Solution 10

1. For any piecewise continuous function $f(x)$, the Legendre expansion is

$$f(x) = \sum_{n=0} a_n P_n(x), \qquad a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx; \quad x \in [-1,1]$$

(i) We can use the above formula. Alternately, using

$$1 = P_0, x = P_1, x^3 = (2P_3 + 3P_1)/5,$$

we get

$$f(x) = (5p_0 + 8p_1 + 2p_3)/5$$

(ii) Using the above formula,

$$a_0 = 1/4, a_1 = 1/2, a_2 = 5/16.$$

Thus,

$$f(x) = P_0/4 + P_1/2 + 5P_2/16 + \cdots$$

2. (i) $x = 1$ regular, $x = 0$ irregular (ii) $x = 0, x = -1/3$ regular

3. (i) $\lambda \leq 0$ leads to trivial solution. Thus, let $\lambda = p^2 > 0$. Then $y = c_1 \cos px + c_2 \sin px$. Using the BCs $c_1 = 0$ and $\sin p + p\cos p = 0$ or $p + \tan p = 0$. This has infinite number of roots (plot the curves $y = -x$ and $y = \tan x$). Thus, the eigen values are the roots of the above equation and the eigen functions are $y_p = \sin px$.

   (ii) This is Euler-Cauchy equation. Using the transformation $x = e^t$, we get $y'' + \lambda y = 0, \quad y(0) = y'(1) = 0$. Again for $\lambda \leq 0$ trivial solution. Thus, $\lambda = p^2 > 0$ and $y = c_1 \cos pt + c_2 \sin pt$. Using BCs, we get $c_1 = 0$ and $p = (n+1/2)\pi,\ n = 0, 1, 2, 3, \cdots$. Thus, $\lambda_n = [(2n+1)\pi/2]^2,\ n = 0, 1, 2, 3, \cdots$ and $y_n = \sin[(n+1/2)\pi \log x]$.

4. Multiplying by y and using integration by parts, we get

$$\lambda \int_a^b r y^2\, dx = \int_a^b q y^2\, dx + \int_a^b p y'^2\, dx - [pyy']_a^b$$

   (i) $p(a) = p(b) = 0 \implies [pyy']_a^b = 0$ (ii) $p(a) = p(b)$ with $y(b) = y(a),\ y'(b) = y'(a) \implies [pyy']_a^b = 0$ (iii) $y(a) - ky'(a) = y(b) + my'(b) = 0, k, m > 0, \implies [pyy']_a^b = -mp(b)y'(b)^2 - kp(a)y'(a)^2$.

   Thus, in (i) & (iii) $[pyy']_a^b = 0$ and in (ii) $[pyy']_a^b \leq 0$. Thus, λ is positive.

5. The BC $y(a) \neq y(b) \implies$ either $y(a) \neq 0$ or $y(b) \neq 0$ and $y'(a) \neq y'(b) \implies$ either $y'(a) \neq 0$ or $y'(b) \neq 0$. Also $y(a) = y'(a) = 0$ is not possible since then we get trivial solution only. Similarly $y(b) = y'(b) = 0$ is not possible. Thus, we can write the BCs as

$$c_1 y(a) + c_2 y'(a) = 0 \quad \text{and} \quad d_1 y(b) + d_2 y'(b) = 0$$

   where c_1 or c_2 not equal to zero and d_1 or d_2 not equal to zero.

   Let u and v are eigen functions corrsponding to an eigen value λ. Then $(pu')' + qu + \lambda ru = 0$ and $(pv')' + qv + \lambda rv = 0$. Multiplying the 1st by v and the second by u and subtracting we get $[pW(u,v)]' = 0$ where W is the Wronskian. Since u and v satisfy the above BCs, $W(u,v) = 0$ at $x = a$ and $x = b$. Thus, $pW(u,v) \equiv 0$ or $W(u,v) \equiv 0$. Hence u and v are linealy dependent.

Laplace Transformation

6.
$$\mathcal{L}\left(f(at)\right) = \int_0^\infty e^{-st} f(at)\,dt = \frac{1}{a}\int_0^\infty e^{-(s/a)\tau} f(\tau)\,d\tau = F(s/a)/a$$

7. (a)

$$\begin{aligned}\mathcal{L}([t]) &= \int_1^2 e^{-st}\,dt + 2\int_2^3 e^{-st}\,dt + 3\int_3^4 e^{-st}\,dt + \cdots \\ &= \frac{e^{-s}}{s}(1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots) = \frac{1}{s(e^s - 1)}\end{aligned}$$

(b)

$$\begin{aligned}\mathcal{L}(t^m) &= \frac{m!}{s^{m+1}} \implies \mathcal{L}(t^m \cosh bt) = \frac{1}{2}\mathcal{L}(e^{bt}t^m + e^{-bt}t^m) \\ &= \frac{(m!}{2}\left[\frac{1}{(s-b)^{m+1}} + \frac{1}{(s+b)^{m+1}}\right]\end{aligned}$$

(c)
$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2} \implies \mathcal{L}(e^t \sin at) = \frac{a}{(s-1)^2 + a^2}$$

(d) Use $\mathcal{L}[f(t)/t] = \int_s^\infty F(s)\,ds$. Now

$$\begin{aligned}\mathcal{L}(\sin at) &= \frac{a}{s^2 + a^2} \implies \mathcal{L}(\sin at/t) = a\int_s^\infty \frac{1}{s^2 + a^2} = \tan^{-1}(a/s) \\ \implies \mathcal{L}(e^t \sin at/t) &= \tan^{-1}\left(\frac{a}{s-1}\right)\end{aligned}$$

(e)

$$\mathcal{L}(\sin t/t) = \tan^{-1}(1/s) \implies \mathcal{L}(\cosh t \sin t/t) = \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{s-1}\right) + \tan^{-1}\left(\frac{1}{s+1}\right)\right]$$

(f)
$$\mathcal{L}[f(t)] = \int_0^\pi e^{-st} \sin 3t\,dt = \frac{3(1 + e^{-\pi s})}{s^2 + 9}$$

8. (a) Consider $g(t) = u(t) - u(t-\pi) + u(t-2\pi)\cos t = u(t) - u(t-\pi) + u(t-2\pi)\cos(t-2\pi)$

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{se^{-2\pi s}}{s^2 + 1}$$

(b) Consider $g(t) = [u(t-1) - u(t-2)]\cos(\pi t) = -u(t-1)\cos\pi(t-1) - u(t-2)\cos\pi(t-2)$

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = -\frac{s(e^{-s} + e^{-2s})}{s^2 + \pi^2}$$

9. (a) Use $\mathcal{L}(-tf(t)) = F'(s)$. Thus,

$$F'(s) = -\frac{a}{s^2 + a^2} \implies \mathcal{L}^{-1}[F'(s)] = -\sin at \implies f(t) = \frac{\sin at}{t}$$

(b)

$$F'(s) = \frac{2s}{s^2+1} - \frac{2}{s+1} \implies \mathcal{L}^{-1}[F'(s)] = 2(\cos t - e^{-t}) \implies f(t) = \frac{2(e^{-t} - \cos t)}{t}$$

(c)

$$F(s) = \frac{s+2}{(s^2+4s-5)^2} = \frac{1}{12}\left(\frac{1}{(s-1)^2} - \frac{1}{(s+5)^2}\right)$$

Thus,

$$f(t) = t\frac{e^t - e^{-5t}}{12}$$

(d)

$$\frac{se^{-\pi s}}{s^2+4} = e^{-\pi s}\mathcal{L}(\cos 2t) = \mathcal{L}(u(t-\pi)\cos[2(t-\pi)])$$

Thus,

$$\mathcal{L}^{-1}\left(\frac{se^{-\pi s}}{s^2+4}\right) = u(t-\pi)\cos 2t$$

(e)

$$\frac{(1-e^{-2s})(1-3e^{-2s})}{s^2} = \frac{1}{s^2} - \frac{4e^{-2s}}{s^2} + \frac{3e^{-4s}}{s^2}$$

Thus,

$$f(t) = t - 4u(t-2)(t-2) + 3(t-4)u(t-4)$$

10(a) Taking Laplace Transform on both sides and simplifying (Y(s)= $\mathcal{L}$[y(t)])

$$Y(s) = s/(s^2+4)^2 + 1/(s^2+4)$$

Using convolution [or any other technique]

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sin(2\tau)\cos(2(t-\tau))d\tau + \frac{\sin 2t}{2} \\ &= \frac{t\sin 2t}{4} + \frac{\sin 2t}{2} \end{aligned}$$

(b) Let $r(t) = 4u(t-0)t + 4u(t-1)(1-(t-1))$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+3s+2)Y(s) = R(s) \implies Y(s) = \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{s-1}{s^2(s+1)(s+2)}$$

Using partial fraction and solving we get

$$y(t) = -3 - e^{-2t} + 4e^{-t} + 2t + u(t-1)\left(5 + 3e^{-2(t-1)} - 8e^{-(t-1)} - 2(t-1)\right)$$

(c) Let $r(t) = 8(u(t-0) - u(t-\pi))\sin t = 8u(t-0)\sin t + u(t-\pi)\sin(t-\pi)$. Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+9)Y(s) = R(s) + 4 \implies Y(s) = \frac{4}{s^2+9} + \frac{R(s)}{s^2+9}$$

We can explicitly write $R(s)$ and then use partial fraction technique. Otherwise, use convolution as follows

$$y(t) = \frac{4}{3}\sin 3t + \frac{1}{3}\int_0^t r(\tau)\sin 3(t-\tau)\,d\tau$$

Thus for $0 < t < \pi$, we get

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^t \sin\tau\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t + \sin t - \frac{1}{3}\sin 3t = \sin 3t + \sin t$$

and for $t > \pi$, we get [since $r(t) = 0$]

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^\pi \sin\tau\sin 3(t-\tau)\,d\tau + \frac{1}{3}\int_\pi^t 0\,\sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t$$