Lecture 3: Continuity of Functions of several variables

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Theorem 3.1 *Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at point $(x_0, y_0) \in D$. Let $E \subset \mathbb{R}$ such that it contains the range of function f. Suppose $g : E \subset \mathbb{R}$ is continuous at the point $f(x_0, y_0)$. Then $g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0).*

Example 3.2 *Discuss the continuity of the function $f(x, y) = \cos(xy^2)$.*

Solution: Since the points at which we are suppose to check continuity is not specified, we are expected to check the continuity at every point of the largest possible domain of the given function.

First of all note that xy^2 is a real number for any point (x, y) in the plane, and cosine function is define for all real numbers hence the largest possible domain of the given function is $\mathbb{R}^2$.

Now let $h(x, y) = xy^2$ and $g(t) = \cos t$. Then $f = g \circ h$. We know that g is continuous everywhere on $\mathbb{R}$ hence we need to check continuity of h only.

Let $(x_0, y_0) \in \mathbb{R}^2$ be given. Suppose $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x_0, y_0)$, i.e., $x_n \to x_0$ and $y_n \to y_0$. Limit theorems for sequences gives us

$$\begin{aligned} y_n \to y_0 &\implies y_n^2 \to y_0 \\ &\implies x_n y_n^2 \to x_0 y_0^2 \end{aligned}$$

Hence $h(x_n, y_n) \to h(x_0, y_0)$. So h is continuous at (x_0, y_0). Since (x_0, y_0) was arbitrary, h is continuous everywhere.

Hence Theorem 3.1 implies that f is continuous everywhere on the plane. ■

Recall the $\epsilon - \delta$ definition of continuity for function of one variable

Definition 3.3 *Let $D \subseteq \mathbb{R}$. Consider a function $f : D \to \mathbb{R}$ and a point $c \in D$. We say that f is continuous at c if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$|f(x) - f(c)| < \epsilon \text{ for all points } x \in D \text{ with } |x - c| < \delta.$$

We want to have a similar definition for functions of two-variables. Let us see what are the changes we need.

Definition 3.4 *Let $D \subset \mathbb{R}^2, f : D \to \mathbb{R}, (x_0, y_0) \in D$. We say that f is continuous at (x_0, y_0) if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$|f(x,y) - f(x_0,y_0)| < \epsilon \textit{ for all points } (x,y) \in D \textit{ with } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

Example 3.5 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous at (x_0, y_0) and $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are defined as*

$$\phi(x) = f(x, y_0), \quad \forall x \in \mathbb{R}, \quad \psi(y) = f(x_0, y), \quad \forall y \in \mathbb{R},$$

Show that ϕ is continuous at x_0 and ψ is continuous at y_0..

Solution:

$\epsilon - \delta$ **Approach:** We establish the continuity of ϕ at x_0. Let $\epsilon > 0$ be given. By continuity of f at (x_0, y_0) there exists a $\delta > 0$ such that

$$|f(x,y) - f(x_0,y_0)| < \epsilon, \quad \text{for all } (x,y) \text{ with } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

In particular the points (x, y_0) with $|x - x_0| < \delta$ satisfies above,

$$|f(x,y_0) - f(x_0,y_0)| < \epsilon, \quad \text{for all } x \text{ with } \sqrt{(x-x_0)^2 + (y_0-y_0)^2} = |x - x_0| < \delta$$

Hence we have

$$|\phi(x) - \phi(x_0)| < \epsilon, \quad \text{for all } x \text{ with } |x - x_0| < \delta.$$

Similarly one can prove the continuity of ψ at y_0.

Sequential Approach: We establish the continuity of ψ at y_0. Let $(y_n)_{n \geq 1}$ be a sequence in $\mathbb{R}$ such that $y_n \to y_0$. Therefore $(x_0, y_n) \to (x_0, y_0)$. Now continuity of f at (x_0, y_0) implies that $f(x_0, y_n) \to f(x_0, y_0)$, i.e., $\psi(y_n) \to \psi(y_0)$. This proves the continuity of ψ at y_0.

Similarly one can prove the continuity of ϕ at x_0. ∎

Example 3.6 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and (x_0, y_0) is a point. Suppose $\phi, \psi : \mathbb{R} \to \mathbb{R}$ defined as*

$$\phi(x) = f(x, y_0), \quad \forall x \in \mathbb{R}, \quad \psi(y) = f(x_0, y), \quad \forall y \in \mathbb{R}.$$

If ϕ is continuous at x_0 and ψ is continuous at y_0, is it true that f is continuous at (x_0, y_0)?

Solution: f need not be continuous at (x_0, y_0).

Continuity of ϕ at x_0 implies that if (x_n, y_n) is a sequence along the path $y = y_0$, which converges to (x_0, y_0) then $f(x_n, y_n)$ will converge to $f(x_0, y_0)$ and continuity of ψ at y_0

implies that if (x_n, y_n) is a sequence along the path $x = x_0$, which converges to (x_0, y_0) then $f(x_n, y_n)$ converges to $f(x_0, y_0)$. And this is not enough to conclude the continuity of f, as it demands that if (x_n, y_n) is a sequence along any path which converges to (x_0, y_0) then $f(x_n, y_n)$ must converge to $f(x_0, y_0)$.

Now for better illustration we look at the following concrete example.

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We have proved that f is not continuous at $(0,0)$. Now see that

$$\phi(x) = f(x,0) = \begin{cases} 0 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \equiv 0, \quad \psi(y) = f(0,y) = 0$$

Both are continuous at $x_0 = 0$ and $y_0 = 0$. ■

3.1 Limits

The set $\{(x,y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\}$ is called an open disk of radius r centered at (x_0, y_0).

The set $\{(x,y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq r\}$ is called a closed disk of radius r centered at (x_0, y_0).

The set $\{(x,y) \in \mathbb{R}^2 : 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\}$ is called a punctured disk of radius r centered at (x_0, y_0).

Definition 3.7 *Suppose that $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a function. Let (x_0, y_0) be a point in the plane (not necessarily in D) such that every punctured disk centered at (x_0, y_0) contains at least one point from the set D (domain of f). We say that the f has a limit as (x,y) approaches (x_0, y_0) if there exists $L \in \mathbb{R}$ such that for every sequence $((x_n, y_n))$ in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to L$. We then write*

$$f(x,y) \to L \text{ as } (x,y) \to (x_0,y_0) \text{ or } \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

Example 3.8 *If $D = \{(1,n) : n \in \mathbb{Z}\}$ and (x_0, y_0) is any point in the plane then D does not satisfy the condition mentioned in Definition 3.7. Hence we can not talk about limit for functions with D as a domain.*

If $D = \{(x,x) : 0 \leq x < 1\}$ and $(x_0, y_0) = (1,1)$ then it does satisfy the condition mentioned in Definition 3.7. Hence we can talk about limit at $(1,1)$ for functions with D as a domain.

If D is an open disk with radius r centered at $(0,0)$ and (x_0, y_0) is any point on circle $x^2 + y^2 = r^2$ then it does satisfy the condition mentioned in Definition 3.7. Where as if (x_0, y_0) is any point belong to the complement of closed disk with radius r centered at $(0,0)$ then D does not satisfy the condition mentioned in Definition 3.7.