Aspherical lens calculations

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Perfect focus

For graphical purposes, since we take a 2D lens surface as a polygonal curve, let us look for a shape like Fig. 1, which with a particular refractive index n deflects every axis-parallel ray through a right-edge midpoint to pass through the same point $\mathbf{f}$.

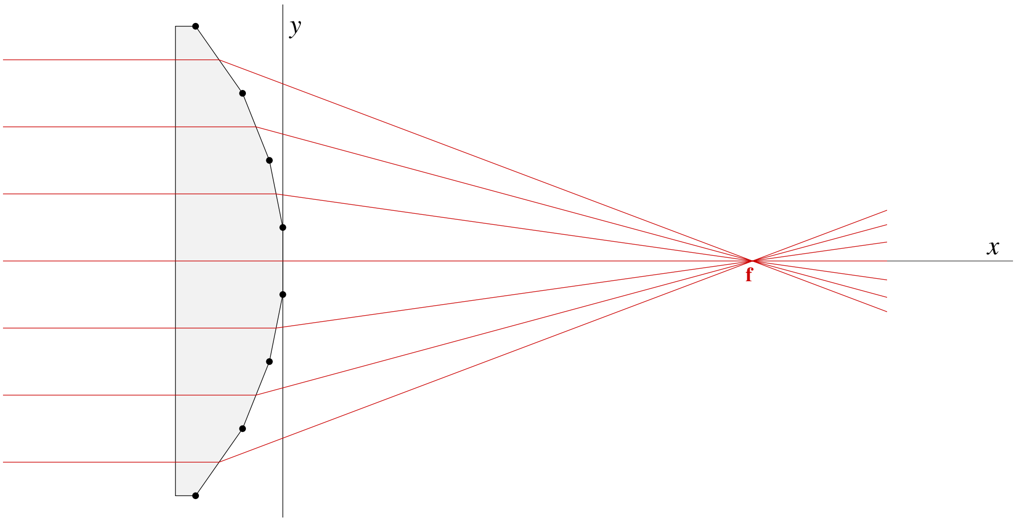

Figure 1: A polygonal 'aspherical lens'.

Evidently a seven-edge curve like Fig. 1 cannot send all rays through $\mathbf{f}$ (at best we could get seven parallel beams, each with $\mathbf{f}$ in the middle), but with many small edges we can get close, and with Phong interpolated normals, much closer.

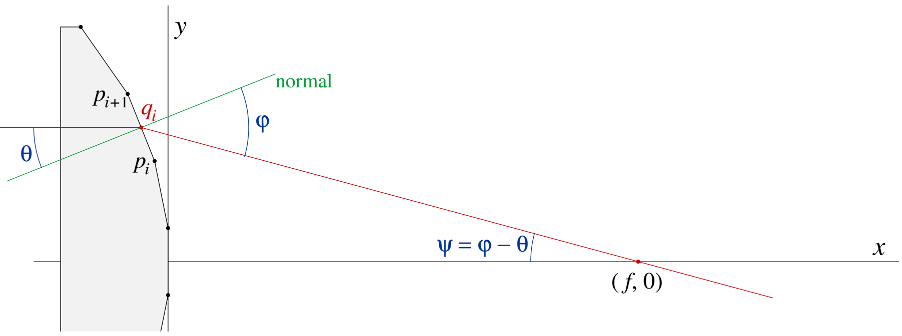

Figure 2: Angles for a particular edge.

To be specific, assume the rays arrive parallel to the x-axis, and first meet the lens on a flat y-direction side, without deflection there. On the other side is a curve of short segments between points $p_i = (x_i, y_i)$, with $y_i = (2i+1)\delta/2$. They are thus equally y-spaced at δ apart in the y-direction, but their x-spacing is more complicated. Our goal here is to find the x_i.

It is convenient to start the count from $i = 0$, and find the p_i with $y_i > 0$. The y-negative vertices have the same x_i (counting downward from $(0,0)$), so we get those by copying.

Consider the ray that reaches the centre

$$q_i \;=\; (\overline{x}_i,\, \overline{y}_i) \;=\; \left(\frac{x_i + x_{i+1}}{2},\ (i+1)\delta\right) \tag{1}$$

of the segment $\overline{p_i p_{i+1}}$ between c and p_{i+1}. The normal is tilted at an angle θ to the horizontal (and the incoming ray), which is also the angle between $\overline{p_i p_{i+1}}$ and the vertical, so

$$\begin{aligned} \tan\theta &= \frac{x_i - x_{i+1}}{\delta} &(2)\\ &= \frac{2(x_i - \overline{x}_i)}{\delta}\,. &(3) \end{aligned}$$

It emerges at an angle φ to the normal, where Snell's Law gives

$$\sin\varphi = n\sin\theta\,. \tag{4}$$

We require that it meets the x-axis in the point $(f, 0)$, which it must do at the angle

$$\psi = \varphi - \theta\,, \tag{5}$$

as shown. We thus have

$$\tan\psi = \frac{\overline{y}_i}{f - \overline{x}_i}\,. \tag{6}$$

We can combine these into an expanded form of (5), giving

$$\arctan\left(\frac{\overline{y}_i}{f - \overline{x}_i}\right) = \arcsin\left(n\sin\left(\arctan\left(\frac{2(x_i - \overline{x}_i)}{\delta}\right)\right)\right) - \arctan\left(\frac{2(x_i - \overline{x}_i)}{\delta}\right) \tag{7}$$

If x_i is known, we can solve for the only unknown, $\overline{x}_i$. Since $x_0 = 0$ *is* known, we begin with that, find $\overline{x}_0$, and set

$$x_1 = x_0 + 2(\overline{x}_0 - x_0) = 2\overline{x}_0 - x_0\,. \tag{8}$$

Iteratively,

$$\begin{aligned} x_2 &= 2\overline{x}_1 - x_1 \\ x_3 &= 2\overline{x}_2 - x_2 &(9) \end{aligned}$$

and so on.

It is hard to give an closed-form solution for (10), but we need only define

$$g(\overline{x}_i) = \arctan\left(\frac{\overline{y}_i}{f - \overline{x}_i}\right) - \arcsin\left(n\sin\left(\arctan\left(\frac{2(x_i - \overline{x}_i)}{\delta}\right)\right)\right) + \arctan\left(\frac{2(x_i - \overline{x}_i)}{\delta}\right) \tag{10}$$

and solve

$$g(\overline{x}_i) = 0 \tag{11}$$

numerically, by Newton's method.