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DAA Assignment-1

(1) $T(n) = T(\lceil n/2 \rceil) + 1$

~~Let~~ We know that

$T(n)$ for the lower bound $\lceil n/2 \rceil$ gives

~~T(n)~~

$T(\lceil n/2 \rceil) \leq c\log(\lceil n/2 \rceil)$ [By Master's Theorem]

Substituting:

$$
\begin{aligned}
T(n) &\leq c\log(\lceil n/2 \rceil) + 1 \\
&\leq c\log(n/2) + 1 \\
&= c\log n - c\log 2 + 1 \\
&= c\log n - c + 1 \\
&= c\log(n)
\end{aligned}
$$

Therefore we get the solⁿ as $O(\log(n))$.

(2) $T(n) = 2T(\lfloor n/2 \rfloor) + n$

By the upper bound of $\lfloor n/2 \rfloor$, we know that

$T(\lfloor n/2 \rfloor) \geq c(\lfloor n/2 \rfloor)\log(\lfloor n/2 \rfloor)$

Substituting:

$$
\begin{aligned}
T(n) &\geq 2c\lfloor n/2 \rfloor \log(\lfloor n/2 \rfloor) + n \\
&\geq cn\log(n/2) + n \\
&= cn\log n - cn\log 2 + n \\
&= c\log n - cn + n \\
&= cn\log n
\end{aligned}
$$

Therefore, for $c \leq 1$, solⁿ is $\Omega(n\log n)$.

By Q.1 & Q.2, as we get $O(\log n)$ bound & $\Omega(n\log n)$ bound for $T(\lfloor n/2 \rfloor)$; we can say that it is a solution of $\underline{O(n \log n)}$.

③ $T(n) = 2T(\sqrt{n}) + 1$

Let $m = \log n$

$$T(2^m) = 2T(2^{m/2}) + 1 \quad \rightarrow ①$$

again let $S(m) = T(2^m)$

Substituting in ① we get

$$S(m) = 2S(m/2) + 1$$

Taking both lower & upper bounds we get

$S(m/2) \leq k\log((m/2)) + 1$

$\leq k \log(m/2) + 1$

$= k\log m - k\log 2 + 1$

$= k\log m - k + 1$

$= k\log m$

$\Rightarrow O(\log m) \rightarrow$ (ii)

$S(m/2) \geq k(m/2)\log(m/2) + 1$

$\cancel{\geq k(m/2)}$

$\geq km\log(m/2) + 1$

$= km\log m - km\log 2 +$

$= km\log m - km + 1$

$= km\log m$

$\Rightarrow \Omega(m\log m) \rightarrow$ (iii)

By (ii) & (iii)

we can say that solution of $S(m/2)$ is $\cancel{O(m/2)}$. $O(\log m)$

on changing back then, we get the sol^n of $T(n)$ as $O(\log n)$.

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4. $T(n) = 3T(n/2) + n$

$= 3\left(3T(n/2^2) + n/2\right) + n$

$= 3^2T(n/2^2) + n(1 + 3/2)$

$= 3^2\left(3T(n/2^3) + n/2^2\right) + n(1+3/2)$

$= 3^3T(n/2^3) + n(1 + 3/2 + 3^2/2^2)$

$\vdots$

$= 3^iT(n/2^i) + n(1 + 3/2 + 3^2/2^2 + \dots + 3^{i-1}/2^{i-1})$

$\boxed{i = \log_2 n}$

$$\left\{ \text{from } \frac{n}{2^i} = 1 \quad n = 2^i \quad i = \log_2 n \right\}$$

$T(n) = 3^{\log_2 n}T(1) + n \cdot \sum_{k=0}^{\log_2 n - 1} (3/2)^k$

$\leq 3^{\log_2 n} \cdot O(1) + n \sum_{k=0}^{\infty} (3/2)^k$

$= \cancel{O(1)}\ O(n^{\log_2 3}) + 2n$

$= O(n) + 4n$

$\boxed{T(n) = O(n)}$

5. any polynomial over n with degree d is $O(n^d)$.

Let $p(n)$ be the polynomial over n.

Then,

$p(n) = a_d n^d + a_{d-1} n^{d-1} + \dots + a_1 n + a_0$ of degree d.

where $a_i \in R$ for $0 \leq i \leq d$ & $a_d \geq 0$.

Now for all ~~$n \geq n_0$~~ $n \geq n_0$;

$$c_1 n^d \leq p(n) \leq c_2 n^d$$

for the limit $\frac{p(n)}{n^d}$ where $n \to 0$;

the sequence $a_d + \frac{a_{d-1}}{n} + \cdots + \frac{a_0}{n^d}$ converges

to a_d.

for $\varepsilon > 0$ when $\varepsilon < a_d$ we get

$$n > N \Rightarrow \left| \frac{p(n)}{n^d} - a_d \right| < \varepsilon$$

for $N > 0$

$$\Rightarrow |p(n) - a_d n^d| < \varepsilon n^d$$

$$\Rightarrow (a_d - \varepsilon) n^d < p(n) < (a_d + \varepsilon) n^d$$

Let $n_0 = N + 1$ $\quad c_1 = (a_d - \varepsilon)$.

$c_2 = (a_d + \varepsilon)$

for $c_1, c_2 \geq 0 \longrightarrow n \geq n_0 \Rightarrow n > N$.

$$\Rightarrow (a_d - \varepsilon) n^d < p(n) < (a_d + \varepsilon) n^d$$

$$c_1 n^d < p(n) \leq c_2 n^d.$$

$$\Rightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} \neq 0, \text{ finite} \Rightarrow f(n) = \Theta(g(n))$$

$$\Rightarrow p(n) = \Theta(n^d)$$

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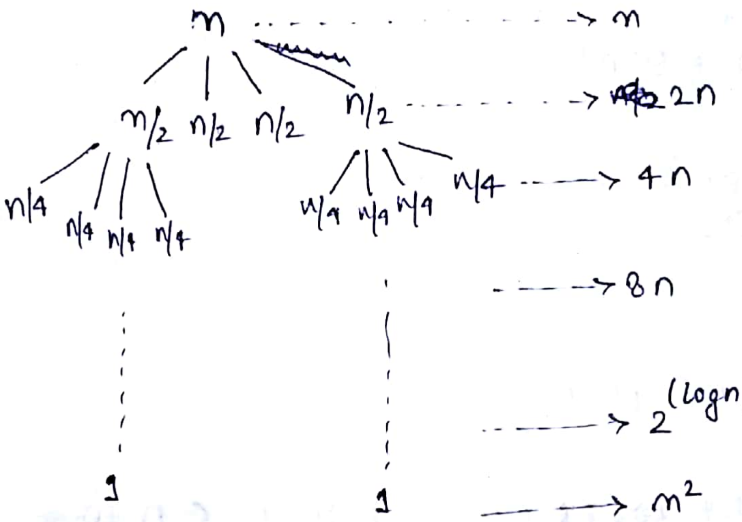
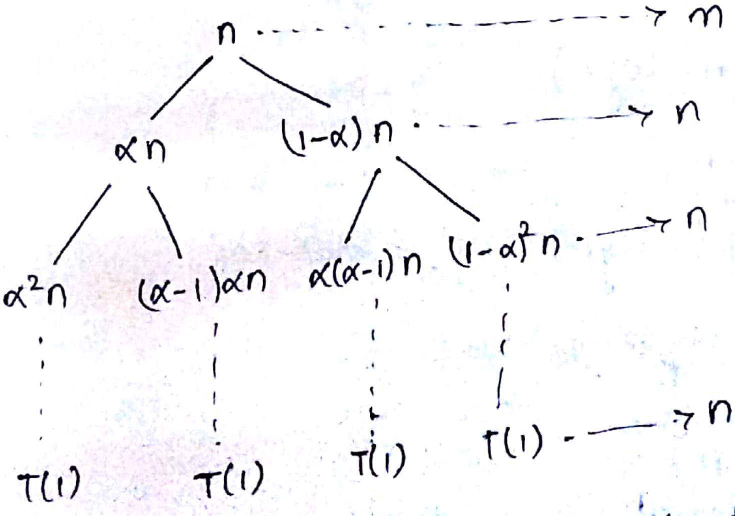
6.) Recursion Tree : $T(n) = 4T(\lfloor n/2 \rfloor) + n$



$$
\begin{aligned}
T(n) &= \text{the sum of recursion tree} \\
&= n^2 + \sum_{i=0}^{(\log n)-1} 2^i \cdot n \\
&= n^2 + n\left(2^{\log n} - 1\right) \\
&= n^2 + n^2 - n \\
&= \Theta(n^2)
\end{aligned}
$$

tight asymptotic bounds.

7.) $T(n) = T(\alpha n) + T((1-\alpha)n) + n \quad : 0 < \alpha < 1$



It can be assumed that $\alpha < 1/2$ & therefore the height of the tree is $\log_{1/\alpha} n$.

$$T(n) = \sum_{i=0}^{\log_{1/x} n} n + O(n)$$

$$= n \log_{1/x} n + O(n)$$

$$= \underline{\Theta(n \log n)}$$

(9.) $\log(n!) = \Theta(n \log n)$

$\longrightarrow$ $\log(n!) = \log 1 + \log 2 + \dots + \log n$ ~~$\log n$~~

$\leq n \log n$

$\Rightarrow \log(n!) = O(n \log n)$ —— (i)

$\longrightarrow$ $\log n! = \sum_{i=1}^{n} \log i$

$\geq \sum_{i=\lceil n/2 \rceil}^{n} \log i$

$\geq \sum_{i=\lceil n/2 \rceil}^{n} \log n/2$

$\geq \frac{n}{2} \log \frac{n}{2}$

$= \frac{n}{2} (\log n - \log 2)$

$= n/2 \log n$

$\Rightarrow \log n! = \Omega(n \log n)$ —— (ii)

(i) + (ii)

$$\boxed{\log n! = \Theta(n \log n)}$$

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(10.) Let $n = e^m$

$\log n = m$

$(\log n)! = m!$

Now for any given k

$e^{km} < m!$ for very large m.

For $km < \sum_{a=1}^{m} \log a$

Let $\log \frac{m}{2} > 2k$

$\Rightarrow m > 2e^{2k}$

$\Rightarrow m$ is even.

Hence we can say that

$(\log n)!$ is not polynomially bounded. —— (i)

let $n = e^{(e^m)}$. ~~$(\log\log n)!$~~

$(\log\log n)! = m!$

For no specific m:

$\log m! < \log m^m$

$= \cancel{\log}\ m\log m < m^2 < e^m$

$= \log e^{(e^m)}$

Hence we can say that

$(\log\log n)!$ is polynomially bounded —— (ii)

(i) & (ii) proves ~~to~~ the given statement.

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11. $T(n) = 7T(n/2) + n^2$

$T'(n) = aT'(n/4) + n^2$

A' is asymptotically faster than A.

$$T(n) = 7T(n/2) + n^2$$

$a = 7 \geq 1 \quad b = 2 > 1 \quad f(n) = n^2$

$n^{\log_b a} = n^{\log_2 7} < n^{\log_2 4} = n^2$

$f(n) = O(n^{\log_b a - \epsilon})$

So, $T(n) = \Theta(n^{\log_b a}) = \Theta(n^{2})$

$T(n) = 7\left(7T(n/4) + 1/4\, n^2\right) + n^2$

$= 49T(n/4) + 11/4\, n^2$

$a = 49 \quad b = 4 \quad f(n) = 11/4\, n^2$

$\Rightarrow f(n) \in O(n^2) \rightarrow c = 2.$

$2 = c < \log_b a = \log_4 49 \approx 2.81$

Applying Master's Theorem

$\hookrightarrow T(n) = \Theta(n^{\log_4 49})$

We know that for

$T(n) \geq T'(n) \Rightarrow a \leq 49$ assuming initial values to be the same.

$T'(n) = aT'(n/4) + n^2$

$a = a \quad b = 4 \quad f(n) = n^2$

$\hookrightarrow f(n) = O(n^2) \longrightarrow c = 2$

$\Rightarrow 2 = c < \log_4 a$

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By Master's Theorem

$T'(n) = \Theta(n^{\log_4 a})$

$\Rightarrow n^{\log_4 49} = o(n^{\log_4 a})$

$\Rightarrow$

$a > 49$.

$T(n) = o(T'(n))$ if $a > 49$

Ans $\rightarrow$ 49

8. (a) $T(n) = 4T(n/2) + n$

$a = 4 \quad b = 2 \quad f(n) = n$

$n^{\log_b a} = n^{\log_2 4} = n^2$

$\therefore f(n) = O(n^{\log_2 4 - \varepsilon})$ where $\varepsilon = 1$

By applying Master's Theorem case 1

$\Rightarrow T(n) = \Theta(n^2)$

(b) $T(n) = 4T(n/2) + n^2$

$a = 4 \quad b = 2 \quad f(n) = n^2$

$n^{\log_b a} = n^2$

$f(n) = \Theta(n^{\log_b a})$

$= \Theta(n^2)$

By applying Master's Theorem case 2.

$\Rightarrow T(n) = \Theta(n^2 \log n)$

(c) $T(n) = 4T(n/2) + n^3$

$a = 4 \quad b = 2 \quad f(n) = n^3$

$n^{\log_b a} = n^{\log_2 4} = n^2$

$\therefore f(n) = \Omega(n^{\log_2 4 + \varepsilon})$ where $\varepsilon = 1/2$

↳ Applying Master's Theorem case 3.

$af(n/b) = 4(n/2)^3$

$= (1/2)(n/2)^3 \leq (2/3)n^3$

$= c f(n)$

↳ for $c = 2$.

$\Rightarrow T(n) = \Theta(f(n)) = \underline{\Theta(n^3)}$.

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