TABLE OF PERCENTAGE POINTS OF KOLMOGOROV STATISTICS

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A simple method for testing the probability that a set of numbers is a sample from a known distribution consists of comparing the empirical cumulative distribution function of the sample, $S_n(x)$, with the known cumulative distribution function $F(x)$. Both $D_n = \text{maximum}\{S_n(x) - F(x)\}$ and $D_n^* = \text{maximum}|S_n(x) - F(x)|$ are random variables, independent of the special form of $F(x)$, if $F(x)$ is continuous. This paper contains more extensive tables of the percentage points in the distributions of D_n and D_n^* than have been published previously. These values are obtained by empirical modification of a known asymptotic formula.

1. INTRODUCTION

In order to test whether a sample of n numbers $(x_1, x_2, \cdots, x_n)$ comes from the cumulative distribution function $F(x)$, we may use either the statistic

$$D_n = \text{maximum } \{S_n(x) - F(x)\},$$

or the statistic

$$D_n^* = \text{maximum } |S_n(x) - F(x)|,$$

where $S_n(x)$ is the sample cumulative distribution function.

As an illustration, suppose that we wish to test the hypothesis that the five numbers .52, .65, .13, .71, and .58 were chosen at random from numbers uniformly distributed between zero and one. Then $S_5(x)$ has the value of zero between $x=0$ and $x=.13$. At $x=.13$, $S_5(x)$ is discontinuous, since its value jumps from 0.0 to 0.2. It is easy to sketch the step-function, $S_5(x)$, and compare it with $F(x)$, which is x for a uniform distribution between 0 and 1.

The maximum difference must occur at one of the jump points of $S_n(x)$. If we tabulate the data of the above illustration, we find that $D_5 = .29$ and $D_n^* = .32$.

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Interval	$S_5(x)$	At right end point $\{S_5(x)-x\}$	$\|S_5(x)-x\|$	
$.00<x<.13$	0.0	$+.07$	.13	
$.13<x<.52$	0.2	$-.12$	.32	$D_5^*=.32$
$.52<x<.58$	0.4	$+.02$	.18	
$.58<x<.65$	0.6	$+.15$	.15	
$.65<x<.71$	0.8	$+.29$	.29	$D_5=.29$
$.71<x<1.0$	1.0	.00	.00	

We shall see that the number .44698, found in Table 1 for $n=5$ and $\alpha=.10$, means that we expect D_5 to exceed .44 more than 10 per cent of the time when a sample of 5 numbers is picked from a uniform distribution. Since $D_5=.29$ in our example, we cannot, at the 10 per cent level, reject the hypothesis that the numbers belong to a uniform distribution. Likewise, the number .50945 found in Table 1 for $n=5$ and $P=.90$, means that D_5^* will exceed .50 about 10 per cent of the time when samples of 5 numbers are picked from a distribution with the continuous cumulative distribution function $F(x)$.

When $(x_1, x_2, \cdots, x_n)$ are mutually independent and all come from the same continuous distribution function $F(x)$, the distribution of D_n does not depend on $F(x)$. This means that a table used to test the hypothesis that numbers come from a uniform distribution may also be used to test the hypothesis that numbers come from a normal distribution, or from any completely specified continuous distribution.

The statistic D_n can be used to test the hypothesis that the observations come from $F(x)$ against the alternative that they come from $G(x)$. If it is assumed that $G(x)=F(x+\Theta)$ where $\Theta>0$, i.e., $G(x)$ is the same as $F(x)$ except shifted to the left, then a test based on large values of D_n would be a reasonable procedure. For $\Theta<0$, small values of D_n would be appropriate, or large values could be used by replacing observations by their negatives and replacing $F(x)$ by $1-F(-x)$. The statistic D_n can be used for testing more general hypotheses, such as $G(x)\geqq F(x)$ for all x.

Let $\alpha=\text{Prob.}\ (D_n\geqq\epsilon)$. Table 1 contains corresponding values of ϵ for $\alpha=.10$, .05, .025, .01, and .005, for values of n from 1 to 100.

Let $P=\text{Prob.}\ (D_n^*\leqq\epsilon)$. Table 1 also contains values of ϵ corresponding to $P=.90$, .95, .98, and .99. For $n=5$ and $P=.98$ the number $\epsilon=.627$ has the following interpretation. If five numbers are picked at random, from a uniform distribution on the interval (0, 1), then it is reasonable to expect that about twice in 100 trials the associated sample cumulative distribution function will cut either the line $x+.627$ or the line $x-.627$.

TABLE 1

PERCENTAGE POINTS IN KOLMOGOROV-SMIRNOV STATISTICS

Values of ϵ for which α = Prob. ($D_n \geqq \epsilon$) and P = Prob. ($D_n^* \leqq \epsilon$)

n	$\alpha = .10$	$\alpha = .05$ ($P = .90$)	$\alpha = .025$ ($P = .95$)	$\alpha = .01$ ($P = .98$)	$\alpha = .005$ ($P = .99$)
1	.90000	.95000	.97500	.99000	.99500
2	.68377	.77639	.84189	.90000	.92929
3	.56481	.63604	.70760	.78456	.82900
4	.49265	.56522	.62394	.68887	.73424
5	.44698	.50945	.56328	.62718	.66853
6	.41037	.46799	.51926	.57741	.61661
7	.38148	.43607	.48342	.53844	.57581
8	.35831	.40962	.45427	.50654	.54179
9	.33910	.38746	.43001	.47960	.51332
10	.32260	.36866	.40925	.45662	.48893
11	.30829	.35242	.39122	.43670	.46770
12	.29577	.33815	.37543	.41918	.44905
13	.28470	.32549	.36143	.40362	.43247
14	.27481	.31417	.34890	.38970	.41762
15	.26588	.30397	.33760	.37713	.40420
16	.25778	.29472	.32733	.36571	.39201
17	.25039	.28627	.31796	.35528	.38086
18	.24360	.27851	.30936	.34569	.37062
19	.23735	.27136	.30143	.33685	.36117
20	.23156	.26473	.29408	.32866	.35241
21	.22617	.25858	.28724	.32104	.34427
22	.22115	.25283	.28087	.31394	.33666
23	.21645	.24746	.27490	.30728	.32954
24	.21205	.24242	.26931	.30104	.32286
25	.20790	.23768	.26404	.29516	.31657
26	.20399	.23320	.25907	.28962	.31064
27	.20030	.22898	.25438	.28438	.30502
28	.19680	.22497	.24993	.27942	.29971
29	.19348	.22117	.24571	.27471	.29466
30	.19032	.21756	.24170	.27023	.28987
31	.18732	.21412	.23788	.26596	.28530
32	.18445	.21085	.23424	.26189	.28094
33	.18171	.20771	.23076	.25801	.27677
34	.17909	.20472	.22743	.25429	.27279
35	.17659	.20185	.22425	.25073	.26897
36	.17418	.19910	.22119	.24732	.26532
37	.17188	.19646	.21826	.24404	.26180
38	.16966	.19392	.21544	.24089	.25843
39	.16753	.19148	.21273	.23786	.25518
40	.16547	.18913	.21012	.23494	.25205

TABLE 1—(*continued*)

n	$\alpha=.10$	$\alpha=.05$ ($P=.90$)	$\alpha=.025$ ($P=.95$)	$\alpha=.01$ ($P=.98$)	$\alpha=.005$ ($P=.99$)
41	.16349	.18687	.20760	.23213	.24904
42	.16158	.18468	.20517	.22941	.24613
43	.15974	.18257	.20283	.22679	.24332
44	.15796	.18053	.20056	.22426	.24060
45	.15623	.17856	.19837	.22181	.23798
46	.15457	.17665	.19625	.21944	.23544
47	.15295	.17481	.19420	.21715	.23298
48	.15139	.17302	.19221	.21493	.23059
49	.14987	.17128	.19028	.21277	.22828
50	.14840	.16959	.18841	.21068	.22604
51	.14697	.16796	.18659	.20864	.22386
52	.14558	.16637	.18482	.20667	.22174
53	.14423	.16483	.18311	.20475	.21968
54	.14292	.16332	.18144	.20289	.21768
55	.14164	.16186	.17981	.20107	.21574
56	.14040	.16044	.17823	.19930	.21384
57	.13919	.15906	.17669	.19758	.21199
58	.13801	.15771	.17519	.19590	.21019
59	.13686	.15639	.17373	.19427	.20844
60	.13573	.15511	.17231	.19267	.20673
61	.13464	.15385	.17091	.19112	.20506
62	.13357	.15263	.16956	.18960	.20343
63	.13253	.15144	.16823	.18812	.20184
64	.13151	.15027	.16693	.18667	.20029
65	.13052	.14913	.16567	.18525	.19877
66	.12954	.14802	.16443	.18387	.19729
67	.12859	.14693	.16322	.18252	.19584
68	.12766	.14587	.16204	.18119	.19442
69	.12675	.14483	.16088	.17990	.19303
70	.12586	.14381	.15975	.17863	.19167
71	.12499	.14281	.15864	.17739	.19034
72	.12413	.14183	.15755	.17618	.18903
73	.12329	.14087	.15649	.17498	.18776
74	.12247	.13993	.15544	.17382	.18650
75	.12167	.13901	.15442	.17268	.18528
76	.12088	.13811	.15342	.17155	.18408
77	.12011	.13723	.15244	.17045	.18290
78	.11935	.13636	.15147	.16938	.18174
79	.11860	.13551	.15052	.16832	.18060
80	.11787	.13467	.14960	.16728	.17949
81	.11716	.13385	.14868	.16626	.17840
82	.11645	.13305	.14779	.16526	.17732

TABLE 1—(*continued*)

n	$\alpha=.10$	$\alpha=.05$ ($P=.90$)	$\alpha=.025$ ($P=.95$)	$\alpha=.01$ ($P=.98$)	$\alpha=.005$ ($P=.99$)
83	.11576	.13226	.14691	.16428	.17627
84	.11508	.13148	.14605	.16331	.17523
85	.11442	.13072	.14520	.16236	.17421
86	.11376	.12997	.14437	.16143	.17321
87	.11311	.12923	.14355	.16051	.17223
88	.11248	.12850	.14274	.15961	.17126
89	.11186	.12779	.14195	.15873	.17031
90	.11125	.12709	.14117	.15786	.16938
91	.11064	.12640	.14040	.15700	.16846
92	.11005	.12572	.13965	.15616	.16755
93	.10947	.12506	.13891	.15533	.16666
94	.10889	.12440	.13818	.15451	.16579
95	.10833	.12375	.13746	.15371	.16493
96	.10777	.12312	.13675	.15291	.16408
97	.10722	.12249	.13606	.15214	.16324
98	.10668	.12187	.13537	.15137	.16242
99	.10615	.12126	.13469	.15061	.16161
100	.10563	.12067	.13403	.14987	.16081

2. FORMULAS ASSOCIATED WITH TABLE 1

Let

$$\alpha = \text{Prob. } (D_n \geqq \epsilon). \tag{1}$$

It was shown by Birnbaum and Tingey [3] that values of n, α, and ϵ which satisfy (1) are connected by the relation

$$\alpha = \epsilon \sum_{j=0}^{[n-n\epsilon]} \binom{n}{j} \left(1 - \epsilon - \frac{j}{n}\right)^{n-j} \left(\epsilon + \frac{j}{n}\right)^{j-1} \tag{2}$$

where $\binom{n}{j}$ is the binomial coefficient and $[n-n\epsilon]$ is the largest integer contained in $n(1-\epsilon)$. They tabulated ϵ for $n=5$, 8, 10, 20, 40, 50 and $\alpha=.10$, .05, .01, .001, and compared these solutions of (2) with values given by the asymptotic formula

$$\bar{\epsilon}(n, \alpha) = \sqrt{\frac{\log_e \frac{1}{\alpha}}{2n}} \tag{3}$$

derived by Smirnov.

It is also well known that the random variable D_n^* has a probability distribution independent of $F(x)$ if $F(x)$ is continuous. The limiting distribution of D_n^* was found by Kolmogorov. Feller [6] has given simplified derivations for the limiting distributions of D_n and D_n^*.

Let

$$P = \text{Prob. } (D_n^* \leqq \epsilon). \tag{4}$$

Tables of P for $n=1, 2, \cdots, 100$ and $\epsilon=j/n$, $j=1, 2, \cdots, 15$ have been constructed by Birnbaum [2], and for $n=5, 10, 15, \cdots, 80$ and $\epsilon=j/n$ for selected values of $j\leqq 9$ by Massey [7]. Both Birnbaum [2] and Massey [8], by inverse interpolation, obtained tables of ϵ, corresponding to given values of n, for certain conventional probability levels P.

Table 1 contains values of ϵ which satisfy (1) for $\alpha=.005, .01, .025, .05, .10$ and for $n=1, 2, \cdots, 100$. For $n\leqq 20$ the values of ϵ are computed solutions of equation (2). For $n>20$ the values of ϵ in Table 1 are given by the formula

$$\epsilon^*(n, \alpha) = \bar{\epsilon}(n, \alpha) - .16693n^{-1} - A(\alpha)n^{-3/2} \tag{5}$$

where $\bar{\epsilon}(n, \alpha)$ is the value given by Smirnov's asymptotic formula (3), and $A(.10)=.00256$, $A(.05)=.05256$, $A(.025)=.11282$, $A(.01)=.20562$, $A(.005)=.28464$. Formula (5) is a special case of (11) which is discussed in section 4.

Evidence discussed in sections 5 and 6 indicates that, for $\alpha\leqq .05$, the values of ϵ shown in Table 1 may also be regarded as values which satisfy (4) for $P=1-2\alpha$.

3. COMPUTATIONAL TECHNIQUES

Finding a value of ϵ which satisfies (2), for specified values of n and α, involves finding a real root of a polynomial equation of degree n. To solve this equation efficiently it is desirable to have methods for: (a) making a good first approximation; (b) assuring the rapid convergence of subsequent approximations; and (c) checking to eliminate numerical errors.

Formula (5) may be used to make a first approximation for a solution of (2). The substitution of an approximation for ϵ in (2) requires the computation of

$$A_j(n, \epsilon) = \binom{n}{j}(1 - \epsilon - j/n)^{n-j}(\epsilon + j/n)^{j-1} \tag{6}$$

which in the present study was evaluated using 7, 8, or 10 place logarithms.

For fixed n, the rate of change of α with respect to ϵ can be expressed in the form

$$\frac{\partial\alpha}{\partial\epsilon} = \epsilon \sum_{j=0}^{[n-n\epsilon]} \left(\frac{1}{\epsilon} + \frac{j-1}{\epsilon + j/n} - \frac{n-j}{1-\epsilon-j/n}\right) A_j(n, \epsilon). \tag{7}$$

Formula (7), which can be evaluated without additional use of logarithms after the products in (6) have been found for a given value of ϵ, provides a method for obtaining a convergent sequence of approximate solutions of (2).

The differential relation

$$dA_j(n, \epsilon) = \left(\frac{j-1}{\epsilon + j/n} - \frac{n-j}{1-\epsilon-j/n}\right) A_j(n, \epsilon) d\epsilon \tag{8}$$

may be used to approximate $A_j(n, \epsilon + d\epsilon)$ when $A_j(n, \epsilon)$ is known. It is useful in checking two computations for slightly different values of ϵ and can be used both to locate numerical errors and to estimate the number of significant digits in computed results.

The formula

$$A_{j-1}(n, \epsilon + 1/n) = \frac{j(n - n\epsilon - j)}{(n - j + 1)(n\epsilon + j)} A_j(n, \epsilon) \tag{9}$$

is useful when evaluating (2) for values of ϵ which differ by $1/n$.

A more detailed discussion of computational methods used and computed solutions of (2) not included in the present paper are included in an unpublished master's thesis by Nelson Prentiss [9].

4. DERIVATION OF A MODIFIED ASYMPTOTIC FORMULA

Using techniques outlined in section 3, numerous values of n, α, ϵ which satisfy (1) were found with values of α and ϵ correct to seven decimal places. Then attempts were made to modify (3) to obtain a formula which agreed closely with computed values of ϵ for a specified range of values for n and α. This general approach has been used previously.

Modified asymptotic formulas were used by Campbell [4] and by Hotelling, Fisher, and Riordan [10]. Aroian [1] used empirically modified asymptotic formulas in finding percentage points of the chi-square distribution.

The starting point in the development of (5) was the conjecture that, for fixed α, the ratio $r = \epsilon/\bar{\epsilon}$, with ϵ and $\bar{\epsilon}$ given by (1) and (3), might be smooth for fairly small values of n. This was first tested for the fixed probability level $\alpha=.10$. Table 2 contains computed values of ϵ and $\bar{\epsilon}$ along with the ratio r, for $\alpha=.10$ and for $n=20$, 25, 30, 35, 40, 45, 50, 80, 100. The ratio $r=\epsilon/\bar{\epsilon}$ was approximated by a function of n to produce the formula

$$\epsilon(.1) = \left(1 - \frac{1}{6.44245\sqrt{n} - .08892 \log_{10} n - .03422}\right)\sqrt{\frac{\log_e 10}{2n}}. \quad (10)$$

Values given by (10) are shown in Table 2.

TABLE 2

EXAMPLE OF A MODIFIED ASYMPTOTIC FORMULA FOR A FIXED PERCENTAGE LEVEL

n	α	ϵ Formula (2)	$\bar{\epsilon}$ Formula (3)	$r=\epsilon/\bar{\epsilon}$	ϵ (.1) Formula (10)
20	.10	.2315554	.2399263	.9651105	.2315554
25	.10	.2079016	.2145966	.9688019	.2079017
30	.10	.1903213	.1958990	.9715277	.1903212
35	.10	.1765872	.1813672	.9736446	.1765872
40	.10	.1654716	.1696535	.9753503	.1654717
45	.10	.1562342	.1599509	.9767635	.1562342
50	.10	.1483981	.1517427	.9779587	.1483981
80	.10	.1178739	.1199631	.9825846	.1178739
100	.10	.1056273	.1072983	.9844266	.1056273

The methods used to obtain (10) can be employed to find an approximate formula for any specified value of α. It is also possible to develop a single formula which can be used for all α in some interval, such as $.005 \leqq \alpha \leqq .10$, but such a formula, in general, would not be expected to be so accurate as (10) for $\alpha=.10$. One such formula is given below

$$\epsilon^*(n, \alpha) = \bar{\epsilon}(n, \alpha) - .16693n^{-1} - A(\alpha)n^{-3/2} \quad (11)$$

where

$$A(\alpha) = .09037(-\log_{10} \alpha)^{3/2} + .01515(\log_{10} \alpha)^2 - .08467\alpha - .11143$$

and $\bar{\epsilon}(n, \alpha)$ is the value given by (3).

The accuracy of (11) is indicated by Table 3 where each value of α shown was obtained by substituting the corresponding values of n and ϵ in (2), and each value of ϵ^* was obtained by substituting n and α in (11).

5. A RELATION CONNECTING P AND α

Using the notation of section 1, let β be the probability that the step polygon $S_n(x)$ will cut both of the bands $F(x)+\epsilon$ and $F(x)-\epsilon$. It follows that, for fixed n and ϵ,

$$P = 1 - 2\alpha + \beta. \tag{12}$$

Since (2) provides an exact formula for α, (12) may be used to find P when β is known and conversely.

If the approximation

$$P^* = 1 - 2\alpha \tag{13}$$

is used as a value for P, the error cannot be greater than an upper bound for the corresponding value of β. For any n less than 100 and for any ϵ, $0<\epsilon<1$, a bound for the corresponding value of β can be

TABLE 3

SPOT CHECKS OF THE ACCURACY OF FORMULA (11)

n	α Formula (2)	ϵ	ϵ^* Formula (11)	$P-P^*$
20	.0688270	.25	.249997	.00002
20	.0215335	.30	.299998	.00000
20	.0053775	.35	.350007	.00000
20	.0010595	.40	.400037	.00000
50	.1282836	.14	.139997	.00049
50	.0692174	.16	.160000	.00003
50	.0343888	.18	.180000	.00000
50	.0157195	.20	.200000	.00000
50	.0066052	.22	.220000	.00000
50	.0025487	.24	.240000	.00000
50	.0009021	.26	.260000	.00000
100	.1265906	.10	.099999	.00049
100	.0825244	.11	.109999	.00008
100	.0516583	.12	.120000	.00002
100	.0310470	.13	.130000	.00000
100	.0179127	.14	.140000	.00000
100	.0099196	.15	.150000	.00000
100	.0052718	.16	.160000	—
100	.0026882	.17	.170000	—
200	.1002216	.075	.074999	—
200	.0524240	.085	.085000	—
200	.0253058	.095	.095000	—
200	.0112640	.105	.105000	—
200	.0046232	.115	.115000	—

obtained from the table of values of P given by Birnbaum [2]. For example, Table 1 of the present paper contains the entry $n=12$, $\epsilon=.33815$, $P=.90$, which was obtained by (13) using $\alpha=.05$.

To find a bound for the error in the approximation, consider the entry in Birnbaum's Table for $n=12$ and for the first value of P smaller than .90. This is $P=.89126$ which corresponds to $\epsilon=c/n=\frac{1}{3}$. If $\epsilon=\frac{1}{3}$ is substituted in (2) for $n=12$, the corresponding value of α is found to be .054372. Substitution of this value in (13) gives $P^*=.891256$ which agrees to five decimal places with Birnbaum's value of P. Since β is a decreasing function of ϵ, the error in P is less than .00001 for each value of ϵ in Table 1 corresponding to $n=12$ and $P=.90$, .95, .98, .99.

For small values of n, and in certain cases for general values of n, it is easy to find an exact expression for β. For example, if n is an even integer

$$\beta = 0, \qquad 1/2 \leqq \epsilon \leqq 1,$$

$$\beta = (1-2\epsilon)^n\left\{1+\frac{1}{2^n}\binom{n}{\frac{n}{2}}\right\}, \qquad \frac{n-1}{2n} \leqq \epsilon \leqq \frac{1}{2} \tag{14}$$

where $(n\ n/2)$ is the binomial coefficient.

6. ACCURACY OF TABLE 1

The author believes that the greatest error in Table 1 does not exceed one unit in the fifth decimal place. For $n \leqq 20$, the values of ϵ shown in Table 1 are solutions of (2), computed to seven decimals before rounding off to five places.

For $n>20$ values of ϵ were obtained from (11). Table 3 indicates that this formula approaches five decimal accuracy for $n=20$, for the range of α used in Table 1, and is a better approximation for $n=50$, 100, and 200. Formulas (10) and (11) were obtained by completely independent methods, yet they agree to five decimal places for $\alpha=.10$ and $20 \leqq n \leqq 100$.

The method illustrated in section 5 can be used to show that the error in using (13) to approximate P is negligible over the range on which the formula was used. The first 17 values of n and ϵ which are shown in Table 3 correspond to entries in Birnbaum's Table [2], where $c=n\epsilon$. In Table 3, $P-P^*$ is the difference between the value for P, found by Birnbaum, and the value for P^* given by (13).

Additional empirical evidence of this type is contained in a master's thesis by Vincent Donato [5]. For each integer $20<n<100$ he chose a value of P given by Birnbaum [2] with P in the range $.875 < P < .995$. He substituted the corresponding values of n and ϵ in (11) and found α by inverse interpolation. He substituted this value of α in (13) to produce P^* as an approximation for the P found by Birnbaum. The difference $P - P^*$ was .00002 in one case, .00001 in 17 cases, and .00000 in 61 cases. The close agreement in this limited range supports the conjecture that for any n, $\beta \leqq \alpha^2$ which is a special form of a conjecture of Wald and Wolfowitz [11].

Relatively few of the values in Table 1 are included in papers previously published. The agreement is excellent except for several values of P for small sample size. For example for $n = 10$ and $P = .99$, Massey gives $\epsilon = .490$, Birnbaum gives $\epsilon = .4864$, and Table 1 shows $\epsilon = .48893$. Massey and Birnbaum found these values by inverse interpolation. For $n = 10$, and $.45 < \epsilon < 1$, formulas (14), (12), and (2) provide a tenth degree polynomial which can be used to find ϵ as accurately as desired. For $P = .99$, $\epsilon = .48893$ is the solution of this equation, correct to five decimal places.

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