Lecture 2: Sequences:II

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Exercise 2.1 *Is sequence $((-1)^n)$ convergent ? Justify your answer.*

Solution: The sequence is not convergent. To prove this, let us assume that it is convergent. Then

$$\exists a \in \mathbb{R}(\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|(-1)^n - a| < \epsilon))))$$

In particular, if we choose $\epsilon = \frac{1}{2}$, there exist n_1 such that we have

$$|(-1)^n - a| < \frac{1}{2}, \forall n \geq n_1$$

That is

$$|-1-a| = |a+1| < \frac{1}{2} \text{ and } |1-a| = |a-1| < \frac{1}{2}$$

That is $a \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$ and $a \in \left(\frac{1}{2}, \frac{3}{2}\right)$, which is absurd. ■

Exercise 2.2 *Prove or disprove: The sequence (n) is convergent.*

Solution: Statement is false. The sequence (n) is divergent. Let us assume contrary, that is we assume sequence (n) is convergent. Then

$$\exists a \in \mathbb{R}(\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|n - a| < \epsilon))))$$

In particular, if we choose $\epsilon = 1$, there exist n_1 such that we have

$$|n - a| < 1, \forall n \geq n_1$$

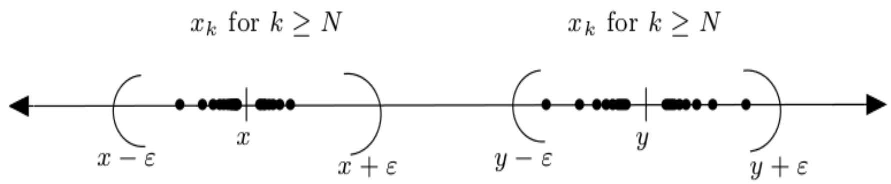
That is $n \in (a-1, a+1)$ for all $n \geq n_1$, which is a contradiction to Archimedian property. ■

Let us recall the definition of a convergent sequence. We say that a real sequence (a_n) is convergent if

$$\exists a \in \mathbb{R}(\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|a_n - a| < \epsilon))))$$

As far as the definition is considered, we are not demanding the uniqueness of a but at least one such a must exists. That's all our expectations.

An observant reader must have noticed that in the cases when (a_n) is convergent, the moment we guessed a possible limit say a, we stopped looking for other real numbers b such that $a_n \to b$. Why did we do so? Is it possible for a sequence a_n to converge to two distinct real numbers a and b? The following picture must convince you that it is not possible.



In fact, unless we prove the uniqueness of the limit of sequence it is not legitimate to write $\lim_{n\to\infty} a_n = a$, because which a we mean here. So let prove the following proposition.

Proposition 2.3 *A convergent sequence has a unique limit.*

Proof: Suppose $a_n \to a$ as well as $a_n \to b$. Suppose $b \neq a$, Then $\epsilon = \frac{|a-b|}{2} > 0$. Since $a_n \to a$, there is $n_1 \in \mathbb{N}$ such that $|a_n - a| < \epsilon \ \forall n \geq n_1$, and since $a_n \to b$, there is $n_2 \in \mathbb{N}$ such that $|a_n - b| < \epsilon \ \forall n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$|a-b| \leq |a - a_{n_0}| + |a_{n_0} - b| < \epsilon + \epsilon = |a-b| \implies |a-b| < |a-b|.$$

which is a contradiction. ■

Definition 2.4 *1. A sequence (x_n) is said to be bounded above if there is $\alpha \in \mathbb{R}$ such that $x_n \leq \alpha$ for all $n \in \mathbb{N}$. For example $(-n)$ is bounded above by zero.*

2. A sequence (x_n) is said to be bounded below if there is $\beta \in \mathbb{R}$ such that $x_n \geq \beta$ for all $n \in \mathbb{N}$. For example (n) is bounded below by 1.

3. The sequence (x_n) is said to be bounded if it is bounded above as well as bounded below. For example $\left(\frac{1}{n}\right)$, (1) are bounded.

Exercise 2.5 *Show that sequence (x_n) is bounded if and only if there is $\gamma \in \mathbb{R}$ such that $|x_n| \leq \gamma$ for all $n \in \mathbb{N}$.*

Solution: Assume sequence (x_n) is bounded, then there exists $\alpha, \beta \in \mathbb{R}$ such that $\beta \leq x_n \leq \alpha$ for all $n \in \mathbb{N}$. Take $\gamma = \max\{|\beta|, |\alpha|\}$. Then $x_n \leq \alpha \leq |\alpha| \leq \gamma$. Using the fact that for any real number x, $x \geq -|x|$, we have $-\gamma \leq -|\beta| < \beta \leq x_n$. converse is trivial. ■

The following result gives a necessary condition for the convergence of a sequence.

Theorem 2.6 *If $a_n \to a$ then (a_n) is bounded.*

Proof: Since $a_n \to a$ so for $\epsilon = 1$ there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq n_0$. Note $|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a| < 1$. This implies $|a_n| < 1 + |a|$ for all $n \geq n_0$. Now take $\gamma = \max\{|a_1|, |a_2|, \cdots, |a_{n_0-1}|, |a| + 1\}$ Then $|a_n| \leq \gamma$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. ∎

Remark 2.7

1. *The above condition is necessary for convergence of a sequence, but it is not sufficient. For example $((-1)^n)$ is bounded but not convergent.*
2. *Theorem 2.6 implies that an unbounded sequence is divergent.*