Q13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$ using integral test

let $f(x) = \dfrac{1}{x(\ln x)^4}$ cont on $[1, \infty)$ since $x \neq 0$

$x > 0$ so $f(x)$ is positive

$$f'(x) = -\frac{(x(\ln x)^4)'}{(x(\ln x)^4)^2} = \frac{\overset{\text{neg}}{-}\left((\ln x)^4 + \left(4(\ln x)^3\right)\frac{x}{x}\right)}{\underset{\text{Pos}}{(x(\ln x)^4)^2}}$$

neg, Pos

$f(x)$ is decreasing

can use integral test

$$\int_2^{\infty} \frac{1}{x(\ln x)^4} = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^4}$$

Solve for indefinite integral first:

$$= \int \frac{1}{x(\ln x)^4}\,dx \qquad u = \ln x \qquad du = \frac{1}{x}dx, \qquad du = x\,du$$

$$= \int \frac{\cancel{x}\,du}{\cancel{x}\cdot u^4} = \int \frac{du}{u^4} = \int u^{-4}\,du = \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{3(\ln x)^3}$$

Now, solve using definite integral:

$$\lim_{t\to\infty}\left[-\frac{1}{3(\ln x)^3}\Big|_2^t\right] = \lim_{t\to\infty}\left[\frac{1}{-3(\ln t)^3}\right] + \frac{1}{3(\ln 2)^3}$$

$$= \frac{1}{\infty} + \frac{1}{3(\ln 2)^3} = 0 + \frac{1}{3(\ln 2)^3} \Rightarrow \text{Convergent}$$

$$\boxed{\text{Since } \int_2^{\infty} \frac{1}{x(\ln x)^4} \text{ is convergent, } \therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4} \text{ is convergent}}$$