Lecture 20: Line Integral:II

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Recall by fundamental theorem of calculus, If $f : [a, b] \to \mathbb{R}$ be such that f' is continuous, then

$$\int_a^b f'(t)dt = f(b) - f(a)$$

This says that the integral of a smooth function, depends only on the end points and not on the points between them. We extend this result to line integrals.

Theorem 20.1 *Let $D \subset \mathbb{R}^3$, $f : D \to \mathbb{R}$ be differentiable on D and the gradient ∇f is continuous on D. Let $A = (x_0, y_0, z_0), B = (x_1, y_1, z_1)$ be two points in D. Let $C = \{\mathbf{r}(t) : t \in [a, b]\}$ be a curve lying in D and joining the points A and B, that is $\mathbf{r}(a) = A$ and $\mathbf{r}(b) = B$. Suppose $\mathbf{r}'(t)$ is continuous on $[a, b]$. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

Proof: Define $g : [a, b] \to \mathbb{R}$ by $g(t) := f(\mathbf{r}(t))$. Then by chain rule g' is differentiable and $g'(t) = \nabla f \cdot \mathbf{r}'(t)$. By continuity of ∇f and $\mathbf{r}'(t)$ we ensure continuity of g'. Applying fundamental theorem of calculus to g, we obtain

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(r(t)) \cdot \mathbf{r}'(t) \quad dt \\ &= \int_a^b g'(t) \quad dt \\ &= g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)\end{aligned}$$

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Example 20.2 *Evaluate the line integral $\int_C xdy + ydx$ where C is a path $\left(t^9, \sin^9\left(\frac{\pi t}{2}\right)\right)$, $0 \le t \le 1$*

Solution: Let $f(x, y) = xy$, then $\nabla f = (y, x)$. Hence the line integral $\int_C xdy + ydx$ can be written as $\int_C \nabla f \cdot dr$, which can evaluated easily by previous theorem $f(1, 1) - f(0, 0) = 1$. ■

Remark 20.3 *The Theorem 20.1 tells us that line integral of a gradient vector field depends only on the endpoints of path C not on the details of the trajectory of the path. The next example shows that for a nongradient vector field the line integral can depend on the path chosen between two points in space.*

Exercise 20.4 *Find the line integral of the vector field $F = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ along each of the following paths joining $(1, 0, 0)$ to $(1, 0, 1)$*

$$\begin{aligned}
r_1(t) &= \cos t\mathbf{i} + \sin t\mathbf{j} + \frac{t}{2\pi}\mathbf{k}, \quad 0 \le t \le 2\pi \\
r_2(t) &= \cos(t^3)\mathbf{i} + \sin(t^3)\mathbf{j} + \frac{t^3}{2\pi}\mathbf{k}, \quad 0 \le t \le (2\pi)^{\frac{1}{3}} \\
r_3(t) &= \cos t\mathbf{i} - \sin t\mathbf{j} + \frac{t}{2\pi}\mathbf{k}, \quad 0 \le t \le 2\pi
\end{aligned}$$

Solution:

$$\int_{C_1} F \cdot d\mathbf{r_1} = \int_{C_2} F \cdot d\mathbf{r_2} = -2\pi + 1 \qquad \text{and} \qquad \int_{C_3} F \cdot d\mathbf{r_1} = 2\pi + 1.$$

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Remark 20.5 *The fact $\displaystyle\int_{C_1} F \cdot d\mathbf{r_1} \neq \int_{C_3} F \cdot d\mathbf{r_3}$ shows that F is not a gradient vector field. If F were equal to ∇f for some f, we would have $\displaystyle\int_C F \cdot d\mathbf{r} = f(1, 0, 1) - f(1, 0, 0)$ for any path C joining $(1, 0, 0)$ to $(1, 0, 1)$.*

Remark 20.6 *The equality $\displaystyle\int_{C_1} F \cdot d\mathbf{r_1} = \int_{C_2} F \cdot d\mathbf{r_2}$ is not a lucky accident. It reflects a general fact that the line integral of a vector field (gradient or nongradient) is unchanged when the curve is reparametrized.*

Exercise 20.7 *Show that the integral $\displaystyle\int_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.*

Solution: If $F = yz\hat{i} + (xz + 1)\hat{j} + xy\hat{k}$, then $F = \nabla\phi$, where $\phi(x, y, z) = xyz + y$. Hence, by the Theorem 20.1, the conclusion follows. ■

Example 20.8 *Let C be the line segment from $(0, 0, 0)$ to $(1, 0, 0)$ and let $r_1(t) = (t, 0, 0)$, where $0 \le t \le 1$. Find line integral of $F(x, y, z) = \mathbf{i}$ along the curve. Also, find the line integral if C is parametrized by $r_2(t) = (1 - t, 0, 0)$, where $0 \le t \le 1$.*

Solution:

$$\int F \cdot d\mathbf{r_1} = 1, \int F \cdot d\mathbf{r_2} = -1$$

Geometrically r_1 and r_2 represents the same curve but they traverse along C in opposite direction so actually r_1 and r_2 does not represent the same curve for line integral. This illustrate the importance of direction in line integral. ∎

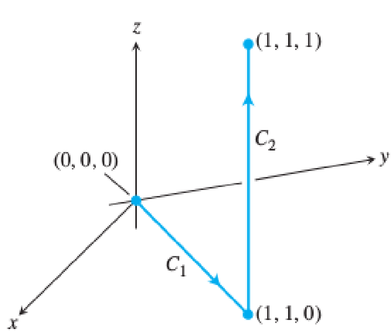
Exercise 20.9 *Let $F(x,y) = -y\hat{i} + x\hat{j}$ and C be the unit circle $x^2 + y^2 = 1$. If $r_1(t) = (\cos t, \sin t)$ where $0 \leq t \leq 2\pi$ and $r_2(t) = (\cos t, \sin t)$ where $0 \leq t \leq 4\pi$. Show that line integral along r_1 is 2π and along r_2 its 4π. Since r_2 traverses circle twice, we get answer along r_2 double.*

Line Integral of scalar functions

Definition 20.10 *Let $f(x,y,z)$ be a continuous real valued function and C be continuously differentiable path (with parameterization $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$) defined on the interval $[a,b]$. The line integral of f over C is defined by*

$$\int_C f ds := \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|dt$$

Example 20.11 *Integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$.*



Solution We choose the simplest parametrizations for C_1 and C_2 we can find, calculating the lengths of the velocity vectors as we go along:

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

With these parametrizations we find that

$$\begin{aligned}
\int_{C_1 \cup C_2} f(x, y, z)\, ds &= \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds && \text{Eq. (3)} \\
&= \int_0^1 f(t, t, 0)\sqrt{2}\, dt + \int_0^1 f(1, 1, t)(1)\, dt && \text{Eq. (2)} \\
&= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\, dt + \int_0^1 (1 - 3 + t)(1)\, dt \\
&= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}.
\end{aligned}$$

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