Lecture 11: Absolute Extrema of functions of two variables

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We are now in a position to identify the points at which an absolute extremum is attained. In practice, the critical points of a function are few in number, whereas the boundary points consist of "one-dimensional pieces". The function, when restricted to a "one-dimensional piece" is effectively a function of one variable. Thus, the methods of one-variable calculus can be applied to determine the absolute extrema of the restrictions of the function to the "one-dimensional pieces".

Thus, we have a plausible recipe to determine the absolute extrema and the points where they are attained: First, determine the critical points of the function and the values of the function at these points. Next, determine the boundary of its domain. Restrict the function to the boundary components and determine the absolute extrema of the restricted function by one-variable methods. Compare the values of the function at all these points. The greatest value among them is the absolute maximum, while the least value is the absolute minimum. This recipe is illustrated by the following example.

Example 11.1 *Let $D := [-2, 2] \times [-2, 2]$ and let $f : D \to \mathbb{R}$ be given by $f(x, y) := 4xy - 2x^2 - y^4$. Find absolute maximum and absolute minimum values of f. Also identify all the points of absolute extreme values.*

Solution:

Step 1 Note that D is a closed and bounded subset of $\mathbb{R}^2$, and f is continuous everywhere on the plane. Thus the absolute extrema of f exist and are attained by f.

Step 2 To locate points of absolute extrema, we first determine the critical points of f. Solve $\nabla f(x, y) = (0, 0)$ for $(x, y) \in (-2, 2) \times (-2, 2)$. We get two simultaneous equation $y - x = 0$ and $x - y^3 = 0$. Substituting $y = x$ in second equation we get $x - x^3 = 0$ which has three solutions in interval $(-2, 2)$. Namely, $x = 0, \pm 1$. Hence critical points are $(x, y) = (0, 0), (1, 1)$, or $(-1, -1)$.

Step 3 Now we identify the points on the boundary of D, where f possibly can have absolute extreme values. The restrictions of f to its boundary components are the four functions from $[-2, 2]$ to $\mathbb{R}$ given by $f(2, y), f(-2, y), f(x, -2)$, and $f(x, 2)$. Due to symmetry $[f(-x, -y) = f(x, y)]$, it suffices to consider only the first and the last of these. So, let us determine the possible candidate of absolute maximum and minimum of $f(2, y)$ for $-2 \le y \le 2$ and of $f(x, 2)$ for $-2 \le x \le 2$.

(a) As for $f(2,y) = 8y - 8 - y^4, y \in [-2,2]$, $f'(2,y) = 8 - 4y^3$. Hence the only critical point is $y = (2)^{\frac{1}{3}}$, and the boundary points are $y = \pm 2$. Hence possible candidate of absolute extrema of f are $(2,-2), (2,(2)^{\frac{1}{3}}), (2,2)$.

(b) Similarly, for $f(x,2) = 8x - 2x^2 - 16, \forall x \in [-2,2]$, and the derivative $f'(x,2) = 8 - 4x$ does not vanish in $(-2,2)$ hence possible candidate of absolute extrema of f are $(-2,2), (2,2)$.

Step 4 We can now tabulate all the relevant values as follows.

(x,y)	$(0,0)$	$(1,1)$	$(2,(2)^{\frac{1}{3}})$	$(2,-2)$	$(2,2)$
$f(x,y)$	0	1	-0.440473701	-40	-8

In list we have disregarded the points $(-1,-1), (-2,2)$, due to symmetry. It follows that the absolute maximum of f on D is 1, which is attained at $(1,1)$ as well as at $(-1,-1)$ (due to symmetry), and the absolute minimum of f on D is -40, which is attained at $(2,-2)$ as well as at $(-2,2)$.

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Example 11.2 *Find the absolute minimum and the absolute maximum of the function f given by $f(x,y) := 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular region bounded by the lines given by $x = 0, y = 2$, and $y = 2x$.*

Solution: The given closed triangular region D has vertices $(0,0), (0,2), (1,2)$. Clearly D is a closed and bounded subset of $\mathbb{R}^2$ and given function f being a polynomial function is continuous everywhere hence in particular on D. Therefore f indeed attains it's absolute extrema on D. Now to identify the points of global extreme values we proceed as discussed above.

1. To locate points of absolute extrema, we first determine the critical points of f. Solve $\nabla f(x,y) = (0,0)$. We get two simultaneous equation $4x - 4 = 0$ and $2y - 4 = 0$. Hence simultaneous solution is $(1,2)$, which is not an interior point of the triangle. Therefore there are not critical points of f inside the triangle. So absolute maximum and minimum will be attained on the boundary of the triangular region.

2. Now we identify the points on the boundary of D, where f possibly can have absolute extreme values.

   (a) We consider $f(0,y) = y^2 - 4y + 1, y \in [0,2]$. So $f'(0,y) = 2y - 4$. Hence the there is no critical point, and the boundary points are $y = 0, 2$. Hence possible candidate of absolute extrema of f are $(0,0), (0,2)$.

(b) We consider $f(x,2) = 2x^2 - 4x + 5, x \in [0,1]$. So $f'(x,2) = 4x - 4$. Hence the there is no critical point, and the boundary points are $x = 0, 1$. Hence possible candidate of absolute extrema of f are $(1,2)$. We already got the point $(0,2)$ in previous step.

(c) We consider $f(x,2x) = 2x^2 - 4x + 4x^2 - 8x + 1 = 6x^2 - 12x + 1, x \in [0,1]$. So $f'(x,2) = 12x - 12$. Hence the there is no critical point, and the boundary points are $x = 0, 1$. Both points are already identified in previous steps.

3. We can now tabulate all the relevant values as follows.

(x,y)	$(0,0)$	$(1,2)$	$(0,2)$
$f(x,y)$	1	-5	-3

It follows that the absolute maximum of f on D is 1, which is attained at $(0,0)$, and the absolute minimum of f on D is -5, which is attained at $(1,2)$.

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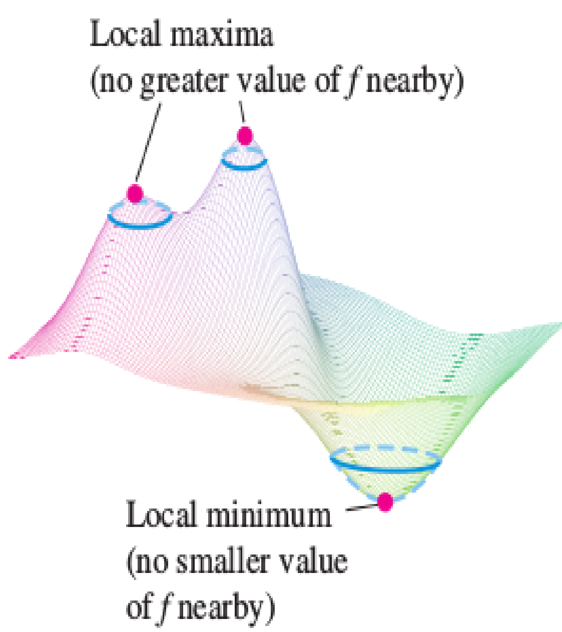
11.1 Saddle Point & Second derivative test

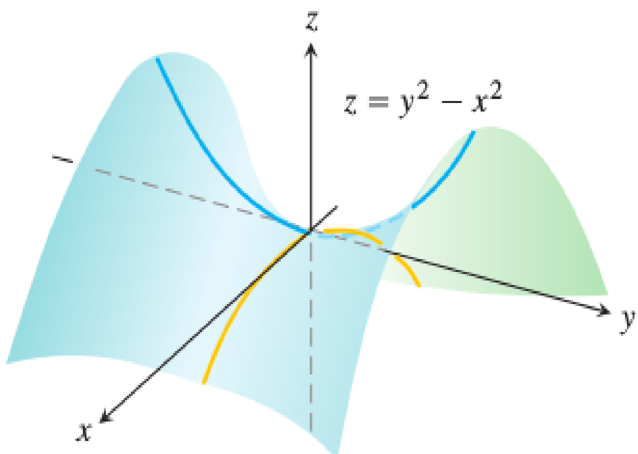
Geometrically, local maxima correspond to mountain peaks on the surface $z = f(x,y)$ and local minima correspond to valley bottoms.



Saddle Point As for functions of a single variable, not every critical point gives rise to a local extremum. A function of a single variable might have a point of inflection. In the same spirit, a function of two variables might have a saddle point.

Definition 11.3 *A function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.*

The name "saddle point" is motivated by the shape of surface near such a point. Look at the following picture.



Surface $f(x, y) = y^2 - x^2$ look like a saddle near origin. Note that $f(r, 0) = -r^2 < 0 = f(0, 0)$ and $f(0, r) = r^2 > 0 = f(0, 0)$ for any $r \in \mathbb{R}$. Therefore, however small open disk centered at $(0, 0)$ we consider, there will always be some points from x-axis and y-axis. Hence, $(0, 0)$ is a saddle point as per Definition 11.3.

Theorem 11.4 (Second Derivative Test or Discriminant Test) *Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then*

1. *f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
2. *f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*
3. *f has a saddle point at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).*

The test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the Hessian or discriminant of f. It is denoted by Δf