Lecture 9: More Differentiability & The Chain Rule

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Proposition 9.1 *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. If $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) then f is continuous at (x_0, y_0).*

Proof: Since f is differentiable at (x_0, y_0), i.e., there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha_1 h - \alpha_2 k|}{\sqrt{h^2+k^2}} = 0$$

Also note that $G(h,k) = \sqrt{h^2+k^2}$ is continuous everywhere hence $\lim\limits_{(h,k)\to(0,0)} G(h,k) = G(0,0) = 0$. Therefore,

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha_1 h - \alpha_2 k|}{\sqrt{h^2+k^2}} G(h,k) = 0$$

That is

$$\begin{aligned} &\lim_{(h,k)\to(0,0)} |f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha_1 h - \alpha_2 k| = 0 \\ \implies &\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha_1 h - \alpha_2 k = 0 \end{aligned}$$

Again $H(h,k) = \alpha_1 h + \alpha_2 k$ is continuous everywhere hence $\lim\limits_{(h,k)\to(0,0)} H(h,k) = H(0,0) = 0$. Therefore,

$$\begin{aligned} &\lim_{(h,k)\to(0,0)} [f(x_0+h, y_0+k) - f(x_0,y_0) - \alpha_1 h - \alpha_2 k] + H(h,k) = 0 \\ \implies &\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) - f(x_0,y_0) = 0 \end{aligned}$$

Which is saying that f is continuous at (x_0, y_0). ■

Example 9.2 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x,y) := \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Discuss the differentiability of f at $(0,0)$

Solution: Note that $f(x,0) = 0$ for all $x \in \mathbb{R}$. Hence

$$f_x(0,0) = 0$$

Similarly $f(0,y) = 0$ for all $y \in \mathbb{R}$. Hence

$$f_y(0,0) = 0$$

Now one should consider the following limit.

$$\begin{aligned}
\lim_{(h,k)\to(0,0)} \frac{|f(0+h,0+k) - f(0,0) - 0.h - 0k|}{\sqrt{h^2+k^2}} &= \lim_{(h,k)\to(0,0)} \frac{\frac{h^2|k|}{h^4+k^2}}{\sqrt{h^2+k^2}} \\
&= \lim_{(h,k)\to(0,0)} \frac{h^2|k|}{\sqrt{h^2+k^2}(h^4+k^2)}
\end{aligned}$$

If $(h,k) \to (0,0)$ along the path $k = mh$, then

$$\begin{aligned}
\lim_{(h,k)\to(0,0):k=mh} \frac{h^2|k|}{\sqrt{h^2+k^2}(h^4+k^2)} &= \lim_{h\to 0} \frac{h^2|mh|}{\sqrt{h^2+m^2h^2}(h^4+m^2h^2)} \\
&= \lim_{h\to 0} \frac{|m|}{\sqrt{1+m^2}(h^2+m^2)} \\
&= \frac{1}{|m|\sqrt{1+m^2}}.
\end{aligned}$$

Alternate Solution: We have proved that f is not continuous at $(0,0)$. Hence, f is not differentiable at $(0,0)$ by previous proposition.

Example 9.3 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x,y) := \sqrt{x^2+y^2}$. We have seen that partial derivatives of f does not exists at the origin. Hence f is not differentiable at origin.*

Theorem 9.4 (Sufficient Condition for Differentiability) *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be such that f_x, f_y exists throughout some open disk $B_r(x_0,y_0) \subseteq D$. If either f_x or f_y is continuous at (x_0,y_0) then f is differentiable at (x_0,y_0).*

Example 9.5 *Note that for f in Example 9.3, for all $(x,y) \neq (0,0)$,*

$$f_x(x,y) = \frac{x}{\sqrt{x^2++y^2}} \quad \textit{and } f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

It is left as an exercise to the students to show that f_x and f_y are continuous for all $(x,y) \neq (0,0)$. Therefore, by previous theorem f is differentiable at all non-zero points.

The Chain Rule

The Chain Rule for functions of a single variable says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, $w = f \circ g$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

For functions of two variables the Chain Rule has three forms, simply for the reason that there are three types of composites are possible.

1. If $f(x,y)$ is differentiable and $g(t)$ is differentiable, then the composite $w(x,y) = g(f(x,y))$ is a differentiable function of (x,y), and

$$\frac{\partial w}{\partial x} = \frac{dg}{dt}\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{dg}{dt}\frac{\partial f}{\partial y}$$

2. If $f(x,y)$ is differentiable and if $x = x(t), y = y(t)$ are is differentiable, then $z = f((x(t), y(t))$ is a is differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

3. If $f(x,y)$ is differentiable and if $x = x(u,v), y = y(u,v)$ are differentiable, then $z(u,v) = f(x(u,v), y(u,v))$ is a differentiable function of (u,v), and

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$