Lecture 12: Let's go to The Extreme...

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Theorem 12.1 *Let $D \subset \mathbb{R}^2$ be closed and bounded, and $f : D \to \mathbb{R}$ is continuous. Then f is bounded on D, and attains its absolute minimum and the absolute maximum on D.*

Next question arises naturally. Knowing the function f, how does one find the absolute extrema and points where they are attained? As in one-variable calculus, it helps to consider the interior points of D at which the partial derivatives vanish or fail to exist, and also the boundary points of D.

Definition 12.2 *Given $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$, a point $(x_0, y_0) \in D$ is called a critical point of f if*

1. *(x_0, y_0) is an interior point of D.*
2. *either $\nabla f(x_0, y_0)$ does not exist, or if $\nabla f(x_0, y_0)$ exists, then $\nabla f(x_0, y_0) = (0, 0)$.*

Definition 12.3 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. A point $(x_0, y_0) \in D$ is said to be a point of local maximum of f if*

1. *(x_0, y_0) is an interior point of D*
2. *there exists $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and $f(x, y) \le f(x_0, y_0)$ for all $(x, y) \in B_\delta(x_0, y_0)$.*

Definition 12.4 *Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. A point $(x_0, y_0) \in D$ is said to be a point of local minimum of f if*

1. *(x_0, y_0) is an interior point of D*
2. *there exists $\delta > 0$ such that $B_\delta(x_0, y_0) \subseteq D$ and $f(x, y) \ge f(x_0, y_0)$ for all $(x, y) \in B_\delta(x_0, y_0)$.*

Theorem 12.5 (Necessary condition for local extrema) *Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) is an interior point of D. Suppose $f : D \to \mathbb{R}$ has a local extremum at (x_0, y_0). If $\nabla f(x_0, y_0)$ exists, then $\nabla f(x_0, y_0) = (0, 0)$.*

Example 12.6 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := -(x^2 + y^2)$. Find the local extreme values (if any) of f.*

Solution: Then f is differentiable and $\nabla f(x, y) = (-2x, -2y)$ for $(x, y) \in \mathbb{R}^2$. Thus the only point where f can possibly have a local extremum is $(0, 0)$. Since $f(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$, we see that f does have a local maximum at $(0, 0)$ and this value is equal to 0. ■

Example 12.7 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := xy$. Find the points of local extrema (if any) of f.*

Solution: Note that f is differentiable everywhere and $\nabla f(x, y) = (y, x)$ for $(x, y) \in \mathbb{R}^2$. Thus the only point where f can possibly have a local extremum is $(0, 0)$. But $f(0, 0) = 0$ and for any $\delta > 0$, there are $(x_1, y_1), (x_2, y_2) \in B_\delta(0, 0)$ such that $f(x_1, y_1) < 0$ and $f(x_2, y_2) > 0$. For example, one can choose any $t \in (0, \delta)$ and let $(x_1, y_1) := (t, -t)$ and $(x_2, y_2) := (t, t)$. It follows that f has neither a local maximum nor a local minimum at $(0, 0)$. ■

We are now in a position to identify the points at which an absolute extremum is attained.

Theorem 12.8 *Let $D \subset \mathbb{R}^2$ be closed and bounded, and let $f : D \to \mathbb{R}$ be a continuous function. Then the absolute minimum as well as the absolute maximum of f are attained either at a critical point of f or at a boundary point of D.*

In practice, the critical points of a function are few in number, whereas the boundary points consist of "one-dimensional pieces". The function, when restricted to a "one-dimensional piece" is effectively a function of one variable. Thus, the methods of one-variable calculus can be applied to determine the absolute extrema of the restrictions of the function to the "one-dimensional pieces".

Thus, we have a plausible recipe to determine the absolute extrema and the points where they are attained: First, determine the critical points of the function and the values of the function at these points. Next, determine the boundary of its domain. Restrict the function to the boundary components and determine the absolute extrema of the restricted function by one-variable methods. Compare the values of the function at all these points. The greatest value among them is the absolute maximum, while the least value is the absolute minimum. This recipe is illustrated by the following example.

Example 12.9 *Let $D := [-2, 2] \times [-2, 2]$ and let $f : D \to \mathbb{R}$ be given by $f(x, y) := 4xy - 2x^2 - y^4$. Find absolute maximum and absolute minimum values of f. Also identify all the points of absolute extreme values.*

Solution:

Step 1 Note that D is a closed and bounded subset of $\mathbb{R}^2$, and f is continuous everywhere on the plane. Thus the absolute extrema of f exist and are attained by f.

Step 2 To locate points of absolute extrema, we first determine the critical points of f. Solve $\nabla f(x, y) = (0, 0)$ for $(x, y) \in (-2, 2) \times (-2, 2)$. We get two simultaneous equation $y - x = 0$ and $x - y^3 = 0$. Substituting $y = x$ in second equation we get $x - x^3 = 0$ which has three solutions in interval $(-2, 2)$. Namely, $x = 0, \pm 1$. Hence critical points are $(x, y) = (0, 0), (1, 1)$, or $(-1, -1)$.

Step 3 Now we identify the points on the boundary of D, where f possibly can have absolute extreme values. The restrictions of f to its boundary components are the four functions from $[-2, 2]$ to $\mathbb{R}$ given by $f(2, y), f(-2, y), f(x, -2)$, and $f(x, 2)$. Due to symmetry $[f(-x, -y) = f(x, y)]$, it suffices to consider only the first and the last of these. So, let us determine the possible candidate of absolute maximum and minimum of $f(2, y)$ for $-2 \leq y \leq 2$ and of $f(x, 2)$ for $-2 \leq x \leq 2$.

(a) As for $f(2, y) = 8y - 8 - y^4, y \in [-2, 2]$, $f'(2, y) = 8 - 4y^3$. Hence the only critical point is $y = (2)^{\frac{1}{3}}$, and the boundary points are $y = \pm 2$. Hence possible candidate of absolute extrema of f are $(2, -2), (2, (2)^{\frac{1}{3}}), (2, 2)$.

(b) Similarly, for $f(x, 2) = 8x - 2x^2 - 16, \forall x \in [-2, 2]$, and the derivative $f'(x, 2) = 8 - 4x$ does not vanish in $(-2, 2)$ hence possible candidate of absolute extrema of f are $(-2, 2), (2, 2)$.

Step 4 We can now tabulate all the relevant values as follows.

(x, y)	$(0, 0)$	$(1, 1)$	$(2, (2)^{\frac{1}{3}})$	$(2, -2)$	$(2, 2)$
$f(x, y)$	0	1	-0.440473701	-40	-8

In list we have disregarded the points $(-1, -1), (-2, 2)$, due to symmetry. It follows that the absolute maximum of f on D is 1, which is attained at $(1, 1)$ as well as at $(-1, -1)$ (due to symmetry), and the absolute minimum of f on D is -40, which is attained at $(2, -2)$ as well as at $(-2, 2)$.

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