Numerical Question Bank for JEE Main

Quadratic Equations – Solutions

1. **(4)** Given $|x|^2 - 3|x| + 2 = 0$

Here we consider two cases *viz.* $x < 0$ and $x > 0$

Case I: $x < 0$ This gives $x^2 + 3x + 2 = 0$

$\Rightarrow (x+2)(x+1) = 0 \Rightarrow x = -2, -1$

Also $x = -1, -2$ satisfy $x < 0$, so $x = -1$, – 2 is solution in this case.

Case II: $x > 0$. This gives $x^2 - 3x + 2 = 0$

$\Rightarrow (x-2)(x-1) = 0 \Rightarrow x = 2, 1$, so $x = 2$, 1 is solution in this case. Hence the number of solutions are four i.*e*. $x = -1, 1, 2, -2$

Aliter: $|x|^2 - 3|x| + 2 = 0$

$\Rightarrow (|x| - 1)(|x| - 2) = 0$

$\Rightarrow |x| = 1$ and $|x| = 2 \Rightarrow x = \pm 1, x = \pm 2$.

2. **(2)** Here two cases arise viz.

Case I: $x^2 + 4x + 3 > 0$

This gives $x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow x^2 + 6x + 8 = 0 \Rightarrow (x+2)(x+4) = 0 \Rightarrow x = -2, -4$

$x = -2$ is not satisfying the condition $x^2 + 4x + 3 > 0$, so $x = -4$ is the only solution of the given equation.

Case II: $x^2 + 4x + 3 < 0$

This gives $-(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow -x^2 - 2x + 2 = 0 \Rightarrow x^2 + 2x - 2 = 0$

$\Rightarrow (x + 1 + \sqrt{3})(x + 1 - \sqrt{3}) = 0$

$\Rightarrow x = -1 + \sqrt{3}, -1 - \sqrt{3}$

Hence $x = -(1 + \sqrt{3})$ satisfy the given condition $x^2 + 4x + 3 < 0$, while $x = -1 + \sqrt{3}$ is not satisfying the condition. Thus, number of real solutions are two.

3. **(6)** $2^{x+2}.3^{3x/(x-1)} = 9$ Taking log, we get

$$(x+2)\log 2 + \left(\frac{3x}{x-1}\right)\log 3 = 2\log 3$$

$$\Rightarrow \quad (x+2)\left(\log 2 + \frac{1}{x-1}\log 3\right) = 0$$

$$\Rightarrow \quad x = -2 \text{ or } \frac{1}{1-x} = \frac{\log 2}{\log 3}$$

$$\Rightarrow 1 - x = \frac{\log 3}{\log 2} \Rightarrow x = 1 - \frac{\log 3}{\log 2}$$

Sum of roots = -1 – $\log_2 3$

a + b + c = 1 + 2 + 3 = 6

4. **(12)** Equation, $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

$$\Rightarrow 2^{2x} + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$$

$$\Rightarrow \quad 2^{2x}\left(1+\frac{1}{2}\right) = 3^{x-\frac{1}{2}}(1+3)$$

$$\Rightarrow 2^{2x}.\frac{3}{2} = 3^{x-\frac{1}{2}}.4 \;\Rightarrow\; 2^{2x-3} = 3^{x-\frac{3}{2}}$$

Taking log both sides

$$\Rightarrow (2x-3)\log 2 = (x - 3/2)\log 3$$

$$\Rightarrow 2x\log 2 - 3\log 2 = x\log 3 - \frac{3}{2}\log 3$$

$$\Rightarrow x\log 4 - x\log 3 = 3\log 2 - \frac{3}{2}\log 3$$

$$\Rightarrow x\log\left(\frac{4}{3}\right) = \log 8 - \log 3\sqrt{3}$$

$$\Rightarrow \quad \left(\frac{4}{3}\right)^x = \frac{8}{3\sqrt{3}} \Rightarrow \left(\frac{4}{3}\right)^x = \left(\frac{4}{3}\right)^{3/2}$$

$$\therefore x = \frac{3}{2}$$

a = 3 and b = 2

$ab^2 = 12$

5. **(2)** Let all four roots are imaginary. Then roots of both equations $P(x)=0$ and $Q(x)=0$ are imaginary.

Thus $b^2 - 4ac < 0; d^2 + 4ac < 0$, So $b^2 + d^2 < 0$, which is impossible unless $b = 0, d = 0$.

So, if $b \neq 0$ or $d \neq 0$ at least two roots must be real.

If $b=0,\ d=0$, we have the equations.

$P(x)=ax^2+c=0$ and $Q(x)=-ax^2+c=0$

or $x^2=-\frac{c}{a}; x^2=\frac{c}{a}$ as one of $\frac{c}{a}$ and $-\frac{c}{a}$ must be positive, so two roots must be real.

6. **(3)** (For the given equation to be meaningful we must have $x>0$. For $x>0$ the given equation can be written as $\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}=\log_x\sqrt{2}=\frac{1}{2}\log_x 2$

$\Rightarrow \frac{3}{4}t^2+t-\frac{5}{4}=\frac{1}{2}\left(\frac{1}{t}\right)$

By putting $t=\log_2 x$ so that $\log_x 2=\frac{1}{t}$ because

$\log_2 x\log_x 2=1$.

$\Rightarrow 3t^3+4t^2-5t-2=0 \quad \Rightarrow (t-1)(t+2)(3t+1)=0$

$\Rightarrow \log_2 x=t=1,-2,-\frac{1}{3}$

$\Rightarrow x=2,2^{-2},2^{-1/3}$ or $x=2,\frac{1}{4},\frac{1}{2^{1/3}}$

Thus, the given equation has exactly three real solutions out of which exactly one is irrational namely $\frac{1}{2^{1/3}}$

A = 2, B = 1

A + B^2 = 3

7. **(8)** Let for real roots are $\alpha,\beta,\gamma,\delta$ then equation is

$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$

$x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\beta\gamma+\gamma\delta+\alpha\delta+\beta\delta+\alpha\gamma)x^2-(\alpha\beta\gamma+\beta\gamma\delta+\alpha\beta\delta+\alpha\gamma\delta)x+\alpha\beta\gamma\delta=0$

$x^4-\sum\alpha.x^3+\sum\alpha\beta.x^2-\sum\alpha\beta\gamma.x+\alpha\beta\gamma\delta=0$

On comparing with $x^4-4x^3+ax^2+bx+1=0$

$\sum\alpha=4,\sum\alpha\beta=a$

$\sum\alpha\beta\gamma=-b,\alpha\beta\gamma\delta=1$

For real roots, A.M. of roots $\geq$ G.M. of roots

$\frac{1}{4}(\sum\alpha)\geq(\alpha\beta\gamma\delta)^{1/4}$; $\sum\alpha=4$

$\therefore\ \frac{1}{4}\sum\alpha = \frac{1}{4}\times 4 = 1$

$(\alpha\beta\gamma\delta)^{1/4} = 1 \Rightarrow\ \alpha\beta\gamma\delta = 1$

$\Sigma\alpha = 4$ and $\alpha\beta\gamma\delta = 1$

$\therefore\ \alpha = \beta = \gamma = \delta$ = 1

Now, $\sum\alpha\beta = a$

$\therefore a = \alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma$

$= 1\times1+1\times1+1\times1+1\times1+1\times1+1\times1$ = 6

$-b = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

$= 1\times1\times1+1\times1\times1+1\times1\times1+1\times1\times1$

$= (1)^3 + (1)^3 + (1)^3 + (1)^3 = 1+1+1+1 = 4$

$\therefore b = -4$; $\therefore a = 6$ and $b = -4$.

2a + b = 8

8. **(2)** $\alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

and $\alpha^2+\beta^2 = \frac{(b^2-2ac)}{a^2}$

Now $\frac{\alpha}{a\beta+b}+\frac{\beta}{a\alpha+b} = \frac{\alpha(a\alpha+b)+\beta(a\beta+b)}{(a\beta+b)(a\alpha+b)}$

$$= \frac{a(\alpha^2+\beta^2)+b(\alpha+\beta)}{\alpha\beta a^2+ab(\alpha+\beta)+b^2} = \frac{a\frac{(b^2-2ac)}{a^2}+b\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)a^2+ab\left(-\frac{b}{a}\right)+b^2}$$

$$= \frac{b^2-ac-b^2}{a^2c-ab^2+ab^2} = \frac{-2ac}{a^2c} = -\frac{2}{a}.$$

9. **(2)** Given equation can be written as

$x^2 + x(p+q-2r) + pq - pr - qr = 0$ ..... (i)

whose roots are α and $-\alpha$, then the product of roots

$-\alpha^2 = pq - pr - qr = pq - r(p+q)$ ..... (ii)

and sum $0 = p+q-2r \Rightarrow r = \frac{p+q}{2}$ ..... (iii)

From (ii) and (iii), we get $-\alpha^2 = pq - \frac{p+q}{2}(p+q) = -\frac{1}{2}\left\{(p+q)^2 - 2pq\right\} = -\frac{(P^2+q^2)}{2}$.

10. **(1)** Given that α and β be the roots of $x^2+x+1=0$, so $\alpha+\beta=-1$ and $\alpha\beta=1$

Again $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are the roots of $x^2+px+q=0$,

so $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-p \Rightarrow -p=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$\Rightarrow -p=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{1-2}{1}\Rightarrow p=1$

11. **(4)** $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(b^2-4ac)/a^2$ ...... (i)

Also $\{(\alpha-k)-(\beta-k)\}^2$

$=\{(\alpha-k)+(\beta-k)\}^2-4(\alpha-k)(\beta-k)$

$=(-B/A)^2-4(C/A)=(B^2-4AC)/A^2$ ..... (ii)

From (i) and (ii), $(b^2-4ac)/a^2=(B^2-4AC)/A^2$

$\therefore \quad \frac{b^2-4AC}{b^2-4ac}=\left(\frac{A}{a}\right)^2$

m = 2, n = -2

m-n = 4

12. **(0)** α,β are the roots of the equation $x^2-3x+1=0$

$\therefore \alpha+\beta=3$ and $\alpha\beta=1$

$S=\frac{1}{\alpha-2}+\frac{1}{\beta-2}=\frac{\alpha+\beta-4}{\alpha\beta-2(\alpha+\beta)+4}$

$=\frac{3-4}{1-2.3+4}=1$

and $P=\frac{1}{(\alpha-2)(\beta-2)}=\frac{1}{\alpha\beta-2(\alpha+\beta)+4}=-1$

Hence the equation whose roots are $\frac{1}{\alpha-2}$ and $\frac{1}{\beta-2}$ are $x^2-Sx+P=0\Rightarrow x^2-x-1=0$.

A = 1, B = -1

A + B = 0

13. **(10)** α,β are the roots of the equation $6x^2-6x+1=0$

$\Rightarrow \alpha+\beta=1,\ \alpha\beta=1/6$

$$\therefore \frac{1}{2}\left[a+b\alpha+c\alpha^2+d\alpha^3\right]+\frac{1}{2}\left[a+b\beta+c\beta^2+d\beta^3\right]$$

$$= a+\frac{1}{2}b(\alpha+\beta)+\frac{1}{2}c(\alpha^2+\beta^2)+\frac{1}{2}d(\alpha^3+\beta^3)$$

$$= a+\frac{1}{2}b+\frac{1}{2}c[(\alpha+\beta)^2-2\alpha\beta]+\frac{1}{2}d[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]$$

$$= a+\frac{b}{2}+\frac{1}{2}c\left[(1)^2-2.\frac{1}{6}\right]+\frac{1}{2}d\left[(1)^3-3.\frac{1}{6}\right]$$

$$= \frac{a}{1}+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}.$$

= A + B + C + D = 1 + 2 + 3 + 4 =10

14. **(34)** Let r be the common ratio of the G.P. $\alpha, \beta, \gamma, \delta$ then $\beta=\alpha r,\ \gamma=\alpha \boldsymbol{r^2}$ and $\delta=\alpha \boldsymbol{r^3}$

$\therefore \alpha+\beta=1 \Rightarrow \alpha+\alpha r=1 \Rightarrow \alpha(1+r)=1$ ….. (i)

$\alpha\beta=p \Rightarrow \alpha(\alpha r)=p \Rightarrow \alpha^2 r=p$ ….. (ii)

$\gamma+\delta=4 \Rightarrow \alpha r^2+\alpha r^3=4 \Rightarrow \alpha r^2(1+r)=4$ ….. (iii)

and $\gamma\delta=q \Rightarrow \alpha r^2.\alpha r^3=q \Rightarrow \alpha^2 r^5=q$ ….. (iv)

Dividing (iii) by (i), we get, $r^2=4 \Rightarrow r=\pm 2$

If we take $r=2$, then α is not integral, so we take $r=-2$,

Substituting $r=-2$ in (i), we get $\alpha=-1$

Now, from (ii), we have $p=\alpha^2 r=(-1)^2(-2)=-2$

and from (iv), we have $q=\alpha^2 r^5=(-1)^2(-2)^5=-32$

$\Rightarrow$ (p, q) = (– 2, – 32).

= |p + q| = 34

15. **(5)** Let roots are α, β so, $\frac{\alpha}{\beta}=\frac{2}{3} \Rightarrow \alpha=\frac{2\beta}{3}$

$$\therefore\ \alpha+\beta=\frac{m}{12}$$

$\Rightarrow \frac{2\beta}{3}+\beta=\frac{m}{12} \Rightarrow \frac{5\beta}{3}=\frac{m}{12}$ …..(i)

and $\alpha\beta=\frac{5}{12} \Rightarrow \frac{2\beta}{3}.\beta=\frac{5}{12} \Rightarrow \beta^2=\frac{5}{8}$

$\Rightarrow \beta = \sqrt{5/8}$

Put the value of β in (i), $\frac{5}{3}.\sqrt{\frac{5}{8}} = \frac{m}{12} \Rightarrow m = 5\sqrt{10}$.

16. **(40)** Let α be a common root, then

$\alpha^2 + a\alpha + 10 = 0$ ….. (i)

and $\alpha^2 + b\alpha - 10 = 0$ ….. (ii)

form (i) – (ii),

$(a-b)\alpha + 20 = 0 \Rightarrow \alpha = -\frac{20}{a-b}$

Substituting the value of α in (i), we get

$$\left(-\frac{20}{a-b}\right)^2 + a\left(-\frac{20}{a-b}\right) + 10 = 0$$

$\Rightarrow 400 - 20\,a(a-b) + 10(a-b)^2 = 0$

$\Rightarrow 40 - 2a^2 + 2ab + a^2 + b^2 - 2ab = 0$

$\Rightarrow a^2 - b^2 = 40$.

17. **(9)** Let $y = \frac{x^2 + 14x + 9}{x^2 + 2x + 3}$

$\Rightarrow y(x^2 + 2x + 3) - x^2 - 14x - 9 = 0$

$\Rightarrow (y-1)x^2 + (2y-14)x + 3y - 9 = 0$

For real *x*, its discriminant ≥ 0

i.e. $4(y-7)^2 - 4(y-1)3(y-3) \geq 0$

$\Rightarrow y^2 + y - 20 \leq 0$ or $(y-4)(y+5) \leq 0$

Now, the product of two factors is negative if these are of opposite signs. So following two cases arise:

Case I: $y - 4 \geq 0$ or $y \geq 4$ and $y + 5 \leq 0$ or $y \leq -5$

This is not possible.

Case II: $y - 4 \leq 0$ or $y \leq 4$ and $y + 5 \geq 0$ or $y \geq -5$ Both of these are satisfied if $-5 \leq y \leq 4$

Hence maximum value of *y* is 4 and minimum value is – 5.

Maximum – minimum = 9

18. **(1)** $f(x) = ax^2 + bx + c$

Let $F(x) = \int f(x)dx = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$

Clearly $F(0) = 0$ and $F(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0$

$\Rightarrow F(0) = F(1) = 0$

There exist at least one point *c* in between 0 and 1 such that $F'(x) = 0$ or $ax^2 + bx + c = 0$ for some $x \in (0, 1)$.

19. **(1)** Given, $\log(-2x) = 2\log(x+1)$

$\Rightarrow -2x = (x+1)^2 \quad \Rightarrow x^2 + 4x + 1 = 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{16-4}}{2} \Rightarrow x = \frac{-4 \pm \sqrt{12}}{2}$

$\Rightarrow x = -2 \pm \sqrt{3} \Rightarrow x = (-2+\sqrt{3}), (-2-\sqrt{3})$.

Product of roots = 4 – 3 = 1

20. **(4)** **Case I:** When $x + 2 \geq 0$ *i.e.* $x \geq -2$,

Then given inequality becomes

$x^2 - (x+2) + x > 0 \quad \Rightarrow x^2 - 2 > 0 \Rightarrow |x| > \sqrt{2}$

$\Rightarrow x < -\sqrt{2}$ or $x > \sqrt{2}$

As $x \geq -2$, therefore, in this case the part of the solution set is $[-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

Case II: When $x + 2 \leq 0$ *i.e.* $x \leq -2$,

Then given inequality becomes $x^2 + (x+2) + x > 0$

$\Rightarrow x^2 + 2x + 2 > 0 \Rightarrow (x+1)^2 + 1 > 0$, which is true for all real *x*

Hence, the part of the solution set in this case is $(-\infty, -2]$. Combining the two cases, the solution set is $(-\infty, -2) \cup ([-2, -\sqrt{2}] \cup (\sqrt{2}, \infty) = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

A = 2, B = 2

AB = 4