Lecture 10: Directional Derivatives

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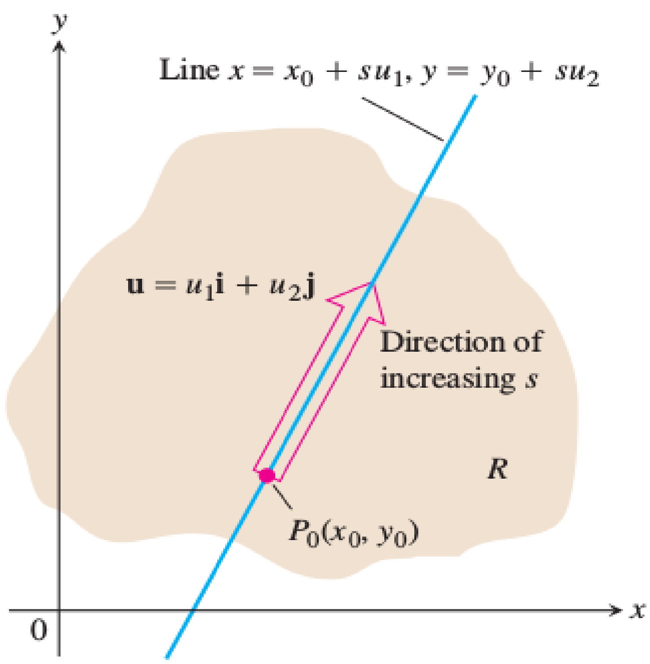
Parametric Equation of a straight line passing through a given point and a given direction



In order to get equation of the line passing through (x_0, y_0) in the direction of (u_1, u_2). First note that if $P = (x, y)$ and $P' = (x_0, y_0)$ then vector from P to P' is $(x - x_0, y - y_0)$. Now we want this vector to be parallel to given unit vector (u_1, u_2), that is there exist some $s \in \mathbb{R}$ such that

$$(x - x_0, y - y_0) = s(u_1, u_2).$$

If we very s over $\mathbb{R}$ are we get all the point on the straight line. So parametric equation of the line passing through (x_0, y_0) in the direction of (u_1, u_2) is

$$x = x_0 + su_1, y = y_0 + su_2,$$

where parameter s varies over set of all real numbers.

Directional Derivatives

The notion of partial derivatives can be easily generalized to that of a directional derivative, which measures the rate of change of a function at a point along a given direction. We specify a direction by specifying a unit vector. Let $u = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$, i.e., $|u| = 1 \iff u_1^2 + u_2^2 = 1$.

Definition 10.1 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be any function.Let $u = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$. Let $(x_0, y_0) \in D$ be such that D contains a segment of the line passing through (x_0, y_0) in the direction of u. We define the directional derivative of f at (x_0, y_0) along u to be the limit*

$$\lim_{t\to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

provided this limit exists. It is denoted by $D_u f(x_0, y_0)$.

Note that if $v = -u$, then

$$\begin{aligned} D_v f(x_0, y_0) &= \lim_{t\to 0} \frac{f(x_0 - tu_1, y_0 - tu_2) - f(x_0, y_0)}{t} \\ &= \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{-h} \quad \text{(substituing } t = -h) \\ &= -D_u f(x_0, y_0) \end{aligned}$$

Note also that if $\mathbf{i} := (1, 0)$ and $\mathbf{j} := (0, 1)$, then $D_i f(x_0, y_0) = f_x(x_0, y_0)$ and $D_j f(x_0, y_0) = f_y(x_0, y_0)$.

Theorem 10.2 (Differentiability and Directional Derivatives) *Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0), then for every unit vector $u = (u_1, u_2)$ in $\mathbb{R}^2$, the directional derivative $D_u f(x_0, y_0)$ exists and moreover,*

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 10.3

1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) := x^2 + y^2$. Since f is a polynomial in x, y hence it is differentiable everywhere on plane. Hence by above theorem, given any unit vector $u = (u_1, u_2)$ in $\mathbb{R}^2$ and any $(x_0, y_0) \in \mathbb{R}^2$, $D_u f(x_0, y_0)$ exists and is equal to $2x_0u_1 + 2y_0u_2$.*

2. *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

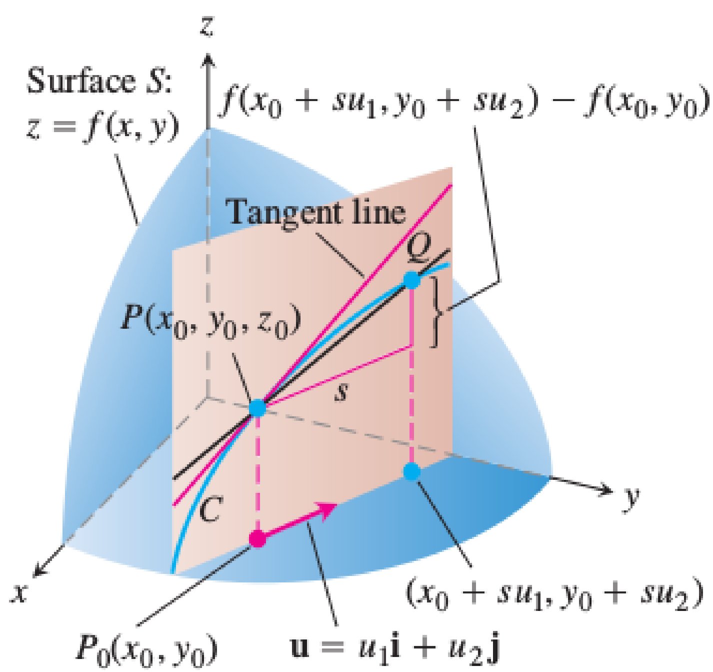
Find the directional derivative of f at $(0,0)$ in the direction of the vector $v = (1,1)$.
Solution: *The given vector is not unit vector hence in order find directional derivative in the direction of the vector $(1,1)$ we find its unit vector. $|v| = \sqrt{2}$ Hence unit vector in direction of v would be the vector $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Also recall that we have shown that function is not differentiable at $(0,0)$, hence we can not apply the theorem to calculate the directional derivative. For $t \neq 0$, we consider*

$$\frac{f\left(0+\frac{t}{\sqrt{2}}, 0+\frac{t}{\sqrt{2}}\right) - f(0,0)}{t} = \frac{\frac{\frac{t^3}{2\sqrt{2}}}{\frac{t^4}{4}+\frac{t^2}{2}} - 0}{t} = \frac{\frac{\frac{t}{2\sqrt{2}}}{\frac{t^2+2}{4}}}{t} = \frac{\frac{t}{2\sqrt{2}} \times \frac{4}{t^2+2}}{t} = \frac{\sqrt{2}t}{t(2+t^2)} = \frac{\sqrt{2}}{2+t^2}$$

Hence $D_u f(0,0) = \frac{1}{\sqrt{2}}$. Also recall that $\nabla f(0,0) = (0,0)$. Hence $D_u f(0,0) \neq \nabla f(0,0) \cdot u$.

Geometrical Interpretation of the Directional Derivative

The equation $z = f(x,y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to u intersects S in a curve C. Then $D_u f(x_0, y_0)$ is the slope of the tangent line to the curve C at the point $P(x_0, y_0, z_0)$.



Geometrical interpretation of the Gradient Vector

The Theorem 10.2 suggests the following geometric interpretation of the gradient. Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0) and suppose $\nabla f(x_0, y_0) \neq (0, 0)$. Given any unit vector $u = (u_1, u_2)$,

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u = |\nabla f(x_0, y_0)| \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between $\nabla f(x_0, y_0)$ and u. Thus, if we keep in mind the fact that $D_u f(x_0, y_0)$ measures the rate of change in f in the direction of u, then we can make the following observations.

1. $D_u f(x_0, y_0)$ is maximum when $\cos\theta = 1$, that is, when $\theta = 0$. Thus near (x_0, y_0), $u = \dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$ is the direction in which f increases most rapidly.
2. $D_u f(x_0, y_0)$ is minimum when $\cos\theta = -1$, that is, when $\theta = \pi$. Thus near (x_0, y_0), $u = \dfrac{-\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$ is the direction in which f decreases most rapidly.
3. $D_u f(x_0, y_0) = 0$ when $\cos\theta = 0$, that is, when $\theta = \frac{\pi}{2}$. Thus near (x_0, y_0), $u = \pm\dfrac{(f_y(x_0, y_0), -f_x(x_0, y_0))}{|\nabla f(x_0, y_0)|}$ are the directions of no change in f.

Example 10.4 *For example, consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 4 - x^2 - y^2$. We have $f_x = -2x$ and $f_y = -2y$. So at $(x_0, y_0) = (1, 1)$, the gradient is given by $\nabla f(1, 1) = (-2, -2)$. Thus, near $(1, 1)$, the steepest ascent on the surface $z = f(x, y)$ is in the direction of $\dfrac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = \left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$, while the steepest descent is in the reverse direction, namely, $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. The directions of no change are $\pm\left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$.*