The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-III
Practice Problems Set #2

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1. If $\lim_{z\to z_0} f(z)$ exist, prove that it must be unique.

Hint: Use method of contradiction, If possible assume that two limits l_1 and l_2 exist, then $|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2| \le |l_1 - f(z)| + |f(z) - l_2| < \epsilon + \epsilon = 2\epsilon$.

2. If $f(z)$ and $g(z)$ are continuous at $z = z_0$, so also are
$(a)\ \ f(z) + g(z)$, $\quad (b)\ \ f(z)g(z)$, $\quad (c)\ \ \frac{f(z)}{g(z)}$ if $g(z_0) \ne 0$.

Hint: (a) For $\epsilon > 0$, there exist $\delta_1,\ \delta_2$ for which $|f(z) - f(z_0)| < \epsilon$ for $|z - z_0| < \delta_1$, $|g(z) - g(z_0)| < \epsilon$ for $|z - z_0| < \delta_2|$. Let δ = Minimum $(\delta_1, delta_2)$. Then for all z satisfying $|z - z_0| < \delta$, $|f(z) + g(z) - f(z_0) - g(z_0)| \le |f(z) - f(z_0))| + |g(z) - g(z_0)| < \epsilon + \epsilon = 2\epsilon$

(b) If $|f(z) - f(z_0)| < 1$ for $|z - z_0| < \delta_3$, then $|f(z)| - |f(z_0)| < |f(z) - f(z_0)| < 1$ i.e. $|f(z)| < |f(z_0)|+1 = P$, a positive constant. For $\epsilon > 0$, there exist $\delta_1,\ \delta_2$ for which $|f(z)-f(z_0)| < \frac{\epsilon}{2(1+|g(z_0)|)}$ for $|z-z_0| < \delta_1$, and $|g(z)-g(z_0)| < \frac{\epsilon}{2P}$ for $|z-z_0| < \delta_2$. Let δ = Minimum $(\delta_1, \delta_2, \delta_3)$. Then for all z satisfying $|z - z_0| < \delta$,$|f(z)g(z) - f(z_0)g(z_0)| = |f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)| \le |g(z) - g(z_0)||f(z)| + |g(z_0)||f(z) - f(z_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

(c) Since $g(z) \to g(z_0)$ as $z \to z_0$, $|g(z) - g(z_0)| < \frac{1}{2}|g(z_0)|$ for $|z - z_0| < \delta_2$. Then $|g(z_0)| \le |g(z)-g(z_0)|+|g(z)|$ i.e. $|g(z)| \ge \frac{1}{2}|g(z_0)|$. For $\epsilon > 0$, there exist δ_1 for which $|g(z)-g(z_0)| < \frac{\epsilon|g(z_0)|^2}{2}$. Let δ = Minimum (δ_1, δ_2) Then for all z satisfying $|z - z_0| < \delta$, $|\frac{1}{g(z)} - \frac{1}{g(z_0)}| = \frac{|g(z)-g(z_0)|}{|g(z)g(z_0)}| < \epsilon$. $\frac{1}{g(z)} \to \frac{1}{g(z_0)}$. Hence $\frac{f(z)}{g(z)} \to \frac{f(z_0)}{g(z_0)}$)

3. For what value of z each of the following functions are continuous?
$(a)\ \ f(z) = \dfrac{1}{z}$, $\quad (b)\ \ g(z) = \dfrac{z}{z^2+1}$.

Ans (a) on entire complex plane except $z = 0$ (b) on entire complex plane except $z = \pm i$.

4. Using the definition of limit, prove that
$(a)\ \ \lim_{z\to 1}(\mathrm{z}^2 - 3z + 3) = 1 \qquad (b)\ \ \lim_{z\to -i} \dfrac{iz}{3} = \dfrac{1}{3}$.

Hint: (a) $0 < |z-1| < \delta$. If $\delta \le 1$, then $|(z^2-3z+3)-1| = |(z-1)(z-2)| \le 2\delta$. Choose δ =minimum$(1, \frac{2}{\epsilon})$. Then for all $\epsilon > 0$, there exist δ such that for all z satisfying $0 < |z - 1| < \delta$, $|(z^2 - 3z + 3) - 1| < \epsilon$

(b) $0 < |z + i| < \delta$. If $\delta \le 1$, $|\frac{iz}{3} - \frac{1}{3}| = \frac{1}{3}|z + i|$ Choose δ =minimum$(1, 3\epsilon)$. Then for all $\epsilon > 0$, there exist δ such that for all z satisfying $0 < |z + i| < \delta$, $|\frac{iz}{3} - \frac{1}{3}| < \epsilon$.

5. Check the existence of

$$(a)\ \ \lim_{z\to z_0} \frac{\overline{z}}{z} \qquad (b)\ \ \lim_{z\to z_0} \frac{z^2}{|z|^2}, \qquad (c)\ \ \lim_{z\to z_0} \frac{[Rez - Imz]^2}{|z|^2}.$$

Hint: Check the limit along the path $y = mx$.

6. Show that the following functions are not continuous at $z = 0$

$$(a) \begin{cases} \dfrac{Im(z)}{|z|}, & \text{if } z \ne 0 \\ 0, & \text{if } z = 0. \end{cases} \qquad (b) \begin{cases} \dfrac{Re(z^2)}{|z|^2}, & \text{if } z \ne 0 \\ 0, & \text{if } z = 0. \end{cases} \qquad (c) \begin{cases} |z|^2, & \text{if } z \ne 0 \\ 1, & \text{if } z = 0. \end{cases}$$

Hint: in (a) and (b) limit does not exist; in (c) limiting vale is not equal to the value at the point.

7. Consider the following functions:
(a) $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $(z \ne 0)$, $f(0) = 0$, $\qquad (b)\ \ f(z) = \sqrt{|xy|}$, $\forall z \in \mathbb{C}$,

$$(c) \begin{cases} f(z) = \dfrac{\overline{z}^2}{z}, & \text{if } z \ne 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the above functions are continuous and satisfied CR equations at the origin but are not differentiable there.

Hint Use the definition of differentiability and C-R equations, to show that the function is not differentiable at $z = 0$.

8. $f(z)$ is analytic in a domain D and $|f(z)|$ is a nonzero constant in D, than $f(z)$ is constant in D.

Hint $uu_x + vv_x = 0,\ uu_y + vv_y = 0$. Using C-R equation, $uu_x - vu_y = 0,\ uu_y + vu_x = 0$. Multiplying u, v respectively and then adding, we get $(u^2 + v^2)u_x = 0$, i.e. $u_x = 0$. similarly show that $u_y = 0$. So u is a constant c_1. Similarly make equations for v_x, v_y and show that $v_x = 0$, $v_y = 0$. That is v is a constant c_2. Hence $f(z)$ is a constant $c_1 + ic_2$.

9. Prove that $|z|^4$ is differentiable, but not analytic at $z = 0$.

Hint Use $C - R$ equations to prove that $|z|^4$ is differentiable, but not analytic at $z = 0$.

10. Let $f(z) = u + iv$ be an analytic function defined in a domain D. Then, $f(z)$ must be constant if one of the following statements is true for all $z \in D$.
(a) $f(z)$ is real valued, (b) v is constant(c) u is constant(d) $\arg(f(z))$ is constant.

Hint (a) $v = 0$. Then using C-R equation $v_x = 0 = -u_y$ and $v_y = 0 = u_x$. So u is constant c. Hence $f(z) = u + iv = c$

(b) If v is constant, using C-R equation $v_x = 0 = -u_y$ and $v_y = 0 = u_x$. So u is constant. Hence $f(z) = u + iv$ is constant $(c_1 + ic_2)$

(c) If u is constant, using C-R equation $u_x = 0 = v_y$ and $u_y = 0 = -v_x$. So v is constant. Hence $f(z) = u + iv$ is constant $(c_1 + ic_2)$

(d) If $\arg(f(z))$ is constant, then $tan^{-1}\frac{v}{u}$ is constant. Them $\frac{uv_x - vu_x}{u^2+v^2} = 0$ and $\frac{uv_y - vu_y}{u^2+v^2} = 0$. Then following Q.8, we can show $u_x = 0 = u_y$ and $v_x = 0 = v_y$.

11. Prove that $f(z) = \sin x \cosh y - i \cos x \sinh y$ is nowhere analytic.

Hint Use $C - R$ equations. It will be sufficient to show that $f(z) = \sin x \cosh y - i \cos x \sinh y$ nowhere satisfies $C - R$ equations.

12. Show that the following functions $u(x, y)$ are harmonic functions. Determine their harmonic conjugate and find the corresponding analytic functions $f(z)$ in terms of z.

$$(a)\ u = 2x(1 - y) \qquad (b)\ u = e^{-x}(x \sin y - y \cos y) \qquad (c)\ u = \cos x \cosh y$$

Ans (a) $v = x^2 - y^2 + 2y$ (b) $f(z) = ize^{-z}$ (c) $f(z) = \cos z$

13. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^x(\cos y - \sin y)$, then find $f(z)$ in terms of z.

Hint $u_x - v_x = e^x(\cos y - \sin y)$, $u_y - v_y = e^x(-\cos y - \sin y)$. Using C-R equation $u_x - v_x = u_x + u_y = e^x(\cos y - \sin y)$, $u_y - v_y = u_y - u_x = e^x(-\cos y - \sin y)$. Then $u = e^x \cos y + c_1$. Using conjugate harmonic of u, $v == e^x \sin y + c_2$. So $f(z) = u + iv = e^z + c_1 + ic_2$