ODE of First Order

ODE

which involve only one independent variable and the differential coefficients w.r.t. it.

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \quad \text{--- (1)}$$

$$y = x\frac{dy}{dx} + \frac{c}{dy/dx} \quad \text{--- (2)}$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\cdot\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \quad \text{--- (3)}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\cdot\frac{d^2y}{dx^2} \quad \text{--- (4)}$$

Order

It is the order of the highest order derivative occurring in the diff. eqn.

e.g. Eqns (1) + (4) are of 2nd order
(2) is of 1st order
(3) is of 3rd order.

Degree

It is the degree of the highest order derivative which occurs in the diff. eqn. provided the eqn. has been made free of the radicals + fractions as far as the derivatives are concerned.

e.g. (1) + (3) are of degree one.

(2) $\Rightarrow \quad y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c \quad \Rightarrow$ Degree $= 2$

(4) $\Rightarrow \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2 \Rightarrow$ Degree = 2

(*) Solution of a D.E.

It is a relation, free from derivatives, b/w the variables which satisfies the eqn.

e.g. $y = c_1\cos x + c_2\sin x$ is soln. of the differential eqn. $\frac{d^2y}{dx^2} + y = 0$. —— (1)

(*) General (Complete) solution

No. of arbitrary constants = Order of DE

e.g. $y = c_1\cos x + c_2\sin x$ is general soln. of $\frac{d^2y}{dx^2} + y = 0$ of 2^{nd} order.

(*) Particular Solution

obtained from its general solution by giving particular values to constants.

e.g. (1) $\Rightarrow$ $\left.\begin{array}{l} y = \cos x \\ y = \cos x - \sin x \end{array}\right]$ is particular soln. of $\frac{d^2y}{dx^2} + y = 0$.

Ques. Obtain the differential eqns :-

(i) $y = A + Bx + Cx^2$

$\frac{dy}{dx} = B + 2Cx$, $\frac{d^2y}{dx^2} = 2C$

$$\boxed{\frac{d^3y}{dx^3} = 0}$$ ⇒ Ans.

(ii) $y = A\cos 2t + B\sin 2t$

$\frac{dy}{dt} = -2A\sin 2t + 2B\cos 2t$

$\frac{d^2y}{dt^2} = -4A\cos 2t - 4B\sin 2t$

$= -4(A\cos 2t + B\sin 2t)$

$= -4y$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + 4y = 0}$$ Ans.

Ques. All circles of radius r whose centre lie on the x-axis.

Soln $(x-a)^2 + y^2 = r^2$ ——— (1)

$2(x-a) + 2y\frac{dy}{dx} = 0$ ——— (2)

$x - a = -y\frac{dy}{dx}$ ——— (3)

(1) ⇒ $y^2\left(\frac{dy}{dx}\right)^2 + y^2 = r^2$

$y^2 = \frac{r^2}{1+\left(\frac{dy}{dx}\right)^2}$ ——— (4)

① $\Rightarrow \quad y^2\left(\frac{dy}{dx}\right)^2 + y^2 = x^2$

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{x^2}{y^2}$$

$$\Rightarrow y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = x^2$$

Ans

✱ Solution of DE of 1st order & 1st degree.

① Variable seperable method

② $\frac{dy}{dx} = f(ax+by+c) \Rightarrow ax+by+c = t$

$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

③ Homogeneous eqns. $\Rightarrow y = vx$

④ Linear eqns.

⑤ Exact eqns.

Ques. $\frac{dy}{dx} = e^{2x+3y}$

Soln. $\frac{dy}{dx} = e^{2x} \cdot e^{3y}$

$$\frac{dy}{e^{3y}} = e^{2x}\,dx \Rightarrow e^{-3y}\,dy = e^{2x}\,dx$$

$$\Rightarrow \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c$$

$$\Rightarrow 3e^{2x} + 2e^{-3y} = c$$

Ans

Ques. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Soln. $\sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$$\frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y}{\tan y} dy$$

Integrating

$$\log \tan x = -\log \tan y + \log c$$

$$\Rightarrow \tan x \tan y = c$$

Ans.

Ques. $\frac{dy}{dx} = (4x + y + 1)^2$

Soln. $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\frac{dt}{t^2 + 4} = dx$$

$$\frac{1}{2} \tan^{-1} \frac{t}{2} = x + c'$$

$$\tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + c$$

$$\Rightarrow 4x + y + 1 = 2\tan(2x + c)$$

Ans.

Ques: $(x+y)\,dx + (y-x)\,dy = 0$.

Soln: $$\frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{———} \quad (1)$$

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$(1) \Rightarrow v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$= \frac{1+v-v+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}$$

$$\frac{dv}{1+v^2} - \frac{1}{2}\frac{2v}{1+v^2}\,dv = \frac{dx}{x}$$

Integrating, we get

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) = \log x + c$$

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2+y^2) + \log x = \log x + c$$

$$\tan^{-1}\frac{y}{x} = \frac{1}{2}\log(x^2+y^2) + c$$

Ans

(*) Exact Differential Eqn.

A D.E. of the form $Mdx + Ndy = 0$ is s.t.b exact if it can be obtained directly by differentiating the eqn. $u(x,y) = c$ which is its primitive i.e. if

$$du = Mdx + Ndy$$

where M and N are fns. of x & y.

Thm:- The necessary and sufficient condition for the differential eqn. $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Soln. Necessay Condition

The eqn. $Mdx + Ndy = 0$ will be exact if $du = Mdx + Ndy$.

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\therefore \frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Since $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

which is the reqd. necessary condition.

Sufficient Condition

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

T.P. $Mdx + Ndy = 0$ is exact.

Let $u = \int_{y \text{ constt}} M dx$

$\Rightarrow M = \frac{\partial u}{\partial x}$ & $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$

Also $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Integrating w.r.t. x keeping y constt.

$$N = \frac{\partial u}{\partial y} + f(y)$$

$$\therefore Mdx + Ndy = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + f(y)\, dy$$

$$= du + d\left[\int f(y)\, dy\right]$$

$$= d\left[u + \int f(y)\, dy\right]$$

which is an exact diff. eqn.

$\therefore Mdx + N dy = 0$ is exact.

Hence proved.

NOTE $\Rightarrow$ Soln. of $M\,dx + N\,dy = 0$.

Since $M\,dx + N\,dy = d\left[u + \int f(y)\,dy\right]$

$\therefore\ u + \int f(y)\,dy = c$ is soln.

$$\boxed{\Rightarrow \int_{y\ \text{constt.}} M\,dx + \int (\text{Terms of } N \text{ free from } x)\,dy = c}$$

Ques. Solve :

$$\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy\,e^{xy^2} - 3y^2\right)dy = 0 \quad \text{—} (1)$$

Soln.

$$M = y^2 e^{xy^2} + 4x^3$$

$$N = 2xy\,e^{xy^2} - 3y^2$$

$$\frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 e^{xy^2}\cdot 2xy$$

$$= 2y\,e^{xy^2} + 2xy^3 e^{xy^2}$$

$$\frac{\partial N}{\partial x} = 2y\,e^{xy^2} + 2xy^3 e^{xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow (1) \text{ is exact.}$$

$\therefore$ Soln. is

$$\int_{y\ \text{constt.}} \left(y^2 e^{xy^2} + 4x^3\right)dx - \int 3y^2\,dy = C$$

$$y^2\,\frac{e^{xy^2}}{y^2} + x^4 - y^3 = C$$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = C$$

Ans.