Lecture 5: Iterated Limits & Partial Derivatives

October 15, 2018

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5.1 Iterated Limits

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and (x_0, y_0) be a point in the plane. Then by iterated limits we mean the following limits:

$$\lim_{x \to x_0} \left[\lim_{y \to y_0} f(x,y) \right] \quad \text{and} \quad \lim_{y \to y_0} \left[\lim_{x \to x_0} f(x,y) \right].$$

In the first iterated limit, treat x as a fixed parameter we find limit of $f(x,y)$ as y approaches y_0, and this we do for each x in a deleted neighborhood of x_0. If this inside limit exists then we find limit as x approaches to x_0.

In the second iterated limit, for each y in a deleted neighborhood of y_0 we find limit if $f(x,y)$ as x approaches to x_0. If this limit exists then find limit as y approaches to y_0.

Example 5.1 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find iterated limits $\lim_{x \to 0} \left[\lim_{y \to 0} f(x,y) \right]$ *and* $\lim_{y \to 0} \left[\lim_{x \to 0} f(x,y) \right]$. *(if exists). Also find* $\lim_{(x,y) \to (0,0)} f(x,y)$ *(if exists).*

Solution: For $x \neq 0$, we compute $\lim_{y \to 0} f(x,y)$.

$$\lim_{y \to 0} \frac{x^2y^2}{x^2y^2 + (x-y)^2} = 0 \quad \text{(by substitution)}$$

This implies $\lim_{x \to 0} \left[\lim_{y \to 0} f(x,y) \right] = 0$.

Similarly for $y \neq 0$, we compute $\lim_{x \to 0} f(x, y)$.

$$\lim_{x \to 0} \frac{x^2 y^2}{x^2 y^2 + (x - y)^2} = 0 \quad \text{(by substitution)}$$

This implies $\lim_{x \to 0} \left[\lim_{y \to 0} f(x, y) \right] = 0$.

If $(x, y) \to (0, 0)$ along x-axis, then $f(x, y) \to 0$, whereas if $(x, y) \to (0, 0)$ along straight line $y = x$ then $f(x, y) \to 1$. Hence double limit does not exists. ■

Hence both the iterated limits may exists and they are equal also but double limit may not exist.

We will use the following result subsequently.

Example 5.2 *Show that* $\lim_{x \to 0} \sin \frac{1}{x}$ *does not exists.*

Solution: Take a sequence $a_k = \frac{2}{(4k+1)\pi}, k = 0, 1, 2, \cdots$, then $a_k \to 0$. But $\sin \frac{1}{a_k} = \sin(4k+1)\frac{\pi}{2} = 1$ for all k. Hence $\sin \frac{1}{a_k} \to 1$.

Take a sequence $b_k = \frac{2}{(4k-1)\pi}, k = 1, 2, \cdots$, then $b_k \to 0$. But $\sin \frac{1}{b_k} = \sin(4k-1)\frac{\pi}{2} = -1$ for all k. Hence $\sin \frac{1}{b_k} \to -1$.

Therefore limit does not exist. ■

Example 5.3 *Consider the function*

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Find iterated limits $\lim_{x \to 0} \left[\lim_{y \to 0} f(x, y) \right]$ *and* $\lim_{y \to 0} \left[\lim_{x \to 0} f(x, y) \right]$. *(if exists). Also find* $\lim_{(x,y) \to (0,0)} f(x, y)$ *(if exists).*

Solution: For $x \neq 0$, we compute $\lim_{y \to 0} f(x, y)$.

$$\lim_{y \to 0} x \sin \frac{1}{y} \text{ does not exists}$$

This implies $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$ does not exists.

Similarly for $y \neq 0$, we compute $\lim_{x\to 0} f(x,y)$.

$$\lim_{x\to 0} x\sin\frac{1}{y} = 0 \text{ (by substituion)}$$

This implies $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0$.

Now we show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$, i.e., $x_n \to 0$ and $y_n \to 0$. Now if $x_n = 0$ but $y_n \neq 0$ then $f(x_n, y_n) = 0$. If $x_n \neq 0$ and $y_n \neq 0$ then $f(x_n, y_n) = x_n \sin\frac{1}{y_n}$. If $x_n \neq 0$ and $y_n = 0$ then $f(x_n, y_n) = 0$. Combining all these possibilities together, we have

$$0 \leq |f(x_n, y_n)| \leq |x_n|, \quad \forall n.$$

Hence by sandwich theorem, we have $f(x_n, y_n) \to 0$. ■

Double limit may exist but iterated limits may not exists.

5.2 Partial Derivatives

Definition 5.4 *Let $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be any function. Let $(x_0, y_0) \in D$ be an interior point of D, i.e., there exist some $r > 0$ such that $B_r(x_0, y_0) \subseteq D$. We say that the partial derivative of f with respect to x at (x_0, y_0) exists if the limit*

$$\lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \quad \textit{exists.}$$

This limit is denoted by $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$. Similarly, We say the partial derivative of f with respect to y at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) := \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

provided limit in right hand side exist.

These partial derivatives are also called the first-order partial derivatives or simply the first partials of f at (x_0, y_0).

In practice, finding the partial derivative of f with respect to x amounts to taking the derivative of $f(x,y)$ as a function of x, treating y as a constant. Indeed if $\phi(x) = f(x, y_0)$, then $f_x(x_0, y_0)$ exists if and only if ϕ is differentiable at x_0. Similarly if $\psi(x) = f(x_0, y)$, then $f_y(x_0, y_0)$ exists if and only if ψ is differentiable at y_0.

As a consequence, the sum rule, difference rule, constant multiple rule, Product Rule, the Quotient Rule, the Chain Rule, and so on-are still valid for partial derivatives because partial differentiation is just the differentiation that we already know, applied one variable at a time.

Example 5.5 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find both the partial derivatives of f at every point (if it exists).

Solution: Partial derivatives at $(0,0)$: Note that

$$f(x,0) = 0, \forall x \in \mathbb{R} \implies f_x(x,0) = 0, \forall x \in \mathbb{R}$$

In particular, $f_x(0,0) = 0$.

Similarly,

$$f(0,y) = 0, \forall y \in \mathbb{R} \implies f_y(0,y) = 0, \forall y \in \mathbb{R}$$

In particular, $f_y(0,0) = 0$.

Partial derivatives at $(x_0, y_0) \neq (0,0)$: If (x_0, y_0) is a non-zero point, then we can find an open disk $B_r(x_0, y_0)$ centered at (x_0, y_0) such that this open disk does not contain $(0,0)$. Hence function f is given by

$$f(x,y) = \frac{xy}{x^2+y^2}, \ \forall (x,y) \in B_r(x_0, y_0)$$

Recall the quotient rule of differentiation of real-valued function of one variable: If f and g are differentiable at x_0 and $g(x_0) \neq 0$ then

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{[g(x_0)]^2}$$

If $(x,y) \neq (0,0)$ then $x^2+y^2 > 0$. Hence we can compute both partial derivatives by quotient rule:

$$f_x(x,y) = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}, f_y(x,y) = \frac{x(x^2+y^2) - 2y(xy)}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2},$$

At all non-zero points. ■