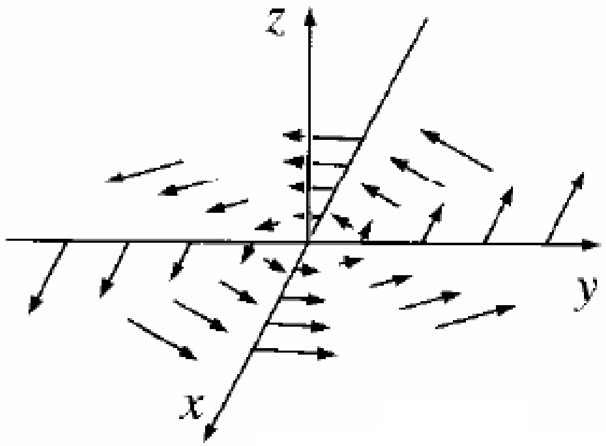


(C) The force field $\mathbf{F} = \langle x, -y \rangle$ with $\operatorname{div}(\mathbf{F}) = 0$. The flux through every circle is zero.

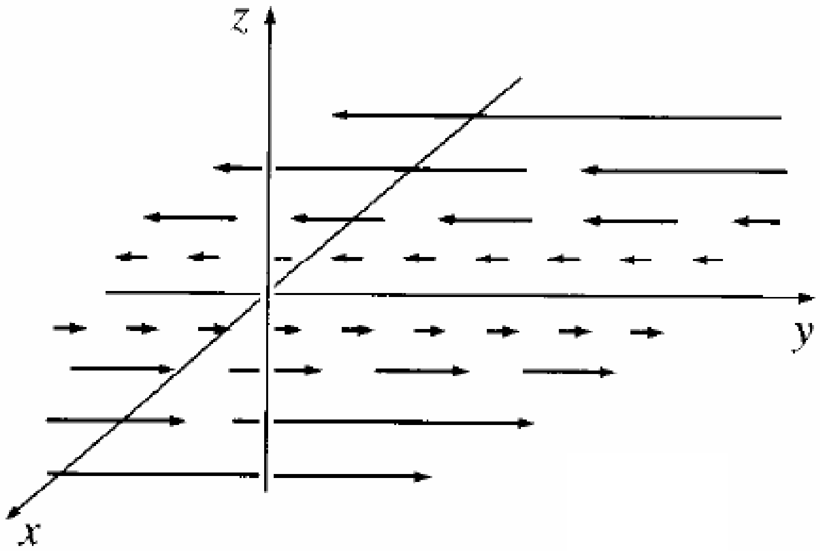
The Curl

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



$\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$



$\mathbf{v}_b = x\hat{\mathbf{y}}$

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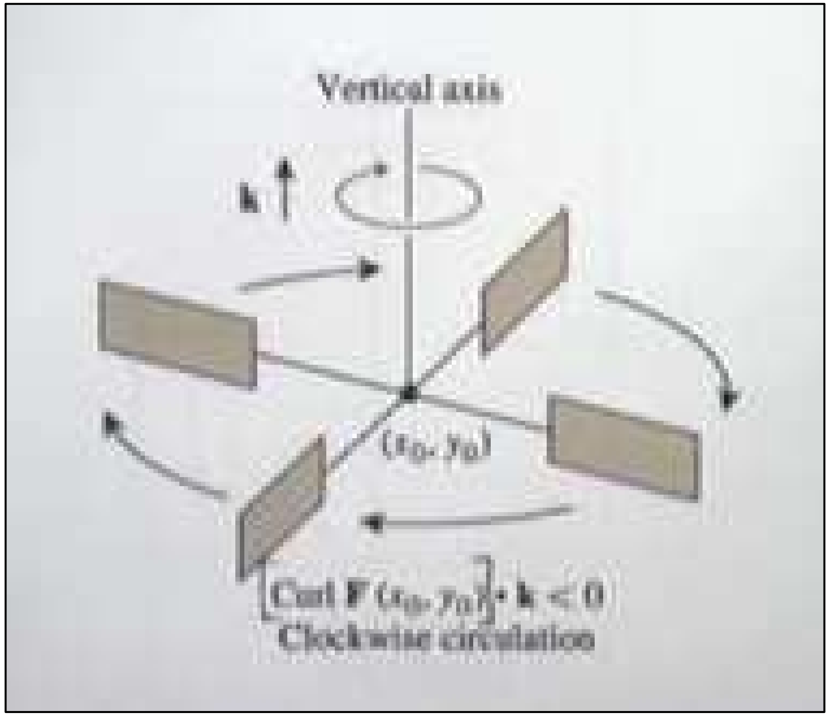
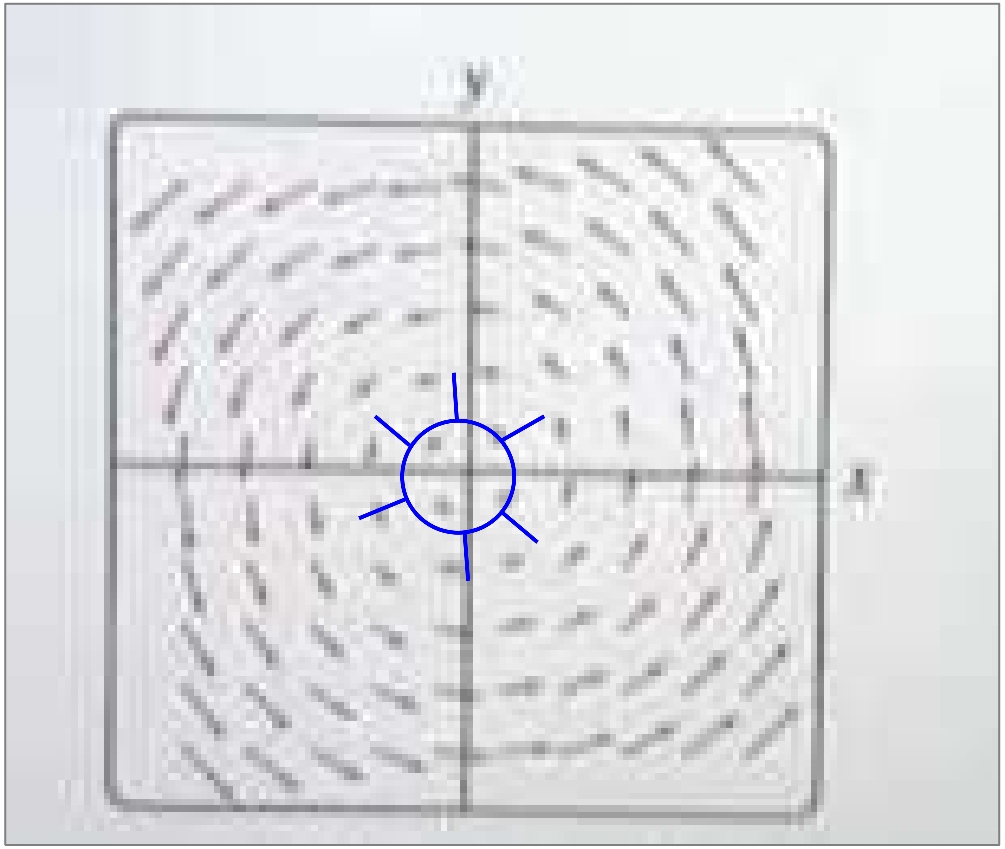
Paddle wheel analysis



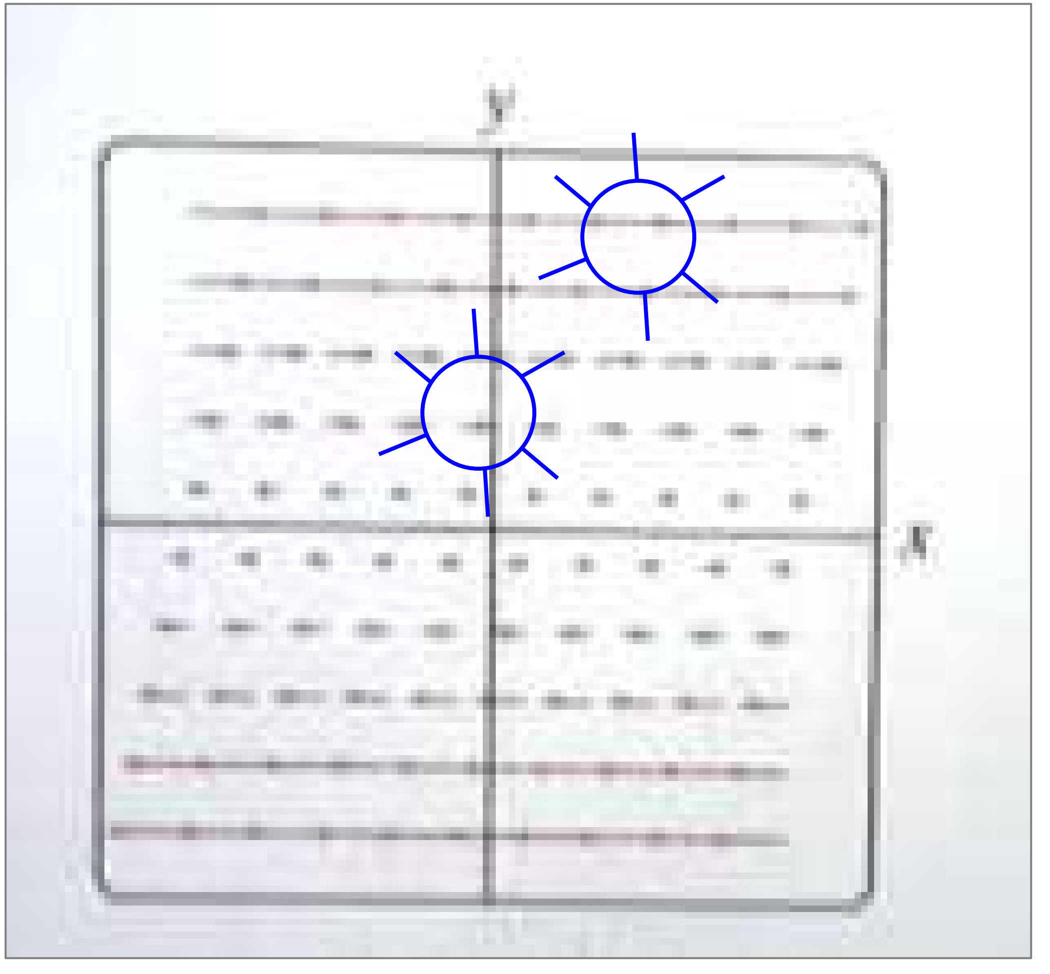
$$[\nabla \times \vec{\boldsymbol{F}}(\boldsymbol{x}_0, \boldsymbol{y}_0)] \cdot \hat{\boldsymbol{k}} < 0$$

$$\vec{F}(x, y) = -y\hat{i} + x\hat{j}$$

$$(\vec{\nabla} \times \vec{F}) \cdot \hat{k} = 2$$



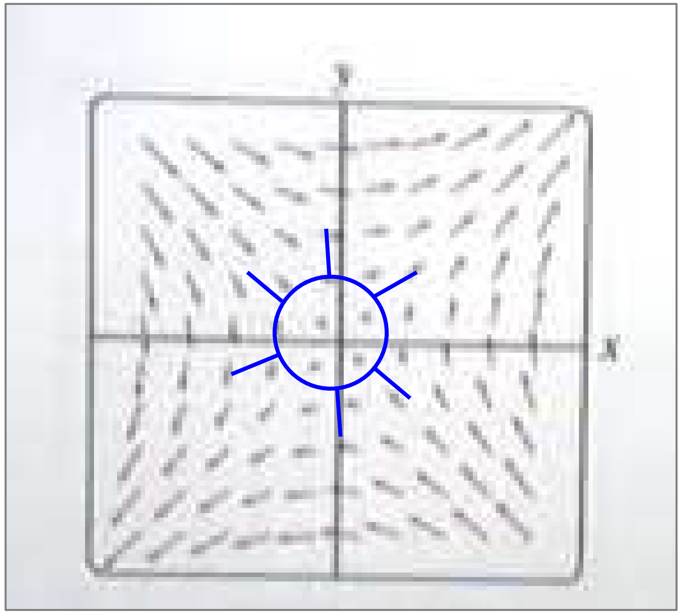
$$\vec{F}(x, y) = y\hat{i}$$

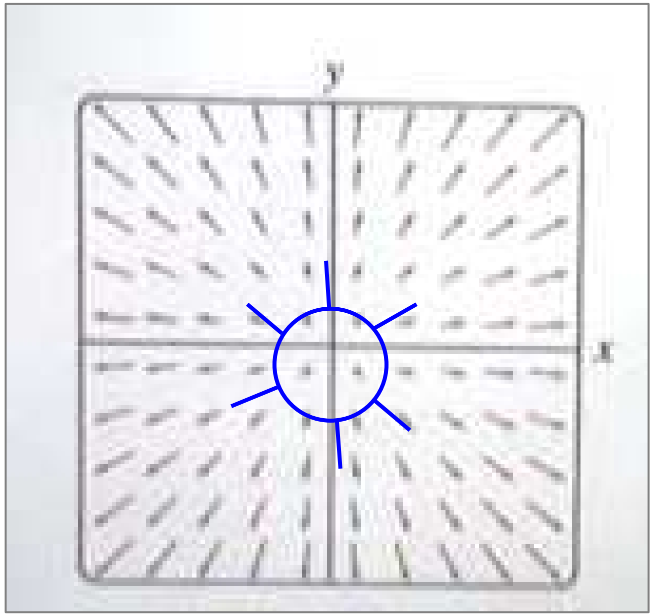


$$(\nabla \times \vec{F}) \cdot \hat{k} = -1$$

$$\vec{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}\hat{\boldsymbol{i}} + \boldsymbol{x}\hat{\boldsymbol{j}}$$

$$\vec{F}(x, y) = x\hat{i} + y\hat{j}$$





$$\nabla \times \vec{\boldsymbol{F}} = 0$$

Summary

Gradient of a Scalar Field

$$\underset{\text{Scalar Field}}{\nabla T} = \underbrace{\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) T}_{\text{Vector Field}}$$

Divergence of a Vector Field ⟶ **Measure of Change of flux per unit volume**

Curl of a Vector Field ⟶ **Measure of degree of rotation of the field**

Identities related to Gradient

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla\cdot(\mathbf{A}+\mathbf{B}) = (\nabla\cdot\mathbf{A}) + (\nabla\cdot\mathbf{B}),$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = (\nabla\times\mathbf{A}) + (\nabla\times\mathbf{B}),$$

$$\nabla(kf) = k\nabla f, \qquad \nabla\cdot(k\mathbf{A}) = k(\nabla\cdot\mathbf{A}), \qquad \nabla\times(k\mathbf{A}) = k(\nabla\times\mathbf{A}),$$

fg (product of two scalar functions),

$\mathbf{A}\cdot\mathbf{B}$ (dot product of two vector functions)

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Identities related to Divergence & Curl

$f\mathbf{A}$ (scalar times vector),

$\mathbf{A} \times \mathbf{B}$ (cross product of two vectors)

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

∇T is a *vector*

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$

(2) Curl of gradient: $\nabla \times (\nabla T)$.

$\nabla \cdot \mathbf{v}$ is a *scalar*

Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$

$\nabla \times \mathbf{v}$ is a *vector*

Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$

Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

$$\begin{aligned}
\nabla \cdot (\nabla T) &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \\
&= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \\
&= \nabla^2 T
\end{aligned}$$

$\nabla^2 \longleftarrow$ Laplacian oparator

Laplacian of a *vector*,

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}$$

The curl of a gradient is always *zero*.

$$\boxed{\nabla \times (\nabla T) = 0}$$

$$\nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$$

The divergence of a curl, like the curl of a gradient, is *always zero*

$$\boxed{\nabla \cdot (\nabla \times \mathbf{v}) = 0}$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$