Lecture 12: Extrema

October 31, 2018

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

Example 12.1 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := 4xy - x^4 - y^4$. Find all the points of local extrema, saddle points (if any) of f.*

Solution: Since f is a polynomial function, f has continuous partial derivatives of all orders. Also, $f_x = 4y - 4x^3$ and $f_y = 4x - 4y^3$, and so $\nabla f(x, y) = (0, 0) \iff y = x^3, x = y^3 \implies x = (x^3)^3 \implies x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \implies x = 0, \pm 1 \implies (x, y) = (x, x^3) = (0, 0), (1, 1), (1, -1)$. Further, $f_{xx} = -12x^2, f_{xy} = 4$, and $f_{yy} = -12y^2$, and so the discriminant is given by $\Delta f = f_{xx} f_{yy} - f_{xy} = 16(9x^2y^2 - 1)$. In particular, $\Delta f(0, 0) = -16 < 0$ and $\Delta f(1, 1) = \Delta f(-1, -1) = 128 > 0$. Also $f_{xx}(1, 1) = f_{xx}(-1, -1) = -12 < 0$. By the Discriminant Test, f has a saddle point at $(0, 0)$ and a local maximum at $(1, 1)$ as well as at $(-1, -1)$.

■

Example 12.2 *Find all points (if any) of local extrema, saddle points for the function $f(x, y) = x^4 + y^3$. Also discuss the points of absolute maxima and minima.*

Solution: Since f is a polynomial function, f has continuous partial derivatives of all orders. Also, $f_x = 4x^3, f_y = 3y^2$. So $(0, 0)$ is the only critical point. $f_{xx} = 12x^2, f_{yy} = 6y, f_{xy} = 0$. Hence $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(0, 0)$. So test fails. We claim that f neither has local maximum nor a local minimum at $(0, 0)$. To see this, note that $f(0, 0) = 0$ and f takes both positive as well as negative values in any open disk centered at the origin. For example, $f(r, 0) = r^4 > 0$ and $f(0, -r) = -r^3 < 0$ for any $r > 0$. It turns out that f does have a saddle point at $(0, 0)$.

Function f does not attain absolute maximum and absolute minimum on $\mathbb{R}^2$, since

$$f(x, 0) = x^4, f(0, y) = y^3$$

So as we move along x-axis away from origin, f values increases arbitrarily large and as we move away from $(0, 0)$ along negative y-axis f values becomes arbitrarily small. ■

Example 12.3 *Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.*

Solution: If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, then $f_{xx}(a,b)f_{yy}(a,b) < 0$ so discriminant is < 0. The surface must therefore have a saddle point at (a,b) by the second derivative test. ■

Example 12.4 *Show that* $(0,0)$ *is a critical point of* $f(x,y) = x^2 + kxy + y^2$ *no matter what value the constant* k *has.*

Solution: Since f is a polynomial function hence $\nabla f(x,y)$ exists everywhere. Also $f_x = 2x + ky, f_y = kx + 2y$. For each $k \in \mathbb{R}$, we have $f_x(0,0) = 0 = f_y(0,0)$. Hence $(0,0)$ is a critical point of f. ■

Example 12.5 *For what values of the constant* k *does the Second Derivative Test guarantee that* $f(x,y) = x^2 + kxy + y^2$ *will have a saddle point at* $(0,0)$*? A local minimum at* $(0,0)$*? For what values of* k *is the Second Derivative Test inconclusive? Can you decide the nature of* $(0,0)$ *when Second Derivative Test is inconclusive? Give reasons for your answers.*

Solution: $f_{xx} = 2, f_{yy} = 2, f_{xy} = k$. Hence discriminant $\Delta f(x,y) = 4 - k^2$. So if $4 - k^2 < 0$, i.e., $|k| > 2$ then the Second Derivative Test guarantee that f has a saddle point at $(0,0)$. If $4 - K^2 > 0$, i.e. $|k| < 2$ then the Second Derivative Test guarantee that f has a local minimum at $(0,0)$. For $k = \pm 2$ the Second Derivative Test inconclusive.

For $k = \pm 2$, $f(x,y) = x^2 \pm 2xy + y^2 = (x \pm y)^2 \geq 0 = f(0,0)$ for all $(x,y) \in \mathbb{R}^2$. Hence f has local and global minimum at $(0,0)$ for $k = \pm 2$. ■