From: http://mathoverflow.net/questions/177574/existence-of-solutions-of-a-polynomial-system

Fix $k \in \mathbb{N}, k \geq 1$. Let $p \in [0, 1]$ and $x = (x_0, \ldots, x_k)$ be a $(k+1)$-dimensional *real* vector, and define

$$S_k(p, x) = -x_0^2 + \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} \cdot (x_i - p)^2.$$

Experiments show that for small values of k

$$\exists x \in \text{ interior of } [0, 1]^{k+1} . \ \forall p \in [0, 1] . \ S_k(p, x) = 0.$$

In other words, there are x_i's such that $S_k(x, p)$ is identically zero as a polynomial in p.
For a given k we can expand $S_k(x, p)$ as a polynomial in p and equate the coefficients to 0. For $k = 2$ we get

and this has two solutions:

$$x = (\frac{1}{2}(-1 - \sqrt{2}), \frac{1}{2}, \frac{1}{2}(3 + \sqrt{2}))$$

and

$$x = (\frac{1}{2}(-1 + \sqrt{2}), \frac{1}{2}, \frac{1}{2}(3 - \sqrt{2})).$$

For $k = 1, 2, 3, 4, 5, 6, 7$ there are $1, 2, 4, 8, 14, 28, 48$ solutions respectively, according to Mathematica. According to OEIS (https://oeis.org/searc q=1%2C%202%2C%204%2C%208%2C%2014%2C%2028%2C%2048) this is A068912 (https://oeis.org/A068912), "the number of n step walks (each step ± 1 starting from 0) wh are never more than 3 or less than -3." This is kind of interesting because the problem arises in statistics, see John Mount's blog post (http://www.w vector.com/blog/2014/07/frequenstist-inference-only-seems-easy/) for background.

Question: Is there a solution for every k?

Addendum: John says he wants soltions in $[0, 1]^{k+1}$...

Here is the relevant Mathematica code:

```
s[k_, p_, x_] := Sum[Binomial[k, i] * p^i* (1 - p)^(k - i)* (Subscript[x, i] - p)^2, {i, 0, k}]  Subscript[x, 0]^2
xs[k_] := Table[Subscript[x, i], {i, 0, k}]
system[k_, p_, x_] := Thread[CoefficientList[s[k, p, x], p] == 0]
solutions[k_] := Solve[system[k, p, x], xs[k], Reals]
```

To see the system of equations for $k = 4$, type

```
system[4, p, x] // ColumnForm
```

To see the solutions for $k = 4$, type

```
solutions[4]
```

To make a table of counts of solutions up to $k = 7$, type

```
Table[{k, Length@solutions[k]}, {k, 1, 7}] // ColumnForm
```

In [0]:

Solution submitted 8-4-2013 by me (John Mount), but not accepted by MathOverflow for reasons of links and formatting. Enough of that submitting it here.

This is some background to the question and the solution (minus one check mentioned at the end).

Define

$$S(k,p,x) = \sum_{i=0}^{k} \binom{k}{i} p^i (1-p)^{k-i} (x_i - p)^2.$$

Define

$$f_k(k) = \text{argmin}_x \max_p S(k,p,x).$$

Then $f_k(k)$ is the minimax square-loss solution to trying to estimate the win rate of a random process by observing k results (Wald wrote on this). The neat thing is: we can show if the is a real solution x in $[0,1]^{k+1}$ to $S(k,p,x) = x_0^2$ then $x = f_k(k)$. Meaning we avoided two nasty quantifiers. See this (https://github.com/WinVector/Examples/blob/master/freq/python/freqMin.rst) for some experimental examples. The proof this is optimal (not just extremal involves using $p - f_k(h)$ show when curves are coincident we have a diversity of signs of gradients in various directions.

From the original problem we expect a lot of symmetries. Also, a change of variables $z = p/(1-p)$ makes collecting terms easier. In fact I now have a conjectured exact solution, I n only need a proof that it always works (cancels the p's, is real and in the interior of $[0,1]^{k+1}$; I already have a proof that such a solution when it exists solves the original estimati problem). The conjectured solution for $k > 1$ (for $k = 1$ the solution is $[1/4, 3/4]$) is:

$$\begin{aligned}
f_k(0) &= (\sqrt{k} - 1)/(2(k-1)) \\
f_k(1) &= \sqrt{f_k(0)^2 + 2f_k(0)/k} \\
&\text{for } h > 1: \\
f_k(h)^2 &= (k+2)(k+1)(f_k(0)^2)/((k+2-h)(k+1-h)) \\
&\quad + 2hf_k(h-1)(1 - f_k(h-1))/(k+1-h) \\
&\quad - h(h-1)((f_k(h-2)-1)^2)/((k+2-h)(k+1-h))
\end{aligned}$$

A Python implementation, demonstration and check of this solution up through $k = 8$ is given h (https://github.com/WinVector/Examples/blob/master/freq/python/explicitSolution.rst). So really all that is left to prove is the right hand side of $f_k(h)^2$ is always positive and in the inter of $[0,1]$ for all k, h.

In [1]:
```
import sympy


# expecting a dictionary solution
def isGoodSoln(si):
   def isGoodVal(x):
      xn = complex(x)
      xr = xn.real
      xi = xn.imag
      return (abs(xi)<1.0e-6) and (xr>0.0) and (xr<1.0)
   return all([ isGoodVal(xi) for xi in si.values() ])


# only good for k>=1
def solveKz(k):
   vars = sympy.symbols(['phi' + str(i) for i in range((k+1)/2)])
   if k%2!=0:
      phis = vars + [1-varsi for varsi in reversed(vars) ]
   else:
      phis = vars + [sympy.Rational(1,2)] + [1-varsi for varsi in reversed(vars) ]
   z = sympy.symbols('z')
   poly = sum([ sympy.binomial(k,h) * z**h * ((1+z)*phis[h] -z)**2 for h in range(k+1)]) - phis[0]**2 * (1+z)**(k+2)
   polyTerms = poly.expand().collect(z,evaluate=False)
   eqns = [ polyTerms[ki] for ki in polyTerms.keys() if (not ki==1) ]
   solns = sympy.solve(eqns,vars,dict=True)
   soln1 = [ si for si in solns if isGoodSoln(si)][0]
   solnv = [ soln1[vi] for vi in vars ]
   if k%2!=0:
      xs = solnv + [1-solni for solni in reversed(solnv) ]
   else:
      xs = solnv + [sympy.Rational(1,2)] + [1-solni for solni in reversed(solnv) ]
   return xs

# only good for k>=1
def conjectureK(k,numeric=False):
   if k<=1:
      return [sympy.Rational(1,4),sympy.Rational(3,4)]
   phi = [ 0 for i in range(k+1) ]
   phi[0] = (sympy.sqrt(k)-1)/(2*(k-1))
   phi[1] = (sympy.sqrt((phi[0]**2+2*phi[0]/k).expand())).simplify()
   if numeric:
      for h in range(2):
         phi[h] = float(phi[h])
   for h in range(2,(k+1)):
      phi[h] = sympy.sqrt(( (k+2)*(k+1)*(phi[0]**2)/((k+2-h)*(k+1-h)) + 2*h*phi[h-1]*(1-phi[h-1])/(k+1-h) - h*(h-1)*((phi[h-2]
1)**2)/((k+2-h)*(k+1-h)) ))
   return phi
```

In [2]:
```
p = sympy.symbols('p')
for k in range(1,9):
   print
   print 'k',k
   solnk = solveKz(k)
   print 'soln       ',solnk
   poly = sum([ p**h * (1-p)**(k-h) * sympy.binomial(k,h) * (solnk[h]-p)**2 for h in range(k+1) ]).expand()
   print 'check poly',poly
   conjk = conjectureK(k,numeric=True)
   print 'conjecture:',conjk
   print 'max difference:',max([ abs(complex(solnk[i]-conjk[i])) for i in range(len(solnk)) ])
   print '1/k for scale:',1/float(k)
   print
```

```
k 1
soln        [1/4, 3/4]
check poly 1/16
conjecture: [1/4, 3/4]
max difference: 0.0
1/k for scale: 1.0


k 2
soln        [-1/2 + sqrt(2)/2, 1/2, -sqrt(2)/2 + 3/2]
check poly -sqrt(2)/2 + 3/4
conjecture: [0.20710678118654752, 0.5, 0.792893218813452]
max difference: 6.26858358953e-17
1/k for scale: 0.5


k 3
soln        [-1/4 + sqrt(3)/4, sqrt(3)/12 + 1/4, -sqrt(3)/12 + 3/4, -sqrt(3)/4 + 5/4]
check poly -sqrt(3)/8 + 1/4
conjecture: [0.18301270189221933, 0.39433756729740643, 0.605662432702594, 0.816987298107781]
max difference: 2.50877105545e-17
1/k for scale: 0.333333333333


k 4
soln        [1/6, 1/3, 1/2, 2/3, 5/6]
check poly 1/36
conjecture: [0.16666666666666666, 0.3333333333333333, 0.500000000000000, 0.666666666666667, 0.833333333333333]
max difference: 1.11022302463e-16
1/k for scale: 0.25


k 5
soln        [-1/8 + sqrt(5)/8, 1/8 + 3*sqrt(5)/40, sqrt(5)/40 + 3/8, -sqrt(5)/40 + 5/8, -3*sqrt(5)/40 + 7/8, -sqrt(5)/8 + 9/8]
check poly -sqrt(5)/32 + 3/32
conjecture: [0.15450849718747373, 0.2927050983124842, 0.430901699437495, 0.569098300562505, 0.707294901687516, 0.8454915028125
7]
max difference: 3.19486619378e-16
1/k for scale: 0.2


k 6
soln        [-1/10 + sqrt(6)/10, 1/10 + sqrt(6)/15, sqrt(6)/30 + 3/10, 1/2, -sqrt(6)/30 + 7/10, -sqrt(6)/15 + 9/10, -sqrt(6)/1
 + 11/10]
check poly -sqrt(6)/50 + 7/100
conjecture: [0.14494897427831782, 0.2632993161855452, 0.381649658092773, 0.500000000000000, 0.618350341907227, 0.7367006838144
5, 0.855051025721682]
max difference: 3.77994123684e-16
1/k for scale: 0.166666666667


k 7
soln        [-1/12 + sqrt(7)/12, 1/12 + 5*sqrt(7)/84, sqrt(7)/28 + 1/4, sqrt(7)/84 + 5/12, -sqrt(7)/84 + 7/12, -sqrt(7)/28 + 3
4, -5*sqrt(7)/84 + 11/12, -sqrt(7)/12 + 13/12]
check poly -sqrt(7)/72 + 1/18
conjecture: [0.13714594258871587, 0.24081853042051135, 0.344491118252307, 0.448163706084102, 0.551836293915898, 0.655508881747
93, 0.759181469579488, 0.862854057411286]
max difference: 1.69186951436e-15
1/k for scale: 0.142857142857


k 8
soln        [-1/14 + sqrt(2)/7, 1/14 + 3*sqrt(2)/28, sqrt(2)/14 + 3/14, sqrt(2)/28 + 5/14, 1/2, -sqrt(2)/28 + 9/14, -sqrt(2)/1
 + 11/14, -3*sqrt(2)/28 + 13/14, -sqrt(2)/7 + 15/14]
check poly -sqrt(2)/49 + 9/196
conjecture: [0.1306019374818707, 0.22295145311140305, 0.315300968740935, 0.407650484370468, 0.500000000000000, 0.5923495156295
2, 0.684699031259065, 0.777048546888597, 0.869398062518130]
max difference: 5.41301093293e-16
1/k for scale: 0.125
```

In [3]:
```
p = sympy.symbols('p')
for k in range(1,21):
   print
   print 'k',k
   conjk = conjectureK(k,numeric=True)
   print 'conjecture:',conjk
   polyc = sum([ p**h * (1-p)**(k-h) * sympy.binomial(k,h) * (conjk[h]-p)**2 for h in range(k+1) ]).expand()
   print 'conjecture check poly',polyc
   print '1/k for scale:',1/float(k)
   print
```

```
k 1
conjecture: [1/4, 3/4]
conjecture check poly 1/16
1/k for scale: 1.0


k 2
conjecture: [0.20710678118654752, 0.5, 0.792893218813452]
conjecture check poly 2.22044604925031e-16*p**3 + 0.0428932188134525
1/k for scale: 0.5


k 3
conjecture: [0.18301270189221933, 0.39433756729740643, 0.605662432702594, 0.816987298107781]
conjecture check poly -1.33226762955019e-15*p**4 - 8.88178419700125e-16*p**3 + 5.55111512312578e-17*p + 0.0334936490538903
1/k for scale: 0.333333333333


k 4
conjecture: [0.16666666666666666, 0.3333333333333333, 0.500000000000000, 0.666666666666667, 0.833333333333333]
conjecture check poly -8.88178419700125e-16*p**5 + 1.77635683940025e-15*p**4 + 0.0277777777777778
1/k for scale: 0.25


k 5
conjecture: [0.15450849718747373, 0.2927050983124842, 0.430901699437495, 0.569098300562505, 0.707294901687516, 0.8454915028125
7]
conjecture check poly -7.105427357601e-15*p**6 + 7.105427357601e-15*p**4 - 3.5527136788005e-15*p**3 + 8.88178419700125e-16*p**
 - 5.55111512312578e-17*p + 0.0238728757031316
1/k for scale: 0.2


k 6
conjecture: [0.14494897427831782, 0.2632993161855452, 0.381649658092773, 0.500000000000000, 0.618350341907227, 0.7367006838144
5, 0.855051025721682]
conjecture check poly -1.37667655053519e-14*p**7 + 7.43849426498855e-15*p**6 - 1.77635683940025e-15*p**5 + 4.44089209850063e-1
*p**2 - 5.55111512312578e-17*p + 0.0210102051443364
1/k for scale: 0.166666666667


k 7
conjecture: [0.13714594258871587, 0.24081853042051135, 0.344491118252307, 0.448163706084102, 0.551836293915898, 0.655508881747
93, 0.759181469579488, 0.862854057411286]
conjecture check poly 2.22044604925031e-15*p**8 + 1.08801856413265e-13*p**7 - 1.13686837721616e-13*p**6 - 2.8421709430404e-14*
**4 + 7.105427357601e-15*p**3 + 5.55111512312578e-17*p + 0.0188090095685473
1/k for scale: 0.142857142857


k 8
conjecture: [0.1306019374818707, 0.22295145311140305, 0.315300968740935, 0.407650484370468, 0.500000000000000, 0.5923495156295
2, 0.684699031259065, 0.777048546888597, 0.869398062518130]
conjecture check poly 7.105427357601e-14*p**8 - 5.6843418860808e-14*p**7 - 1.27897692436818e-13*p**6 + 5.6843418860808e-14*p**
 - 1.4210854715202e-14*p**4 + 1.06581410364015e-14*p**3 - 8.88178419700125e-16*p**2 + 1.11022302462516e-16*p + 0.0170568660740
85
1/k for scale: 0.125


k 9
conjecture: [0.125, 0.20833333333333334, 0.291666666666667, 0.375000000000000, 0.458333333333333, 0.541666666666667, 0.6250000
0000000, 0.708333333333333, 0.791666666666667, 0.874999999999993]
conjecture check poly 2.27373675443232e-13*p**8 - 4.54747350886464e-13*p**7 + 4.54747350886464e-13*p**6 + 5.6843418860808e-14*
**4 - 7.105427357601e-15*p**3 - 4.44089209850063e-16*p**2 + 5.55111512312578e-17*p + 0.015625
1/k for scale: 0.111111111111


k 10
conjecture: [0.12012653667602108, 0.19610122934081686, 0.272075922005613, 0.348050614670408, 0.424025307335204, 0.500000000000
00, 0.575974692664796, 0.651949385329591, 0.727924077994388, 0.803898770659182, 0.879873463323994]
```

```
conjecture check poly -4.54747350886464e-13*p**10 - 7.95807864051312e-13*p**9 + 1.53477230924182e-12*p**8 + 5.82645043323282e-
3*p**7 - 2.27373675443232e-13*p**5 + 1.13686837721616e-13*p**4 - 2.1316282072803e-14*p**3 + 1.77635683940025e-15*p**2 + 0.0144
03848137754
1/k for scale: 0.1


k 11
conjecture: [0.11583123951777, 0.18568010505999363, 0.255528970602217, 0.325377836144441, 0.395226701686665, 0.465075567228888
 0.534924432771112, 0.604773298313335, 0.674622163855559, 0.744471029397782, 0.814319894940009, 0.884168760482201]
conjecture check poly -9.09494701772928e-13*p**12 - 1.81898940354586e-12*p**11 + 2.72848410531878e-12*p**10 + 1.36424205265939
-12*p**9 + 1.02318153949454e-12*p**8 + 4.59010607301025e-12*p**7 - 1.36424205265939e-12*p**6 + 4.40536496171262e-13*p**5 - 1.1
686837721616e-13*p**4 + 2.8421709430404e-14*p**3 - 8.88178419700125e-16*p**2 - 5.55111512312578e-17*p + 0.013416876048223
1/k for scale: 0.0909090909091


k 12
conjecture: [0.11200461886989793, 0.17667051572491493, 0.241336412579932, 0.306002309434949, 0.370668206289966, 0.435334103144
83, 0.500000000000000, 0.564665896855017, 0.629331793710034, 0.693997690565051, 0.758663587420068, 0.823329484275084, 0.887995
81130121]
conjecture check poly -4.54747350886464e-13*p**13 - 3.63797880709171e-12*p**11 + 1.45519152283669e-11*p**10 - 1.45519152283669
-11*p**9 + 1.45519152283669e-11*p**8 - 7.27595761418343e-12*p**7 - 9.09494701772928e-13*p**5 + 5.6843418860808e-14*p**4 - 2.84
1709430404e-14*p**3 + 3.5527136788005e-15*p**2 + 0.0125450346481911
1/k for scale: 0.0833333333333


k 13
conjecture: [0.10856463647766622, 0.16878546163494834, 0.229006286792230, 0.289227111949513, 0.349447937106795, 0.409668762264
77, 0.469889587421359, 0.530110412578641, 0.590331237735923, 0.650552062893205, 0.710772888050488, 0.770993713207768, 0.831214
38365061, 0.891435363522209]
conjecture check poly 1.81898940354586e-12*p**14 - 3.63797880709171e-12*p**13 - 1.45519152283669e-11*p**12 + 5.82076609134674e
11*p**11 + 1.45519152283669e-11*p**10 + 1.45519152283669e-11*p**9 - 1.45519152283669e-11*p**8 + 7.105427357601e-15*p**3 - 1.77
35683940025e-15*p**2 + 5.55111512312578e-17*p + 0.0117862802935278
1/k for scale: 0.0769230769231


k 14
conjecture: [0.10544836102976697, 0.1618128808826574, 0.218177400735548, 0.274541920588438, 0.330906440441329, 0.3872709602942
9, 0.443635480147110, 0.500000000000000, 0.556364519852890, 0.612729039705781, 0.669093559558672, 0.725458079411560, 0.7818225
9264458, 0.838187119117306, 0.894551638970746]
conjecture check poly -3.63797880709171e-12*p**15 + 4.36557456851006e-11*p**14 - 4.36557456851006e-11*p**13 + 1.16415321826935
-10*p**12 + 1.45519152283669e-11*p**11 + 1.57342583406717e-10*p**10 - 8.5265128291212e-12*p**9 - 4.36557456851006e-11*p**8 + 7
27595761418343e-12*p**7 - 9.09494701772928e-13*p**6 + 1.36424205265939e-12*p**5 + 3.41060513164848e-13*p**4 + 0.01111935684386
1
1/k for scale: 0.0714285714286


k 15
conjecture: [0.10260654807883632, 0.1555923416683248, 0.208578135257813, 0.261563928847302, 0.314549722436790, 0.3675355160262
9, 0.420521309615767, 0.473507103205256, 0.526492896794744, 0.579478690384233, 0.632464483973721, 0.685450277563210, 0.7384360
1152698, 0.791421864742188, 0.844407658331663, 0.897393451921343]
conjecture check poly -1.09139364212751e-11*p**16 + 5.82076609134674e-11*p**15 + 8.73114913702011e-11*p**14 + 7.27595761418343
-11*p**13 + 1.45519152283669e-10*p**12 + 1.60071067512035e-10*p**11 - 1.28466126625426e-10*p**10 + 9.25410859053954e-11*p**9 +
2.5465851649642e-11*p**7 + 1.02318153949454e-12*p**6 + 1.59161572810262e-12*p**5 + 1.4210854715202e-13*p**4 - 7.105427357601e-
5*p**3 + 0.0105281037086545
1/k for scale: 0.0666666666667


k 16
conjecture: [0.1, 0.15, 0.200000000000000, 0.250000000000000, 0.300000000000000, 0.350000000000000, 0.400000000000000, 0.45000
000000000, 0.500000000000000, 0.550000000000000, 0.600000000000000, 0.650000000000000, 0.700000000000000, 0.749999999999999, 0
800000000000001, 0.849999999999987, 0.900000000000195]
conjecture check poly -1.81898940354586e-11*p**17 - 1.01863406598568e-10*p**16 - 8.73114913702011e-11*p**15 + 1.60071067512035
-10*p**14 - 1.14960130304098e-9*p**13 + 1.8007995095104e-10*p**12 + 8.84483597474173e-11*p**11 - 1.32786226458848e-10*p**10 +
.69166014529765e-10*p**9 - 4.00177668780088e-11*p**8 - 2.91038304567337e-11*p**7 - 5.91171556152403e-12*p**6 - 1.5916157281026
e-12*p**5 + 1.70530256582424e-13*p**4 - 7.105427357601e-15*p**3 + 1.77635683940025e-15*p**2 - 5.55111512312578e-17*p + 0.01
1/k for scale: 0.0625


k 17
conjecture: [0.0975970508005519, 0.14493857423578108, 0.192280097671010, 0.239621621106239, 0.286963144541469, 0.3343046679766
8, 0.381646191411927, 0.428987714847156, 0.476329238282385, 0.523670761717615, 0.571012285152844, 0.618353808588073, 0.6656953
2023302, 0.713036855458531, 0.760378378893763, 0.807719902328977, 0.855061425764318, 0.902402949197767]
conjecture check poly 5.82076609134674e-11*p**18 - 3.49245965480804e-10*p**17 + 2.3283064365387e-10*p**16 - 3.14321368932724e-
*p**15 + 8.73114913702011e-10*p**14 - 2.56113708019257e-9*p**13 + 1.38243194669485e-10*p**12 + 2.20006768358871e-9*p**11 + 3.2
142135024071e-10*p**10 + 1.16415321826935e-10*p**9 + 2.47382558882236e-10*p**8 + 2.18278728425503e-11*p**7 + 3.63797880709171e
12*p**6 - 9.09494701772928e-13*p**5 + 2.27373675443232e-13*p**4 - 2.8421709430404e-14*p**3 + 1.77635683940025e-15*p**2 - 1.110
2302462516e-16*p + 0.00952518432496551
1/k for scale: 0.0588235294118
```

```
k 18
conjecture: [0.0953717849152731, 0.14033047548024274, 0.185289166045212, 0.230247856610182, 0.275206547175152, 0.3201652377401
1, 0.365123928305091, 0.410082618870061, 0.455041309435030, 0.500000000000000, 0.544958690564970, 0.589917381129939, 0.6348760
1694909, 0.679834762259878, 0.724793452824850, 0.769752143389811, 0.814710833954823, 0.859669524519452, 0.904628215090205]
conjecture check poly -8.02060640126001e-11*p**19 + 2.29277929975069e-10*p**18 - 1.90539140021428e-10*p**17 + 3.85398379876278
-9*p**16 - 9.00399754755199e-10*p**15 - 6.15546014159918e-9*p**14 + 4.82395989820361e-9*p**13 + 5.75528247281909e-9*p**12 + 2.
1916707232594e-9*p**11 - 2.40834197029471e-9*p**10 + 8.14907252788544e-10*p**9 - 8.36735125631094e-11*p**8 + 5.45696821063757e
11*p**7 + 3.63797880709171e-12*p**6 - 4.54747350886464e-13*p**5 - 2.8421709430404e-13*p**4 - 2.8421709430404e-14*p**3 - 8.8817
419700125e-16*p**2 + 0.00909577735792511
1/k for scale: 0.0555555555556


k 19
conjecture: [0.09330274843168537, 0.1361129854388764, 0.178923222446067, 0.221733459453258, 0.264543696460449, 0.3073539334676
0, 0.350164170474831, 0.392974407482022, 0.435784644489213, 0.478594881496404, 0.521405118503595, 0.564215355510786, 0.6070255
2517978, 0.649835829525168, 0.692646066532360, 0.735456303539549, 0.778266540546746, 0.821076777553906, 0.863887014561363, 0.9
6697251563763]
conjecture check poly 2.9849189786546e-10*p**20 + 9.40303834795486e-10*p**19 + 5.04905983689241e-9*p**18 - 1.49611878441647e-9
p**17 + 1.15578586701304e-8*p**16 - 9.6588337328285e-9*p**15 + 7.99627741798759e-9*p**14 + 7.93079379945993e-9*p**13 - 2.16823
36902666e-9*p**12 - 7.56699591875076e-10*p**11 - 1.29512045532465e-9*p**10 - 6.54836185276508e-11*p**9 + 4.07453626394272e-10*
**8 - 9.09494701772928e-11*p**7 - 3.18323145620525e-12*p**5 + 5.6843418860808e-14*p**4 + 3.5527136788005e-14*p**3 - 8.88178419
00125e-16*p**2 + 5.55111512312578e-17*p + 0.00870540286490637
1/k for scale: 0.0526315789474


k 20
conjecture: [0.0913719988157784, 0.13223479893420056, 0.173097599052623, 0.213960399171045, 0.254823199289467, 0.2956859994078
9, 0.336548799526311, 0.377411599644734, 0.418274399763156, 0.459137199881578, 0.500000000000000, 0.540862800118422, 0.5817256
0236844, 0.622588400355266, 0.663451200473689, 0.704314000592110, 0.745176800710534, 0.786039600828950, 0.826902400947407, 0.8
7765201065522, 0.908628001189783]
conjecture check poly -4.65661287307739e-10*p**21 + 3.72529029846191e-9*p**20 - 1.11758708953857e-8*p**19 + 2.98023223876953e-
*p**17 + 7.45058059692383e-8*p**15 - 2.98023223876953e-8*p**14 - 9.31322574615479e-9*p**13 + 3.25962901115417e-9*p**12 - 3.732
662560761e-9*p**10 + 6.98491930961609e-10*p**9 - 5.23868948221207e-10*p**8 + 1.8007995095104e-10*p**7 - 5.82076609134674e-11*p
*6 + 3.63797880709171e-12*p**5 + 2.8421709430404e-14*p**3 - 8.88178419700125e-16*p**2 + 5.55111512312578e-17*p + 0.00834884216
59061
1/k for scale: 0.05
```