Lecture 27: Marginal PMFs and Random Vector with density

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Example 27.1 *Suppose X be a random variable taking two values 1 and 2, and let Y be a random variable that assume four values $1, 2, 3, 4$. Their joint probabilities are given by the following table.*

$X \backslash Y$	1	2	3	4
1	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{8}$

Determine the marginal pmf of random variables X and Y.

Solution:

$$f_X(1) = \sum_{y=1}^{4} f(1, y) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}$$

$$f_X(2) = \sum_{y=1}^{4} f(2, y) = \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

$$f_Y(1) = \sum_{x=1}^{2} f(x, 1) = \frac{1}{4} + \frac{1}{16} = \frac{5}{16} = f_Y(3)$$

$$f_Y(2) = \sum_{x=1}^{2} f(x, 2) = \frac{1}{8} + \frac{1}{16} = \frac{3}{16} = f_Y(4)$$

Clearly X is uniformly distributed but Y is not. ■

The marginal pmfs f_X and f_Y do not completely determines the joint pmf of X and Y. Indeed there are many different joint pmfs that have same marginal pmfs.

Example 27.2 *Define a joint pmf by*

$$f(0, 0) = \frac{1}{12}, f(1, 0) = \frac{5}{12}, f(0, 1) = f(1, 1) = \frac{3}{12}$$

The marginal pmf of Y *is* $f_Y(0) = f_Y(1) = \frac{1}{2}$. *The marginal pmf of* X *is* $f_X(0) = \frac{1}{3}, f_X(1) = \frac{2}{3}$.

Define a joint pmf by

$$f(0,0) = f(0,1) = \frac{1}{6}, f(1,0) = f(1,1) = \frac{1}{3}$$

The marginal pmf of Y *is* $f_Y(0) = f_Y(1) = \frac{1}{2}$. *The marginal pmf of* X *is* $f_X(0) = \frac{1}{3}, f_X(1) = \frac{2}{3}$.

Thus, it is hopeless to try to determine the joint pmf from the knowledge of only the marginal pmfs. The marginals does not capture the information how X and Y are interrelated.

27.1 Random Vectors with density

Definition 27.3 *A random vector* (X, Y) *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *is called absolutely continuous if there is a nonnegative function* $f(x, y)$ *defined on* $\mathbb{R}^2$*, called the joint pdf of* (X, Y) *(sometimes just joint density of* (X, Y)*), such that*

$$P((X,Y) \in S) = \iint_S f(x,y)dxdy,$$

for every Borel subset S *of* $\mathbb{R}^2$.

Example of Borel subsets of $\mathbb{R}^2$: polygons, disks, ellipses, and finite or countably unions of such shapes. Open set, closed set, their (finite or counatble) union or intersections etc.

In particular, the probability that the value of (X, Y) falls within an rectangle $[a, b] \times [c, d]$ is

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)dxdy.$$

and can be interpreted as the volume of region lying below the surface $z = f(x, y)$ and above the rectangle $[a, b] \times [c, d]$.

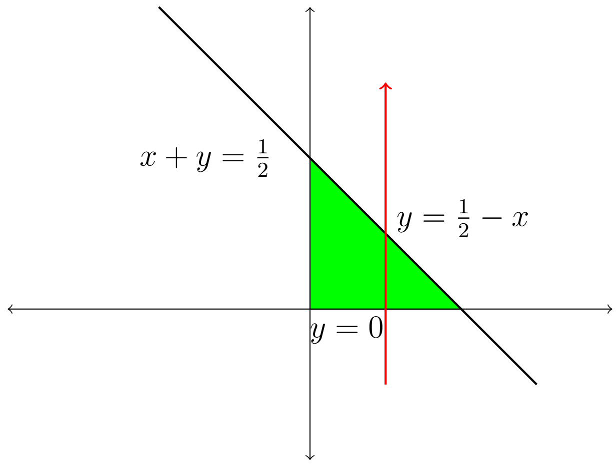
Example 27.4 *The joint probability density function of* X *and* Y *is*

$$f(x,y) = \begin{cases} 2 & \text{if} \quad x > 0, y > 0, 0 < x + y < 1 \\ 0 & \quad \textit{elsewhere} \end{cases}$$

Then find $P\left(X + Y < \dfrac{1}{2}\right)$.

Solution:

Define $A := \left\{(x, y) \in \mathbb{R}^2 : x + y < \dfrac{1}{2}\right\}$. Then



$$
\begin{aligned}
P\left(X + Y < \frac{1}{2}\right) &= \iint_A f(x, y)dxdy \\
&= \int_0^{\frac{1}{2}} \left(\int_0^{\frac{1}{2}-x} 2dy\right) dx = \int_0^{\frac{1}{2}} [2y] \Big|_0^{\frac{1}{2}-x} dx \\
&= \int_0^{\frac{1}{2}} (1 - 2x)dx = \left[x - x^2\right] \Big|_0^{\frac{1}{2}} = \frac{1}{4}
\end{aligned}
$$

■

27.2 Properties of Joint Density

Let f be the joint pdf of random variable X and Y. Then

$$P(-\infty < X < \infty, -\infty < Y < \infty) = P(\Omega \cap \Omega) = 1. \tag{27.1}$$

But by definition we have

$$P(-\infty < X < \infty, -\infty < Y < \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy \tag{27.2}$$

Hence joint pdf integrate to 1 on the entire plane.

Theorem 27.5 (Characterization of joint pdf) *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying*

(a) $f(x,y) \geq 0$ *for all* $(x,y) \in \mathbb{R}^2$.

(b) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1.$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ *and a random vector* (X,Y) *defined on it such that* f *is the joint pdf of* (X,Y).

Example 27.6 *Let* $f(x,y) = ce^{-\frac{x^2-xy+4y^2}{2}}, x, y \in \mathbb{R}$. *Find the value of* c *such that* f *is a joint pdf.*

Solution:

If f is a joint pdf then $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)dtds = 1.$

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2-xy+4y^2}{2}}dxdy \\
&= c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2-2\cdot x\cdot\frac{y}{2}+\frac{y^2}{4}-\frac{y^2}{4}+4y^2}{2}}dxdy \\
&= c\int_{-\infty}^{\infty} e^{-\frac{15y^2}{8}}\left(\int_{-\infty}^{\infty} e^{-\frac{(x-\frac{y}{2})^2}{2}}dx\right)dy \\
&= c\int_{-\infty}^{\infty} e^{-\frac{15y^2}{8}}\left(\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du\right)dy \quad (\text{put } x = u + \tfrac{y}{2}) \\
&= c\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\frac{15y^2}{8}}dy \quad \left(\because \int_{-\infty}^{\infty}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}du = 1\right) \\
&= c\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\frac{2du}{\sqrt{15}} \quad (\text{put } y = \tfrac{2u}{\sqrt{15}}) \\
&= c\sqrt{2\pi}\frac{2}{\sqrt{15}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}du \\
&= c\sqrt{2\pi}\frac{2}{\sqrt{15}}\sqrt{2\pi}
\end{aligned}
$$

Hence $c = \dfrac{\sqrt{15}}{4\pi}$. ■