MATH 3: Mid-Semester Examination: Part-A
(To be returned after 30 mins..)

R.No.: **Section:** **Name:**

Instructions:

- Attempt all questions. Use the main sheet for rough work.
- Only the answers should be written on this sheet.
- ***Answers will be rejected if there is any overwriting or cutting***.
  No partial credits. Each question carries 4 marks.
- **One mark will be deducted for each wrong answer.**

Fill in the Blanks

1. Simplify $\left(\frac{\sqrt{3}+i}{\sqrt{2}}\right)^{61} = (2)^{61/2}e^{\frac{\pi i}{6}} = (2)^{30}\frac{\sqrt{3}+i}{\sqrt{2}}$.

Justification:

$$\begin{aligned}\left(\frac{\sqrt{3}+i}{\sqrt{2}}\right)^{61} &= (2)^{61/2}\left(e^{\frac{\pi i}{6}}\right)^{61} \\ &= (2)^{61/2}\left(e^{10\pi i+\frac{\pi i}{6}}\right) \\ &= (2)^{61/2}e^{\frac{\pi i}{6}} = (2)^{30}\frac{\sqrt{3}+i}{\sqrt{2}}\end{aligned}$$

2. Let the closed path $\Gamma : \{z \in \mathbb{C} : |z-1| = \frac{1}{2}\}$ with positive orientation. Then, the value of $\oint_\Gamma \frac{\cot z}{(4z-5)(3z-5)(3z-8)}dz$ is $\frac{8\pi i \cot\frac{5}{4}}{85}$

Justification: Inside curve $\Gamma : \{z \in \mathbb{C} : |z-1| = \frac{1}{2}\}$, only simple pole $z = \frac{5}{4}$,
We can evaluate using Cauchy residue theorem
$\oint_\Gamma \frac{\cot z}{(4z-5)(3z-5)(3z-8)}dz = 2\pi i \left\{\text{Residue of } \frac{\cot z}{(4z-5)(3z-5)(3z-8)} \text{ at } z = \frac{5}{4}\right\} = \frac{8\pi i \cot\frac{5}{4}}{85}$

3. Number of zeroes and poles of the function $f(z) = \tan \pi z$ in the region $|z| < 4$.
   Poles $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}$, i.e., $P = 8$
   Zeros $0, \pm1, \pm2, \pm3, i.e., N = 7$

4. Region of convergence of the series $\sum_{n=0}^{\infty} \frac{(n!)^4}{(4n)!}(z-3i)^{2n} = |z-3i| < 16$

Justification: As $\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \frac{|z-3i|^2}{4^4}$

5. $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$

Justification:

$$\begin{aligned} I = \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta &= \int_0^{2\pi} e^{\cos\theta}(\frac{e^{i\sin\ \theta}+e^{-i\sin\ \theta}}{2})d\theta \\ &= \frac{1}{2}\int_0^{2\pi} e^{\cos\theta+i\sin\ \theta}+e^{\cos\theta-i\sin\ \theta}d\theta \end{aligned}$$

Take $z = e^{i\theta}$, then $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \frac{1}{2}\oint_{|z|=1}(e^z+e^{\frac{1}{z}})\frac{dz}{iz}$

By applying Cauchy residue theorem,

$\oint_{|z|=1}(e^z+e^{\frac{1}{z}})\frac{dz}{iz} = 4\pi$

So, $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$