Rays through the eye

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September 4, 2014

1 Structure

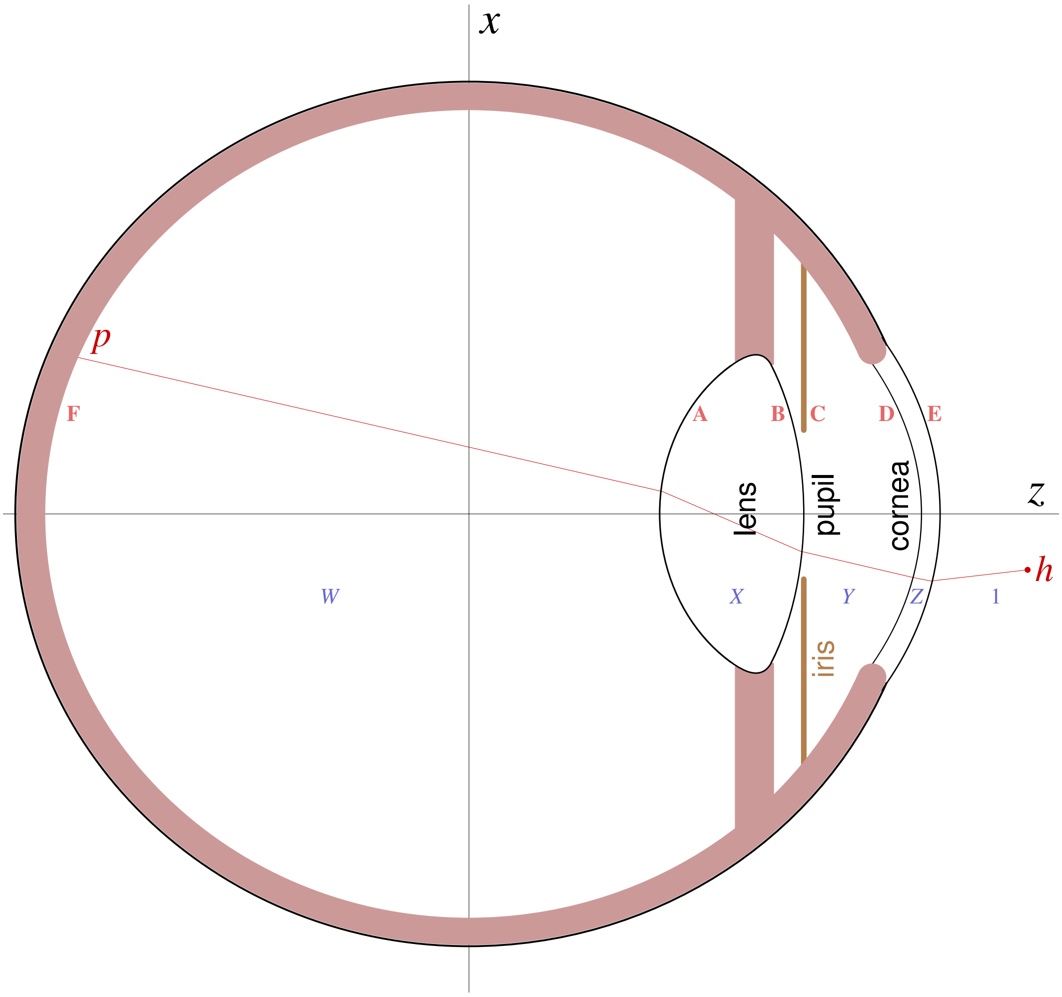

Figure 1: Parts, surfaces and regions of the optical structure of the eye.

We take the fundus F as part of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \quad = \quad 1\,, \tag{1}$$

c units long (in the z direction), b units across (along x) and c units vertically (along y). Usually, we assume $a = b$, but c may be a little different (leading to myopia, *etc.*) Cross-terms like εxy can give more distorted shapes.

We seek to track the ray passing through point h outside the eye, in a known direction, and find the point p where it meets the fundus F. To get there it passes through the cornea and lens, and the two liquid spaces between them. Since the refractive indices vary, it deflects as it goes. The refractive index of the lens actually varies within it, from about 1.406 in the core to 1.386 in the outermost layer, so the ray follows a slightly curved path through it, but for fast computation we treat it as a constant. Deflection then occurs only at the rear A and front B of the lens, and the rear D and front e of the cornea. (A ray that meets the iris C is blocked, so the point p does not exist.)

Each deflection at a point q on a surface S obeys Snell's Law, and so depends on the normal to S at q, and the refractive indices of the materials that S separates. Representative indices are

Material	**Symbol**	**Value**
Vitreous humour	W	1.322
Lens	X	1.401
Aqueous humour	Y	1.322
Cornea	Z	1.362

2 Geometry and normals

None of the surfaces in Fig. 1 are spherical, but they are well approximated by widely-used quadric surfaces (in two dimensional slices, conic curves like ellipses and hyperbolas). We follow the specifics of [1], except that for now we treat the lens as homogeneous: and we ignore scatter and chromatic aberration. For use in 2D diagrams, we include (x, y) as well as (x, y, z) geometry.

A convenient way to describe a axis-symmetrical quadric $\mathcal{Q}$ with an apex P at the origin is

$$(1+Q)\zeta^2 - 2\zeta R + \omega^2 \;=\; 0\,, \tag{2}$$

where ω^2 is short for x^2 or $x^2 + y^2$ according to dimension, R is the[1] radius of curvature at P, and Q is the 'conic constant' that fixes the shape. A negative R means that a surface bends toward negative z, while $Q > 0$ tells us that curvature increases away from P; with $Q < 0$, it decreases. We have a circle or a sphere for $Q = 0$, an ellipse or ellipsoid for $Q > -1$, and so on.

Each boundary $\mathcal{Q}$ has an apex, not at the origin but at a distance $\alpha \neq 0$ along the z-axis. In the shared coordinate system, (2) becomes

$$\begin{aligned}
(1+Q)(z-\alpha)^2 - 2(z-\alpha)R + \omega^2 &= 0 && (3)\\
\text{or for short}\quad qz^2 + 2rz + s + \omega^2 &= 0 && (4)\\
\text{where}\quad q &= 1+Q\\
r &= -\,(q\alpha + R)\\
s &= q\alpha^2 + 2\alpha R\,.
\end{aligned}$$

We store q, r and s for use in intersection calculations. Tabulating, for a representative adult we may (after re-checking!) use the values

$\mathcal{Q}$	Q	R	α	q	r	s	radius of aperture
A	-2.30	5.38	5.03	-1.30	1.159	21.232	$10mm$
B	-3.88	-10.38	8.90	-2.88	36.012	-412.889	$10mm$
D	-0.60	-6.40	12.06	0.40	1.376	-96.191	$11mm$
E	-0.18	-7.77	12.56	0.82	-2.592	-65.824	$11mm$

[1] Symmetry implies that there is only one radius of curvature at P

The aperture radius (a rougher number than the others) limits what can be seen from an external point, by eye or camera. The iris is (for our purposes) simply a flat obstruction, at the same z value α as the front B of the lens. Its aperture, the pupil, is centred about $0.5mm$ from the shared axis of the cornea and lens.

If a ray through a point $H = (H_x,\, H_z)$ or $(H_x,\, H_y,\, H_z)$ is in the direction of a vector $\mathbf{v}_0 = (v_x,\, v_z)$ or $(v_x,\, v_y,\, v_z)$, we can parametrise a general point on it as

$$\begin{aligned} \rho(t) &= H + t\mathbf{v} \\ &= (H_x + tv_x,\, H_z + tv_z) \ \text{ or } \ (H_x + tv_x,\, H_y + tv_y,\, H_z + tv_z) \end{aligned} \tag{5}$$

which lies on$\mathcal{Q}$ if

$$q\,(H_z + tv_z)^2 + 2r\,(H_z + tv_z) + s + \left((H_x + tv_x)^2 + (H_x + tv_x)^2\right) \;=\; 0 \tag{6}$$

$$\begin{aligned} \left(qv_z^2 + v_x^2 + v_y^2\right)t^2 \;+\; 2\left(qH_zv_z + rv_z + sH_xv_x + sH_yv_y\right)t & \\ + \left(qH_z^2 + 2rH_z + \left(H_x^2 + H_x\right)^2\right)\right) &\;=\; 0\,, \end{aligned} \tag{7}$$

or in 2D the same without the y terms.

We can solve (7) by the quadratic formula, usually getting two roots. For a ray starting from $H_0 = h$ external to the eye, and $\mathcal{Q}$ as the outer surface E of the cornea, complex roots occur only if the ray misses the cornea. It is not of interest here whether it hits the eyeball elsewhere, or the surrounding structure like eyelids, or misses the face completely, so we disregard such a ray (and exclude it from diagrams). If the roots are real, substituting the smaller t into (5) gives a point $H_E = (x, y, z)$ or (x, z) on the outward side of $\mathcal{Q}$. If $x^2 + y^2 >$ aperture^2, the ray is blocked.

If the ray is not blocked, it goes from H_E into the cornea in a new direction $\mathbf{v}_E$, computed as in §3 below. We repeat the procedure with D's parameters to find whether it meets D at a real point H_D within the aperture. If not, we end it. If it does, we continue to find whether it passes the pupil. With the α for the front B of the lens, we solve

$$\begin{aligned} H_z + tv_z &= \alpha \\ t &= \frac{\alpha - H_z}{v_z}\,, \end{aligned} \tag{8}$$

substituting t into (5) gives a point $H_C = (x, y, z)$ or (x, z) in the iris plane. If

$$(x - 0.5)^2 + y^2 > \text{(iris aperture)}^2,$$

the ray is blocked. If it is not blocked, we go on to find H_B in the same manner. If H_B exists (as it almost always must), we deflect and continue in the same way to H_A, with the difference that we choose the larger root for t, because the ray is arriving from inside. From there, we refract again, and pursue the ray to find p on F by the same logic.

3 Normals and refraction

At a point $H = (\overline{x},\, \overline{y},\, \overline{z})$ on a boundary $\mathcal{Q}$ defined by (4) we have the gradient

$$\nabla_H\left(qz^2 + 2rz + s + \omega^2\right) \;=\; 2\,(\overline{x},\, \overline{y}, q\overline{z} + r) \tag{9}$$

giving a normal

$$\mathbf{n}_H \;=\; (\overline{x},\, \overline{y}, q\overline{z} + r) \tag{10}$$

to $\mathcal{Q}$ at H. An incident direction vector $\mathbf{v}$ has the normal component

$$\mathbf{v}_N \;=\; \frac{\mathbf{v} \bullet \mathbf{n}_H}{\mathbf{n}_H \bullet \mathbf{n}_H}\,\mathbf{n}_H \tag{11}$$

and hence the tangential component

$$\mathbf{v}_T = \mathbf{v} - \mathbf{v}_N \,. \tag{12}$$

For a fixed normal component the sine of the angle to the normal is proportional to the tangential component, so by Snell's Law we have the emerging tangential component

$$\mathbf{v}_T^{\text{refracted}} = \left(\frac{\text{incident refractive index}}{\text{departing refractive index}}\right)\mathbf{v}_T^{\text{incident}} \tag{13}$$

and the refracted direction vector

$$\mathbf{v}^{\text{refracted}} = \mathbf{v}_N + \mathbf{v}_T^{\text{refracted}} \,. \tag{14}$$

References

[1] Y C Chen, C J Jiang, T H Yang, C C Sun, *Development of a human eye model incorporated with intraocular scattering for visual performance assessment*, J Biomed Opt. 2012 Jul;17(7).