Lecture 7: Partial Derivatives

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Definition 7.1 *Let $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be any function. Let $(x_0, y_0) \in D$ be an interior point of D, i.e., there exist some $r > 0$ such that $B_r(x_0, y_0) \subseteq D$. We say that the partial derivative of f with respect to x at (x_0, y_0) exists if the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \quad \textit{exists.}$$

This limit is denoted by $f_x(x_0, y_0)$ or $\dfrac{\partial f}{\partial x}(x_0, y_0)$. Similarly, We say the partial derivative of f with respect to y at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) := \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

provided limit in right hand side exist.

These partial derivatives are also called the first-order partial derivatives or simply the first partials of f at (x_0, y_0).

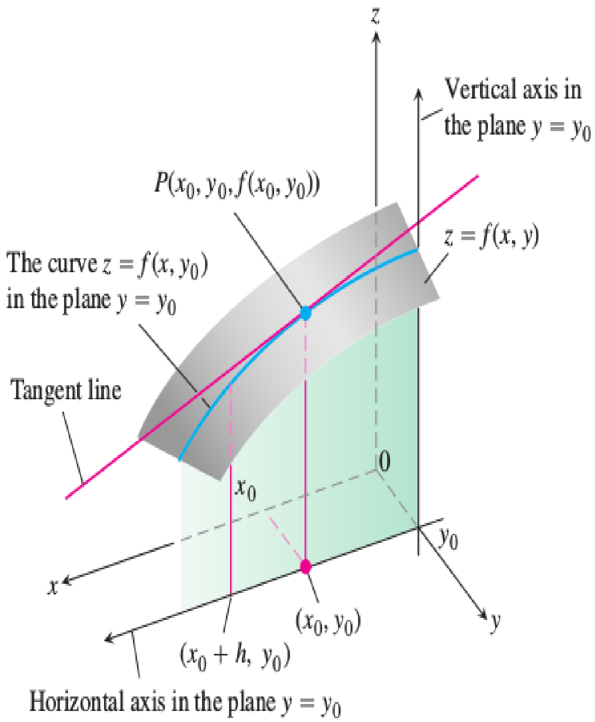
Definition 7.2 *If $f_x(x_0, y_0), f_y(x_0, y_0)$ exists, then the ordered pair $(f_x(x_0, y_0), f_y(x_0, y_0))$ is called the gradient of f at (x_0, y_0) and is denoted by $\nabla f(x_0, y_0)$. Thus*

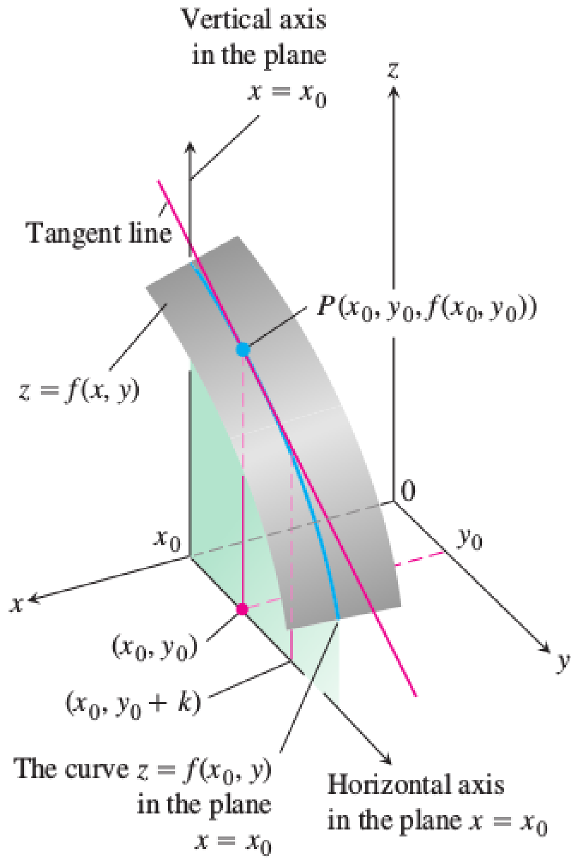
$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

Physical Interpretation of Partial Derivatives:

The partial derivative $f_x(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the x-axis, whereas $f_y(x_0, y_0)$ gives the rate of change in f at (x_0, y_0) along the y-axis. In practice, finding the partial derivative of f with respect to x amounts to taking the derivative of $f(x, y)$ as a function of x, treating y as a constant.

Geometrical Interpretation of Partial Derivatives:





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Example 7.3 *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find partial derivative of f *at* $(0,0)$.

Solution: For $h \neq 0$, we have

$$\frac{f(0+h,0)-f(0,0)}{h} = \frac{0-0}{h} = 0$$

Hence $f_x(0,0) = 0$. Similarly, for $k \neq 0$, we have

$$\frac{f(0,0+k)-f(0,0)}{k} = \frac{0-0}{k} = 0$$

Hence $f_y(0,0) = 0$. ∎

Remark 7.4 *We have seen that f is not continuous at $(0,0)$. So existence of partial derivatives does not imply the continuity. This tell us that partial derivative is not equivalent to existence of derivative of f.*

Example 7.5 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x,y) := \sqrt{x^2+y^2}$. Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$ except the origin; in fact, for any $(x_0,y_0) \neq (0,0)$,*

$$f_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2 + +y_0^2}} \quad \text{and } f_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2 + +y_0^2}}$$

To examine whether any of the partial derivatives exist at $(0,0)$, we look at $f(x,0) = \sqrt{x^2} = |x|$. We know that it is not differentiable at $x = 0$. Hence $f_x(0,0)$ does not exists. Similarly, to find $f_y(0,0)$ look at $f(0,y) = \sqrt{y^2} = |y|$, which is again not differentiable at 0.

Question: *Is f continuous at $(0,0)$?*
Answer: *Yes. We give two methods.*

1. *Note that $g(x,y) = x^2 + y^2 \geq 0$ is a polynomial function hence continuous everywhere. Also $h(t) = \sqrt{t}$ is a continuous function for all $t \geq 0$. Hence composition $f(x,y) = (h \circ g)(x,y)$ is continuous everywhere on the plane. So in particular, f is continuous at $(0,0)$.*

2. *Note that $g(x,y) = x^2 + y^2 \geq 0$ is a polynomial function hence continuous everywhere, therefore $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$. Now applying the "root rule" of limits of functions of two variable,*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt{g(x,y)} = \sqrt{0} = 0 = f(0,0)$$

   Hence f is continuous at $(0,0)$.

This example tells us continuity does not imply existence of partial derivatives.

Theorem 7.6 *If the partial derivatives of $f(x,y)$ exist throughout $B_r(x_0,y_0)$ for some $r > 0$ and if either f_x or f_y is bounded on the disk $B_r(x_0,y_0)$ then f is continuous at (x_0,y_0).*

Exercise 7.7 *Let f be as in Example 7.3. Show that none of the partial derivatives is bounded on any disk centered at $(0,0)$.*