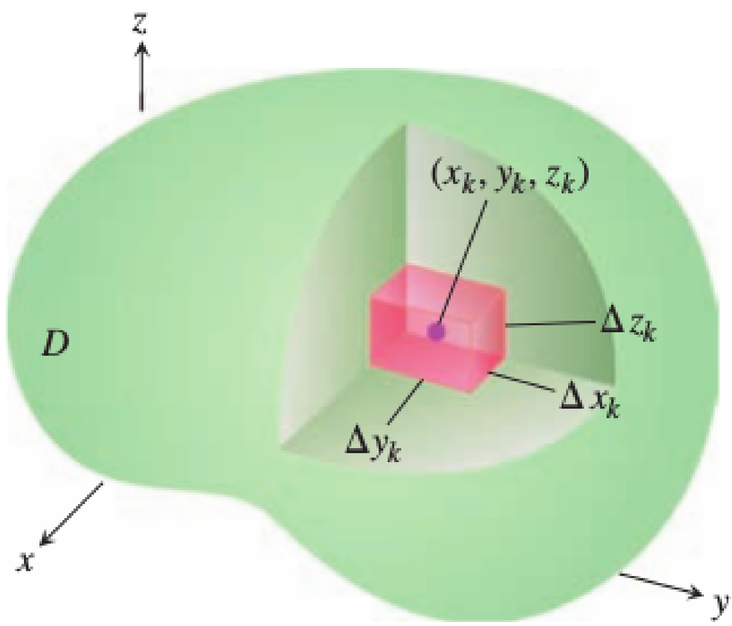
Lecture 18: Triple integral

November 18, 2018

Sunil Kumar Gauttam

Department of Mathematics, LNMIIT

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such as the region occupied by a solid ball, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes.



We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k) \Delta V_k$$

When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is integrable over D. The limit is called the triple integral of f

over D, written as

$$\iiint_D F(x,y,z)dV, \quad \text{or} \iiint_D F(x,y,z)dxdydz$$

Volume of a Region in Space If F is the constant function whose value is 1, then the sums reduce to

$$S_n = \sum_{k=1}^{n} \Delta V_k$$

As Δx_k, Δy_k and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$V = \iiint_D dV.$$

Evaluation of the triple integral

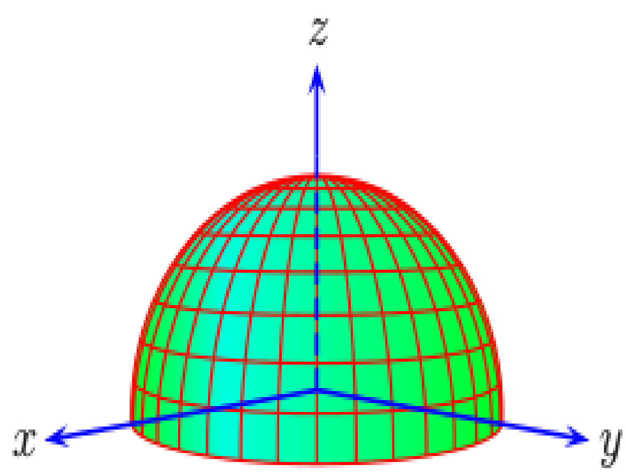
If $D = [a,b] \times [c,d] \times [p,q]$ then

$$\begin{aligned}\iiint_D F(x,y,z)dV &= \int_p^q \int_c^d \int_a^b F(x,y,z)dxdydz \\ &= \int_a^b \int_c^d \int_p^q F(x,y,z)dzdydx\end{aligned}$$

There are six possible orders altogether. Now we focus on elementary region in the space.

Example 18.1 *Find the volume of the region* $D = \{(x,y,z)|z \geq 0, x^2 + y^2 + z^2 \leq 1\}$.

Solution: It is clear that D is the region bounded by hemisphere.



So now we need to find the limit of integration. We integrate first with respect to z, then with respect to y, and finally with respect to x

1. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. In this case shadow is unit circle $x^2 + y^2 = 1$.

2. Find the z-limits of integration. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = 0$ and leaves at $z = \sqrt{1 - x^2 - y^2}$. These are the z-limits of integration.

3. Find the y-limits of integration. $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$

Hence desired volume is given by

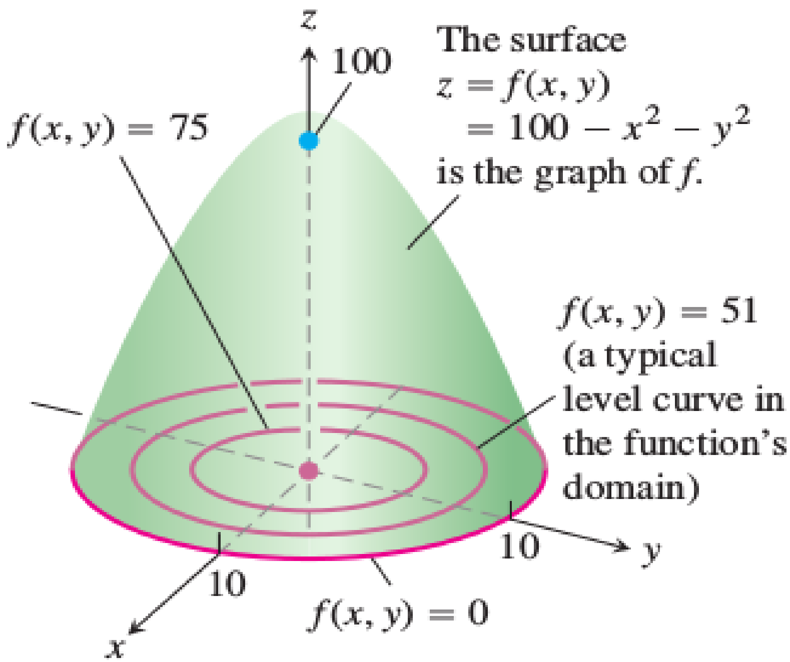
$$
\begin{aligned}
\iiint_D dV &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dzdydx\\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\; dydx\\
&= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\; dydx \quad \left[\int \sqrt{a^2-t^2}dt = \frac{t}{2}\sqrt{a^2-t^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right)\right]\\
&= \int_{-1}^{1} \frac{1-x^2}{2}\pi dx\\
&= \pi\int_0^1 (1-x^2)dx\\
&= \frac{2}{3}\pi
\end{aligned}
$$

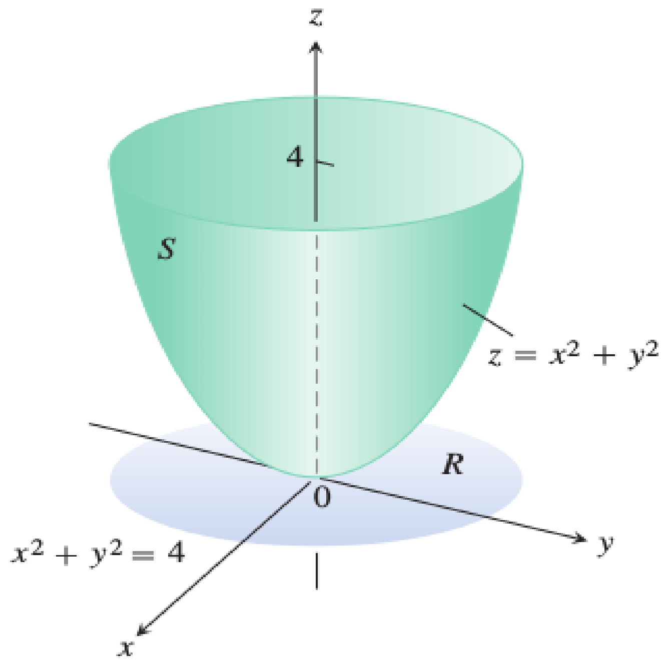
Alternatively, one can use the polar coordinates to evaluate the iterated integral

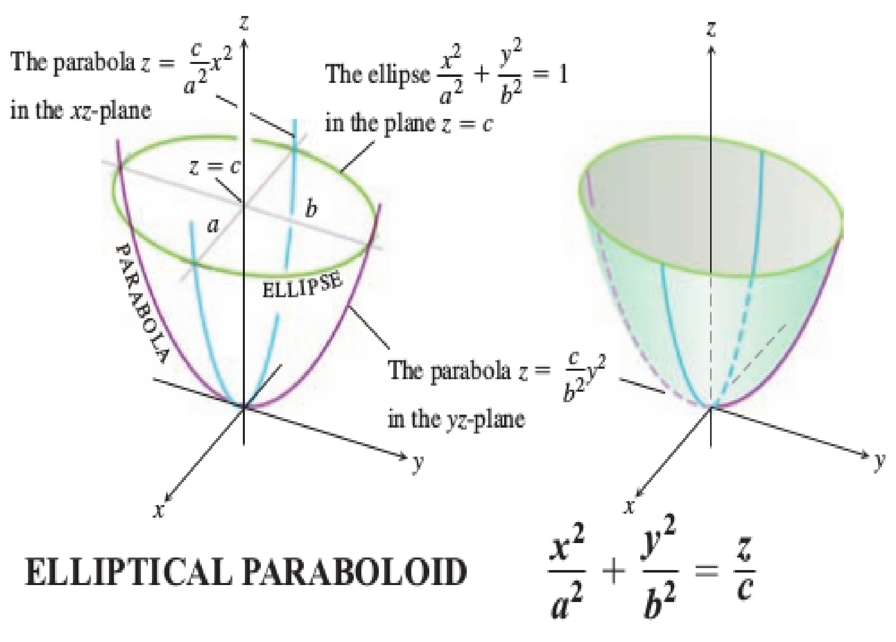
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\; dydx.$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\; dydx = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2} r dr d\theta = 2\pi \int_0^1 \sqrt{u}\frac{du}{2} = \frac{2}{3}\pi$$

In evaluating triple integrals, we need to identify limits of integration and for that the we need have to some idea about how geometrically surface looks like. We will not go in detail how to plot a surface but discuss the idea with some standard surfaces.

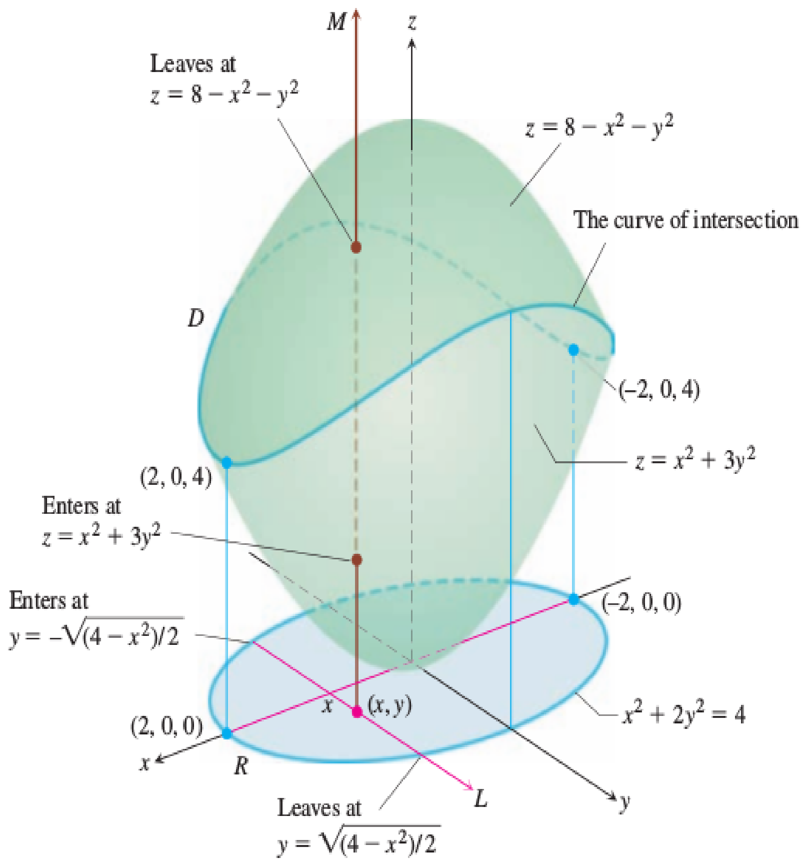

z
100
The surface
z = f(x, y)
= 100 − x² − y²
is the graph of f.
f(x, y) = 75
f(x, y) = 51
(a typical
level curve in
the function's
domain)
10
y
10
x
f(x, y) = 0





Example 18.2 *Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.*

Solution:



We find intersection curve of both the surfaces. Equate both the equations:

$$x^2 + 3y^2 = 8 - x^2 - y^2 \implies 2x^2 + 4y^2 = 8 \implies \frac{x^2}{4} + \frac{y^2}{2} = 1$$

The surfaces intersect on the elliptical cylinder $\frac{x^2}{4} + \frac{y^2}{2} = 1$, $z > 0$.

Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4-x^2)/2}$ and leaves at $y = \sqrt{(4-x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$
\begin{aligned}
V &= \iiint_D dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8-2x^2-4y^2)\,dy\,dx \\
&= \int_{-2}^{2}\left[(8-2x^2)y-\frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right) dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx \\
&= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u
\end{aligned}
$$

■