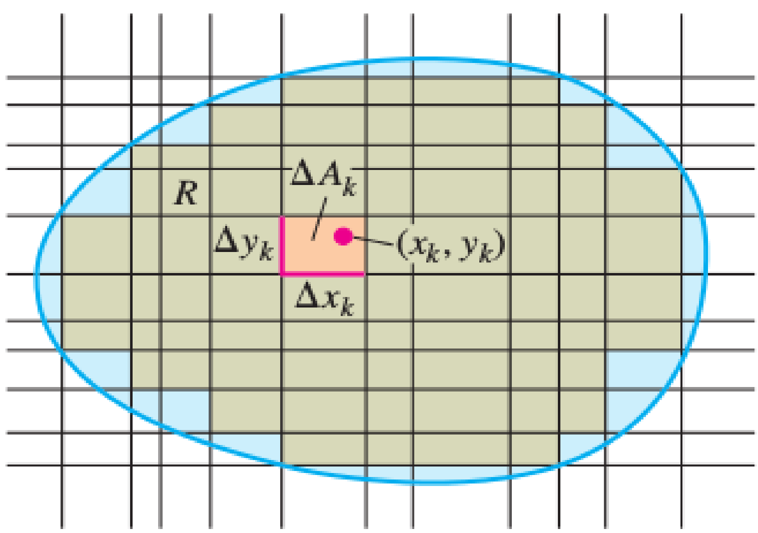
Lecture 16: Double Integrals over General Regions

November 5, 2016

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

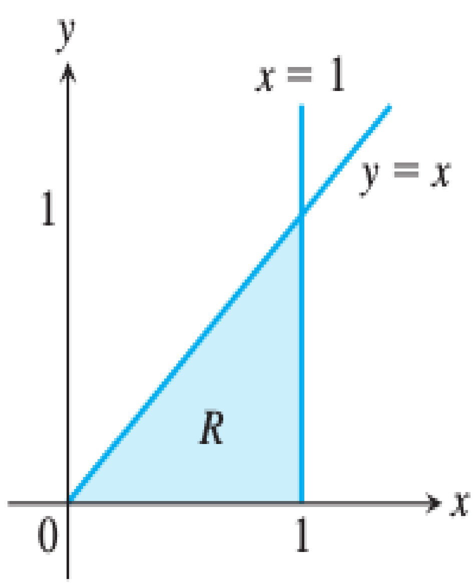


How to evaluate them ?

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$
2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 2 Calculate
$$\iint_R \frac{\sin x}{x}\, dA,$$
where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.



Solution: The Region R can be identified as: $0 \leq y \leq 1$, $y \leq x \leq 1$. Draw a line parallel to x-axis. Then by Fubini's theorem
$$\iint_A \frac{\sin x}{x} dA = \int_0^1 \int_y^1 \frac{\sin x}{x} dx dy$$

Now we run into a problem because $\frac{\sin x}{x}$ has no closed from of antiderivative. So let us change the order of integration. The same region R can be identified as: $0 \le x \le 1$, $0 \le y \le x$. Draw a line parallel to y-axis. Then by Fubini's theorem

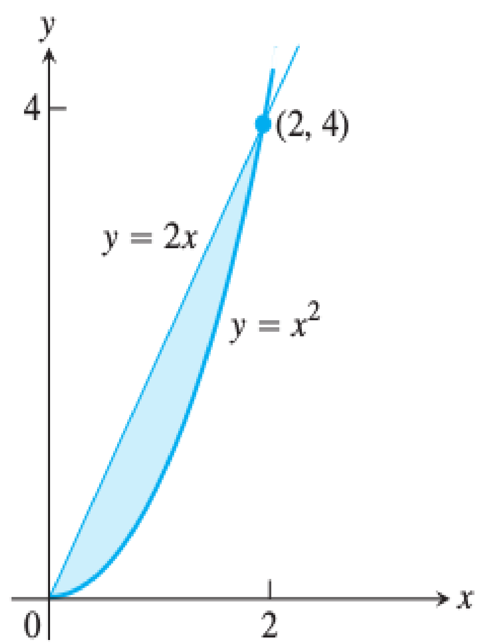
$$\begin{aligned}\iint_A \frac{\sin x}{x} dA &= \int_0^1 \int_0^x \frac{\sin x}{x} dy dx \\ &= \int_0^1 \frac{\sin x}{x} [y]_{y=0}^{y=x} dx \\ &= \int_0^1 \sin x dx \\ &= -\cos x|_{x=0}^{x=1} = 1 - \cos 1\end{aligned}$$

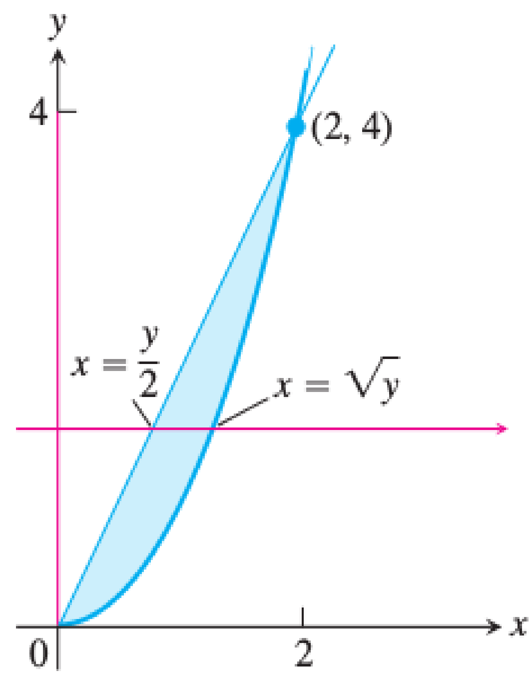
Example 16.1 *Sketch the region of integration for the integral*

$$\int_0^2 \int_{x^2}^{2x} (4x+2) dy dx.$$

and write an equivalent integral with the order of integration reversed.

Solution: It is clear that the region R is : $0 \le x \le 2, x^2 \le y \le 2x$.





To find limits for integrating in the reverse order, draw a line parallel to the x-axis.

$$\int_0^2 \int_{\frac{y}{2}}^{\sqrt{y}} (4x+2) dx dy.$$

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

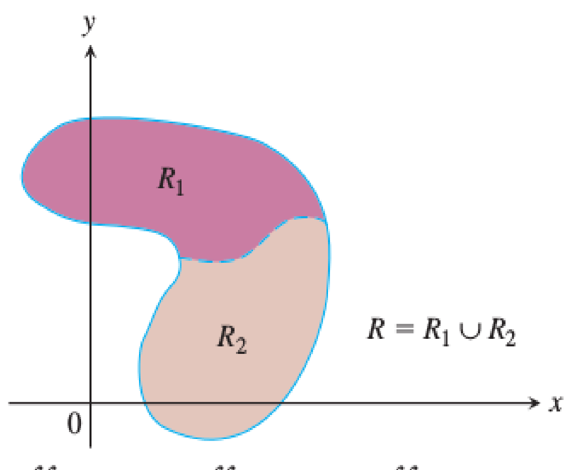
3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

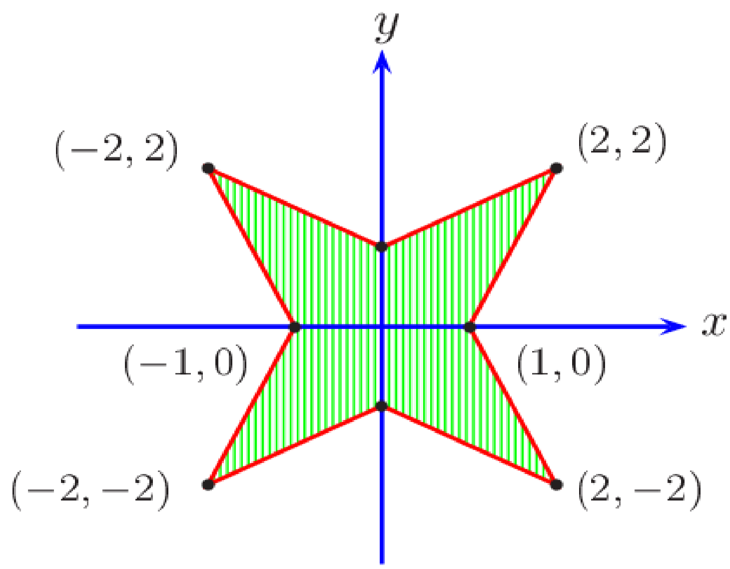
4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

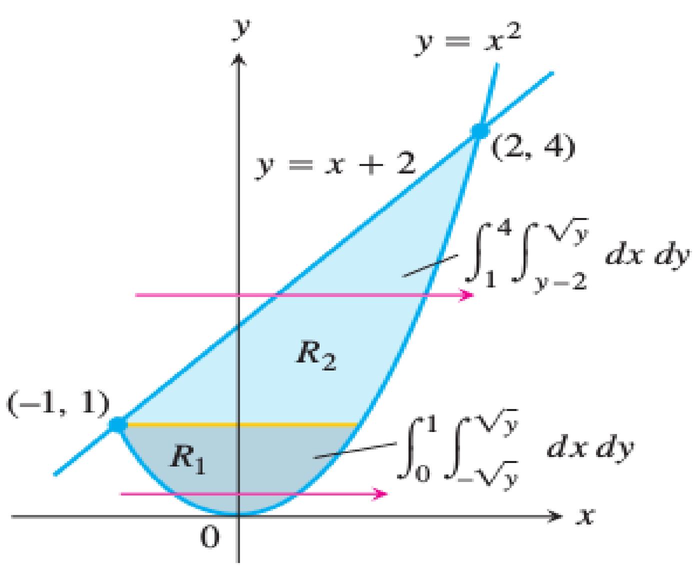


$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

This additivity property is very useful for evaluating double integral over the following regions.



y
$(-2, 2)$
$(2, 2)$
x
$(-1, 0)$
$(1, 0)$
$(-2, -2)$
$(2, -2)$



y
$y = x^2$
$(2, 4)$
$y = x + 2$
$\int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$
R_2
$(-1, 1)$
R_1
$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy$
x
0

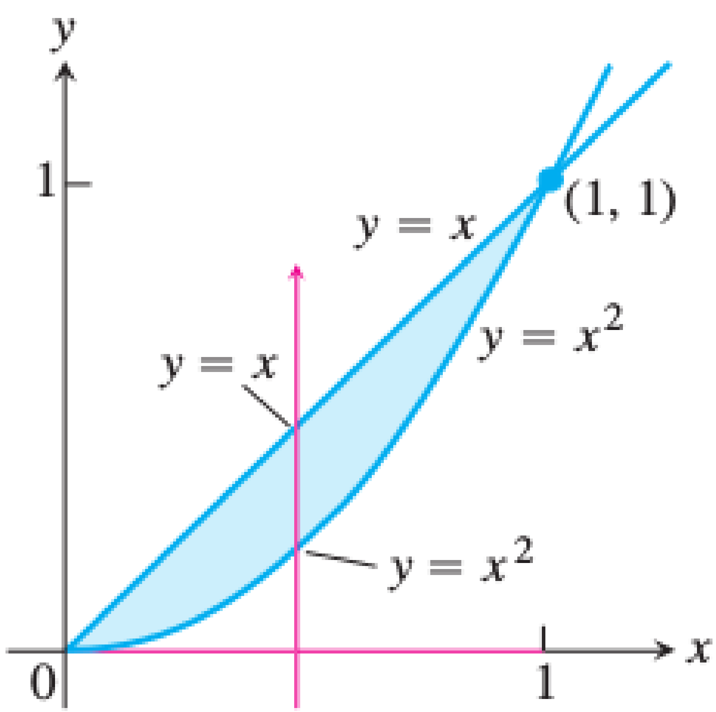
Areas of Bounded Regions in the Plane as a double integral

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, the Riemann sums reduce to $S_n = \sum_{k=1}^{n} \Delta A_k$ This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R. The area of a closed, bounded plane region R is the double integral

$$\iint_R dA$$

Example 16.2 *Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.*

Solutions:



Hence

$$Area = \int_0^1 \int_{x^2}^{x} dydx = \frac{1}{6}$$