The LNM Institute of Information Technology, Jaipur

Mathematics - II (MidTerm) Part-B March 09, 2017

Max.Duration: 90 mins. Max.Marks: 20

Name: ____________________ Roll No.: __________ Signature:

Instructions: There are two parts. Part A carries 10 marks and Part B carries 20 marks. **Part A will be collected after 30 minutes of the start of examination.** Attempt all questions.

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1. If a vector space $\mathbb{V}$ is the set of all real valued continuous functions over the field of real number $\mathbb{R}$, then show that [2 + 2 Marks]

   (a) The set $\mathbb{W}$ of solutions of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$ **is a subspace** of $\mathbb{V}$.

   Sol Consider S is the solution space for the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$, Consider $y_1, y_2 \in S$. That means y_1 and y_2 satisy the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$.
   Verify that $y_1+\alpha y_2$ satisfy the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$ for all $\alpha \in \mathbb{R}$ and $y_1+\alpha y_2 \in S$ and hence S is a subspace of $\mathbb{V}$. [2]

   (b) The set $\mathbb{U}$ of solutions of the differential equation $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + y = 1$ is **not a subspace** of $\mathbb{V}$.

   Sol Clearly, the additive identity of $\mathbb{V}$ i.e. $y = 0$ is not a solution of the DE $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + y = 1$ and hence the solution set is not a subsapce of $\mathbb{V}$. [2]

2. Verify that the following defines an inner product in the vector space $\mathbb{P}(t) = \{a_0 + a_1t + a_2t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$: [2+3 Marks]

$$< f, g >= \int_{-1}^{1} f(t)g(t)\ dt, \tag{1}$$

where $f(t), g(t) \in \mathbb{P}(t)$. If (1) defines an inner product, apply the Gram-Schmitz Orthogonalization process to $\{1, t, t^2\}$ to find orthogonal basis with integer coefficient for $\mathbb{P}(t)$.

Sol For $f, g, h \in \mathbb{P}(t)$ and $a \in \mathbb{R}$

$$< af + g, h >= \int_{-1}^{1} (af(t) + g(t))h(t)\ dt = a < f, h > + < g, h >$$

$$< f, g >=< g, f >$$

[1]

For all $f \in \mathbb{P}(t), f \neq 0$,

$$< f, f >= \int_{-1}^{1} |f(t)|^2\ dt = 2\int_{0}^{1} |f(t)|^2\ dt > 0$$

as we are integrating a positive quantity from 0 to 1.
$< f, f >= \int_{-1}^{1} |f(t)|^2 \, dt = 0$ if and only if $f = 0$. [1]

Hence, $< \cdot, \cdot >$ is an inner product in the vector space $\mathbb{P}(t)$.

In order to find an orthogonal basis with integer coefficient for $\mathbb{P}(t)$, we apply the Gram-Schmitz Orthogonalization process.

Take $w_0 = 1$.

$$w_1 = t - \frac{< t, 1 >}{< 1, 1 >} 1 = t$$

[1]

$$w_2 = t^2 - \frac{< t^2, 1 >}{< 1, 1 >} 1 - \frac{< t^2, t >}{< t, t >} t = t^2 - 1/3$$

[1]

Multiply with 3 to obtain a vector with integer coefficient i.e. $\overline{w_2} = 3t^2 - 1$ and hence $\{1, t, 3t^2 - 1\}$ forms an orthogonal basis with integer coefficient for $\mathbb{P}(t)$ [1]

3. Find the row space, null space, and nullity of the following matrix [2 + 2 + 1 Marks]

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Ans $\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \sim \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

So, row space = span $\left\{ \begin{bmatrix} -3 \\ 6 \\ -1 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right\}$

Null space is given by the solution of $AX = 0$

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

It gives

$-3x_1 + 6x_2 + -x_3 + x_4 - 7x_5 = 0$
$0x_1 + 0x_2 + x_3 + 2x_4 - 2x_5 = 0$

Gives the following null space

$$\text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Nullity = dimension of nulll space = 3

4. For the initial value problem (IVP) [2 + 2 + 2 Marks]

$$\frac{dy}{dx} = y + y^2, \ y(0) = 1 \tag{2}$$

(a) Verify existence and uniqueness theorem.

(b) If existence and uniqueness theorem holds, find three successive approximations using Picard's iteration method, and compare with exact solution.

Ans: $f(x,y) = y+y^2$. Take some rectangle $D = \{(x,y) : |x-0| < a \ |y-1| < b\}$. Take $a = 1$, $b = 1$. Then $|f(x,y)| = |y+y^2| \le |y| + |y|^2 = 1+1 = 2 = k$. So $f(x,y)$ is bounded. Cleary $f(x,y)$ is also continuous in D. As continuous and bounded, it satisfies the condition of existence theorem. Now $\frac{\partial f}{\partial y} = 1+2y$, then $|\frac{\partial f}{\partial y}| = |1+2y| \le 3 = L$. Cleary $\frac{\partial f}{\partial y}$ is continuous and bounded. So IVP satisfies Existence and Uniquness in D. Hence unique solution exists in $|x| < \frac{1}{2}$.

Thus we can apply Picard's iteration method $y_{n+1} = y_0 + \int_{x_0}^{x} f[t, y_n(t)]dt = 1 + \int_0^2 (y_n(t) + y_n^2(t)]dt$. $y_1 = y_0 + \int_0^x (1+1)dt = 1+2x$, $y_2 = y_0 + \int_0^x [1+2x+(1+2x)^2]dt = 1 + 2x + 3x^2 + \frac{4x^3}{3}$, $y_3 = y_0 + \int_0^x [1 + 2x + 3x^2 + \frac{4x^3}{3} + (1 + 2x + 3x^2 + \frac{4x^3}{3})^2]dt = 1 + 2x + 3x^2 + \frac{13x^3}{3} + \cdots$

Again, $\frac{dy}{dx} - y = y^2$ or, $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 1$. Take $\frac{1}{y} = v$, then $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$. Then $\frac{dv}{dx} + v = -1$. I.F.$= e^{\int 1dx} = e^x$. So $ve^x = -\int e^x dx + c$, or, $\frac{e^x}{y} = -e^x + c$. Given $y(0) = 1$. Then $1 = -1 + c$ or, $c = 2$. So solution $\frac{e^x}{y} = -e^x + 2$ or, $y = \frac{1}{2e^{-x}-1} = [1 - \{2x - x^2 + \frac{x^3}{3} \cdots\}]^{-1} = 1 + (2x - x^2 + \frac{x^3}{3} - \cdots) + (2x - x^2 + \frac{x^3}{3} \cdots)^2 + \cdots = 1 + 2x + 3x^2 + \cdots$, whose first three terms are agree with the first three terms of y_3.