ASSIGNMENT 4

① $y''+y'-xy=0$ — ① $\sum a_n x^n$

$p(x)=1$, $q(x)=-x$

continuous, analytic

$$y=\sum_{n=0}^{\infty} a_n x^n$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$=a_1+2a_2x+3a_3x^2+\dots$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

By putting in ①, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=1}^{\infty} n a_n x^{n-1}-\sum_{n=0}^{\infty} a_n x^{n+1}=0$$

Let $n-2=m$, $n=m+2$, $m=0$ | $n-1=m$, $n=1$, $n=m+1$, $m=0$ | $n+1=m$, $n=0$, $n=m-1$, $n=1$

$n-2\to m$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n}-\sum_{n=1}^{\infty}a_{n-1}x^{n}=0$$

Put $n=0$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}x^{n}-\sum_{n=1}^{\infty}a_{n-1}x^{n}=0$$

$$2a_2+a_1+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-a_{n-1}\right]x^{n}=0$$

$$2a_2+a_1=0$$

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$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} - a_{n-1} = 0$$

$n = 1, 2$

$$a_2 = -a_1/2$$

$$a_{n+2} = \frac{a_{n-1} - (n+1)a_{n+1}}{(n+2)(n+1)\,a_0} \qquad n = 1, 2, 3$$

$n=1$

$$a_3 = \frac{a_0 - 2a_2}{6} = \frac{a_0}{6} - \frac{1}{3}a_2 = \frac{a_0}{6} - \frac{1}{3}\left(\frac{-a_1}{2}\right)$$

$$= \frac{a_1}{6} + \frac{a_0}{6} = \frac{a_0}{3}$$

$n=2$

$$a_4 = \frac{a_1 - 3a_3}{4\cdot 3} = \frac{a_1}{2\cdot 3\cdot 4} - \frac{a_0}{2\cdot 3\cdot 4}$$

Now putting in $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$$= a_0 + a_1 x - \frac{a_1}{2}x^2 + \left(\frac{a_1}{2\cdot 3} + \frac{a_0}{2\cdot 3}\right)x^3 + \left(\frac{a_1}{2\cdot 3\cdot 4} - \frac{a_0}{2\cdot 3\cdot 4}\right)x^4 + \cdots$$

$$= a_0\Big[\underbrace{1 + \frac{x^3}{2\cdot 3} - \frac{x^4}{2\cdot 3\cdot 4} + \cdots}_{y_1(x)}\Big] + a_1\Big[\underbrace{x - \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \cdots}_{y_2(x)}\Big]$$

where $y_1(x) = 1 + \frac{x^3}{2\cdot 3} - \frac{x^4}{2\cdot 3\cdot 4} + \cdots$

$$y_2(x) = x - \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \cdots$$

$$y = a_0 y_1(x) + a_1 y_2(x)$$

$$y(0) = a_0 y_1(0) + a_1 y_2(0)$$

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$y_1(0) = 1, \; y_2(0) = 0$

(3)

$$y_1 = \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 + a_2 x^2 + \cdots$$

$$y_2 = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = a_1 x + a_3 x^3 + \cdots$$

a_{2n+2} ~~= p=2,~~ $, p=0$

$$a_{2n+2} = \frac{2n(2u+1)}{(2u+1)(2u+2)} a_{2n} \quad n=0,1,2$$

$a_2 = 0,$

$$a_{2n+1} = -\frac{(p-2n+1)(p+2n)}{2n(2n+1)} a_{2n},$$

$n = 1, 2 \ldots$

$$= \frac{(2n+1)2n}{2n(2n+1)} a_{2n-1} \quad n=1,2,$$

$$= \frac{(2-1}{2+1)(3)}$$

$$a_3 = \frac{1}{3} \cdot a_1$$

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Unit vectors

$\hat{s} = \dfrac{\frac{\delta \vec{r}}{\delta s}}{\left|\frac{\delta \vec{r}}{\delta s}\right|}$ $\hat{\phi} = \dfrac{\frac{\delta \vec{r}}{\delta s}}{\left|\frac{\delta \vec{r}}{\delta s}\right|}$ $\hat{z} = \dfrac{\frac{\delta \vec{r}}{\delta z}}{\left|\frac{\delta \vec{r}}{\delta z}\right|}$

4\. (i) $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2-1)^n$

$x \to -x$

$\downarrow$

(ii) Differentiate

Generating f^n

$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

$\downarrow$ $x \to -x$ $\downarrow$ Legendre eq^n

$(1+2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(-x)$

$(1-2x(-t)t^2)^{-1/2} = \sum_{n=0} t^n P_n(-x)$

$\sum_{n=0}^{\infty} (-t)^n P_n(x) = \sum_{n=0} t^n P_n(-x)$

(5) from (ii)

$I = \int_{-1}^{1} x^m P_n(x)\,dx$

$\downarrow$ $\hookrightarrow$ apply by parts

legendre eq^m

$= \dfrac{m(m-1)\cdots(m-n+1)}{2^m!} \int_{-1}^{1} x^{m-n}(1-x^2)^n dx$

where $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2-1)^n$

Since $m-n$ is odd $\Rightarrow I = 0$

Since $m-n$ is even $\Rightarrow m-n = 2k$.

If ~~(m-n) is even~~ $I = \dfrac{2m(m-1)\cdots(m-n+1)}{2^n n!} \int_0^{\pi/2} m^{2k}\theta \cos^{2n+1}\theta\, d\theta$

$\downarrow$

take $x = \sin\theta$

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$$= \frac{2m(y-1) \cdots (m-n+1)}{2^n n!} I_{k,n}$$

where $I_{k,n} = \int_0^{\pi/2} \sin^{2k}\theta \cos^{2k+1}\theta \, d\theta$.

$$= \frac{2n}{2k+1} I_{k+1;n-1}$$

$$I_{k,n} = \frac{2n \cdot 2(n-1) \, 2 \cdot 1}{(2k+1)(2k+3) \cdot (2(k+n-1)+1)} I_{k,0}$$

(6) Q. Diff w.r.t t

(i)
$$\frac{-2x+2t}{2(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (-x+t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (-x+t) \sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \quad \text{--- (2)}$$

{Equating the coeff of t^n

$$\Rightarrow -x P_n(x) + P_{n-1}(x) = (n+1) P_{n+1}(x) - 2x P_n(x)$$
$$+ (n-1) P_{n-1}(x) = 0$$

(ii) Diff w.r.t to x.

$$\frac{-xt}{2(1-2xt+t^2)^{3/2}} = \sum_{n=0} t^n P_n'(x)$$

∴ Equating t^n

(iii) $P_n'(x) - 2x(P_{n-1}(x)) = P_{n-1}$

replace $n \rightarrow n+1$

$$P_{n+1}' - 2x P_n'(x) + P_{n-1}'(x) = P_n(x) \quad \text{---} (*)$$

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Diff. eq^n (2) w.r.t x & equate

$$(n+1)P'_{n+1}(x) - \big((2n+1)P_n + xP'_m(x)\big) + nP'_{n-1}(x) = 0 \quad (**)$$

By eliminating $P'_{(n+1)}(x)$ from (*) & (**)

$$nP_n(x) = xP'_0(x) - P'_{n-1}(x)$$

Eliminating $P'_{(n-1)}(x)$ from (*)

6(v) take $x = 0$

n is odd

n is even

take n is even

↓

form of 2n

equating t^{2n}

eqn t^{2n+1} n is odd.

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