Lecture 8: Differentiability & Directional Derivative

October 24, 2018

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Proposition 8.1 (Necessary Condition for Differentiability) *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) then f is continuous at (x_0, y_0).*

Example 8.2 *We have seen that the function*

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$. Hence by Proposition 8.1, f is not differentiable at $(0,0)$.

Theorem 8.3 (Sufficient Condition for Differentiability) *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be such that f_x, f_y exists throughout some open disk $B_r(x_0, y_0) \subseteq D$. If either f_x or f_y is continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).*

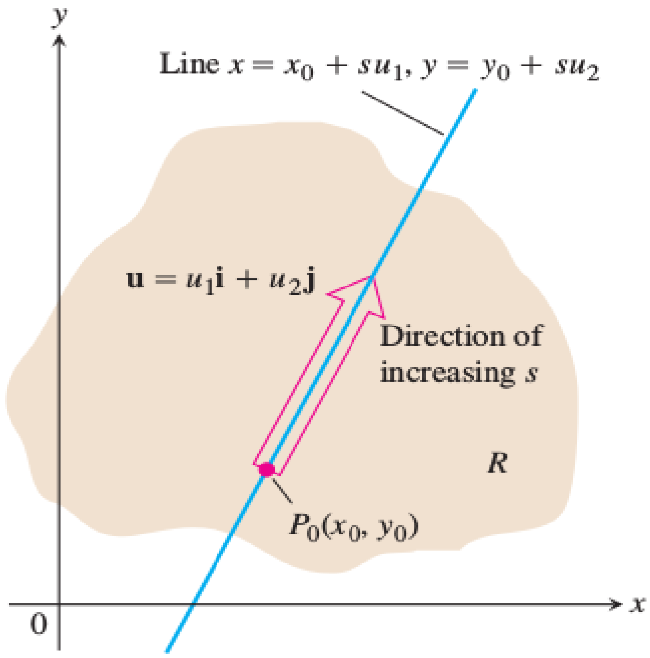
Example 8.4 *For the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = |xy|$, in order to check the differentiability at interior points of all four quadrant we may use Theorem 8.3.*

$$f(x,y) = \begin{cases} xy, & x > 0, y > 0 \\ -xy, & x < 0, y > 0 \\ xy, & x < 0, y < 0 \\ -xy, & x > 0, y < 0 \end{cases}$$

If (x_0, y_0) is an interior point of first quadrant then $f(x,y) = xy$ on some open disk centered at (x_0, y_0). Hence $f_x = y, f_y = x$ throughout that open disk. Both f_x and f_y are polynomial function in the open disk hence continuous everywhere in the open disk, hence by Theorem 8.3, f is differentiable at (x_0, y_0).

If (x_0, y_0) is an interior point of second quadrant then $f(x,y) = -xy$ on some open disk centered at (x_0, y_0). Hence $f_x = -y, f_y = -x$ throughout that open disk. Both f_x and f_y are polynomial function in the open disk hence continuous everywhere in the open disk, hence by Theorem 8.3, f is differentiable at (x_0, y_0).

Directional Derivatives



In order to get equation of the line passing through (x_0, y_0) in the direction of (u_1, u_2). First note that if $P = (x, y)$ and $P' = (x_0, y_0)$ then vector from P to P' is $(x - x_0, y - y_0)$. Now we want this vector to be parallel to given unit vector (u_1, u_2), that is there exist some $s \in \mathbb{R}$ such that

$$(x - x_0, y - y_0) = s(u_1, u_2).$$

If we very s over $\mathbb{R}$ are we get all the point on the straight line. So parametric equation of the line passing through (x_0, y_0) in the direction of (u_1, u_2) is

$$x = x_0 + su_1, y = y_0 + su_2,$$

where parameter s varies over set of all real numbers.

The notion of partial derivatives can be easily generalized to that of a directional derivative, which measures the rate of change of a function at a point along a given direction. We specify a direction by specifying a unit vector. Let $u = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$, i.e., $|u| = 1 \iff u_1^2 + u_2^2 = 1$.

Definition 8.5 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be any function.Let $u = (u_1, u_2)$ be a unit vector in $\mathbb{R}^2$. Let $(x_0, y_0) \in D$ be such that D contains a segment of the line passing through (x_0, y_0) in the direction of u. We define the directional derivative of f at (x_0, y_0) along u to be the limit*

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

provided this limit exists. It is denoted by $D_u f(x_0, y_0)$.

Note also that if $\mathbf{i} := (1,0)$ and $\mathbf{j} := (0,1)$, then $D_i f(x_0, y_0) = f_x(x_0, y_0)$ and $D_j f(x_0, y_0) = f_y(x_0, y_0)$.

Theorem 8.6 (Differentiability and Directional Derivatives) *Let* $D \subset \mathbb{R}^2$ *and let* (x_0, y_0) *be an interior point of* D. *If* $f : D \to \mathbb{R}$ *is differentiable at* (x_0, y_0), *then for every unit vector* $u = (u_1, u_2)$ *in* $\mathbb{R}^2$, *the directional derivative* $D_u f(x_0, y_0)$ *exists and moreover,*

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 8.7 1. *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(x,y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Find the directional derivative of f *at* $(0,0)$ *in the direction of the vector* $v = (1,1)$. *Also discuss the differentiability of* f *at* $(0,0)$.

Solution: *The given vector is not unit vector hence in order find directional derivative in the direction of the vector* $(1,1)$ *we find its unit vector.* $|v| = \sqrt{2}$ *Hence unit vector in direction of* v *would be the vector* $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. *Also recall that we have shown that function is not differentiable at* $(0,0)$, *hence we can not apply the theorem to calculate the directional derivative. For* $t \neq 0$, *we consider*

$$\frac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - f(0,0)}{t} = \frac{\frac{\frac{t^3}{2\sqrt{2}}}{\frac{t^4}{4} + \frac{t^2}{2}} - 0}{t} = \frac{\frac{\frac{t}{2\sqrt{2}}}{\frac{t^2+2}{4}}}{t} = \frac{\frac{t}{2\sqrt{2}} \times \frac{4}{t^2+2}}{t} = \frac{\sqrt{2}t}{t(2+t^2)} = \frac{\sqrt{2}}{2+t^2}$$

Hence $D_u f(0,0) = \frac{1}{\sqrt{2}}$. *Also recall that* $\nabla f(0,0) = (0,0)$. *Hence* $D_u f(0,0) \neq \nabla f(0,0) \cdot u$. *Therefore by Theorem 8.6,* f *is not differentiable at* $(0,0)$.

2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by* $f(x,y) := x^2 + y^2$. *Since* f *is a polynomial in* x, y *hence it is differentiable everywhere on plane. Hence by Theorem 8.6, given any unit vector* $u = (u_1, u_2)$ *in* $\mathbb{R}^2$ *and any* $(x_0, y_0) \in \mathbb{R}^2$, $D_u f(x_0, y_0)$ *exists and is equal to* $2x_0 u_1 + 2y_0 u_2$.