MATH 3: Mid-Semester Examination: Part-B

Duration: 90 Minitus. Max.Marks: 30

Instructions:

- There are two parts. Part A carries 20 marks and Part B carries 30 marks.
- **Part A will be collected after 30 minutes of the start of examination.**
- Attempt all questions. Usual notations are used. No queries will be entertained during the conduct of examination.
- **If you use a theorem, make sure to state it.**

Part B

1. (a) Define analytic function. Check for analyticity of the following complex functions at $z = 0$ [1 + 2 +2]
(i) $f(z) = 2z + (\overline{z})^2$ (ii) $g(z) = |z|^2$.
Give reasons for your answer.

Ans **Analytic function:** A function $f(z)$ is said to be analytic at a point $z_1 \in \mathbb{C}$ if $f(z)$ is differentiable at every point of some neighborhood of z_1.

(i)

$$f(z) = 2z + (\overline{z})^2 = (2x + i2y) + (x - iy)^2 = (x^2 + 2x - y^2) + i(2y - 2xy) \quad (1)$$

Then,

$$u = (x^2 + 2x - y^2) \quad (2)$$

$$v = (2y - 2xy) \quad (3)$$

$u_x = 2 + 2x, v_y = 2 - 2y$
It is clear that C-R equations does not satisfy, *i.e.* $u_x \neq v_y$ in the neighborhood of $z = 0$.

(ii)

$$g(z) = |z|^2 = x^2 + y^2 \quad (4)$$

$$u = (x^2 + y^2) \quad (5)$$

$$v = 0 \tag{6}$$

Thus, $u_x = 2x, v_y = 0, u_y = 2y, v_x = 0$

It is clear that C-R equations does not satisfy, *i.e.* $u_x \neq v_y$ and $u_y \neq -v_x$ in the neighborhood of $z = 0$.

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(b) Show that $u = xe^x \cos y - ye^x \sin y$ is harmonic. If u is harmonic, find v such that $f(z) = u + iv$ is analytic, and $f(z)$ in terms of z. [1+3+1]

Ans 1b u is harmonic as it satisfies laplace equation $u_{xx} + v_{yy} = 0$

Now using Cauchy Riemann equations

$$v_y = u_x = e^x \cos y + xe^x \cos y - e^x y \sin y \tag{7}$$

Integrating,

$$v = xe^x \sin y + ye^x \cos y + F(x) \tag{8}$$

Differentiate w.r.t. x, we get

$$v_x = xe^x \sin y + e^x \sin y + ye^x \cos y + F'(x) \tag{9}$$

Using C-R equations, we can also get

$$v_x = -u_y = xe^x \sin y + e^x \sin y + ye^x \cos y \tag{10}$$

From (9) and (10), we get $F'(x) = 0$ so $F(x) = c$.

Hence, $v = ye^x \cos y + xe^x \sin y + c$

As $f(z) = u + iv$, Putting $y = 0$, $f(x) = u(x, 0) + iv(x, 0)$

Replacing x by z, we get $f(z) = u(z, 0) + iv(z, 0) = ze^z + ic$

2. (a) Evaluate the integral $\displaystyle\oint_\gamma \frac{\cosh z}{z^2(z+2)} dz$, where γ is [2 + 3]

   i. $|z - 2| = 1$ traveled counterclockwise.

   ii. $|z - 1| = 2$ traveled clockwise.

Ans Function $f(z) = \frac{\cosh z}{z^2(z+2)}$ have two points of singularity $z = 0, -2$. (i) In the region $|z - 2| = 1$, $f(z)$ does not have singularity, i.e., $f(z)$ is analytic in the closed curve $\gamma : |z-2| = 1$. Then, by The Cauchy-Goursat Theorem $\displaystyle\oint_\gamma \frac{\cosh z}{z^2(z+2)} dz = 0$

Note: Statement of theorem is required to support the argument.

Cauchy-Goursat Theorem: Let $f(z)$ be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then $\oint_C f(z)dz = 0$.

(i) In region $|z-1| = 2$, $f(z) = \frac{\cosh z}{z^2(z+2)}$ has singularity at $z = 0$.
As $z = 0$ is a pole of order 2, using Cauchy Integral Formula (state Cauchy Integral Formula if used)

$$\oint_\gamma \frac{\cosh z}{z^2(z+2)} dz = -2\pi i f'(0) = \frac{\pi i}{2}$$

(b) Prove that an analytic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying $|f(z)| \leq 3$ for any $z \in \mathbb{C}$, must be constant. [5]

Ans By hypothesis, $|f(z)| \leq 3$ for any $z \in \mathbb{C}$.
Let z_1 be an arbitrary complex number and $\Gamma = \{z : z = Re^{i\theta}, 0 \leq \theta \leq 2\pi\}$, where $|z_1| < R < \infty$. Then, by Cauchy integral formula, we have

$f(z_1) - f(0) = \frac{1}{2\pi i} \oint_\Gamma \{\frac{1}{z-z_1} - \frac{1}{z}\} f(z)dz = \frac{z_1}{2\pi i} \oint_\Gamma \frac{f(z)}{z(z-a)} dz$
Thus, for each fixed z_1, we have
$|f(z_1) - f(0)| \leq |\frac{z_1}{2\pi i}| \oint_\Gamma |\frac{f(z)}{z(z-a)}| dz \leq \frac{3|z_1|}{R-|z_1|}$

Which approaches to zero as $R \to \infty$. Thus, $f(z_1) = f(0)$ for each $z_1 \in \mathbb{C}$. Hence, $f(z)$ is constant.

Alternate sol. Given that $f(z)$ is analytic. Additionally, $|f(z)| \leq 3$ for any $z \in \mathbb{C}$, i.e., $f(z)$ is bounded. For an analytic and bounded function, with any choice of z_1 and R, we can apply Cauchy inequality when $n = 1$ and get

$|f'(z_1)| \leq \frac{3}{R}$, where R is arbitrary large.
$|f'(z_1)|$ approaches to zero as $R \to \infty$ for any choice of z_1. This means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function.

3. (a) Find the Laurent series expansions of the function $\dfrac{2z+4}{z^2+4z+3}$ about the point $z = 1$ for the following regions. [3+3]

i. $2 < |z-1| < 4$

ii. $|z-1| > 4$

Ans

$$f(z) = \frac{2z+4}{z^2+4z+3} = \frac{1}{z+1} + \frac{1}{z+3} = \frac{1}{2+z-1} + \frac{1}{4+z-1} \tag{11}$$

(i) Laurent series expansions in $2 < |z-1| < 4$ is

$$\begin{aligned} f(z) &= \frac{1}{z-1}\left(1+\frac{2}{z-1}\right)^{-1} + \frac{1}{4}\left(1+\frac{z-1}{4}\right)^{-1} && (12)\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{(z-1)^{n+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{4^{n+1}} && (13)\\ &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{2^n}{(z-1)^{n+1}} + \frac{(z-1)^n}{4^{n+1}}\right) && (14) \end{aligned}$$

(ii) Laurent series expansions in $|z-1| > 4$ is

$$\begin{aligned} f(z) &= \frac{1}{z-1}\left(1+\frac{2}{z-1}\right)^{-1} + \frac{1}{z-1}\left(1+\frac{4}{z-1}\right)^{-1} && (15)\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{(z-1)^{n+1}} + \sum_{n=0}^{\infty}(-1)^n \frac{4^n}{(z-1)^{n+1}} && (16)\\ &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{2^n}{(z-1)^{n+1}} + \frac{4^n}{(z-1)^{n+1}}\right) && (17) \end{aligned}$$

(b) Evaluate $\oint_\gamma \frac{z^{-2}e^{2z} + 3z^3 \sin\frac{2}{z} - 2z^3\cos\frac{1}{2z}}{2z}dz$, where γ is a unit circle with center at origin ($\gamma : |z| = 1$). [1+1+1+1]

Ans The function $\frac{z^{-2}e^{2z} + 3z^3 \sin\frac{2}{z} - 2z^3\cos\frac{1}{2z}}{2z}$ has singularity at $z = 0$ inside the circle γ. Now,

$$\oint_\gamma \frac{z^{-2}e^{2z} + 3z^3 \sin\frac{2}{z} - 2z^3\cos\frac{1}{2z}}{2z}dz = \frac{1}{2}\oint_\gamma \frac{e^{2z}}{z^3}dz + \frac{3}{2}\oint_\gamma z^2 \sin\frac{2}{z}dz + \oint_\gamma z^2\cos\frac{1}{2z}dz \tag{18}$$

Then, by using residue theorem

$$\oint_\gamma \frac{e^{2z}}{z^3}dz = 2\pi i\{\texttt{Residue of } \frac{e^{2z}}{z^3}\ at z = 0\} = 4\pi i \tag{19}$$

[1]

$$\oint_\gamma z^2 \sin\frac{2}{z}dz = 2\pi i\{\texttt{Residue of } z^2\sin\frac{2}{z}\ at z = 0\} = -\frac{8\pi i}{3} \tag{20}$$

[1]

$$\oint_\gamma z^2 \cos\frac{1}{2z}dz = 2\pi i\{\texttt{Residue of } z^2\cos\frac{1}{2z}\ at z = 0\} = 0 \tag{21}$$

[1]

Then, $\oint_\gamma \frac{z^{-2}e^{2z} + 3z^3 \sin\frac{2}{z} - 2z^3\cos\frac{1}{2z}}{2z}dz = 2\pi i + -4\pi i + 0 = -2\pi i$ [1]