Lecture 6: Partial Derivatives

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Recall that in the last lecture for the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We have obtained $f_x(0,0) = 0 = f_y(0,0)$. Also, we have seen that f is not continuous at $(0,0)$. **So existence of partial derivatives at a point does not imply the continuity at that point**.

Example 6.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the norm function given by $f(x,y) := \sqrt{x^2+y^2}$. Then both the partial derivatives of f exist at every point of $\mathbb{R}^2$ except the origin; in fact, for any $(x_0, y_0) \neq (0,0)$,*

$$f_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2 + +y_0^2}} \quad \textit{and } f_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2 + +y_0^2}}$$

To examine whether any of the partial derivatives exist at $(0,0)$, we look at $f(x,0) = \sqrt{x^2} = |x|$. We know that it is not differentiable at $x = 0$. Hence $f_x(0,0)$ does not exists. Similarly, to find $f_y(0,0)$ look at $f(0,y) = \sqrt{y^2} = |y|$, which is again not differentiable at 0.

Question: *Is f continuous at $(0,0)$?*
Answer: *Yes. We give two methods.*

1. *Note that $g(x,y) = x^2 + y^2 \geq 0$ is a polynomial function hence continuous everywhere. Also $h(t) = \sqrt{t}$ is a continuous function for all $t \geq 0$. Hence composition $f(x,y) = (h \circ g)(x,y)$ is continuous everywhere on the plane. So in particular, f is continuous at $(0,0)$.*

2. *Note that $g(x,y) = x^2 + y^2 \geq 0$ is a polynomial function hence continuous everywhere, therefore $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0 = g(0,0)$. Now applying the "root rule" of limits of functions of two variable,*

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt{g(x,y)} = \sqrt{0} = 0 = f(0,0)$$

   Hence f is continuous at $(0,0)$.

This example tells us continuity does not imply existence of partial derivatives.

The set $B_r(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < r\}$ is called an open disk of radius r centered at (x_0, y_0). Suppose f is a real-valued function of two variables such that it's domain contains $B_r(x_0, y_0)$. We say f is bounded on $B_r(x_0, y_0)$ if there exists a positive constant $K > 0$ such that

$$|f(x, y)| \leq K, \quad \forall (x, y) \in B_r(x_0, y_0).$$

Theorem 6.2 *If both the first-order partial derivatives of $f(x, y)$ exist throughout $B_r(x_0, y_0)$ for some $r > 0$ and if either f_x or f_y is bounded on the disk $B_r(x_0, y_0)$ then f is continuous at (x_0, y_0).*

Example 6.3 *For the function*

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

we have

$$f_x(x, y) = \begin{cases} \dfrac{y^3 - x^2 y}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}, \quad f_y(x, y) = \begin{cases} \dfrac{x^3 - y^2 x}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Claim 6.4 *For any $r > 0$, f_x and f_y are not bounded on the disk $B_r(0, 0)$.*

To see this, suppose $r > 0$ is given. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < r$ for all $n \geq n_0$. Hence for all $n \geq n_0$, points $\left(\frac{1}{n}, 0\right), \left(0, \frac{1}{n}\right)$ belongs to $B_r(0, 0)$. But

$$f_x\left(0, \frac{1}{n}\right) = n, f_y\left(\frac{1}{n}, 0\right) = n.$$

Hence for any $K > 0$, we can find $n \geq \max\{n_0, K\}$ such that

$$f_x\left(0, \frac{1}{n}\right) = n \geq K, f_y\left(\frac{1}{n}, 0\right) = n \geq K.$$

This completes the proof of the claim.

Claim 6.5 *For each $(x_0, y_0) \neq (0, 0)$, there exists $r > 0$ such that both f_x and f_y are bounded on $B_r(x_0, y_0)$.*

To see this, suppose $(x_0, y_0) \neq (0,0)$ *be given. Then* $d := \sqrt{x_0^2 + y_0^2} > 0$ *is the distance of point* (x_0, y_0) *from the origin* $(0,0)$*. Choose* $r = \dfrac{d}{2} > 0$*. Then* $(0,0) \notin B_r(x_0, y_0)$*. Now for every point* $(x,y) \in B_r(x_0, y_0)$ *we have*

$$|x| - |x_0| \leq ||x| - |x_0|| \leq |x - x_0| \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} < r \implies |x| < r + |x_0|$$
$$|y| - |y_0| \leq ||y| - |y_0|| \leq |y - y_0| \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} < r \implies |y| < r + |y_0|$$

Sum of length two sides of any triangle is $\geq$ *the sum of the length of rest two sides. Hence*

$$\sqrt{x^2+y^2} + r > \sqrt{x^2+y^2} + \sqrt{(x-x_0)^2 + (y-y_0)^2} \geq \sqrt{x_0^2 + y_0^2} = d,$$
$$\implies \sqrt{x^2+y^2} > d - r = r > 0.$$

Hence for all $(x,y) \in B_r(x_0, y_0)$ *we have,*

$$|f_x(x,y)| = \left|\frac{y^3 - x^2 y}{(x^2+y^2)^2}\right| = |y|\frac{|y^2 - x^2|}{(x^2+y^2)^2} \leq (r + |y_0|)\frac{|y|^2 + |x|^2}{r^4} \leq (r + |y_0|)\frac{(r+|y_0|)^2 + (r+|x_0|)^2}{r^4},$$
$$|f_y(x,y)| = \left|\frac{x^3 - y^2 x}{(x^2+y^2)^2}\right| = |x|\frac{|x^2 - y^2|}{(x^2+y^2)^2} \leq (r + |x_0|)\frac{|y|^2 + |x|^2}{r^4} \leq (r + |x_0|)\frac{(r+|y_0|)^2 + (r+|x_0|)^2}{r^4}.$$

Higher-Order Partial Derivatives

Let $f : B_r(x_0, y_0) \to \mathbb{R}$ be a function such that such that $f_x(x_0, y_0)$ exists at every $(x_0, y_0) \in B_r(x_0, y_0)$, then we obtain a function from $B_r(x_0, y_0)$ to $\mathbb{R}$ given by $(x,y) \longmapsto f_x(x,y)$. It is denoted by f_x and called the partial derivative of f with respect to x on $B_r(x_0, y_0)$. In case f_x is defined on $B_r(x_0, y_0)$, we can consider its partial derivatives at any point of $B_r(x_0, y_0)$. The partial derivative of $f_x : \mathbb{R}^2 \to \mathbb{R}$ with respect to x at (x_0, y_0), if it exists, is denoted by $f_{xx}(x_0, y_0)$. Also, the partial derivative of f_x with respect to y at (x_0, y_0), if it exists, is denoted by $f_{xy}(x_0, y_0)$. We can similarly define $f_{yx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$. Collectively, these are referred to as the second-order partial derivatives or simply the second partials of f at (x_0, y_0). Among these, $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ are called the mixed (second-order) partial derivatives of f , or simply the mixed partials of f.

Example 6.6 *Consider* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(x,y) := \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Solution: Note that

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}, \quad f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

Hence it suffices to calculate f_x and f_y only along y-axis and x-axis, respectively. For any $y_0 \in \mathbb{R}$ we have

$$f_x(0,y_0) = \lim_{h\to 0} \frac{f(0+h,y_0) - f(0,y_0)}{h} = \lim_{h\to 0} \frac{hy_0\frac{h^2-y_0^2}{h^2+y_0^2} - 0}{h} = \lim_{h\to 0} y_0\frac{h^2-y_0^2}{h^2+y_0^2} = -y_0$$

Hence $\quad f_{xy}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$

Similarly for any $x_0 \in \mathbb{R}$ we have

$$f_y(x_0,0) = \lim_{k\to 0} \frac{f(x_0,0+k) - f(x_0,0)}{k} = \lim_{k\to 0} \frac{x_0k\frac{x_0^2-k^2}{x_0^2+k^2} - 0}{k} = \lim_{k\to 0} x_0\frac{x_0^2-k^2}{x_0^2+k^2} = x_0$$

Hence $\quad f_{yx}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$

Theorem 6.7 (Mixed Partials Theorem) *Let f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in some open disk with center (x_0,y_0). If either f_{xy} or f_{yx} are continuous at (x_0,y_0), then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.*

Using above theorem, we can say that for function f in Example 6.6, none of f_{yx} and f_{xy} is continuous at $(0,0)$.