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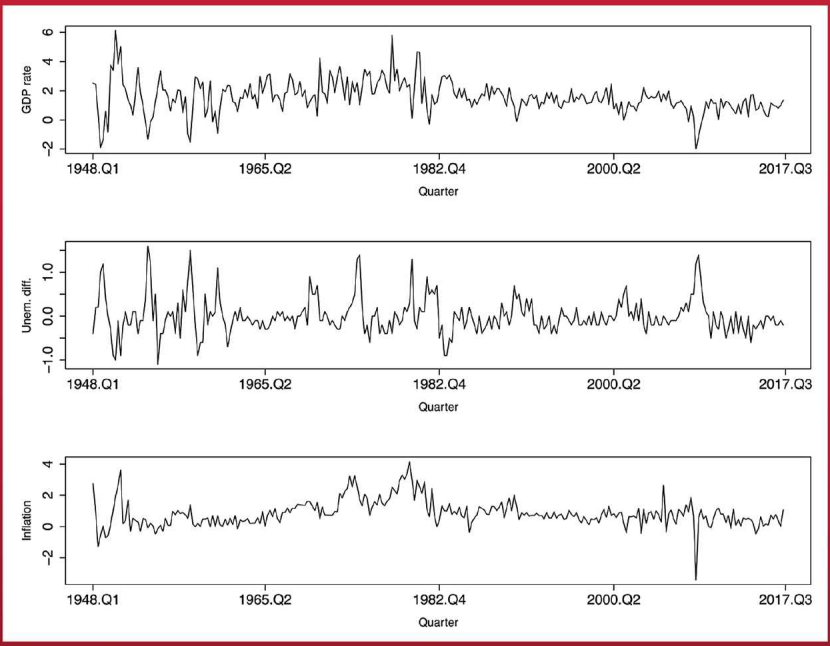
The Analysis of Time Series

An Introduction with R

Seventh Edition



Chris Chatfield
Haipeng Xing

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An Introduction with R

Seventh Edition

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To Liz and Ying

Alice sighed wearily. ‘I think you might do something better with the time,’ she said, ‘than waste it in asking riddles that have no answers.’

‘If you knew time as well as I do,’ said the Hatter, ‘you wouldn’t talk about wasting *it*. It’s *him*.’

‘I don’t know what you mean,’ said Alice.

‘Of course you don’t!’ the Hatter said, tossing his head contemptuously. ‘I dare say you never even spoke to Time!’

‘Perhaps not,’ Alice cautiously replied, ‘but I know I have to beat time when I learn music.’

‘Ah! That accounts for it,’ said the Hatter. ‘He won’t stand beating.’

Lewis Carroll, *Alice’s Adventures in Wonderland*

Contents

Preface to the Seventh Edition

The first six editions of this book highlight basic concepts, models, and methods in time series analysis, and have been used as a text for undergraduate and graduate-level time series courses in many universities during the past three decades. Although the previous editions successfully introduce time series analysis in an accessible way, there is a small gap between presenting time series theory and discussing its implementation, especially given the fact that, recently, many statistical analyses can be easily carried out with the aid of statistical software. I gradually realized this when I used the sixth edition of the book for my time series courses during the past decade, and felt that it would be much more convenient for both instructors and students to have an introductory time series textbook that highlights not only basic time series theory but the implementation of time series analysis as well.

Obviously, Chris Chatfield shared the same view with me. The concrete idea of having this edition of the book arose in 2016 when Chris Chatfield and Rob Calver, Executive Editor in Mathematics, Statistics, and Physics at Taylor and Francis, asked me if I was interested in revising the sixth edition of the book. By then, I had been teaching undergraduate and graduate-level time series courses for over ten years, and had collected a set of examples on real data analysis with `R` implementation, so I expressed my interest to both Rob and Chris and started working on the new edition.

Similar to the sixth edition, this edition assumes knowledge of basic probability theory and elementary statistical inference. As the sixth edition of the book covers a broad range of topics at the introductory level, this edition keeps most of the material from the sixth edition. However, several changes are made in this edition. First, a new chapter (Chapter 12) and a new section (Section 13.7) are added to introduce uni- and multi-variate volatility models in finance, respectively. Necessary updates are also made in different chapters and sections. Second, many examples and real data are added in this edition. Specifically, I added examples of real data analysis in most chapters except for Chapters 9 and 10. Third, all examples in the book are implemented with `R`, and `R` codes for most examples are provided in the book so that the reader can easily replicate the result. The data and scripts in the book are available at

`http://www.ams.sunysb.edu/~xing/tsRbook/index.html`.

I would like to thank Chris Chatfield for his invitation and authorization for revising the book. I also thank all the students who took my time series course for their interest in the subject and comments on the earlier draft

of the book. Besides, I want to express my gratitude to my colleagues for being supportive and helpful over the years. At last, I want to thank the U.S. National Science Foundation for providing support for my research and teaching during the past years. Any errors, omissions, or obscurities in this edition are my responsibility.

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Abbreviations and Notation

AR	Autoregressive
MA	Moving average
ARMA	Autoregressive moving average
ARIMA	Autoregressive integrated moving average
SARIMA	Seasonal ARIMA
TAR	Threshold autoregressive
GARCH	Generalized autoregressive conditionally heteroscedastic
SWN	Strict white noise
WN	White noise
MMSE	Minimum mean square error
P.I.	Prediction interval
FFT	Fast Fourier transform
ac.f.	Autocorrelation function
acv.f.	Autocovariance function
N	Sample size, or length of observed time series
$\widehat{x}_N(h)$	Forecast of x_{N+h} made at time N
B	Backward shift operator such that $BX_t = X_{t-1}$
B^d	$B^d X_t = X_{t-d}$
∇	First differencing operator, $(1 - B)$, such that $\nabla X_t = X_t - X_{t-1}$
∇_d	$\nabla_d = 1 - B^d$
∇_d^r	$\nabla_d^r = (1 - B^d)^r$
E	Expectation or expected value
Var	Variance
I	Identity matrix – a square matrix with ones on the diagonal and zeros otherwise
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
χ^2_ν	Chi-square distribution with ν degrees of freedom
$\{Z_t\}$ or $\{\varepsilon_t\}$	Purely random process of independent random variables, usually $N(0, \sigma^2)$-distributed
A^T or $\boldsymbol{X}^T$	Transpose of a matrix A or vector $\boldsymbol{X}$ — vectors are indicated by boldface, but not scalars or matrices

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Chapter 1

Introduction

A time series is a collection of observations made sequentially through time. Examples occur in a variety of fields, ranging from economics to engineering, and methods of analysing time series constitute an important area of statistics.

1.1 Some Representative Time Series

We begin with some examples of the sort of time series that arise in practice.

Economic and financial time series

Many time series are routinely recorded in economics and finance. Examples include share prices on successive days, export totals in successive months, average incomes in successive months, company profits in successive years and so on.

The classic Beveridge wheat price index series consists of the average wheat price in nearly 50 places in various countries measured in successive years from 1500 to 1869 (Beveridge, 1921). This series is of particular interest to economics historians, and is available in many places (e.g. in the `tseries` package of `R`). Figure 1.1 shows this series and some apparent cyclic behaviour can be seen. The trend of the series will be studied in Section 2.5.2.

To plot the data using the `R` statistical package, you can load the data `bev` in the `tseries` package and plot the time series (the `>` below are prompts):

```
> library(tseries)     # load the library
> data(bev)     # load the dataset
> plot(bev, xlab="Year", ylab="Wheat price index", xaxt="n")
> x.pos<-c(1500, 1560, 1620, 1680, 1740, 1800, 1869)
          # define x-axis labels
> axis(1, x.pos, x.pos)
```

As an example of financial time series, Figure 1.2 shows the daily returns (or percentage change) of the adjusted closing prices of the Standard & Poor's 500 (S&P500) Index from January 4, 1995 to December 30, 2016. The data shown in Figure 1.2 are typical of return data. The mean of the return series seems to be stable with an average return of approximately zero, but the volatility of data changes over time. This series will be analyzed in Chapter 12.

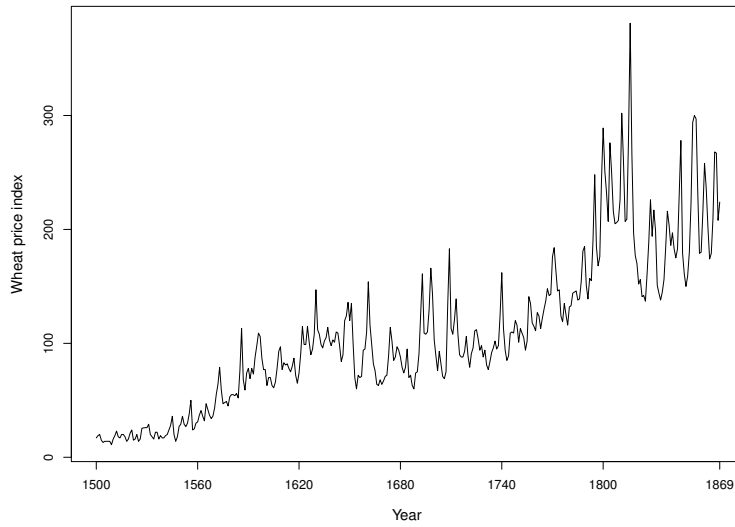

Figure 1.1 *The Beveridge wheat price annual index series from 1500 to 1869.*

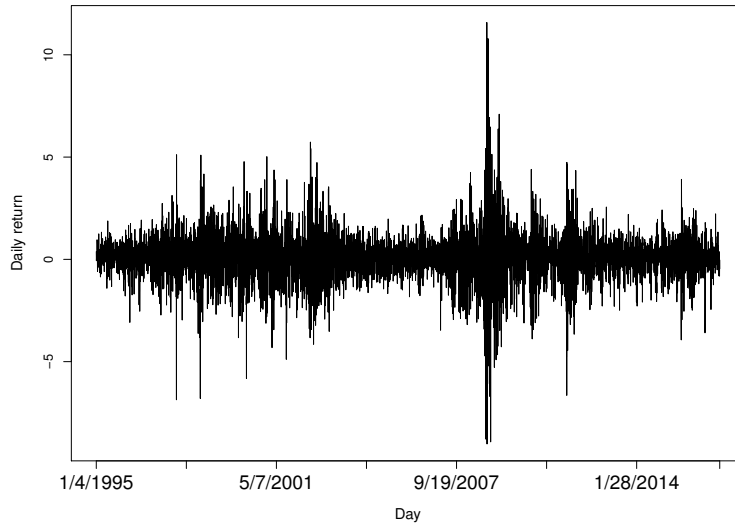

Figure 1.2 *Daily returns of the adjusted closing prices of the S&P500 index from January 4, 1995 to December 30, 2016.*

To reproduce Figure 1.2 in R, suppose you save the data as sp500_ret_1995-2016.csv in the directory mydata. Then you can use the following command to read the data and plot the time series.

```
> sp500<-read.csv("mydata/sp500_ret_1995-2016.csv")
> n<-nrow(sp500)
> x.pos<-c(seq(1,n,800),n)
> plot(sp500$Return, type="l", xlab="Day",
     ylab="Daily return", xaxt="n")
> axis(1, x.pos, sp500$Date[x.pos])
```

Physical time series

Many types of time series occur in the physical sciences, particularly in meteorology, marine science and geophysics. Examples are rainfall on successive days, and air temperature measured in successive hours, days or months. Figure 1.3 shows the average air temperature in Anchorage, Alaska in the United States in successive months over a 16-year period. The series can be downloaded from the U.S. National Centers for Environmental Information (https://www.ncdc.noaa.gov/cag/). Seasonal fluctuations can be clearly seen in the series.

Some mechanical recorders take measurements continuously and produce a continuous trace rather than observations at discrete intervals of time. For example, in some laboratories it is important to keep temperature and humidity as constant as possible and so devices are installed to measure these variables continuously. Action may be taken when the trace goes outside pre-specified limits. Visual examination of the trace may be adequate for many purposes, but, for more detailed analysis, it is customary to convert the continuous trace to a series in discrete time by sampling the trace at appropriate equal intervals of time. The resulting analysis is more straightforward and can readily be handled by standard time series software.

Marketing time series

The analysis of time series arising in marketing is an important problem in commerce. Observed variables could include sales figures in successive weeks or months, monetary receipts, advertising costs and so on. As an example, Figure 1.4 shows the domestic sales of Australian fortified wine by winemakers in successive quarters over a 30-year period, which are available at the Australian Bureau of Statistics (http://www.abs.gov.au/AUSSTATS/). This series will be analysed in Sections 4.8 and 4.9. Note the trend and seasonal variation which is typical of sales data. It is often important to forecast future sales so as to plan production. It may also be of interest to examine the relationship between sales and other time series such as advertising expenditure.

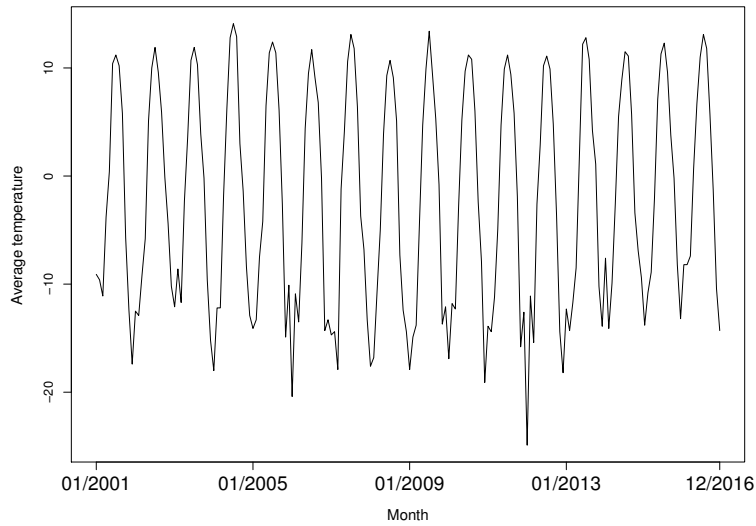

Figure 1.3 *Monthly average air temperature (deg C) in Anchorage, Alaska, the United States, in successive months from 2001 to 2016.*

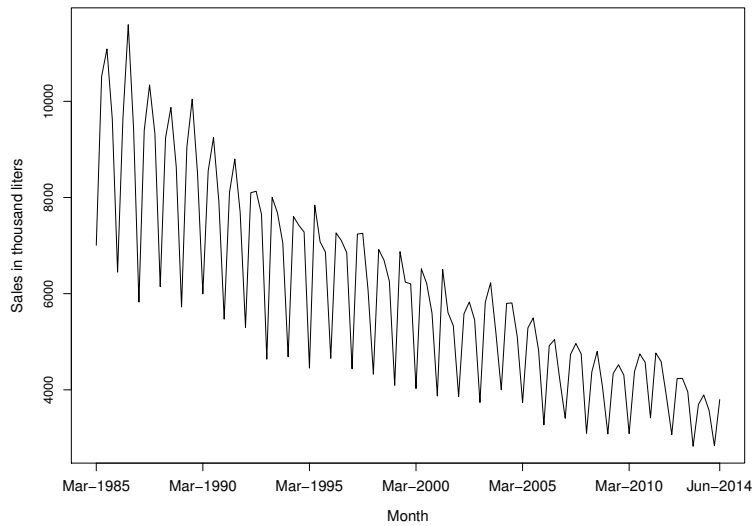

Figure 1.4 *Domestic sales (unit: thousand liters) of Australian fortified wine by winemakers in successive quarters from March 1985 to June 2014.*

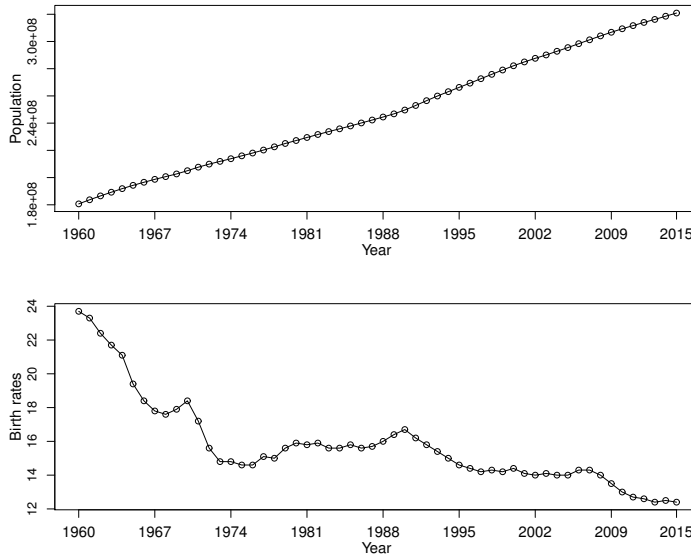

Figure 1.5 *Total population and birth rate (per 1,000 people) for the United States from 1965 to 2015.*

Demographic time series

Various time series occur in the study of population change. Examples include the total population of Canada measured annually, and monthly birth totals in England. Figure 1.5 shows the total population and crude birth rate (per 1,000 people) for the United States from 1965 to 2015. The data are available at the U.S. Federal Reserve Bank of St. Louis (https://fred.stlouisfed.org/). Demographers want to predict changes in population for as long as 10 or 20 years into the future, and are helped by the slowly changing structure of a human population. Standard time series methods can be applied to study this problem.

To reproduce Figure 1.5 in R, you can use the following command to read the data and plot the time series.

```
> pop<-read.csv("mydata/US_pop_birthrate.csv", header=T)
> x.pos<-c(seq(1, 56, 7), 56)
> x.label<-c(seq(1960, 2009, by=7), 2015)

> par(mfrow=c(2,1), mar=c(3,4,3,4))
> plot(pop[,2], type="l", xlab="", ylab="", xaxt="n")
> points(pop[,2])
> axis(1, x.pos, x.label, cex.axis=1.2)
> title(xlab="Year", ylab="Population", line=2, cex.lab=1.2)
```

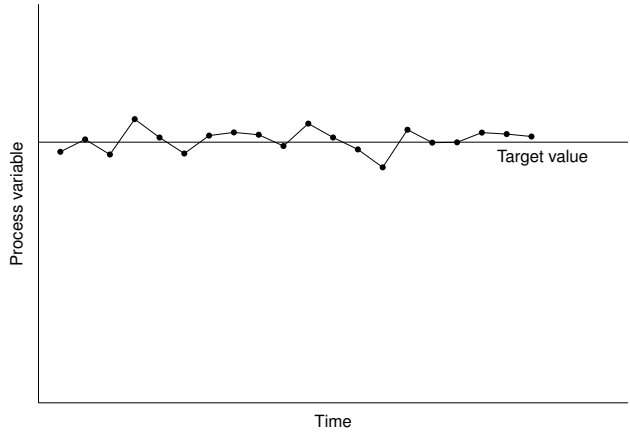

Figure 1.6 *A process control chart.*

```
> plot(pop[,3], type="l", xlab="", ylab="", xaxt="n")
> points(pop[,3])
> axis(1, x.pos, x.label, cex.axis=1.2)
> title(xlab="Year", ylab="Birth rates", line=2, cex.lab=1.2)
```

Process control data

In process control, a problem is to detect changes in the performance of a manufacturing process by measuring a variable, which shows the quality of the process. These measurements can be plotted against time as in Figure 1.6. When the measurements stray too far from some target value, appropriate corrective action should be taken to control the process. Special techniques have been developed for this type of time series problems, and the reader is referred to a book on statistical quality control (e.g. Montgomery, 1996).

Binary processes

A special type of time series arises when observations can take one of only two values, usually denoted by 0 and 1 (see Figure 1.7). For example, in computer science, the position of a switch, either 'on' or 'off', could be recorded as one or zero, respectively. Time series of this type, called **binary processes**, occur in many situations, including the study of communication theory. A problem here is to predict when the process will take a different value. One way to solve this problem is to use regime-switching models, which will be discussed in Chapter 11 (Section 11.6).

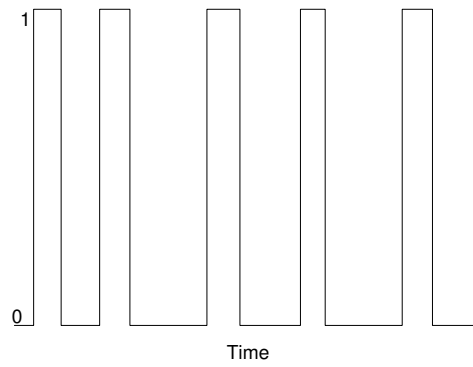

Figure 1.7 *A realization of a binary process.*

Point processes

A completely different type of time series occurs when we consider a series of events occurring 'randomly' through time. For example, we could record the dates of major railway disasters. A series of events of this type is usually called a **point process**. As an example, Figure 1.8 shows the intraday transaction data of the International Business Machines Corporation (IBM) from 9:35:00 to 9:38:00 on January 4, 2010. When a trade event occurs, the corresponding trading price and trading volume are observed. However, trades do not occur equally spaced in time; hence time intervals between trades (or trade durations) are considered as random variables. For observations of this type, we are interested in such quantities as the distribution of the number of events occurring in a given time period and distribution of time intervals between events. Methods of analysing point process data are generally very different from those used for analysing standard time series data and the reader is referred, for example, to Cox and Isham (1980).

To reproduce Figure 1.8 in R, you can use the following command to read the data and plot the time series.

```
> ibm<-read.table("mydata/taq_trade_ibm_100104.txt",
    header=T, sep="\t")
> ibm.new<-ibm[,c(1,2,7)]
> ibm[,2]<-as.numeric(as.character(ibm[,2]))

> ### take 9:35:00-9:37:59am trading record
> data<-ibm.new[1458:2371,]
> newtime<-rep(0, nrow(data))
> for (i in 1:nrow(data)){
   min<-as.numeric(substr(as.character(data$TIME[i]),3,4))
   sec<-as.numeric(substr(as.character(data$TIME[i]),6,7))
```

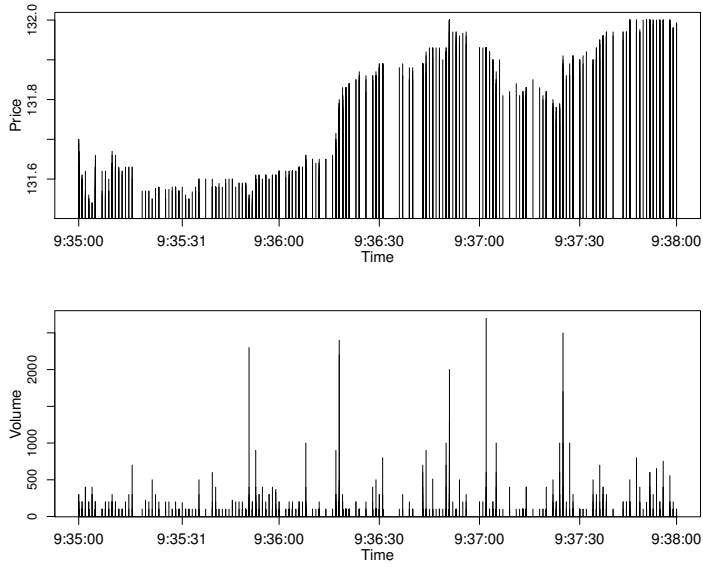

Figure 1.8 *Transaction prices and volumes of IBM stocks from 9:35:00 to 9:38:00 on January 4, 2010.*

```
   newtime[i]<- (min-30)*60+sec
  }

> x.label<-c("9:35:00", "9:35:31", "9:36:00", "9:36:30",
  "9:37:00", "9:37:30", "9:38:00")
> x.pos<-c(1, 139, 249, 485, 619, 776, 914)

> par(mfrow=c(2,1), mar=c(2,4,2,4))
> plot(newtime, data[,2],xlab="",ylab="",xaxt="n",type="h")
> axis(1, newtime[x.pos], x.label, cex.axis=1.2)
> title(xlab="Time", ylab="Price", line=2, cex.lab=1.2)
> plot(newtime, data[,3],xlab="",ylab="",xaxt="n",type="h")
> axis(1, newtime[x.pos], x.label, cex.axis=1.2)
> title(xlab="Time", ylab="Volume", line=2, cex.lab=1.2)
```

1.2 Terminology

A time series is said to be **continuous** when observations are made continuously through time as in Figure 1.7. The adjective 'continuous' is used for series of this type even when the measured variable can only take a discrete set of values, as in Figure 1.7. A time series is said to be **discrete** when observations are taken only at specific times, usually equally spaced.

The term 'discrete' is used for series of this type even when the measured variable is a continuous variable.

This book is mainly concerned with discrete time series, where the observations are taken at equal intervals. We also consider continuous time series briefly, while Section 14.4.4 gives some references regarding the analysis of discrete time series taken at unequal intervals of time.

Discrete time series can arise in several ways. Given a continuous time series, we could read off (or digitise) the values at equal intervals of time to give a discrete time series, sometimes called a **sampled** series. The sampling interval between successive readings must be carefully chosen so as to lose little information (see Section 7.7). A different type of discrete series arises when a variable does not have an instantaneous value but we can **aggregate** (or accumulate) the values over equal intervals of time. Examples of this type are monthly exports and daily rainfalls. Finally, some time series are inherently discrete, an example being the dividend paid by a company to shareholders in successive years.

Much statistical theory is concerned with random samples of independent observations. The special feature of time series analysis is the fact that successive observations are usually *not* independent and that the analysis must take into account the *time order* of the observations. When successive observations are dependent, future values may be predicted from past observations. If a time series can be predicted exactly, it is said to be **deterministic**. However, most time series are **stochastic** in that the future is only partly determined by past values, so that exact predictions are impossible and must be replaced by the idea that future values have a probability distribution, which is conditioned on a knowledge of past values.

1.3 Objectives of Time Series Analysis

There are several possible objectives in analysing a time series. These objectives may be classified as description, explanation, prediction, and control, and will be considered in turn.

(i) *Description*

When presented with a time series, the first step in the analysis is usually to plot the observations against time to give what is called a **time plot**, and then to obtain simple descriptive measures of the main properties of the series. This is described in detail in Chapter 2. The power of the time plot as a descriptive tool is illustrated by Figure 1.4, which clearly shows that there is a regular seasonal effect, with wine sales 'high' in the third quarter and 'low' in the first quarter of each year. The time plot also shows that annual sales are decreasing (i.e. there is an downward trend). For some series, the variation is dominated by such 'obvious' features, and a fairly simple model, which only attempts to describe trend and seasonal variation, may be perfectly adequate to describe the variation in the time series. For other series, more sophisticated techniques

will be required to provide an adequate analysis. Then a more complex model will be constructed, such as the various types of stochastic processes described in Chapter 3.

This book devotes a greater amount of space to the more advanced techniques, but this does not mean that elementary descriptive techniques are unimportant. Anyone who tries to analyse a time series without plotting it first is asking for trouble. A graph will not only show up trend and seasonal variation, but will also reveal missing observations and any 'wild' observations or **outliers** that do not appear to be consistent with the rest of the data. The treatment of outliers is a complex subject in which common sense is as important as theory (see Section 14.4.5). An outlier may be a perfectly valid, but extreme, observation, which could, for example, indicate that the data are not normally distributed. Alternatively, an outlier may be a freak observation arising, for example, when a recording device goes wrong or when a strike severely affects sales. In the latter case, the outlier needs to be adjusted in some way before further analysis of the data. Robust methods are designed to be insensitive to outliers.

Other features to look for in a time plot include sudden or gradual changes in the properties of the series. For example, a step change in the level of the series would be very important to notice, if one exists. Any changes in the seasonal pattern should also be noted. The analyst should also look out for the possible presence of turning points, where, for example, an upward trend suddenly changes to a downward trend. If there is some sort of discontinuity in the series, then different models may need to be fitted to different parts of the series.

(ii) *Explanation*

When observations are taken on two or more variables, it may be possible to use the variation in one time series to explain the variation in another series. This may lead to a deeper understanding of the mechanism that generated a given time series.

Although regression models are occasionally helpful here, they are not really designed to handle time series data, with all the correlations inherent therein, and so we will see that alternative classes of models should be considered. Chapter 9 considers the analysis of what are called **linear systems**. A linear system converts an input series to an output series by a linear operation. Given observations on the input and output to a linear system (see Figure 1.9), the analyst wants to assess the properties of the linear system. For example, it is of interest to see how sea level is affected by temperature and pressure, and to see how sales are affected by price and economic conditions. A class of models, called transfer function models, enables us to model time series data in an appropriate way.

(iii) *Prediction*

Given an observed time series, one may want to predict the future values of the series. This is an important task in sales forecasting, and in the analysis

Figure 1.9 *Schematic representation of a linear system.*

of economic and industrial time series. Many writers, including ourselves, use the terms 'prediction' and 'forecasting' interchangeably, but note that some authors do not. For example, Brown (1963) uses 'prediction' to describe subjective methods and 'forecasting' to describe objective methods.

(iv) *Control*

Time series are sometimes collected or analysed so as to improve control over some physical or economic system. For example, when a time series is generated that measures the 'quality' of a manufacturing process, the aim of the analysis may be to keep the process operating at a 'high' level. Control problems are closely related to prediction in many situations. For example, if one can predict that a manufacturing process is going to move off target, then appropriate corrective action can be taken.

Control procedures vary considerably in style and sophistication. In statistical quality control, the observations are plotted on control charts and the controller takes action as a result of studying the charts. A more complicated type of approach is based on modelling the data and using the model to work out an 'optimal' control strategy — see, for example, Box et al. (1994). In this approach, a stochastic model is fitted to the series, future values of the series are predicted, and then the input process variables are adjusted so as to keep the process on target.

Many other contributions to control theory have been made by control engineers and mathematicians rather than statisticians. This topic is rather outside the scope of this book but is briefly introduced in Section 14.3.

1.4 Approaches to Time Series Analysis

This book will describe various approaches to time series analysis. Chapter 2 describes **simple descriptive techniques**, which include plotting the data and looking for trends, seasonal fluctuations and so on. Chapter 3 introduces a variety of probability models for time series, while Chapter 4 discusses ways of fitting these models to time series. The major diagnostic tool that is used in Chapter 4 is a function called the **autocorrelation** function, which helps to describe the evolution of a process through time. Inference based on this function is often called an **analysis in the time domain**. Chapter 5 goes

on to discuss a variety of forecasting procedures, but this chapter is not a prerequisite for the rest of the book.

Chapter 6 introduces a function called the **spectral density** function, which describes how the variation in a time series may be accounted for by cyclic components at different frequencies. Chapter 7 shows how to estimate the spectral density function by means of a procedure called **spectral analysis**. Inference based on the spectral density function is often called an **analysis in the frequency domain**.

Chapter 8 discusses the analysis of bivariate time series, while Chapter 9 extends this work by considering **linear systems** in which one series is regarded as the input, while the other series is regarded as the output. Chapter 10 introduces an important class of models, called **state-space models**. It also describes the **Kalman filter**, which is a general method of updating the best estimate of the 'signal' in a time series in the presence of noise. Chapters 11 and 12 introduce **non-linear** and **volatility** time series models, respectively. Chapter 13 introduce **multivariate** time series models, while Chapter 14 briefly reviews some more advanced topics.

1.5 Review of Books on Time Series

This section gives a brief review of some alternative books on time series that may be helpful to the reader. The literature has expanded considerably in recent years and so a selective approach is necessary.

Alternative general introductory texts include Brockwell and Davis (2002), Harvey (1993), Kendall and Ord (1990) and Wei (1990). Diggle (1990) is aimed primarily at biostatisticians, while Enders (1995), Mills (1990, 1999), and Harris and Sollis (2003) are aimed at economists.

There are many more advanced texts including Anderson (1971), Brockwell and Davis (1991), Fuller (1996) and Priestley (1981). The latter is particularly strong on spectral analysis and multivariate time series modelling. Brillinger (2001) is a classic reprint of a book concentrating on the frequency domain. Kendall et al. (1983), now in the fourth edition, is a valuable reference source, but note that earlier editions are somewhat dated. Hamilton (1994) is aimed at econometricians. Tsay (2010), now in the third edition, provides a comprehensive introduction to financial econometric models and their applications to modeling and prediction of financial time series data.

The classic book by Box and Jenkins (1970) describes an approach to time series analysis, forecasting and control that is based on a particular class of models, called autoregressive integrated moving average (ARIMA) models. This important book is not really suitable for the beginner, who is recommended to read Chapters 3–5 in this book, Vandaele (1983) or the relevant chapters of Wei (1990). The 1976 revised edition of Box and Jenkins (1970) was virtually unchanged, but the third edition (Box et al., 1994), with G. Reinsel as third author, has changed substantially.

Some time series books introduce the subject with statistical computing languages. For example, Cowpertwit and Metcalfe (2009) shows how to use `R` to apply time series methods in some practical applications. Shumway and Stoffer (2010) introduces time series models in time and frequency domains with `R` codes provided for some examples. Tsay (2010) demonstrates the application of time series models in financial econometrics with examples in `R` and `S-Plus`.

In this edition of the book, the first 11 chapters of the new edition have a very similar structure to the original edition. We added a new chapter (Chapter 12) to introduce volatility models in finance, and part of Chapter 13 was rewritten to briefly describe multivariate volatility models. We added more examples and exercises in this edition. For illustration purposes, we also present `R` code for data examples in the book.

Additional books will be referenced, as appropriate, in later chapters.

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Chapter 2

Basic Descriptive Techniques

Statistical techniques for analysing time series range from relatively straightforward descriptive methods to sophisticated inferential techniques. This chapter introduces the former, which will often clarify the main properties of a given series. Descriptive methods should generally be tried before attempting more complicated procedures, because they can be vital in 'cleaning' the data, and then getting a 'feel' for them, before trying to generate ideas as regards to a suitable model.

Before doing anything, the analyst should make sure that the practical problem being tackled is properly understood. In other words, the **context** of a given problem is crucial in time series analysis, as in all areas of statistics. If necessary, the analyst should ask questions so as to get appropriate background information and clarify the objectives. These preliminary questions should not be rushed. In particular, make sure that appropriate data have been, or will be, collected. If the series are too short, or the wrong variables have been measured, it may not be possible to solve the given problem.

For a chapter on descriptive techniques, the reader may be expecting the first section to deal with **summary statistics**. Indeed, in most areas of statistics, a typical analysis begins by computing the sample mean (or median or mode) and the standard deviation (or interquartile range) to measure 'location' and 'dispersion'. However, *Time-series analysis is different!* If a time series contains trend, seasonality or some other systematic component, the usual summary statistics can be seriously misleading and should not be calculated. Moreover, even when a series does *not* contain any systematic components, the summary statistics do not have their usual properties (see Section 4.1.2). Thus, this chapter focuses on ways of understanding typical time-series effects, such as trend, seasonality, and correlations between successive observations.

2.1 Types of Variation

Traditional methods of time-series analysis are mainly concerned with decomposing the variation in a series into components representing trend, seasonal variation and other cyclic changes. Any remaining variation is attributed to 'irregular' fluctuations. This approach is not always the best but is particularly valuable when the variation is dominated by trend and

seasonality. However, it is worth noting that a decomposition into trend and seasonal variation is generally not unique unless certain assumptions are made. Thus some sort of modelling, either explicit or implicit, may be involved in carrying out these descriptive techniques, and this demonstrates the blurred borderline that always exists between descriptive and inferential techniques in statistics.

The different sources of variation will now be described in more detail.

Seasonal variation

Many time series, such as sales figures and temperature readings, exhibit variation that is annual in period. For example, unemployment is typically 'high' in winter but lower in summer. This yearly variation is easy to understand, and can readily be estimated if seasonality is of direct interest. Alternatively, seasonal variation can be removed from the data, to give deseasonalized data, if seasonality is not of direct interest.

Other cyclic variation

Apart from seasonal effects, some time series exhibit variation at a fixed period due to some other physical cause. An example is daily variation in temperature. In addition some time series exhibit oscillations, which do not have a fixed period but which are predictable to some extent. For example, economic data are sometimes thought to be affected by business cycles with a period varying from about 3 or 4 years to more than 10 years, depending on the variable measured. However, the existence of such business cycles is the subject of some controversy, and there is increasing evidence that any such cycles are not symmetric. An economy usually behaves differently when going into recession, rather than emerging from recession.

Trend

This may be loosely defined as 'long-term change in the mean level'. A difficulty with this definition is deciding what is meant by 'long term'. For example, climatic variables sometimes exhibit cyclic variation over a very long time period such as 50 years. If one just had 20 years of data, this long-term oscillation may look like a trend, but if several hundred years of data were available, then the long-term cyclic variation would be visible. Nevertheless in the short term it may still be more meaningful to think of such a long-term oscillation as a trend. Thus in speaking of a 'trend', we must take into account the number of observations available and make a subjective assessment of what is meant by the phrase 'long term'. As for seasonality, methods are available either for estimating the trend, or for removing it so that the analyst can look more closely at other sources of variation.

Other irregular fluctuations

After trend and cyclic variations have been removed from a set of data, we are left with a series of residuals that may or may not be 'random'. In

due course, we will examine various techniques for analysing series of this type, either to see whether any cyclic variation is still left in the residuals, or whether apparently irregular variation may be explained in terms of probability models, such as moving average (MA) or autoregressive (AR) models, which will be introduced in Chapter 3.

2.2 Stationary Time Series

A mathematical definition of a stationary time series model will be given in Section 3.2. However, it may be helpful to introduce here the idea of stationarity from an intuitive point of view. Broadly speaking a time series is said to be **stationary** if there is no systematic change in mean (no trend), if there is no systematic change in variance and if strictly periodic variations have been removed. In other words, the properties of one section of the data are much like those of any other section. Strictly speaking, it is very often that time series data violate the stationarity property. However, the phrase is often used for time series data meaning that they exhibit characteristics that suggest a stationary model can sensibly be fitted.

Much of the probability theory of time series is concerned with stationary time series, and for this reason time series analysis often requires one to transform a non-stationary series into a stationary one so as to use this theory. For example, it may be of interest to remove the trend and seasonal variation from a set of data and then try to model the variation in the residuals by means of a stationary stochastic process. However, it is also worth stressing that the non-stationary components, such as the trend, may be of more interest than the stationary residuals.

2.3 The Time Plot

The first, and most important, step in any time-series analysis is to plot the observations against time. This graph, called a **time plot**, will show up important features of the series such as trend, seasonality, outliers and discontinuities. The plot is vital, both to describe the data and to help in formulating a sensible model, and several examples have already been given in Chapter 1.

Plotting a time series is not as easy as it sounds. The choice of scales, the size of the intercept and the way that the points are plotted (e.g. as a continuous line or as separate dots or crosses) may substantially affect the way the plot 'looks', and so the analyst must exercise care and judgement. In addition, the usual rules for drawing 'good' graphs should be followed: a clear title must be given, units of measurement should be stated and axes should be properly labelled.

Nowadays, graphs are usually produced by computers. Some are well drawn but packages sometimes produce rather poor graphs and the reader must be prepared to modify them if necessary or, better, give the computer appropriate

instructions to produce a clear graph in the first place. For example, the software will usually print out the title you provide, and so it is your job to provide a clear title. It cannot be left to the computer.

2.4 Transformations

Plotting the data may suggest that it is sensible to consider transforming them, for example, by taking logarithms or square roots. The three main reasons for making a transformation are as follows.

(i) To stabilize the variance

If there is a trend in the series and the variance appears to increase with the mean, then it may be advisable to transform the data. In particular, if the standard deviation is directly proportional to the mean, a logarithmic transformation is indicated. On the other hand, if the variance changes through time *without* a trend being present, then a transformation will not help. Instead, a model that allows for changing variance should be considered.

(ii) To make the seasonal effect additive

If there is a trend in the series and the size of the seasonal effect appears to increase with the mean, then it may be advisable to transform the data so as to make the seasonal effect constant from year to year. The seasonal effect is then said to be **additive**. In particular, if the size of the seasonal effect is directly proportional to the mean, then the seasonal effect is said to be **multiplicative** and a logarithmic transformation is appropriate to make the effect additive. However, this transformation will only stabilize the variance if the error term is also thought to be multiplicative (see Section 2.6), a point that is sometimes overlooked.

(iii) To make the data normally distributed

Model building and forecasting are usually carried out on the assumption that the data are normally distributed. In practice this is not necessarily the case; there may, for example, be evidence of skewness in that there tend to be 'spikes' in the time plot that are all in the same direction (either up or down). This effect can be difficult to eliminate with a transformation and it may be necessary to model the data using a different 'error' distribution.

The logarithmic and square-root transformations, mentioned above, are special cases of a general class of transformations called the Box–Cox transformation. Given an observed time series $\{x_t\}$ and a transformation parameter λ, the transformed series is given by

$$y_t = \begin{cases} (x_t^\lambda - 1)/\lambda & \lambda \neq 0 \\ \log x_t & \lambda = 0. \end{cases}$$

This is effectively just a power transformation when $\lambda \neq 0$, as the constants are introduced to make y_t a continuous function of λ at the value $\lambda = 0$. The 'best' value of λ can be 'guesstimated', or alternatively estimated by a proper inferential procedure, such as maximum likelihood.

It is instructive to note that Nelson and Granger (1979) found little improvement in forecast performance when a general Box–Cox transformation was tried on a number of series. There are problems in practice with transformations in that a transformation, which makes the seasonal effect additive, for example, may fail to stabilize the variance. Thus it may be impossible to achieve all the above requirements at the same time. In any case a model constructed for the transformed data may be less than helpful. It is more difficult to interpret and forecasts produced by the transformed model may have to be 'transformed back' in order to be of use. This can introduce biasing effects. Usually, transformations should be avoided wherever possible except where the transformed variable has a direct physical interpretation. For example, when percentage increases are of interest, then taking logarithms makes sense. Further general remarks on transformations are given by Granger and Newbold (1986, Section 10.5).

2.5 Analysing Series that Contain a Trend and No Seasonal Variation

In Section 2.1, we loosely defined trend as a 'long-term change in the mean level'. It is much more difficult to give a precise definition of trend and different authors may use the term in different ways. The simplest type of trend is the familiar 'linear trend + noise', for which the observation at time t is a random variable X_t, given by

$$X_t = \alpha + \beta t + \varepsilon_t, \tag{2.1}$$

where α, β are constants and ε_t denotes a random error term with zero mean. The mean level at time t is given by $m_t = (\alpha + \beta t)$; this is sometimes called 'the trend term'. Other writers prefer to describe the slope β as the trend, so that trend is the *change* in the mean level per unit time. It is usually clear from the context as to what is meant by 'trend'.

The trend in Equation (2.1) is a deterministic function of time and is sometimes called a **global** linear trend. In practice, this generally provides an unrealistic model, and nowadays there is more emphasis on models that allow for **local** linear trends. One possibility is to fit a **piecewise linear** model where the trend line is locally linear but with change points where the slope and intercept change (abruptly). It is usually arranged that the lines join up at the change points, but, even so, the sudden changes in slope often seem unnatural. Thus, it often seems more sensible to look at models that allow a smooth transition between the different submodels. Extending this idea, it seems even more natural to allow the parameters α and β in Equation (2.1) to evolve through time. This could be done deterministically, but it is more common to assume that α and β evolve stochastically giving rise to

what is called a **stochastic trend**. Some examples of suitable models, under the general heading of state-space models, are given in Chapter 10. Another possibility, depending on how the data look, is that the trend has a *non-linear* form, such as quadratic growth. Exponential growth can be particularly difficult to handle, even if logarithms are taken to transform the trend to a linear form. Even with present-day computing aids, it can still be difficult to decide what form of trend is appropriate in a given context (see Ball and Wood (1996) and the discussion that followed).

The analysis of a time series that exhibits trend depends on whether one wants to (1) measure the trend and/or (2) remove the trend in order to analyse local fluctuations. It also depends on whether the data exhibit seasonality (see Section 2.6). With seasonal data, it is a good idea to start by calculating successive yearly averages, as these will provide a simple description of the underlying trend. An approach of this type is sometimes perfectly adequate, particularly if the trend is fairly small, but sometimes a more sophisticated approach is desired.

We now describe some different general approaches to describing trend.

2.5.1 Curve fitting

A traditional method of dealing with non-seasonal data that contain a trend, particularly yearly data, is to fit a simple function of time such as a polynomial curve (linear, quadratic, etc.), a Gompertz curve or a logistic curve (e.g. see Meade, 1984; Franses, 1998, Chapter 4). The global linear trend in Equation (2.1) is the simplest type of polynomial curve. The Gompertz curve can be written in the form

$$\log x_t = a + br^t,$$

where a, b, r are parameters with $0 < r < 1$, or in the alternative form of

$$x_t = \alpha \exp[\beta \exp(-\gamma t)],$$

which looks quite different, but is actually equivalent, provided $\gamma > 0$. The logistic curve is given by

$$x_t = a/(1 + b\ e^{-ct}).$$

Both these curves are S-shaped and approach an asymptotic value as $t \to \infty$, with the Gompertz curve generally converging slower than the logistic. Fitting the curves to data may lead to non-linear simultaneous equations.

For all curves of this type, the fitted function provides a measure of the trend, and the residuals provide an estimate of local fluctuations, where the residuals are the differences between the observations and the corresponding values of the fitted curve.

2.5.2 Filtering

A second procedure for dealing with a trend is to use a **linear filter**, which converts one time series, $\{x_t\}$, into another, $\{y_t\}$, by the linear operation

$$y_t = \sum_{r=-q}^{+s} a_r x_{t+r},$$

where $\{a_r\}$ is a set of weights. In order to smooth out local fluctuations and estimate the local mean, we should clearly choose the weights so that $\sum a_r = 1$, and then the operation is often referred to as a **moving average**. Moving averages are discussed in detail by Kendall et al. (1983, Chapter 46), and we will only provide a brief introduction.

Moving averages are often symmetric with $s = q$ and $a_j = a_{-j}$. The simplest example of a symmetric smoothing filter is the simple moving average, for which $a_r = 1/(2q+1)$ for $r = -q, \ldots, +q$, and the smoothed value of x_t is given by

$$\mathrm{Sm}(x_t) = \frac{1}{2q+1} \sum_{r=-q}^{+q} x_{t+r}. \tag{2.2}$$

The simple moving average is not generally recommended by itself for measuring trend, although it can be useful for removing seasonal variation. Another symmetric example is provided by the case where the $\{a_r\}$ are successive terms in the expansion of $(\frac{1}{2} + \frac{1}{2})^{2q}$. Thus when $q = 1$, the weights are $a_{-1} = a_{+1} = \frac{1}{4}, a_0 = \frac{1}{2}$. As q gets large, the weights approximate to a normal curve.

A third example is **Spencer's 15-point moving average**, which is used for smoothing mortality statistics to get life tables. This covers 15 consecutive points with $q = 7$, and the symmetric weights are

$$\frac{1}{320}[-3, -6, -5, 3, 21, 46, 67, 74, 67, 46, 21, \ldots].$$

A fourth example, called the **Henderson moving average**, is described by Kenny and Durbin (1982) and is widely used, for example, in the X-11 and X-12 seasonal packages (see Section 2.6). This moving average aims to follow a cubic polynomial trend without distortion, and the choice of q depends on the degree of irregularity. The symmetric nine-term moving average, for example, is given by

$$[-0.041, -0.010, 0.119, 0.267, 0.330, \ldots].$$

The general idea is to fit a polynomial curve, not to the whole series, but to a local set of points. For example, a polynomial fitted to the first $(2q+1)$ data points can be used to determine the interpolated value at the middle of the range where $t = (q+1)$, and the procedure can then be repeated using the data from $t = 2$ to $t = (2q+2)$ and so on.

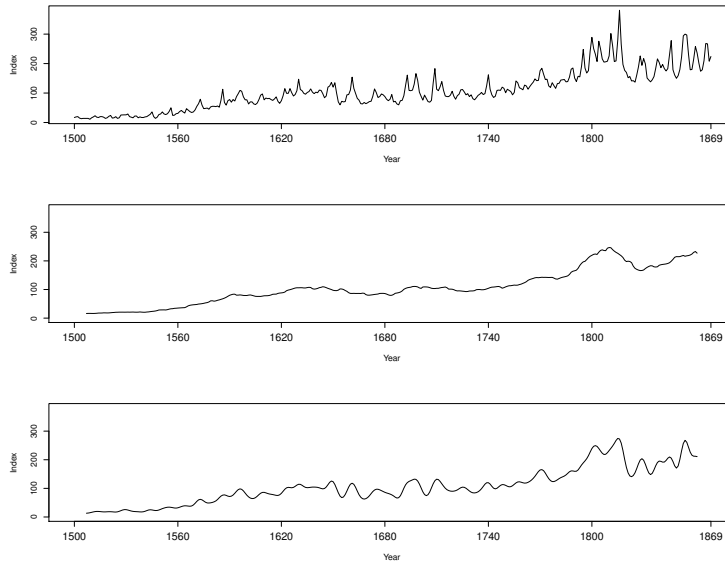

Figure 2.1 *The original and smoothed Beveridge wheat price annual index series from 1500 to 1869 (Top: The raw data; Middle: Smoothed series with filter (2.2) and $q = 7$; Bottom: Smoothed series with Spencer's 15-point moving average).*

To demonstrate the effect of these moving averages, we use the Beveridge wheat price annual index series from 1500 to 1869 as an example (see Figure 1.1). For comparison purpose, the top panel of Figure 2.1 shows the original Beveridge wheat price annual index series from 1500 to 1869, and the middle and bottom panels show the smoothed series by using the moving average (2.2) with $q = 7$ and Spencer's 15-point moving average. Note that unequal weights in Spencer's 15-point moving average allow us to capture more local variation in the original series.

To reproduce Figure 2.1 and the two smoothed series in R, use the following command:

```
> smooth.sym<-function(my.ts, window.q){
    window.size<-2*window.q+1
    my.ts.sm<-rep(0, length(my.ts)-window.size)
    for (i in 1:length(my.ts.sm)) {
      my.ts.sm[i]<-mean(my.ts[i:(i+window.size-1)])
    }
    my.ts.sm
  }     # moving average with equal and symmetric weights
> smooth.spencer<-function(my.ts){
    weight<-c(-3,-6,-5,3,21,46,67,74,67,46,21,3,-5,-6,-3)/320
    my.ts.sm<-rep(0, length(my.ts)-15)
    for (i in 1:length(my.ts.sm)) {
```

```
      my.ts.sm[i]<-sum(my.ts[i:(i+14)]*weight)
    }
   my.ts.sm
  }    # Spencer's 15-point moving average

> library(tseries)
> data(bev)
> bev.sm<-smooth.sym(bev, 7)
> bev.spencer<-smooth.spencer(bev)
> x.pos<-c(1500, 1560, 1620, 1680, 1740, 1800, 1869)
> par(mfrow=c(3,1), mar=c(4,4,4,4))
> plot(bev, type="l", xlab="Year", ylab="Index", xaxt="n")
> axis(1, x.pos, x.pos)
> plot(c(1, length(bev)), c(0, max(bev)), type="n", xlab="Year",
                ylab="Index", xaxt="n")
> lines(seq(8, length(bev)-8), bev.sm)
> axis(1, x.pos-1500+1, x.pos)
> plot(c(1, length(bev)), c(0, max(bev)), type="n", xlab="Year",
                ylab="Index", xaxt="n")
> lines(seq(8, length(bev)-8), bev.spencer)
> axis(1, x.pos-1500+1, x.pos)
```

Whenever a symmetric filter is chosen, there is likely to be an **end-effects** problem (e.g. Kendall et al., 1983, Section 46.11), since $\mathrm{Sm}(x_t)$ can only be calculated for $t = (q+1)$ to $t = N - q$. In some situations this may not be important, as, for example, in carrying out some retrospective analyses. However, in other situations, such as in forecasting, it is particularly important to get smoothed values right up to $t = N$. The analyst can project the smoothed values by eye or, alternatively, can use an asymmetric filter that only involves present and past values of x_t. For example, the popular technique known as **exponential smoothing** (see Section 5.2.2) effectively assumes that

$$\mathrm{Sm}(x_t) = \sum_{j=0}^{\infty} \alpha(1-\alpha)^j x_{t-j},$$

where α is a constant such that $0 < \alpha < 1$. Here we note that the weights $a_j = \alpha(1-\alpha)^j$ decrease geometrically with j.

Having estimated the trend, we can look at the local fluctuations by examining

$$\begin{aligned} \mathrm{R}es(x_t) &= \text{residual from smoothed value} \\ &= x_t - \mathrm{S}m(x_t) \\ &= \sum_{r=-q}^{+s} b_r x_{t+r}. \end{aligned}$$

This is also a linear filter with $b_0 = 1 - a_0$, and $b_r = -a_r$ for $r \neq 0$. If $\sum a_r = 1$, then $\sum b_r = 0$ and the filter is a trend remover.

How do we choose the appropriate filter? The answer to this question really requires considerable experience plus a knowledge of the frequency aspects of time-series analysis, which will be discussed in later chapters. As the name implies, filters are usually designed to produce an output with emphasis on variation at particular frequencies. For example, to get smoothed values we want to remove the local fluctuations that constitute what is called the high-frequency variation. In other words we want what is called a **low-pass** filter. To get $\mathrm{Res}(x_t)$, we want to remove the long-term fluctuations or the low-frequency variation. In other words we want what is called a **high-pass** filter. The **Slutsky** or **Slutsky–Yule effect** is related to this problem. Slutsky showed that by operating on a completely random series with both averaging and differencing procedures one could induce sinusoidal variation in the data. Slutsky went on to suggest that apparently periodic behaviour in some economic time series might be accounted for by the smoothing procedures used to form the data. We will return to this question later.

Filters in series

A smoothing procedure may be carried out in two or more stages.

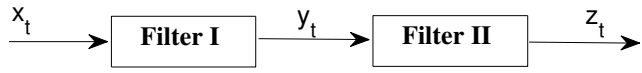

Figure 2.2 *Two filters in series.*

As an example, two filters in series may be represented as on the opposite page. It is easy to show that a series of linear operations is still a linear filter overall. Suppose filter I, with weights $\{a_r\}$, acts on $\{x_t\}$ to produce $\{y_t\}$. Then filter II with weights $\{b_j\}$ acts on $\{y_t\}$ to produce $\{z_t\}$. Now

$$\begin{aligned} z_t &= \sum_j b_j y_{t+j} \\ &= \sum_j b_j \sum_r a_r x_{t+j+r} \\ &= \sum_k c_k x_{t+k} \end{aligned}$$

where

$$c_k = \sum_r a_r b_{(k-r)}$$

are the weights for the overall filter. The weights $\{c_k\}$ are obtained by a procedure called **convolution**, and we may write

$$\{c_k\} = \{a_r\} * \{b_j\},$$

where the symbol $*$ represents the convolution operator. For example, the filter $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ may be written as

$$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) * \left(\frac{1}{2}, \frac{1}{2}\right).$$

The Spencer 15-point moving average is actually a convolution of four filters, namely

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) * \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) * \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) * \left(-\frac{3}{4}, \frac{3}{4}, 1, \frac{3}{4}, -\frac{3}{4}\right)$$

and this may be the best way to compute it.

2.5.3 Differencing

A special type of filtering, which is particularly useful for removing a trend, is simply to difference a given time series until it becomes stationary. We will see that this method is an integral part of the so-called Box-Jenkins procedure. For non-seasonal data, first-order differencing is usually sufficient to attain apparent stationarity. Here a new series, say $\{y_2, \ldots, y_N\}$, is formed from the original observed series, say $\{x_1, \ldots, x_N\}$, by $y_t = x_t - x_{t-1} = \nabla x_t$ for $t = 2, 3, \ldots, N$. Occasionally second-order differencing is required using the operator ∇^2, where

$$\nabla^2 x_t = \nabla x_t - \nabla x_{t-1} = x_t - 2x_{t-1} + x_{t-2}.$$

First differencing is widely used and often works well. For example, Franses and Kleibergen (1996) show that better out-of-sample forecasts are usually obtained with economic data by using first differences rather than fitting a deterministic (or global) trend. *Seasonal differencing*, to remove seasonal variation, will be introduced in the next section.

2.5.4 Other approaches

Some alternative, more complicated, approaches to handling trend will be introduced later in the book. In particular, several state-space models involving trend terms will be introduced in Chapter 10.

2.6 Analysing Series that Contain a Trend and Seasonal Variation

In Section 2.1 we introduced seasonal variation, which is generally annual in period, while Section 2.4 distinguished between additive seasonality, which

is constant from year to year, and multiplicative seasonality. Three seasonal models in common use are

$$\begin{aligned} &\text{A} \qquad X_t = m_t + S_t + \varepsilon_t \\ &\text{B} \qquad X_t = m_t S_t + \varepsilon_t \\ &\text{C} \qquad X_t = m_t S_t \varepsilon_t \end{aligned}$$

where m_t is the deseasonalized mean level at time t, S_t is the seasonal effect at time t and ε_t is the random error.

Model A describes the **additive** case, while models B and C both involve **multiplicative** seasonality. In model C the error term is also multiplicative, and a logarithmic transformation will turn this into a (linear) additive model, which may be easier to handle. The time plot should be examined to see which model is likely to give the better description. The seasonal indices $\{S_t\}$ are usually assumed to change slowly through time so that $S_t \simeq S_{t-s}$, where s is the number of observations per year. The indices are usually normalized so that they sum to zero in the additive case, or average to one in the multiplicative case. Difficulties arise in practice if the seasonal and/or error terms are not exactly multiplicative or additive. For example, the seasonal effect may increase with the mean level but not at such a fast rate so that it is somewhere 'in between' being multiplicative or additive. A mixed additive-multiplicative seasonal model is described by Durbin and Murphy (1975).

The analysis of time series, which exhibit seasonal variation, depends on whether one wants to (1) measure the seasonal effect and/or (2) eliminate seasonality. For series showing little trend, it is usually adequate to estimate the seasonal effect for a particular period (e.g. January) by finding the average of each January observation minus the corresponding yearly average in the additive case, or the January observation divided by the yearly average in the multiplicative case.

For series that do contain a substantial trend, a more sophisticated approach is required. With monthly data, the most common way of eliminating the seasonal effect is to calculate

$$\mathrm{Sm}(x_t) = \frac{\frac{1}{2}\,x_{t-6} + x_{t-5} + x_{t-4} + \cdots + x_{t+5} + \frac{1}{2}\,x_{t+6}}{12}.$$

Note that the two end coefficients are different from the rest but that the coefficients sum to unity. A simple moving average cannot be used, as this would span 12 months and would not be centered on an integer value of t. A simple moving average over 13 months cannot be used, as this would give twice as much weight to the month appearing at both ends. For quarterly data, the seasonal effect can be eliminated by calculating

$$\mathrm{Sm}(x_t) = \frac{\frac{1}{2}\,x_{t-2} + x_{t-1} + x_t + x_{t+1} + \frac{1}{2}\,x_{t+2}}{4}.$$

For 4-weekly data, one *can* use a simple moving average over 13 successive observations.

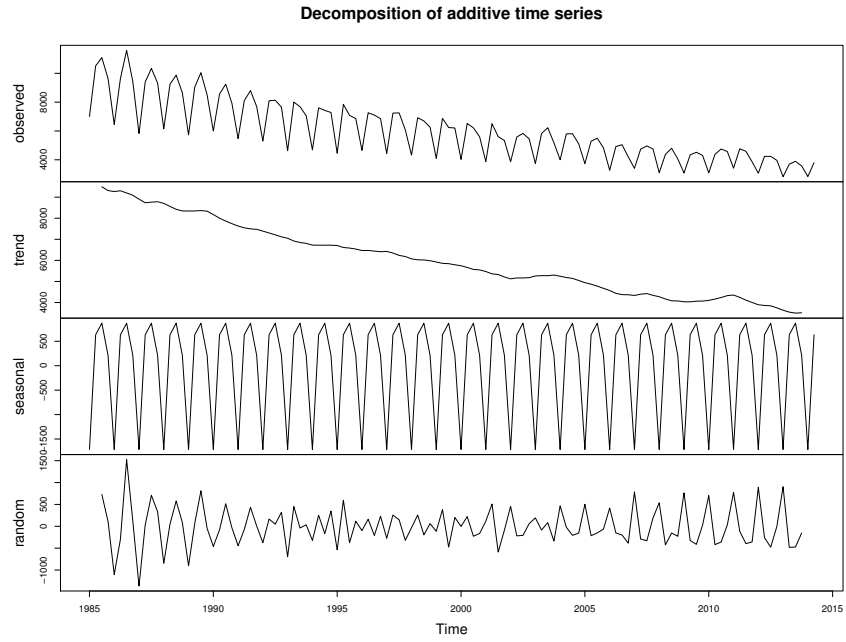

Figure 2.3 *The decomposition of the domestic sales of Australian fortified wine.*

These smoothing procedures all effectively estimate the local (deseasonalized) level of the series. The seasonal effect itself can then be estimated by calculating $x_t - \mathrm{Sm}(x_t)$ or $x_t/\mathrm{Sm}(x_t)$ depending on whether the seasonal effect is thought to be additive or multiplicative. A check should be made that the seasonals are reasonably stable, and then the average monthly (or quarterly etc.) effects can be calculated.

As an example, Figure 2.3 decomposes the domestic monthly sales series of Australian fortified wine by winemakers into trend, additive, seasonal, and remainder series. The decomposition is carried out by using the `R` command `decompose`, which decomposes a time series into trend, (additive or multiplicative) seasonal, and remainder components using moving averages.

```
> wine<-read.csv("../data/aus_wine_sales.csv", header=F)
> wine.ts<-ts(wine[,2], frequency=4, start=c(1985,1))
        # create a time series object
> wine.de<-decompose(wine.ts, type="additive")
> plot(wine.de)
```

A seasonal effect can also be eliminated by a simple linear filter called **seasonal differencing**. For example, with monthly data one can employ the

operator ∇_{12} where

$$\nabla_{12} x_t = x_t - x_{t-12}.$$

Further details on seasonal differencing will be given in Sections 4.8 and 5.3.

Two general reviews of methods for seasonal adjustment are Butter and Fase (1991) and Hylleberg (1992). Without going into great detail, we should mention the widely used **X-11 method**, now updated as the **X-12 method** (Findley et al., 1998), which is used for estimating or removing both trend and seasonal variation. It is a fairly complicated procedure that employs a series of linear filters and adopts a recursive approach. Preliminary estimates of trend are used to get preliminary estimates of seasonal variation, which in turn are used to get better estimates of trend and so on. The new software for X-12 gives the user more flexibility in handling outliers, as well as providing better diagnostics and an improved user interface. X-12 also allows the user to deal with the possible presence of **calendar effects**, which should always be considered when dealing with seasonal data (e.g. Bell and Hillmer, 1983). For example, if Easter falls in March one year, and April the next, then this will alter the seasonal effect on sales for both months. The X-11 or X-12 packages are often combined with ARIMA modelling, as introduced in the next three chapters, to interpolate the values near the end of the series and avoid the end-effects problem arising from using symmetric linear filters alone. The package is called **X-12-ARIMA**. On mainland Europe, many governments use an alternative approach, based on packages called **SEATS** (Signal Extraction in ARIMA Time Series) and **TRAMO** (Time-Series Regression with ARIMA Noise). They are described in Gómez and Maravall (2001).

2.7 Autocorrelation and the Correlogram

An important guide to the properties of a time series is provided by a series of quantities called the **sample autocorrelation coefficients**. They measure the correlation, if any, between observations at different distances apart and provide useful descriptive information. In Chapter 4, we will see that they are also an important tool in model building, and often provide valuable clues to a suitable probability model for a given set of data.

We assume that the reader is familiar with the ordinary correlation coefficient.[1] Given N pairs of observations on two variables x and y, say $\{(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)\}$, the sample correlation coefficient is given by

$$r = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{N}(x_i - \bar{x})^2 \sum_{i=1}^{N}(y_i - \bar{y})^2}}. \tag{2.3}$$

[1] This topic is briefly covered in Appendix C.

This quantity lies in the range $[-1, 1]$ and measures the strength of the linear association between the two variables. It can easily be shown that the value does not depend on the units in which the two variables are measured. The correlation is negative if 'high' values of x tend to go with 'low' values of y. If the two variables are independent, then the true correlation is zero. Here, we apply an analogous formula to time-series data to measure whether successive observations are correlated.

Given N observations $x_1, \ldots, x_N$, on a time series, we can form $N-1$ pairs of observations, namely, $(x_1, x_2), (x_2, x_3), \ldots, (x_{N-1}, x_N)$, where each pair of observations is separated by one time interval. Regarding the first observation in each pair as one variable, and the second observation in each pair as a second variable, then, by analogy with Equation (2.3), we can measure the correlation coefficient between adjacent observations, x_t and x_{t+1}, using the formula

$$r_1 = \frac{\sum_{t=1}^{N-1}(x_t - \bar{x}_{(1)})(x_{t+1} - \bar{x}_{(2)})}{\sqrt{\sum_{t=1}^{N-1}(x_t - \bar{x}_{(1)})^2 \sum_{t=1}^{N-1}(x_{t+1} - \bar{x}_{(2)})^2}}, \tag{2.4}$$

where

$$\bar{x}_{(1)} = \sum_{t=1}^{N-1} x_t/(N-1)$$

is the mean of the first observation in each of the $(N-1)$ pairs and so is the mean of the first $N-1$ observations, while

$$\bar{x}_{(2)} = \sum_{t=2}^{N} x_t/(N-1)$$

is the mean of the last $N-1$ observations. As the coefficient given by Equation (2.4) measures correlation between successive observations, it is called the **sample autocorrelation coefficient** or a **serial correlation coefficient** at lag one.

Equation (2.4) is rather complicated, and so, as $\bar{x}_{(1)} \simeq \bar{x}_{(2)}$, it is usually approximated by

$$r_1 = \frac{\sum_{t=1}^{N-1}(x_t - \bar{x})(x_{t+1} - \bar{x})}{(N-1)\sum_{t=1}^{N}(x_t - \bar{x})^2/N} \tag{2.5}$$

where $\bar{x} = \sum_{t=1}^{N} x_t/N$ is the overall mean. It is often convenient to further simplify this expression by dropping the factor $N/(N-1)$, which is close to

one for large N. This gives the even simpler formula

$$r_1 = \frac{\sum_{t=1}^{N-1}(x_t - \bar{x})(x_{t+1} - \bar{x})}{\sum_{t=1}^{N}(x_t - \bar{x})^2} \tag{2.6}$$

and this is the form that will be used in this book.

In a similar way, we can find the correlation between observations that are k steps apart, and this is given by

$$r_k = \frac{\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{N}(x_t - \bar{x})^2}, \quad k = 0, 1, 2, \ldots. \tag{2.7}$$

This is called the sample autocorrelation coefficient at lag k. The r_k-values are always in $[-1, 1]$. Note that $r_0 = 1$.

In practice the autocorrelation coefficients are usually calculated by computing the series of autocovariance coefficients, $\{c_k\}$, which we define by analogy with the usual covariance formula[2] as

$$c_k = \frac{1}{N}\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x}). \tag{2.8}$$

This is the sample autocovariance coefficient at lag k.

We then compute

$$r_k = c_k/c_0 \tag{2.9}$$

for $k = 0, 1, 2, \ldots, M$, where $M < N$.

Note that some authors prefer to use

$$c_k = \frac{1}{N-k}\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x})$$

rather than Equation (2.8), and this gives Equation (2.5) when $k = 1$, but we prefer to use Equations (2.8) and (2.6) for reasons explained later in Section 4.1. We note, once again, that there are only small differences between the different formulae for large N.

2.7.1 The correlogram

A useful aid in interpreting a set of autocorrelation coefficients is a graph called a **correlogram** in which the sample autocorrelation coefficients r_k are

[2]The covariance is the average cross-product of two variables – see Appendix C.

plotted against the lag k for $k = 0, 1, \ldots, M$, where M is usually much less than N. For example if $N = 200$, then the analyst might look at the first 20 or 30 coefficients. Examples are given in Figures 2.4-2.8. A visual inspection of the correlogram is often very helpful. Of course, r_0 is always unity, but is still worth plotting for comparative purposes. The correlogram may alternatively be called the sample autocorrelation function (ac.f.).

2.7.2 Interpreting the correlogram

Interpreting the meaning of a set of autocorrelation coefficients is not always easy. Here we offer some general advice.

Random series

A time series is said to be completely random (or i.i.d.) if it consists of a series of independent observations having the same distribution. Figure 2.4 plots a completely random series and its correlogram. Then, for large N, we expect to find that $r_k \simeq 0$ for all non-zero values of k. In fact we will see later that, for a random time series, r_k, $k \geq 1$, is approximately $\mathrm{N}(0, 1/N)$. Thus, if a time series is random, we can expect 19 out of 20 of the values of r_k to lie between $\pm 1.96/\sqrt{N}$. As a result, it is common practice to regard any values of r_k outside these limits as being 'significant'. However, if one plots say the first 20 values of r_k, then one can expect to find one 'significant' value on average even when the time series really is random. This spotlights one of the difficulties in interpreting the correlogram, in that a large number of coefficients is quite likely to contain one (or more) 'unusual' results, even when no real effects are present. (See also Section 4.1.)

Figure 2.4 can be reproduced using the following `R` code.

```
> set.seed(1)
> x<-rnorm(400)
> par(mfrow=c(2,1), mar=c(3,4,3,4))
> plot(x, type="l", xlab="", ylab="")
> title(xlab="Time", ylab="Series", line=2, cex.lab=1.2)
> acf(x, ylab="", main="")
> title(xlab="Lag", ylab="ACF", line=2)
```

Short-term correlation

Stationary series often exhibit short-term correlation characterized by a fairly large value of r_1 followed by one or two further coefficients, which, while greater than zero, tend to get successively smaller. Values of r_k for longer lags tend to be approximately zero. An example of such a correlogram is shown in Figure 2.5. A time series that gives rise to such a correlogram is one for which an observation above the mean tends to be followed by one or more further observations above the mean, and similarly for observations below the mean.

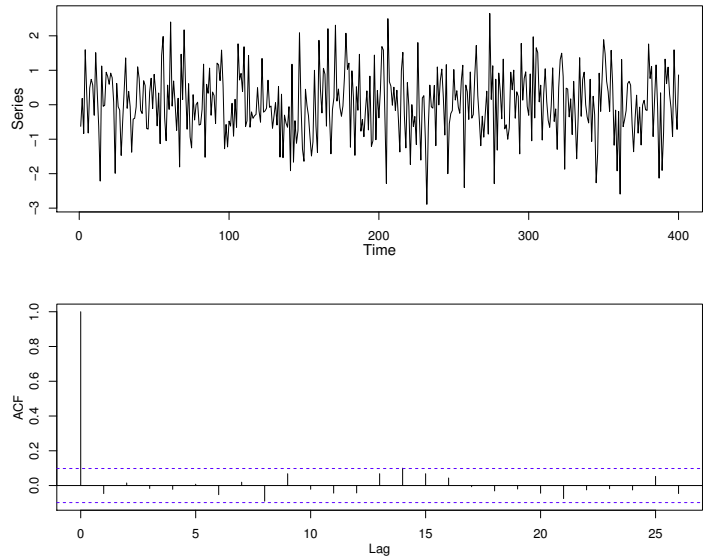

Figure 2.4 *A completely random series together with its correlogram. The dotted lines in the correlogram are at* $\pm 1.96/\sqrt{N}$*. Values outside these lines are said to be significantly different from zero.*

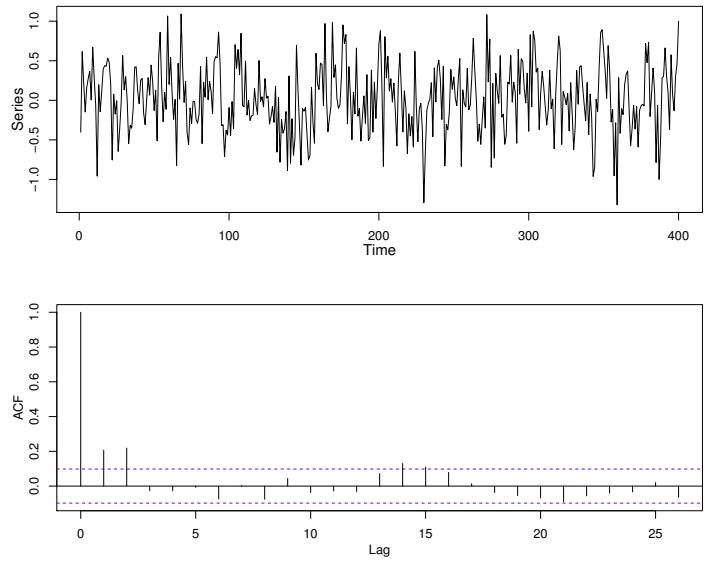

Figure 2.5 *A time series showing short-term correlation together with its correlogram.*

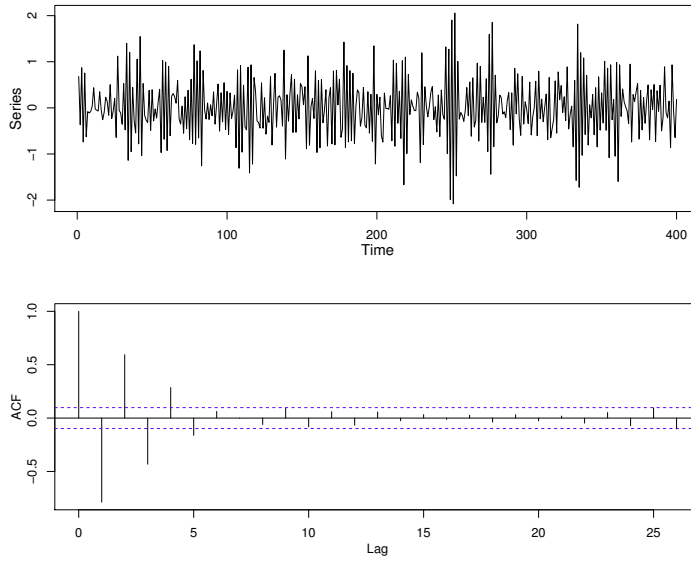

Figure 2.6 *An alternating time series together with its correlogram.*

Alternating series

If a time series has a tendency to alternate, with successive observations on different sides of the overall mean, then the correlogram also tends to alternate. With successive values on opposite sides of the mean, the value of r_1 will naturally be negative, but the value of r_2 will be positive, as observations at lag 2 will tend to be on the same side of the mean. An alternating time series together with its correlogram is shown in Figure 2.6. The time series plots in Figures 2.5 and 2.6 also suggest that it is not necessarily easy to visually distinguish a series with short-term correlation from the one with alternating correlation.

Non-stationary series

If a time series contains a trend, then the values of r_k will not come down to zero except for very large values of the lag. This is because an observation on one side of the overall mean tends to be followed by a large number of further observations on the same side of the mean because of the trend. A typical non-stationary time series together with its correlogram is shown in Figure 2.7. Little can be inferred from a correlogram of this type as the trend dominates all other features. In fact the sample ac.f. $\{r_k\}$ is only meaningful for data from a **stationary** time-series model (see Chapters 3 and 4) and so any trend should be removed before calculating $\{r_k\}$. Of course, if the trend

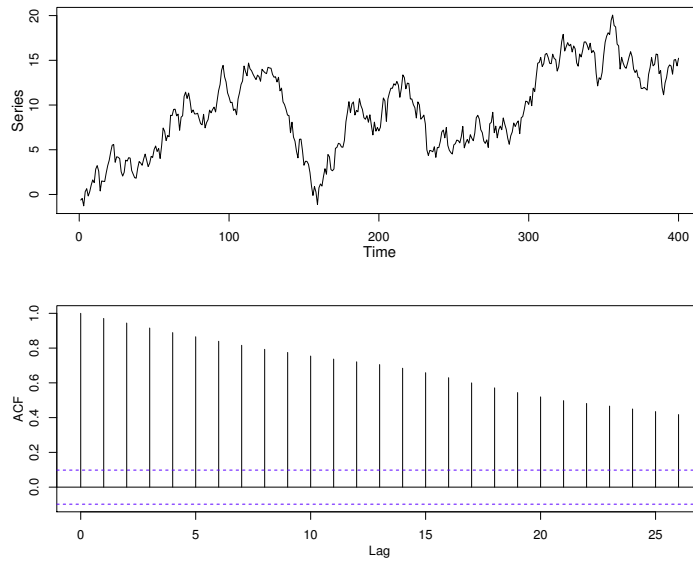

Figure 2.7 *A non-stationary time series together with its correlogram.*

itself is of prime interest, then it should be modelled, rather than removed, and then the correlogram is not helpful.

Figure 2.7 can be reproduced using the following `R` code.

```
> set.seed(1)
> ts.sim3<-cumsum(rnorm(400))
> par(mfrow=c(2,1), mar=c(3,4,3,4))
> plot(ts.sim3, type="l", xlab="", ylab="")
> title(xlab="Time", ylab="Series", line=2, cex.lab=1.2)
> acf(ts.sim3, ylab="",main="")
> title(xlab="Lag", ylab="ACF", line=2)
```

Seasonal series

If a time series contains seasonal variation, then the correlogram will also exhibit oscillation at the same frequency. For example, with monthly observations, we can expect r_6 to be 'large' and negative, while r_{12} will be 'large' and positive. In particular if x_t follows a sinusoidal pattern, then so does r_k. For example, if

$$x_t = a\cos\ t\omega$$

where a is a constant and the frequency ω is such that $0 < \omega < \pi$, then it can be shown (see Exercise 2.3) that

$$r_k \simeq \cos\ k\omega \qquad \text{for large } N.$$

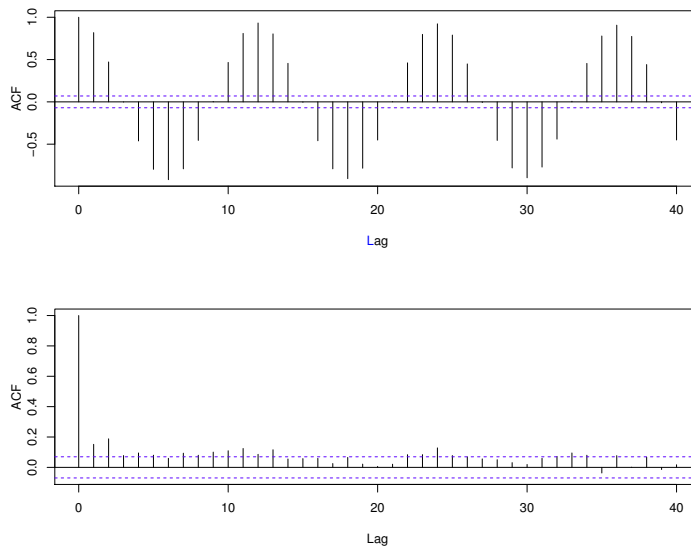

Figure 2.8 *The correlograms of monthly observations on air temperature in Anchorage, Alaska for the raw data (top) and for the seasonally adjusted data (bottom).*

The top panel of Figure 2.8 shows the correlogram of the monthly air temperature data shown in Figure 1.3. The sinusoidal pattern of the correlogram is clearly evident, but for seasonal data of this type the correlogram provides little extra information, as the seasonal pattern is usually displayed in the time plot of the data (e.g., Figure 1.3). Note that it is generally wise to look at coefficients covering at least three seasons. For example, coefficients up to lag 36 may be useful for monthly data.

If the seasonal variation is removed from seasonal data, then the correlogram may provide useful information. The seasonal variation was removed from the air temperature data by the simple, but rather crude, procedure of calculating the 12 monthly averages and subtracting the appropriate one from each individual observation. The correlogram of the resulting series (The bottom panel of Figure 2.8) shows that the first three coefficients are significantly different from zero. This indicates short-term correlation in that a month, which is say colder than the average for that month, will tend to be followed by one or two further months that are colder than average.

Outliers

If a time series contains one or more outliers, the correlogram may be seriously affected and it may be advisable to adjust outliers in some way before starting

the formal analysis. For example, if there is one outlier in the time series at, say, time t_0, and if it is not adjusted, then the plot of x_t against x_{t+k} will contain **two** 'extreme' points, namely, (x_{t_0-k}, x_{t_0}) and (x_{t_0}, x_{t_0+k}). The effect of these two points will be to depress the sample correlation coefficients towards zero. If there are two outliers, this effect is even more noticeable, except when the lag equals the distance between the outliers when a spuriously large correlation may occur.

General remarks

Considerable experience is required to interpret sample autocorrelation coefficients. In addition it is necessary to study the probability theory of stationary series and learn about the classes of models that may be appropriate. It is also necessary to investigate the sampling properties of r_k. These topics will be covered in the next two chapters and we will then be in a better position to interpret the correlogram of a given time series.

2.8 Other Tests of Randomness

In most cases, a visual examination of the graph of a time series is enough to see that the series is **not** random, as, for example, if trend or seasonality is present or there is obvious short-term correlation. However, it is occasionally desirable to assess whether an apparently stationary time series is 'random'. One type of approach is to carry out what is called a **test of randomness** in which one tests whether the observations $x_1, \ldots, x_N$ could have arisen in that order by chance by taking a simple random sample size N from a population assumed to be stationary but with unknown characteristics. Various tests exist for this purpose as described, for example, by Kendall et al. (1983, Section 45.15) and by Kendall and Ord (1990, Chapter 2). It is convenient to briefly mention such tests here.

One type of test is based on counting the number of *turning points*, meaning the number of times there is a local maximum or minimum in the time series. A local maximum is defined to be any observation x_t such that $x_t > x_{t-1}$ and also $x_t > x_{t+1}$. A converse definition applies to local minima. If the series really is random, one can work out the expected number of turning points and compare it with the observed value. An alternative type of test is based on *runs* of observations. For example, the analyst can count the number of runs where successive observations are all greater than the median or all less than the median. This may show up short-term correlation. Alternatively, the analyst can count the number of runs where successive observations are (monotonically) increasing or are (monotonically) decreasing. This may show up trend. Under the null hypothesis of randomness, the expected number of such runs can be found and compared with the observed value, giving tests that are non-parametric or distribution-free in character.

Tests of the above types will not be described here, as we have generally found it more convenient to simply examine the correlogram (and possibly the

spectral density function) of a given time series to see whether it is random. This can often be done visually, but, if a test is required, then the so-called **portmanteau test** can be used (see Section 4.7). The latter test can also be used when assessing models by means of a residual analysis, where the residual of an observation is the difference between the observation and its fitted value from the model. Thus, having fitted a model to a non-random series, one wants to see if the residuals are random, as they should be if the correct model has been fitted. Testing residuals for randomness will be discussed in Section 4.7.

2.9 Handling Real Data

We close this chapter with some important comments on how to handle real data. Analysts generally like to think they have 'good' data, meaning that the data have been carefully collected with no outliers or missing values. In reality, this does not always happen, so that an important part of the initial examination of the data is to assess the quality of the data and consider modifying them, if necessary. An even more basic question is whether the most appropriate variables have been measured in the first place, and whether they have been measured to an appropriate accuracy. Assessing the structure and format of the data is a key step. Practitioners will tell you that these types of questions often take longer to sort out than might be expected, especially when data come from a variety of sources. It really is important to avoid being driven to bad conclusions by bad data.

The process of checking through data is often called *cleaning* the data, or *data editing*. It is an essential precursor to attempts at modelling data. Data cleaning could include **modifying outliers**, identifying and correcting obvious **errors** and filling in (or *imputing*) any **missing observations**. This can sometimes be done using fairly crude devices, such as downweighting outliers to the next most extreme value or replacing missing values with an appropriate mean value. However, more sophisticated methods may be needed, requiring a deeper understanding of time-series models, and so we defer further remarks until Sections 14.4.4 and 14.4.5, respectively. The analyst should also deal with any other known peculiarities, such as a change in the way that a variable is defined during the course of the data-collection process. Data cleaning often arises naturally during a simple preliminary descriptive analysis. In particular, in time-series analysis, the construction of a time plot for each variable is the most important tool for revealing any oddities such as outliers and discontinuities.

After cleaning the data, the next step for the time-series analyst is to determine whether trend and seasonality are present. If so, how should such effects be modelled, measured or removed? In our experience, the treatment of such effects, together with the treatment of outliers and missing values, is often *more* important than any subsequent choices as regards analysing and modelling time-series data.

The *context* of the problem is crucial in deciding how to modify data, if at all, and how to handle trend and seasonality. This explains why it is essential to get background knowledge about the problem, and in particular to clarify the study objectives. A corollary is that it is difficult to make any general remarks or give general recommendations on data cleaning. It is essential to combine statistical theory with sound common sense and knowledge of the particular problem being tackled.

We close by giving the following checklist of possible actions, while noting that the list is not exhaustive and needs to be adapted to the particular problem under consideration.

- Do you understand the context? Have the 'right' variables been measured?
- Have all the time series been plotted?
- Are there any missing values? If so, what should be done about them?
- Are there any outliers? If so, what should be done about them?
- Are there any obvious discontinuities in the data? If so, what does this mean?
- Does it make sense to transform any of the variables?
- Is trend present? If so, what should be done about it?
- Is seasonality present? If so, what should be done about it?

Exercises

2.1 The following data show the coded sales of company X in successive 4-week periods over 1995–1998.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
1995	153	189	221	215	302	223	201	173	121	106	86	87	108
1996	133	177	241	228	283	255	238	164	128	108	87	74	95
1997	145	200	187	201	292	220	233	172	119	81	65	76	74
1998	111	170	243	178	248	202	163	139	120	96	95	53	94

(a) Plot the data.

(b) Assess the trend and seasonal effects.

2.2 Sixteen successive observations on a stationary time series are as follows: 1.6, 0.8, 1.2, 0.5, 0.9, 1.1, 1.1, 0.6, 1.5, 0.8, 0.9, 1.2, 0.5, 1.3, 0.8, 1.2

(a) Plot the observations.

(b) Looking at the graph plotted in (a), guess an approximate value for the autocorrelation coefficient at lag 1.

(c) Plot x_t against x_{t+1}, and again try to guess the value of r_1.

(d) Calculate r_1.

2.3 If $x_t = a\cos t\omega$ where a is a constant and ω is a constant in $(0, \pi)$, show that $r_k \to \cos k\omega$ as $N \to \infty$.

(Hint: You will need to use the trigonometrical results listed in Section 7.2. Using Equation (7.2) it can be shown that $\bar{x} \to 0$ as $N \to \infty$, so that $r_k \to \sum \cos \omega t \cos \omega(t+k) / \sum \cos^2 \omega t$. Now use the result that $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ together with the result that $\sum \cos^2 \omega t = N/2$ for a suitably chosen N.)

2.4 A computer generates a series of 400 observations that are supposed to be random. The first 10 sample autocorrelation coefficients of the series are $r_1 = 0.02$, $r_2 = 0.05$, $r_3 = -0.09$, $r_4 = 0.08$, $r_5 = -0.02$, $r_6 = 0.00$, $r_7 = 0.12$, $r_8 = 0.06$, $r_9 = 0.02$, $r_{10} = -0.08$. Is there any evidence of non-randomness?

2.5 Suppose we have a seasonal series of monthly observations $\{X_t\}$, for which the seasonal factor at time t is denoted by $\{S_t\}$. Further suppose that the seasonal pattern is constant through time so that $S_t = S_{t-12}$ for all t. Denote a stationary series of random deviations by $\{\varepsilon_t\}$.

(a) Consider the model $X_t = a + bt + S_t + \varepsilon_t$ having a global linear trend and additive seasonality. Show that the seasonally differenced series $\nabla_{12} X_t := X_t - X_{t-12}$ is stationary.

(b) Consider the model $X_t = (a + bt)S_t + \varepsilon_t$ having a global linear trend and multiplicative seasonality. Does the operator ∇_{12} transform X_t to stationarity? If not, find a differencing operator that does.

(Note: As stationarity is not formally defined until Chapter 3, you should use heuristic arguments. A stationary process may involve a constant mean value (that could be zero) plus any linear combination of the stationary series $\{\varepsilon_t\}$, but should not include terms such as trend and seasonality.)

2.6 Consider the process $X_t = t^3 + (1 + t + t^2)S_t + \epsilon_t$, in which S_t is a seasonal component with period d (i.e., $S_{t-d} = S_t$) and $\{\epsilon_t\}$ is a stationary series of random devations.

(a) Compute the seasonally differenced series $\nabla_d X_t = X_t - X_{t-d}$.

(b) Define the following operator ∇_d^r for X_t ($d > 1$, $r \geq 2$, d and r are positive integers),

$$\nabla_d^r X_t = \nabla(\nabla_d^{r-1} X_t).$$

For example, when $d = 12$ and $r = 2$, we have

$$\begin{aligned} \nabla_{12}^2 X_t &= \nabla_{12}(\nabla_{12} X_t) = \nabla_{12}(X_t - X_{t-12}) \\ &= (X_t - X_{t-12}) - (X_{t-12} - X_{t-24}) \\ &= X_t - 2X_{t-12} + X_{t-24}. \end{aligned}$$

What is the minimum value of r so that $\nabla_d^r X_t$ does not contain trend or seasonal components?

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Chapter 3

Some Linear Time Series Models

This chapter introduces various probability models for time series. Some tools for describing the properties of such models are specified and the important notion of stationarity is formally defined.

3.1 Stochastic Processes and Their Properties

We concentrate on various types of time-series models that collectively come under the general title of 'stochastic processes'. Most physical processes in the real world involve a random element in their structure and a **stochastic process** can be described as 'a statistical phenomenon that evolves in time according to probabilistic laws'. Examples include the length of a queue, the number of accidents in a particular town in successive months and the air temperature at a particular site on successive days. The word 'stochastic', which is of Greek origin, is used to mean 'pertaining to chance', and many writers use 'random process' as a synonym for stochastic process.

Mathematically, a stochastic process may be defined as a collection of random variables that are ordered in time and defined at a set of time points, which may be continuous or discrete. We will denote the random variable at time t by $X(t)$ if time is continuous (usually $-\infty < t < \infty$), and by X_t if time is discrete (usually $t = 0, \pm 1, \pm 2, \ldots$). Some stochastic processes, such as the size of a bacterial colony, are studied using specialist tools, and we concentrate here on models suitable for time-series variables measured at equal intervals of time.

The theory of stochastic processes has been extensively developed and is discussed in many books including Cox and Miller (1968, especially Chapter 7), Grimmett and Stirzaker (2001), Papoulis (1984), written primarily for engineers, and Ross (1997). In this chapter we concentrate on those aspects particularly relevant to time-series analysis.

Most statistical problems are concerned with estimating the properties of a population from a sample. The properties of the latter are typically determined by the investigator, including the sample size and whether randomness is incorporated into the selection procedure. In time-series analysis, there is a rather different situation in that the order of observations is determined by time and it is usually impossible to make more than one observation at any given time. Thus, although it may be possible to increase the sample size by varying the *length* of the observed time series, there will only be a single

outcome of the process and a single observation on the random variable at time t. Nevertheless we may regard the observed time series as just one example of the infinite set of time series that might have been observed. This infinite set of time series is sometimes called the **ensemble**. Every member of the ensemble is a possible **realization** of the stochastic process. The observed time series can be thought of as one particular realization, and will be denoted by $x(t)$ for $(0 \leq t \leq T)$ if time is continuous, and by x_t for $t = 1, \ldots, N$ if time is discrete. Time-series analysis is essentially concerned with evaluating the properties of the underlying probability model from this observed time series, even though this single realization is the only one we will ever observe.

Many models for stochastic processes are expressed by means of an algebraic formula relating the random variable at time t to past values of the process, together with values of an unobservable 'error' process. From this model, it may be possible to specify the joint probability distribution of $X(t_1), \ldots, X(t_k)$ for any set of times $t_1, \ldots, t_k$ and any value of k. However, this is rather complicated and is not usually attempted in practice. A simpler, more useful way of describing a stochastic process is to give the *moments* of the process, particularly the first and second moments that are called the **mean** and **autocovariance function** (acv.f.), respectively. The **variance function** is a special case of the acv.f. These functions will now be defined for continuous time, with similar definitions applying in discrete time.

Mean. The mean function $\mu(t)$ is defined for all t by

$$\mu(t) = E[X(t)].$$

Variance. The variance function $\sigma^2(t)$ is defined for all t by

$$\sigma^2(t) = \mathrm{Var}[X(t)] = E[(X(t) - \mu(t))^2].$$

Autocovariance. The variance function alone is not enough to specify the second moments of a sequence of random variables. More generally, we define the acv.f. $\gamma(t_1, t_2)$ to be the covariance[1] of $X(t_1)$ with $X(t_2)$, namely

$$\gamma(t_1, t_2) = E\{[X(t_1) - \mu(t_1)][X(t_2) - \mu(t_2)]\}.$$

Clearly, the variance function is a special case of the acv.f. when $t_1 = t_2$.

Higher moments of a stochastic process may be defined in an obvious way, but are rarely used in practice.

3.2 Stationary Processes

An important class of stochastic processes are those that are stationary. A heuristic idea of stationarity was introduced in Section 2.2.

[1] Readers who are unfamiliar with the term 'covariance' should read Appendix C. When applied to a sequence of random variables, it is called an autocovariance.

A time series is said to be **strictly stationary** if the joint distribution of $X(t_1), \ldots, X(t_k)$ is the same as the joint distribution of $X(t_1+\tau), \ldots, X(t_k+\tau)$ for all $t_1, \ldots, t_k, \tau$. In other words, shifting the time origin by an amount τ has no effect on the joint distributions, which must therefore depend only on the intervals between $t_1, t_2, \ldots, t_k$. The above definition holds for any value of k. In particular, if $k = 1$, strict stationarity implies that the distribution of $X(t)$ is the same for all t, so that, provided the first two moments are finite, we have

$$\begin{aligned} \mu(t) &= \mu, \\ \sigma^2(t) &= \sigma^2 \end{aligned}$$

are both constants, which do not depend on the value of t.

Furthermore, if $k = 2$ the joint distribution of $X(t_1)$ and $X(t_2)$ depends only on the time difference $(t_2 - t_1) = \tau$, which is called the **lag**. Thus the acv.f. $\gamma(t_1, t_2)$ also depends only on $(t_2 - t_1)$ and may be written as $\gamma(\tau)$, where

$$\begin{aligned} \gamma(\tau) &= E\{[X(t) - \mu][X(t+\tau) - \mu]\} \\ &= \mathrm{Cov}[X(t), X(t+\tau)] \end{aligned}$$

is called the autocovariance coefficient at lag τ.

The size of an autocovariance coefficient depends on the units in which $X(t)$ is measured. Thus, for interpretative purposes, it is helpful to standardize the acv.f. to produce a function called the **autocorrelation function** (ac.f.), which is defined by

$$\rho(\tau) = \gamma(\tau)/\gamma(0).$$

This quantity measures the correlation between $X(t)$ and $X(t + \tau)$. Its empirical counterpart was introduced in Section 2.7. Note that the argument τ of $\gamma(\tau)$ and $\rho(\tau)$ is discrete if time is discrete, but continuous if time is continuous. We typically use $\gamma(k)$ and $\rho(k)$ to denote these functions in the discrete-time case.

At first sight it may seem surprising to suggest that there are processes for which the distribution of $X(t)$ should be the same for all t. However, readers with some knowledge of stochastic processes will know that there are many processes $\{X(t)\}$, which have what is called an **equilibrium** distribution as $t \to \infty$, whereby the probability distribution of $X(t)$ tends to a limit that does *not* depend on the initial conditions. Thus once such a process has been running for some time, the distribution of $X(t)$ will change very little. Indeed if the initial conditions are specified to be identical to the equilibrium distribution, the process is stationary in time and the equilibrium distribution is then the stationary distribution of the process. Of course the **conditional** distribution of $X(t_2)$ given that $X(t_1)$ has taken a particular value, say $x(t_1)$, may be quite different from the stationary distribution, but this is perfectly consistent with the process being stationary.

In practice it is often useful to define stationarity in a less restricted way than that described above. A process is called **second-order stationary** (or **weakly stationary**) if its mean is constant and its acv.f. depends only on the lag, so that

$$E[X(t)] = \mu$$

and

$$\mathrm{Cov}[X(t), X(t+\tau)] = \gamma(\tau).$$

No requirements are placed on moments higher than second order. By letting $\tau = 0$, we note that the form of a stationary acv.f. implies that the variance, as well as the mean, is constant. The definition also implies that both the variance and the mean must be finite.

This weaker definition of stationarity will generally be used from now on, as many of the properties of stationary processes depend only on the structure of the process as specified by its first and second moments. One important class of processes where this is particularly true is the class of **normal** processes where the joint distribution of $X(t_1), \ldots, X(t_k)$ is multivariate normal for all $t_1, \ldots, t_k$. The multivariate normal distribution is completely characterized by its first and second moments, and hence by $\mu(t)$ and $\gamma(t_1, t_2)$, and so it follows that second-order stationarity implies strict stationarity for normal processes. However, μ and $\gamma(\tau)$ may not adequately describe stationary processes, which are very 'non-normal'.

3.3 Properties of the Autocorrelation Function

We have already noted in Section 2.7 that the sample autocorrelation coefficients of an observed time series are an important set of statistics for describing the time series. Similarly the (theoretical) ac.f. of a stationary stochastic process is an important tool for assessing its properties. This section investigates the general properties of the ac.f.

Suppose a stationary stochastic process $X(t)$ has mean μ, variance σ^2, acv.f. $\gamma(\tau)$ and ac.f. $\rho(\tau)$. Then

$$\rho(\tau) = \gamma(\tau)/\gamma(0) = \gamma(\tau)/\sigma^2.$$

Note that $\rho(0) = 1$.

Property 1: The ac.f. is an even function of lag, so that $\rho(\tau) = \rho(-\tau)$.

This property simply says that the correlation between $X(t)$ and $X(t+\tau)$ is the same as that between $X(t)$ and $X(t-\tau)$. The result is easily proved using $\gamma(\tau) = \rho(\tau)\sigma^2$ by

$$\begin{aligned} \gamma(\tau) &= \mathrm{Cov}[X(t), X(t+\tau)] \\ &= \mathrm{Cov}[X(t-\tau), X(t)] \qquad \text{since } X(t) \text{ is stationary.} \\ &= \gamma(-\tau) \end{aligned}$$

Property 2: $|\rho(\tau)| \leq 1$.

This is the 'usual' property of correlation, namely, that it lies between ± 1. It is proved by noting that

$$\mathrm{Var}[\lambda_1 X(t) + \lambda_2 X(t+\tau)] \geq 0$$

for any constants λ_1, λ_2, since a variance is always non-negative. The variance is equal to $\lambda_1^2 \mathrm{Var}[X(t)] + \lambda_2^2 \mathrm{Var}[X(t+\tau)] + 2\lambda_1\lambda_2 \mathrm{Cov}[X(t), X(t+\tau)]$

$$= (\lambda_1^2 + \lambda_2^2)\sigma^2 + 2\lambda_1\lambda_2\gamma(\tau)$$

When $\lambda_1 = \lambda_2 = 1$, we find $\gamma(\tau) \geq -\sigma^2$, so that $\rho(\tau) \geq -1$.
When $\lambda_1 = 1, \lambda_2 = -1$, we find $\sigma^2 \geq \gamma(\tau)$, so that $\rho(\tau) \leq +1$.
Thus $|\rho(\tau)| \leq 1$ as required.

The standardized nature of a correlation coefficient means that the value of $\rho(\tau)$ does not depend on the units in which the time series is measured, as can readily be demonstrated by multiplying all values in a series by the same constant and showing that the resulting autocorrelations are unchanged.

Property 3: The ac.f. does not uniquely identify the underlying model.

Although a given stochastic process has a unique covariance structure, the converse is not in general true. It is usually possible to find many normal and non-normal processes with the same ac.f. and this creates further difficulty in interpreting sample ac.f.s. Jenkins and Watts (1968, p. 170) give an example of two different stochastic processes, which have exactly the same ac.f. Even for stationary normal processes, which are completely determined by the mean, variance and ac.f., we will see in Section 3.6 that a requirement, called the invertibility condition, is needed to ensure uniqueness.

3.4 Purely Random Processes

A discrete-time process is called a purely random process if it consists of a sequence of random variables, $\{Z_t\}$, which are mutually *independent and identically distributed* (i.i.d.). We normally further assume that the random variables are normally distributed with mean zero and variance σ_Z^2. From the definition it follows that the process has constant mean and variance. Moreover, the independence assumption means that

$$\gamma(k) = \mathrm{Cov}(Z_t, Z_{t+k}) = \begin{cases} \sigma_Z^2 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots . \end{cases} \tag{3.1}$$

This means that different values are uncorrelated so that the ac.f. is given by

$$\rho(k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots . \end{cases} \tag{3.2}$$

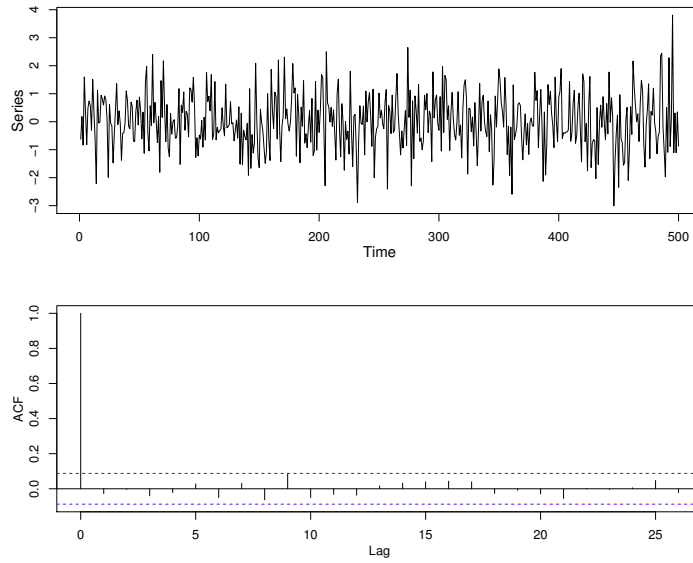

Figure 3.1 *A purely random process with* $\sigma_Z^2 = 1$ *(top) and its correlogram (bottom).*

As the mean and acv.f. do not depend on time, the process is second-order stationary. In fact the independence assumption implies that the process is also strictly stationary.

Purely random processes are useful in many situations, particularly as building blocks for more complicated processes such as moving average processes – see Section 3.6. In practice, if all sample ac.f.'s of a series are close to zero, then the series is considered as a realization of a purely random process. Figure 3.1 shows an example of a purely random process, $Z_t \sim N(0,1)$, $1 \leq t \leq 500$, and its correlogram, which can be reproduced by the following command in R:

```
> z<-rnorm(500, 0, 1)
> par(mfrow=c(2,1), mar=c(3,4,3,4))
> plot(z, type="l", xlab="Time", ylab="Series")
> acf(z, xlab="Lag",ylab="ACF", main="")
```

Some authors prefer to make the weaker assumption that the Z_t's are mutually uncorrelated, rather than independent. This is adequate for linear, normal processes, but the stronger independence assumption is needed when considering non-linear models (see Chapter 11). Note that a purely random process is sometimes called **white noise**, particularly by engineers.

The possibility of defining a continuous-time purely random process is discussed in Section 3.12.

3.5 Random Walks

Suppose that $\{Z_t\}$ is a discrete-time, purely random process with mean μ and variance σ_Z^2. A process $\{X_t\}$ is said to be a random walk if

$$X_t = X_{t-1} + Z_t. \tag{3.3}$$

The process is customarily started at zero when $t = 0$, so that

$$X_1 = Z_1$$

and

$$X_t = \sum_{i=1}^{t} Z_i.$$

Then we find that $E(X_t) = t\mu$ and that $\mathrm{Var}(X_t) = t\sigma_Z^2$ since the Z_i's are independent. As the mean and variance change with t, the process is non-stationary.

However, it is interesting to note that the first differences of a random walk, given by

$$\nabla X_t = X_t - X_{t-1} = Z_t$$

form a purely random process, which is therefore stationary. This feature can be used to construct a random walk. For instance, Figure 3.2 shows a random walk series and its correlogram. The random walk series is generated by cumulative sums of a white noise series via the following command in R.

```
> n<-500
> z<-rnorm(n, 0, 1)
> x.rw<-cumsum(z)
> par(mfrow=c(2,1), mar=c(4,4,4,4))
> plot(x.rw, type="l", xlab="Time", ylab="Series")
> acf(x.rw, xlab="Lag", ylab="ACF", main="")
```

The best-known examples of time series, which behave like random walks, are share prices on successive days. A model, which often gives a good approximation to such data, is

share price on day t = share price on day $(t-1)$ + random error.

3.6 Moving Average Processes

Suppose that $\{Z_t\}$ is a purely random process with mean zero and variance σ_Z^2. Then a process $\{X_t\}$ is said to be a moving average process of order q (abbreviated to a MA(q) process) if

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}, \tag{3.4}$$

where $\{\beta_i\}$ are constants. The Zs are usually scaled so that $\beta_0 = 1$.

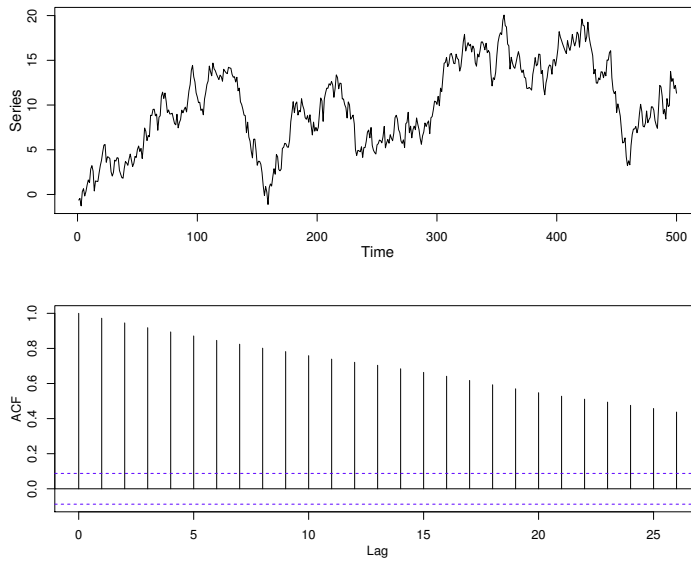

Figure 3.2 *Simulated random walk (top) and its correlogram (bottom). The random walk series is generated from the white noise series in Figure 3.1.*

3.6.1 Stationarity and autocorrelation function of an MA process

We find immediately that

$$\begin{aligned} E(X_t) &= 0, \\ \mathrm{Var}(X_t) &= \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2, \end{aligned}$$

since the Zs are independent. We also have

$$\begin{aligned} \gamma(k) &= \mathrm{Cov}(X_t, X_{t+k}) \\ &= \mathrm{Cov}(\beta_0 Z_t + \cdots + \beta_q Z_{t-q}, \beta_0 Z_{t+k} + \cdots + \beta_q Z_{t+k-q}) \\ &= \begin{cases} 0 & k > q \\ \sigma_Z^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k} & k = 0, 1, \ldots, q \\ \gamma(-k) & k < 0 \end{cases} \end{aligned} \tag{3.5}$$

since

$$\mathrm{Cov}(Z_s, Z_t) = \begin{cases} \sigma_Z^2 & s = t \\ 0 & s \neq t. \end{cases}$$

As $\gamma(k)$ does not depend on t, and the mean is constant, the process is second-order stationary for all values of the $\{\beta_i\}$. Furthermore, if the Zs are normally distributed, then so are the Xs, and we have a strictly stationary normal process.

The ac.f. of the above MA(q) process is given by

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \sum_{i=0}^{q-k} \beta_i \beta_{i+k} \Big/ \sum_{i=0}^{q} \beta_i^2 & k = 1, \ldots, q \\ 0 & k > q \\ \rho(-k) & k < 0. \end{cases} \tag{3.6}$$

Note that the ac.f. 'cuts off' at lag q, which is a special feature of MA processes. In particular, the MA(1) process with $\beta_0 = 1$ has an ac.f. given by

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \beta_1/(1+\beta_1^2) & k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of the MA(q) process, one can construct a MA series. For example, we can simulate a white noise series, $Z_t \sim N(0,1)$, and then construct a MA(1) process, $X_t = Z_t - 0.8Z_{t-1}$, and a MA(2) process, $X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$ using Z_t. The resulting series and their correlograms are shown in Figure 3.3, which can be reproduced by the following command in R. Note that the r_k's are significant only for $k = 0$ and 1, and $k = 0, 1, 2$, respectively.

```
> n<-500
> z<-rnorm(n)
> x.ma1<-z[2:n]-0.8*z[1:(n-1)]
> x.ma2<-z[3:n]+0.7*z[2:(n-1)]-0.2*z[1:(n-2)]

> par(mfrow=c(2,2), mar=c(4,4,4,4))
> plot(x.ma1, type="l", xlab="Time", ylab="Series")
> acf(x.ma1, xlab="Lag", ylab="ACF", main="")
> plot(x.ma2, type="l", xlab="Time", ylab="Series")
> acf(x.ma2, xlab="Lag", ylab="ACF", main="")
```

3.6.2 Invertibility of an MA process

No restrictions on the $\{\beta_i\}$ are required for a (finite-order) MA process to be stationary, but it is generally desirable to impose restrictions on the $\{\beta_i\}$ to ensure that the process satisfies a condition called **invertibility**. This

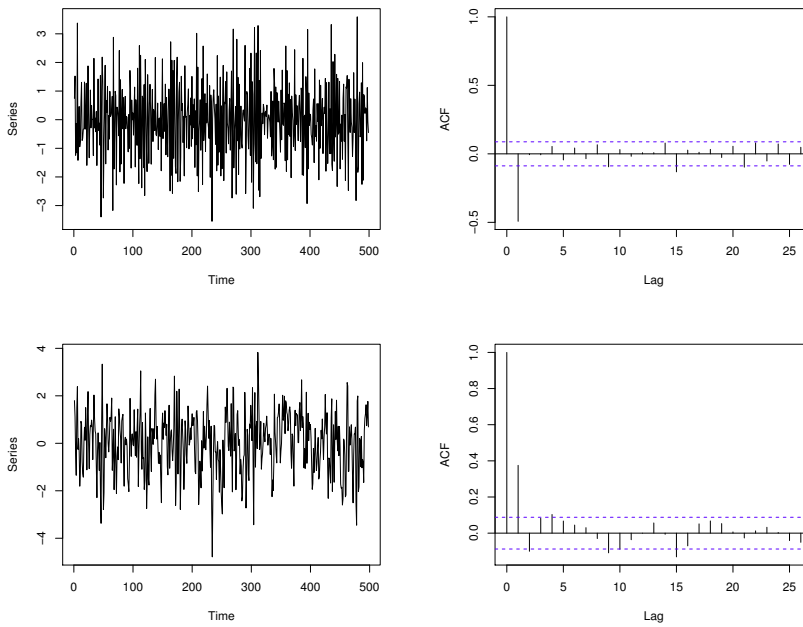

Figure 3.3 *Simulated MA(1) (top left) and MA(2) (bottom left) processes and their corresponding correlograms (top right and bottom right).*

condition may be explained in the following way. Consider the following first-order MA processes:

$$\text{A:} \qquad X_t = Z_t + \theta Z_{t-1}.$$

$$\text{B:} \qquad X_t = Z_t + \frac{1}{\theta} Z_{t-1}.$$

It can easily be shown that these two different processes have exactly the same ac.f. (Are you surprised? Then check $\rho(k)$ for models A and B.) Thus we cannot identify a MA process uniquely from a given ac.f., as we noted in Property 3 in Section 3.3. Now, if we 'invert' models A and B by expressing Z_t in terms of $X_t, X_{t-1}, \ldots$, we find by successive substitution that

$$\text{A:} \qquad Z_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \cdots.$$

$$\text{B:} \qquad Z_t = X_t - \frac{1}{\theta} X_{t-1} + \frac{1}{\theta^2} X_{t-2} - \cdots.$$

If $|\theta| < 1$, the series of coefficients of X_{t-j} for model A converges whereas that of B does not. Thus model B cannot be 'inverted' in this way. More generally,

a process, $\{X_t\}$, is said to be invertible if the random disturbance at time t, sometimes called the *innovation*, can be expressed, as a convergent sum of present and past values of X_t in the form

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \tag{3.7}$$

where $\sum |\pi_j| < \infty$. This effectively means that the first-order MA process can be rewritten in the form of an autoregressive process, possibly of infinite order, whose coefficients form a convergent sum. Autoregressive processes are introduced below in Section 3.7. It turns out that an estimation procedure for a MA process, which involves estimating the residuals (see Section 4.3.1), will lead naturally to model A. Thus, if $|\theta| < 1$, model A is said to be invertible whereas model B is not. The imposition of the invertibility condition ensures that there is a unique invertible first-order MA process for a given ac.f.

The invertibility condition for a MA process of any order is best expressed by using the backward shift operator, denoted by B, which is defined by

$$B^j X_t = X_{t-j} \qquad \text{for all } j$$

Then Equation (3.4) may be written as

$$\begin{aligned} X_t &= (\beta_0 + \beta_1 B + \cdots + \beta_q B^q) Z_t \\ &= \theta(B) Z_t \end{aligned} \tag{3.8}$$

where $\theta(B)$ is a polynomial of order q in B. It can be shown that a MA(q) process is invertible if the roots of the equation

$$\theta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q = 0 \tag{3.9}$$

all lie outside the unit circle, where we regard B as a complex variable and not as an operator. This means that the roots, which may be complex, have modulus greater than unity.

How the invertibility condition ensures X_t to be invertible? This can be explained by utilizing the backward shift operator. In particular, Equation (3.8) can be written equivalently as

$$Z_t = \frac{1}{\theta(B)} X_t. \tag{3.10}$$

In the first-order case for model A, we have $\theta(B) = 1 + \theta B$, which has root $B = -1/\theta$. Provided that $|\theta| < 1$, the root $B = -1/\theta$ is real and lies "outside the unit circle". So again we see that model A is invertible if $|\theta| < 1$. Furthermore, if we regard B as a complex variable, the operator $1/\theta(B)$ can be expanded as an infinite series, i.e.,

$$\frac{1}{1+\theta B} = 1 + \sum_{i=1}^{\infty} (-\theta)^i B^i. \tag{3.11}$$

When $|\theta| < 1$, this infinite series is convergent since

$$|1 + \sum_{i=1}^{\infty}(-\theta)^i| \leq 1 + \sum_{i=1}^{\infty}|\theta|^i = \frac{1}{1-|\theta|}.$$

Hence Equation (3.10) becomes

$$Z_t = \Big(1 + \sum_{i=1}^{\infty}(-\theta)^i B^i\Big)X_t = X_t + \sum_{i=1}^{\infty}(-\theta)^i X_{t-i},$$

and thus $\{X_t\}$ is invertible.

The argument above can be extended to MA(q) processes. Suppose that $\theta(B)$ can be decomposed as the following form

$$\theta(B) = (1 + \theta_1 B)\cdots(1 + \theta_q B),$$

where $\theta_1, \ldots, \theta_q$ could possibly take complex values. Then the operator $1/\theta(B)$ can be written as

$$\frac{1}{\theta(B)} = \prod_{j=1}^{q}\frac{1}{1+\theta_j B} = \prod_{j=1}^{q}\Big(1 + \sum_{i=1}^{\infty}(-\theta_j)^i B^i\Big). \tag{3.12}$$

When all the roots, $-1/\theta_1, \ldots, -1/\theta_q$, are outside the unit circle, the product of infinite series in (3.12) is convergent, and hence $Z_t = 1/\theta(B)X_t$ can be written in the form of (3.7), and therefore $\{X_t\}$ is invertible.

MA processes have been used in many areas, particularly econometrics. For example, economic indicators are affected by a variety of 'random' events such as strikes, government decisions, shortages of key materials and so on. Such events will not only have an immediate effect but may also affect economic indicators to a lesser extent in several subsequent periods, and so it is at least plausible that an MA process may be appropriate.

Note that an arbitrary constant, μ say, may be added to the right-hand side of Equation (3.4) to give a process with mean μ. This does not affect the ac.f. (see Exercise 3.5) and has been omitted for simplicity.

3.7 Autoregressive Processes

Suppose that $\{Z_t\}$ is a purely random process with mean zero and variance σ_Z^2. Then a process $\{X_t\}$ is said to be an autoregressive process of order p (abbreviated to an AR(p) process) if

$$X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t. \tag{3.13}$$

This is rather like a multiple regression model, but X_t is regressed on past values of X_t rather than on separate predictor variables. This explains the prefix 'auto'.

3.7.1 First-order process

For simplicity, we begin by examining the first-order case, where $p = 1$. Then

$$X_t = \alpha X_{t-1} + Z_t. \tag{3.14}$$

The AR(1) process is sometimes called the Markov process, after the Russian A.A. Markov. By successive substitution into Equation (3.14) we may write

$$\begin{aligned} X_t &= \alpha(\alpha X_{t-2} + Z_{t-1}) + Z_t \\ &= \alpha^2(\alpha X_{t-3} + Z_{t-2}) + \alpha Z_{t-1} + Z_t \end{aligned}$$

and eventually we find that X_t may be expressed as an infinite-order MA process in the form

$$X_t = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots$$

provided $-1 < \alpha < +1$, so that the sum converges.

The possibility that AR processes may be written in MA form, and vice versa — see also Section 3.6 — means that there is a duality between AR and MA processes, which is useful for a variety of purposes. Rather than use successive substitution to explore this duality, it is simpler to use the backward shift operator B. Thus Equation (3.14) may be written

$$(1 - \alpha B)X_t = Z_t,$$

so that

$$\begin{aligned} X_t &= Z_t/(1 - \alpha B) \\ &= (1 + \alpha B + \alpha^2 B^2 + \cdots)Z_t \\ &= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots. \end{aligned}$$

When expressed in this form it is clear that

$$\begin{aligned} E(X_t) &= 0 \\ \mathrm{Var}(X_t) &= \sigma_Z^2(1 + \alpha^2 + \alpha^4 + \cdots) \end{aligned}$$

by independence of the Z's. Thus the variance is finite provided that $|\alpha|^2 < 1$, so if $|\alpha| < 1$, in which case

$$\mathrm{Var}(X_t) = \sigma_X^2 = \sigma_Z^2/(1 - \alpha^2).$$

The acv.f. is given by

$$\begin{aligned} \gamma(k) &= E[X_t X_{t+k}] \\ &= E[(\Sigma\alpha^i Z_{t-i})(\Sigma\alpha^j Z_{t+k-j})] \\ &= \sigma_Z^2 \sum_{i=0}^{\infty} \alpha^i \alpha^{k+i} \qquad \text{for } k \geq 0, \\ &= \alpha^k \sigma_Z^2/(1 - \alpha^2) \qquad \text{provided } |\alpha| < 1, \\ &= \alpha^k \sigma_X^2. \end{aligned}$$

For $k < 0$, we find $\gamma(k) = \gamma(-k)$. Since $\gamma(k)$ does not depend on t, an AR process of order 1 is second-order stationary provided that $|\alpha| < 1$, and the ac.f. is then given by

$$\rho(k) = \alpha^k \qquad k = 0, 1, 2, \ldots.$$

To get an even function defined for all integer k we can use the modulus operator to write

$$\rho(k) = \alpha^{|k|} \qquad k = 0, \pm 1, \pm 2, \ldots.$$

The ac.f. may also be obtained more easily by assuming *a priori* that the process is stationary. Then $E(X_t)$ must be a constant, μ say, that turns out to be zero using Equation (3.14). If we then multiply through Equation (3.14) by X_{t-k} (not X_{t+k}!) and take expectations, we find, for $k > 0$, that

$$\gamma(-k) = \alpha\gamma(-k + 1),$$

since $E(Z_t X_{t-k}) = 0$ for $k > 0$, by independence of the Z's. Since $\gamma(k)$ is an even function of k, we must also have

$$\gamma(k) = \alpha\gamma(k - 1) \qquad \text{for } k > 0.$$

Now $\gamma(0) = \sigma_X^2$, and so $\gamma(k) = \alpha^k \sigma_X^2$ for $k \geq 0$. Thus $\rho(k) = \alpha^k$ for $k \geq 0$. Now since $|\rho(k)| \leq 1$, we must have $|\alpha| \leq 1$. However, if $|\alpha| = 1$, then $|\rho(k)| = 1$ for all k, which is a degenerate case. Thus we must have $|\alpha|$ strictly less than 1 to have a proper stationary process.

The above method of obtaining the ac.f. is often used, even though it involves 'cheating' a little by making an initial assumption of stationarity.

Three examples of the ac.f. of a first-order AR process are shown in Figure 3.4 for $\alpha = 0.8$, -0.8, and 0.3. All three series are constructed from the same noise series by the `R` script below. Note how quickly the ac.f. decays when $\alpha = 0.3$, and note how the ac.f. alternates when α is negative.

```
> n<-500
> z<-rnorm(n, 0, 1)
> x.ar1.1<-x.ar1.2<-x.ar1.3<-rep(0,n)
> x.ar1.1[1]<-x.ar1.2[1]<-x.ar1.3[1]<-z[1]
> for (i in 2:n){
    x.ar1.1[i]<- 0.8*x.ar1.1[i-1]+z[i]
    x.ar1.2[i]<- -0.8*x.ar1.2[i-1]+z[i]
    x.ar1.3[i]<- 0.3*x.ar1.3[i-1]+z[i]
   }
```

3.7.2 General-order process

As in the first-order case, we can express an AR process of finite order as an MA process of infinite order. This may be done by successive substitution, or,

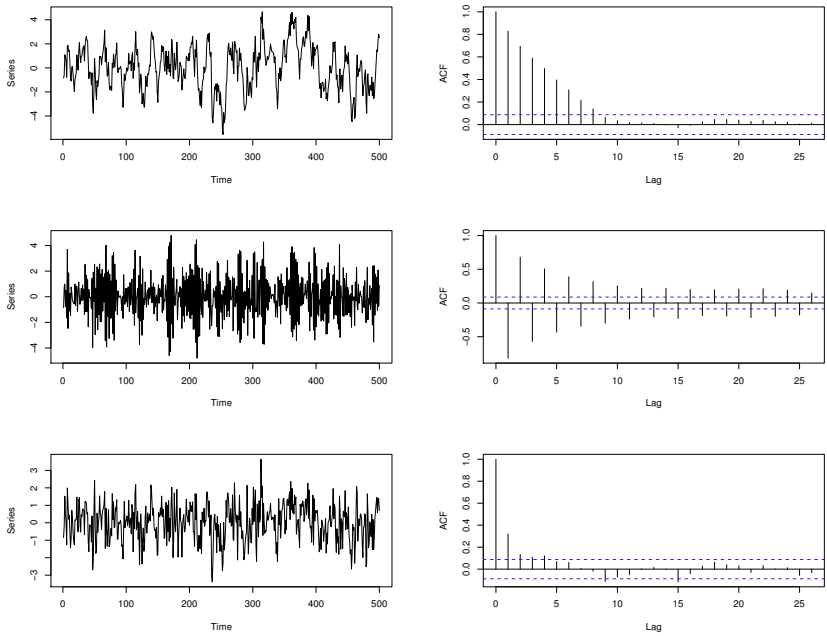

Figure 3.4 *Three simulated AR(1) processes and their correlograms. Top:* $X_t = 0.8X_{t-1} + Z_t$, $Z_t \sim N(0,1)$*; Middle:* $X_t = -0.8X_{t-1} + Z_t$, $Z_t \sim N(0,1)$*; Bottom:* $X_t = 0.3X_{t-1} + Z_t$, $Z_t \sim N(0,1)$.

more easily, by utilizing the backward shift operator. Thus Equation (3.13) may be written as

$$(1 - \alpha_1 B - \cdots - \alpha_p B^p) X_t = Z_t \tag{3.15}$$

or equivalently as

$$\begin{aligned} X_t &= Z_t/(1 - \alpha_1 B - \cdots - \alpha_p B^p) \\ &= f(B) Z_t, \end{aligned}$$

where

$$\begin{aligned} f(B) &= (1 - \alpha_1 B - \cdots - \alpha_p B^p)^{-1} \\ &= (1 + \beta_1 B + \beta_2 B^2 + \cdots). \end{aligned}$$

The relationship between the αs and the βs may then be found. Having expressed X_t as an MA process, it follows that $E(X_t) = 0$. The variance is finite provided that $\Sigma \beta_i^2$ converges, and this is a necessary condition for stationarity. The acv.f. is given by

$$\gamma(k) = \sigma_Z^2 \sum_{i=0}^{\infty} \beta_i \beta_{i+k} \qquad \text{where } \beta_0 = 1.$$

A sufficient condition for this to converge, and hence for stationarity, is that $\Sigma|\beta_i|$ converges.

Yule-Walker equations

We can in principle find the ac.f. of the general-order AR process using the above procedure, but the $\{\beta_i\}$ may be algebraically hard to find. The alternative simpler way is to *assume* the process is stationary, multiply through Equation (3.13) by X_{t-k}, take expectations and divide by σ_X^2, assuming that the variance of X_t is finite. Then, using the fact that $\rho(k) = \rho(-k)$ for all k, we find

$$\rho(k) = \alpha_1\rho(k-1) + \cdots + \alpha_p\rho(k-p) \qquad \text{for all } k > 0. \tag{3.16}$$

This set of equations is called the Yule–Walker equations after G.U. Yule and Sir Gilbert Walker. It is a set of difference equations and has the general solution

$$\rho(k) = A_1\pi_1^{|k|} + \cdots + A_p\pi_p^{|k|},$$

where $\{\pi_i\}$ are the roots of the so-called auxiliary equation

$$y^p - \alpha_1 y^{p-1} - \cdots - \alpha_p = 0.$$

The constants $\{A_i\}$ are chosen to satisfy the initial conditions depending on $\rho(0) = 1$, which means that $\Sigma A_i = 1$. The first $(p-1)$ Yule–Walker equations provide $(p-1)$ further restrictions on the $\{A_i\}$ using $\rho(0) = 1$ and $\rho(k) = \rho(-k)$.

Stationarity conditions

From the general form of $\rho(k)$, it is clear that $\rho(k)$ tends to zero as k increases provided that $|\pi_i| < 1$ for all i, and this is a necessary and sufficient condition for the AR(p) process to be stationary.

It can be shown that an equivalent way of expressing the stationarity condition is to say that the roots of the equation

$$\phi(B) = 1 - \alpha_1 B - \cdots - \alpha_p B^p = 0 \tag{3.17}$$

must lie outside the unit circle (where we again regard B as a complex variable, rather than as an operator, so that the roots, which may be complex, are greater than one in modulus).

Of particular interest is the AR(2) process, when π_1, π_2 are the roots of the quadratic equation

$$y^2 - \alpha_1 y - \alpha_2 = 0.$$

Here $|\pi_i| < 1$ if

$$\left| \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \right| < 1$$

from which it can be shown (Exercise 3.6) that the stationarity region is the triangular region satisfying

$$\begin{aligned} \alpha_1 + \alpha_2 &< 1 \\ \alpha_1 - \alpha_2 &> -1 \\ \alpha_2 &> -1. \end{aligned}$$

The roots are real if $\alpha_1^2 + 4\alpha_2 > 0$, in which case the ac.f. decreases exponentially with k, but the roots are complex if $\alpha_1^2 + 4\alpha_2 < 0$, in which case the ac.f. turns out to be a damped sinusoidal wave. (See Example 3.1 at the end of this section.)

When the roots are real, we have $\rho(k) = A_1\pi_1^{|k|} + A_2\pi_2^{|k|}$ where the constants A_1, A_2 are also real and may be found as follows. Since $\rho(0) = 1$, we have

$$A_1 + A_2 = 1$$

while the first Yule–Walker equation gives

$$\begin{aligned} \rho(1) &= \alpha_1\rho(0) + \alpha_2\rho(-1) \\ &= \alpha_1 + \alpha_2\rho(1). \end{aligned}$$

This equation may be solved to give $\rho(1) = \alpha_1/(1-\alpha_2)$, which in turn must equal

$$A_1\pi_1 + A_2\pi_2 = A_1\pi_1 + (1 - A_1)\pi_2.$$

Hence we find

$$\begin{aligned} A_1 &= [\alpha_1/(1-\alpha_2) - \pi_2]/(\pi_1 - \pi_2) \\ A_2 &= 1 - A_1 \end{aligned}$$

and this enables us to write down the general form of the ac.f. of an AR(2) process with real roots. As an example, consider an AR(2) process, $X_t = 1/3X_{t-1} + 2/9X_{t-2} + Z_t$. The top panels of Figure 3.5 show a realization of this process and its correlogram. Note that the roots of $1 - 1/3B - 2/9B^2 = 0$ are -3 and $3/2$; both have absolute values larger than 1. This indicates X_t is stationary and its ac.f. can be calculated with the method above (see Exercise 3.6). For complex roots, the general solution can also be found and an example is given below.

Example 3.1 Consider the AR(2) process given by

$$X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t.$$

Is this process stationary? If so, what is its ac.f.?

In order to answer the first question we find the roots of Equation (3.17), which, in this case, is

$$\phi(B) = 1 - B + \frac{1}{2}B^2 = 0.$$

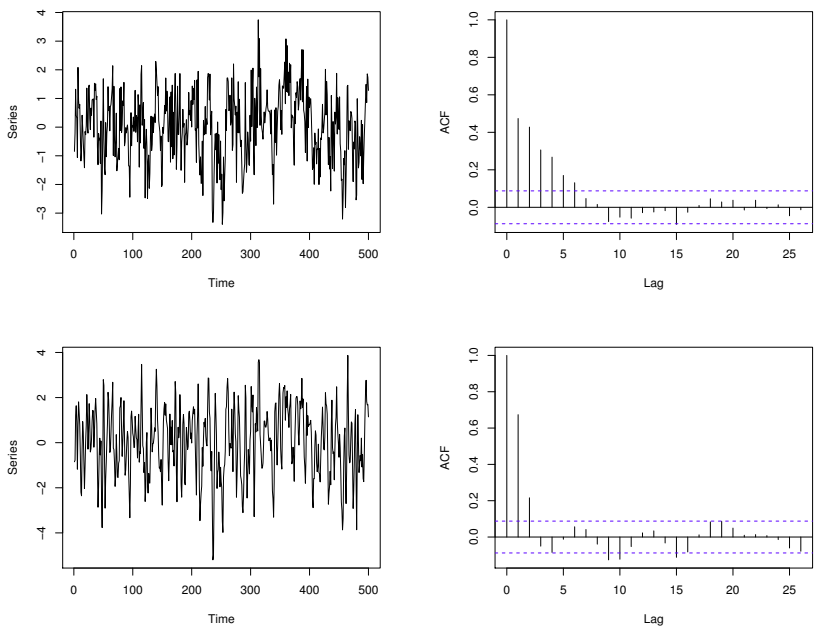

Figure 3.5 *Two simulated AR(2) processes and their correlograms. Top:* $X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + Z_t$, $Z_t \sim N(0,1)$*; Bottom:* $X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t$, $Z_t \sim N(0,1)$

The roots of this equation (regarding B as a variable) are complex, namely, $1 \pm \mathrm{i}$. As the modulus of both roots exceeds one, the roots are both outside the unit circle and so the process is stationary.

In order to find the ac.f. of the process, we use the first Yule–Walker equation to give

$$\begin{aligned} \rho(1) &= \rho(0) - \frac{1}{2}\rho(-1) \\ &= 1 - \frac{1}{2}\rho(1) \end{aligned}$$

giving $\rho(1) = 2/3$.

For $k \geq 2$, the Yule–Walker equations are

$$\rho(k) = \rho(k-1) - \frac{1}{2}\rho(k-2).$$

We could use these equations to find $\rho(2)$, then $\rho(3)$ and so on by successive substitution, but it is easier to find the general solution by solving the set of Yule–Walker equations as a set of difference equations. The general form of

the above Yule–Walker equation has the auxiliary equation

$$y^2 - y + \frac{1}{2} = 0$$

with roots $y = (1 \pm \mathrm{i})/2$. They may be rewritten as $[\cos(\pi/4) \pm \mathrm{i}\ \sin(\pi/4)]/\sqrt{2}$ or as $\mathrm{e}^{\pm \mathrm{i}\pi/4}/\sqrt{2}$. Since $\alpha_1^2 + 4\alpha_2 = (1 - 2)$ is less than zero, and the roots are complex, the ac.f. is a damped sinusoidal wave. Using $\rho(0) = 1$ and $\rho(1) = 2/3$, some messy trigonometry and algebra involving complex numbers gives

$$\rho(k) = \left(\frac{1}{\sqrt{2}}\right)^k \left(\cos\frac{\pi k}{4} + \frac{1}{3}\sin\frac{\pi k}{4}\right)$$

for $k = 0, 1, 2, \ldots$. Note the values of the ac.f. are all real, even though the roots of the auxiliary equation are complex. The bottom two panels of Figure 3.5 show a realization of X_t and its correlogram. □

AR processes have been applied to many situations in which it is reasonable to assume that the present value of a time series depends linearly on the immediate past values together with a random error. For simplicity we have only considered processes with mean zero, but non-zero means may be dealt with by rewriting Equation (3.13) in the form

$$X_t - \mu = \alpha_1(X_{t-1} - \mu) + \cdots + \alpha_p(X_{t-p} - \mu) + Z_t.$$

This does not affect the ac.f. (see Exercise 3.4).

3.8 Mixed ARMA Models

A useful class of models for time series is formed by combining MA and AR processes. A mixed autoregressive/moving-average process containing p AR terms and q MA terms is said to be an ARMA process of order (p, q). It is given by

$$X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}, \tag{3.18}$$

where $\{Z_t\}$ is a purely random process with mean zero and variance σ_Z^2. Using the backward shift operator B, Equation (3.18) may be written in the form

$$\phi(B)X_t = \theta(B)Z_t, \tag{3.19}$$

where $\phi(B), \theta(B)$ are polynomials of order p, q, respectively, such that

$$\phi(B) = 1 - \alpha_1 B - \cdots - \alpha_p B^p$$

and

$$\theta(B) = 1 + \beta_1 B + \cdots + \beta_q B^q.$$

3.8.1 Stationarity and invertibility conditions

The conditions on the model parameters to make the process stationary and invertible are the same as for a pure AR or pure MA process, namely, that the values of $\{\alpha_i\}$, which make the process stationary, are such that the roots of

$$\phi(B) = 0$$

lie outside the unit circle, while the values of $\{\beta_i\}$, which make the process invertible, are such that the roots of

$$\theta(B) = 0$$

lie outside the unit circle. It is straightforward in principle, though algebraically rather tedious, to calculate the ac.f. of an ARMA process, but this will not be discussed here. (See Exercise 3.11, and Box et al., 1994, Section 3.4.)

The importance of ARMA processes lies in the fact that a stationary time series may often be adequately modelled by an ARMA model involving fewer parameters than a pure MA or AR process by itself. This is an early example of what is often called the **Principle of Parsimony**. This says that we want to find a model with as few parameters as possible, but which gives an adequate representation of the data at hand.

3.8.2 Yule-Walker equations and autocorrelations

The ac.f. of the general ARMA process can be found using similar procedures as for AR processes. First, multiply through Equation (3.18) by X_{t-k} and take expectations. Note that, for $k \geq q+1$, $Z_t, \ldots, Z_{t-q}$ are independent of X_{t-k}. Hence the expected values of $Z_t X_{t-k}, \ldots, Z_{t-q} X_{t-k}$ are all zero. If $k \geq p$, we can further divide both sides by $\gamma(0)$. Then we find the following Yule-Walker equations for general ARMA(p, q) processes

$$\rho(k) = \alpha_1 \rho(k-1) + \cdots + \alpha_p \rho(k-p), \qquad k \geq \max(p, q+1). \tag{3.20}$$

Note that (3.20) has the same form as that of an AR process except that their initial conditions are different. The initial conditions of the Yule-Walker equations are usually computed seperately.

The following example shows how the ac.f. of an ARMA(1,1) process is derived.

Example 3.2

Consider the ARMA(1,1) process

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}, \tag{3.21}$$

where $|\alpha| < 1$ and $|\beta| < 1$. To derive the ac.f. of the process, the Yule-Walker equations are

$$\rho(k) = \alpha \rho(k-1), \qquad k \geq 2.$$

To obtain the initial conditions, we note that $\gamma(1)$ can be computed as follows.

$$\begin{aligned}\gamma(1) &= \text{Cov}(X_t, X_{t-1}) = \text{Cov}(\alpha X_{t-1} + Z_t + \beta Z_{t-1}, X_{t-1}) \\ &= \alpha\gamma(0) + \beta\text{Cov}(Z_{t-1}, X_{t-1}) \\ &= \alpha\gamma(0) + \beta\text{Cov}(Z_{t-1}, \alpha X_{t-2} + Z_{t-1} + \beta Z_{t-2}) \\ &= \alpha\gamma(0) + \beta\sigma_Z^2. \end{aligned} \tag{3.22}$$

The variance of X_t, or $\gamma(0)$, can be calculated as follows

$$\begin{aligned}\gamma(0) &= \text{Var}(\alpha X_{t-1} + Z_t + \beta Z_{t-1}) \\ &= \text{Cov}(\alpha X_{t-1} + Z_t + \beta Z_{t-1}, \alpha X_{t-1} + Z_t + \beta Z_{t-1}) \\ &= \alpha^2\gamma(0) + (1+\beta^2)\sigma_Z^2 + 2\alpha\beta\text{Cov}(X_{t-1}, Z_{t-1}) \\ &= \alpha^2\gamma(0) + (1 + 2\alpha\beta + \beta^2)\sigma_Z^2. \end{aligned}$$

Solving for $\gamma(0)$ yields

$$\gamma(0) = \frac{1 + 2\alpha\beta + \beta^2}{1 - \alpha^2}\sigma_Z^2. \tag{3.23}$$

Plug $\gamma(0)$ into Equation (3.22) to obtain an explicit expression for $\gamma(1)$. Then

$$\gamma(1) = \frac{(1 + \alpha\beta)(\alpha + \beta)}{1 - \alpha^2}\sigma_Z^2.$$

Hence

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(1 + \alpha\beta)(\alpha + \beta)}{1 + 2\alpha\beta + \beta^2}.$$

Using the Yule-Walker equations, we then have

$$\rho(k) = \frac{(1 + \alpha\beta)(\alpha + \beta)}{1 + 2\alpha\beta + \beta^2}\alpha^{k-1}, \qquad k \geq 1. \tag{3.24}$$

□

In principle, the ac.f. for a general ARMA process can be computed, similarly as in Example 3.2, but are usually more complicated than for AR processes. Figure 3.6 shows two simulated ARMA processes (ARMA(1,1) and ARMA(2,2)) and their correlograms using the following `R` script.

```
> x1<-arima.sim(n=500, list(ar=0.7, ma=-0.4))
> x2<-arima.sim(n=500, list(ar=c(0.9,-0.5),
     ma=c(-0.2,0.25)), sd=sqrt(0.5))
```

Note that, though the correlogram of the ARMA(2, 2) process in Figure 3.6 looks similar to the bottom correlogram of the AR(2) process in Figure 3.5, these two processes are completely different.

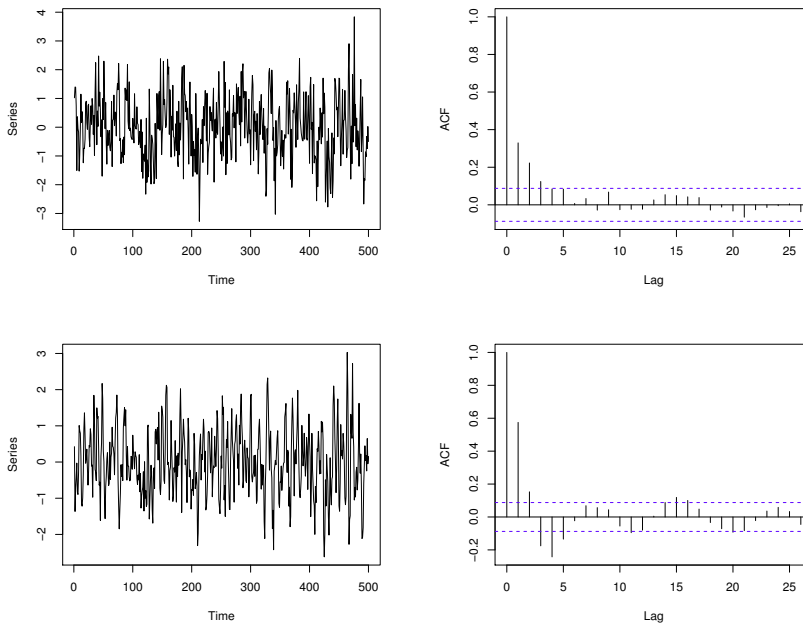

Figure 3.6 *Two simulated ARMA processes and their correlograms. Top:* $X_t = 0.7X_{t-1} + Z_t - 0.4Z_{t-1}$, $Z_t \sim N(0,1)$; *Bottom:* $X_t = 0.9X_{t-1} - 0.5X_{t-2} + Z_t - 0.2Z_{t-1} + 0.25Z_{t-2}$, $Z_t \sim N(0, 0.5)$.

3.8.3 AR and MA representations

It is sometimes helpful to express an ARMA model as a pure MA process in the form

$$X_t = \psi(B)Z_t, \tag{3.25}$$

where $\psi(B) = \sum_{i\geq 0} \psi_i B^i$ is the MA operator, which may be of infinite order. The ψ weights, $\{\psi_i\}$, can be useful in calculating forecasts (see Chapter 5) and in assessing the properties of a model. By comparison with Equation (3.19), we see that $\psi(B) = \theta(B)/\phi(B)$. Alternatively, it can be helpful to express an ARMA model as a pure AR process in the form

$$\pi(B)X_t = Z_t, \tag{3.26}$$

where $\pi(B) = \phi(B)/\theta(B)$. By convention we write $\pi(B) = 1 - \sum_{i\geq 1} \pi_i B^i$, since the natural way to write an AR model is in the form

$$X_t = \sum_{i=1}^{\infty} \pi_i X_{t-i} + Z_t.$$

By comparing (3.25) and (3.26), we see that

$$\pi(B)\psi(B) = 1.$$

The ψ weights or π weights may be obtained directly by division or by equating powers of B in an equation such as

$$\psi(B)\phi(B) = \theta(B).$$

Example 3.3 Find the ψ weights and π weights for the ARMA$(1, 1)$ process given by

$$X_t = 0.5X_{t-1} + Z_t - 0.3Z_{t-1}.$$

Here $\phi(B) = (1 - 0.5B)$ and $\theta(B) = (1 - 0.3B)$. It follows that the process is stationary and invertible, because both equations have roots greater than one (or are outside the unit circle). Then

$$\begin{aligned} \psi(B) = \theta(B)/\phi(B) &= (1 - 0.3B)(1 - 0.5B)^{-1} \\ &= (1 - 0.3B)(1 + 0.5B + 0.5^2B^2 + \cdots) \\ &= 1 + 0.2B + 0.1B^2 + 0.05B^3 + \cdots. \end{aligned}$$

Hence

$$\psi_0 = 1, \qquad \psi_i = 0.2 \times 0.5^{i-1} \qquad \text{for } i = 1, 2, \ldots.$$

Similarly we find

$$\pi_0 = 1, \qquad \pi_i = 0.2 \times 0.3^{i-1} \qquad \text{for } i = 1, 2, \ldots.$$

Note that both the ψ weights and π weights die away quickly, and this also indicates a stationary, invertible process. □

3.9 Integrated ARMA (or ARIMA) Models

In practice most time series are non-stationary. In order to fit a stationary model, such as those discussed in Sections 3.6–3.8, it is necessary to remove non-stationary sources of variation. If the observed time series is non-stationary in the mean, then we can difference the series, as suggested in Section 2.5.3. Differencing is widely used for econometric data. If X_t is replaced by $\nabla^d X_t$ in Equation (3.18), then we have a model capable of describing certain types of non-stationary series. Such a model is called an 'integrated' model because the stationary model that is fitted to the differenced data has to be summed or 'integrated' to provide a model for the original non-stationary data. Writing

$$W_t = \nabla^d X_t = (1 - B)^d X_t, \qquad d = 0, 1, 2, \ldots$$

the general autoregressive integrated moving average (ARIMA) process is of the form

$$W_t = \alpha_1 W_{t-1} + \cdots + \alpha_p W_{t-p} + Z_t + \cdots + \beta_q Z_{t-q}. \tag{3.27}$$

ARIMA models are mainly used to remove a trend from the data. Note that if $d = 0$, the process is an ARMA(p, q) process. By analogy with Equation (3.19), we may write Equation (3.27) in the form

$$\phi(B)W_t = \theta(B)Z_t \tag{3.28}$$

or

$$\phi(B)(1 - B)^d X_t = \theta(B)Z_t. \tag{3.29}$$

Thus we have an ARMA(p, q) model for W_t, while the model in Equation (3.29), describing the dth differences of X_t, is said to be an ARIMA process of order (p, d, q).

The model for X_t is clearly non-stationary, as the AR operator $\phi(B)(1 - B)^d$ has d roots on the unit circle (since putting $B = 1$ makes the AR operator equal to zero). In practice, first differencing is often found to be adequate to make a series stationary, and so the value of d is often taken to be one. Note that the random walk in Section 3.5 can be regarded as an ARIMA$(0, 1, 0)$ process. As we saw in Section 3.5, the random walk is a non-stationary process, and applying the difference operator once makes the process stationary.

Figure 3.7 shows a simulated ARIMA(1,1,1) and a simulated ARIMA(1,1,2) process and their correlograms using the following commands in R:

```
> y1<-arima.sim(list(order=c(1,1,1), ar=-0.5, ma=-0.3),
      n=500)
> y2<-arima.sim(list(order=c(1,1,2), ar=0.3,
      ma=c(-0.3,0.5)), n=500)
```

Note that the ac.f. of the two integrated processes in Figure 3.7 is slowly decaying because of the trend, as already observed in Section 3.5.

ARIMA models can be generalized to include seasonal terms, as will be discussed in Section 4.8.

3.10 Fractional Differencing and Long-Memory Models

An interesting variant of ARIMA modelling arises with the use of what is called **fractional differencing**, leading to a fractional integrated ARMA (abbreviated to ARFIMA) model. The ARIMA(p, d, q) model from Section 3.9 is usually written as

$$\phi(B)(1 - B)^d X_t = \theta(B)Z_t,$$

where $\phi(B)$ and $\theta(B)$ are polynomials of order p, q, respectively, in the backward shift operator B. Here p, q and d are integers and, in particular, the order of differencing d is an integer that is typically one or two. We further assume here that $\phi(B)$ and $\theta(B)$ have all their roots outside the unit circle, so that when $d = 0$ the process is stationary and invertible. Fractional ARIMA

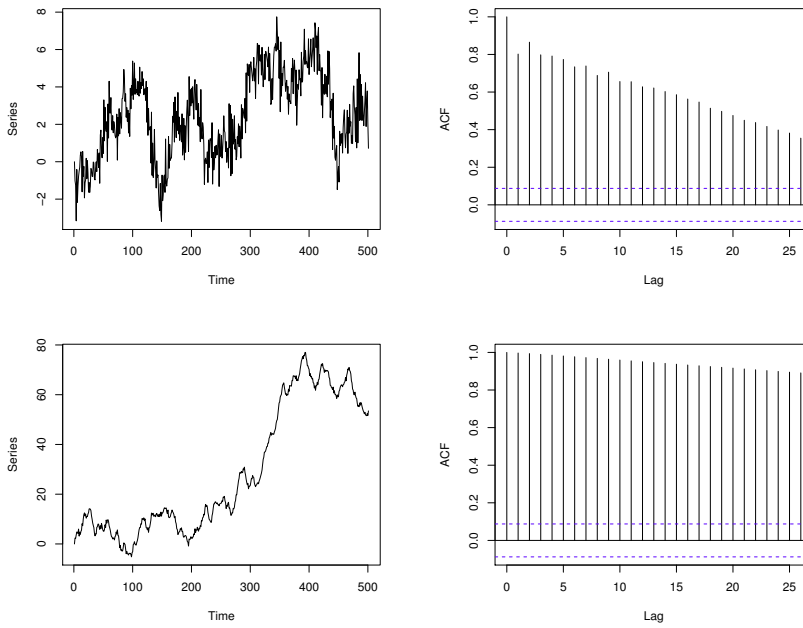

Figure 3.7 *Two simulated ARIMA processes and their correlograms. Top:* $(1 + 0.5B)(1-B)X_t = (1+0.3B)Z_t$*; Bottom:* $(1-0.6B)(1-B)X_t = (1-0.3B+0.5B^2)Z_t$.

models extend the above class of models by allowing d to be a non-integer. In other words, the formula for an ARFIMA model is exactly the same as for an ARIMA(p, d, q) model, except that d is no longer restricted to being an integer.

When d is not an integer, then the dth difference $(1 - B)^d X_t$ becomes a fractional difference, and may be represented by its binomial expansion, namely

$$(1 - B)^d X_t = \Big[1 - dB + \frac{d(d-1)}{2!}B^2 - \frac{d(d-1)(d-2)}{3!}B^3 + \dots\Big]X_t.$$

As such, it is an infinite weighted sum of past values. This contrasts with the case where d is an integer when a finite sum is obtained.

It can be shown (e.g. Brockwell and Davis, 1991, Section 13.2) that an ARFIMA process is stationary provided that $-0.5 < d < 0.5$. For $d > \frac{1}{2}$, the process is not stationary in the usual sense, but further integer differencing can be used to give a stationary ARFIMA process. For example, if an observed series is ARFIMA$(p, d = 1.3, q)$, then the first differences of the series will follow a stationary ARFIMA$(p, d = 0.3, q)$ process.

A drawback to fractional differencing is that it is difficult to give an intuitive interpretation to a non-integer difference. It is also more difficult to

calculate them, given that the binomial expansion will need to be truncated, and so the parameter d is often estimated in the frequency domain. Details will not be given here and the reader is referred, for example, to Beran (1994), Crato and Ray (1996) and the earlier references therein, though the reader is warned that the literature is technically demanding.

A stationary ARFIMA model, with $0 < d < 0.5$, is of particular interest as such a process is not only stationary, but is also an example of what is called a **long-memory** model. For most stationary time-series models (including stationary ARMA models), the autocorrelation function (ac.f.) decreases 'fairly fast', as demonstrated, for example, by the exponential decay in the ac.f. of the AR(1) model. However, for some models the correlations decay to zero very slowly, implying that observations far apart are still related to some extent. An intuitive way to describe such behaviour is to say that the process has a long memory, or that there is long-range dependence.

A working definition of a long-memory process is as follows: A stationary process with ac.f. $\rho(k)$ is said to be a long-memory process if $\sum_{k=0}^{\infty} |\rho(k)|$ does not converge. In particular, the latter condition applies when the ac.f. $\rho(k)$ is of the form $\rho(k) \sim Ck^{2d-1}$ as $k \to \infty$, where C is a constant, not equal to zero, and $0 < d < 0.5$. It can be shown that a stationary ARFIMA model, with differencing parameter d in the range $0 < d < 0.5$, has an ac.f. ρ_k whose limiting form as $k \to \infty$ has the required structure. This means that the correlations decay slowly at a hyperbolic rate and are not absolutely summable. Thus an ARFIMA model is a long-memory model for $0 < d < 0.5$.

As an example, Figure 3.8 shows a simulated ARFIMA(1,$d = 0.4$,2) series, $(1 - 0.3B)(1 - B)^d X_t = (1 - 0.3B + 0.5B^2)Z_t$ with $Z_t \sim N(0, 0.2^2)$, and its correlogram using the following command in `R`. Note that a specific `R` package, `fracdiff`, needs to be loaded before calling the function `fracdiff.sim`.

```
> library("fracdiff")
> x<-fracdiff.sim(2000, ar=0.3, ma=c(-0.3, 0.5), d=0.4,
     sd=0.2)
> names(x)
[1] "series"  "ar"    "ma"    "d"     "mu"    "n.start"
> par(mfrow=c(2,1), mar=c(4,4,4,4))
> plot(x$series, type="l", xlab="Time", ylab="Series")
> acf(x$series, 50, xlab="Lag", ylab="ACF", main="")
```

In contrast, the ac.f. of a stationary ARMA process satisfies the condition that $|\rho(k)| < C\lambda^k$, where C and λ are constants with $0 < \lambda < 1$. Thus these correlations are absolutely summable and such processes may be called **short-memory** models.

Further details on ARFIMA and other long-memory models are given by Beran (1994). It is also possible to find non-linear models that have

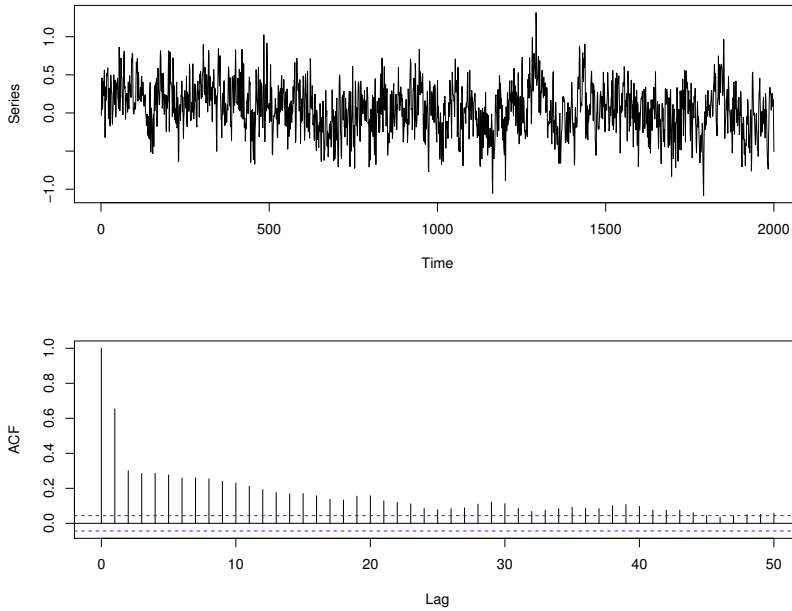

Figure 3.8 *A simulated ARFIMA(1, $d = 0.4$, 2) process,* $(1 - 0.3B)(1 - B)^d X_t = (1 - 0.3B + 0.5B^2)Z_t$ *with* $Z_t \sim N(0, 0.04)$*, and its correlogram.*

the long-memory property (e.g. Robinson and Zaffaroni, 1998). For example, while ordinary ARCH models are short memory, some variants for modelling volatility in stock and exchange rate returns are such that the squared returns are long memory (see Section 12.1 for a definition of the term 'returns').

Long-memory models have a number of interesting features. Although it can be difficult to get good estimates of some parameters of a long-memory model, notably the mean, it is usually possible to make better forecasts, at least in theory. As regards estimating the mean, the usual formula for the variance of a sample mean is σ_Z^2/N, but this applies to the case of N independent observations having constant variance σ^2. In time-series analysis, successive observations are generally correlated and the variance of a sample mean can be expressed as

$$\frac{\sigma_Z^2}{N}\Big[1 + 2\sum_{k=1}^{N-1}(1 - \frac{k}{n})\rho(k)\Big];$$

see Section 4.1.2. When the correlations are positive, as they usually are, the latter expression can be much larger than σ^2/N, especially for long-memory processes where the correlations die out slowly. In contrast to this result, it is intuitively clear that the larger and longer lasting the autocorrelations, the

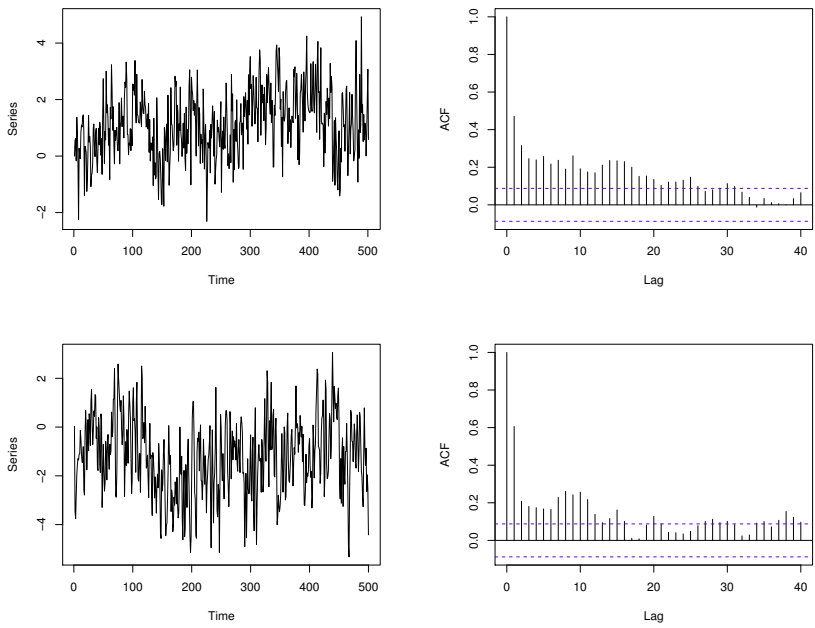

Figure 3.9 *A simulated non-stationary process and a simulated ARFIMA(1, d = 0.45, 2) process and their correlograms. Top:* $(1-0.3B)(1-B)X_t = (1+0.9B)Z_t$, $Z_t \sim N(0,1)$. *Bottom:* $(1-0.3B)(1-B)^d X_t = (1-0.3B+0.5B^2)Z_t$, $Z_t \sim N(0,1)$.

better will be the forecasts from the model. This can readily be demonstrated, both theoretically and practically (Beran, 1994, Section 8.7), but this topic will not be pursued here.

A long-memory (stationary) process and a non-stationary process

A major problem in practice is distinguishing between a long-memory (stationary) process and a non-stationary process. A feature of both models is that the empirical ac.f. will die out slowly and the spectrum will be large at zero frequency. Figure 3.9 shows a simulated non-stationary process, a simulated long memory (stationary) process, and their correlograms. The non-stationary process is specified as $(1-0.3B)(1-B)X_t = (1+0.9B)Z_t$, $Z_t \sim N(0,1)$, and the long memory process is given by $(1-0.3B)(1-B)^d X_t = (1-0.3B+0.5B^2)Z_t$ with $d = 0.45$ and $Z_t \sim N(0,1)$. Note that both the series themselves and their correlograms look quite similar in Figure 3.9.

Therefore, given a set of data with these properties, so that it appears to be non-stationary, or at least 'nearly non-stationary', then it may be worth considering a fractional ARIMA model, with $0 < d < 1$, as well as an ordinary ARIMA model with $d = 1$. The question then arises as to whether the resulting

forecasts are likely to be better than those from alternative models. While there are some encouraging results (e.g. Sutcliffe, 1994), the results in Smith and Yadav (1994) suggest that little will be lost by taking first, rather than fractional, differences. Research continues on this interesting class of models.

Note that the fractional autoregressive (FAR) model defined by Tong (1990, Section 3.5) is unrelated to ARFIMA models. Also note that a completely different definition of the long-memory property is given by Granger and Teräsvirta (1993, Chapter 5). It concerns the asymptotic properties of the expected value of the forecast for long lead times, and applies to many non-stationary processes.

3.11 The General Linear Process

Expanding Equation (3.25), a general class of processes may be written as an MA process, of possibly infinite order, in the form

$$X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}. \tag{3.30}$$

A sufficient condition for the sum to converge, and hence for the process to be stationary, is that $\sum_{i=0}^{\infty} |\psi_i| < \infty$, in which case we also have $\sum_{i=0}^{\infty} \psi_i^2 < \infty$ so that $\mathrm{Var}(X_t)$ is finite. A stationary process described by Equation (3.30) is sometimes called a **general linear process**, although some authors use this term when the Z's are merely uncorrelated, rather than independent. Although the model is in the form of an MA process, it is interesting to note that stationary AR and ARMA processes can also be expressed as a general linear process using the duality between AR and MA processes arising, for example, from Equations (3.25) and (3.26).

3.12 Continuous Processes

So far, we have only considered stochastic processes in discrete time, because models of this type are nearly always used by the statistician in practice. This subsection[2] gives a brief introduction to processes in continuous time. The latter have been used in some applications, notably in the study of control theory by electrical engineers. Here we are mainly concerned with indicating some of the mathematical problems that arise when time is continuous.

By analogy with a discrete-time purely random process, we might expect to define a continuous-time purely random process as having an ac.f. given by

$$\rho(\tau) = \begin{cases} 1 & \tau = 0, \\ 0 & \tau \neq 0. \end{cases}$$

[2]This subsection may be omitted at a first reading.

However, this is a discontinuous function, and it can be shown that such a process would have an infinite variance and hence be a physically unrealizable phenomenon. Nevertheless, some processes that arise in practice do appear to have the properties of continuous-time white noise even when sampled at quite small discrete intervals. We may approximate continuous-time white noise by considering a purely random process in discrete time at intervals Δt, and letting $\Delta t \to 0$, or by considering a process in continuous time with ac.f. $\rho(\tau) = \mathrm{e}^{-\lambda|\tau|}$ and letting $\lambda \to \infty$ so that the ac.f. decays very quickly.

As an example of the difficulties involved with continuous-time processes, we briefly consider a first-order AR process in continuous time. A first-order AR process in discrete time may readily be rewritten[3] in terms of $X_t, \nabla X_t = (1 - B)X_t$ and Z_t, rather than X_t, X_{t-1} and Z_t. Now differencing in discrete time corresponds to differentiation in continuous time, so that an apparently natural way of trying to define a first-order AR process in continuous time is by means of the general equation

$$aX(t) + \frac{\mathrm{d}X(t)}{\mathrm{d}t} = Z(t), \tag{3.31}$$

where a is a constant, and $Z(t)$ denotes continuous white noise. However, as $\{Z(t)\}$ cannot physically exist, it is more legitimate to write Equation (3.31) in a form involving infinitesimal small changes as

$$\mathrm{d}X(t) = -aX(t)\ \mathrm{d}t + \mathrm{d}U(t), \tag{3.32}$$

where $\{U(t)\}$ is a process with orthogonal increments such that the random variables $[U(t_2) - U(t_1)]$ and $[U(t_4) - U(t_3)]$ are uncorrelated for any two non-overlapping intervals (t_1, t_2) and (t_3, t_4). In the theory of Brownian motion, Equation (3.32) arises in the study of the Ornstein–Uhlenbeck model and is sometimes called the Langevin equation. It can be shown that the process $\{X(t)\}$ defined in Equation (3.32) has ac.f.

$$\rho(\tau) = \mathrm{e}^{-a|\tau|},$$

which is similar to the ac.f. of a first-order AR process in discrete time in that both decay exponentially. However, the rigorous study of continuous processes, such as that defined by Equation (3.32), requires considerable mathematical machinery, including a knowledge of stochastic integration. Thus we will not pursue this topic here. The reader is referred to a specialist book on probability theory such as Rogers and Williams (1994).

3.13 The Wold Decomposition Theorem

This section[4] gives a brief introduction to a famous result, called the Wold decomposition theorem, which is of mainly theoretical interest. This essentially

[3] For example, if $X_t = \alpha X_{t-1} + Z_t$, then $(1 - \alpha)X_t + \alpha \nabla X_t = Z_t$.

[4] This section may be omitted at a first reading.

says that any discrete-time stationary process can be expressed as the sum of two uncorrelated processes, one purely deterministic and one purely indeterministic. The terms 'deterministic' and 'indeterministic' are defined as follows. We can regress X_t on $(X_{t-q}, X_{t-q-1}, \ldots)$ and denote the residual variance from the resulting linear regression model by τ_q^2. As $\tau_q^2 \leq \text{Var}(X_t)$, it is clear that, as q increases, τ_q^2 is a non-decreasing bounded sequence and therefore tends to a limit as $q \to \infty$. If $\lim_{q\to\infty} \tau_q^2 = \text{Var}(X_t)$ then linear regression on the remote past is useless for prediction purposes, and we say that $\{X_t\}$ is **purely indeterministic**. Stationary linear stochastic processes, such as AR, MA, and ARMA processes, are of this type. However, if $\lim_{q\to\infty} \tau_q^2$ is zero, then the process can be forecast exactly, and we say that $\{X_t\}$ is purely **deterministic**. Note that this definition of 'deterministic' involves linear models and bears rather little relation to the usual meaning of the word as used in philosophical debate or common parlance.

The Wold decomposition theorem also says that the purely indeterministic component can be written as a linear sum of a sequence of uncorrelated random variables, say $\{Z_t\}$. This has the same form as the general linear process in Equation (3.30) except that the Zs are merely uncorrelated, rather than independent. Of course, if the Zs are normally distributed, then zero correlation implies independence and we really do have a general linear process.

While the concept of a purely indeterministic process may sometimes be helpful, the Wold decomposition itself can be of little assistance. For a linear purely indeterministic process, such as an AR or ARMA model, it is inappropriate to try to model it as an MA(∞) process, as there will be too many parameters to estimate (although the MA(∞) form may be helpful for computing forecast error variances — see Section 5.3.1). Rather, we normally seek a model that gives an adequate approximation to the given data with as few parameters as possible, and that is where an ARMA representation can help.

For processes generated in a non-linear way, the Wold decomposition is usually of even less interest, as the best predictor may be quite different from the best linear predictor. Consider, for example, a sinusoidal process (see Exercise 3.14), such as

$$X_t = g\cos(\omega t + \theta), \tag{3.33}$$

where g is a constant, ω is a constant in $(0, \pi)$ called the frequency of the process, and θ is a random variable, called the phase, which is uniformly distributed on $(0, 2\pi)$ but which is fixed for a single realization. Note that we must include the term θ so that

$$E(X_t) = 0 \qquad \text{for all } t.$$

If this is not done, Equation (3.33) would not define a stationary process. As θ is fixed for a single realization, once enough values of X_t have been observed

to evaluate θ, all subsequent values of X_t are completely determined. It is then obvious that (3.33) defines a deterministic process. However, it is not 'purely deterministic' as defined above because (3.33) is not linear. In fact the process is 'purely indeterministic' using a linear predictor, even though it *is* deterministic using an appropriate non-linear predictor. These are deep waters!

Exercises

In all the following questions, $\{Z_t\}$ is a discrete-time, purely random process, such that $E(Z_t) = 0$, $\mathrm{Var}(Z_t) = \sigma_Z^2$, and successive values of Z_t are independent so that $\mathrm{Cov}(Z_t, Z_{t+k}) = 0, k \neq 0$. Exercise 3.14 is harder than the other exercises and may be omitted.

3.1 Show that the ac.f. of the second-order MA process

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2}$$

is given by

$$\rho(k) = \begin{cases} 1 & k = 0 \\ 0.37 & k = \pm 1 \\ -0.13 & k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

3.2 Consider the MA(m) process, with equal weights $1/(m+1)$ at all lags (so it is a *real* moving average), given by

$$X_t = \sum_{k=0}^{m} Z_{t-k}/(m+1)$$

Show that the ac.f. of this process is

$$\rho(k) = \begin{cases} (m+1-k)/(m+1) & k = 0, 1, \ldots, m, \\ 0 & k > m, \\ \rho(-k) & k < 0. \end{cases}$$

3.3 Consider the infinite-order MA process $\{X_t\}$, defined by

$$X_t = Z_t + C(Z_{t-1} + Z_{t-2} + \cdots)$$

where C is a non-zero constant. Show that the process is non-stationary. Also show that the series of first differences $\{Y_t\}$ defined by

$$Y_t = X_t - X_{t-1}$$

is a first-order MA process and is stationary. Find the ac.f. of $\{Y_t\}$.

3.4 Find the ac.f. of the first-order AR process with $E(X_t) = \mu$, defined by

$$X_t - \mu = 0.7(X_{t-1} - \mu) + Z_t.$$

Plot $\rho(k)$ for $k = -6, -5, \ldots, -1, 0, +1, \ldots, +6$.

3.5 If $X_t = \mu + Z_t + \beta Z_{t-1}$, where μ is a constant, show that the ac.f. does not depend on μ.

3.6 Find the values of λ_1, λ_2, such that the second-order AR process defined by

$$X_t = \lambda_1 X_{t-1} + \lambda_2 X_{t-2} + Z_t$$

is stationary. If $\lambda_1 = 1/3, \lambda_2 = 2/9$, show that the ac.f. of X_t is given by

$$\rho(k) = \frac{16}{21}\left(\frac{2}{3}\right)^{|k|} + \frac{5}{21}\left(-\frac{1}{3}\right)^{|k|} \qquad k = 0, \pm 1, \pm 2, \ldots.$$

3.7 Show that the ac.f. of the stationary second-order AR process

$$X_t = \frac{1}{12}X_{t-1} + \frac{1}{12}X_{t-2} + Z_t$$

is given by

$$\rho(k) = \frac{45}{77}\left(\frac{1}{3}\right)^{|k|} + \frac{32}{77}\left(-\frac{1}{4}\right)^{|k|} \qquad k = 0, \pm 1, \pm 2, \ldots.$$

3.8 Suppose the process $\{X_t\}$ is stationary and has acv.f. $\gamma_X(k)$. A new process $\{Y_t\}$ is defined by $Y_t = X_t - X_{t-1}$. Show that $\{Y_t\}$ is stationary and obtain the acv.f. of $\{Y_t\}$ in terms of $\gamma_X(k)$ and find $\gamma_Y(k)$ when $\gamma_X(k) = \lambda^{|k|}$.

3.9 For each of the following models:

(a) $X_t = 0.3X_{t-1} + Z_t$

(b) $X_t = Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$

(c) $X_t = 0.5X_{t-1} + Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$

express the model using B notation and determine whether the model is stationary and/or invertible. For model (a) find the equivalent MA representation.

3.10 Suppose that a MA(q) process, $\{X_t\}$, can be represented in the form $X_t = \psi(B)Z_t$. The autocovariance generating function is defined by

$$\Gamma(s) = \sum_{k=-\infty}^{\infty} \gamma_X(k)s^k,$$

where $\gamma_X(k)$ is the autocovariance coefficient of X_t at lag k, and s is a dummy variable. Show that $\Gamma(s) = \sigma_Z^2 \psi(s)\psi(1/s)$. (Hint: Equate coefficients of s^k.)

3.11 Compute the ac.f. of the ARMA(1, 1) model

$$X_t = 0.7X_{t-1} + Z_t + 0.3Z_{t-1}.$$

3.12 For the model $(1 - 0.2B)(1 - B)X_t = (1 - 0.5B)Z_t$:

(a) Classify the model as an ARIMA(p, d, q) process (i.e. find p, d, q).

(b) Determine whether the process is stationary and invertible.

(c) Evaluate the first three ψ weights of the model when expressed as a MA(∞) model.

(d) Evaluate the first four π weights of the model when expressed as an AR(∞) model.

Is the behaviour of the ψ and π weights what you would expect, given the type of model?

3.13 Show that the AR(2) process

$$X_t = X_{t-1} + cX_{t-2} + Z_t$$

is stationary provided $-1 < c < 0$. Find the autocorrelation function when $c = -3/16$.

Show that the AR(3) process

$$X_t = X_{t-1} + cX_{t-2} - cX_{t-3} + Z_t$$

is non-stationary for all values of c.

3.14 For a complex-valued process $X(t)$, with (possibly complex) mean μ, the acv.f. is defined by

$$\gamma(\tau) = E\{[X(t) - \mu][\bar{X}(t + \tau) - \bar{\mu}]\}$$

where the overbar denotes the complex conjugate. If Y is a complex random variable which does not depend on t with mean zero, show that the process $X(t) = Y\mathrm{e}^{\mathrm{i}\omega t}$ is second-order stationary, where ω is a real constant. One useful form for the random variable Y occurs when it takes the form $g\ \mathrm{e}^{\mathrm{i}\theta}$, where g is a constant and θ is a random variable that is uniformly distributed on $(0,\ 2\pi)$. Show that $E(Y) = 0$ in this case (see Yaglom, 1962, Section 2.8; but note that the autocovariance function is called the correlation function by Yaglom).

3.15 Consider the stationary process $X_t = Z_t - \theta Z_{t-1} - 6\theta^2 Z_{t-2}$, where θ is a real non-zero number.

(a) Under which condition does X_t become invertible?

(b) Find the ac.f. of $\{X_t\}$.

3.16 Consider the AR(2) process (a is a real number):

$$X_t - \frac{2}{a(a+2)}X_{t-1} - \frac{1}{a(a+2)}X_{t-2} = Z_t.$$

(a) Under which conditions, is the process X_t stationary?

(b) Find the ac.f. of $\{X_t\}$.

3.17 Consider the ARMA(2,2) process $X_t - \alpha X_{t-2} = Z_t - \beta Z_{t-2}$, where α, β are different real non-zero numbers.

(a) Under which conditions, is process X_t stationary and invertible?

(b) Compute $\gamma(0), \gamma(1)$, and $\gamma(2)$, where $\gamma(\cdot)$ is the acv.f. of X_t. Find the values of $\rho(1)$ and $\rho(2)$, where $\rho(\cdot)$ is the ac.f. of X_t.

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Chapter 4

Fitting Time Series Models in the Time Domain

Chapter 3 introduced several different types of probability models that may be used to describe time series. This chapter discusses the problem of fitting a suitable model to an observed time series. We restrict attention to the discrete-time case and the major diagnostic tool used in this chapter is the sample autocorrelation function (ac.f.). Inference based on this function is often called an **analysis in the time domain**.

4.1 Estimating Autocovariance and Autocorrelation Functions

We have already noted in Section 3.3 that the theoretical ac.f. is an important tool for describing the properties of a stationary stochastic process. In Section 2.7 we heuristically introduced the sample ac.f. of an observed time series, and this is an intuitively reasonable estimate of the theoretical ac.f., provided the series is stationary. This section investigates the properties of the sample ac.f. more closely.

Let us look first at the autocovariance function (acv.f.). Suppose we have N observations on a stationary process, say $x_1, x_2, \ldots, x_N$. Then the sample autocovariance coefficient at lag k (see Equation (2.8)) given by

$$c_k = \sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})/N \tag{4.1}$$

is the usual estimator for the theoretical autocovariance coefficient $\gamma(k)$ at lag k. It can be shown (e.g. Priestley, 1981, Chapter 5) that this estimator is biased, but that the bias is of order $1/N$. Moreover

$$\lim_{N\to\infty} E(c_k) = \gamma(k),$$

so that the estimator is asymptotically unbiased.

It can also be shown that

$$\mathrm{Cov}(c_k, c_m) \simeq \sum_{r=-\infty}^{\infty} \{\gamma(r)\gamma(r+m-k) + \gamma(r+m)\gamma(r-k)\}/N. \tag{4.2}$$

When $m = k$, Equation (4.2) gives us the variance of c_k. Equation (4.2) also highlights the fact that successive values of c_k may be (highly) correlated and this increases the difficulty of interpreting the correlogram.

The estimator (4.1) may be compared with the alternative estimator

$$c'_k = \sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})/(N-k), \tag{4.3}$$

where the divisor is $(N-k)$, rather than N. This is used by some authors because it is claimed to have a smaller bias, though this is, in fact, not always the case. Indeed some authors call c'_k the 'unbiased estimator' even though it is biased when, as here, the population mean, μ, is unknown and replaced by the sample mean, $\bar{x}$, as is usually the case in practice. It turns out that the earlier estimator in Equation (4.1) leads to a sample acv.f. having a useful property called positive semi-definiteness, which means that its finite Fourier transform is non-negative, among other consequences. This property is useful in estimating the spectrum (see Chapter 7) and so we generally prefer to use Equation (4.1), rather than (4.3). Note that when $k = 0$, we get the same estimate of variance using both formulae, but that this estimate, involving $\bar{x}$, will generally still be biased (Percival, 1993).

Having estimated the acv.f., we then take

$$r_k = c_k / c_0 \tag{4.4}$$

as an estimator for $\rho(k)$. The properties of r_k are rather more difficult to find than those of c_k because it is the ratio of two random variables. It can be shown that r_k is generally biased. A general formula for the variance of r_k is given by Kendall et al. (1983, Section 48.1) and depends on *all* the autocorrelation coefficients of the process. We will only consider the properties of r_k when sampling from a purely random process, when all the theoretical autocorrelation coefficients are zero except at lag zero. These results help us to decide if the observed values of r_k from a given time series are significantly different from zero.

Suppose that $x_1, \ldots, x_N$ are observations on independent and identically distributed random variables with arbitrary mean. Then it can be shown (Kendall et al., 1983, Chapter 48) that

$$\begin{aligned} E(r_k) &\simeq -1/N, \\ Var(r_k) &\simeq 1/N \end{aligned}$$

and that r_k is asymptotically normally distributed under weak conditions. Thus having plotted the correlogram, as described in Section 2.7, we can check for randomness by plotting approximate 95% confidence limits at

$$-1/N \pm 1.96/\sqrt{N} = \frac{-1 \pm 1.96\sqrt{N}}{N} \approx \frac{\pm 1.96\sqrt{N}}{N} = \pm \frac{1.96}{N}.$$

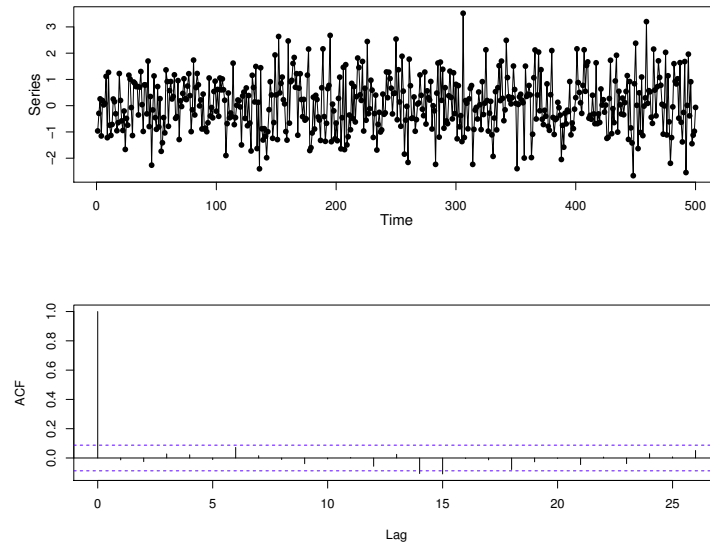

Figure 4.1 *500 "independent" normally distributed observations and their correlogram. The dashed lines in the bottom panel are at* $\pm 1.96/\sqrt{500}$.

Observed values of r_k which fall outside these limits are 'significantly' different from zero at the 5% level. However, when interpreting a correlogram, it must be remembered that the overall probability of getting at least one coefficient outside these limits, given that the data really are random, increases with the number of coefficients plotted. For example, if the first 20 values of r_k are plotted, then one expects one 'significant' value (at the 5% level) on average even if the data really are random. Thus, if only one or two coefficients are 'significant', the size and lag of these coefficients must be taken into account when deciding if a set of data is random. A single coefficient just outside the 'null' 95% confidence limits may be ignored, but two or three values well outside the 'null' limits will be taken to indicate non-randomness. A single 'significant' coefficient at a lag which has some physical interpretation, such as lag 1 or a lag corresponding to seasonal variation, will also provide plausible evidence of non-randomness.

Figure 4.1 shows the correlogram for 400 observations, generated on a computer, which are independent normally distributed variables. The 'null' 95% confidence limits are approximately $\pm\ 1.96/\sqrt{500} \approx \pm 0.088$. We see that 2 of the first 20 values of r_k are just 'significant'. However, they occur at apparently arbitrary lags (namely, 14 and 15). Thus we conclude that there is no firm evidence to reject the hypothesis that the observations are independent and identically distributed. This in turn means that the way that the computer generates 'random' numbers appears to be satisfactory.

4.1.1 Using the correlogram in modelling

We have already given some general advice on interpreting correlograms in Section 2.7.2, while their use in assessing randomness was considered above. The correlogram is also helpful in trying to identify a suitable class of models for a given time series, and, in particular, for selecting the most appropriate type of autoregressive integrated moving average (ARIMA) model. A correlogram like that in Figure 2.7, where the values of r_k do not come down to zero reasonably quickly, indicates non-stationarity and so the series needs to be differenced. For stationary series, the correlogram is compared with the theoretical ac.f.s of different ARMA processes in order to choose the one which seems to be the 'best' representation. In Section 3.6, we saw that the ac.f. of a MA(q) process is easy to recognize as it 'cuts off' at lag q. However, the ac.f. of an AR(p) process is rather more difficult to categorize, as it is a mixture of damped exponentials and sinusoids and dies out slowly (or attenuates). The ac.f. of a mixed ARMA model will also generally attenuate rather than 'cut off'.

Suppose, for example, that we find that r_1 is significantly different from zero but that subsequent values of r_k are all close to zero. Then an MA(1) model is indicated because its theoretical ac.f. is of this form. Alternatively, if $r_1, r_2, r_3, \ldots$ appear to be decreasing exponentially, then an AR(1) model may be appropriate.

The interpretation of correlograms is one of the hardest aspects of time-series analysis and practical experience is a 'must'. Inspection of the partial autocorrelation function (see Section 4.2.2) can provide additional help.

4.1.2 Estimating the mean

The first inferential problem considered in most statistics texts is the estimation of a population mean using a sample mean. However, in time-series analysis, the topic is often overlooked and omitted, because of the special problems that relate to using a time-series sample mean as an estimate of some underlying population mean. Although we have used the sample mean in computing the sample acv.f. and ac.f. — see Equations (4.1) and (4.4) — we have already noted, at the start of Chapter 2, that the sample mean is a potentially misleading summary statistic unless all systematic components have been removed. Thus the sample mean should only be considered as a summary statistic for data thought to have come from a stationary process. Even when this is so, it is important to realize that the statistical properties of the sample mean are quite different from those that usually apply.

Suppose we have time-series data $\{x_i \text{ for } i = 1, \ldots, N\}$ from a stationary process having mean μ, variance σ^2 and theoretical ac.f. $\rho(k)$. Let $\overline{X} = \sum_{i=1}^{N} X_i/N$ denote the sample mean value expressed as a random variable. Then the usual result for independent observations is that $\mathrm{Var}(\overline{X}) = \sigma^2/N$.

However, for correlated observations, it can be shown (e.g. Priestley, 1981, p. 319) that

$$\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{N}\Big[1 + 2\sum_{r=1}^{N-1}(1 - \frac{r}{N})\rho(r)\Big]$$

and this quantity can differ considerably from σ^2/N when autocorrelations are substantial. In particular, for an AR(1) process with parameter α, the formula reduces to $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{N}(\frac{1+\alpha}{1-\alpha})$ for large N (see Exercise 4.1). Put another way, the equivalent number of independent observations is $N(1-\alpha)/(1+\alpha)$. When $\alpha > 0$ (the usual case), the positive autocorrelation means that there is less information in the data than might be expected in regard to estimating the mean. On the other hand, when $\alpha < 0$, there is *more* information. Any tests on the sample mean should remember to adjust for this effect.

Another way that estimating a time-series mean differs from the rest of Statistics is that any results depend upon the underlying process having a property called ergodicity. The difficult ideas involved are introduced briefly in the next optional subsection.

4.1.3 Ergodicity

This subsection[1] gives a brief introduction to the idea of ergodicity. It is not immediately obvious that one can obtain consistent estimates of the properties of a stationary process from a single finite realization. This requires that an average over time for a single time series, like $\bar{x} = \sum_{t=1}^{N} x_t/N$, can be used to estimate the ensemble properties of the underlying process at a particular time, like $E(X_t)$. This means that, as we only ever get one observation actually at time t, the properties of the series at time t are estimated using data collected at other time points. Fortunately, some theorems, called **ergodic theorems**, have been proved, showing that for most stationary processes, which are likely to be met in practice, the sample moments of an observed time series do indeed converge to the corresponding population moments. In other words, a time average like $\sum_{t=1}^{N} x_t/N$ converges to a population quantity like $E(X_t)$ as $N \to \infty$. A sufficient condition for this to happen is that $\rho_k \to 0$ as $k \to \infty$ and the process is then called 'ergodic in the mean'.

We will not pursue the topic here but rather simply assume that appropriate ergodic properties are satisfied when estimating the properties of stationary processes. More details may be found, for example, in Hamilton (1994, p. 46).

4.2 Fitting an Autoregressive Process

Having estimated the ac.f. of a given time series, we should have some idea as to which stochastic process will provide a suitable model. If an autoregressive (AR) process is thought to be appropriate, there are two related questions:

[1] This subsection may be omitted at a first reading.

1. What is the order of the process?
2. How can we estimate the parameters of the process?

It is convenient to consider the second question first.

4.2.1 Estimating parameters of an AR process

Suppose we have an AR process of order p, with mean μ, given by

$$X_t - \mu = \alpha_1(X_{t-1} - \mu) + \cdots + \alpha_p(X_{t-p} - \mu) + Z_t. \tag{4.5}$$

Given N observations $x_1, \ldots, x_N$, the parameters μ, $\alpha_1, \ldots, \alpha_p$ may be estimated by least squares by minimizing

$$S = \sum_{t=p+1}^{N} \Big[x_t - \mu - \alpha_1(x_{t-1} - \mu) - \cdots - \alpha_p(x_{t-p} - \mu)\Big]^2$$

with respect to $\mu, \alpha_1, \ldots, \alpha_p$. If the Z_t process is normal, then the least squares estimates are also maximum likelihood estimates (Jenkins and Watts, 1968, Section 5.4) conditional on the first p values in the time series being fixed.

In the first-order case, with $p = 1$, we find (see Exercise 4.1)

$$\widehat{\mu} = \frac{\bar{x}_{(2)} - \widehat{\alpha}_1 \bar{x}_{(1)}}{1 - \widehat{\alpha}_1} \tag{4.6}$$

and

$$\widehat{\alpha}_1 = \frac{\sum_{t=1}^{N-1}(x_t - \widehat{\mu})(x_{t+1} - \widehat{\mu})}{\sum_{t=1}^{N-1}(x_t - \widehat{\mu})^2} \tag{4.7}$$

where $\bar{x}_{(1)}, \bar{x}_{(2)}$ are the means of the first and last $(N-1)$ observations. Now

$$\bar{x}_{(1)} \simeq \bar{x}_{(2)} \simeq \bar{x}$$

and so we have approximately that

$$\widehat{\mu} = \bar{x}. \tag{4.8}$$

This approximate estimator is intuitively appealing and is nearly always preferred to Equation (4.6). Substituting this estimator into Equation (4.7) gives

$$\widehat{\alpha}_1 \approx \frac{\sum_{t=1}^{N-1}(x_t - \bar{x})(x_{t+1} - \bar{x})}{\sum_{t=1}^{N-1}(x_t - \bar{x})^2}. \tag{4.9}$$

It is interesting to note that this is exactly the same estimator that would arise if we were to treat the autoregressive equation

$$X_t - \bar{x} = \alpha_1(x_{t-1} - \bar{x}) + Z_t$$

as an ordinary regression with $(x_{t-1} - \bar{x})$ as the 'independent' variable (which of course it isn't). In fact Mann and Wald (1943) showed that, asymptotically, much classical regression theory can be applied to AR models.

A further approximation, which is often used, is obtained by changing the denominator of (4.9) slightly to

$$\sum_{t=1}^{N} (x_t - \bar{x})^2$$

$$\begin{aligned} \text{so that} \qquad \widehat{\alpha}_1 &\simeq c_1/c_0 \\ &= r_1. \end{aligned}$$

This approximate estimator for $\widehat{\alpha}_1$ is also intuitively appealing since r_1 is an estimator for $\rho(1)$ and $\rho(1) = \alpha_1$ for a first-order AR process. A confidence interval for α_1 may be obtained from the fact that the asymptotic standard deviation of $\widehat{\alpha}_1$ is $\sqrt{(1-\alpha_1^2)/N}$, although the confidence interval will not be symmetric for $\widehat{\alpha}_1$ away from zero. When $\alpha_1 = 0$, the standard deviation of $\widehat{\alpha}_1$ is $1/\sqrt{N}$, and so a test for $\alpha_1 = 0$ is given by seeing whether $\widehat{\alpha}_1 \approx 1$ lies within the range $\pm\, 1.96/\sqrt{N}$. This is equivalent to the test for $\rho(1) = 0$ already noted in Section 4.1.

For a second-order AR process, where $p = 2$, similar approximations may be made to give

$$\widehat{\mu} \simeq \bar{x}$$

$$\widehat{\alpha}_1 \simeq r_1(1 - r_2)/(1 - r_1^2) \tag{4.10}$$

$$\widehat{\alpha}_2 \simeq (r_2 - r_1^2)/(1 - r_1^2). \tag{4.11}$$

These results are also intuitively reasonable in that if we fit a second-order model to what is really a first-order process, then as $\alpha_2 = 0$ we have $\rho(2) = \rho(1)^2 = \alpha_1^2$ and so $r_2 \simeq r_1^2$. Thus Equations (4.10) and (4.11) become $\widehat{\alpha}_1 \simeq r_1$ and $\widehat{\alpha}_2 \simeq 0$, as we would hope. The coefficient $\widehat{\alpha}_2$ is called the (sample) **partial autocorrelation coefficient** of order two, as it measures the excess correlation between observations two steps apart (e.g. between X_t and X_{t+2}) not accounted for by the autocorrelation at lag 1, namely, r_1 – see Section 4.2.2 below.

In addition to point estimates of α_1 and α_2 it is also possible to find a confidence region in the (α_1, α_2) plane (Jenkins and Watts, 1968, p. 192).

Higher-order AR processes may also be fitted by least squares in a straightforward way. Two alternative approximate methods are commonly used, which both involve taking $\widehat{\mu} = \bar{x}$. The first method fits the data to the model

$$X_t - \bar{x} = \alpha_1(x_{t-1} - \bar{x}) + \cdots + \alpha_p(x_{t-p} - \bar{x}) + Z_t$$

treating it as if it were an ordinary regression model. A standard multiple regression computer program may be used with appropriate modification.

The second method involves substituting the sample autocorrelation coefficients into the first p Yule–Walker equations (see Section 3.7) and solving for $(\widehat{\alpha}_1, \ldots, \widehat{\alpha}_p)$. In matrix form these equations are

$$R\widehat{\boldsymbol{\alpha}} = \mathbf{r} \tag{4.12}$$

where $\widehat{\boldsymbol{\alpha}}^{\mathrm{T}} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_p)$, $\mathbf{r}^{\mathrm{T}} = (r_1, \ldots, r_p)$ and

$$\mathrm{R} = \begin{pmatrix} 1 & r_1 & r_2 & \cdots & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & \cdots & r_{p-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{pmatrix}$$

is a $(p \times p)$ matrix. For N reasonably large, both methods will give estimated values 'very close' to the exact least squares estimates for which $\widehat{\mu}$ is close to, but not necessarily equal to, $\bar{x}$. However, for smaller N (e.g. less than about 50), the Yule–Walker estimates are not so good, especially when the parameter values are such that the model is 'close' to being non-stationary.

4.2.2 Determining the order of an AR process

It is often difficult to assess the order of an AR process from the sample ac.f. alone. For a first-order process the theoretical ac.f. decreases exponentially and the sample function should have a similar shape. However, for higher-order processes, the ac.f. may be a mixture of damped exponential or sinusoidal functions and is difficult to identify. One approach is to fit AR processes of progressively higher order, to calculate the residual sum of squares for each value of p and to plot this against p. It may then be possible to see the value of p where the curve 'flattens out' and the addition of extra parameters gives little improvement in fit.

Another aid to determining the order of an AR process is the **partial autocorrelation function**, which is defined as follows. When fitting an AR(p) model, the last coefficient α_p will be denoted by π_p and measures the excess correlation at lag p which is not accounted for by an AR($p-1$) model. It is called the pth partial autocorrelation coefficient and, when plotted against p, gives the partial ac.f. The first partial autocorrelation coefficient π_1 is simply equal to $\rho(1)$, and this is equal to α_1 for an AR(1) process. It can be shown (see Exercise 4.3 and the discussion following Equation (4.11) above) that the second partial correlation coefficient is $[\rho(2) - \rho(1)^2]/[1 - \rho(1)^2]$, and we note that this is zero for an AR(1) process where $\rho(2) = \rho(1)^2$.

The sample partial ac.f. is usually estimated by fitting AR processes of successively higher order and taking $\widehat{\pi}_1 = \widehat{\alpha}_1$ when an AR(1) process is fitted, taking $\widehat{\pi}_2 = \widehat{\alpha}_2$ when an AR(2) process is fitted, and so on. Values of $\widehat{\pi}_j$, which are outside the range $\pm\, 1.96/\sqrt{N}$, are significantly different from zero at the 5% level. It can be shown that the partial ac.f. of an AR(p) process 'cuts off'

at lag p so that the 'correct' order is assessed as that value of p beyond which the sample values of $\{\pi_j\}$ are not significantly different from zero.

In contrast, the partial ac.f. of an MA process will generally attenuate (or die out slowly). Thus the partial ac.f. has reverse properties to those of the ac.f. in regard to identifying AR and MA models. Note that McCullough (1998) recommends that Yule–Walker estimates should *not* be used to estimate partial autocorrelations.

Some additional tools to aid model identification are discussed in Section 4.5.

Example 4.1

We now show an example of using the above procedure to fit an AR process to two simulated series. The first series follows an AR(2) process

$$X_t = -0.4X_{t-1} + 0.3X_{t-2} + Z_t, \qquad Z_t \sim N(0,1),$$

and the second follows an AR(3) process

$$X_t = \frac{1}{4}X_{t-1} + \frac{1}{2}X_{t-2} - \frac{1}{8}X_{t-3} + Z_t, \qquad Z_t \sim N(0,1).$$

Both series have $N = 500$ observations, and are shown in the top left and top right panels of Figure 4.2, respectively. To see the dependence of time series observations at different lags, their sample ac.f.'s and sample partial ac.f.'s are calculated and shown in the middle and bottom panels of Figure 4.2, respectively. Note that both sample ac.f.'s decay rapidly, and the sample partial ac.f.'s cut off at lags 2 and 3, respectively. Figure 4.2 can be reproduced using the following `R` command.

```
> set.seed(4)
> n<-500
> x1<-arima.sim(list(order=c(2,0,0), ar=c(-0.4, 0.3)), n)
> x2<-arima.sim(list(order=c(3,0,0), ar=c(1/4,1/2,-1/8)),n)

> par(mfrow=c(3,2), mar=c(4,4,4,4))
> plot(x1, type="l", xlab="Time", ylab="Series")
> plot(x2, type="l", xlab="Time", ylab="Series")
> acf(x1, 25, xlab="Lag", ylab="ACF", main="")
> acf(x2, 25, xlab="Lag", ylab="ACF", main="")
> acf(x1,25,type="partial",xlab="Lag",ylab="Partial ACF",
     main="", ylim=c(-1,1),xlim=c(0,25))
> acf(x2,25,type="partial",xlab="Lag",ylab="Partial ACF",
     main="", ylim=c(-1,1),xlim=c(0,25))
```

If AR models are thought to be appropriate for these two series, one may consider fitting an AR(2) process and an AR(3) process, respectively, to the data because the sample partial acf's cut off at lags 2 and 3, respectively.

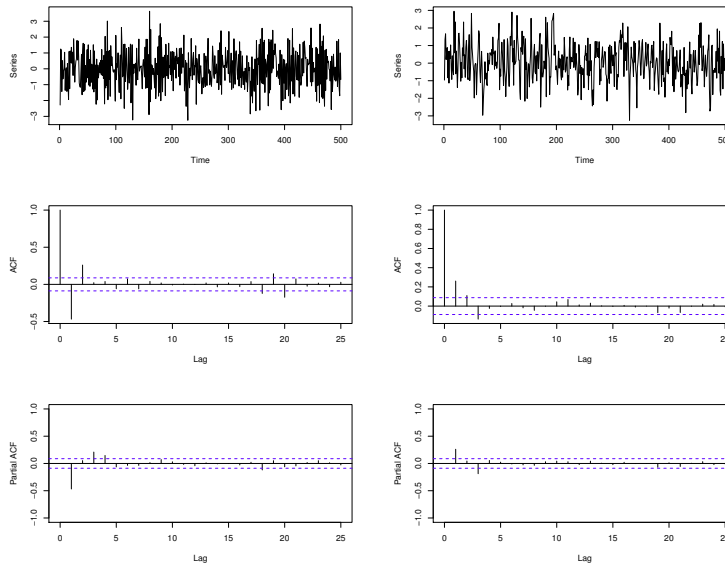

Figure 4.2 *Two simulated AR processes (Left:* $(1 + 0.4B - 0.3B^2)X_t = Z_t$*; Right:* $(1 - 1/4B - 1/2B^2 + 1/8B^3)X_t = Z_t$*) and their sample ac.f.'s and sample partial ac.f.'s. (Top: Simulated series; Middle: Sample ac.f.'s; Bottom: Sample partial ac.f.'s.)*

Depending on whether one assumes the mean of the series is zero or not, parameters of an AR process with zero or nonzero mean should be estimated. This can be easily done by using the `arima` function in `R`, which estimates model parameters by maximum likelihood or minimizing conditional sum-of-squares. The following `R` commands demonstrate the usage of this function.

```
> x1.fit1<-arima(x1, order=c(2,0,0))
> x1.fit1
Call:
arima(x = x1, order = c(2, 0, 0))

Coefficients:
          ar1     ar2  intercept
      -0.4450  0.2959    -0.0600
s.e.   0.0426  0.0426     0.0374

sigma^2 estimated as 0.9254: log likelihood=-690.42,aic=1388.84

> x2.fit1<-arima(x2, order=c(3,0,0))
> x2.fit1
```

```
Call:
arima(x = x2, order = c(3, 0, 0))

Coefficients:
         ar1     ar2      ar3  intercept
      0.1737  0.5043  -0.1175    -0.0946
s.e.  0.0445  0.0393   0.0448     0.0988

sigma^2 estimated as 0.9495: log likelihood=-696.82,aic=1403.65
```

Hence the fitted models are

$$X_t = -0.0600_{(.0426)} - 0.4450_{(.0426)}X_{t-1} + 0.2959_{(.0426)}X_{t-2} + Z_t,$$

with $Z_t \sim N(0, 0.9254)$, for the first series, and

$$X_t = -0.0946_{(.0988)} + 0.1737_{(.0445)}X_{t-1} + 0.5043_{(.0393)}X_{t-2} \\ -0.1175_{(.0448)}X_{t-3} + Z_t,$$

with $Z_t \sim N(0, 0.9495)$, for the second series. Note that the 95% confidence intervals of the intercepts in the two fitted models, $-0.0600 \pm 1.96 \cdot 0.0426$ and $-0.0946 \pm 1.96 \cdot 0.0988$, cover zero, indicating that the means of two AR processes might be zero. An AR model with zero mean can be estimated by adding an argument `include.mean=FALSE` to the `arima` command. In particular, fitting AR models with zero means to the two series can be done by the following `R` commands.

```
> x1.fit2<-arima(x1, order=c(2,0,0), include.mean=FALSE)
> x1.fit2

Call:
arima(x = x1, order = c(2, 0, 0), include.mean = FALSE)

Coefficients:
          ar1     ar2
      -0.4384  0.3025
s.e.   0.0425  0.0425

sigma^2 estimated as 0.93: log likelihood=-691.69, aic=1389.37

> x2.fit2<-arima(x2, order=c(3,0,0), include.mean=FALSE)
> x2.fit2

Call:
arima(x = x2, order = c(3, 0, 0), include.mean = FALSE)
```

```
Coefficients:
         ar1     ar2      ar3
      0.1755  0.5069  -0.1156
s.e.  0.0445  0.0392   0.0448

sigma^2 estimated as 0.9511: log likelihood=-697.28, aic=1402.55
```

Therefore, the fitted AR models with zero means can be summarized as

$$X_t = -0.4384_{(.0425)}X_{t-1} + 0.3025_{(.0425)}X_{t-2} + Z_t,$$

with $Z_t \sim N(0, 0.93)$, for the first series, and

$$X_t = 0.1755_{(.0445)}X_{t-1} + 0.5069_{(.0392)}X_{t-2} - 0.1156_{(.0448)}X_{t-3} + Z_t,$$

with $Z_t \sim N(0, 0.9511)$, for the second series. We further note that, for the first series, the 95% confidence intervals of coefficients of X_{t-1} and X_{t-2} in the fitted model are $-0.4384 \pm 1.96 \cdot 0.0425$ and $0.3025 \pm 1.96 \cdot 0.0425$, respectively, and hence cover the true values of the coefficients -0.4 and 0.3, respectively. For the second series, the 95% confidence intervals of coefficients of X_{t-1}, X_{t-2} and X_{t-3} in the fitted model are $0.1755 \pm 1.96 \cdot 0.0445$, $0.5069 \pm 1.96 \cdot 0.0392$, and $-0.1156 \pm 1.96 \cdot 0.0448$, respectively, and hence cover the true values of the coefficients 0.25, 0.5, and -0.125, respectively.

4.3 Fitting a Moving Average Process

Suppose now that a moving average (MA) process is thought to be an appropriate model for a given time series. As for an AR process, we have two problems:

(1) Finding the order of the process

(2) Estimating the parameters of the process.

As for an AR process, it is convenient to consider the second problem first.

4.3.1 Estimating parameters of an MA process

Estimation is more difficult for an MA process than an AR process, because efficient explicit estimators cannot be found. Instead some form of numerical iteration must be performed, albeit greatly facilitated by modern computers.

Let us begin by considering the first-order MA process, with mean μ, given by

$$X_t = \mu + Z_t + \beta_1 Z_{t-1} \tag{4.13}$$

where μ, β_1 are constants and Z_t denotes a purely random process. We would like to be able to write the residual sum of squares ΣZ_t^2 solely in terms of the observed xs and the parameters μ, β_1, as we did for the AR process. Then we could differentiate with respect to μ and β_1, and hence find the least

squares estimates. Unfortunately this cannot be done and so explicit least squares estimates cannot be found; nor is it wise to simply equate sample and theoretical first-order autocorrelation coefficients by

$$r_1 = \widehat{\beta}_1/(1 + \widehat{\beta_1^2}) \tag{4.14}$$

and choose the solution $\widehat{\beta}_1$ such that $|\widehat{\beta}_1| < 1$, because it can be shown that this gives rise to an inefficient estimator.

One possible alternative approach is as follows: (i) Select suitable starting values for μ and β_1, such as $\mu = \bar{x}$ and the value of β_1 given by the solution of Equation (4.14) (see Box et al., 1994, Part 5, Table A). (ii) Calculate the corresponding residual sum of squares using Equation (4.13) recursively in the form

$$Z_t = X_t - \mu - \beta_1 Z_{t-1}. \tag{4.15}$$

Taking $z_0 = 0$, we calculate $z_1 = x_1 - \widehat{\mu}$, and then $z_2 = x_2 - \widehat{\mu} - \widehat{\beta}_1 z_1$, and so on until $z_N = x_N - \widehat{\mu} - \widehat{\beta}_1 z_{N-1}$. Then the residual sum of squares $\sum_{t=1}^{N} z_t^2$ is calculated conditional on the given values of the parameters μ and β_1 and $z_0 = 0$. (iii) Repeat this procedure for other neighbouring values of μ and β_1 so that the residual sum of squares Σz_t^2 is computed on a grid of points in the (μ, β_1) plane. (iv) Determine by inspection the values of μ and β_1 that minimize Σz_t^2.

The above procedure gives least squares estimates, which are also maximum likelihood estimates conditional on $z_0 = 0$ provided that Z_t is normally distributed. The procedure can be further refined by **back-forecasting** the value of z_0 (see Box et al., 1994, Section 6.4.3), but this is unnecessary except when N is small or when β_1 is 'close' to plus or minus one. Nowadays the values of μ and β, which minimize Σz_t^2, would normally be found by some iterative optimization procedure, such as hill-climbing, although a grid search can still sometimes be useful to see what the sum of squares surface looks like.

For higher-order processes a similar type of iterative procedure to that described above could be used. For example, with a second-order MA process one would guess starting values for μ, β_1, β_2, compute the residuals recursively using

$$z_t = x_t - \widehat{\mu} - \widehat{\beta}_1 z_{t-1} - \widehat{\beta}_2 z_{t-2}$$

and compute Σz_t^2. Then other values of $\widehat{\mu}, \widehat{\beta}_1, \widehat{\beta}_2$ could be tried, perhaps over a grid of points, until the minimum value of Σz_t^2 is found. Nowadays computers can take advantage of an appropriate numerically efficient optimization procedure to minimize the residual sum of squares. Box et al. (1994, Section 7.2) describe such a procedure, which they call 'non-linear estimation', because the residuals are non-linear functions of the parameters. However, this term could give rise to confusion.

For a completely new set of data, it may be a good idea to use the method based on evaluating the residual sum of squares at a grid of points. A visual examination of the sum of squares surface will sometimes provide useful information. In particular, it is interesting to see how 'flat' the surface is; whether the surface is approximately quadratic; and whether the parameter estimates are approximately uncorrelated (because the major axes of the 'hill' are parallel to the coordinate axes).

In addition to point estimates, an approximate confidence region for the model parameters may be found as described by Box et al. (1994, Chapter 7) by assuming that the Z_t are normally distributed. However, there is some doubt as to whether the asymptotic normality of maximum likelihood estimators will apply even for moderately large sample sizes (e.g. $N = 200$).

It should now be clear that it is much harder to estimate the parameters of an MA model than those of an AR model, because the 'errors' in an MA model are non-linear functions of the parameters and iterative methods are required to minimize the residual sum of squares. Because of this, many analysts prefer to fit an AR model to a given time series even though the resulting model may contain more parameters than the 'best' MA model. Indeed the relative simplicity of AR modelling is the main reason for its use in the stepwise autoregression forecasting technique (see Section 5.2.3) and in autoregressive spectrum estimation (see Section 14.4.1).

4.3.2 Determining the order of an MA process

Although parameter estimation is harder for a given MA process than for a given AR process, the reverse is true as regards determining the order of the process. If an MA process is thought to be appropriate for a given set of data, the order of the process is usually evident from the sample ac.f. The theoretical ac.f. of an MA(q) process has a very simple form in that it 'cuts off' at lag q (see Section 3.6.1), and so the analyst should look for the lag beyond which the values of r_k are close to zero. The partial ac.f. is generally of little help in identifying pure MA models because of its attenuated form.

Example 4.2

We now show an example of using the above procedure to fit a MA process to two simulated series. The first series follows an MA(2) process

$$X_t = Z_t - 0.5Z_{t-1} + 0.4Z_{t-2}, \qquad Z_t \sim N(0, 1),$$

and the second follows an MA(3) process

$$X_t = Z_t + \frac{1}{3}Z_{t-1} + \frac{1}{5}Z_{t-2} - \frac{1}{9}Z_{t-3}, \qquad Z_t \sim N(0, 1).$$

Both series have $N = 500$ observations, and are shown in the top left and top right panels of Figure 4.3, respectively. To see the dependence of time series

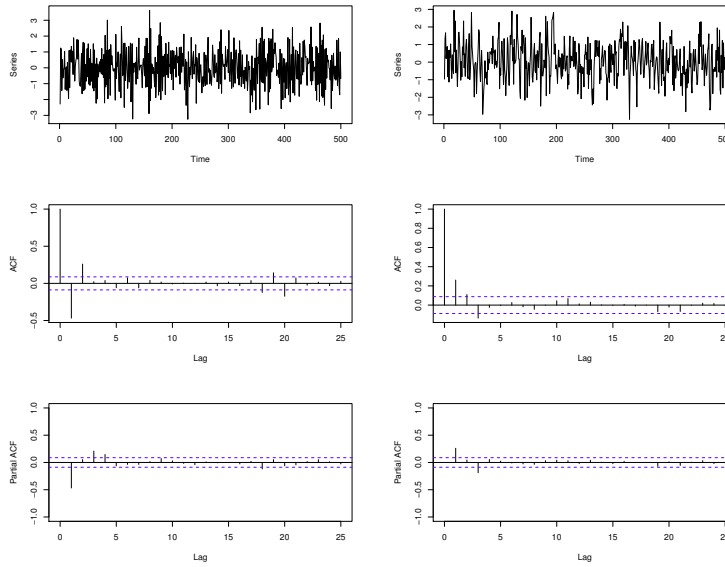

Figure 4.3 *Two simulated MA processes (Left: $X_t = Z_t - 0.5Z_{t-1} + 0.4Z_{t-2}$; Right: $X_t = Z_t + \frac{1}{3}Z_{t-1} + \frac{1}{5}Z_{t-2} - \frac{1}{9}Z_{t-3}$) and their sample ac.f.'s and partial ac.f.'s. (Top: Simulated series; Middle: Sample ac.f.'s; Bottom: Sample partial ac.f.'s.)*

observations at different lags, their sample ac.f.'s and sample partial ac.f.'s are calculated and shown in the middle and bottom panels of Figure 4.3, respectively. Note that the sample ac.f.'s cut off at lags 2 and 3, respectively, while it is difficult to see patterns in their sample partial ac.f.'s. Figure 4.3 can be reproduced using the following R command.

```
> set.seed(5)
> n<-500
> x3<-arima.sim(list(order=c(0,0,2), ma=c(-0.5, 0.4)), n)
> x4<-arima.sim(list(order=c(0,0,3), ma=c(1/3,1/5,-1/9)),n)

> par(mfrow=c(3,2), mar=c(2,2,2,2))
> plot(x3, type="l", xlab="", ylab="")
> plot(x4, type="l", xlab="", ylab="")
> acf(x3, 25, ylab="", main="")
> acf(x4, 25, ylab="", main="")
> acf(x3, 25, type="partial", ylab="", ylim=c(-1,1),
      xlim=c(0,25))
> acf(x4, 25, type="partial", ylab="", ylim=c(-1,1),
      xlim=c(0,25))
```

If MA models are thought to be appropriate for these two series, the correlograms in Figure 4.3 suggest an MA(2) process and an MA(3) process

respectively, to the data since the sample acf's cut off at lags 2 and 3, respectively. Depending on whether one assumes the mean of the series is zero or not, parameters of an MA process with zero or nonzero mean should be estimated. This can be easily done by using the `arima` function in `R`, which estimates model parameters by maximum likelihood or minimizing conditional sum-of-squares. The following `R` commands demonstrate the usage of this function.

```
> x3.fit1<-arima(x1, order=c(0,0,2))
> x3.fit1
Call:
arima(x = x1, order = c(0, 0, 2))

Coefficients:
          ma1     ma2  intercept
      -0.4324  0.3517    -0.0598
s.e.   0.0459  0.0366     0.0430

sigma^2 estimated as 1.093: log likelihood=-731.87, aic=1471.73

> x4.fit1<-arima(x2, order=c(0,0,3))
> x4.fit1
Call:
arima(x = x2, order = c(0, 0, 3))

Coefficients:
         ma1     ma2      ma3  intercept
      0.1659  0.5062  0.0284    -0.0978
s.e.  0.0430  0.0366  0.0396     0.0751

sigma^2 estimated as 0.9784: log likelihood=-704.32, aic=1418.63
```

Hence the fitted models are

$$X_t = -0.0598_{(.0430)} + Z_t - 0.4324_{(.0459)} Z_{t-1} + 0.3517_{(.0366)} Z_{t-2},$$

with $Z_t \sim N(0, 1.093)$, for the first series, and

$$X_t = -0.0978_{(.0751)} + Z_t + 0.1659_{(.0430)} Z_{t-1} + 0.5062_{(.0366)} Z_{t-2} - 0.0284_{(.0396)} Z_{t-3}$$

with $Z_t \sim N(0, 0.9784)$, for the second series. Note that the 95% confidence intervals of the intercepts in the two fitted models, $-0.0598 \pm 1.96 \cdot 0.0430$ and $-0.0978 \pm 1.96 \cdot 0.0751$, cover zero, indicating that the means of the two MA processes might be zero. MA models with zero means can be fitted to the series using the following `R` commands.

```
> x3.fit2<-arima(x1, order=c(0,0,2), include.mean=FALSE)
> x3.fit2
Call:
arima(x = x1, order = c(0, 0, 2), include.mean = FALSE)

Coefficients:
          ma1     ma2
      -0.4278  0.3547
s.e.   0.0460  0.0365

sigma^2 estimated as 1.097: log likelihood=-732.83, aic=1471.65

> x4.fit2<-arima(x2, order=c(0,0,3), include.mean=FALSE)
> x4.fit2

Call:
arima(x = x2, order = c(0, 0, 3), include.mean = FALSE)

Coefficients:
         ma1     ma2      ma3
      0.1687  0.5081  0.0309
s.e.  0.0430  0.0365  0.0395

sigma^2 estimated as 0.9817: log likelihood=-705.16, aic=1418.32
```

Therefore, the fitted MA models with zero means can be summarized as

$$X_t = Z_t - 0.4278_{(.0460)} Z_{t-1} + 0.3547_{(.0365)} Z_{t-2},$$

with $Z_t \sim N(0, 1.097)$, for the first series, and

$$X_t = Z_t + 0.1687_{(.0430)} Z_{t-1} + 0.5081_{(.0365)} Z_{t-2} + 0.0309_{(.0395)} Z_{t-3},$$

with $Z_t \sim N(0, 0.9817)$, for the second series. For the first series, the 95% confidence intervals of coefficients of Z_{t-1} and Z_{t-2} are $-0.4278 \pm 1.96 \cdot 0.0460$ and $0.3547 \pm 1.96 \cdot 0.0365$, respectively, and hence cover the true values of the coefficients -0.5 and 0.4, respectively. For the second series, the 95% confidence intervals of coefficients of Z_{t-1}, Z_{t-2}, and Z_{t-3} are $0.1687 \pm 1.96 \cdot 0.0430$, $0.5081 \pm 1.96 \cdot 0.0365$, and $0.0309 \pm 1.96 \cdot 0.0395$, respectively, or equivalently, $(0.0844, 0.2530)$, $(0.4366, 0.5796)$, and $(-0.0465, 0.1083)$, respectively. Note that neither of these intervals does not cover the true values of the corresponding coefficients, 1/3, 1/5, and $-1/9$, indicating that the fitted MA(3) model is different from the true MA(3) model. We shall point out that such consistency does not necessarily imply we obtain a "wrong" model; in fact, when the true model becomes complicated, it is possible to obtain a different model that fits the observed time series as well as the true model does.

4.4 Estimating Parameters of an ARMA Model

Suppose now that a model with both AR and MA terms is thought to be appropriate for a given time series. The estimation problems for an ARMA model are similar to those for an MA model in that an iterative procedure has to be used. The residual sum of squares can be calculated at every point on a suitable grid of the parameter values, and the values, which give the minimum sum of squares may then be assessed. Alternatively some sort of optimization procedure may be used.

As an example, consider the ARMA$(1,1)$ process whose ac.f. decreases exponentially after lag 1 (see Example 3.2). This model may be recognized as appropriate if the sample ac.f. has a similar form. The model is given by

$$X_t - \mu = \alpha_1(X_{t-1} - \mu) + Z_t + \beta_1 Z_{t-1}.$$

Given N observations, $x_1, \ldots, x_N$, we guess values for $\widehat{\mu}, \widehat{\alpha}_1, \widehat{\beta}_1$, set $z_0 = 0$ and $x_0 = \widehat{\mu}$, and then calculate the residuals recursively by

$$\begin{aligned} z_1 &= x_1 - \widehat{\mu} \\ z_2 &= x_2 - \widehat{\mu} - \widehat{\alpha}_1(x_1 - \mu) - \widehat{\beta}_1 z_1 \\ &\quad \text{up to} \\ z_N &= x_N - \widehat{\mu} - \widehat{\alpha}_1(x_{N-1} - \mu) - \widehat{\beta}_1 z_{N-1}. \end{aligned}$$

The residual sum of squares $\sum_{t=1}^{N} z_t^2$ may then be calculated. Then other values of $\widehat{\mu}, \widehat{\alpha}_1, \widehat{\beta}_1$ may be tried until the minimum residual sum of squares is found.

Many variants of the above estimation procedure have been studied – see, for example, the reviews by Priestley (1981, Chapter 5) and Kendall et al. (1983, Chapter 50). Nowadays exact maximum likelihood estimates are often preferred, since the extra computation involved is no longer a limiting factor. The conditional least squares estimates introduced above are conceptually easier to understand and can also be used as starting values for exact maximum likelihood estimation. The Hannan–Rissanen recursive regression procedure (e.g. see Granger and Newbold, 1986) is primarily intended for model identification but can additionally be used to provide starting values as well. The Kalman filter (see Section 10.1.4) may be used to calculate exact maximum likelihood estimates to any desired degree of approximation. We will say no more about this important, but rather advanced, topic here. Modern computer software for time-series modelling should incorporate sound estimation procedures for fitting MA and ARMA models, though it is worth finding out exactly what method is used, provided this information is given in the software documentation (as it should be, but often isn't!). For short series, and models with roots near the unit circle, it is worth stressing that different estimation procedures can give noticeably different results. It is also worth noting that least squares estimates can be biased for short series (Ansley and Newbold, 1986)

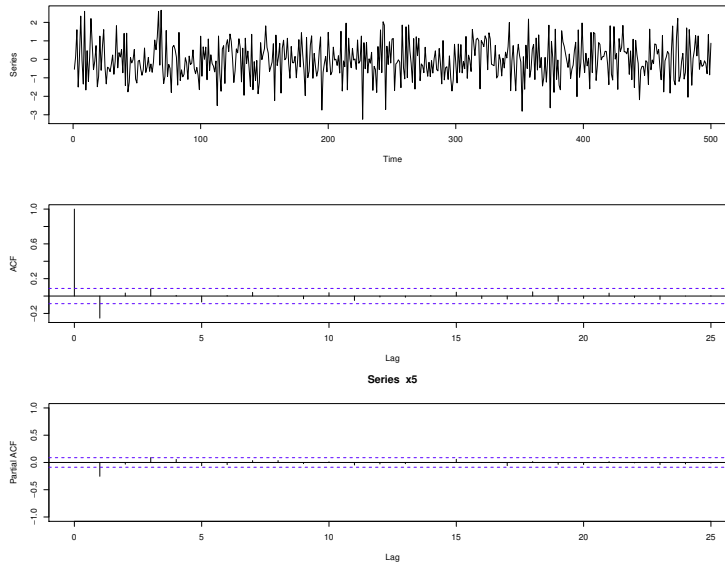

Figure 4.4 *A simulated ARMA(2,2) process (Top),* $X_t = 0.4X_{t-1} - 0.3X_{t-2} + Z_t - 0.6Z_{t-1} + 0.5Z_{t-2}$*, and its sample ac.f.'s (Middle) and partial ac.f.'s (Bottom).*

Example 4.3

We simulate an ARMA(2,2) process with $N = 500$ observations,

$$X_t = 0.4X_{t-1} - 0.3X_{t-2} + Z_t - 0.6Z_{t-1} + 0.5Z_{t-2}, \quad Z_t \sim N(0,1).$$

We then use the simulated series to compute its sample ac.f. and the sample partial ac.f. Figure 4.4 shows the simulated series and its sample ac.f. and sample partial ac.f.

```
> set.seed(6)
> x5<-arima.sim(list(order=c(2,0,2), ar=c(0.4,-0.3),
      ma=c(-0.6,0.5)), 500)
```

To fit an ARMA model to the series, we need to specify the orders of AR and MA components. However, it is difficult to determine the orders of the AR and MA components from Figure 4.4. As model identification tools are introduced in the next section, we consider the issue of estimating model parameters when the orders of AR and MA components are given. The following commands in R show how to fit an ARMA(1,1) model to the series.

```
> x5.fit1<-arima(x5, order=c(1,0,1))
> x5.fit1
```

```
Call:
arima(x = x5, order = c(2, 0, 2))

Coefficients:
          ar1      ar2      ma1     ma2  intercept
      -0.0404  -0.4311  -0.2188  0.4872    -0.0445
s.e.   0.2843   0.1498   0.2838  0.1162     0.0376

sigma^2 estimated as 0.9503: log likelihood=-696.8, aic=1405.6
```

Then we may conclude the fitted ARMA(1,1) model is

$$X_t = -0.0445_{(.0376)} - 0.0404_{(.2843)}X_{t-1} - 0.4311_{(.1498)}X_{t-2} + Z_t \\ -0.2188_{(.2838)}Z_{t-1} + 0.4872_{(.1162)}Z_{t-2}$$

with $Z_t \sim N(0, 0.9503)$. Note that the 95% confidence intervals of the intercept in the fitted model, $-0.0445 \pm 1.96 \cdot 0.0376$, covers zero, suggesting that the means of the ARMA(2,2) process might be zero. ARMA models with zero means can be fitted to the series using the following R commands.

```
> x5.fit2<-arima(x5, order=c(2,0,2), include.mean=FALSE)
> x5.fit2

Call:
arima(x = x5, order = c(2, 0, 2), include.mean = FALSE)

Coefficients:
          ar1      ar2      ma1     ma2
      -0.0267  -0.4300  -0.2305  0.4915
s.e.   0.2928   0.1514   0.2919  0.1147

sigma^2 estimated as 0.953: log likelihood=-697.5, aic=1405
```

Hence the fitted ARMA(2,2) model with zero means is

$$X_t = -0.0267_{(.2928)}X_{t-1} - 0.4300_{(.1514)}X_{t-2} + Z_t \\ -0.2305_{(.2919)}Z_{t-1} + 0.4915_{(.1147)}Z_{t-2},$$

with $Z_t \sim N(0, 0.953)$. The 95% confidence intervals of the coefficients of X_{t-1}, X_{t-2}, Z_{t-1}, and Z_{t-2} are $-0.0267 \pm 1.96 \cdot 0.2928$, $-0.4300 \pm 1.96 \cdot 0.1514$, $-0.2305 \pm 1.96 \cdot 0.2919$, and $0.4915 \pm 1.96 \cdot 0.1147$, respectively, which are equivalents to $(-0.6006, 0.5472)$, $(-0.7267, -0.1332)$, $(-0.8026, 0.3416)$, and $(0.2667, 0.7163)$, respectively. These intervals cover the true values of the coefficients of X_{t-1}, X_{t-2}, Z_{t-1}, and Z_{t-2}, which are 0.4, -0.3, -0.6, and 0.5, respectively, suggesting the fitted model is not very different from the true model.

4.5 Model Identification Tools

Model building is a key element of most statistical work. Some general remarks on how to formulate an appropriate model for a given time series are given in Section 4.8. As well as finding out about any background knowledge, the analyst will typically look at the time plot of the data and at various diagnostic tools. The two standard tools, which are used to identify an appropriate autoregressive moving average (ARMA) model for a given stationary time series (see Section 4.4), are the sample autocorrelation function (ac.f.) and the sample partial ac.f. This choice is often made subjectively, using the analyst's experience to match an appropriate model to the observed characteristics. However, various additional diagnostic tools are also available, and this section gives a brief introduction to some of them. Detailed reviews of methods for determining the order of an ARMA process are given by de Gooijer et al. (1985) and Choi (1992); see also Newbold (1988).

An alternative to the partial ac.f. is the **inverse ac.f.**, whose use in identifying ARMA models is described by Chatfield (1979). The inverse ac.f. of the general ARMA model, as written in Equation (3.19), namely

$$\phi(B)X_t = \theta(B)Z_t$$

is exactly the same as the ordinary ac.f. of the corresponding inverse ARMA model given by

$$\theta(B)X_t = \phi(B)Z_t,$$

where θ and ϕ are interchanged. It turns out that the inverse ac.f. has similar properties as the partial ac.f. of the ARMA process in that it 'cuts off' at lag p for an AR(p) process but generally dies out slowly for MA and ARMA processes. The inverse ac.f. often contains more information than the partial ac.f., extends easily to seasonal ARMA models, and is a viable competitor to the partial ac.f.

Instead of (or in addition to) subjectively examining functions like the ac.f., two alternative approaches to model selection are to carry out a series of **hypothesis tests** or to use a **model-selection criterion**. Econometricians tend to favour the former approach and test null hypotheses such as normality, constant variance and non-linearity. Statisticians tend to favour the latter approach and several such model-selection criteria have been proposed. Given a class of models, such as the ARIMA class, the idea is to choose a model from that class so as to optimize a suitably chosen function of the data.

What criterion should we use to select a model in the 'best' way? It is not sensible to simply choose a model to give the best fit by minimizing the residual sum of squares, as the latter will generally decrease as the number of parameters is increased regardless of whether the additional complexity is really worthwhile. There is an alternative fit statistic, called adjusted-R^2, which makes some attempt to take account of the number of parameters fitted, but more sophisticated model-selection statistics are generally preferred.

Akaike's Information Criterion (AIC) is the most commonly used and is given (approximately) by:

$$\text{AIC} \quad = \quad -2\ \ln(\text{max. likelihood}) + 2r$$

where r denotes the number of independent parameters that are fitted for the model being assessed. Thus the AIC essentially chooses the model with the best fit, as measured by the likelihood function, subject to a penalty term, to prevent over-fitting, that increases with the number of parameters in the model. For an ARMA(p, q) model, note that $r = p + q + 1$ as the residual variance is included as a parameter. Ignoring arbitrary constants, the first (likelihood) term is usually approximated by $N \ln(S/N)$, where S denotes the residual sum of squares, and N is the number of observations. It turns out that the AIC is biased for small samples, and a bias-corrected version, denoted by AIC_C, is increasingly used. The latter is given (approximately) by replacing the quantity $2r$ in the ordinary AIC with the expression $2rN/(N-r-1)$. The AIC_C is recommended, for example, by Brockwell and Davis (1991, Section 9.3) and Burnham and Anderson (2002).

An alternative, widely used criterion is the **Bayesian Information Criterion** (BIC) that essentially replaces the term $2r$ in the AIC with the expression $(r + r \ln N)$. Thus it penalizes the addition of extra parameters more severely than the AIC. **Schwartz's Bayesian criterion** is yet another alternative that is similar to BIC in its dependence on $\ln N$, but replaces $(r + r \ln N)$ with $r \ln N$.

Several other possible criteria have been proposed including Parzen's **autoregressive transfer function criterion (CAT)** and Akaike's **final prediction error (FPE)** criterion, which are both primarily intended for choosing the order of an AR process. Note that all the above criteria may not have a unique minimum and depend on assuming that the data are (approximately) normally distributed. Priestley (1981, Chapter 5) and Burnham and Anderson (2002) give a general review of these criteria. Following the results in Faraway and Chatfield (1998), we recommend to use the AIC_C or BIC. Computer packages routinely produce numerical values for several such criteria so that analysts can pick the one they like best. The guiding principle throughout is to apply the Principle of Parsimony, introduced in Section 3.8. In brief this says, '*Adopt the simplest acceptable model*'. Of course, there is a real danger that the analyst will try many different models, pick the one that appears to fit best according to one of these criteria, but then make predictions as if certain that the best-fit model is the true model. Further remarks on this problem are made in Section 14.2.

As noted above, an alternative approach to model selection relies on carrying out a series of hypothesis tests. However, little will be said here about this approach, because the author prefers to rely on the subjective interpretation of diagnostic tools, such as the ac.f., allied to the model-selection criteria given above. There are three reasons for this preference:

1. A model-selection criterion gives a numerical-valued ranking of all models, so that the analyst can see whether there is a clear winner or, alternatively, several close competing models.
2. Some model-selection criteria can be used to compare non-nested[2] models, as would arise, for example, when trying to decide whether to compute forecasts using an ARIMA, neural network or econometric model. It is very difficult to use hypothesis tests to compare non-nested models.
3. A hypothesis test requires the specification of an appropriate null hypothesis, and effectively assumes the existence of a true model that is contained in the set of candidate models.

Example 4.4

We take the ARMA(2,2) series simulated in Example 4.3 as an example. The following `R` commands show how to select a best model using AIC. We fit a set of ARMA models and compute the AIC for each fitted model.

```
> x5.aic<-matrix(0,4,4)
> for (i in 0:3) for (j in 0:3) {
    x5.fit<-arima(x5, order=c(i,0,j))
    x5.aic[i+1,j+1]<-x5.fit$aic
  }
> x5.aic
         [,1]     [,2]     [,3]     [,4]
[1,] 1439.570 1409.792 1408.332 1406.792
[2,] 1408.238 1409.955 1408.699 1408.510
[3,] 1409.748 1410.947 1405.603 1407.509
[4,] 1407.752 1409.147 1407.489 1409.022
```

The model with the minimum AIC (1405.603) is ARMA(p, q) with $p = 2$ and $q = 2$. Given the values of p and q, the inference on the ARMA(2, 2) model in the next step is same as that in Section 4.4. We refer to the reader to Example 4.3 for further analysis of the series "`x5`". Model selection procedures with criteria other than AIC can be similarly carried out as this example.

4.6 Testing for Unit Roots

In the previous subsection, we referred to the difficulty in distinguishing between a long-memory and a non-stationary process. More generally, a major problem in practice, especially with relatively short series, is distinguishing between a non-stationary process and one that is 'nearly non-stationary' in some general sense. There is no simple mathematical definition of the latter

[2] Two models are nested when one is a subset of the other. For example, a first-order AR model is a special case of a second-order AR model when the coefficient at lag two is zero. This makes it relatively easy to test the null hypothesis that a first-order model is adequate.

description, but we have in mind processes whose ac.f. decays slowly. With finite samples, the sample ac.f. of non-stationary and nearly non-stationary processes will both decay (very) slowly towards zero. This makes it hard to distinguish between these different types of processes on the basis of sample data.

Long-memory stationary processes are 'nearly non-stationary' in the above sense, as are ordinary AR stationary processes with roots near the unit circle. For example, the (stationary) AR(1) process with parameter 0.95, namely, $X_t = 0.95X_{t-1} + Z_t$, will yield data that, for short series, will have properties that look much like those of data generated by a (non-stationary) random walk, namely, $X_t = X_{t-1} + Z_t$. The time plots and sample ac.f.s will have similar characteristics.

For some purposes, it won't matter too much if we identify the wrong model. For example, the short-term forecasts from an AR(1) model with parameter 0.95 and from a random walk are quite close to each other. However, the long-term forecasts will be substantially different. For the AR(1) model the long-run forecasts revert to the overall mean, whereas those for the random walk are all equal to the most recent observed value. Moreover, the forecast error variance remains finite for a stationary model but increases without bound for a non-stationary model. Thus it can be very important to distinguish between the two different types of processes.

One way of tackling this problem in regard to ARIMA models, is to ask whether the differencing parameter d is exactly equal to one. If it is, then that would mean that a *unit root* is present. Put another way, if the original data are non-stationary but the first differences are stationary, then a unit root is said to be present. One way of trying to answer this question is to carry out an appropriate significance test, and there is a large and growing literature in econometrics (e.g. DeJong and Whiteman, 1993; Hamilton, 1994, Chapter 15; Diebold and Kilian, 2000; Diebold, 2001, Chapter 12) on **testing for a unit root**.

Several types of tests designed for assessing different null and alternative hypotheses are available. Perhaps the most important example is the so-called **augmented Dickey–Fuller test**, details of which may be found, for example, in Enders (1995, Chapter 4) or Harvey (1993, Section 5.4). The tests generally take the *null* hypothesis to be that there *is* a unit root (so that $d = 1$), partly on the grounds that many economic series are known to be close to a random walk or ARIMA$(0, 1, 0)$ process. The following commands in `R` show how to use the Augmented Dickey-Fuller test, `adf.test` (in `tseries` library), to test the null hypothesis that a series has a unit root.

```
> set.seed(8)
> y1<-cumsum(rnorm(500))
> library("tseries")
> y1.adf<-adf.test(y1, alternative="stationary")
> y1.adf
```

```
Augmented Dickey-Fuller Test

data:  y1
Dickey-Fuller = -2.6149, Lag order = 7, p-value = 0.318
alternative hypothesis: stationary
```

However, it is not obvious that this is always a sensible choice of null hypothesis, and the statistician is likely to be interested in assessing an appropriate value for d, as well as for p and q, with no prior assumptions other than assuming that some member of the ARIMA class of models is appropriate. Whereas the econometrician tends to rely on a series of tests, not only for the presence of a unit root but also for other features such as constant variance and uncorrelated residuals, the statistician is more likely to choose a model from a general class of models by looking at appropriate diagnostic tools and perhaps minimizing a criterion such as AIC — see Section 14.1.

Unfortunately, tests for unit roots generally have poor power, even for moderate size samples, quite apart from the question as to whether and when it is sensible to take $d = 1$ as the null hypothesis. The alternative hypothesis is typically 'close' to the null hypothesis and the testing procedure can be sensitive to the way that lag structure is modelled. Newbold et al. (1993) go so far as to say that "testing for unit autoregressive roots is misguided", although there is some simulation evidence (Diebold and Kilian, 2000) that unit root tests can help to select models that give superior forecasts. The topic is too complex and unresolved to make a definitive assessment, but a formal test for a unit root can only ever be a small contribution to the important task of modelling (near) non-stationary behaviour. The fact that we cannot reject a unit root, does not mean that we should necessarily impose one, as, for example, if we want an explicit estimate of any trend. Conversely, there could still be practical reasons why we might wish to difference our data, even when a unit root is rejected, as, for example, when a model for the differenced data appears to be more robust to unexpected changes. The key question is not whether a unit-root test helps select the 'true model' (that probably doesn't exist), but whether the chosen model (that we fully expect to be misspecified in some respects) helps to solve the given problem better than alternatives.

It may be helpful at this point to make some further remarks on different types of trend. Section 2.5 distinguished between a global and a local linear trend. Econometricians make a similar distinction but use a different vocabulary, by referring to what they call **difference-stationary** series, where stationarity can be induced by first differencing, and **trend-stationary** series where the deviations from a deterministic trend are stationary. The random walk is a simple example of a difference-stationary series, whereas a global trend model with stationary residuals is trend-stationary. In the latter case, the trend is deterministic, while econometricians generally say that there is a **stochastic trend** for difference-stationary series. Most economic series are difference-stationary rather than trend-stationary, and there is empirical

evidence that difference-stationary models tend to give better out-of-sample forecasts for non-stationary data.

Spurious autocorrelations can readily be induced in a series showing trend, either by mistakenly removing a deterministic trend from difference-stationary data, or by differencing trend-stationary data. This illustrates the importance of identifying the appropriate form of trend so that an appropriate level of differencing may be applied. However, from what is said above, it is clear that it is generally rather difficult to distinguish between the cases (1) $d = 1$, (2) $0 < d < 1$, (3) $d = 0$ and (4) trend-stationarity. This is perhaps not too surprising given that it is possible to construct examples where models with different orders of differencing or with different trend structures can be made in some sense arbitrarily close. We illustrate this problem by considering the AR(1) model as the coefficient ϕ approaches one. For $\phi = 0.99$, we have a stationary model with $d = 0$, but, when $\phi = 1$, then the model reduces to a difference-stationary random walk model with $d = 1$.

Alternatively, consider the model

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}$$

When $\alpha = 0.95$ and $\beta = -0.9$, the process is a stationary ARMA$(1, 1)$ model so that $d = 0$ when expressed as an ARIMA$(1, 0, 1)$ model. When the value of α is changed slightly to $\alpha = 1$, the operator $(1-B)$ appears on the left-hand side of the equation so that the process becomes a non-stationary ARIMA$(0, 1, 1)$ model with $d = 1$. However, if in addition we now change the value of β slightly to -1, then the operator $(1 - B)$ will appear on both sides of the equation and can be cancelled. Then X_t is white noise and we are back to the stationary case ($d = 0$). With a sample size of say 100, it would be nearly impossible to distinguish between these three cases.

The implications of the above discussion are that identification of an appropriate trend model is a key element of time-series modelling. However, it can be very difficult and a unit root test may not help much. Rather than agonize about the 'correct' order of differencing to use, there is much to be said for fitting a model that makes few assumptions about the form of the trend, but rather is designed to be adaptive in form and to be *robust* to changes in the underlying model. For this reason, state-space models may be preferred to ARIMA models, and a local trend model may be preferred to a global trend model.

4.7 Estimating Parameters of an ARIMA Model

In practice, many time series are clearly non-stationary, and so the stationary models we have studied so far cannot be applied directly. In Box–Jenkins ARIMA modelling, the general approach is to *difference* an observed time series until it appears to come from a stationary process. An AR, MA or ARMA model may then be fitted to the differenced series as described in Sections 4.2–4.4. It is often found that first-order differencing of non-seasonal

data is adequate — see Section 3.9 and Example 14.2 — although second-order differencing is occasionally required. The resulting model for the undifferenced series is the fitted ARIMA model. For seasonal data, seasonal differencing may also be required (see Section 4.8 and Example 14.3).

It is unnecessary to say anything further about fitting ARIMA models because the possible use of differencing is the only feature that changes from fitting ARMA models.

4.8 Box–Jenkins Seasonal ARIMA Models

In practice, many time series contain a seasonal periodic component, which repeats every s observations. For example, with monthly observations, where $s = 12$, we may typically expect X_t to depend on values at annual lags, such as X_{t-12}, and perhaps X_{t-24}, as well as on more recent non-seasonal values such as X_{t-1} and X_{t-2}. Box and Jenkins (1970) generalized the ARIMA model to deal with seasonality, and defined a general multiplicative seasonal ARIMA (SARIMA) model as

$$\phi_p(B)\Phi_P(B^s)W_t = \theta_q(B)\Theta_Q(B^s)Z_t, \tag{4.16}$$

where B denotes the backward shift operator, $\phi_p, \Phi_P, \theta_q, \Theta_Q$ are polynomials of order p, P, q, Q, respectively, Z_t denotes a purely random process and

$$W_t = \nabla^d \nabla_s^D X_t \tag{4.17}$$

denotes the differenced series. If the integer D is not zero, then seasonal differencing is involved (see Section 2.6). The above model is called a SARIMA model of order $(p, d, q) \times (P, D, Q)_s$.

The SARIMA model looks rather complicated at first sight, so let us look at the different components of the model in turn. First, the differenced series $\{W_t\}$ is formed from the original series $\{X_t\}$ by appropriate differencing to remove non-stationary terms. If d is non-zero, then there is simple differencing to remove trend, while seasonal differencing ∇_s may be used to remove seasonality. For example, if $d = D = 1$ and $s = 12$, then

$$\begin{aligned} W_t &= \nabla\nabla_{12}X_t = \nabla_{12}X_t - \nabla_{12}X_{t-1} \\ &= (X_t - X_{t-12}) - (X_{t-1} - X_{t-13}). \end{aligned}$$

In practice, the values of d and D are usually zero or one, and rarely two.

Now let us look at the seasonal AR term, $\Phi_P(B^s)$. Suppose for simplicity that $P = 1$. Then $\Phi_1(B^s)$ will be of the form $(1 - C \times B^s)$, where C denotes a constant, which simply means that W_t will depend on W_{t-s}, since $B^s W_t = W_{t-s}$. Similarly, a seasonal MA term of order one means that W_t will depend on Z_{t-s} as well as on Z_t.

As an example, consider a SARIMA model of order $(1, 0, 0) \times (0, 1, 1)_{12}$, where we note $s = 12$. Here we have one non-seasonal AR term, one seasonal

MA term and one seasonal difference. Then Equations (4.16) and (4.17) can be written as

$$(1 - \alpha B)W_t = (1 + \Theta B^{12})Z_t,$$

where $W_t = \nabla_{12} X_t$ and Θ is a constant parameter (rather than a function as in Equation (4.16)). It may help to write this out in terms of the original observed variable X_t as

$$X_t = X_{t-12} + \alpha(X_{t-1} - X_{t-13}) + Z_t + \Theta Z_{t-12},$$

so that X_t depends on X_{t-1}, X_{t-12} and X_{t-13} as well as the innovations at times t and $(t - 12)$. You may be surprised to see that X_{t-13} is involved, but this is a consequence of mixing first-order autocorrelation with seasonal differencing.

When fitting a seasonal model to data, the first task is to assess values of d and D, which remove most of the trend and seasonality and make the series appear to come from a stationary process. Then the values of p, P, q and Q need to be assessed by looking at the ac.f. and partial ac.f. of the differenced series and choosing a SARIMA model whose ac.f. and partial ac.f. are of similar form. Finally, the model parameters may be estimated by some suitable iterative procedure. Full details are given by Box et al. (1994, Chapter 9), but the wide availability of suitable computer software means that the average analyst does not need to worry about the practical details of fitting a particular model. Instead, the analyst can concentrate on choosing a sensible model to fit and ensuring that appropriate diagnostic checks are carried out.

Example 4.5

Figure 1.4 shows the domestic sales of Australian fortified wine by winemakers in successive quarters over a 28-year period. For convenience, we show the time series again in the top panel of Figure 4.5. Also shown in the middle and bottom panels of Figure 4.5 are the sample ac.f. and sample partial ac.f. of the sales series. Note that the series consists of quarterly observations, so we denote X_t the sales at quarter t. The sale series demonstrates both a trend and seasonal variation, and the analysis below shows how to use Box-Jenkins models to fit a time series model to the data.

```
> library("tseries")
> wine<-read.csv("mydata/aus_wine_sales.csv", header=F)
> wine.ts<-ts(wine[,2], start=c(1985,1), frequency=4)

> par(mfrow=c(3,1), mar=c(4,4,4,4))  ### Figure 4.5
> plot(wine.ts, type="l", xlab="Year",
    ylab="Sales in thousand liters")
> acf(wine[,2], 25, xlab="Lag", ylab="ACF", main="")
> acf(wine[,2], 25, type="partial", xlab="Lag",
    ylab="Partial ACF", main="")
```

First, regardless of the trend effect, we find that the sale series shows cyclic behavior every year. As the series consists of quarterly observations, it is reasonable to conclude the period of the seasonal effect is $d = 4$. This motivates us to remove the seasonal effect by taking seasonal difference. Consider the seasonal differencing operator $\nabla_d = 1 - B^d$ with $d = 4$, and we use it to obtain the seasonal differenced series Y_t,

$$Y_t = \nabla_d X_t = (1 - B^d)X_t, \qquad d = 4.$$

```
> dwine<-diff(wine[,2], lag=4)
> par(mfrow=c(3,1), mar=c(4,4,4,4))  ### Figure 4.6
> plot(diff(wine.ts, lag=4), type="l", xlab="Year",
    ylab="Differenced Series")
> acf(dwine, 25, xlab="Lag", ylab="ACF", main="")
> acf(dwine, 25, type="partial", xlab="Lag",
    ylab="Partial ACF", main="")
```

Figures 4.6 show the time series plot, the sample ac.f., and the sample partial ac.f. of the differenced series Y_t. Note that the differenced series does not show any seasonal effect in the time series plot; furthermore, the sample ac.f. and sample partial ac.f. of Y_t are significantly less than those of the original series X_t. We then try to fit an ARMA model to Y_t. Since the order of the ARMA model is not given, we fit a set of ARMA models and compute their AICs.

```
> dwine.aic<-matrix(0,5,5)
> for (i in 0:4) for (j in 0:4) {
    dwine.fit<-arima(dwine, order=c(i,0,j))
    dwine.aic[i+1,j+1]<-dwine.fit$aic
  }
> dwine.aic
         [,1]     [,2]     [,3]     [,4]     [,5]
[1,] 1650.682 1652.504 1653.521 1650.854 1637.755
[2,] 1652.489 1654.140 1655.125 1644.301 1638.960
[3,] 1654.209 1651.112 1643.001 1630.410 1632.405
[4,] 1651.993 1645.663 1632.891 1632.401 1634.360
[5,] 1638.412 1639.506 1631.802 1633.801 1633.935
```

The model with minimum AIC (1630.410) is ARMA(p, q) with $p = 2$ and $q = 3$. Hence we fit an ARMA(2,3) model to the differenced series Y_t.

```
> dwine.fit<-arima(dwine, order=c(2,0,3))
> dwine.fit
```

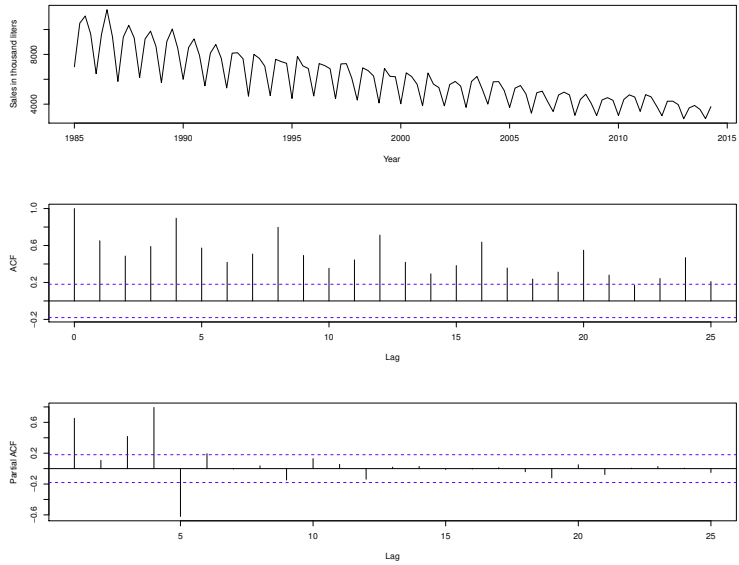

Figure 4.5 *The series of domestic sales of Australian fortified wine (Top) and its sample ac.f. (Middle) and sample partial ac.f. (Bottom).*

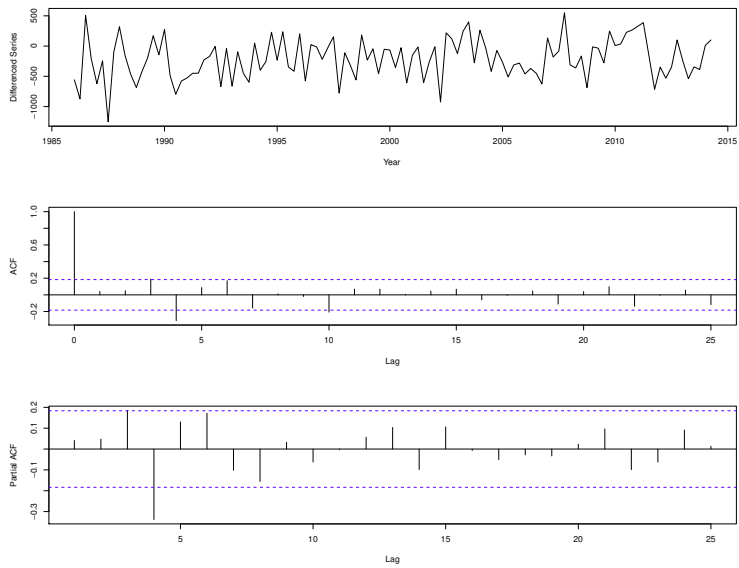

Figure 4.6 *The seasonal differenced series of domestic sales of Australian fortified wine (Top) and its sample ac.f. (Middle) and sample partial ac.f. (Bottom).*

```
Call:
arima(x = dwine, order = c(2, 0, 3))

Coefficients:
          ar1      ar2     ma1     ma2     ma3  intercept
      -1.0226  -0.8662  1.1775  1.1854  0.4210  -213.3700
s.e.   0.0691   0.0717  0.1011  0.1348  0.0915    35.2152

sigma^2 estimated as 82830: log likelihood=-808.21, aic=1630.41
```

The fitted result suggests the following ARMA(2,3) model for Y_t.

$$Y_t = -213.3700_{(35.2152)} - 1.0226_{(.0691)}Y_{t-1} - 0.8662_{(0.0717)}Y_{t-2} + Z_t \\ +1.1775_{(0.1011)}Z_{t-1} + 1.1854_{(0.1348)}Z_{t-2} + 0.4210_{(0.0915)}Z_{t-3},$$

where Z_t are i.i.d. series with $Z_t \sim N(0, 82830)$. Note that the next step one should perform is residual diagnostics, which will be discussed in the next section.

4.9 Residual Analysis

When a model has been fitted to a time series, it is advisable to check that the model really does provide an adequate description of the data. As with most statistical models, this is usually done by looking at the **residuals**, which are generally defined by

$$\text{residual} \;=\; \text{observation} \;-\; \text{fitted value}.$$

For a univariate time-series model, the fitted value is the one-step-ahead forecast so that the residual is the one-step-ahead forecast error. For example, for the AR(1) model, $X_t = \alpha X_{t-1} + Z_t$ where α is estimated by least squares, the fitted value at time t is $\widehat{\alpha} x_{t-1}$ so that the residual corresponding to the observed value, x_t, is

$$\widehat{z}_t = x_t - \widehat{\alpha} x_{t-1}.$$

Note the 'hat' on z_t as the residual is the estimated error term. Of course if α were known exactly, then the exact error $z_t = x_t - \alpha x_{t-1}$ could be calculated, but this situation never arises in practice (except in simulated exercises).

If we have a 'good' model, then we expect the residuals to be 'random' and 'close to zero', and model validation usually consists of plotting residuals in various ways to see whether this is the case. With time-series models we have the added feature that the residuals are ordered in time and it is natural to treat them as a time series.

Two obvious steps are to plot the residuals as a time plot, and to calculate the correlogram of the residuals. The time plot will reveal any outliers and any obvious autocorrelation or cyclic effects. The correlogram of the residuals will enable autocorrelation effects to be examined more closely. Let $r_{z,k}$ denote

the sample autocorrelation coefficient at lag k of the residuals $\{\hat{z}_t\}$. If we could fit the true model[3], with the correct model parameter values, then the true errors $\{z_t\}$ form a purely random process and, from Section 4.1, their correlogram is such that each autocorrelation coefficient is approximately normally distributed, with mean 0 and variance $1/N$, for reasonably large values of N. Of course, in practice, the true model is unknown and the correlogram of the residuals from the fitted model has somewhat different properties. For example, suppose we know the form of the model is an AR(1) process, but have to estimate the parameter. If the true value of the parameter is $\alpha = 0.7$, it can be shown that approximate 95% confidence limits for the true residual autocorrelations are at $\pm\ 1.3/\sqrt{N}$ for $r_{z,1}$, at $\pm\ 1.7/\sqrt{N}$ for $r_{z,2}$ and at $\pm\ 2/\sqrt{N}$ for values of $r_{z,k}$ at higher lags. Thus for lags greater than 2, the confidence limits *are* the same as for the correlogram of the true errors. On the other hand, if we fit the wrong form of the model, then the distribution of residual autocorrelations will be quite different, and we hope to get some 'significant' values so that the wrong model is rejected. The analysis of residuals from ARMA processes is discussed more generally by Box et al. (1994, Chapter 8). As in the AR(1) example above, it turns out that $1/\sqrt{N}$ supplies an *upper bound* for the standard error of the residual autocorrelations, so that values, which lie outside the range $\pm\ 2\sqrt{N}$, are significantly different from zero at the 5% level and give evidence that the wrong form of model has been fitted.

Instead of looking at the residual autocorrelations one at a time, it is possible to carry out what is called a **portmanteau lack-of-fit test**. This looks at the first K values of the residual correlogram all at once. The test statistic is

$$Q = N \sum_{k=1}^{K} r_{z,k}^2, \tag{4.18}$$

where N is the number of terms in the differenced series and K is typically chosen in the range 15 to 30. If the fitted model is appropriate, then Q should be approximately distributed as χ^2 with $(K - p - q)$ degrees of freedom, where p, q are the number of AR and MA terms, respectively, in the model. Unfortunately the χ^2 approximation can be rather poor for $N < 100$, and various alternative statistics have been proposed. For example, the modified Ljung–Box–Pierce statistic, given by $N(N+2)\sum_{k=1}^{K} r_{z,k}^2/(N-k)$, is often used — see Box et al. (1994, Section 8.2.2). However, these tests have rather poor power properties (e.g. Davies and Newbold, 1979) and rarely give significant results, except when the model is obviously inadequate. Several other procedures for looking at residuals have also been proposed (e.g. Newbold, 1988, Section 4), but we can just 'look' at the first few values of $r_{z,k}$, particularly at lags 1, 2 and the first seasonal lag (if any), and see if any are significantly different from zero using the crude limits of $\pm\ 1.96/\sqrt{N}$. If they

[3] Whether, or not, a 'true model' really exists, is the subject of debate — see Section 14.2.

are, then the model can be modified in an appropriate way by putting in extra terms to account for the significant autocorrelation(s). However, if only one (or two) values of $r_{z,k}$ are just significant at lags having no obvious physical interpretation, (e.g. $k = 5$), then this should not be regarded as compelling evidence to reject the model.

Another statistic used for testing residuals is the Durbin–Watson statistic (e.g. Granger and Newbold, 1986, Section 6.2). This often appears in computer output. The statistic is defined by

$$d = \sum_{t=2}^{N} (\widehat{z}_t - \widehat{z}_{t-1})^2 / \sum_{t=1}^{N} \widehat{z}_t^2. \tag{4.19}$$

Now since

$$\sum_{t=2}^{N} (\widehat{z}_t - \widehat{z}_{t-1})^2 \simeq 2 \sum_{t=1}^{N} \widehat{z}_t^2 - 2 \sum_{t=2}^{N} \widehat{z}_t \widehat{z}_{t-1},$$

we find $d \simeq 2(1-r_{z,1})$, where $r_{z,1} = \Sigma \widehat{z}_t \widehat{z}_{t-1} / \Sigma \widehat{z}_t^2$ is the sample autocorrelation coefficient of the residuals at lag 1 (since the mean residual should be virtually zero). Thus, in this sort of application, the Durbin–Watson statistic is really the value $r_{z,1}$ in a slightly different guise. If the true model has been fitted, then we expect $r_{z,1} \simeq 0$, so that $d \simeq 2$. Thus a 'typical' value for d is around two and not zero. Furthermore, a test on d is asymptotically equivalent to a test on the value of $r_{z,1}$ for the residuals.

The Durbin–Watson statistic was originally proposed for use with multiple regression models as applied to time-series data. Suppose we have N observations on a dependent variable y, and m explanatory variables, say $x_1, \ldots, x_m$, and we fit the model

$$Y_t = \beta_1 x_{1t} + \cdots + \beta_m x_{mt} + Z_t \qquad t = 1, \ldots, N.$$

Having estimated the parameters $\{\beta_i\}$ by least squares, we want to see whether the error terms are really independent. The residuals are therefore calculated by

$$\widehat{z}_t = y_t - \widehat{\beta}_1 x_{1t} - \cdots - \widehat{\beta}_m x_{mt} \qquad t = 1, \ldots, N.$$

The statistic d may now be calculated, and the distribution of d under the null hypothesis that the z_t are independent has been investigated. Tables of critical values are available (e.g. Kendall et al., 1983) and they depend on the number of explanatory variables. Since d corresponds to the value of r_1 for the residuals, this test implies that we are only considering an AR(1) process as an alternative to a purely random process for z_t. Although it may be possible to modify the use of the Durbin–Watson statistic for models other than multiple regression models, it is usually better to look at the correlogram of the residuals as described earlier.

If the residual analysis indicates that the fitted model is inadequate in some way, then alternative models may need to be tried, and there are various

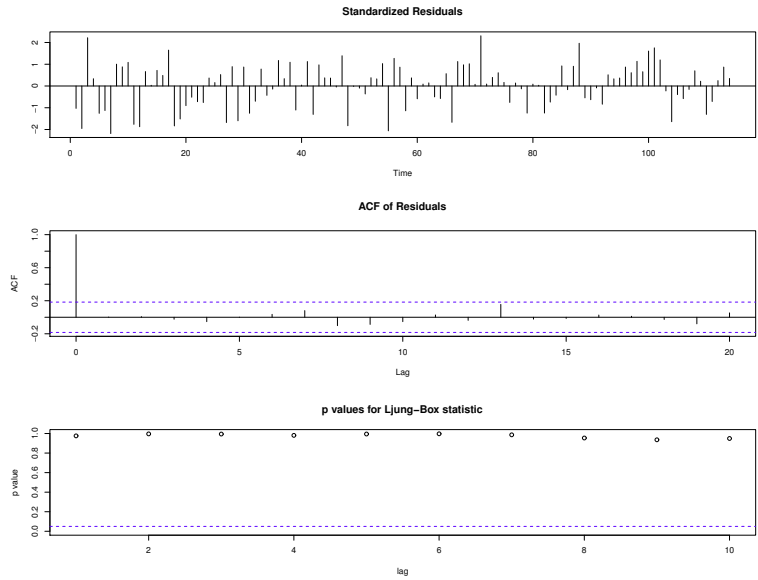

Figure 4.7 *The standardized residual series (Top), the sample ac.f. of residuals (Middle) and p-values for Ljung-Box statistic (Bottom).*

tools for comparing the fit of several competing models (see Section 4.5). An iterative strategy for building time-series models, which is an integral part of the Box–Jenkins approach, is discussed more fully in Sections 4.10 and 5.3.

Example 4.5 (Continued)

For the estimated ARMA(2,2) model in Example 4.5, we show the standardized residual series, the sample ac.f. of residuals, and p-values for Ljung-Box statistic. in Figure 4.7. We also carry out a portmanteau test for residuals. In particular, the p-value of the following test is much larger than 0.05, indicating the fitted model is appropriate.

```
> tsdiag(dwine.fit)
> Box.test(dwine.fit$resid)

        Box-Pierce test

data:  dwine.fit$resid
X-squared = 9e-04, df = 1, p-value = 0.976
```

4.10 General Remarks on Model Building

How do we find a suitable model for a given time series? The answer depends on various considerations, including the properties of the series as assessed by

a visual examination of the data, the number of observations available, the context and the way the model is to be used.

It is important to understand that model building has three main stages, which can be described as:

(1) Model formulation (or model specification)

(2) Model estimation (or model fitting)

(3) Model checking (or model verification).

Textbooks often concentrate on estimation, but say rather little about the more important topic of formulating the model. This is unfortunate because modern computer software makes model fitting straightforward for many types of models, so that the real problem is knowing which model to fit in the first place. For example, it is relatively easy nowadays to fit an ARIMA model, but it can still be difficult to know *when* ARIMA modelling is appropriate and *which* ARIMA model to fit.

Model checking is also very important, and the assessment of residuals is an essential step in the analysis. Modern software makes this relatively painless and may result in an initial model being discredited. Then an alternative model or models will be tried. Sometimes there are several cycles of model fitting as a model is modified and improved in response to residual checks or in response to additional data. Thus model building is an iterative, interactive process (see Section 14.2 and Chatfield, 1995a, Chapter 5).

This section concentrates on model formulation. The analyst should consult appropriate 'experts' about the given problem, ask questions to get relevant background knowledge, look at a time plot of the data to assess their more important features, and make sure that a proposed model is consistent with empirical and/or theoretical knowledge and with the objectives of the investigation.

There are many classes of time-series models to choose from. Chapter 3 introduced a general class of (univariate) models called ARIMA models, which includes AR, MA and ARMA models as special cases. This useful class of processes provides a good fit to many different types of time series and should generally be considered when more than about 50 observations are available. Another general class of models is the trend and seasonal type of model introduced in Chapter 2. Later in this book several more classes of models will be introduced, including multivariate models of various types, and structural models.

In areas such as oceanography and electrical engineering, long stationary series may occur. If a parametric model is required, an ARMA model should be considered. As well as the time plot, the correlogram and the partial ac.f. should be examined in order to identify an appropriate ARMA model. The model can then be fitted, checked and possibly modified in the usual way. However, as we will see later in Chapters 6 and 7, we may be more interested in the frequency properties of the time series, in which case an ARMA model may not be very helpful.

In many other areas, such as economics and marketing, non-stationary series often arise and in addition may be fairly short. It is possible to treat non-stationarity by differencing the observed time series until it becomes stationary and then fitting an ARMA model to the differenced series. For seasonal series, the seasonal ARIMA model may be used. However, it should be clearly recognized that when the variation of the systematic part of the time series (i.e. the trend and seasonality) is dominant, the effectiveness of the ARIMA model is mainly determined by the initial differencing operations and not by the subsequent fitting of an ARMA model to the differenced series, even though the latter operation is much more time-consuming. Thus the simple models discussed in Chapter 2 may be preferable for time series with a pronounced trend and/or large seasonal effect. Models of this type have the advantage of being simple, easy to interpret and fairly robust. In addition they can be used for short series where it is impossible to fit an ARIMA model. An alternative approach is to model the non-stationary effects explicitly, rather than to difference them away, and this suggests the use of a state-space or structural model as will be described in Chapter 10.

Sometimes the analyst may have several competing models in mind and then it may help to look at a *model-selection statistic* such as *Akaike's Information Criterion* (AIC). These statistics try to strike a balance between the need for a 'parsimonious' model, which uses as few parameters as possible, and a model that is too simple and overlooks important effects. A useful reference on model building, in general, and model-selection statistics and the Principle of Parsimony, in particular, is Burnham and Anderson (2002).

Whatever model is fitted, it is important to realise that it is only an approximation to the 'truth', and the analyst should always be prepared to modify a model in the light of new evidence. The effects of model uncertainty are discussed later in Section 14.2.

Exercises

4.1 Suppose that time series data $\{x_i\ ;\ i = 1, \ldots, N\}$ are generated from an AR(1) process with parameter α. Show that the variance of their sample mean can be approximated as follows for large N,

$$\mathrm{Var}(\overline{X}) \approx \frac{\sigma^2}{N}\Big(\frac{1+\alpha}{1-\alpha}\Big).$$

4.2 Derive the least squares estimates for an AR(1) process having mean μ (i.e. derive Equations (4.6) and (4.7), and check the approximations in Equations (4.8) and (4.9)).

4.3 Derive the least squares normal equations for an AR(p) process, taking $\widehat{\mu} = \bar{x}$, and compare with the Yule–Walker equations (Equation (4.12)).

4.4 Show that the (theoretical) partial autocorrelation coefficient of order 2, π_2, is given by

$$[\rho(2) - \rho(1)^2]/[1 - \rho(1)^2].$$

Compare with Equation (4.11).

4.5 Find the partial ac.f. of the AR(2) process given by

$$X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + Z_t$$

(see Exercise 3.6).

4.6 Suppose that the correlogram of a time series consisting of 100 observations has $r_1 = 0.31, r_2 = 0.37, r_3 = -0.05, r_4 = 0.06, r_5 = -0.21, r_6 = 0.11, r_7 = 0.08, r_8 = 0.05, r_9 = 0.12, r_{10} = -0.01$. Suggest an ARMA model, which may be appropriate.

4.6 Sixty observations are taken on a quarterly economic index, x_t. The first eight values of the sample ac.f., r_k, and the sample partial ac.f., $\widehat{\pi}_k$, of x_t, and of the first differences, ∇x_t, are shown below:

	Lag	1	2	3	4	5	6	7	8
x_t	r_k	0.95	0.91	0.87	0.82	0.79	0.74	0.70	0.67
	$\widehat{\pi}_k$	0.95	0.04	−0.05	0.07	0.00	0.07	−0.04	−0.02
∇x_t	r_k	0.02	0.08	0.12	0.05	−0.02	−0.05	−0.01	0.03
	$\widehat{\pi}_k$	0.02	0.08	0.06	0.03	−0.05	−0.06	−0.04	−0.02

Identify a model for the series. What else would you like to know about the data in order to make a better job of formulating a 'good' model?

4.7 Use the Box-Jenkins seasonal ARIMA models to analyze the Beverage price index series shown in Figure 1.1.

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Chapter 5

Forecasting

Forecasting is the art of saying what will happen, and then explaining why it didn't! – Anonymous

5.1 Introduction

Forecasting the future values of an observed time series is an important problem in many areas, including economics, production planning, sales forecasting and stock control.

Suppose we have an observed time series $x_1, x_2, \ldots, x_N$. Then the basic problem is to estimate future values such as x_{N+h}, where the integer h is called the **lead time** or **forecasting horizon** – h for horizon. The forecast of x_{N+h} made at time N for h steps ahead is typically denoted by $\widehat{x}_N(h)$. Note that some authors still use the imprecise notation $\widehat{x}_{N+h}$, which assumes implicitly that the forecast is made at time N.

A wide variety of different forecasting procedures is available and it is important to realize that no single method is universally applicable. Rather, the analyst must choose the procedure that is most appropriate for a given set of conditions. It is also worth bearing in mind that forecasting is a form of extrapolation, with all the dangers that it entails. Forecasts are conditional statements about the future based on specific assumptions. Thus forecasts are not sacred and the analyst should always be prepared to modify them as necessary in the light of any external information. For long-term forecasting, it can be helpful to produce a range of forecasts based on different sets of assumptions so that alternative 'scenarios' can be explored.

Forecasting methods may be broadly classified into three groups as follows:

(1) *Subjective*

Forecasts can be made on a subjective basis using judgement, intuition, commercial knowledge and any other relevant information. Methods range widely from bold freehand extrapolation to the Delphi technique, in which a group of forecasters tries to obtain a consensus forecast with controlled feedback of other analysts' predictions and opinions as well as other relevant information. These methods will not be described here, as most statisticians will want their forecasts to be at least partly objective. The interested reader is referred, for example, to Webby and O'Connor (1996), Rowe and Wright (1999) and the relevant sections of Armstrong (2001). However, note that

some subjective judgement is often used in a more statistical approach, for example, to choose an appropriate model and perhaps make adjustments to the resulting forecasts.

(2) *Univariate*

Forecasts of a given variable are based on a model fitted only to present and past observations of a given time series, so that $\widehat{x}_N(h)$ depends only on the values of $x_N, x_{N-1}, \ldots$, possibly augmented by a simple function of time, such as a global linear trend. This would mean, for example, that univariate forecasts of the future sales of a given product would be based entirely on past sales, and would not take account of other economic factors. Methods of this type are sometimes called naive or projection methods, and will be discussed in Sections 5.2 and 5.3.

(3) *Multivariate*

Forecasts of a given variable depend at least partly on values of one or more additional series, called **predictor** or **explanatory** variables[1]. For example, sales forecasts may depend on stocks and/or on economic indices. Models of this type are sometimes called causal models, and will be introduced in Section 5.4.

In practice, a forecasting procedure may involve a *combination* of the above approaches. For example, marketing forecasts are often made by combining statistical predictions with the subjective knowledge and insight of people involved in the market. A more formal type of combination is to compute a weighted average of two or more objective forecasts, as this often proves superior on average to the individual forecasts. Unfortunately, an informative model may not result.

An alternative way of classifying forecasting methods is between an **automatic** approach requiring no human intervention, and a **non-automatic** approach requiring some subjective input from the forecaster. The latter applies to subjective methods and most multivariate methods. Most univariate methods can be made fully automatic but can also be used in a non-automatic form, and there can be a surprising difference between the results.

The choice of method depends on a variety of considerations, including:

- How the forecast is to be used.
- The type of time series (e.g. macroeconomic series or sales figures) and its properties (e.g. are trend and seasonality present?). Some series are very regular and hence 'very predictable', but others are not. As always, a time plot of the data is very helpful.
- How many past observations are available.

[1] They are also sometimes called *independent variables* but this terminology is misleading, as they are typically *not* independent of each other.

- The length of the forecasting horizon. This book is mainly concerned with short-term forecasting. For example, in stock control the lead time for which forecasts are required is the time between ordering an item and its delivery.
- The number of series to be forecast and the cost allowed per series.
- The skill and experience of the analyst. Analysts should select a method with which they feel 'happy' and for which relevant computer software is available. They should also consider the possibility of trying more than one method.

It is particularly important to *clarify the objectives* (as in any statistical investigation). This means finding out how a forecast will actually be used, and whether it may even influence the future. In the latter case, some forecasts turn out to be self-fulfilling. In a commercial environment, forecasting should be an integral part of the management process leading to what is sometimes called a *systems approach*.

This chapter concentrates on calculating **point forecasts**, where the forecast for a particular future time period consists of a single number. Point forecasts are adequate for many purposes, but a **prediction interval** is often helpful to give a better indication of future uncertainty. Instead of a single value, a prediction interval consists of upper and lower limits between which a future value is expected to lie with a prescribed probability. Some methods for calculating prediction intervals are considered in Section 5.3.3. Taking one more step away from a point forecast, it may be desirable to calculate the entire probability distribution of a future value of interest. This is called **density forecasting**. The reader is referred to Tay and Wallis (2000). A practical halfway house between prediction intervals and density forecasting is the use of *fan charts*. The latter essentially plot prediction intervals at several different probability levels, by using darker shades for central values, and lighter shades for outer bands, which cover less likely values. These graphs can be very effective for presenting the future range of uncertainty in a simple, visually-effective way. The reader is referred to Wallis (1999).

Whatever forecasting method is used, some sort of *forecast monitoring scheme* is often advisable, particularly with large numbers of series, to ensure that forecast errors are not systematically positive or negative. A variety of *tracking signals* for detecting 'trouble' are discussed, for example, by Gardner (1983) and McLain (1988).

5.2 Extrapolation and Exponential Smoothing

This section introduces the many projection methods that are now available. Further details may be found in Abraham and Ledolter (1983), Chatfield (2001, Chapters 3–4), Diebold (2001), Granger and Newbold (1986) and Montgomery et al. (1990).

5.2.1 Extrapolation of trend curves

For long-term forecasting of non-seasonal data, it is often useful to fit a **trend curve** (or **growth curve**) to successive values and then extrapolate. This approach is most often used when the data are yearly totals, and hence clearly non-seasonal. A variety of curves may be tried including polynomial, exponential, logistic and Gompertz curves (see also Section 2.5.1). When the data are annual totals, at least 7 to 10 years of historical data are required to fit such curves. The method is worth considering for short annual series where fitting a complicated model to past data is unlikely to be worthwhile. Although primarily intended for long-term forecasting, it is inadvisable to make forecasts ahead for a longer period than about half the number of past years for which data are available.

A drawback to the use of trend curves is that there is no logical basis for choosing among the different curves except by goodness-of-fit. Unfortunately it is often the case that one can find several curves that fit a given set of data almost equally well but which, when projected forward, give widely different forecasts. Further details about trend curves are given by Meade (1984).

5.2.2 Simple exponential smoothing

Exponential smoothing (ES) is the name given to a general class of forecasting procedures that rely on simple updating equations to calculate forecasts. The most basic form, introduced in this subsection, is called **simple exponential smoothing** (SES), but this should only be used for non-seasonal time series showing no systematic trend. Of course many time series that arise in practice *do* contain a trend or seasonal pattern, but these effects can be measured and removed to produce a stationary series for which simple ES is appropriate. Alternatively, more complicated versions of ES are available to cope with trend and seasonality – see Section 5.2.3 below. Thus adaptations of exponential smoothing are useful for many types of time series – see Gardner (1985) for a detailed general review of these popular procedures.

Given a non-seasonal time series, say $x_1, x_2, \ldots, x_N$, with no systematic trend, it is natural to forecast x_{N+1} by means of a weighted sum of the past observations:

$$\widehat{x}_N(1) = c_0 x_N + c_1 x_{N-1} + c_2 x_{N-2} + \cdots, + c_{N-1} x_1, \tag{5.1}$$

where the $\{c_i\}$ are weights. It seems sensible to give more weight to recent observations and less weight to observations further in the past. An intuitively appealing set of weights are *geometric* weights, which decrease by a constant ratio for every unit increase in the lag. In order that the weights sum to one, we take

$$c_i = \alpha(1-\alpha)^i \qquad i = 0, 1, \ldots, N-1,$$

where α is a constant such that $0 < \alpha < 1$. Then Equation (5.1) becomes

$$\widehat{x}_N(1) = \alpha x_N + \alpha(1-\alpha)x_{N-1} + \alpha(1-\alpha)^2 x_{N-2} + \cdots + \alpha(1-\alpha)^{N-1} x_1. \tag{5.2}$$

Strictly speaking, Equation (5.2) implies an infinite number of past observations, but in practice there will only be a finite number. Thus Equation (5.2) is customarily rewritten in the **recurrence** form as

$$\begin{aligned} \widehat{x}_N(1) &= \alpha x_N + (1-\alpha)[\alpha x_{N-1} + \alpha(1-\alpha)x_{N-2} + \cdots] \\ &= \alpha x_N + (1-\alpha)\widehat{x}_{N-1}(1). \end{aligned} \tag{5.3}$$

If we set $\widehat{x}_1(1) = x_1$, then Equation (5.3) can be used recursively to compute forecasts. Equation (5.3) also reduces the amount of arithmetic involved since forecasts can easily be updated using only the latest observation and the previous forecast of that latest observation.

The procedure defined by Equation (5.3) is called simple exponential smoothing. The adjective 'exponential' arises from the fact that the geometric weights lie on an exponential curve, but the procedure could equally well have been called geometric smoothing.

Equation (5.3) is sometimes rewritten in the equivalent **error-correction** form

$$\begin{aligned} \widehat{x}_N(1) &= \alpha[x_N - \widehat{x}_{N-1}(1)] + \widehat{x}_{N-1}(1) \\ &= \alpha e_N + \widehat{x}_{N-1}(1), \end{aligned} \tag{5.4}$$

where $e_N = x_N - \widehat{x}_{N-1}(1)$ is the prediction error at time N. Equations (5.3) and (5.4) look different at first sight, but give identical forecasts, and it is a matter of practical convenience as to which one should be used.

Although intuitively appealing, it is natural to ask when SES is a 'good' method to use. It can be shown (see Exercise 5.6) that SES is optimal if the underlying model for the time series is given by

$$X_t = \mu + \alpha \sum_{j<t} Z_j + Z_t,$$

where $\{Z_t\}$ denotes a purely random process. This infinite-order moving average (MA) process is non-stationary, but the first differences $(X_{t+1} - X_t)$ form a stationary first-order MA process (see Exercise 3.3). Thus X_t is an ARIMA process of order $(0, 1, 1)$. In fact it can be shown (Chatfield et al., 2001) that there are many other models for which SES is optimal. This helps to explain why SES appears to be such a robust method.

The value of the smoothing constant α depends on the properties of the given time series. Values between 0.1 and 0.3 are commonly used and produce a forecast that depends on a large number of past observations. Values close to one are used rather less often and give forecasts that depend much more on recent observations. When $\alpha = 1$, the forecast is equal to the most recent observation.

The value of α may be estimated from past data by a similar procedure to that used for estimating the parameters of an MA process. Given a particular value of α, one-step-ahead forecasts are produced iteratively through the

series, and then the sum of squares of the one-step-ahead prediction errors is computed. This can be repeated for different values of α so that the value, which minimizes the sum of squares, can be found. In more detail, for a given value of α, calculate

$$\begin{aligned} \widehat{x}_1(1) &= x_1 \\ e_2 &= x_2 - \widehat{x}_1(1) \\ \widehat{x}_2(1) &= \alpha e_2 + \widehat{x}_1(1) \\ e_3 &= x_3 - \widehat{x}_2(1) \end{aligned}$$

and so on until

$$e_N = x_N - \widehat{x}_{N-1}(1)$$

and then compute $\sum_{i=2}^{N} e_i^2$. Repeat this procedure for other values of α between 0 and 1, say in steps of 0.1, and select the value that minimizes $\sum e_i^2$, either by inspection or using an algorithmic numerical procedure. Modern computers make this all easy to do. Usually the sum of squares surface is quite flat near the minimum and so the choice of α is not critical.

5.2.3 The Holt and Holt–Winters forecasting procedures

Exponential smoothing may readily be generalized to deal with time series containing trend and seasonal variation. The version for handling a trend with non-seasonal data is usually called Holt's (two-parameter) exponential smoothing, while the version that also copes with seasonal variation is usually referred to as the Holt–Winters (three-parameter) procedure. These names honour the pioneering work of C.C. Holt and P.R. Winters around 1960. The general idea is to generalize the equations for SES by introducing trend and seasonal terms, which are also updated by exponential smoothing.

We first consider Holt's ES. In the absence of trend and seasonality, the one-step-ahead forecast from simple ES can be thought of as an estimate of the local mean level of the series, so that simple ES can be regarded as a way of updating the local **level** of the series, say L_t. This suggests rewriting Equation (5.3) in the form

$$L_t = \alpha x_t + (1 - \alpha) L_{t-1}.$$

Suppose we now wish to include a trend term, T_t say, which is the expected increase or decrease per unit time period in the current level. Then a plausible pair of equations for updating the values of L_t and T_t in recurrence form are the following

$$\begin{aligned} L_t &= \alpha x_t + (1 - \alpha)(L_{t-1} + T_{t-1}) \\ T_t &= \gamma(L_t - L_{t-1}) + (1 - \gamma) T_{t-1}. \end{aligned}$$

Then the h-step-ahead forecast at time t will be of the form

$$\widehat{x}_t(h) = (L_t + hT_t)$$

for $h = 1, 2, 3, \ldots$. There are now two updating equations, involving two smoothing parameters, α and γ, which are generally chosen to lie in the range $(0, 1)$. It is natural to call this the two-parameter version of ES.

The above procedure may readily be generalized again to cope with seasonality. Let L_t, T_t, I_t denote the local level, trend and seasonal index, respectively, at time t. The interpretation of I_t depends on whether seasonality is thought to be additive or multiplicative – see Section 2.6. In the former case, $x_t - I_t$ is the deseasonalized value, while in the multiplicative case, it is x_t/I_t. The values of the three quantities, L_t, T_t and I_t, all need to be estimated and so we need three updating equations with three smoothing parameters, say α, γ and δ. As before, the smoothing parameters are usually chosen in the range (0,1). The form of the updating equations is again intuitively plausible. Suppose the observations are monthly, and that the seasonal variation is multiplicative. Then the (recurrence form) equations for updating L_t, T_t, I_t, when a new observation x_t becomes available, are

$$\begin{aligned} L_t &= \alpha(x_t/I_{t-12}) + (1-\alpha)(L_{t-1} + T_{t-1}) \\ T_t &= \gamma(L_t - L_{t-1}) + (1-\gamma)T_{t-1} \\ I_t &= \delta(x_t/L_t) + (1-\delta)I_{t-12} \end{aligned}$$

and the forecasts from time t are then

$$\widehat{x}_t(h) = (L_t + hT_t)I_{t-12+h}$$

for $h = 1, 2, \ldots, 12$. There are analogous formulae for the additive seasonal case. There are also analogous formulae for the case where the seasonality is of length s say, rather than 12 as for monthly observations. In particular, $s = 4$ for quarterly data, when we would, for example, compare I_t with I_{t-4}. Unfortunately, the literature is confused by many different notations and by the fact that the updating equations may be presented in an equivalent error-correction form, which can look quite different. For example, the above formula for updating the trend in the monthly multiplicative case can be rewritten (after some algebra) in the form

$$T_t = T_{t-1} + \alpha\gamma e_t/I_{t-12}$$

where $e_t = x_t - \widehat{x}_{t-1}(1)$ denotes the one-step-ahead forecast error as before. This formula looks quite different and, in particular, it looks as though $\alpha\gamma$ is the smoothing parameter. Clearly, great care needs to be taken when comparing formulae from different sources expressed in different ways.

In order to apply Holt–Winters smoothing to seasonal data, the analyst should carry out the following steps:

(1) Examine a graph of the data to see whether an additive or a multiplicative seasonal effect is the more appropriate.

(2) Provide starting values for L_1, and T_1 as well as seasonal values for the first year, say $I_1, I_2, \ldots, I_s$, using the first few observations in the series in a fairly simple way; for example, the analyst could choose $L_1 = \sum_1^s x_i/s$.

(3) Estimate values for α, γ, δ by minimizing $\sum e_t^2$ over a suitable fitting period for which historical data are available.

(4) Decide whether to normalize the seasonal indices at regular intervals by making them sum to zero in the additive case or have an average of one in the multiplicative case.

(5) Choose between a fully automatic approach (for a large number of series) and a non-automatic approach. The latter allows subjective adjustments for particular series, for example, by allowing the removal of outliers and a careful selection of the appropriate form of seasonality.

Further details on the Holt–Winters method are given by Chatfield and Yar (1988). The method is straightforward and is widely used in practice.

Another variation of ES that deserves mention here is the use of a **damped trend** (Gardner and McKenzie, 1985). This procedure can be used with the Holt and Holt–Winters methods and introduces another smoothing parameter, say ϕ, where $0 < \phi < 1$, such that the estimate of the trend or growth rate at time t, namely, T_t, is damped to ϕT_t in the subsequent time period. The intuitive justification for this is that most trends do not in fact go on forever, but rather damp down towards zero over time. For Holt's method, the h-steps-ahead forecast at time t then becomes

$$\widehat{x}_t(h) = [L_t + (\sum_{i=1}^{h} \phi^i) T_t)]$$

rather than $(L_t + hT_t)$. There are analogous formulae for damped Holt–Winters. The method involves estimating one extra smoothing parameter, but can give more accurate forecasts (Gardner and McKenzie, 1985) and so may justify the extra effort. A successful empirical example is given by Grubb and Mason (2001) using U.K. airline passenger data from 1949 to 1998, but note that they damp towards the average trend rather than towards zero.

We have still not exhausted the many variants of ES. Brown (1963) has suggested a technique called **general exponential smoothing**, which consists of fitting polynomial, sinusoidal or exponential functions to the data and finding appropriate updating formulae. This method is more complicated than earlier ES methods, even though there is rather little emphasis on identifying the correct functional forms. It should be noted that sinusoids are used to describe any seasonality rather than a set of seasonal indices as in the Holt–Winters method. One special case of this approach is **double exponential smoothing**, which is applicable to series containing a linear trend. Unlike Holt's method, double ES uses a single smoothing parameter.

This makes the method simpler, but may lead to poorer forecast performance. Note that Brown suggests fitting by *discounted* least squares, in which more weight is given to recent observations, rather than using (global) least squares.

Further details on the many variants of ES are given by Gardner (1985).

5.3 The Box–Jenkins Methodology

This section gives a brief outline of the forecasting procedure, based on autoregressive integrated moving average (ARIMA) models, which is usually known as the Box–Jenkins approach. The beginner may find it easier to get further information from books such as Vandaele (1983), Granger and Newbold (1986) or Jenkins (1979), rather than the original 1970 Box–Jenkins book now revised as Box et al. (1994), although the latter is still an essential reference source.

5.3.1 The Box-Jenkins procedures

Now AR, MA and ARMA models have been around for many years and are associated, in particular, with early work by G.U. Yule and H.O. Wold. A major contribution of Box and Jenkins has been to provide a general *strategy* for time-series forecasting, which emphasizes the importance of identifying an appropriate model in an iterative way as outlined briefly in Section 4.8. Indeed the iterative approach to model building that they suggested has since become standard in many areas of statistics. Furthermore, Box and Jenkins showed how the use of differencing can extend ARMA models to ARIMA models and hence cope with non-stationary series. In addition, Box and Jenkins show how to incorporate seasonal terms into seasonal ARIMA (SARIMA) models. Because of all these fundamental contributions, ARIMA models are often referred to as Box–Jenkins models.

In brief, the main stages in setting up a Box–Jenkins forecasting model are as follows:

(1) *Model identification*

Examine the data to see which member of the class of ARIMA processes appears to be most appropriate.

(2) *Estimation*

Estimate the parameters of the chosen model as described in Chapter 4.

(3) *Diagnostic checking*

Examine the residuals from the fitted model to see if it is adequate.

(d) *Consideration of alternative models if necessary*

If the first model appears to be inadequate for some reason, then alternative ARIMA models may be tried until a satisfactory model is found. When such

a model has been found, it is usually relatively straightforward to calculate forecasts as conditional expectations.

We now consider these stages in more detail. In order to identify an appropriate ARIMA model, the first step in the Box–Jenkins procedure is to *difference* the data until they are stationary. This is achieved by examining the correlograms of various differenced series until one is found that comes down to zero 'fairly quickly' and from which any seasonal cyclic effect has been largely removed, although there could still be some 'spikes' at the seasonal lags $s, 2s$, and so on, where s is the number of observations per year. For non-seasonal data, first-order differencing is usually sufficient to attain stationarity. For monthly data (of period 12), the operator $\nabla\nabla_{12}$ is often used if the seasonal effect is additive, while the operator ∇_{12}^2 may be used if the seasonal effect is multiplicative. Sometimes the operator ∇_{12} by itself will be sufficient. Over-differencing should be avoided. For a seasonal period of length s, the operator ∇_s may be used, and, in particular, for quarterly data we may use ∇_4.

The differenced series will be denoted by $\{w_t; t = 1, \ldots, N - c\}$, where c terms are 'lost' by differencing. For example, if the operator $\nabla\nabla_{12}$ is used, then $c = 13$.

For non-seasonal data, an ARMA model can now be fitted to $\{w_t\}$ as described in Chapter 4. If the data are seasonal, then the SARIMA model defined in Equation (4.16) may be fitted as follows. Plausible values of p, P, q, Q are selected by examining the correlogram and the partial autocorrelation function (ac.f.) of the differenced series $\{w_t\}$. Values of p and q are selected by examining the first few values of r_k, as outlined in Chapter 4. Values of P and Q are selected primarily by examining the values of r_k at $k = 12, 24, \ldots$, when the seasonal period is given by $s = 12$. If, for example, r_{12} is 'large' but r_{24} is 'small', this suggests one seasonal moving average term, so we would take $P = 0, Q = 1$, as this SARIMA model has an ac.f. of similar form. Box et al. (1994, Table A9.1) list the autocovariance functions of various SARIMA models.

Having tentatively identified what appears to be a reasonable SARIMA model, least squares estimates of the model parameters may be obtained by minimizing the residual sum of squares in a similar way to that proposed for ordinary ARMA models. In the case of seasonal series, it is advisable to estimate initial values of a_t and w_t by backforecasting (or backcasting) rather than setting them equal to zero. This procedure is described by Box et al. (1994, Section 9.2.4). In fact, if the model contains a seasonal MA parameter that is close to one, several cycles of forward and backward iteration may be needed. Nowadays several alternative estimation procedures are available, based on, for example, the exact likelihood function, on conditional or unconditional least squares, or on a Kalman filter approach (see references in Section 4.4).

For both seasonal and non-seasonal data, the adequacy of the fitted model should be checked by what Box and Jenkins call 'diagnostic checking'.

This essentially consists of examining the residuals from the fitted model to see whether there is any evidence of non-randomness. The correlogram of the residuals is calculated and we can then see how many coefficients are significantly different from zero and whether any further terms are indicated for the ARIMA model. If the fitted model appears to be inadequate, then alternative ARIMA models may be tried until a satisfactory one is found.

When a satisfactory model is found, forecasts may readily be computed. Given data up to time N, these forecasts will involve the observations and the fitted residuals (i.e. the one-step-ahead forecast errors) up to and including time N. The minimum mean square error forecast of X_{N+h} at time N is the conditional expectation of X_{N+h} at time N, namely,

$$\widehat{x}_N(h) = E(X_{N+h}|X_N, X_{N-1}, \ldots).$$

In evaluating this conditional expectation, we use the fact that the 'best' forecast of all future Zs is simply zero (or more formally that the conditional expectation of Z_{N+h}, given data up to time N, is zero for all $h > 0$). Box et al. (1994) describe three general approaches to computing forecasts.

(1) *Using the model equation directly*

Point forecasts are usually computed most easily directly from the ARIMA model equation, which Box et al. (1994) call the difference equation form. Assuming that the model equation is known exactly, then $\widehat{x}_N(h)$ is obtained from the model equation by replacing (i) future values of Z by zero, (ii) future values of X by their conditional expectation and (iii) present and past values of X and Z by their observed values.

As an example, consider the SARIMA$(1,0,0) \times (0,1,1)_{12}$ model used as an example in Section 4.8, where

$$X_t = X_{t-12} + \alpha(X_{t-1} - X_{t-13}) + Z_t + \theta Z_{t-12}.$$

Then we find

$$\begin{aligned} \widehat{x}_N(1) &= x_{N-11} + \alpha(x_N - x_{N-12}) + \theta z_{N-11} \\ \widehat{x}_N(2) &= x_{N-10} + \alpha[\widehat{x}_N(1) - x_{N-11}] + \theta z_{N-10}. \end{aligned}$$

Forecasts further into the future can be calculated recursively in an obvious way. It is also possible to find ways of updating the forecasts as new observations become available. For example, when x_{N+1} becomes known, we have

$$\begin{aligned} \widehat{x}_{N+1}(1) &= x_{N-10} + \alpha(x_{N+1} - x_{N-11}) + \theta z_{N-10} \\ &= \widehat{x}_N(2) + \alpha[x_{N+1} - \widehat{x}_N(1)] \\ &= \widehat{x}_N(2) + \alpha z_{N+1}. \end{aligned}$$

(2) *Using the ψ weights*

An ARMA model can be rewritten as an infinite-order MA process and the resulting ψ weights, as defined in Equation (3.25), could also be used to compute forecasts, but are primarily helpful for calculating forecast error variances. Since

$$X_{N+h} = Z_{N+h} + \psi_1 Z_{N+h-1} + \psi_2 Z_{N+h-2} + \cdots \tag{5.5}$$

and future Zs are unknown at time N, it is clear that $\widehat{x}_N(h)$ is equal to $\sum_{j=0}^{\infty} \psi_{h+j} z_{N-j}$ (since future zs cannot be included). Thus the h-steps-ahead forecast error is $(Z_{N+h} + \psi_1 Z_{N+h-1} + \cdots + \psi_{h-1} Z_{N+1})$. Hence the variance of the h-steps-ahead forecast error is $(1 + \psi_1^2 + \cdots + \psi_{h-1}^2)\sigma_z^2$, by independence of the Z's.

(3) *Using the π weights*

An ARMA model can also be rewritten as an infinite-order AR process and the resulting π weights, as defined in Equation (3.26), can also be used for compute point forecasts. Since

$$X_{N+h} = \pi_1 X_{N+h-1} + \cdots + \pi_h X_N + \cdots + Z_{N+h}$$

it is intuitively clear that $\widehat{x}_N(h)$ is given by

$$\begin{aligned} \widehat{x}_N(h) \quad = \quad & \pi_1 \widehat{x}_N(h-1) + \pi_2 \widehat{x}_N(h-2) + \cdots + \pi_{h-1} \widehat{X}_N(1) \\ & + \pi_h X_N + \pi_{h+1} X_{N-1} + \cdots . \end{aligned}$$

These forecasts can be computed recursively, replacing future values of X with predicted values as necessary.

In general, methods (1) or (3) are used for point forecasts while method (2) is used for forecast error variances.

In practice, the model will not be known exactly, and we have to estimate the model parameters (and hence the ψs and πs if they are needed); we also have to estimate the past observed values of Z, namely, z_t, by the observed residuals or one-step-ahead forecasts errors, namely, $\widehat{z}_t$. Thus for the SARIMA$(1,0,0) \times (0,1,1)_{12}$ model given above, we would have, for example, that

$$\widehat{x}_N(1) = x_{N-11} + \widehat{\alpha}(x_N - x_{N-12}) + \widehat{\theta}\widehat{z}_{N-11}.$$

Except for short series, this generally makes little difference to forecast error variances.

Although some packages have been written to carry out ARIMA modelling and forecasting in an automatic way, the Box–Jenkins procedure is primarily intended for a non-automatic approach where the analyst uses subjective judgement to select an appropriate model from the large family of ARIMA models according to the properties of the individual series being analysed.

Thus, although the procedure is more versatile than many competitors, it is also more complicated and considerable experience is required to identify an appropriate ARIMA model. Unfortunately, the analyst may find several different models, which fit the data equally well but give rather different forecasts, while sometimes it is difficult to find any sensible model. Of course, an inexperienced analyst will sometimes choose a 'silly' model. Another drawback is that the method requires several years of data (e.g. at least 50 observations for monthly seasonal data).

5.3.2 Other methods

Many other univariate forecasting procedures have been proposed, and we briefly mention a few of them.

Granger and Newbold (1986, Section 5.4) describe a procedure called **stepwise autoregression**, which can be regarded as a subset of the Box–Jenkins procedure. It has the advantage of being fully automatic and relies on the fact that AR models are much easier to fit than MA or ARMA models even though an AR model may require extra parameters to give as good a representation of the data. The first step is to take first differences of the data to allow for non-stationarity in the mean. Then a maximum possible lag, say p, is chosen and the best AR model with just one lagged variable at a lag between 1 and p, is found, namely

$$W_t = \mu + \alpha_k^{(1)} W_{t-k} + e_t^{(1)},$$

where $W_t = X_t - X_{t-1}$, $1 \leq k \leq p$, $\alpha_k^{(1)}$ is the autoregression coefficient at lag k when fitting one lagged variable only, and $e_t^{(1)}$ is the corresponding error term. Then the best AR model with $2, 3, \ldots$ lagged variables is found. The procedure is terminated when the reduction in the sum of squared residuals at the jth stage is less than some preassigned quantity. Thus an integrated AR model is fitted, which is a special case of the Box–Jenkins ARIMA class. Granger and Newbold suggest choosing $p = 13$ for quarterly data and $p = 25$ for monthly data.

Harrison (1965) has proposed a modification of seasonal exponential smoothing, which consists essentially of performing a Fourier analysis of the seasonal factors and replacing them by smoothed factors. Parzen's **ARARMA** approach (Parzen, 1982; Meade and Smith, 1985) relies on fitting an AR model to remove the trend (rather than just differencing the trend away) before fitting an ARMA model. An apparently new method, called the **theta** method, gave promising results (Makridakis and Hibon, 2000), but subsequent research has shown that it is actually equivalent to a form of exponential smoothing.

There are two general forecasting methods, called **Bayesian forecasting** (West and Harrison, 1997) and **structural modelling** (Harvey, 1989), which rely on updating model parameters by a technique called **Kalman filtering**.

The latter is introduced in Chapter 10, and so we defer consideration of these methods until then.

5.3.3 Prediction intervals

Thus far, we have concentrated on calculating point forecasts, but it is sometimes better to calculate an *interval forecast* to give a clearer indication of future uncertainty. A **prediction interval** (P.I.) consists of upper and lower limits between which a future value is expected to lie with a prescribed probability. This interval can be calculated in several ways, depending on the forecasting method used, the properties of the data and so on.

Most P.I.s used in practice are essentially of the following general form. A $100(1-\alpha)\%$ P.I. for X_{N+h} is given by :

$$\widehat{x}_N(h) \pm z_{\alpha/2}\sqrt{\text{Var}[e_N(h)]}, \tag{5.6}$$

where $e_N(h) = X_{N+h} - \widehat{x}_N(h)$ denotes the forecast error made at time N when forecasting h steps ahead. Here $z_{\alpha/2}$ denotes the percentage point of a standard normal distribution with a proportion $\alpha/2$ above it. Equation (5.6) assumes that an appropriate expression for $\text{Var}[e_N(h)]$ can be found for the method or model being used. As the P.I. in Equation (5.6) is symmetric about $\widehat{x}_N(h)$, it effectively assumes that the forecast is unbiased. The formula also assumes that the forecast errors are normally distributed.

In practice, the forecast errors are unlikely to be exactly normal, because the estimation of model parameters produces small departures from normality, while the assumption of a known, invariant model with normal errors is also unlikely to be exactly true. Nevertheless, because of its simplicity, Equation (5.6) is the formula that is generally used to compute P.I.s, though preferably after checking that the underlying assumptions (e.g. forecast errors are normally distributed) are at least approximately satisfied. For any given forecasting method, the main problem will then lie with evaluating $\text{Var}[e_N(h)]$. Fortunately, formulae for this variance are available for many classes of model. Perhaps the best-known formula is for Box–Jenkins ARIMA forecasting, where the variance may be evaluated by writing an ARIMA model in infinite-moving-average form as

$$X_t = Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \cdots \tag{5.7}$$

— see Equation (5.5). Then the best forecast at time N of X_{N+h} can only involve the values of Z_t up to time $t = N$, so that $e_N(h) = [X_{N+h} - \widehat{X}_N(h)] = Z_{N+h} + \sum_{j=1}^{h-1} \psi_j Z_{N+h-j}$. Thus

$$\text{Var}[e_N(h)] = [1 + \psi_1^2 + \cdots + \psi_{h-1}^2]\sigma_z^2.$$

Formulae can also be found for most variations of exponential smoothing, by assuming that the given method is optimal. For example, for simple exponential smoothing, with smoothing parameter α, it can be shown that

$$\text{Var}[e_N(h)] = [1 + (h-1)\alpha^2]\sigma_e^2,$$

where σ_e^2 denotes the variance of the one-step-ahead forecast errors. Analogous formulae are available in the literature for some other methods and models — see Chatfield (1993; 2001, Chapter 7).

A completely different alternative approach to calculating P.I.s is to work empirically by using the 'forecast' errors obtained by fitting a model to past data, finding the observed variance of the within-sample 'forecast' errors at different lead times and using these values in Equation (5.6). A detailed review of different methods of calculating P.I.s, including simulation, resampling and Bayesian methods, is given by Chatfield (2001, Chapter 7).

Unfortunately, whichever way P.I.s are calculated, they tend to be too narrow in practice. The empirical evidence for this is discussed by Chatfield (2001, Section 7.7), who also suggests various reasons why this phenomenon occurs. The main reason seems to be that the underlying model may change in the future. The forecaster should bear this in mind and should not think that a narrow P.I. is necessarily 'good'.

Example 5.1

We now use the Box-Henkins procedure to analyze the monthly unemployment rates in the U.S. from January 1948 to October 2018. The series can be dowloaded from the U.S. Federal Reserve at St. Louis. Figure 5.1 plots the series and its sample ac.f. and sample partial ac.f. We note that all unemployment rates vary from 2.1% to 10.4%, and the sample ac.f. of the series decays slowly with the increase of lags.

```
> data<-read.csv("unemrate_LNU04000024.csv", header=T)
> unem<-data[,2]
> unem.ts<-ts(unem, start=c(1948,1), frequency=12)

> ### Figure 5.1
> par(mfrow=c(3,1), mar=c(4,4,4,4))
> plot(unem, type="l", xlab="Month", ylab="Rates", xaxt="n")
> x.pos<-c(1,145,289,433,577,721,850)
> x.label<-c("1948/01", "1960/01", "1972/01", "1984/01",
   "1996/01", "2008/01", "2018/10")
> axis(1, x.pos, x.label)
> acf(unem, 25, xlab="Lag", ylab="ACF", main="")
> acf(unem, 25, type="partial", xlab="Lag",
   ylab="Partial ACF",main="")
```

To find a time series model with good out-of-sample performance, we split the time series into a training sample of historical data from January 1948 to December 2017 and a second sample of "test data" from January 2018 to October 2018. Denote $\{X_t^{\text{in}}\}$ the unemployment rates in the training period, and $\{X_t^{\text{out}}\}$ the rates in the test period. We will use $\{X_t^{\text{out}}\}$ to measure the performance of the out-of-sample forecasts developed from the training sample $\{X_t^{\text{in}}\}$.

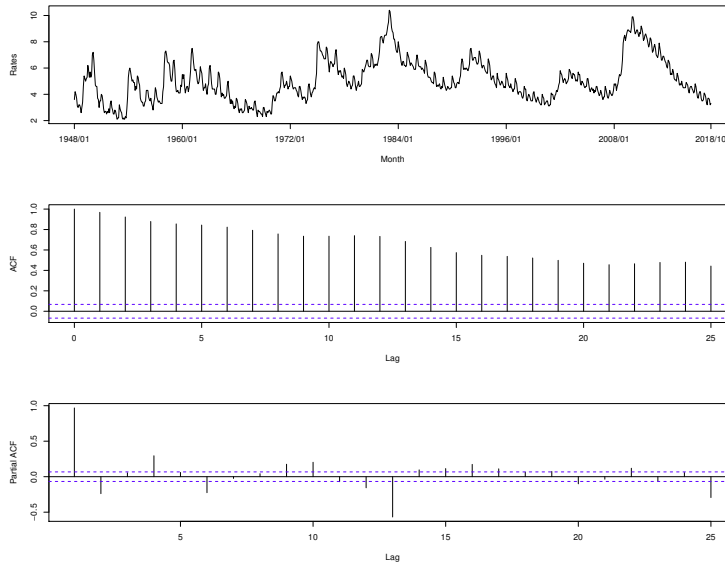

Figure 5.1 *The unemployment rate series (Top) and its sample ac.f. (Middle) and sample partial ac.f. (Bottom).*

Since there are obvious seasonal effects on unemployment, we use the `R` function `stl` to decompose X_t^{in} in the training period into a trend, a seasonal component, and remainder series, with a period of 12 months for the seasonal component. Let S_t be the seasonal component in X_t, and denote the deseasonalized series as $\widetilde{X}_t := X_t^{\text{in}} - S_t$. Figure 5.2 shows the deseasonalized series $\widetilde{X}_t$ in the training period and its sample ac.f. and sample partial ac.f.

```
> ### Partition the series into training and test data
> unem.ts<-ts(unem, start=c(1948,1), frequency=12)
> unem.train<-ts(unem[1:840], start=c(1948,1),
    end=c(2017,12), frequency=12)
> unem.test<-ts(unem[841:850], start=c(2018,1),
    end=c(2018,10), frequency=12)

> unem.stl<-stl(unem.train,"periodic")
> unem.sea<-unem.stl$time[,1]
> unem.desea<- unem.train-unem.sea
> unem.desea.series<-as.numeric(unem.desea)

> ### Figure 5.2
> par(mfrow=c(3,1), mar=c(4,4,4,4))
> plot(unem.desea.series, type="l", xlab="Month",
```

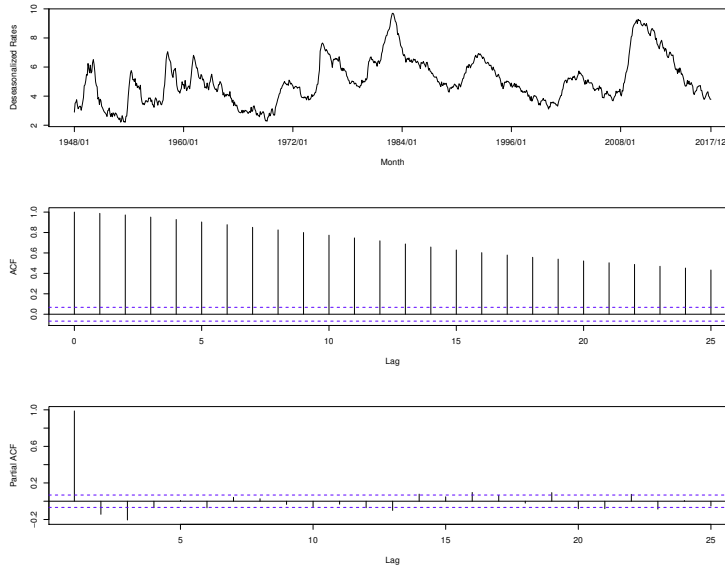

Figure 5.2 *The deseasonalized series* $\widetilde{X}_t$ *and its sample ac.f. and sample partial ac.f.*

```
    ylab="Deseasonalized Rates", xaxt="n")
> x.pos<-c(1,145,289,433,577,721,840)
> x.label<-c("1948/01", "1960/01", "1972/01", "1984/01",
    "1996/01", "2008/01", "2017/12")
> axis(1, x.pos, x.label)
> acf(unem.desea.series, 25, xlab="Lag",
    ylab="ACF", main="")
> acf(unem.desea.series, 25, type="partial",
    xlab="Lag", ylab="Partial ACF", main="")
```

The sample ac.f. of the deseasonalized series $\widetilde{X}_t$ in Figure 5.2 shows that the deseasonalized series is very persistent. As this might suggest the series is not stationary, we consider a unit-root test for the deseasonalized series. There are several unit-root tests developed in the literature; we consider the augmented Dickey-Fuller test introduced in Section 4.6. The following `R` implementation shows that the p-value of the test is 0.03078, smaller than 5%; hence the null hypothesis that the series $\widetilde{X}_t$ has a unit root should not be rejected and an ARIMA$(p, 1, q)$ model might be appropriate for series $\widetilde{X}_t$.

```
> # ADF test indicates that there is a unit root
> library(tseries)
> adf.test(unem.desea)
```

```
Augmented Dickey-Fuller Test

data:  unem.desea
Dickey-Fuller = -3.6198, Lag order = 9, p-value = 0.03078
alternative hypothesis: stationary
```

Since the orders p and q are unknown, we fit a set of ARIMA$(p, 1, q)$ models to series $\widetilde{X}_t$ and compute their AICs.

```
> # Select ARIMA models via AICs
> unem.desea.aic<-matrix(0, 4, 4);
> for (i in 0:3) for (j in 0:3) {
   fit.arima <- arima(unem.desea, order=c(i,1,j))
   unem.desea.aic[i+1,j+1]<-fit.arima$aic
> }
> unem.desea.aic
          [,1]      [,2]      [,3]      [,4]
[1,] -102.6192 -121.3318 -164.0700 -168.9881
[2,] -130.6713 -159.3643 -172.0880 -170.2128
[3,] -171.1796 -171.1427 -170.1949 -169.0363
[4,] -172.0392 -174.2756 -173.2259 -171.7914
```

The model with minimum AIC (-174.2756) is ARIMA$(p, 1, q)$ with $p = 3$ and $q = 1$. Hence we fit an ARIMA$(3, 1, 1)$ model to the series $\widetilde{X}_t$.

```
> fit.arima311<-arima(unem.desea.series, order=c(3,1,1))
> fit.arima311

Call:
arima(x = unem.desea.series, order = c(3, 1, 1))

Coefficients:
          ar1     ar2     ar3     ma1
      -0.4935  0.3039  0.2109  0.6296
s.e.   0.1517  0.0434  0.0406  0.1516

sigma^2 estimated as 0.04699:  log likelihood=92.14, aic=-174.28
```

The fitted result suggests the following ARIMA(3,1,1) model for $\widetilde{X}_t$,

$$\begin{aligned} Y_t &= \widetilde{X}_t - \widetilde{X}_{t-1} \\ Y_t &= -0.4935_{(.1517)} Y_{t-1} + 0.3039_{(.0434)} Y_{t-2} \\ &\quad +0.2109_{(.0406)} Y_{t-3} + Z_t + 0.6296_{(.1516)} Z_{t-1}, \end{aligned}$$

where $Z_t \sim N(0, 0.0470)$. Note that the fitted model can also be expressed as

$$\begin{aligned} &(1 - B)(1 + 0.4935_{(.1517)} B - 0.3039_{(.0434)} B^2 - 0.2109_{(.0406)} B^3)\widetilde{X}_t \\ &\qquad = Z_t + 0.6296_{(.1516)} Z_{t-1}. \end{aligned}$$

To perform a residual diagnostic, we show in Figure 5.3 the time series plot of standardized $\widehat{Z}_t$, the sample ac.f. of $\widehat{Z}_t$, and p-values for the Ljung-Box statistic (see Section 4.9). We also carry out a portmanteau test for the residuals. Note that the p-value of the test is much larger than 0.05, indicating the fitted model is appropriate.

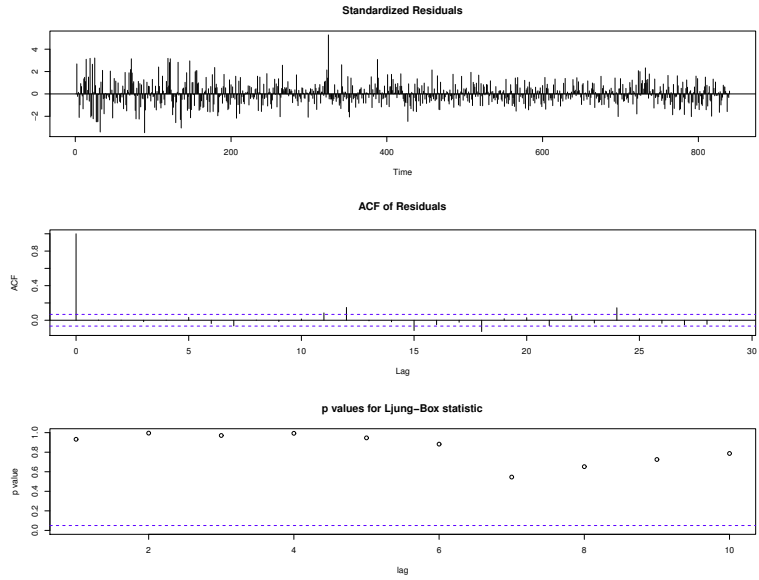

Figure 5.3 *The standardized residual series (Top), the sample ac.f. of residuals (Middle) and p-values for Ljung-Box statistic (Bottom).*

```
> # Figure 5.3
> tsdiag(fit.arima311)
> Box.test(fit.arima311$resid)

Box-Pierce test

data:  fit.arima311$resid
X-squared = 0.0072558, df = 1, p-value = 0.9321
```

We then use the fitted model to forecast the rates at January, February, ..., October in 2018. We compute the k-step-ahead forecast and its 95% prediction intervals for $\widetilde{X}_t$ at $k = 1, \ldots, 10$. Note that to obtain predicted values for $\{X_t^{(\text{out}}\}$, seasonal effects need to be added back. Figure 5.4 shows the true and predicted values of $\{X_t^{(\text{out}}\}$, and prediction intervals. We can see the following. First, all values of $\{X_t^{(\text{out}}\}$ lie in the corresponding 95%

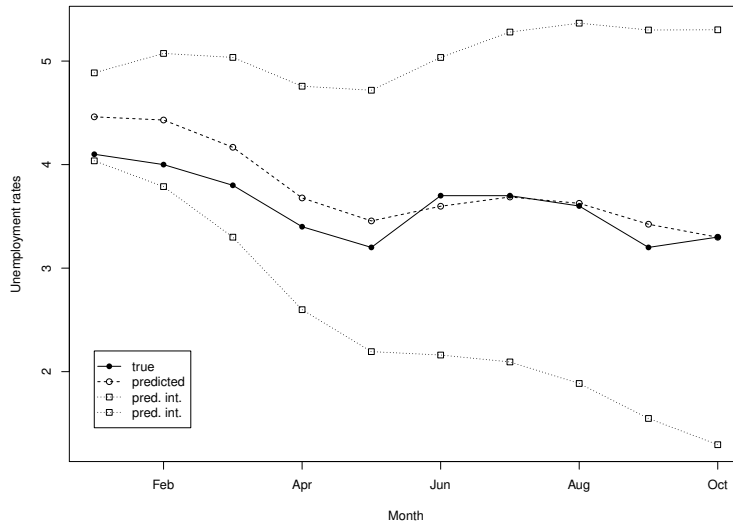

Figure 5.4 *The true and predicted unemployment rates with 95% prediction intervals at January, February, ..., October in 2018.*

prediction intervals. Second, the predicted values for $X_t^{(\text{out}}$ are quite close to the values of $X_t^{(\text{out}}$. This indicates that the estimated model also achieves a good performance in the out-of-sample analysis.

```
> # Prediction
> fit.arima311<-arima(unem.desea, order=c(3,1,1))
> unem.pred<-predict(fit.arima311, n.ahead=10)
> unem.pred.summary<-cbind(unem.test,
    unem.pred$pred+unem.sea[1:10],
    unem.pred$pred-1.96*unem.pred$se+unem.sea[1:10],
    unem.pred$pred+1.96*unem.pred$se+unem.sea[1:10])
> colnames(unem.pred.summary)<-c("true", "predict",
    "ci.lower", "ci.upper")

> ### Figure 5.4
> plot(c(1,10), range(unem.pred.summary), type="n",
    xlab="Month", ylab="Unemployment rates", xaxt="n")
> lines(seq(1,10),unem.pred.summary[,1], lty=1)
> points(seq(1,10),unem.pred.summary[,1], pch=16)
> lines(seq(1,10),unem.pred.summary[,2], lty=2)
> points(seq(1,10),unem.pred.summary[,2], pch=1)
> lines(seq(1,10),unem.pred.summary[,3], lty=3)
```

```
> points(seq(1,10),unem.pred.summary[,3], pch=0)
> lines(seq(1,10),unem.pred.summary[,4], lty=3)
> points(seq(1,10),unem.pred.summary[,4], pch=0)
> legend(1,2.2,c("true", "predicted", "pred. int.",
    "pred. int."), lty=c(1,2,3,3), pch=c(16,1,0,0))
> x.pos<-c(2,4,6,8,10)
> x.label<-c("Feb", "Apr", "Jun", "Aug", "Oct")
> axis(1, x.pos, x.label)
```

5.4 Multivariate Procedures

This section provides a brief introduction to some multivariate forecasting procedures. Further details on multivariate modelling are given later in this book, especially in Section 9.4.2 and in Chapter 12. More detailed coverage is given by Chatfield (2001, Chapter 5). The concepts are much more advanced than for univariate modelling and more sophisticated tools, such as the cross-correlation function, need to be developed in later chapters before we can make much progress.

5.4.1 Multiple regression

One common forecasting method makes use of the multiple regression model, which will be familiar to many readers. This model assumes that the response variable of interest, say y, is linearly related to p explanatory variables, say $x_1, x_2, \ldots, x_p$. The usual multiple regression model can be written as

$$y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + u, \tag{5.8}$$

where $\{\beta_i\}$ are constants, and u denotes the 'error' term. This equation is linear in terms of the parameters $\{\beta_i\}$, but could involve non-linear functions of observed variables. When building a regression model, it is helpful to distinguish between explanatory variables that can be controlled (like stock levels) and those that cannot (like air temperature). Now Equation (5.8) does not specifically involve time. Of course, we could regard time as a predetermined variable and introduce it as one of the explanatory variables, but regression on time alone would normally be regarded as a univariate procedure. More generally, we need to specify when each of the variables in Equation (5.8) is measured and so each variable really needs a subscript indicating when the variable is measured. When lagged values of the explanatory variables are included, they may be called **leading indicators**. Such variables are much more useful for forecasting. If lagged values of the response variable y are included, they are of an autoregressive nature and change the character of the model.

Multiple regression is covered in numerous statistics texts and the details need not be repeated here. The models are widely used and sometimes work

well. However, there are many dangers in applying such models to time-series data. Modern computer software makes it easy (perhaps too easy!) to fit regression models, and the ease of computation makes it tempting to include lots of explanatory variables, even though including too many may yield dubious results.

In fact applying regression models to time-series data is really *not* straightforward, especially as standard results assume that successive values of the 'errors' $\{u\}$ are independent, which is unlikely to be the case in practice. Although a high multiple correlation coefficient R^2 may result from fitting a regression model to time-series data, this apparent good fit may be spurious and does not mean that good forecasts necessarily result. This may be demonstrated both theoretically (Phillips, 1986; Hamilton, 1994, Section 18.3) and empirically (Granger and Newbold, 1974; 1986, Section 6.4) for non-stationary data. It is advisable to restrict the value of p to perhaps 3 or 4, and keep back part of the data to check forecasts from the fitted model. When doing this, it is important to distinguish between *ex ante* forecasts of y, which replace future values of explanatory variables by their forecasts (and so are true out-of-sample forecasts), and *ex post* forecasts, which use the true values of explanatory variables. The latter can look misleadingly good.

Problems can arise when the explanatory variables are themselves correlated, as often happens with time-series data. It is advisable to begin by looking at the correlations between explanatory variables so that, if necessary, selected explanatory variables can be removed to avoid possible singularity problems. The quality and characteristics of the data also need to be checked. For example, if a crucial explanatory variable has been held more or less constant in the past, then it is impossible to assess its effect using past data. Another type of problem arises when the response variable can, in turn, affect values of the explanatory variables to give what is called a closed-loop system. This is discussed later in Sections 9.4.3 and 13.1.

Perhaps the most important danger arises from mistakenly assuming that the 'error' terms form an independent sequence. This assumption is often inappropriate and can lead to a badly misspecified model and poor forecasts (Box and Newbold, 1971). The residuals from a regression model should always be checked for possible autocorrelation – see Section 4.7. A standard regression model, with independent errors, is usually fitted by Ordinary Least Squares (OLS), but this is seldom applicable directly to time-series data without suitable modification. Several alternative estimation procedures, such as Generalized Least Squares (GLS), have been developed over the years to cope with autocorrelated errors, but in such a way as still to be able to use OLS software. It can be shown that GLS and OLS are sometimes equivalent asymptotically, but such results may have little relevance for short series. Moreover, it is disturbing that autocorrelated errors may arise because certain lagged variables have been omitted from the model so that efforts to overcome such problems (e.g. by using GLS) are likely to lead to failure in the presence of a mis-specified model (e.g. Mizon, 1995). Mis-specifying the error

structure also causes problems. Further details about methods, such as GLS, may be found, for example, in an econometrics text such as Hamilton (1994). Nowadays, full maximum likelihood is likely to be used once an appropriate model for the errors (e.g. AR, MA or ARMA) has been identified (Choudhury et al., 1999).

In summary, the use of multiple regression can be dangerous except when there are clear contextual reasons why one or more series should explain variation in another. There are various precautions that should be taken, and various alternative strategies that should be considered. They include: (1) Using the context to choose the explanatory variables with care, and limiting their total number to perhaps 3 or 4; (2) Including appropriate lagged values of variables as variables in their own right; (3) Removing obvious sources of non-stationarity before fitting a regression model; (4) Carrying out careful diagnostic checks on any fitted model; (5) Allowing for correlated errors in the fitting procedure and (6) Considering alternative families of models such as transfer function models, vector AR models or a model allowing for co-integration — see Sections 9.4.2, 13.3 and 13.6, respectively.

5.4.2 Econometric models

Econometric models (e.g. Harvey, 1990) often assume that an economic system can be described, not by a single equation, but by a set of simultaneous equations. For example, not only do wage rates depend on prices but also prices depend on wage rates. Economists distinguish between *exogenous* variables, which affect the system but are not themselves affected, and *endogenous* variables, which interact with each other. The simultaneous equation system involving k dependent (endogenous) variables $\{Y_i\}$ and g predetermined (exogenous) variables $\{X_i\}$ may be written

$$Y_i \;=\; f_i(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_k, X_1, \ldots, X_g) \;+\; \text{error}$$

for $i = 1, 2, \ldots, k$. Some of the exogenous variables may be lagged values of the Y_i. The above set of equations, often called the *structural form* of the system, can be solved to give what is called the *reduced form* of the system, namely

$$Y_i = F_i(X_1, \ldots, X_g) \;+\; \text{error} \qquad \text{for } i = 1, 2, \ldots, k.$$

The principles and problems involved in constructing econometric models are too broad to be discussed in detail here (see, for example, Granger and Newbold, 1986, Section 6.3). A key issue is the extent to which the form of the model should be based on judgement, on economic theory and/or on empirical data. While some econometricians have been scornful[2] of univariate time-series models, which do not 'explain' what is going on, statisticians have been generally sceptical of some econometric model building in which the

[2]This occurs especially when the univariate forecasts are 'better' than alternatives!

structure of the model is determined *a priori* by economic theory and little attention is paid to identifying an appropriate 'error' structure or to using empirical data. Fortunately, mutual understanding has improved in recent years as developments in multivariate time-series modelling have brought statisticians and econometricians closer together to the benefit of both. In fact, the uncontrolled nature of much economic data makes it difficult to identify econometric models solely on an empirical statistical basis, while overreliance on economic theory should also be avoided. It is now widely recognized that a balanced middle way is sensible, and that econometric model building should be an iterative process involving both theory *and* data. See Chapter 13 for more detailed discussions on multivariate time-series modelling in recent years.

5.4.3 Other multivariate models

There are many other types of multivariate models that may be used to produce forecasts. The multivariate generalization of ARIMA models is considered in Chapter 13. One special case is the class of *vector autoregressive* models, while another useful class of models is that called *transfer function models* (see Section 9.4.2). The latter concentrates on describing the relationship between one 'output' variable and one or more 'input' or explanatory variables. It is helpful to understand the interrelationships between all these classes of multivariate models (e.g. see Granger and Newbold, 1986, Chapters 6–8; Priestley, 1981, Chapter 9; Chatfield, 2001, Chapter 5).

Of course, more specialized multivariate models may occasionally be required. For example, forecasts of births must take account of the number and age of women of child-bearing age. Common sense and background knowledge of the problem context should indicate what is required.

5.5 Comparative Review of Forecasting Procedures

We noted in Section 5.1 that there is no such thing as a 'best' forecasting procedure, but rather that the choice of method depends on a variety of factors such as the objective in producing forecasts, the degree of accuracy required and the properties of the given time series. This section gives a brief review of relevant research and makes recommendations on which method to use and when.

The context and the reason for making a forecast are, of course, of prime importance. Forecasts may be used, for example, for production planning in industry. They can also be used to assess the effects of different strategies by producing several forecasts. They may also be used as a 'norm' (or yardstick) against which the actual outcome may be assessed to see whether anything is changing. Univariate forecasts are usually used to provide such a 'norm' and are also particularly suitable when there are large numbers of series to be forecast (e.g. in stock control) so that a relatively simple method has to

be used. They are also suitable when the analyst's skill is limited or when they are otherwise judged appropriate for the client's needs and level of understanding. Multivariate models are particularly appropriate for assessing the effects of explanatory variables, for understanding the economy, and for evaluating alternative economic policy proposals by constructing a range of 'what if' forecasts. The latter strategy of computing more than one forecast is often very helpful in assessing the effects of different assumptions or strategies.

5.5.1 Forecasting competitions

First, we review the empirical evidence as to which method is 'best'. In order to clarify the choice between different univariate methods, there have been several 'competitions' to compare the forecasting accuracy of different methods on a given collection of time series. Some of the more important competitions are described by Newbold and Granger (1974), Makridakis and Hibon (1979), Makridakis et al. (1984), Makridakis et al. (1993) and Makridakis and Hibon (2000). The last three studies are commonly known as the M-competition, the M2-competition and the M3-competition, respectively. The M-competition was designed to be more wide ranging than earlier studies, and compared 24 methods on 1001 series, while the M3-competition compared 24 methods on no fewer than 3003 series – a mammoth task! The M2 competition was much smaller, just 29 series, but this allowed non-automatic procedures to be investigated.

Given different analysts and data sets, it is perhaps not too surprising that the results from different competitions have not always been consistent. For example, Newbold and Granger (1974) found that Box–Jenkins tended to give more accurate forecasts than other univariate methods, but this was not the case in the M- and M3-competitions. A detailed assessment of the strengths and weaknesses of forecasting competitions is given by Chatfield (1988; 2001, Section 6.4). It is essential that results be replicable and that appropriate criteria are used. Moreover, accuracy is only one aspect of forecasting, and practitioners think that cost, ease of use and ease of interpretation are of almost equal importance. Furthermore, competitions mainly analyse large numbers of series in a completely automatic way. Thus although they tell us something, competitions only tell part of the story and are mainly concerned with comparing automatic forecasts. In practice, large gains in accuracy can often be made by applying a carefully-chosen method that is tailored to a particular context for a particular type of series – see, for example, Tashman and Kruk (1996) and Section 5.4.2 below. This will not be evident in the results from automatic forecasting competitions, where, for example, simple exponential smoothing (SES) has sometimes been applied to all series in a group regardless of whether a particular series exhibits trend or not. This is unfair to SES, which does not pretend to be able to cope with trend.

One important question is how forecasting accuracy should be assessed – see, for example, Armstrong and Collopy (1992), Fildes (1992) and Chatfield

(2001, Section 6.3). While average prediction mean square error is the obvious measure, care needs to be taken when averaging across series with widely differing variances. Because of this, the mean absolute percentage error is often used, rather than mean square error. Of course, a model that is 'best' under one criterion need not be best under some other criterion. In particular, it can be useful to use a robust measure that is not inflated too much by the occasional large forecast error.

When comparing results from different competitions, it is also essential to ensure that genuine out-of-sample forecasts have been used throughout. When one method appears to give much better forecasts than all alternative methods, there must be a suspicion that it has an unfair advantage in some way. Usually, a method that seems to be a clear winner may not be using genuine out-of-sample forecasts, but rather may be 'cheating' in some way, perhaps unwittingly. For example, multivariate forecasts sometimes outperform univariate methods only by dubiously using future values of explanatory variables.

It is also worth noting that unexciting results tend to be suppressed. For example, forecasters seem eager to publish results that show a new method to be better, but not to publish results that show the reverse – see Chatfield (1995c) for examples. This sort of publication bias is endemic in other areas of statistical investigation, as people like to publish 'significant' results, but not the reverse. As a result, new methods tend to have an unfair advantage. Finally, it is worth saying that empirical results tend to be ignored (Fildes and Makridakis, 1995), especially when they run counter to orthodox thinking by showing, for example, that simple forecasting methods do as well as more complicated ones.

Choosing an automatic method. If an automatic approach is desirable or unavoidable, perhaps because a large number of series is involved, then my interpretation of the competition results is as follows. While there could be significant gains in being selective, most users will want to apply the same method to all series for obvious practical reasons. Some methods should be discarded, but there are several automatic methods for which average differences in accuracy are small. Thus the choice between them may depend on other practical considerations such as availability of computer programs. The methods include Holt's exponential smoothing (applied to seasonally adjusted data where appropriate), Holt–Winters and Bayesian forecasting. In particular, the Holt–Winters method provides a generally reliable, easy to understand method for seasonal data. The results in Chen (1997) have shown that Holt–Winters is robust to departures from the model for which the method is optimal. Holt's method can be used for non-seasonal data or data that have already been deseasonalized. The use of a damped trend term is often advisable in both the Holt and Holt–Winters procedures.

5.5.2 Choosing a non-automatic method

Suppose instead that a non-automatic approach is indicated because the number of series is small and/or because external information is available that cannot be ignored. Then sensible forecasters will use their skill and knowledge to interact with their clients, incorporate background knowledge, plot the data and generally use all relevant information to build an appropriate model and compute sensible forecasts. The choice then lies between some form of multivariate method and a non-automatic univariate procedure. Here forecasting competitions are of limited value and it is easy to cite case studies where subjective adjustment of automatic forecasts leads to improvements (e.g. Chatfield, 1978). Moreover, the differences in accuracy for different methods when averaged over many series are relatively small compared with the large differences in accuracy that can arise when different methods are applied to individual series. The potential rewards in selecting an appropriate, perhaps non-automatic, method indicate that the distinction between an automatic and a non-automatic *approach* may be more fundamental than the differences between different forecasting *methods*.

We look first at *multivariate methods*. It is possible to cite case studies (e.g. Jenkins and McLeod, 1982) where statistician and client collaborate to develop a successful multivariate model. However, it is difficult to make general statements about the relative accuracy of multivariate methods. Many people expect multivariate forecasts to be at least as good as univariate forecasts, but this is not true either in theory or in practice, and univariate methods outperform multivariate ones in many studies. One reason for this is that the computation of multivariate forecasts of a response variable may require the prior computation of forecasts of explanatory variables, and the latter must be sufficiently accurate to make this viable (Ashley, 1988). Multivariate models work best when one or more variables are leading indicators for the response variables of interest and so do not need to be forecast, at least for shorter lead times. Two other general points are that multivariate models are more difficult to identify than univariate models, and that they are generally less robust to departures from model assumptions. For all these reasons, simple methods and models are often better.

The empirical evidence is reviewed by Chatfield (1988, 2001). Regression models do rather better on average than univariate methods, though not by any means in every case (Fildes, 1985). Econometric simultaneous equation models have a patchy record and it is easy to cite cases where univariate forecasts are more accurate (e.g. Makridakis and Hibon, 1979, Section 2). There have been some encouraging case studies using transfer function models (e.g. Jenkins, 1979; Jenkins and McLeod, 1982), but such models assume there is no feedback, and this state of affairs will not apply to much multivariate economic data. Vector AR models, introduced later in Chapter 13, have a mixed record with some successes, but some failures as well. Moreover, many researchers fail to compare their multivariate forecasts with those from

simpler alternatives, perhaps because they do not wish to be embarrassed by the possibility that univariate forecasts turn out better. It is arguable that multivariate models are more useful for understanding relationships than for forecasting. Of course multivariate models can usually be made to give a better *fit* to given data than univariate models, but this superiority does not necessarily translate into better forecasts, perhaps because multivariate models are more sensitive to changes in structure.

It has to be realized that the nature of economic time-series data is such as to make it difficult to fit reliable multivariate time-series models. Most economic variables are simply observed, rather than controlled, and there are usually high autocorrelations within each series. In addition there may be high correlations between series, not necessarily because of a real relationship but simply because of a mutual correlation with time. Feedback between 'output' and 'input' variables is another problem. There are special difficulties in fitting regression models to time-series data anyway, as already noted in Section 5.3.1, and an apparent good fit may be spurious. Simultaneous equation and vector AR models are also difficult to construct, and their use seems likely to be limited to the analyst who is as interested in the modelling process as in forecasting. Thus although the much greater effort required to construct multivariate models will sometimes prove fruitful, there are many situations where a univariate method will be preferred.

With a non-automatic *univariate* approach, the main choice is between the Box–Jenkins approach and the non-automatic use of a simple method, such as Holt–Winters, which is more often used in automatic mode. The Box–Jenkins approach has been one of the most influential developments in time-series analysis. However, the accuracy of the resulting forecasts has been rather mixed in practice, particularly when one realizes that forecasting competitions are biased in favour of Box–Jenkins by implementing other methods in a completely automatic way. The advantage of being able to choose from the broad class of ARIMA models is clear, but, as noted in Section 5.3, there are also dangers in that considerable experience is needed to interpret correlograms and other indicators. Moreover, when the variation in a series is dominated by trend and seasonality, the effectiveness of the fitted ARIMA model is mainly determined by the differencing procedure rather than by the identification of the autocorrelation structure of the differenced (stationary) series. Yet the latter is what is emphasized in the Box–Jenkins approach. Nevertheless, some writers have suggested that all exponential smoothing models should be regarded as special cases of Box–Jenkins, the implication being that one might as well use Box–Jenkins. However, this view is now discredited (Chatfield and Yar, 1988) because exponential smoothing methods are actually applied in a different way to Box–Jenkins.

In some situations, a large expenditure of time and effort can be justified and then Box–Jenkins is worth considering. However, for routine sales forecasting, simple methods are more likely to be understood by managers and workers who have to utilize or implement the results. Thus that Box–Jenkins is

only recommended when the following conditions are satisfied: (1) the analyst is competent to implement the method and has appropriate software; (2) the objectives justify the additional complexity; and (3) the variation in the series is *not* dominated by trend and seasonality. If these conditions are not satisfied, then a non-automatic version of a simple method, such as Holt–Winters, may be 'best'.

Of course, another alternative to the Box–Jenkins approach is to use a *more* complicated method, such as one of the multivariate methods discussed earlier in this subsection. This can be especially rewarding when there is an obvious leading indicator to include. However, there are many situations when it is advisable to restrict attention to univariate forecasts.

5.5.3 A strategy for non-automatic univariate forecasting

If circumstances suggest that a non-automatic univariate approach is appropriate, then the following eight steps could generally provide a sensible strategy that covers many forecasting situations:

1. Get appropriate background information and carefully define the objectives.
2. Plot the data and look for trend, seasonal variation, outliers and any changes in structure such as slow changes in variance, sudden discontinuities and anything else that looks unusual.
3. 'Clean' the data if necessary, for example, by adjusting any suspect observations, preferably after taking account of external information. Consider the possibility of transforming the data.
4. Decide whether the seasonal variation is non-existent, multiplicative, additive or something else.
5. Decide whether the trend is non-existent, global linear, local linear or non-linear.
6. Fit an appropriate model where possible. It can be helpful to distinguish the following four types of series (although the reader may, of course, come across series that do not fit into any of these categories. Then common sense has to be applied):

   (i) *Discontinuities present.* For a series having a major discontinuity, it is generally unwise to produce any univariate forecasts. There is further discussion of this series in Section 14.1.

   (ii) *Trend and seasonality present.* Figures 1.3 shows series whose variation is dominated by trend and seasonality. Here the Holt–Winters exponential smoothing method is a suitable candidate. The correct seasonal form must be chosen and the smoothing parameters can then be estimated by optimizing one-step-ahead forecasts over the period of fit. Full details are given by Chatfield and Yar (1988). For data showing trend, but no seasonality, Holt's method may be used.

(iii) *Short-term correlation present.* Many economic indicator series are of this form and it is essential to try to understand the autocorrelation structure. Here, the Box–Jenkins approach is recommended.

(iv) *Exponential growth present.* Series of this type are difficult to handle because exponential forecasts are inherently unstable. No one really believes that economic growth or population size, for example, can continue forever to increase exponentially. Two alternative strategies are to fit a model that explicitly includes exponential (or perhaps quadratic) growth terms, or to fit a model to the logarithms of the data (or to some other suitable transformation of the data). It may help to damp any trend that is fitted to the transformed data as in Gardner and McKenzie (1985).

7. Check the adequacy of the fitted model. In particular, study the one-step-ahead forecast errors over the period of fit to see whether they have any undesirable properties such as significant autocorrelation. Modify the model if necessary.
8. Compute forecasts. Decide whether the forecasts need to be adjusted subjectively because of anticipated changes in other variables, or because of any other reason.

Of course, much of the above procedure could be automated, making the dividing line between automatic and non-automatic procedures rather blurred. We should also mention **rule-based forecasting** (Collopy and Armstrong, 1992), which is a form of **expert system** that integrates skilled judgement with domain knowledge and also takes account of the features of the data. The rule base, consisting of as many as 99 rules, entertains a variety of forecasting procedures, which may be combined in an appropriate way. This rule base may be modified (e.g. Adya et al., 2000) depending on the particular situation.

5.5.4 Summary

It is difficult to summarize the many empirical findings and other general principles that have been established in regard to forecasting, but the following general observations and recommendations can be made:

- As in all statistical work, it is essential to formulate the forecasting problem carefully, clarify objectives and be clear exactly how a forecast will be used.
- The more frequent and the greater the number of forecasts required, the more desirable it is to use a simple approach.
- Fitting the 'best' model to historical data does not necessarily lead to the most accurate out-of-sample forecast errors in the future. In particular, complex models usually give a better fit than simpler models but the resulting forecasts need not be more accurate. It is essential that all comparisons between different forecasting models and methods should be made using genuine out-of-sample forecasts.

- Prediction intervals, calculated on the assumption that the model fitted to past data will also be true in the future, are generally too narrow.
- If an automatic univariate method is required, then the Holt and Holt–Winters versions of exponential smoothing are suitable candidates, but there are several close competitors.
- When a non-automatic approach is appropriate, there is a wide choice from judgemental and multivariate methods through to (univariate) Box–Jenkins and the 'thoughtful' use of univariate methods that are usually regarded as being automatic. A general strategy for non-automatic univariate forecasting has been proposed that involves looking carefully at the data and selecting an appropriate method from a set of candidates including the Box–Jenkins, Holt and Holt–Winters methods. Whatever approach is used, the analyst should be prepared to improvise and modify 'objective' forecasts using subjective judgement.
- It is often possible to get more accurate results by combining forecasts from different methods, perhaps by using a weighted average, but this does not lead to an informative model.

Comprehensive coverage of the general principles involved in time-series forecasting, together with further guidance on empirical results, is given by Armstrong (2001) and Chatfield (2001).

5.6 Prediction Theory

This section[3] gives a brief introduction to the general theory of linear prediction, which has been developed by Kolmogorov, Wiener (1949), Yaglom (1962) and Whittle (1983) among others. All these authors avoid the use of the word 'forecasting', although most of the univariate methods considered in Section 5.2 are in the general class of linear predictors. The theory of linear prediction has applications in control and communications engineering and is of considerable theoretical interest, but readers who wish to tackle the sort of forecasting problem we have been considering earlier in this chapter will find this literature less accessible, and less relevant to real-life problems, than earlier references, such as those describing the Box–Jenkins approach.

Two types of problems are often distinguished. In the first type of problem we have data up to time N, say $\{x_N, x_{N-1}, \ldots\}$, and wish to predict the value of x_{N+h}. One approach is to use the predictor

$$\widehat{x}_N(h) = \sum_{j \geq 0} c_j x_{N-j}$$

which is a linear function of the available data. The weights $\{c_j\}$ are chosen so as to minimize the expected mean square prediction error $E(X_{N+h} - \widehat{x}_N(h))^2$. This is often called the *prediction* problem (e.g. Cox and Miller, 1968), while

[3]This section may be omitted at first reading.

Yaglom (1962) refers to it as the *extrapolation* problem and Whittle (1983) calls it *pure prediction*. As an example of the sort of result that has been obtained, Wiener (1949) has considered the problem of evaluating the weights $\{c_j\}$, so as to find the best linear predictor, when the underlying process is assumed to be stationary with a known autocorrelation function, and when the entire past sequence of observations, namely, $\{x_t\}$ for $t \leq N$, is assumed known. It is interesting to compare this sort of approach with the forecasting techniques proposed earlier in this chapter. Wiener, for example, says little or nothing about identifying an appropriate model and then estimating the model parameters from a finite sample, and yet these are the sort of problems that have to be faced in real-life applications. The Box–Jenkins approach does tackle this sort of problem, and aims to employ a linear predictor that is optimal for a particular ARIMA process, while recognizing that the form of the model and the model parameters are usually *not* known beforehand and have to be inferred from the data.

The second type of problem tackled in the more theoretical literature arises when the process of interest, called the *signal*, is contaminated by *noise*, and we actually observe the process

$$y(t) = s(t) + n(t),$$

where $s(t)$ and $n(t)$ denote the signal and noise, respectively. In some situations the noise is simply measurement error; in engineering applications the noise could be an interference process of some kind. The problem now is to separate the signal from the noise. Given measurements on $y(t)$ up to time T we may want to reconstruct the signal up to time T or alternatively make a prediction of $s(T+\tau)$. The problem of reconstructing the signal is often called *smoothing* or *filtering*. The problem of predicting the signal is also sometimes called *filtering* (Yaglom, 1962; Cox and Miller, 1968), but is sometimes called *prediction* (Astrom, 1970). To make progress, it is often assumed that the signal and noise processes are uncorrelated and that $s(t)$ and $n(t)$ have known autocorrelation functions. These assumptions, which are unlikely to hold in practice, make the results of limited practical value.

It is clear that both the above types of problems are closely related to the control problem because, if we can predict how a process will behave, then we can adjust the process so that the achieved values are, in some sense, as close as possible to the target value. Control theory is generally outside the scope of this book, although we do make some further brief remarks on the topic later in Section 14.3 after we have studied the theory of linear systems.

Exercises

5.1 For the MA(1) model given by

$$X_t = Z_t + \theta Z_{t-1},$$

show that $\widehat{x}_N(1) = \theta z_N$ and that $\widehat{x}_N(h) = 0$ for $h = 2, 3, \ldots$.

Show that the variance of the h-steps-ahead forecast error is given by σ_Z^2 for $h = 1$, and by $(1+\theta^2)\sigma_Z^2$ for $h \geq 2$, provided the true model is known. (In practice we would take $\widehat{x}_N(1) = \widehat{\theta}\widehat{z}_N$, where $\widehat{\theta}$ is the least squares estimate of θ and $\widehat{z}_N$ is the observed residual at time N.)

5.2 For the AR(1) model given by

$$X_t = \alpha X_{t-1} + Z_t,$$

show that $\widehat{x}_N(h) = \alpha^h x_N$ for $h = 1, 2, \ldots$. Show that the variance of the h-steps-ahead forecast error is given by $(1 - \alpha^{2h})\sigma_Z^2/(1 - \alpha^2)$.
For the AR(1) model, with non-zero mean μ, given by

$$X_t - \mu = \alpha(X_{t-1} - \mu) + Z_t$$

show that $\widehat{x}_N(h) = \mu + \alpha^h(x_N - \mu)$ for $h = 1, 2, \ldots$. (In practice the least squares estimates of α and μ would need to be substituted into the above formulae.)

5.3 Consider the SARIMA$(1, 0, 0) \times (0, 1, 1)_{12}$ model used as an example in Section 5.2.4. Show that

$$\widehat{x}_N(2) = x_{N-10} + \alpha^2(x_N - x_{N-12}) + \theta\alpha z_{N-11} + \theta z_{N-10}.$$

5.4 For the SARIMA$(0, 0, 1) \times (1, 1, 0)_{12}$ model, find forecasts at time N for up to 12 steps ahead in terms of observations and estimated residuals up to time N.

5.5 For the model $(1 - B)(1 - 0.2B)X_t = (1 - 0.5B)Z_t$ in Exercise 3.12, find forecasts for one and two steps ahead, and show that a recursive expression for forecasts three or more steps ahead is given by

$$\widehat{x}_N(h) = 1.2\widehat{x}_N(h-1) - 0.2\widehat{x}_N(h-2).$$

Find the variance of the one-, two- and three-steps-ahead forecast errors. If $z_N = 1, x_N = 4, x_{N-1} = 3$ and $\sigma_Z^2 = 2$, show that $\widehat{x}_N(2) = 3.64$ and that the standard error of the corresponding forecast error is 1.72.

5.6 Consider the ARIMA$(0, 1, 1)$ process

$$(1 - B)X_t = (1 - \theta B)Z_t.$$

Show that $\widehat{x}_N(1) = x_N - \theta z_N$, and $\widehat{x}_N(h) = \widehat{x}_N(h-1)$ for $h \geq 2$. Express $\widehat{x}_N(1)$ in terms of x_N and $\widehat{x}_{N-1}(1)$ and show that this is equivalent to exponential smoothing. By considering the ψ weights of the process, show that the variance of the h-steps-ahead prediction error is

$$[1 + (h-1)(1-\theta)^2]\sigma_Z^2.$$

5.7 Consider the AR(2) process (a is real)

$$X_t + \frac{1}{a(a+1)}X_{t-1} - \frac{1}{a(a+1)}X_{t-2} = Z_t.$$

(a) For what values of real number a, is the process stationary?

(b) What is the autocorrelation of the process at lag k, i.e., $\rho(k)$? ($k \geq 0$).

(c) What are the 1-step and 2-step ahead forecasts of the process at forecast origin X_n?

5.8 Consider the ARMA(2,1) process $X_t - \alpha X_{t-2} = Z_t - \beta Z_{t-1}$, where α, β are real and $\alpha, \beta \neq 0, \alpha \neq \beta$.

(a) Under which conditions is the process X_t stationary and invertible?

(b) Given the observations $X_1, \ldots, X_n$, what is your best prediction for X_{n+1} and X_{n+2} at the forecast origin X_n? What is the corresponding forecasting error?

Chapter 6

Stationary Processes in the Frequency Domain

6.1 Introduction

Chapter 3 described several types of stationary stochastic processes, such as autoregressive processes. When discussing their properties, we emphasised the autocovariance (or autocorrelation) function, as being the natural tool for considering the evolution of a process through time. This chapter introduces a complementary function, called the **spectral density function**, which is the natural tool for considering the frequency properties of a time series. Inference regarding the spectral density function is called an analysis in the **frequency domain**.

Some statisticians initially have difficulty in understanding the frequency approach, but the advantages of frequency methods are widely appreciated in such fields as electrical engineering, geophysics and meteorology. These advantages will become apparent in the next few chapters.

We confine attention to real-valued processes. Some authors consider the more general problem of complex-valued processes, and this has some mathematical advantages. However, in our view, the reader is more likely to understand an approach restricted to real-valued processes. The vast majority of practical problems are covered by this approach.

6.2 The Spectral Distribution Function

In order to introduce the idea of a spectral density function, we first consider a function called the **spectral distribution function**. The approach adopted is heuristic and not mathematically rigorous, but will hopefully give the reader a better understanding of the subject than a more theoretical approach.

Suppose we suspect that a time series contains a periodic sinusoidal component with a known wavelength. Then a natural model is

$$X_t = R\cos(\omega t + \phi) + Z_t \tag{6.1}$$

where ω is called the **frequency** of the sinusoidal variation, R is called the **amplitude** of the variation, ϕ is called the **phase** and $\{Z_t\}$ denotes a stationary random series. Note that the angle $(\omega t + \phi)$ is usually measured in units called *radians*, where π radians $= 180°$. As ω is the number of radians per unit time, it is sometimes called the **angular** frequency, but in keeping

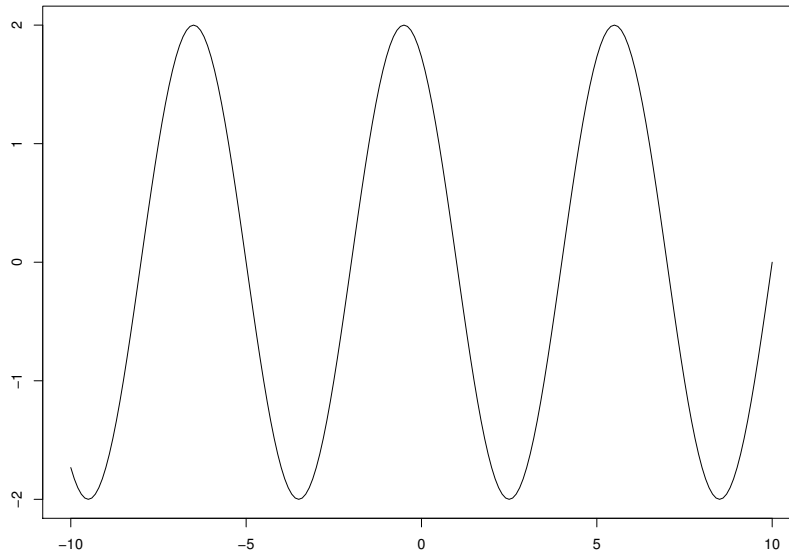

Figure 6.1 *A graph of* $R\cos(\omega t + \phi)$ *with* $R = 2, \omega = \pi/3$ *and* $\phi = \pi/6$.

with most authors we simply call ω the frequency. However, some authors, notably Jenkins and Watts (1968), use the term 'frequency' to refer to the number of cycles per unit time, namely, $f = \omega/2\pi$, and this form of frequency is easier to interpret from a physical point of view. We usually use the angular frequency ω in mathematical formulae, because it makes them more concise, but we will often use the frequency $f = \omega/2\pi$ to interpret the results of a data analysis. The **period** of a sinusoidal cycle, sometimes called the **wavelength**, is clearly $1/f$ or $2\pi/\omega$. Figure 6.1 shows an example of a sinusoidal function with angular frequency $\omega = \pi/3$, so that $f = \omega/2\pi = 1/6$. The wavelength is the reciprocal of f, namely, 6, and inspection of Figure 6.1 shows that this is indeed the number of time units between successive peaks or successive troughs of the sinusoid.

Figure 6.1 is a very simple model, but in practice the variation in a time series may be caused by variation at several different frequencies. For example, sales figures may contain weekly, monthly, yearly and other cyclical variation. In other words the data show variation at high, medium and low frequencies. It is natural therefore to generalize Equation (6.1) to

$$X_t = \sum_{j=1}^{k} R_j \cos(\omega_j t + \phi_j) + Z_t, \tag{6.2}$$

where R_j, ϕ_j denote the amplitude and phase, respectively, at frequency ω_j. Figure 6.2 shows an example of the mixture (6.2) with $k = 3$ constructed in the following way. First, for $t = 1, \ldots, 100$), we generate four series

$$X_{t1} = \cos\left(\frac{10\pi t}{150} + \frac{\pi}{8}\right), \qquad X_{t2} = 3\cos\left(\frac{30\pi t}{150} + \frac{3\pi}{8}\right),$$
$$X_{t3} = 5\cos\left(\frac{60\pi t}{150} + \frac{5\pi}{8}\right), \qquad Z_t \sim N(0,1).$$

Then the series X_t is constructed as

$$X_t = X_{t1} + X_{t2} + X_{t3} + Z_t.$$

The series of X_{t1}, X_{t2}, X_{t3} and X_t are displayed in Figure 6.2, which can be reproduced by the following R code.

```
> y<-seq(0, 100, 0.3)
> x1<- 1*cos(pi*y*10/150 + pi/8)
> x2<- 3*cos(pi*y*30/150 + 3*pi/8)
> x3<- 5*cos(pi*y*60/150 + 5*pi/8)
> set.seed(1)
> z<- rnorm(length(y), 0, 1)
> x<-x1+x2+x3+z
> par(mfrow=c(2,2), mar=c(2,2,2,2))
> plot(x1, type="l", xlab="", ylab="x1")
> plot(x2, type="l", xlab="", ylab="x1")
> plot(x3, type="l", xlab="", ylab="x1")
> plot(x, type="l", xlab="", ylab="x1")
```

The reader will notice that the models in Equations (6.1) and (6.2) are *not* stationary if parameters such as $\{R_j\}$, $\{\omega_j\}$ and $\{\phi_j\}$ are all fixed constants, because $E(X_t)$ will change with time. In order to apply the theory of stationary processes to models like Equation (6.2), it is customary to assume that $\{R_j\}$ are (uncorrelated) random variables with mean zero, or that $\{\phi_j\}$ are random variables with a uniform distribution on $(0, 2\pi)$, which are fixed for a single realization of the process (see Section 3.13 and Exercise 3.14). This is something of a 'mathematical trick', but it does enable us to treat time series containing one or more deterministic sinusoidal components as stationary series.

Since $\cos(\omega t + \phi) = \cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi$, the model in Equation (6.2) can alternatively be expressed as a sum of sine and cosine terms in the form

$$X_t = \sum_{j=1}^{k} (a_j \cos(\omega_j t) + b_j \sin(\omega_j t)) + Z_t, \tag{6.3}$$

where $a_j = R_j \cos\phi_j$ and $b_j = -R_j \sin\phi_j$.

However, we may now ask why there should only be a finite number of frequencies involved in Equations (6.2) or (6.3). In fact, letting $k \to \infty$, the

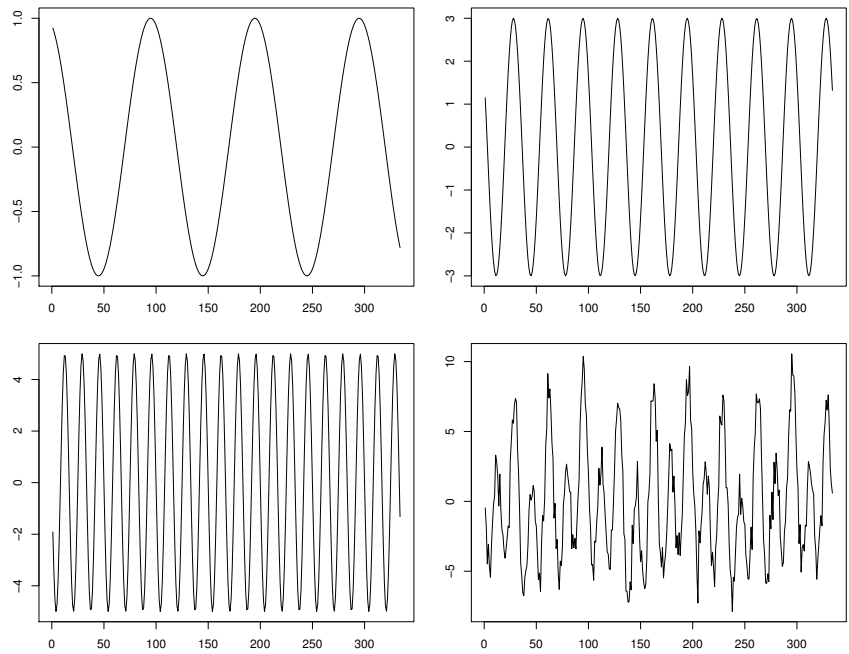

Figure 6.2 *Periodic components and their sum (Top left: X_{t1}; Top right: X_{t2}; Bottom left: X_{t3}; Bottom right: X_t).*

work of Wiener and others has shown that any discrete-time stationary process measured at unit intervals may be represented in the form

$$X_t = \int_0^\pi \cos\,\omega t\,\mathrm{d}u(\omega) + \int_0^\pi \sin\,\omega t\,\mathrm{d}v(\omega), \tag{6.4}$$

where $u(\omega), v(\omega)$ are uncorrelated continuous processes, with orthogonal increments (see Section 3.12), which are defined for all ω in the range $(0, \pi)$. Equation (6.4) is called the **spectral representation** of the process; it involves stochastic integrals, which require considerable mathematical skill to handle properly. It is intuitively more helpful to ignore these mathematical problems and simply regard X_t as a linear combination of orthogonal sinusoidal terms. Thus the derivation of the spectral representation will not be considered here (see, for example, Cox and Miller, 1968, Chapter 8).

The reader may wonder why the upper limits of the integrals in Equation (6.4) are π rather than ∞. For a continuous process the upper limits would indeed be ∞, but for a discrete-time process measured at unit intervals of time there is no loss of generality in restricting ω to the range $(0, \pi)$, since

$$\cos\,[(\omega + k\pi)t] = \begin{cases} \cos\,\omega t & \text{for } k,\ t \text{ integers with } k \text{ even} \\ \cos\,(\pi - \omega)t & \text{for } k,\ t \text{ integers with } k \text{ odd} \end{cases}$$

and so variation at frequencies higher than π cannot be distinguished from variation at a corresponding frequency in $(0, \pi)$. The frequency $\omega = \pi$ is called the **Nyquist frequency**. We will say more about this in Section 7.2.1. For a discrete-time process measured at equal intervals of time of length Δt, the Nyquist frequency is $\pi/\Delta t$. In the next two sections we consider discrete-time processes measured at unit intervals of time, but the arguments carry over to any discrete-time process, measured at an arbitrary time interval Δt if we replace π by $\pi/\Delta t$.

The main point of introducing the spectral representation in Equation (6.4) is to show that every frequency in the range $(0, \pi)$ may contribute to the variation of the process. However, the processes $u(\omega)$ and $v(\omega)$ in Equation (6.4) are of little direct practical interest. Instead we introduce a single function, $F(\omega)$, called the **(power) spectral distribution function**. This function is related to the autocovariance function and provides an intuitively understandable description of the frequency properties of a stationary process. It arises from a theorem, called the Wiener–Khintchine theorem, named after N. Wiener and A.Y. Khintchine, which says that for any real-valued stationary stochastic process with autocovariance function $\gamma(k)$, there exists a monotonically[1] increasing function $F(\omega)$ such that

$$\gamma(k) = \int_0^{\pi} \cos \omega k \, \mathrm{d}F(\omega). \tag{6.5}$$

Equation (6.5) is called the spectral representation of the autocovariance function, and involves a type of integral (called Stieltjes) that may be unfamiliar to some readers. However, it can be shown that the function $F(\omega)$ has a direct physical interpretation: it is the contribution to the variance of the series, which is accounted for by frequencies in the range $(0, \omega)$. If variation at negative frequencies is not allowed, then

$$F(\omega) = 0 \quad \text{for } \omega < 0.$$

For a discrete-time process measured at unit intervals of time, the highest possible frequency is the Nyquist frequency π and so all the variation is accounted for by frequencies less than π. Thus

$$F(\pi) = \mathrm{Var}(X_t) = \sigma_X^2.$$

This last result also comes directly from Equation (6.5) by putting $k = 0$, when we have

$$\gamma(0) = \sigma_X^2 = \int_0^{\pi} \mathrm{d}F(\omega) = F(\pi).$$

In between $\omega = 0$ and $\omega = \pi$, $F(\omega)$ is monotonically increasing.

If the process contains a deterministic sinusoidal component at frequency

[1] A function, say $f(x)$, is said to be monotonically increasing with x if $f(x)$ never decreases as x increases. Thus $f(x)$ will either increase or stay constant.

ω_0, say $R\cos(\omega_0 t + \phi)$ where R is a constant and ϕ is uniformly distributed on $(0, 2\pi)$, then there will be a step increase in $F(\omega)$ at ω_0 equal to the contribution to variance of this particular component. As the component has mean zero, the contribution to variance is just the average squared value, namely, $E[R^2\cos^2(\omega_0 t + \phi)] = \frac{1}{2}R^2$.

As $F(\omega)$ is monotonically increasing, it can be decomposed into two functions, $F_1(\omega)$ and $F_2(\omega)$, such that

$$F(\omega) = F_1(\omega) + F_2(\omega), \tag{6.6}$$

where $F_1(\omega)$ is a non-decreasing continuous function and $F_2(\omega)$ is a non-decreasing step function. This decomposition usually corresponds to the Wold decomposition, with $F_1(\omega)$ relating to the purely indeterministic component of the process and $F_2(\omega)$ relating to the deterministic component. We are mainly concerned with purely indeterministic processes, where $F_2(\omega) \equiv 0$, in which case $F(\omega)$ is a continuous function on $(0, \pi)$.

The adjective 'power', which is sometimes prefixed to 'spectral distribution function', derives from the engineer's use of the word in connection with the passage of an electric current through a resistance. For a sinusoidal input, the power is directly proportional to the squared amplitude of the oscillation. For a more general input, the power spectral distribution function describes how the power is distributed with respect to frequency. In the case of a time series, the variance may be regarded as the total power.

Note that some authors use a normalized form of $F(\omega)$ given by

$$F^*(\omega) = F(\omega)/\sigma_X^2. \tag{6.7}$$

Thus $F^*(\omega)$ is the *proportion* of variance accounted for by frequencies in the range $(0, \omega)$. Since $F^*(0) = 0$, $F^*(\pi) = 1$, and $F^*(\omega)$ is monotonically increasing in $(0, \pi)$, $F^*(\omega)$ has similar properties to the cumulative distribution function of a random variable.

6.3 The Spectral Density Function

For a purely indeterministic discrete-time stationary process, the spectral distribution function is a continuous (monotone bounded) function in $[0, \pi]$, and may therefore be differentiated[2] with respect to ω in $(0, \pi)$. We will denote the derivative by $f(\omega)$, so that

$$f(\omega) = \frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}. \tag{6.8}$$

[2]Strictly speaking, $F(\omega)$ may not be differentiable on a set of measure zero, but this is usually of no practical importance. In particular, $F(\omega)$ may not be differentiable at the endpoints 0 and π and so it is more accurate to define the derivative $f(\omega)$ over the open interval $(0, \pi)$ that excludes the endpoints. This does not affect any integrals involving $f(\omega)$. In fact the formula in Equation (6.12) for $f(\omega)$, which is given later, usually can be evaluated at the end-points.

This is the **power spectral density function**. The term 'spectral density function' is often shortened to **spectrum**, and the adjective 'power' is sometimes omitted.

When $f(\omega)$ exists, Equation (6.5) can be expressed in the form

$$\gamma(k) = \int_0^{\pi} \cos\, \omega k\, f(\omega)\, \mathrm{d}\omega. \tag{6.9}$$

This is an ordinary (Riemann) integral and therefore much easier to handle than (6.5). Putting $k = 0$, we have

$$\gamma(0) = \sigma_X^2 = \int_0^{\pi} f(\omega)\, \mathrm{d}\omega = F(\pi). \tag{6.10}$$

The physical interpretation of the spectrum is that $f(\omega)\, \mathrm{d}\omega$ represents the contribution to variance of components with frequencies in the range $(\omega, \omega + \mathrm{d}\omega)$. When the spectrum is drawn, Equation (6.10) indicates that the total area underneath the curve is equal to the variance of the process. A peak in the spectrum indicates an important contribution to variance at frequencies near the value that corresponds to the peak. An example of a spectrum is shown in Figure 6.3, together with the corresponding normalized spectral distribution function.

It is important to realise that the autocovariance function (acv.f.) and the power spectral density function are equivalent ways of describing a stationary stochastic process. From a practical point of view, they are complementary to each other. Both functions contain the same information but express it in different ways. In some situations a time-domain approach based on the acv.f. is more useful, while in other situations a frequency-domain approach is preferable.

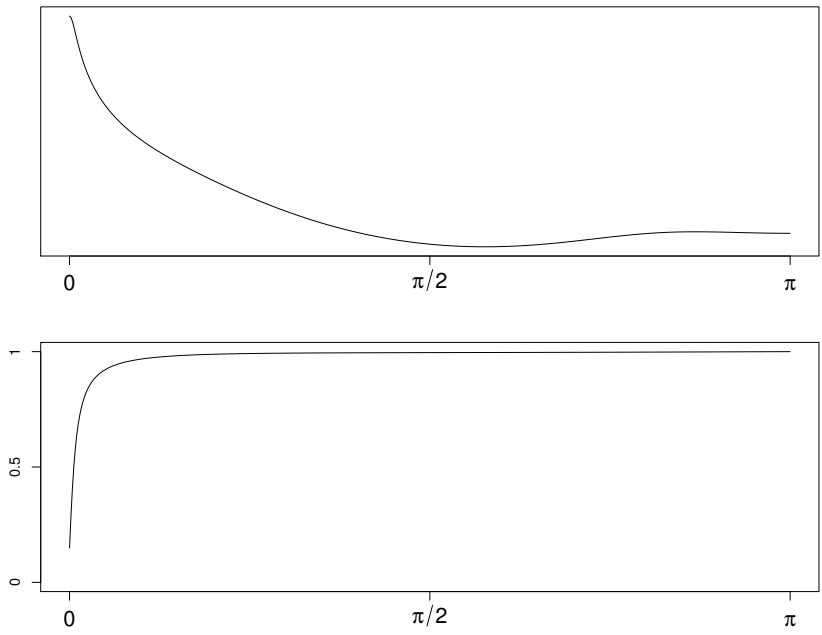

Figure 6.3 *An example of a spectrum, together with the corresponding normalized spectral distribution function.*

Equation (6.9) expresses $\gamma(k)$ in terms of $f(\omega)$ as a cosine transform. It can be shown that the corresponding inverse relationship is given by

$$f(\omega) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \gamma(k) \mathrm{e}^{-\mathrm{i}\omega k}, \tag{6.11}$$

so that the spectrum is the **Fourier transform**[3] of the acv.f. Since $\gamma(k)$ is an even function of k, Equation (6.11) is often written in the equivalent form

$$f(\omega) = \frac{1}{\pi} \left[\gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k) \cos \omega k \right]. \tag{6.12}$$

Note that if we try to apply Equation (6.12) to a process containing a deterministic component at a particular frequency ω_0, then $\sum \gamma(k) \cos \omega k$ will not converge when $\omega = \omega_0$. This arises because $F(\omega)$ has a step change at ω_0 and so will not be differentiable at ω_0. Thus its derivative $f(\omega)$ will not be defined at $\omega = \omega_0$.

[3] See Appendix A for details on the Fourier transform.

The reader should note that several other definitions of the spectrum are given in the literature, most of which differ from Equation (6.12) by a constant multiple and by the range of definition of $f(\omega)$. The most popular approach is to define the spectrum in the range $(-\pi, \pi)$ by

$$f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) \mathrm{e}^{-\mathrm{i}\omega k}, \tag{6.13}$$

whose inverse relationship (see Appendix A) is

$$\gamma(k) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega k} f(\omega) \mathrm{d}\omega. \tag{6.14}$$

Jenkins and Watts (1968) use these equations, except that they take $f = \omega/2\pi$ as the frequency variable (see Equations (A.3) and (A.4)). Equations (6.13) and (6.14), which form a Fourier transform pair, are the more usual form of the Wiener–Khintchine relations. The formulation is slightly more general in that it can be applied to complex-valued time series. However, for real time series we find that $f(\omega)$ is an even function of ω, and then we need only consider $f(\omega)$ for $\omega > 0$. As we are concerned only with real-valued processes, we prefer Equation (6.11) defined on $(0, \pi)$.

It is sometimes useful to use a normalized form of the spectral density function, given by

$$f^*(\omega) = f(\omega)/\sigma_X^2 = \frac{\mathrm{d}F^*(\omega)}{\mathrm{d}\omega}. \tag{6.15}$$

This is the derivative of the normalized spectral distribution function (see Equation (6.7)). Then we find that $f^*(\omega)$ is the Fourier transform of the *autocorrelation* function (ac.f.), namely,

$$f^*(\omega) = \frac{1}{\pi} \left[1 + 2 \sum_{k=1}^{\infty} \rho(k) \cos \omega k \right]. \tag{6.16}$$

This means that $f^*(\omega)\,\mathrm{d}\omega$ is the *proportion*[4] of variance in the interval $(\omega, \omega + \mathrm{d}\omega)$.

6.4 The Spectrum of a Continuous Process

For a continuous purely indeterministic stationary process, $X(t)$, the autocovariance function, $\gamma(\tau)$, is defined for all τ and the (power) spectral density function, $f(\omega)$, is defined for all positive ω. The relationship between

[4] As an example of the difficulties in comparing formulae from different sources, Kendall et al. (1983, Equation 47.20) define the spectral density function in terms of the autocorrelation function over the same range, $(0, \pi)$, as we do, but omit the constant $1/\pi$ from Equation (6.16). This makes it more difficult to give the function a physical interpretation. Instead they introduce an intensity function that corresponds to our power spectrum.

these functions is very similar to that in the discrete-time case except that there is no upper bound to the frequency. We have

$$\begin{aligned} f(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\gamma(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau \\ &= \frac{2}{\pi}\int_{0}^{\infty}\gamma(\tau)\cos\,\omega\tau\,\mathrm{d}\tau \end{aligned} \tag{6.17}$$

for $0 < \omega < \infty$, with the inverse relationship

$$\gamma(\tau) = \int_{0}^{\infty} f(\omega)\cos\,\omega\tau\,\mathrm{d}\omega. \tag{6.18}$$

6.5 Derivation of Selected Spectra

This section derives the spectral density functions of six simple, but important, types of discrete-time stationary processes.

(1) *Purely random processes*

A purely random process $\{Z_t\}$ as defined in Section 3.4, is a sequence of independent random variables. Thus, if $\mathrm{Var}(Z_t) = \sigma_Z^2$, then the acv.f. is given by

$$\gamma(k) = \begin{cases} \sigma_Z^2 & k = 0 \\ 0 & \text{otherwise,} \end{cases}$$

so that the power spectral density function (or spectrum) is given by

$$f(\omega) = \sigma_Z^2/\pi \tag{6.19}$$

using Equation (6.12). In other words the spectrum is constant in the range $(0, \pi)$.

In continuous time, we have already pointed out that a continuous white noise process is physically unrealizable. A process is regarded as a practical approximation to continuous white noise if its spectrum is substantially constant over the frequency band of interest, even if it then approaches zero at high frequency.

(2) *First-order moving average processes*

The first-order moving average (MA) process (see Section 3.6)

$$X_t = Z_t + \beta Z_{t-1}$$

has an ac.f. given by

$$\rho(k) = \begin{cases} 1 & k = 0 \\ \beta/(1+\beta^2) & k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

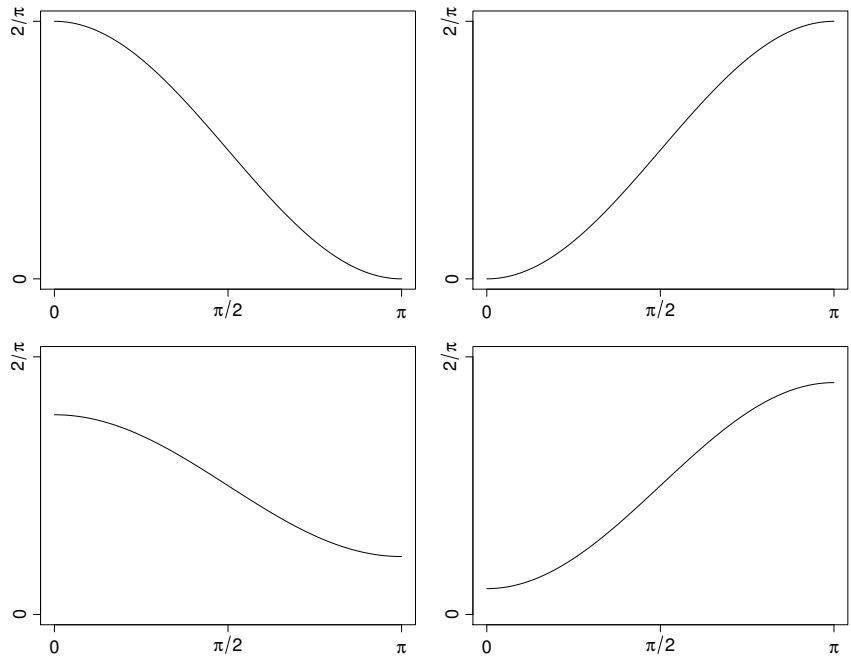

Figure 6.4 *Examples of normalized spectra, $f^*(\omega)$, of first-order moving average processes with $\beta = 1$ (Top left), $\beta = -1$ (Top right), $\beta = 0.3$ (Bottom left), and $\beta = -0.5$ (Bottom right).*

Thus, using Equation (6.16), the normalized spectral density function is given by

$$f^*(\omega) = \frac{1}{\pi}\left[1 + (2\beta \cos \omega)/(1 + \beta^2)\right] \tag{6.20}$$

for $0 < \omega < \pi$. The power spectral density function is then

$$f(\omega) = \sigma_X^2 f^*(\omega),$$

where $\sigma_X^2 = (1 + \beta^2)\sigma_Z^2$.

The shape of the spectrum depends on the value of β. When $\beta > 0$ the power is concentrated at low frequencies, giving what is called a **low-frequency spectrum**; if $\beta < 0$ the power is concentrated at high frequencies, giving a **high-frequency spectrum**. Examples are shown in Figure 6.4.

(3) *First-order autoregressive processes*

The first-order autoregressive (AR) process (see Section 3.7)

$$X_t = \alpha X_{t-1} + Z_t \tag{6.21}$$

has an acv.f. given by

$$\gamma(k) = \sigma_X^2 \alpha^{|k|} \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Then, using Equation (6.11), the power spectral density function is given by

$$\begin{aligned} f(\omega) &= \frac{\sigma_X^2}{\pi}\left(1+\sum_{k=1}^{\infty}\alpha^k \mathrm{e}^{-\mathrm{i}k\omega}+\sum_{k=1}^{\infty}\alpha^k \mathrm{e}^{\mathrm{i}k\omega}\right) \\ &= \frac{\sigma_X^2}{\pi}\left(1+\frac{\alpha \mathrm{e}^{-\mathrm{i}\omega}}{1-\alpha \mathrm{e}^{-\mathrm{i}\omega}}+\frac{\alpha \mathrm{e}^{\mathrm{i}\omega}}{1-\alpha \mathrm{e}^{\mathrm{i}\omega}}\right) \end{aligned}$$

which, after some algebra, gives

$$\begin{aligned} f(\omega) &= \sigma_X^2(1-\alpha^2)/[\pi(1-2\alpha\cos\omega+\alpha^2)] \qquad (6.22) \\ &= \sigma_Z^2/[\pi(1-2\alpha\cos\omega+\alpha^2)] \qquad (6.23) \end{aligned}$$

since $\sigma_Z^2 = \sigma_X^2(1-\alpha^2)$.

The shape of the spectrum depends on the value of α. When $\alpha > 0$, the spectral density function is 'large' when ω is 'small', so that power is concentrated at low frequencies – a low-frequency spectrum. On the other hand, when $\alpha < 0$, the power is concentrated at high frequencies – a high-frequency spectrum. Examples are shown in Figure 6.5.

It is hoped that the reader finds the shapes of the spectra in Figure 6.4 intuitively reasonable. For example, if α is negative, then it is clear from Equation (6.21) that values of X_t will tend to change sign at every time point, and rapid oscillations like this correspond to high-frequency variation.

(4) *ARMA(1, 1) processes*

The ARMA(1, 1) process (see Section 3.8)

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}$$

has an acv.f. given by

$$\begin{aligned} \sigma_X^2 = \gamma(0) &= \frac{1+2\alpha\beta+\beta^2}{1-\alpha^2}\sigma_Z^2, \\ \gamma(k) = \gamma(0)\rho(k) &= \frac{(1+\alpha\beta)(\alpha+\beta)}{1+2\alpha\beta+\beta^2}\sigma_X^2\alpha^{k-1}, \qquad k \geq 1. \end{aligned}$$

Then, using Equation (6.11), the power spectral density function is given by

$$\begin{aligned} f(\omega) &= \frac{\sigma_X^2}{\pi}\left[1+\frac{(1+\alpha\beta)(\alpha+\beta)}{\alpha(1+2\alpha 2\beta+\beta^2)}\left(\sum_{k=1}^{\infty}\alpha^k \mathrm{e}^{-\mathrm{i}k\omega}+\sum_{k=1}^{\infty}\alpha^k \mathrm{e}^{\mathrm{i}k\omega}\right)\right] \\ &= \frac{\sigma_X^2}{\pi}\left[1+\frac{(1+\alpha\beta)(\alpha+\beta)}{\alpha(1+2\alpha 2\beta+\beta^2)}\left(\frac{\alpha \mathrm{e}^{-\mathrm{i}\omega}}{1-\alpha \mathrm{e}^{-\mathrm{i}\omega}}+\frac{\alpha \mathrm{e}^{\mathrm{i}\omega}}{1-\alpha \mathrm{e}^{\mathrm{i}\omega}}\right)\right]. \end{aligned}$$

After some algebra, we have

$$\begin{aligned} f(\omega) &= \frac{\sigma_X^2}{\pi}\cdot\frac{1-\alpha^2}{1+2\alpha\beta+\beta^2}\cdot\frac{1+2\beta\cos\omega+\beta^2}{1-2\alpha\cos\omega+\alpha^2} \qquad (6.24) \\ &= \frac{\sigma_Z^2}{\pi}\cdot\frac{1+2\beta\cos\omega+\beta^2}{1-2\alpha\cos\omega+\alpha^2}. \qquad (6.25) \end{aligned}$$

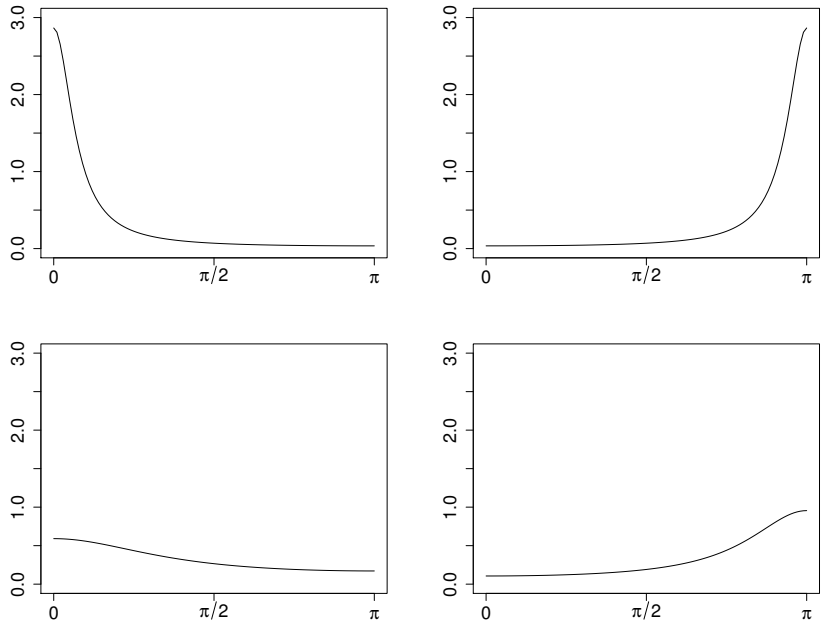

Figure 6.5 *Examples of normalized spectra, $f^*(\omega)$, of first-order autoregressive processes with $\alpha = 0.8$ (Top left), $\alpha = -0.8$ (Top right), $\alpha = 0.3$ (Bottom left), and $\alpha = -0.5$ (Bottom right).*

The shape of the spectrum depends on the values of α and β. When $\alpha > 0$ and $\beta > 0$, the power is concentrated at low frequencies. When $\alpha < 0$ and $\beta < 0$, the power is concentrated at high frequencies. For other values of α and β, the shape of the spectrum becomes complicated. Examples are shown in Figure 6.6.

(5) *Higher order AR, MA and ARMA processes*

For a second-order AR process

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t,$$

after some algebra, it can be shown that its spectrum is given by

$$\begin{aligned} f(\omega) &= \sigma_Z^2/\pi[1 + \alpha_1^2 + \alpha_2^2 - 2\alpha_1(1 - \alpha_2)\cos\omega - 2\alpha_2\cos 2\omega] \\ &= \frac{\sigma_Z^2}{\pi} \cdot \frac{1}{(1 - \alpha_1 e^{-i\omega} - \alpha_2 e^{-i2\omega})(1 - \alpha_1 e^{i\omega} - \alpha_2 e^{i2\omega})} \\ &= \frac{\sigma_Z^2}{\pi} \cdot \frac{1}{|1 - \alpha_1 e^{i\omega} - \alpha_2 e^{i2\omega}|^2} \end{aligned}$$

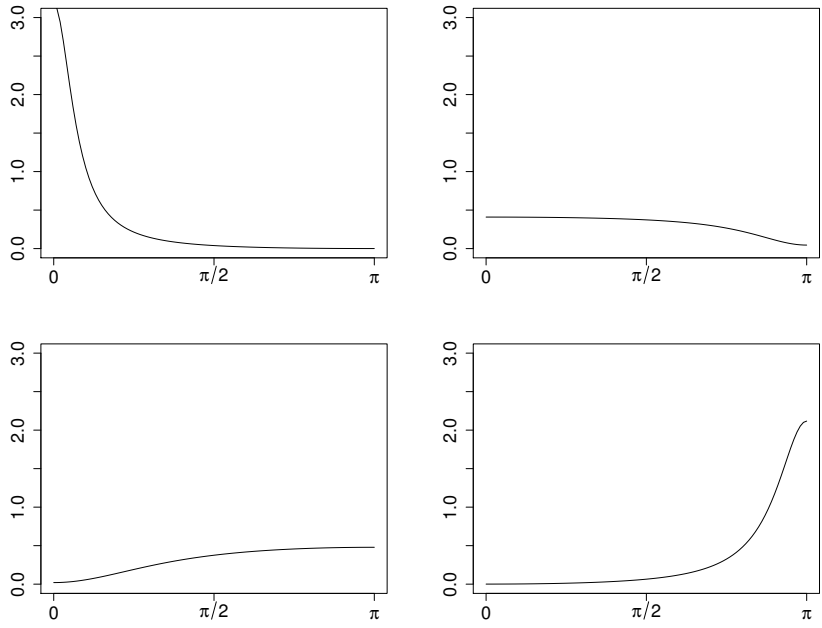

Figure 6.6 *Examples of normalized spectra, $f^*(\omega)$, of ARMA(1, 1) processes (Top left: $\alpha = 0.8, \beta = 0.8$; Top right: $\alpha = -0.5, \beta = 0.8$; Bottom left: $\alpha = 0.3, \beta = -0.8$; Bottom right: $\alpha = -0.7, \beta = -0.8$).*

for $0 < \omega < \pi$, where $|\cdot|$ represents the modulus of a complex number. The shape of the spectrum depends on the values of α_1 and α_2. It is possible to find values of α_1 and α_2 that give a spectrum with a peak at zero frequency (a low-frequency spectrum), a peak at frequency π (a high-frequency spectrum), a peak *between* 0 and π, or a *minimum* between 0 and π.

For AR processes of higher order than first order, their spectrum can be similarly computed. For an AR process of order p

$$\phi(B)X_t = Z_t, \qquad \phi(B) = 1 - \alpha_1 B - \ldots - \alpha_p B^p,$$

it can be shown that its spectrum is given by

$$f(\omega) = \frac{\sigma_Z^2}{\pi} \cdot \frac{1}{|\phi(\mathrm{e}^{\mathrm{i}\omega})|^2}.$$

For a MA process of order q

$$X_t = \theta(B)Z_t, \qquad \theta(B) = 1 + \beta_1 B + \ldots + \beta_q B^q,$$

it can be shown that its spectrum is given by

$$f(\omega) = \frac{\sigma_Z^2}{\pi} \cdot |\theta(\mathrm{e}^{\mathrm{i}\omega})|^2.$$

Similarly, for an ARMA process of order (p, q),

$$\phi(B)X_t = \theta(B)Z_t,$$

its spectrum can be expressed as

$$f(\omega) = \frac{\sigma_Z^2}{\pi} \cdot \frac{|\theta(\mathrm{e}^{\mathrm{i}\omega})|^2}{|\phi(\mathrm{e}^{\mathrm{i}\omega})|^2}.$$

(6) *Deterministic sinusoidal perturbations*

Suppose that

$$X_t = \cos(\omega_0 t + \phi) \tag{6.26}$$

where ω_0 is a constant in $(0, \pi)$ and ϕ is a random variable having a uniform distribution on $(0, 2\pi)$. As explained in Section 3.13, ϕ is fixed for a single realization of the process and Equation (6.26) defines a purely deterministic process.

The acv.f. of the process is given by

$$\gamma(k) = \frac{1}{2}\cos\omega_0 k,$$

which we note does *not* tend to zero as k increases. This contrasts with the behaviour of the ac.f. of a stationary linearly indeterministic process, which does tend to zero as k increases. Put loosely, we can say that the above deterministic process has a long memory.

From the model Equation (6.24), it is obvious that all the 'power' of the process is concentrated at the frequency ω_0. Since $E(X_t) = 0$, we find that $\mathrm{Var}(X_t) = E(X_t^2) = \frac{1}{2}$, so that the power spectral distribution function is given by

$$F(\omega) = \begin{cases} 0 & \omega < \omega_0, \\ \frac{1}{2} & \omega \geq \omega_0 \end{cases}.$$

Since this is a step function, it has no derivative at ω_0 and so the spectrum is not defined at ω_0. If we nevertheless try to use Equation (6.12) to obtain the spectrum as the Fourier transform of the acv.f., then we find that

$$f(\omega) = 0 \qquad \omega \neq \omega_0$$

as expected, but that $\sum \gamma(k)\cos\omega k$ does not converge at $\omega = \omega_0$. This confirms that the spectrum is not defined at ω_0.

(7) *A mixture of deterministic and stochastic components*

Our final example contains a mixture of deterministic and stochastic components, namely,

$$X_t = \cos(\omega_0 t + \phi) + Z_t,$$

where ω_0, ϕ are as defined in Example 5 above, and $\{Z_t\}$ is a purely random process with mean zero and variance σ_Z^2. Then we find that the acv.f. is given by

$$\gamma(k) = \begin{cases} \frac{1}{2} + \sigma_Z^2 & k = 0 \\ \frac{1}{2}\cos\omega_0 k & k = \pm 1, \pm 2, \ldots . \end{cases}$$

As in Example 5 above, note that $\gamma(k)$ does *not* tend to zero, as k increases, because X_t contains a periodic deterministic component.

We can obtain the power spectral distribution function of X_t by using Equation (6.6), since the deterministic component $\cos(\omega_0 t + \phi)$ has power spectral distribution function

$$F_1(\omega) = \begin{cases} 0 & \omega < \omega_0, \\ \frac{1}{2} & \omega \geq \omega_0. \end{cases}$$

while the stochastic component Z_t has power spectral distribution function

$$F_2(\omega) = \sigma_Z^2\, \omega/\pi \qquad 0 < \omega < \pi$$

on integrating Equation (6.19). Thus the combined power spectral distribution function is given by

$$F(\omega) = \begin{cases} \sigma_Z^2 \omega/\pi & 0 < \omega < \omega_0 \\ \frac{1}{2} + \sigma_Z^2 \omega/\pi & \omega_0 \leq \omega < \pi \end{cases}$$

As in Example 5, the distribution function has a step at ω_0 and so the power spectrum is not defined at $\omega = \omega_0$.

Exercises

In the following questions $\{Z_t\}$ denotes a purely random process, mean zero and variance σ_Z^2.

6.1 Find (a) the power spectral density function, (b) the normalized spectral density function of the first-order AR process

$$X_t = \lambda X_{t-1} + Z_t$$

with $|\lambda| < 1$. (Note: (a) is covered in the text, but see if you can do it without looking it up.)

6.2 Find the power spectral density functions of the following MA processes:
(a) $X_t = Z_t + Z_{t-1} + Z_{t-2}$
(b) $X_t = Z_t + 0.5Z_{t-1} - 0.3Z_{t-2}$

6.3 If μ denotes a constant, show that the second-order MA process

$$X_t = \mu + Z_t + 0.8Z_{t-1} + 0.5Z_{t-2}$$

is second-order stationary. Find the acv.f. and ac.f. of $\{X_t\}$ and show that its normalized spectral density function is given by

$$f^*(\omega) = (1 + 1.27\cos\omega + 0.53\cos 2\omega)/\pi \qquad 0 < \omega < \pi.$$

6.4 A stationary time series $(X_t; t = \ldots, -1, 0, +1, \ldots)$ has normalized spectral density function

$$f^*(\omega) = 2(\pi - \omega)/\pi^2 \qquad 0 < \omega < \pi.$$

Show that its ac.f. is given by

$$\rho(k) = \begin{cases} 1 & k = 0, \\ (2/\pi k)^2 & k \text{ odd}, \\ 0 & k \text{ even } (\neq 0). \end{cases}$$

6.5 A two-state Markov process may be set up as follows. Alpha particles from a radioactive source are used to trigger a flip-flop device that takes the states $+1$ and -1 alternately. The times t_i at which changes occur constitute a Poisson process, with mean event rate λ. Let $X(t)$ denote the state variable at time t. If the process is started at time $t = 0$ with the two possible states having equal probability (so that $P[X(0) = 1] = P[X(0) = -1] = \frac{1}{2}$, show that the process is second-order stationary, with autocorrelation function

$$\rho(u) = \mathrm{e}^{-2\lambda|u|} \qquad -\infty < u < \infty$$

and spectral density function

$$f(\omega) = 4\lambda/[\pi(4\lambda^2 + \omega^2)] \qquad 0 < \omega < \infty.$$

6.6 Show that if $\{X_t\}$ and $\{Y_t\}$ are independent, stationary processes with power spectral density functions $f_x(\omega)$ and $f_y(\omega)$, then $\{V_t\} = \{X_t + Y_t\}$ is also stationary with power spectral density function $f_v(\omega) = f_x(\omega) + f_y(\omega)$. If X_t is a first-order AR process

$$X_t = \alpha X_{t-1} + W_t \qquad -1 < \alpha < +1$$

and $\{Y_t\}, \{W_t\}$ are independent purely random processes with zero mean and common variance σ^2, show that the power spectral density function of $\{V_t\}$ is given by

$$f_v(\omega) = \sigma^2(2 - 2\alpha\cos\omega + \alpha^2)/\pi(1 - 2\alpha\cos\omega + \alpha^2) \qquad 0 < \omega < \pi.$$

6.7 Show that the normalized spectral density function of the ARMA$(1, 1)$ process

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}$$

(with $|\alpha| < 1$ and $|\beta| < 1$) is given by

$$f^*(\omega) = \frac{1}{\pi}\,[1 + 2\rho(1)(\cos\omega - \alpha)/(1 - 2\alpha\cos\omega + \alpha^2)] \qquad 0 < \omega < \pi.$$

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Chapter 7

Spectral Analysis

Spectral analysis is the name given to methods of estimating the spectral density function, or spectrum, of a given time series.

Before about 1900, research workers such as A. Schuster were essentially concerned with looking for 'hidden periodicities' in data at one or two specific frequencies. Spectral analysis as we know it today is concerned with estimating the spectrum over the whole range of frequencies. The techniques are widely used by many scientists, particularly in electrical engineering, physics, meteorology and marine science.

We are mainly concerned with purely indeterministic processes, which have a continuous spectrum, but the techniques can also be used for deterministic processes to pick out periodic components in the presence of noise.

7.1 Fourier Analysis

Traditional spectral analysis is essentially a modification of Fourier analysis so as to make it suitable for stochastic rather than deterministic functions of time. **Fourier analysis** (e.g. Priestley, 1981) is essentially concerned with approximating a function by a sum of sine and cosine terms, called the Fourier series representation. Suppose that a function $f(t)$ is defined on $(-\pi, \pi]$[1] and satisfies the so-called Dirichlet conditions. These conditions ensure that $f(t)$ is reasonably 'well behaved', meaning that, over the range $(-\pi, \pi]$, $f(t)$ is absolutely integrable, has a finite number of discontinuities, and has a finite number of maxima and minima. Then $f(t)$ may be approximated by the Fourier series

$$\frac{a_0}{2} + \sum_{r=1}^{k} (a_r \cos rt + b_r \sin rt),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\ \mathrm{d}t, \\ a_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos rt\ \mathrm{d}t \qquad r = 1, 2, \ldots, \end{aligned}$$

[1] The different-shaped brackets indicate that the lower limit $-\pi$ is *not* included in the interval, while the square bracket indicates that the upper limit $+\pi$ *is* included.

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin rt \; \mathrm{d}t \qquad r = 1, 2, \ldots.$$

It can be shown that this Fourier series converges to $f(t)$ as $k \to \infty$ except at points of discontinuity, where it converges to halfway[2] up the step change.

In order to apply Fourier analysis to discrete time series, we need to consider the Fourier series representation of $f(t)$ when $f(t)$ is defined only on the integers $1, 2, \ldots, N$. Rather than write down the formula, we demonstrate that the required Fourier series emerges naturally by considering a simple sinusoidal model.

7.2 A Simple Sinusoidal Model

Suppose we suspect that a given time series, with observations made at unit time intervals, contains a deterministic sinusoidal component at a known frequency ω, together with a random error term. Then we will consider the model

$$X_t = \mu + \alpha\cos\omega t + \beta\sin\omega t + Z_t, \tag{7.1}$$

where Z_t denotes a purely random process, and μ, α, β are parameters to be estimated from the data.

The observations will be denoted by $(x_1, x_2, \ldots, x_N)$. The algebra in the next few sections is somewhat simplified if we confine ourselves to the case where N is *even*. There is no real difficulty in extending the results to the case where N is odd (e.g. Anderson, 1971), and indeed many of the later estimation formulae apply for both odd and even N, but some results require one to consider odd N and even N separately. Thus, if N happens to be odd and a spectral analysis is required, computation can be made somewhat simpler by removing the first observation so as to make N even. If N is reasonably large, little information is lost.

Expected values for the model in Equation (7.1) can be represented in matrix notation by

$$E(\boldsymbol{X}) = A\,\boldsymbol{\theta}$$

where

$$\begin{aligned} \boldsymbol{X}^T &= (X_1, \ldots, X_N) \\ \boldsymbol{\theta}^T &= (\mu, \alpha, \beta) \\ A &= \begin{pmatrix} 1 & \cos\omega & \sin\omega \\ 1 & \cos 2\omega & \sin 2\omega \\ \cdots & \cdots & \cdots \\ 1 & \cos N\omega & \sin N\omega \end{pmatrix}. \end{aligned}$$

[2] Mathematicians say that this is the average of the limit from below and the limit from above, sometimes written as $\frac{1}{2}[f(t-0) + f(t+0)]$.

As this model is linear in the parameters μ, α and β, it is an example of a general linear model. In that case the least squares estimate of $\boldsymbol{\theta}$, which minimizes $\sum_{t=1}^{N}(x_t - \mu - \alpha\cos\omega t - \beta\sin\omega t)^2$, is 'well known' to be

$$\widehat{\boldsymbol{\theta}} = (A^T A)^{-1} A^T \boldsymbol{x},$$

where

$$\boldsymbol{x}^T = (x_1, \ldots, x_N).$$

The above formulae hold for any value of the frequency ω, but they only make practical sense for values of ω that are not too high or too low. As noted in Section 6.2, the highest frequency we can uniquely fit to the data is the Nyquist frequency, given by $\omega = \pi$, which completes one cycle every two observations. At the other end of the spectrum, the lowest frequency we can reasonably fit completes one cycle in the whole length of the time series. These upper and lower limits will be explained further in Section 7.2.1 below. By equating the cycle length $2\pi/\omega$ to N, we find that the lowest frequency is given by $2\pi/N$. The formulae for the least squares estimates of $\widehat{\boldsymbol{\theta}}$ turn out to be particularly simple if ω is restricted to one of the values

$$\omega_p = 2\pi p/N \quad p = 1, \ldots, N/2,$$

which lie in equal steps from the lowest frequency $2\pi/N$ to the Nyquist frequency π. In this case, it turns out that $(A^T A)$ is a diagonal matrix in view of the following 'well-known' trigonometric results (all summations are for $t = 1$ to N):

$$\sum \cos\omega_p t = \sum \sin\omega_p t = 0, \tag{7.2}$$

$$\sum \cos\omega_p t \cos\omega_q t = \begin{cases} 0 & p \neq q, \\ N & p = q = N/2, \\ N/2 & p = q \neq N/2, \end{cases} \tag{7.3}$$

$$\sum \sin\omega_p t \sin\omega_q t = \begin{cases} 0 & p \neq q, \\ 0 & p = q = N/2, \\ N/2 & p = q \neq N/2, \end{cases} \tag{7.4}$$

$$\sum \cos\omega_p t \sin\omega_q t = 0 \qquad \text{for all } p, q. \tag{7.5}$$

With $(A^T A)$ diagonal, it is easy to evaluate the least squares estimate of $\boldsymbol{\theta}$, as the inverse $(A^T A)^{-1}$ will also be diagonal. For ω_p such that $p \neq N/2$, we find (Exercise 7.2)

$$\begin{aligned} \widehat{\mu} &= \sum x_t/N = \bar{x}, \\ \widehat{\alpha} &= 2\left[\sum x_t \cos\omega_p t\right]/N, \\ \widehat{\beta} &= 2\left[\sum x_t \sin\omega_p t\right]/N. \end{aligned} \tag{7.6}$$

If $p = N/2$, we ignore the term in $\beta \sin \omega t$, which is zero for all t, and find

$$\begin{aligned} \widehat{\mu} &= \bar{x}, \\ \widehat{\alpha} &= \textstyle\sum (-1)^t x_t / N. \end{aligned} \tag{7.7}$$

The model in Equation (7.1) is essentially the one used before about 1900 to search for hidden periodicities, but this model has now gone out of fashion. However, it can still be useful if there is reason to suspect that a time series does contain a deterministic periodic component at a known frequency and it is desired to isolate this component (e.g. Bloomfield, 2000, Chapters 2, 3).

Readers who are familiar with the analysis of variance (ANOVA) technique will be able to work out that the total corrected sum of squared deviations, namely,

$$\sum_{t=1}^{N} (x_t - \bar{x})^2$$

can be partitioned into two components, namely, the residual sum of squares and the sum of squares 'explained' by the periodic component at frequency ω_p. This latter component is given by

$$\sum_{t=1}^{N} (\widehat{\alpha} \cos \omega_p t + \widehat{\beta} \sin \omega_p t)^2$$

which, after some algebra (Exercise 7.2), can be shown to be

$$\begin{array}{ll} (\widehat{\alpha}^2 + \widehat{\beta}^2) N/2 & p \neq N/2 \\ \widehat{\alpha}^2 N & p = N/2 \end{array} \tag{7.8}$$

using Equations (7.2)–(7.5).

When fitting the simple sinusoidal model in Equation (7.1), we restricted the frequency ω to one of the values $(2\pi/N, 4\pi/N, \ldots, \pi)$, assuming that N is even. Here we examine the practical rationale for the upper and lower limits, namely, π and $2\pi/N$.

In Section 6.2, we pointed out that, for a discrete-time process measured at unit intervals of time, there is no loss of generality in restricting the spectral distribution function to the range $(0, \pi)$. We now demonstrate that the upper bound π, called the **Nyquist frequency**, is indeed the highest frequency about which we can get meaningful information from a set of data.

First, we give a more general form for the Nyquist frequency. If observations are taken at equal intervals of time of length Δt, then the Nyquist (angular) frequency is given by $\omega_N = \pi/\Delta t$. The equivalent frequency expressed in cycles per unit time is $f_N = \omega_N/2\pi = 1/2\Delta t$.

Consider the following example. Suppose that temperature readings are taken every day in a certain town at noon. It is clear that these observations will tell us nothing about temperature variation *within* a day. In particular,

they will not tell us whether nights are hotter or cooler than days. With only one observation per day, the Nyquist frequency is $\omega_N = \pi$ radians per day or $f_N = \frac{1}{2}$ cycle per day (or 1 cycle per 2 days). This is lower than the frequencies, which correspond to variation within a day. For example, variation with a period of 1 day has (angular) frequency $\omega = 2\pi$ radians per day or $f = 1$ cycle per day. In order to get information about variation within a day at these higher frequencies, we must increase the sampling rate and take two or more observations per day.

A similar example is provided by yearly sales figures. These will obviously give no information about any seasonal effects, whereas monthly or quarterly observations *will* give information about seasonality.

At the other end of the spectrum, we will now explain why there is a *lowest* frequency below which it is not sensible to try to fit to a set of data. If we had just 6 months of temperature readings from winter to summer, the analyst would not be able to decide, from the data alone, whether there is an upward trend in the observations or whether winters are colder than summers. However, with 1 year's data, it *would* become clear that winters are colder than summers. Thus if we are interested in variation at the low frequency of 1 cycle per year, then we should have at least 1 year's data, in which case the lowest frequency we can fit is at 1 cycle per year. With weekly observations, for example, 1 year's data have $N = 52$, $\Delta t = 1$ week, and the lowest angular frequency of $2\pi/N\Delta t$ corresponds to a frequency of $1/N\Delta t$ cycles per week. (Note that all time units must be expressed in terms of the same period, here a week.) The lowest frequency is therefore $1/52$ cycles per week, which can now be converted to 1 cycle per year.

The lowest frequency, namely, $2\pi/N\Delta$, is sometimes called the **fundamental Fourier frequency**, because the Fourier series representation of the data is normally evaluated at the frequencies $\omega_p = 2\pi p/N\Delta$ for $p = 1, \ldots, N/2$, which are all integer multiples of the fundamental frequency. These integer multiples are often called **harmonics**. The phrase *fundamental frequency* is perhaps more typically, and more helpfully, used when a function, $f(t)$ say, is **periodic** with period T so that $f(t + nT) = f(t)$ for all integer values of n. Then $f = 1/T$, or $\omega = 2\pi/T$, is called the fundamental frequency and the Fourier series representation of $f(t)$ is a sum over integer multiples, or harmonics, of the fundamental frequency. When $T = N\Delta =$ (the length of the observed time series), the fundamental frequencies coincide. This raises a practical point, in regard to choosing the length of a time series. Suppose, for example, that you are collecting weekly data and are particularly interested in annual variation. As noted above, you should collect at least 1 year's data. If you collect exactly 52 weeks of data[3], then the fundamental frequency will be at exactly 1 cycle per year. We will see that this makes it much easier to interpret the results of a spectral analysis. The fundamental frequency is at 1 cycle per year and the harmonics are at 2 cycles

[3]For simplicity, ignore day 365, and day 366 if a leap year.

per year, 3 cycles per year and so on. However, if you have say an extra 12 weeks of data making 64 weeks, then it will be much harder to interpret the results at frequencies $\omega_p = 2\pi p/N\Delta$. Wherever possible, you should choose the length of the time series so that the harmonics cover the frequencies of particular interest. The easiest option is to collect observations covering an integer multiple of the lowest wavelength of particular interest. This ensures that this frequency is an integer multiple of the fundamental frequency. Thus, to investigate annual variation, 2 years of data is good and 3 or 4 years of data even better.

The reader will notice that the Nyquist frequency does *not* depend on N, but rather only on the sampling frequency, whereas the lowest frequency *does* depend on N. Put another way, the lower the frequency we are interested in, the longer the time period over which we need to take measurements, whereas the higher the frequency we are interested in, the more frequently must we take observations.

7.3 Periodogram Analysis

Early attempts at discovering hidden periodicities in a given time series basically consisted of repeating the analysis of Section 7.2 at all the frequencies $2\pi/N, 4\pi/N, \ldots, \pi$. In view of Equations (7.3)–(7.5), the different terms are orthogonal and we end up with the finite Fourier series representation of the $\{x_t\}$, namely

$$x_t = a_0 + \sum_{p=1}^{(N/2)-1} [a_p \cos(2\pi pt/N) + b_p \sin(2\pi pt/N)] + a_{N/2} \cos \pi t \qquad (7.9)$$

for $t = 1, 2, \ldots, N$, where the coefficients $\{a_p, b_p\}$ are of the same form as Equations (7.6) and (7.7), namely

$$\left.\begin{aligned} a_0 &= \bar{x} \\ a_{N/2} &= \sum (-1)^t x_t/N \\ a_p &= 2\left[\sum x_t \cos(2\pi pt/N)\right]/N \\ b_p &= 2\left[\sum x_t \sin(2\pi pt/N)\right]/N \end{aligned}\right\} \quad p = 1, \ldots, (N/2) - 1. \qquad (7.10)$$

An analysis along these lines is sometimes called a **Fourier analysis** or a **harmonic analysis**. The Fourier series representation in Equation (7.9) has N parameters to describe N observations and so can be made to fit the data exactly (just as a polynomial of degree $N - 1$ involving N parameters can be found that goes exactly through N observations in polynomial regression). This explains why there is no error term in Equation (7.9) in contrast to Equation (7.1). Also note that there is no term in $\sin \pi t$ in Equation (7.9) as $\sin \pi t$ is zero for all integer t.

It is worth stressing that the Fourier series coefficients in Equation (7.10) at a given frequency ω are exactly the same as the least squares estimates for the coefficients of the model in Equation (7.1).

The overall effect of the Fourier analysis of the data is to partition the variability of the series into components at frequencies $2\pi/N, 4\pi/N, \ldots, \pi$. The component at frequency $\omega_p = 2\pi p/N$ is called the pth harmonic. For $p \neq N/2$, it can be useful to write the pth harmonic in the equivalent form

$$a_p \cos \omega_p t + b_p \sin \omega_p t = R_p \cos(\omega_p t + \phi_p) \tag{7.11}$$

where

$$R_p = \sqrt{a_p^2 + b_p^2} \tag{7.12}$$

is the **amplitude** of the pth harmonic, and

$$\phi_p = \tan^{-1}(-b_p/a_p) \tag{7.13}$$

is the **phase** of the pth harmonic.

We have already noted in Section 7.2 that, for $p \neq N/2$, the contribution of the pth harmonic to the total sum of squares is given by $N(a_p^2 + b_p^2)/2$. Using Equation (7.12), this is equal to $NR_p^2/2$. Extending this result using Equations (7.2)–(7.5) and (7.9), we have, after some algebra (Exercise 7.3), that

$$\sum_{t=1}^{N} (x_t - \bar{x})^2 = N \sum_{p=1}^{(N/2)-1} R_p^2/2 + N a_{N/2}^2.$$

Dividing through by N we have

$$\sum (x_t - \bar{x})^2/N = \sum_{p=1}^{(N/2)-1} R_p^2/2 + a_{N/2}^2 \tag{7.14}$$

which is known as **Parseval's theorem**. The left-hand side of Equation (7.14) is effectively the variance[4] of the observations. Thus $R_p^2/2$ is the contribution of the pth harmonic to the variance, and Equation (7.14) shows how the total variance is partitioned.

If we plot $R_p^2/2$ against $\omega_p = 2\pi p/N$, we obtain a line spectrum. A different type of line spectrum occurs in the physical sciences when light from molecules in a gas discharge tube is viewed through a spectroscope. The light has energy at discrete frequencies and this energy can be seen as bright lines. However, most time series have continuous spectra, and then it is inappropriate to plot a line spectrum. If we regard $R_p^2/2$ as the contribution to variance in the range $\omega_p \pm \pi/N$, we can plot a histogram whose height in the range $\omega_p \pm \pi/N$ is such that

$$\begin{aligned} R_p^2/2 &= \text{area of histogram rectangle} \\ &= \text{height of histogram } \times 2\pi/N. \end{aligned}$$

[4]The divisor is N rather than the more usual $(N-1)$, but this makes little difference for large N.

Thus the height of the histogram at ω_p, denoted by $I(\omega_p)$, is given by

$$I(\omega_p) = NR_p^2/4\pi. \tag{7.15}$$

As usual, Equation (7.15) does not apply for $p = N/2$; we may regard $a_{N/2}^2$ as the contribution to variance in the range $[(N-1)\pi/N, \pi]$ so that

$$I(\pi) = Na_{N/2}^2/\pi.$$

The plot of $I(\omega)$ against ω is usually called the **periodogram**, even though $I(\omega)$ is a function of frequency rather than period. It follows from Parseval's theorem in Equation (7.14) that the total area under the periodogram is equal to the variance of the time series.

Note that the formula for R_p^2, and hence for $I(\omega_p)$, can be written in several equivalent ways that look quite different. For example, after some algebra, it can be shown that

$$I(\omega_p) = \frac{1}{\pi N}\left|\sum_{t=1}^{N} x_t \mathrm{e}^{\mathrm{i}t\omega_p}\right|^2 \tag{7.16}$$

or we can replace $\mathrm{e}^{\mathrm{i}t\omega}$ with $\mathrm{e}^{-\mathrm{i}t\omega}$ in Equation (7.16). The usual way to actually calculate the periodogram directly from the data uses the expression

$$I(\omega_p) = \left[\left(\sum x_t \cos 2\pi pt/N\right)^2 + \left(\sum x_t \sin 2\pi pt/N\right)^2\right] \Big/ N\pi. \tag{7.17}$$

Equation (7.17) also applies for $p = N/2$.

Other authors define the periodogram in what appear to be slightly different ways, but the differences usually arise from allowing negative frequencies or using the cyclic frequency $f = \omega/2\pi$, rather than ω. The expressions generally turn out to be some other multiple of $I(\omega_p)$ or R_p^2. For example, Hannan (1970, Equation (3.8)) and Koopmans (1995, Equation (8.7)) give expressions that correspond to $\frac{1}{2}$ × expression (7.16). As to terminology, Anderson (1971, Section 4.3.2) describes the graph of R_p^2 against the *period* N/p, as the periodogram, and suggests the term **spectrogram** to describe the graph of R_p^2 against frequency. Jenkins and Watts (1968) define a similar expression to Equation (7.17) in terms of the variable $f = \omega/2\pi$, but call it the 'sample spectrum'. As always, when comparing terms and formulae from different sources, the reader needs to take great care.

The periodogram appears to be a natural way of estimating the power spectral density function, but Section 7.3.2 shows that, for a process with a **continuous** spectrum, it provides a poor estimate and needs to be modified. First, we derive the relationship between the periodogram of a given time series and the corresponding autocovariance function (acv.f.).

7.3.1 The relationship between the periodogram and the autocovariance function

The periodogram ordinate $I(\omega_p)$ and the autocovariance coefficient c_k are both quadratic forms of the data $\{x_t\}$. It is therefore natural to enquire how they are related. In fact, we will show that the periodogram is the finite Fourier transform of $\{c_k\}$.

Using Equation (7.2), we may rewrite Equation (7.17) for $p \neq N/2$ as

$$\begin{aligned} I(\omega_p) &= \left\{ \Big[\sum (x_t - \bar{x}) \cos \omega_p t \Big]^2 + \Big[\sum (x_t - \bar{x}) \sin \omega_p t \Big]^2 \right\} \Big/ N\pi \\ &= \sum_{s,t=1}^{N} (x_t - \bar{x})(x_s - \bar{x})(\cos \omega_p t \cos \omega_p s + \sin \omega_p t \sin \omega_p s) \Big/ N\pi. \end{aligned}$$

However, (see Equation (4.1))

$$\sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})/N \;=\; c_k$$

and $\cos \omega_p t \cos \omega_p (t+k) + \sin \omega_p t \sin \omega_p (t+k) = \cos \omega_p (t+k-t) = \cos \omega_p k$ so that

$$I(\omega_p) = \left(c_0 + 2 \sum_{k=1}^{N-1} c_k \cos \omega_p k \right) \Big/ \pi \tag{7.18}$$

$$= \sum_{k=-(N-1)}^{N-1} c_k \mathrm{e}^{-\mathrm{i}\omega_p k} \Big/ \pi. \tag{7.19}$$

The formula in Equation (7.19) is an expression called a **discrete finite Fourier transform** (assuming that $c_k = 0$ for $|k| \geq N$). Any reader not familiar with the Fourier transform, is recommended to read Appendix A – see especially Equation (A.5).

7.3.2 Properties of the periodogram

When the periodogram is expressed in the form of Equation (7.18), it appears to be the 'obvious' estimate of the power spectrum

$$f(\omega) = \left(\gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos \omega k \right) \Big/ \pi$$

simply replacing γ_k by its estimate c_k for values of k up to $(N-1)$, and putting subsequent estimates of γ_k equal to zero. However, although we find

$$\underset{N \to \infty}{E} [I(\omega)] \to f(\omega) \tag{7.20}$$

so that the periodogram is asymptotically unbiased, we see below that the variance of $I(\omega)$ does not decrease as N increases. Thus $I(\omega)$ is *not a consistent estimator* for $f(\omega)$. The lack of consistency is perhaps not too surprising when one realizes that the Fourier series representation in Equation (7.9) requires one to evaluate N parameters from N observations, however long the series may be. Thus in Section 7.4 we will consider alternative ways of estimating a power spectrum that are essentially ways of *smoothing* the periodogram.

We complete this section by proving that $I(\omega)$ is not a consistent estimator for $f(\omega)$ in the case where the observations are assumed to be independent $\mathrm{N}(\mu, \sigma^2)$ variates, so that they form a discrete-time purely random process with a uniform spectrum. This result can be extended to other stationary processes with continuous spectra, but this does not need to be demonstrated here. If the periodogram estimator does not 'work' for a uniform spectrum, it cannot be expected to 'work' for more complicated spectra. Given the above assumptions, Equation (7.10) shows that a_p and b_p are linear combinations of normally distributed random variables and so will themselves be normally distributed. Using Equations (7.2)–(7.4), it can be shown (Exercise 7.4) that a_p and b_p each have mean zero and variance $2\sigma^2/N$ for $p \neq N/2$. Furthermore we have

$$\begin{aligned} \mathrm{Cov}(a_p, b_p) &= 4\mathrm{Cov}\left[\Big(\sum X_t \cos\omega_p t\Big), \Big(\sum X_t \sin\omega_p t\Big)\right] \Big/ N^2 \\ &= 4\sigma^2\Big(\sum \cos\omega_p t \sin\omega_p t\Big)/N^2 \end{aligned}$$

since the observations are assumed to be independent. Thus, using Equation (7.5), we see that a_p and b_p are uncorrelated. Since (a_p, b_p) are bivariate normal, zero correlation implies that a_p and b_p are independent. The variables a_p and b_p can be standardized by dividing by $\sqrt{2}\sigma/\sqrt{N}$ to give standard $N(0, 1)$ variables. Now a result from distribution theory says that if Y_1, Y_2 are independent N(0, 1) variables, then $(Y_1^2 + Y_2^2)$ has a χ^2 distribution with two degrees of freedom, which is written χ_2^2. Thus

$$\frac{N(a_p^2 + b_p^2)}{2\sigma^2} = \frac{I(\omega_p)2\pi}{\sigma^2}$$

is χ_2^2. Put another way, this means that $2I(\omega)/f(\omega)$ is χ_2^2 when $f(\omega) = \sigma^2/\pi$ as in this case, although this result does, in fact, generalize to spectra that are not constant. Now the variance of a χ^2 distribution with ν degrees of freedom is 2ν, so that

$$\mathrm{Var}[I(\omega_p)2\pi/\sigma^2] = 4$$

and

$$\mathrm{Var}[I(\omega_p)] = \sigma^4/\pi^2.$$

As this variance is a constant, it does *not* tend to zero as $N \to \infty$, and hence $I(\omega_p)$ is not a consistent estimator for $f(\omega_p)$. Furthermore it can be shown that neighbouring periodogram ordinates are asymptotically independent, which

further explains the very irregular form of an observed periodogram. This all means that the periodogram needs to be modified in order to obtain a good estimate of a continuous spectrum.

7.4 Some Consistent Estimation Procedures

This section describes several alternative ways of estimating a spectrum. The different methods will be compared in Section 7.6. Each method provides a *consistent* estimator for the (power) spectral density function, in contrast to the (raw) periodogram. However, although the periodogram is itself an inconsistent estimator, the procedures described in this section are essentially based on smoothing the periodogram in some way.

Throughout the section we will assume that any obvious trend and seasonal variation have been removed from the data. If this is not done, the results of the spectral analysis are likely to be dominated by these effects, making any other effects difficult or impossible to see. Trend produces a peak at zero frequency, while seasonal variation produces peaks at the seasonal frequency and at integer multiples of the seasonal frequency – the seasonal **harmonics** (see Section 7.2.1). For a nonstationary series, the estimated spectrum of the detrended, deseasonalized data will depend to some extent on the method chosen to remove trend and seasonality. We assume throughout that a 'good' method is used to do this.

The methods described in this chapter are essentially *non-parametric* in that no model fitting is involved. It is possible to use a model-based approach and an alternative, parametric approach, called **autoregressive spectrum estimation**, will be introduced later in Section 13.6.1.

7.4.1 Transforming the truncated autocovariance function

One type of estimation procedure consists of taking a Fourier transform of the truncated weighted sample acv.f. From Equation (7.18), we know that the periodogram is the discrete finite Fourier transform of the complete sample acv.f. However, it is clear that the precision of the values of c_k decreases as k increases, because the coefficients are based on fewer and fewer terms. Thus, it would seem intuitively reasonable to give less weight to the values of c_k as k increases. An estimator, which has this property is

$$\widehat{f}(\omega) = \frac{1}{\pi}\left\{\lambda_0 c_0 + 2\sum_{k=1}^{M} \lambda_k c_k \cos \omega k\right\} \tag{7.21}$$

where $\{\lambda_k\}$ are a set of weights called the **lag window**, and $M(< N)$ is called the **truncation point**. Comparing Equation (7.21) with (7.18) we see that values of c_k for $M < k < N$ are no longer used, while values of c_k for $k \leq M$ are weighted by a factor λ_k. The latter are chosen so as to get smaller as k approaches M.

In order to use the above estimator, the analyst must choose a suitable lag window and a suitable truncation point. The two best-known lag windows are as follows.

Tukey window

$$\lambda_k = \frac{1}{2}\left(1 + \cos\frac{\pi k}{M}\right) \quad k = 0, 1, \ldots, M.$$

This window is sometimes called the Tukey–Hanning or Blackman–Tukey window.

Parzen window

$$\lambda_k = \begin{cases} 1 - 6\left(\frac{k}{M}\right)^2 + 6\left(\frac{k}{M}\right)^3 & 0 \le k \le M/2 \\ 2(1 - k/M)^3 & M/2 \le k \le M. \end{cases}$$

These two windows are illustrated in Figure 7.1 with $M = 20$.

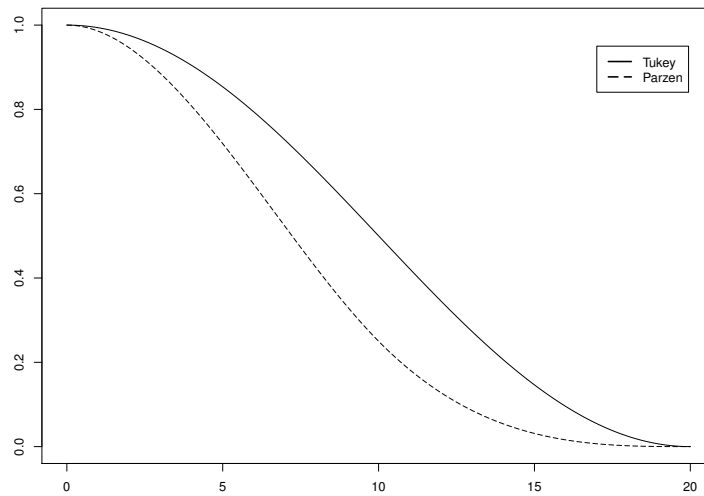

Figure 7.1 *The Tukey and Parzen lag windows with* $M = 20$.

The Tukey and Parzen windows give very similar estimated spectra for a given time series, although the Parzen window has a slight advantage in that it cannot give negative estimates. Many other lag windows have been suggested and 'window carpentry' was a popular research topic in the 1950s. Ways of comparing different windows will be discussed in Section 7.6. The well-known **Bartlett window**, with $\lambda_k = 1 - k/M$ for $k = 0, 1, \ldots, M$, is very simple

but is now rarely used as its properties are inferior to those of the Tukey and Parzen windows.

The choice of the truncation point M is more difficult and it is not easy to give clear-cut advice. It has to be chosen subjectively so as to *balance 'resolution' against 'variance'*. The smaller the value of M, the smaller will be the variance of $\widehat{f}(\omega)$ but the larger will be the bias. If M is too small, important features of $f(\omega)$ may be smoothed out, but if M is too large the behaviour of $\widehat{f}(\omega)$ becomes more like that of the periodogram with erratic variation. Thus a compromise value must be chosen. A useful rough guide is to choose M to be about $2\sqrt{N}$, so that if, for example, N is 200, then M will be round about the value 28. This choice of M ensures the asymptotic situation that as $N \to \infty$, so also does $M \to \infty$ but in such a way that $M/N \to 0$. A somewhat larger value of M is required for the Parzen window than for the Tukey window. Other writers have suggested $\sqrt{N}$, rather than $2\sqrt{N}$, while results from density estimation suggest that a different power of N may be appropriate. Percival and Walden (1993, Chapter 6) point out that an appropriate value of M depends on the properties of the underlying process and give more detailed guidance. However, our advice is to try three or four different values of M. A low value will give an idea where the large peaks in $f(\omega)$ are, but the curve is likely to be too smooth. A high value is likely to produce a curve showing a large number of peaks, some of which may be spurious. A compromise can then be achieved with an in-between value of M.

In principle, Equation (7.21) may be evaluated at any value of ω in $(0, \pi)$, but it is usually evaluated at equal intervals at $\omega = \pi j/Q$ for $j = 0, 1 \ldots, Q$, where Q is chosen sufficiently large to show up all features of $\widehat{f}(\omega)$. Often Q is chosen to be equal to M. The graph of $\widehat{f}(\omega)$ against ω can then be plotted and examined. An example is given later in Figure 7.5, for the data plotted in Figure 1.2, using the Tukey window with $M = 24$.

7.4.2 Hanning

This procedure, named after Julius Von Hann, is equivalent to the use of the Tukey window as described in Section 7.4.1, but adopts a different computational procedure. The estimated spectrum is calculated in two stages. First, a truncated unweighted cosine transform of the acv.f. of the data is taken to give

$$\widehat{f}_1(\omega) = \frac{1}{\pi}\left(c_0 + 2\sum_{k=1}^{M} c_k \cos \omega k\right). \tag{7.22}$$

This is the same as Equation (7.21) except that the lag window is taken to be unity (i.e. $\lambda_k = 1$). The estimates given by Equation (7.22) are calculated at $\omega = \pi j/M$ for $j = 0, 1, \ldots, M$. These estimates are then smoothed using the

weights $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ to give the Hanning estimates

$$\widehat{f}(\omega) = \frac{1}{4}\widehat{f}_1(\omega - \pi/M) + \frac{1}{2}\widehat{f}_1(\omega) + \frac{1}{4}\widehat{f}_1(\omega + \pi/M) \tag{7.23}$$

at $\omega = \pi j/M$ for $j = 1, 2, \ldots, (M-1)$. At zero frequency, and at the Nyquist frequency π, we take

$$\begin{aligned} \widehat{f}(0) &= \frac{1}{2}[\widehat{f}_1(0) + \widehat{f}_1(\pi/M)] \\ \widehat{f}(\pi) &= \frac{1}{2}[\widehat{f}_1(\pi) + \widehat{f}_1(\pi(M-1)/M)]. \end{aligned}$$

It can easily be shown algebraically that this procedure is equivalent to the use of the Tukey window. Substituting Equation (7.22) into (7.23) we find

$$\widehat{f}(\omega) = \frac{1}{\pi}\left\{ c_o + 2\sum_{k=1}^{M} c_k \left[\frac{1}{4}\cos(\omega - \pi/M)k + \frac{1}{2}\cos\omega k + \frac{1}{4}\cos(\omega + \pi/M)k \right] \right\}$$

and, using $\cos(\omega - \pi/M)k + \cos(\omega + \pi/M)k = 2\cos\omega k \cos(\pi k/M)$, a comparison with Equation (7.21) shows that the lag window is indeed the Tukey window.

There is relatively little difference in the computational efficiency of Hanning and the straightforward use of the Tukey window. Both methods should yield the same estimates and so it does not matter which of the two procedures is used in practice.

7.4.3 Hamming

This technique is very similar to Hanning and has a very similar title, which sometimes leads to confusion. In fact Hamming is named after a quite different person, namely R.W. Hamming. The technique is nearly identical to Hanning except that the weights $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ in Equation (7.23) are changed to (0.23, 0.54, 0.23). At the frequencies $\omega = 0$ and $\omega = \pi$, the weights are 0.54 (at the 'end' frequency) and 0.46. The procedure gives similar estimates to those produced by Hanning.

7.4.4 Smoothing the periodogram

The methods of Sections 7.4.1–7.4.3 are based on transforming the truncated sample acv.f. An alternative type of approach is to smooth the periodogram ordinates in some way, the simplest approach being to group the periodogram ordinates in sets of size m and find their average value. The latter approach is based on a suggestion made by P.J. Daniell as long ago as 1946. However, the use of lag window estimators was standard for many years because less computation was involved. Nowadays, some form of smoothed periodogram

is used much more widely, particularly with the advent of the fast Fourier transform — see Section 7.4.5.

The basic idea of the simple smoothed periodogram can be expressed in the following formula:

$$\widehat{f}(\omega) = \frac{1}{m}\sum_{j} I(\omega_j) \tag{7.24}$$

where $\omega_j = 2\pi j/N$ and j varies over m consecutive integers so that the ω_j are symmetric about the frequency of interest, namely, ω. In order to estimate $f(\omega)$ at the end-points $\omega = 0$ and $\omega = \pi$, Equation (7.24) has to be modified in an obvious way, treating the periodogram as being symmetric about 0 and π. Then, taking m to be odd with $m^* = (m-1)/2$, we have

$$\begin{aligned}
\widehat{f}(0) &= I(0) + 2\sum_{j=1}^{m^*} I(2\pi j/N)/m \\
\widehat{f}(\pi) &= \left[I(\pi) + 2\sum_{j=1}^{m^*} I(\pi - 2\pi j/N)\right] \Big/ m.
\end{aligned}$$

The expression for $\widehat{f}(0)$ can be simplified as the first term $I(0)$ is zero.

Now we know that the periodogram is asymptotically unbiased but inconsistent for the true spectrum. Since neighbouring periodogram ordinates are asymptotically uncorrelated, it is clear that the variance of Equation (7.24) will be of order $1/m$. It is also clear that the estimator in Equation (7.24) may be biased since

$$E[\widehat{f}(\omega)] \simeq \frac{1}{m}\sum_{j} f(\omega_j),$$

which is only equal to $f(\omega)$ if the spectrum is linear over the relevant interval. However, the bias will be 'small' provided that $f(\omega)$ is a reasonably smooth function and m is not too large compared with N.

The consequence of the above remarks is that the choice of group size m is rather like the choice of the truncation point M in Section 7.4.1 in that it has to be chosen so as to balance resolution against variance. However, the choice is different in that changes in m and in M act in opposite directions. An increase in m has a similar effect to a reduction in M. The larger the value of m the smaller will be the variance of the resulting estimate but the larger will be the bias. If m is too large, then interesting features of $f(\omega)$, such as peaks, may be smoothed out. Of course, as N increases, we can in turn allow m to increase, just as we allowed M to increase with N in Section 7.4.1.

There is relatively little advice in the literature on the choice of m. As in Section 7.4.1, it seems advisable to try several values for m. A 'high' value should give some idea as to whether large peaks in $f(\omega)$ exist, but the curve is likely to be too smooth and some real peaks may be hidden. A 'low' value is likely to produce a much more uneven curve showing many peaks, some of

which will be spurious. A compromise between the effects of bias and variance can then be made. A rule of thumb is to choose values of m near $2\sqrt{N}$.

Although the procedure described in this section is computationally quite different from that of Section 7.4.1, there are in fact close theoretical links between the two procedures. In Section 7.3.1 we derived the relationship between the periodogram and the sample acv.f., and, if we substitute Equation (7.18) into (7.24), we can express the smoothed periodogram estimate of the spectrum in terms of the sample acv.f. in a similar form to Equation (7.21). After some algebra (Exercise 7.5), it can be shown that the truncation point is $(N-1)$ and the lag window is given by

$$\lambda_k = \begin{cases} 1 & k = 0 \\ \sin(m\pi k/N)/[m\sin(\pi k/N)] & k = 1, 2, \ldots, N-1. \end{cases}$$

Thus, the formula uses values of c_k right up to $k = (N-1)$, rather than having a truncation point much lower than N. Moreover, the lag window has the undesirable property that it does not tend to zero as k tends to N. The smoothed periodogram effectively uses a rectangular window in the frequency domain and the resulting lag window shows that a sudden cut-off in the frequency domain can give rise to 'nasty' effects in the time domain (and vice versa). The smoothed periodogram often works reasonably well, but its window properties suggest that it may be possible to find a way of smoothing the periodogram, using a non-uniform averaging procedure, that has better time-domain properties. In fact, the simple smoothed periodogram is rarely used today, but rather a windowed form of averaging is used instead. Various alternative smoothing procedures have been suggested, with the idea of giving more weight to the periodogram ordinate at the particular frequency of interest and progressively less weight to periodogram ordinates further away. The analyst can think of this as applying a window in the frequency domain rather than in the time domain, but in a way that corresponds to the use of a lag window as in Section 7.4.1. It is possible to use a triangular (Bartlett window) or aim for a bell-shaped curve, perhaps by applying a simple smoother, such as Hanning, more than once. These approaches will not be considered here and the reader is referred, for example, to Hayes (1996, Chapter 8) or Bloomfield (2000, Chapter 8).

Historically, the smoothed periodogram was not much used until the 1990s because it apparently requires much more computational effort than transforming the truncated acv.f. Calculating the periodogram using Equation (7.17) at ω_p for $p = 1, 2, \ldots, N/2$ would require about N^2 arithmetic operations (each one a multiplication and an addition), whereas using Equation (7.21) fewer than MN operations are required to calculate the $\{c_k\}$ so that the total number of operations is only of order $M(N+M)$ if the spectrum is evaluated at M frequencies. Two factors have led to the increasing use of the smoothed periodogram. First, the advent of high-speed computers means that it is unnecessary to restrict attention to the method requiring fewest calculations. The second factor has been the widespread use of an

algorithm called the **fast Fourier transform**, which makes it much quicker to compute the periodogram. This procedure will now be described.

7.4.5 The fast Fourier transform (FFT)

The computational procedure described in this section is usually abbreviated to FFT[5] and we adopt this abbreviation. For long series, the technique can substantially reduce the time required to perform a Fourier analysis of a set of data on a computer, and can also give more accurate results.

The history of the FFT dates back to the early 1900s. However, it was the work of J.W. Cooley, J.W. Tukey and G. Sande in about 1965 coupled with the arrival of faster computers that stimulated the application of the technique to time-series analysis. Much of the early work was published in the various Transactions of the IEEE, but more recent coverage is given, for example, by Bendat and Piersol (2000), Bloomfield (2000) and Priestley (1981). We will only give a broad outline of the technique here.

The FFT requires that the value of N should be composite, meaning that N is not a prime number and so can be factorized. The basic idea of the FFT will be illustrated for the case when N can be factorized in the form $N = rs$, where r and s are integers. If we assume that N is even, then at least one of the factors, say r, will be even. Using complex numbers for mathematical simplicity, the Fourier coefficients from Equation (7.10) can be expressed in the form

$$a_p + \mathrm{i}b_p = 2\Big[\sum x_t \, \mathrm{e}^{2\pi \mathrm{i} p t/N}\Big]/N \tag{7.25}$$

for $p = 0, 1, 2, \ldots, (N/2) - 1$. For mathematical convenience, we denote the observations by $x_0, x_1, \ldots, x_{N-1}$, so that the summation in Equation (7.25) is from $t = 0$ to $N - 1$. Now we can write t in the form

$$t = rt_1 + t_0$$

where $t_1 = 0, 1 \ldots, s - 1$, and $t_0 = 0, 1, \ldots, r - 1$, as t goes from 0 to $N - 1$, in view of the fact that $N = rs$. Similarly we can decompose p in the form

$$p = sp_1 + p_0$$

where $p_1 = 0, 1, \ldots, (r/2) - 1$, and $p_0 = 0, 1, \ldots, s - 1$, as p goes from 0 to $(N/2) - 1$. Then the summation in Equation (7.25) may be written

$$\sum_{t_0=0}^{r-1} \mathrm{e}^{2\pi \mathrm{i} p t_0/N} \sum_{t_1=0}^{s-1} x_t \, \mathrm{e}^{2\pi \mathrm{i} p r t_1/N}.$$

However,

$$\mathrm{e}^{2\pi \mathrm{i} p r t_1/N} = \mathrm{e}^{2\pi \mathrm{i}(sp_1+p_0) r t_1/N} = \mathrm{e}^{2\pi \mathrm{i} p_0 r t_1/N}$$

[5] Some authors have used this abbreviation to denote the *finite Fourier transform*.

since $e^{2\pi i s p_1 r t_1/N} = e^{2\pi i p_1 t_1} = 1$ for all p_1, t_1. Thus $\sum_{t_1=0}^{s-1} x_t e^{2\pi i p r t_1/N}$ does not depend on p_1 and is therefore a function of t_0 and p_0 only, say $A(p_0, t_0)$. Then Equation (7.25) may be written

$$a_p + ib_p = 2\left[\sum_{t_0=0}^{r-1} A(p_0, t_0)\, e^{2\pi i p t_0/N}\right] \Big/ N$$

Now there are $N = rs$ functions of type $A(p_0, t_0)$ to be calculated, each requiring s complex multiplications and additions. There are $N/2$ values of $(a_p + ib_p)$ to be calculated, each requiring r further complex multiplications and additions. This gives a grand total of $Ns + \frac{N}{2}r = N(s + r/2)$ calculations instead of the $N \times N/2 = N^2/2$ calculations required to use Equation (7.25) directly. By a suitable choice of s and r, we can usually arrange for $(s + r/2)$ to be (much) less than $N/2$.

Much bigger reductions in computing can be made by an extension of the above procedure when N is highly composite (i.e. has many small factors). In particular, if N is of the form 2^k, then we find that the number of operations is of order Nk (or $N \log_2 N$) instead of $N^2/2$. Substantial gains can also be made when N has several factors (e.g. $N = 2^p 3^q 5^r \ldots$).

In practice it is unlikely that N will naturally be of a simple form such as 2^k, unless the value of N can be chosen before measurement starts. However, there are other things we can do. It may be possible to make N highly composite by the simple expedient of omitting a few observations from the beginning or end of the series. For example, with 270 observations, we can omit the last 14 to make $N = 256 = 2^8$. More generally we can *increase* the length of the series by adding zeros to the (mean-corrected) sample record until the value of the revised N becomes a suitable integer. Then a procedure called **tapering** or **data windowing** (e.g. Percival and Walden, 1993; Priestley, 1981) is often recommended[6] to avoid a discontinuity at the end of the data. Suppose, for example, that we happen to have 382 observations. This value of N is *not* highly composite and we might proceed as follows:

- Remove any linear trend from the data, and keep the residuals (which should have mean zero) for subsequent analysis. If there is no trend, simply subtract the overall mean from each observation.
- Apply a linear taper to about 5% of the data at each end. In this example, if we denote the detrended mean-corrected data by $x_0, x_1, \ldots, x_{381}$, then the tapered series is given by

$$x_t^* = \begin{cases} (t+1)x_t/20 & t = 0, 1, \ldots, 18, \\ (382 - t)x_t/20 & t = 363, \ldots, 381, \\ x_t & t = 19, 20, \ldots, 362. \end{cases}$$

[6] Note that some researchers view tapering with suspicion as the data are modified — see, for example, the discussion in Percival and Walden (1993, p. 215).

- Add $512 - 382 = 130$ zeros at one end of the tapered series, so that $N = 512 = 2^9$.
- Carry out an FFT on the data, calculate the Fourier coefficients $a_p + \mathrm{i}b_p$ and average the values of $(a_p^2 + b_p^2)$ in groups of about 10.

In fact with N as low as 382, the computational advantage of the FFT is limited and we could equally well calculate the periodogram directly, which avoids the need for tapering and adding zeros. The FFT really comes into its own when there are several thousand observations.

It is also worth explaining that the FFT is still useful when the analyst prefers to look at the autocorrelation function (ac.f.) *before* carrying out a spectral analysis, either because inspecting the ac.f. is thought to be an invaluable preliminary exercise or because the analyst prefers to transform the truncated weighted acv.f. rather than smooth the periodogram. It can be quicker to calculate the sample acv.f. by performing *two* FFTs (e.g. Priestley, 1981, Section 7.6), rather than directly as a sum of lagged products. The procedure is as follows. Compute the Fourier coefficients (a_p, b_p) with an FFT of the mean-corrected data at $\omega_p = 2\pi p/N$ for $p = 0, 1, \ldots, N-1$ rather than for $p = 0, 1, \ldots, N/2$ as we usually do. The extra coefficients are normally redundant for real-valued processes since $a_{N-k} = a_k$ and $b_{N-k} = -b_k$. However, for calculating the autocovariances, we can compute $R_p^2 = a_p^2 + b_p^2$ at these values of p and then fast Fourier *re*transform the sequence (R_p^2) to get the mean lagged products. We will not give the algebra here. For several thousand observations, this can be much faster than calculating them directly. However, when using the FFT in this way, the analyst should take care to add enough zeros to the data (*without* tapering) to make sure that **non-circular** sums of lagged products are calculated, as defined by Equation (4.1) and used throughout this book. **Circular** coefficients result if zeros are not added where, for example, the circular autocovariance coefficient at lag 1 is

$$c_1^* = \left[\sum_{t=1}^{N} (x_t - \bar{x})(x_{t+1} - \bar{x}) \right] \Big/ N.$$

where x_{N+1} is taken to be equal to x_1 to make the series 'circular'. Note that, if $x_1 = \bar{x}$, then the circular and non-circular coefficients at lag 1 are the same. If we use mean-corrected data, which will have mean zero, then adding zeros will make circular and non-circular coefficients be the same. In order to calculate all the non-circular autocovariance coefficients of a set of N mean-corrected observations, the analyst should add N zeros, to make $2N$ 'observations' in all.

7.5 Confidence Intervals for the Spectrum

The methods of Section 7.4 all produce *point* estimates of the spectral density function, and hence give no indication of their likely accuracy. This section shows how to find appropriate confidence intervals.

In Section 7.3.2, we showed that data from a white noise process, with constant spectrum $f(\omega) = \sigma^2/\pi$, yields a periodogram ordinate $I(\omega)$ at frequency ω, which is such that $2I(\omega)/f(\omega)$ is distributed as χ^2_2. Note that this distibution does not depend on N, which explains why $I(\omega)$ is not a consistent estimator for $f(\omega)$. Wide confidence intervals would result if $I(\omega)$ was used as an estimator. Suppose instead that we use the estimator of Section 7.4.1, namely

$$\widehat{f}(\omega) = \left[\sum_{k=-M}^{M} \lambda_k c_k \cos \omega k \right] \Big/ \pi.$$

Then, it can be shown (Jenkins and Watts, 1968, Section 6.4.2) that $v\widehat{f}(\omega)/f(\omega)$ is asymptotically distributed as an approximate χ^2_ν random variable, where

$$\nu = 2N \Big/ \sum_{k=-M}^{M} \lambda_k^2 \tag{7.26}$$

is called the number of *degrees of freedom* of the lag window. It follows that

$$P(\chi^2_{\nu,1-\alpha/2} < \nu\widehat{f}(\omega)/f(\omega) < \chi^2_{\nu,\alpha/2}) = 1 - \alpha,$$

so that a $100(1-\alpha)\%$ confidence interval for $f(\omega)$ is given by

$$\frac{\nu\widehat{f}(\omega)}{\chi^2_{\nu,\alpha/2}} \quad \text{to} \quad \frac{\nu\widehat{f}(\omega)}{\chi^2_{\nu,1-\alpha/2}}.$$

Some simple algebra shows that the degrees of freedom for the Tukey and Parzen windows turn out to be $2.67N/M$ and $3.71N/M$, respectively. Although relying on asymptotic results, Neave (1972a) has shown that the above formulae are also quite accurate for short series.

For the smoothed periodogram estimator of Section 7.4.4, there is no need to apply Equation (7.26), because smoothing the periodogram in groups of size m is effectively the same as averaging independent χ^2_2 random variables. Thus, it is clear that the smoothed periodogram will have $\nu = 2m$ degrees of freedom, and we can then apply the same formula for the confidence interval as given above.

7.6 Comparison of Different Estimation Procedures

Several factors need to be considered when comparing the different estimation procedures that were introduced in Section 7.4. Although we concentrate on the theoretical properties of the different procedures, the analyst will also need to consider practical questions such as computing time and the availability of suitable computer software. Alternative comparative discussions are given by Jenkins and Watts (1968), Neave (1972b), Priestley (1981, Section 7.5) and Bloomfield (2000).

It is helpful to introduce a function called the **spectral window** or **kernel**, which is defined to be the Fourier transform of the lag window $\{\lambda_k\}$ introduced in Equation (7.21). Assuming that λ_k is zero for $k > M$, and symmetric, so that $\lambda_{-k} = \lambda_k$, then the spectral window is defined by

$$K(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \lambda_k \mathrm{e}^{-\mathrm{i}k\omega} \tag{7.27}$$

for $(-\pi < \omega < \pi)$[7]. The corresponding inverse Fourier transform is given by

$$\lambda_k = \int_{-\pi}^{\pi} K(\omega)\mathrm{e}^{\mathrm{i}\omega k}\mathrm{d}\omega. \tag{7.28}$$

All the estimation procedures for the spectrum that we have studied so far can be put in the general form

$$\begin{aligned}
\widehat{f}(\omega_0) &= \frac{1}{\pi} \sum_{k=-N+1}^{N-1} \lambda_k c_k \mathrm{e}^{-\mathrm{i}\omega_0 k} \\
&= \frac{1}{\pi} \sum \left[\int_{-\pi}^{\pi} K(\omega)\mathrm{e}^{\mathrm{i}\omega k}\mathrm{d}\omega \right] c_k \mathrm{e}^{-\mathrm{i}\omega_0 k} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} K(\omega) \left[\sum c_k \mathrm{e}^{\mathrm{i}k(\omega-\omega_0)} \right] \mathrm{d}\omega \\
&= \int_{-\pi}^{\pi} K(\omega) I(\omega_0 - \omega)\mathrm{d}\omega
\end{aligned} \tag{7.29}$$

using Equation (7.19). Equation (7.29) shows that all the estimation procedures are essentially smoothing the periodogram using the weight function $K(\omega)$. The value of the lag window at lag zero is usually specified to be one, so that from Equation (7.28) we have

$$\lambda_0 = 1 = \int_{-\pi}^{\pi} K(\omega)\mathrm{d}\omega$$

which is a desirable property for a smoothing function.

Taking expectations in Equation (7.29) we have asymptotically that

$$E[\widehat{f}(\omega_0)] = \int_{-\pi}^{\pi} K(\omega) f(\omega_0 - \omega)\mathrm{d}\omega. \tag{7.30}$$

Thus the spectral window is a weight function expressing the contribution of the spectral density function at each frequency to the expectation of $\widehat{f}(\omega_0)$. The name 'window' arises from the fact that $K(\omega)$ determines the part of the periodogram that is 'seen' by the estimator.

[7] Note that we cannot avoid negative frequencies here, as the spectral window looks at differences in frequency from some specified frequency. If the lag window is symmetric about zero, then so is $K(\omega)$.

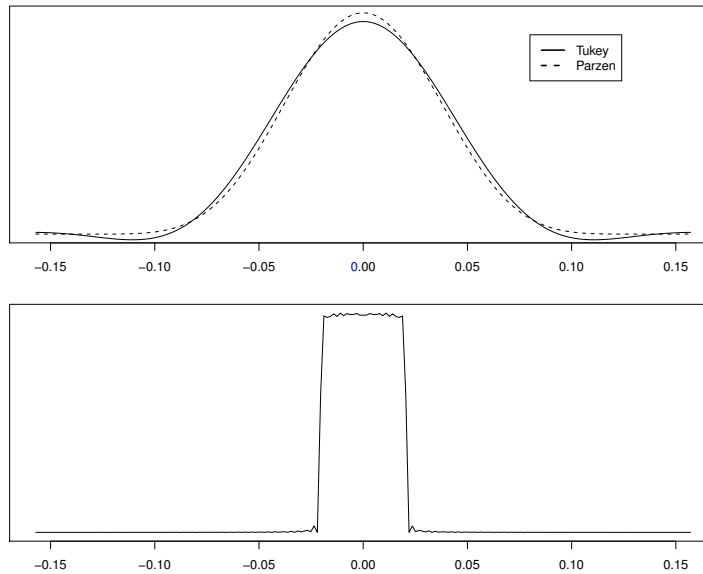

Figure 7.2 *The spectral windows for three common methods of spectral analysis. Top: Parzen ($M = 93$), Tukey ($M = 67$); Bottom: Smoothed periodogram $N = 1000, m = 20$.*

Examples of the spectral windows for three common methods of spectral analysis are shown in Figure 7.2. Taking $N = 1000$, the spectral window for the smoothed periodogram with $m = 20$ is shown in the bottom panel of Figure 7.2. The other two windows are the Parzen and Tukey windows, shown as dashed and solid lines in the top panel. The values of the truncation point M were chosen to be 93 for the Parzen window and 67 for the Tukey window. These values of M were chosen so that the Parzen and Tukey windows gave estimators with equal variance. Formulae for variances will be given later in this section.

Inspecting Figure 7.2, we see that the Parzen and Tukey windows look very similar, although the Parzen window has the advantage of being non-negative and of having smaller side lobes. The shape of the periodogram window is quite different. It is approximately rectangular with a sharp cut-off and is close to the 'ideal' band-pass filter, which would be exactly rectangular but which is unattainable in practice. The periodogram window also has the advantage of being non-negative.

In comparing different windows, we should consider both the bias and the variance of the estimator. This is sometimes called the *variance-bias trade-off* question, as well as *balancing resolution against variance*. By taking a wider window, we generally get a lower variance but a larger bias and some sort of compromise has to be made in practice. This is often achieved by using

trial and error, as, for example, in the choice of the truncation point for a lag window as discussed earlier in Section 7.4.1. It is not easy to get general formulae for the bias produced by the different procedures. However, it is intuitively clear from Equation (7.30) and from earlier remarks that the wider the window, the larger will be the bias. In particular, it is clear that all the smoothing procedures will tend to lower peaks and raise troughs.

As regards variance, we noted in Section 7.5 that $v\widehat{f}(\omega)/f(\omega)$ is approximately distributed as χ^2_ν, where $\nu = 2m$, for the smoothed periodogram, and, using Equation (7.26), $3.71N/M$ and $8N/3M$ for the Parzen and Tukey windows, respectively. Since

$$\mathrm{Var}(\chi^2_\nu) = 2\nu$$

and

$$\mathrm{Var}[\nu\widehat{f}(\omega)/f(\omega)] = \nu^2\mathrm{Var}[\widehat{f}(\omega)/f(\omega)]$$

we find $\mathrm{Var}[\widehat{f}(\omega)/f(\omega)]$ turns out to be $1/m, 2M/3.71N$, and $3M/4N$, respectively, for the three windows. Equating these expressions gives the values of M chosen for Figure 7.2.

When comparing the different estimators, the notion of a **bandwidth** may be helpful. Roughly speaking, the bandwidth is the width of the spectral window, as might be expected. Various formal definitions are given in the literature, but we adopt the one given by Jenkins and Watts (1968), namely, the width of the 'ideal' rectangular window that would give an estimator with the same variance. The window of the smoothed periodogram is so close to being rectangular for m 'large' that it is clear from Figure 7.2 that the bandwidth will be approximately $2m\pi/N$ (as area must be unity and height is $N/2m\pi$). The bandwidths for the Bartlett, Parzen and Tukey windows turn out to be $3/2M$, $2\pi(1.86/M)$ and $8\pi/3M$, respectively. When plotting a graph of an estimated spectrum, it is a good idea to indicate the bandwidth that has been used.

The choice of bandwidth is equivalent to the choice of m or M, depending on the method used. This choice is an important step in spectral analysis, though it is important to remember that the effects of changing m and M act in opposite directions. For the Bartlett, Parzen and Tukey windows, the bandwidth is inversely proportional to M. Figure 7.3 shows how the window changes as M varies, using the Bartlett window as a representative example. As M gets larger, the window gets narrower, the bias gets smaller but the variance of the resulting estimator gets larger. For the smoothed periodogram, the reverse happens. The bandwidth is directly proportional to m, and as m gets larger, the window gets wider, the bias increases but the variance reduces. For the unsmoothed periodogram, with $m = 1$, the window is very tall and narrow giving an estimator with large variance as we have already shown. All in all, the choice of bandwidth is rather like the choice of class interval when constructing a histogram.

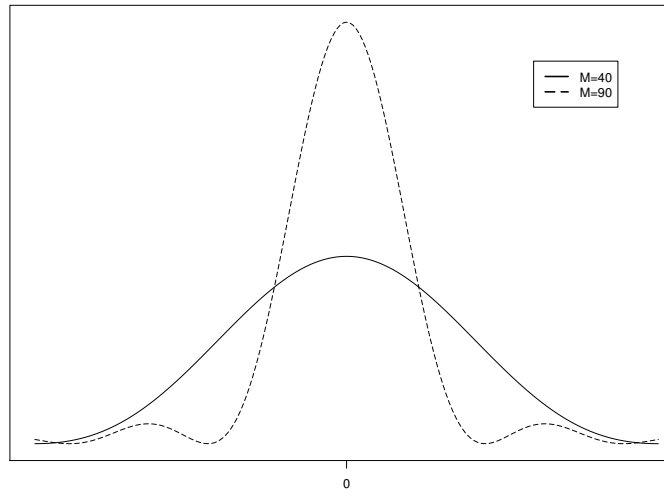

Figure 7.3 *The Bartlett spectral window for different values of M.*

We are now in a position to give guidance on the relative merits of the different estimation procedures. As regards theoretical properties, it is arguable that the smoothed periodogram has the better-shaped spectral window in that it is approximately rectangular, although there are some side lobes. For the transformed acv.f., the Parzen and Tukey windows are preferred to the Bartlett window. Computationally, the smoothed periodogram can be much slower for large N unless the FFT is used. However, if the FFT *is* used, then the smoothed periodogram can be faster. Moreover it is possible to calculate the ac.f. quickly using *two* FFTs. One drawback to the use of the FFT is that it may require data-tapering, whose use is still somewhat controversial. Of course, for small N, computing time is a relatively unimportant consideration. As regards computer software, it is much easier to write a program for the Parzen or Tukey windows, but programs and algorithms for the FFT are becoming readily available. Thus the use of the smoothed periodogram has become more general, either with equal weights as discussed in Section 7.4.4, or with a more 'bell-shaped' set of weights.

All the above assumes that a *non-parametric* approach is used in that no model fitting is attempted prior to carrying out a spectral analysis. As noted earlier, an alternative approach gaining ground is to use a *parametric* approach, fitting an autoregressive (AR) or ARMA model to the data. The spectrum of the fitted model is then used to estimate the spectrum. This approach will be described later in Section 13.7.1.

7.7 Analysing a Continuous Time Series

Up to now, we have been concerned with the spectral analysis of time series recorded at *discrete* time intervals. However, time series are sometimes recorded as a *continuous trace*. For example, variables such as air temperature, humidity and the moisture content of tobacco emerging from a processing plant are often recorded by machines that give continuous-time readings. For series that contain components at very high frequencies, such as those arising in acoustics and speech processing, it may be possible to analyse such records mechanically using tuned filters, but the more usual procedure is to **digitize** the series by reading off the values of the trace at discrete intervals. If values are taken at equal time intervals of length Δt, we have converted a continuous time series into a standard discrete-time time series and can use the methods already described.

In sampling a continuous time series, the main question is how to choose the sampling interval Δt. It is clear that sampling leads to some loss of information and that this loss gets worse as Δt increases. However, sampling costs increase as Δt gets smaller and so a compromise value must be sought.

For the sampled series, the Nyquist frequency is $\pi/\Delta t$ radians per unit time, and we can get no information about variation at higher frequencies. Thus we clearly want to choose Δt so that variation in the continuous series is negligible at frequencies higher than $\pi/\Delta t$. In fact most measuring instruments are **band-limited** in that they do not respond to frequencies higher than a certain maximum frequency. If this maximum frequency, say ω_{max}, is known or can be guessed, then the choice of Δt is straightforward in that it should be less than π/ω_{max}. However, if Δt is chosen to be too large, then a phenomenon called **aliasing** may occur. This can be illustrated by the following theorem.

Theorem 7.1 Suppose that a continuous time series, with spectrum $f_{\mathrm{c}}(\omega)$ for $0 < \omega < \infty$, is sampled at equal time intervals of length Δt. The resulting discrete time series will have a somewhat different spectrum, say $f_{\mathrm{d}}(\omega)$ defined over $0 < \omega < \pi/\Delta t$. We will see that the two spectra will only be equal if $f_{\mathrm{c}}(\omega)$ is zero for $\omega > \pi/\Delta t$. More generally, it can be shown that $f_{\mathrm{d}}(\omega)$ and $f_{\mathrm{c}}(\omega)$ are related by

$$f_{\mathrm{d}}(\omega) = \sum_{s=0}^{\infty} f_{\mathrm{c}}(\omega + 2\pi s/\Delta t) + \sum_{s=1}^{\infty} f_{\mathrm{c}}(-\omega + 2\pi s/\Delta t). \tag{7.31}$$

Proof The proof will be given for the case $\Delta t = 1$. The extension to other values of Δt is straightforward. Suppose that the acv.f.s of the continuous and sampled series are given by $\gamma(\tau)$ and γ_k, respectively. Here $\gamma(\tau)$ is defined for all τ, while γ_k is only defined for integer k. Of course if τ takes an integer value, say k, then the two functions are equal as in

$$\gamma(k) = \gamma_k. \tag{7.32}$$

Now from Equation (6.18) we have

$$\gamma(\tau) = \int_0^\infty f_{\mathrm{c}}(\omega)\, \cos\, \omega\tau\, \mathrm{d}\omega,$$

while, from Equation (6.9), we have

$$\gamma_k = \int_0^\pi f_{\mathrm{d}}(\omega)\, \cos\, \omega k\, \mathrm{d}\omega.$$

Thus, using Equation (7.32), we have

$$\int_0^\pi f_{\mathrm{d}}(\omega)\, \cos\, \omega k\, \mathrm{d}\omega = \int_0^\infty f_{\mathrm{c}}(\omega)\, \cos\, \omega k\, \mathrm{d}\omega$$

for $k = 0, \pm 1, \pm 2, \ldots$. The next step is to split the infinite integral into sections of length 2π, and then of π, using $\cos \omega k = \cos\,(\omega + 2\pi s)k = \cos\,(2\pi s - \omega)k$ for all integers s. We get

$$\begin{aligned}
\int_0^\infty f_{\mathrm{c}}(\omega)\, \cos\, \omega k\, \mathrm{d}\omega &= \sum_{s=0}^{\infty} \int_{2\pi s}^{2\pi(s+1)} f_{\mathrm{c}}(\omega)\, \cos\, \omega k\, \mathrm{d}\omega \\
&= \sum_{s=0}^{\infty} \int_0^{2\pi} f_{\mathrm{c}}(\omega + 2\pi s)\, \cos\, \omega k\, \mathrm{d}\omega \\
&= \sum_{s=0}^{\infty} \int_0^{\pi} \{f_{\mathrm{c}}(\omega + 2\pi s) + f_{\mathrm{c}}[2\pi(s+1) - \omega]\}\, \cos\, \omega k\, \mathrm{d}\omega \\
&= \int_0^\pi \left\{ \sum_{s=0}^{\infty} f_{\mathrm{c}}(\omega + 2\pi s) + \sum_{s=1}^{\infty} f_{\mathrm{c}}(2\pi s - \omega) \right\} \cos\, \omega k\ \mathrm{d}\omega
\end{aligned}$$

and the result follows. □

The implications of this theorem may now be considered. First, as noted earlier, if the continuous series contains no variation at frequencies above the Nyquist frequency, so that $f_{\mathrm{c}}(\omega) = 0$ for $\omega > \pi/\Delta t$, then $f_{\mathrm{d}}(\omega) = f_{\mathrm{c}}(\omega)$. In this case no information is lost by sampling. However, the more general result is that sampling *will* have an effect in that variation at frequencies above the Nyquist frequency in the continuous series will be 'folded back' to produce apparent variation in the sampled series at a frequency lower than the Nyquist frequency. If we denote the Nyquist frequency $\pi/\Delta t$ by ω_{N}, then the frequencies ω, $2\omega_{\mathrm{N}} - \omega, 2\omega_{\mathrm{N}} + \omega, 4\omega_{\mathrm{N}} - \omega, \ldots$ are called **aliases** of one another. Variation at all these frequencies in the continuous series will appear as variation at frequency ω in the sampled series.

From a practical point of view, aliasing will cause trouble unless Δt is chosen to be sufficiently small so that $f_{\mathrm{c}}(\omega) \simeq 0$ for $\omega > \pi/\Delta t$. If we have no advance knowledge about $f_{\mathrm{c}}(\omega)$, then we have to guesstimate a value for Δt. If the resulting estimate of $f_{\mathrm{d}}(\omega)$ approaches zero near the Nyquist frequency

$\pi/\Delta t$, then our choice of Δt is almost certainly sufficiently small. However, if $f_{\mathrm{d}}(\omega)$ does not approach zero near the Nyquist frequency, then it is probably wise to try a smaller value of Δt. Alternatively, if the analyst is only interested in the low-frequency components, then it may be easier to filter the continuous series so as to remove the high-frequency components and remove the need for selecting a small value of Δt.

7.8 Examples and Discussion

Spectral analysis can be a useful exploratory diagnostic tool in the analysis of many types of time series. With the aid of examples, this section discusses how to interpret an estimated spectrum, and tries to indicate when spectral analysis is likely to be most useful and when it is likely to be unhelpful. We also discuss some of the practical problems arising in spectral analysis.

We begin with an example to give the reader some 'feel' for the sorts of spectrum shapes that may arise. Figure 7.4 shows three series and their corresponding spectra. In the top panel, the series shows cyclic behaviors over different periods. For example, the series shows peaks for a period around 60, and it also has fluctuations over the whole sample period, indicating the spectrum is concentrated on different frequencies. In the middle panel, the series has more ocillations than the first series. The largest variation is in the period $(1, 50)$, which never occurs again afterwards. As this feature dominates other oscillations of the series, it indicates the spectrum has more concentration on the low frequency. In the bottom panel, the series shows strong seasonal effects, and the seasonal effect in one period seems been aggregated from oscillations over a different frequency. Actually, its spectrum on the right shows two modes, indicating the third series is essentially an aggregration of signals over two or several frequencies.

Figure 7.4 can be reproduced using the following R commands.

```
> x<-seq(0,400)
> for (i in 1:30)  y1<-y1+cos(x*(i+1)/10)/(i+1)
> for (i in 1:30)  y2<-y2+cos(pi*x/(i+3))
> y3<-cos(x)+2*cos(1.5*x)

> par(mfrow=c(3,2), mar=c(2,2,2,2))
> plot(y1, type="l");
> spec.pgram(y1,spans=c(3,3),main="")
> plot(y2, type="l");
> spec.pgram(y2,spans=c(3,3),main="")
> plot(y3, type="l");
> spec.pgram(y3,spans=c(3,3),main="")
```

An example using air temperature

The next example considers the air temperature series at Alaska in the United States, plotted in Figure 1.3. There the regular seasonal variation is

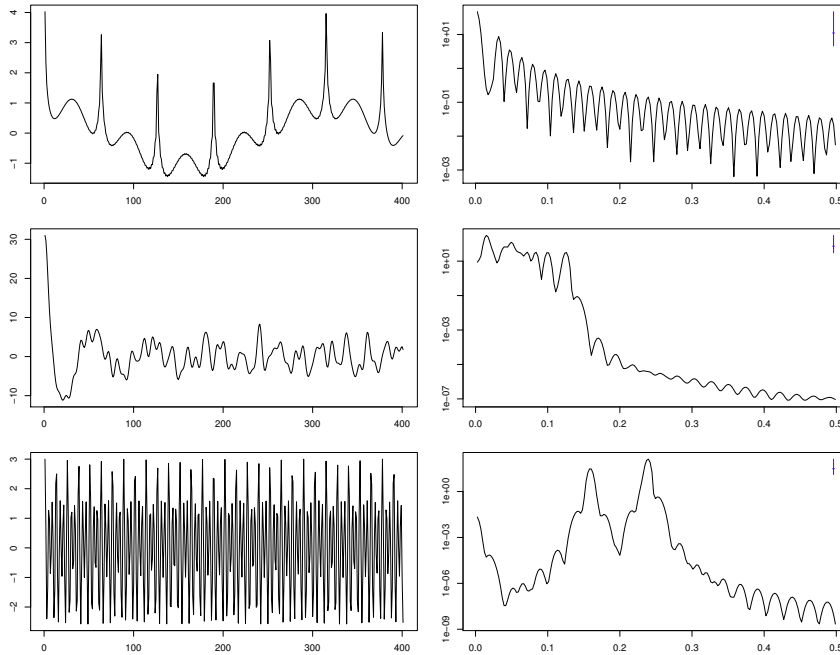

Figure 7.4 *Three time series (Left) and their spectra (Right).*

quite obvious from a visual inspection of the time plot, but in this case the deterministic component accounts for about 94% of the total variation. If we nevertheless carry out a spectral analysis of the air temperature series, we get the spectrum shown in the top panel of Figure 7.5 with a large peak at a frequency of one cycle per year. However, it is arguable that the spectral analysis is not really necessary here, as the seasonal effect is so obvious anyway. In fact, if the analyst has a series containing an obvious trend or seasonal component, then it is advisable to remove such variation from the data **before** carrying out a spectral analysis, as any other effects will be relatively small and may not be visible in the spectrum of the raw data.

The middle panel of Figure 7.5 shows the spectrum of the Alaska air temperature data when the seasonal variation has been removed. The variance is concentrated at low frequencies, indicating either a trend, which is not apparent in Figure 1.3, or short-term correlation as in a first-order AR process with a positive coefficient (cf. Figure 6.4). The latter seems the more likely explanation here, given that there is no contextual reason to expect a trend in temperature (other than global warming, which is relatively small compared with other effects). As noted earlier, the corresponding periodogram in the bottom panel of Figure 7.5 shows a graph that oscillates up and down very quickly and is not helpful for interpreting the properties of the data.

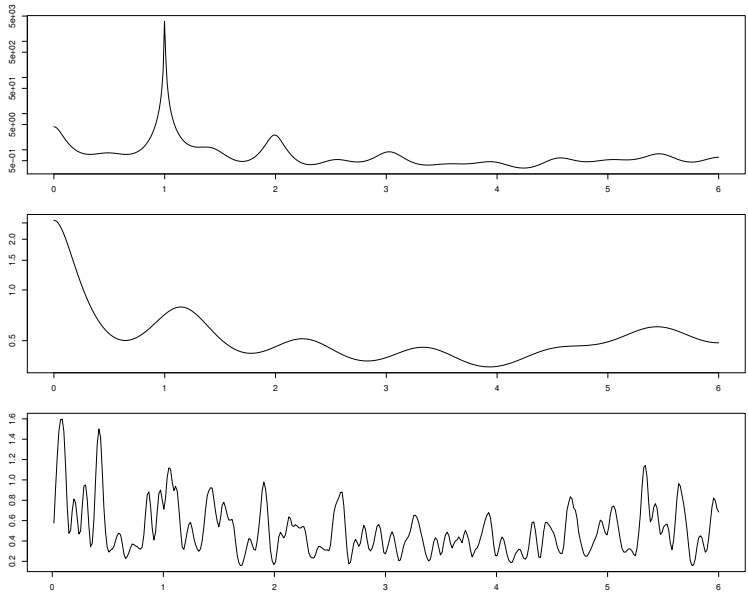

Figure 7.5 *Spectra for average monthly air temperature readings in Alaska. Top: The raw data; Middle: Spectrum for the seasonally adjusted data; Bottom: The periodogram of the seasonally adjusted data is shown for comparison.*

This demonstrates again that the periodogram has to be smoothed to get a consistent estimate of the underlying spectrum.

Removing trend and seasonality is a simple form of the general procedure usually called **prewhitening**. As the name suggests, this aims to construct a series having properties which are closer to those of white noise. In spectral analysis, this is useful because it is easier to estimate the spectrum of a series having a relatively flat spectrum, than one with sharp peaks and troughs. Prewhitening is often carried out by making a linear transformation of the raw data. Then the spectrum of the transformed series can be found, after which the spectrum of the original series can be found, if desired, by using the properties of the linear transformation[8] used to carry out the prewhitening. In spectral analysis, this procedure is often limited to removing trend and seasonality, though in other applications more sophisticated model fitting is often used[9].

Having estimated the spectrum of a given time series, how do we interpret the results? There are various features to look for. First, are there any peaks in

[8]The frequency response function of the linear transformation is defined later in Chapter 9 and leads directly to the required spectrum.

[9]For example, with two time series it is advisable to prewhiten the series by removing as much autocorrelation as possible before calculating quantities called cross-correlations – see Chapter 8.

the spectrum and, if so, at what frequency? Can we find a contextual reason for a peak at this frequency? Second, what is the general shape of the spectrum? In particular, does the spectrum get larger as the frequency tends to zero? This often happens with economic variables and indicates a business cycle with a very long period or an underlying long-term non-stationarity in the mean that has not been removed by prior filtering. Economists, who expect to find a clear peak at low frequency, will usually be disappointed, especially if looking for business cycles with a period around 5–7 years. There is usually little evidence of anything so clear-cut. Rather the low-frequency variation is typically spread over a range of frequencies.

The general shape of the spectrum could in principle be helpful in indicating an appropriate parametric model. For example, the shape of the spectrum of various ARMA models could be found and listed in a similar way to that used for specifying ac.f.s for different ARMA models. The use of the correlogram is a standard diagnostic tool in the Box–Jenkins procedure for identifying an appropriate ARIMA process, but the observed spectrum has rarely been used in this way. Why is this? Spectral analysis, as described in this chapter, is essentially a non-parametric procedure in which a finite set of observations is used to estimate a function defined over the whole range from $(0, \pi)$. The function is not constrained to any particular functional form and so one is effectively trying to estimate more items than in a correlogram analysis, where the analyst may only look at values for a few low lags. Being non-parametric, spectral analysis is in one sense more general than inference based on a particular parametric class of models, but the downside is that it is likely to be less accurate if a parametric model really is appropriate. In our experience, spectral analysis is typically used when there is a suspicion that cyclic variation may be present at some unknown frequency, and the spectrum shape is rarely used for diagnosing a parametric model.

Spectral analysis is arguably at its most useful for series that has no obvious trend or 'seasonal' variation. Such series arise mostly in the physical sciences. In economics, spectral techniques have perhaps not proved as useful as was first hoped, although there have been a few successes. Attempts have also been made to apply spectral analysis to marketing data, but it can be argued (Chatfield, 1974) that marketing series are usually too short and the seasonal variation too large for spectral analysis to give useful results. In meteorology and oceanography, spectral analysis can be very useful (e.g. Craddock, 1965; Snodgrass et al., 1966) but, even in these sciences, spectral analysis may produce no worthwhile results, other than those that are obvious anyway. It is often the case that, once obvious cyclic effects have been removed (e.g. annual variation from monthly rainfall data; daily variation from hourly temperature data), the spectrum will show no clear peaks, but rather a tendency to get larger as the frequency tends to zero. The spectrum in Figure 7.5(b) is a case in point. The two examples in Percival and Walden (1993, Chapter 6), featuring ocean wave data and ice profile data, yield similar

results. Sometimes a small peak is observed but tests usually show that this has dubious significance.

We conclude this section by commenting on some practical aspects of spectral analysis.

As most aspects, such as the choice of truncation point, have already been discussed, one problem that has not been discussed, is whether to plot the estimated spectrum on a linear or logarithm scale. An advantage of using a logarithmic scale is that the asymptotic variance of the estimated spectrum is then independent of the level of the spectrum, and so confidence intervals for the spectrum are of constant width on a logarithmic scale. For spectra showing large variations in power, a logarithmic scale also makes it possible to show more detail over a wide range. A similar idea is used by engineers when measuring sound in decibels, as the latter take values on a logarithmic scale. Jenkins and Watts (1968, p. 266) suggest that spectrum estimates should always be plotted on a logarithmic scale. However, Anderson (1971, p. 547) points out that this exaggerates the visual effects of variations where the spectrum is small. It may be easier to interpret a spectrum plotted on an arithmetic scale, as the area under the graph corresponds to power and this makes it easier to assess the relative importance of different peaks. Thus, while it is often useful to plot $\widehat{f}(\omega)$ on a logarithmic scale in the initial stages of a spectral analysis, especially when trying different truncation points and testing the significance of peaks, it is often better to plot the final version of the estimated spectrum on a linear scale in order to get a clearer interpretation of the final result.

It is also generally easier to interpret a spectrum if the frequency scale is measured in cycles per unit time (f) rather than radians per unit time (ω). This has been done in Figures 7.4 and 7.5. A linear transformation of frequency does not affect the *relative* heights of the spectrum at different frequencies, though it does change the absolute heights by a constant multiple.

Another point worth mentioning is the possible presence in estimated spectra of **harmonics**. As noted earlier, when a series has a strong cyclic component at some frequency ω, then the estimated spectrum may additionally show related peaks at $2\omega, 3\omega, \ldots$. These multiples of the fundamental frequency are called harmonics and generally speaking simply indicate that the main cyclical component is not exactly sinusoidal in character.

Finally, a question that is often asked is how large a value of N is required to get a reasonable estimate of the spectrum. It is often recommended that between 100 and 200 observations is the minimum. With smaller values of N, only very large peaks can be found. However, if the data are prewhitened to make the spectrum fairly flat, then reasonable estimates may be obtained even with values of N around 100, as we have shown in the middle panel of Figure 7.5. However, much longer series are to be preferred and are the norm when spectral analysis is contemplated.

Exercises

7.1 *Revision of Fourier series.* Show that the Fourier series, which represents the function

$$f(x) = x^2 \qquad \text{for } -\pi \leq x \leq \pi$$

is given by

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right).$$

7.2 Derive Equations (7.6) and (7.8).

7.3 Derive Parseval's theorem, given by Equation (7.14).

7.4 If $X_1, \ldots, X_N$ are independent $N(\mu, \sigma^2)$ variates show that

$$a_p = 2\Big[\sum X_t \cos(2\pi pt/N)\Big]/N$$

is $N(0, 2\sigma^2/N)$ for $p = 1, 2, \ldots, (N/2) - 1$.

7.5 Derive the lag window for smoothing the periodogram in sets of size m. For algebraic simplicity take m odd, with $m = 2m^* + 1$, so that

$$\widehat{f}(\omega_p) = \frac{1}{m} \sum_{j=-m^*}^{m^*} I\left(\omega_p + \frac{2\pi j}{N}\right).$$

(Hint: The answer is given in Section 7.4.4. The algebra is rather messy. Use Equation (7.18) and the following two trigonometric results:

$$\cos[2\pi k(p+j)/N] + \cos[2\pi k(p-j)/N] = 2\cos[2\pi kp/N]\cos[2\pi kj/N]$$

$$\sin A - \sin B = 2\sin[(A-B)/2]\cos[(A+B)/2]\,).$$

Chapter 8

Bivariate Processes

Thus far, we have been concerned with analysing a single time series. We now turn our attention to the situation where we have observations on *two* time series and we are interested in the relationship between them.

We may distinguish two types of situations. First, we may have two series that arise 'on an equal footing'. For example, it is often of interest to analyse seismic signals received at two recording sites. Here, we are not usually interested in trying to predict one variable from the other, but rather are primarily interested in measuring the correlations between the two series. In the second type of situation, the two series are thought to be 'causally related', in that one series is regarded as being the **input** to some sort of processor or system, while the second series is regarded as the **output**; we are then interested in finding the properties of the system that converts the input into the output. The two types of situations are roughly speaking the time-series analogues of **correlation** and **regression**.

The first type of situation is considered in this chapter, where the cross-correlation function and the cross-spectrum are introduced, and again in Chapter 13, where vector ARMA models are introduced. The second type of situation is discussed in Chapter 9 where it is assumed that the system can be described as a **linear system**.

8.1 Cross-Covariance and Cross-Correlation

Suppose we make N observations on two variables at unit time intervals over the same period and denote the observations by $(x_1, y_1), \ldots, (x_N, y_N)$. We assume that these observations may be regarded as a finite realization of a discrete-time bivariate stochastic process (X_t, Y_t).

In order to describe a bivariate process it is useful to know the first- and second-order moments. For a univariate process, the first-order moment is the mean while the second-order moment is the autocovariance function (acv.f.), which includes the variance as a special case at lag zero. For a bivariate process, the moments up to second order consist of the mean and acv.f.s for each of the two components plus a new function, called the cross-covariance function. We will only consider bivariate processes that are *second-order stationary*, meaning that all moments up to second order do not change with time (as in the univariate case).

We use the following notation:

Means.	$E(X_t) = \mu_X;$	$E(Y_t) = \mu_Y.$
Autocovariances.	$\text{Cov}(X_t, X_{t+k}) = \gamma_X(k);$	$\text{Cov}(Y_t, Y_{t+k}) = \gamma_Y(k).$

Then the **cross-covariance function** is defined by

$$\text{Cov}(X_t, Y_{t+k}) = E[(X_t - \mu_X)(Y_{t+k} - \mu_Y) = \gamma_{XY}(k) \tag{8.1}$$

and is a function of the lag only, because the processes are assumed to be stationary.

Note that some authors define the cross-covariance function in the 'opposite direction' by

$$\text{Cov}(X_t, Y_{t-k}) = \gamma^*_{XY}(k).$$

Comparing with Equation (8.1) we see that

$$\gamma_{XY}(k) = \gamma^*_{XY}(-k).$$

It doesn't matter which definition is used as long as it is stated clearly and used consistently.

The cross-covariance function differs from the acv.f. in that it is *not* an even function, since in general

$$\gamma_{XY}(k) \neq \gamma_{XY}(-k).$$

Instead we have the relationship

$$\gamma_{XY}(k) = \gamma_{YX}(-k),$$

where the subscripts are reversed.

The size of the cross-covariance coefficients depends on the units in which X_t and Y_t are measured. Thus for interpretative purposes, it is useful to standardize the cross-covariance function to produce a function called the **cross-correlation function**, $\rho_{XY}(k)$, which is defined by

$$\rho_{XY}(k) = \gamma_{XY}(k)/\sqrt{\gamma_X(0)\gamma_Y(0)} = \gamma_{XY}(k)/\sigma_X\sigma_Y, \tag{8.2}$$

where $\sigma_X = \sqrt{\gamma_X(0)}$ denotes the standard deviation of the X-process, and similarly for σ_Y. This function measures the correlation between X_t and Y_{t+k} and has these two properties:

(1) $\rho_{XY}(k) = \rho_{YX}(-k)$ (Note the subscripts reverse in order.)

(2) $|\,\rho_{XY}(k)\,| \leq 1$ (See Exercise 8.2.)

Whereas $\rho_X(0)$, $\rho_Y(0)$ are both equal to one, the value of $\rho_{XY}(0)$ is usually *not* equal to one, a fact that is sometimes overlooked.

8.1.1 Examples

Before discussing the estimation of cross-covariance and cross-correlation functions, we will derive the theoretical functions for two examples of bivariate processes. The first example is rather 'artificial', but the model in Example 8.2 can be useful in practice.

Example 8.1 Suppose that $\{X_t\}, \{Y_t\}$ are both formed from the same purely random process $\{Z_t\}$, which has mean zero, variance σ_Z^2, by

$$\begin{aligned} X_t &= Z_t, \\ Y_t &= 0.5Z_{t-1} + 0.5Z_{t-2}. \end{aligned}$$

Then, using (8.1), we have

$$\gamma_{XY}(k) = \begin{cases} 0.5\sigma_Z^2 & k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Now the variances of the two components are given by

$$\begin{aligned} \gamma_X(0) &= \sigma_Z^2, \\ \gamma_Y(0) &= \sigma_Z^2/2, \end{aligned}$$

so that, using (8.2), we have

$$\rho_{XY}(k) = \begin{cases} 0.5\sqrt{2} & k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 8.2 Suppose that

$$\begin{aligned} X_t &= Z_{1,t}, \\ Y_t &= X_{t-d} + Z_{2,t}, \end{aligned} \tag{8.3}$$

where $\{Z_{1,t}\}, \{Z_{2,t}\}$ are uncorrelated purely random processes with mean zero and variance σ_Z^2, and where d is a positive integer. Then we find

$$\begin{aligned} \gamma_{XY}(k) &= \begin{cases} \sigma_Z^2 & k = d, \\ 0 & \text{otherwise,} \end{cases} \\ \rho_{XY}(k) &= \begin{cases} 1/\sqrt{2} & k = d, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $\sigma_X = \sigma_Z$ and $\sigma_Y = \sqrt{2}\sigma_Z$.

In Chapter 9 we will see that Equation (8.3) corresponds to putting noise into a linear system, which consists of a simple delay of lag d and then adding more noise. The cross-correlation function has a peak at lag d corresponding to the delay in the system, a result that the reader should find intuitively reasonable.

8.1.2 Estimation

The 'obvious' way of estimating the cross-covariance and cross-correlation functions is by means of the corresponding sample functions. Suppose we have N pairs of observations $\{(x_i, y_i); i = 1 \text{ to } N\}$, on two series labelled x and y. Then the **sample cross-covariance function** is

$$c_{XY}(k) = \begin{cases} \sum_{t=1}^{N-k} (x_t - \bar{x})(y_{t+k} - \bar{y})/N & k = 0, 1, \ldots, N-1 \\ \sum_{t=1-k}^{N} (x_t - \bar{x})(y_{t+k} - \bar{y})/N & k = -1, -2, \ldots, -(N-1) \end{cases} \tag{8.4}$$

and the **sample cross-correlation function** is

$$r_{XY}(k) = c_{XY}(k)/s_X s_Y, \tag{8.5}$$

where s_X, s_Y are the sample standard deviations of observations on x_t and y_t, respectively.

It can be shown that these estimators are asymptotically unbiased and consistent. However, it can also be shown that estimators at neighbouring lags are themselves autocorrelated. Furthermore, it can be shown that the variances of sample cross-correlations depend on the autocorrelation functions of the two components. In general, the variances will be inflated. Thus, even for moderately large values of N up to about 200, or even higher, it is possible for two series, which are actually unrelated, to give rise to apparently 'large' cross-correlation coefficients, which are spurious, in that they arise solely from autocorrelations within the two series. Thus, if a test is required for non-zero correlation between two time series, then (at least) one of the series should first be filtered to convert it to (approximate) white noise. The same filter should then be applied to the second series before computing the cross-correlation function – see also Section 9.4.2.

For example, suppose that one series appears to be a first-order autoregressive process with sample mean $\bar{x}$ and estimated parameter $\widehat{\alpha}$. Then the filtered series is given by

$$x'_t = (x_t - \bar{x}) - \widehat{\alpha}(x_{t-1} - \bar{x}).$$

In other words, the new filtered series consists of the residuals from the fitted model. Applying the same filter to the second series, we get $y'_t = (y_t - \bar{y}) - \widehat{\alpha}(y_{t-1} - \bar{y})$.

If the two series are actually uncorrelated, the above procedure will not affect the expected value of the sample cross-correlation coefficients, but should reduce their variance. This makes it much easier to interpret the cross-correlations, or indeed to use them to fit a linear model as in Section 9.4.2.

Note that some authors (e.g. Brockwell and Davis, 1991, Chapter 11) recommend prewhitening *both* series, sometimes called double prewhitening, before calculating cross-correlations.

For two uncorrelated series, of which one is white noise, it can be shown that

$$\begin{aligned} E[r_{XY}(k)] &\simeq 0 \\ \mathrm{Var}[r_{XY}(k)] &\simeq 1/N. \end{aligned}$$

Thus values outside the interval $\pm 2/\sqrt{N}$ are significantly different from zero, and this result can be used to test for zero cross-correlation.

8.1.3 Interpretation

Sometimes, it is rather difficult to interpret a sample cross-correlation function. If the series are properly prewhitened, we have seen that it is easy to test whether any of the cross-correlation coefficients are significantly different from zero. Example 8.2 suggests that a significant peak in the estimated cross-correlation function at lag d may indicate that one series is related to the other when delayed by time d. However, you are more likely to find a series of significant coefficients at neighbouring lags, and they are more difficult to interpret.

The interpretation is even more difficult, and indeed fraught with danger, if the prefiltering procedure, described in Section 8.1.2, is not used. For example, Coen et al. (1969) calculated cross-correlation functions between variables such as the (detrended) *Financial Times (FT)* share index and (detrended) U.K. car production, and this resulted in a fairly smooth, roughly sinusoidal function with 'large' coefficients at lags 5 and 6 months. Coen et al. used this information to set up a regression model to 'explain' the variation in the *FT* share index in terms of car production 6 months earlier. However, Box and Newbold (1971) have shown that the 'large' cross-correlation coefficients are spurious as the two series had not been properly filtered. Rather than having models with *independent* error terms, the appropriate models for the given series were close to being random walks. This meant there were very high autocorrelations within each series and this inflated the variances of the cross-correlations. When the series were properly modelled, it was found that cross-correlations were negligible.

Box and Newbold (1971) also presented some interesting simulations. They constructed two *independent* random walks and then computed the cross-correlations. The latter should have an expectation close to zero, but the high autocorrelations inflated the variance so that several spuriously high coefficients were observed. The general sinusoidal shape was similar to that found for the real data. The best general advice is 'Beware'!

8.2 The Cross-Spectrum

The cross-correlation function is the natural tool for examining the relationship between two time series in the time domain. This section introduces a complementary function, called the cross-spectral density function or cross-spectrum, which is the natural tool in the frequency domain.

By analogy with Equation (6.11), we will define the **cross-spectrum** of a discrete-time bivariate stationary process, measured at unit intervals of time, as the Fourier transform of the cross-covariance function, namely

$$f_{XY}(\omega) = \frac{1}{\pi}\left[\sum_{k=-\infty}^{\infty} \gamma_{XY}(k)\mathrm{e}^{-\mathrm{i}\omega k}\right] \tag{8.6}$$

over the range $0 < \omega < \pi$. The physical interpretation of the cross-spectrum is more difficult than for the autospectrum (see Priestley, 1981, p. 657). Indeed a physical understanding of cross-spectra will probably not become clear until we have studied linear systems.

Note that $f_{XY}(\omega)$ is a *complex* function, unlike the autospectrum, which is real. This is because $\gamma_{XY}(k)$ is not an even function.

The reader should note that many authors define the cross-spectrum in the range $(-\pi, \pi)$ by analogy with Equation (6.13) as

$$f_{XY}(\omega) = \frac{1}{2\pi}\left[\sum_{k=-\infty}^{\infty} \gamma_{XY}(k)\mathrm{e}^{-\mathrm{i}\omega k}\right]. \tag{8.7}$$

This definition has certain mathematical advantages, notably that it can handle complex-valued processes and that it has a simple inverse relationship of the form

$$\gamma_{XY}(k) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\omega k} f_{XY}(\omega)\mathrm{d}\omega \tag{8.8}$$

whereas Equation (8.6) does not have a simple inverse relationship. However, Equation (8.7) introduces negative frequencies, and for ease of understanding we prefer (8.6). In any case, (8.7) means that $f_{XY}(-\omega)$ is the complex conjugate[1] of $f_{XY}(\omega)$ and so provides no extra information. Authors who use Equation (8.7) only examine the cross-spectrum at positive frequencies.

We now describe several functions derived from the cross-spectrum, which are helpful in interpreting the cross-spectrum. From Equation (8.6), the real part of the cross-spectrum, called the **co-spectrum**, is given by

$$\begin{aligned} c(\omega) &= \frac{1}{\pi}\left[\sum_{k=-\infty}^{\infty} \gamma_{XY}(k)\cos\omega k\right] \\ &= \frac{1}{\pi}\left\{\gamma_{XY}(0) + \sum_{k=1}^{\infty}\left[\gamma_{XY}(k) + \gamma_{YX}(k)\right]\cos\omega k\right\}. \end{aligned} \tag{8.9}$$

[1] If $z = a + \mathrm{i}b$, then its complex conjugate is given by $\bar{z} = a - \mathrm{i}b$.

The complex part of the cross-spectrum, with a minus sign attached, is called the **quadrature** spectrum and is given by

$$\begin{aligned} q(\omega) &= \frac{1}{\pi}\left[\sum_{k=-\infty}^{\infty} \gamma_{XY}(k)\sin\omega k\right] \\ &= \frac{1}{\pi}\left\{\sum_{k=1}^{\infty}\left[\gamma_{XY}(k)-\gamma_{YX}(k)\right]\sin\omega k\right\} \end{aligned} \tag{8.10}$$

so that

$$f_{XY}(\omega) = c(\omega) - \mathrm{i}q(\omega). \tag{8.11}$$

An alternative way of expressing the cross-spectrum is in the form

$$f_{XY}(\omega) = \alpha_{XY}(\omega)\mathrm{e}^{\mathrm{i}\phi_{XY}(\omega)}, \tag{8.12}$$

where

$$\alpha_{XY}(\omega) = \sqrt{c^2(\omega)+q^2(\omega)} \tag{8.13}$$

is the **cross-amplitude** spectrum, and

$$\phi_{XY}(\omega) = \tan^{-1}[-q(\omega)/c(\omega)] \tag{8.14}$$

is the **phase** spectrum. From Equation (8.14), it appears that $\phi_{XY}(\omega)$ is undetermined by a multiple of π. However, if the cross-amplitude spectrum is required to be positive so that we take the positive square root in Equation (8.13), then the phase is actually undetermined by a multiple of 2π using the equality of Equations (8.11) and (8.12). This apparent non-uniqueness makes it difficult to evaluate the phase. However, when we consider linear systems in Chapter 9, we will see that there are physical reasons why the phase *is* often uniquely determined and does not need to be confined to the range $\pm\pi$. The phase is usually zero at $\omega = 0$ and it makes sense to treat it as a continuous function of ω, as ω goes from 0 to π. Thus, if the phase grows to $+\pi$ say, then it can be allowed to continue to grow rather than revert to $-\pi$.

Another useful function derived from the cross-spectrum is the (squared) **coherency**, which is given by

$$\begin{aligned} C(\omega) &= [c^2(\omega)+q^2(\omega)]/[f_X(\omega)f_Y(\omega)] \\ &= \alpha_{XY}^2(\omega)/f_X(\omega)f_Y(\omega) \end{aligned} \tag{8.15}$$

where $f_X(\omega), f_Y(\omega)$ are the power spectra of the individual processes, $\{X_t\}$ and $\{Y_t\}$. It can be shown that

$$0 \le C(\omega) \le 1.$$

This quantity measures the square of the linear correlation between the two components of the bivariate process at frequency ω and is analogous to the

square of the usual correlation coefficient. The closer $C(\omega)$ is to one, the more closely related are the two processes at frequency ω.

Finally, we will define a function called the **gain** spectrum, which is given by

$$\begin{aligned} G_{XY}(\omega) &= \sqrt{f_Y(\omega)C(\omega)/f_X(\omega)} \\ &= \alpha_{XY}(\omega)/f_X(\omega) \end{aligned} \tag{8.16}$$

which is essentially the regression coefficient of the process Y_t on the process X_t at frequency ω. A second gain function can also be defined by $G_{YX}(\omega) = \alpha_{XY}(\omega)/f_Y(\omega)$, where we divide by f_Y rather than f_X. In the terminology of linear systems – see Chapter 9 – this definition corresponds to regarding Y_t as the input and X_t as the output.

By this point, the reader will probably be rather confused by all the different functions that have been introduced in relation to the cross-spectrum. Whereas the cross-correlation function is a relatively straightforward development from the autocorrelation function (in theory at least, if not always in practice), statisticians often find the cross-spectrum much harder to understand than the autospectrum. It is no longer possible to interpret the results as some sort of contribution to power (variance) at different frequencies. Usually *three* functions have to be plotted against frequency to describe the relationship between two series in the frequency domain. Sometimes the co-, quadrature and coherency spectra are most suitable. Sometimes the coherency, phase and cross-amplitude are more appropriate, while another possible trio is coherency, phase and gain. Each trio can be determined from any other trio. The physical interpretation of these functions will probably not become clear until we have studied linear systems in Chapter 9.

8.2.1 Examples

This subsection derives the cross-spectrum and related functions for the two examples discussed in Section 8.1.1.

Example 8.3 For Example 8.1, we may use Equation (8.6) to derive the cross-spectrum from the cross-covariances by

$$f_{XY}(\omega) = (0.5\sigma_Z^2 \mathrm{e}^{-\mathrm{i}\omega} + 0.5\sigma_Z^2 \mathrm{e}^{-2\mathrm{i}\omega})/\pi$$

for $0 < \omega < \pi$. Using (8.9), the co-spectrum is given by

$$c(\omega) = 0.5\sigma_Z^2(\cos\omega + \cos 2\omega)/\pi.$$

Using (8.10), the quadrature spectrum is given by

$$q(\omega) = 0.5\sigma_Z^2(\sin\omega + \sin 2\omega)/\pi.$$

Using (8.13), the cross-amplitude spectrum is given by

$$\alpha_{XY}(\omega) = \frac{0.5\sigma_Z^2}{\pi}\sqrt{(\cos\omega + \cos 2\omega)^2 + (\sin\omega + \sin 2\omega)^2}$$

which, after some algebra, gives

$$\alpha_{XY}(\omega) = \sigma_Z^2 \cos(\omega/2)/\pi.$$

Using (8.14), the phase spectrum is given by

$$\tan\phi_{XY}(\omega) = -(\sin\omega + \sin 2\omega)/(\cos\omega + \cos 2\omega).$$

Figure 8.1 shows the co-spectrum, the quadrature, the cross-amplitude, and the phase spectra. In order to evaluate the coherency, we need to find the power spectra of the two individual processes. Since $X_t = Z_t$ is white noise, it has constant spectrum $f_X(\omega) = \sigma_Z^2/\pi$. The second process, Y_t, is a form of moving average process and it can readily be shown that its spectrum is given by

$$\begin{aligned} f_Y(\omega) &= 0.5\sigma_Z^2(1+\cos\omega)/\pi \\ &= \sigma_Z^2\cos^2(\omega/2)/\pi. \end{aligned}$$

Thus, using Equation (8.15), the coherency spectrum is given by

$$C(\omega) = 1 \quad \text{for all } \omega \text{ in } (0,\pi).$$

This latter result may appear surprising at first sight. However, both X_t and Y_t are generated from the *same* noise process and this explains why there is perfect correlation between the components of the two processes at any given frequency.

Finally, using Equation (8.16), the gain spectrum is given by

$$G_{XY}(\omega) = \sqrt{f_Y(\omega)C(\omega)/f_X(\omega)} = \cos(\omega/2)$$

since the coherency is unity.

Example 8.4 For the bivariate process of Example 8.2, Equation (8.6) gives the cross-spectrum as $f_{XY}(\omega) = \sigma_Z^2 e^{-i\omega d}/\pi$, and this leads in turn to the following functions:

$$\begin{aligned} c(\omega) &= \sigma_Z^2\cos\omega d/\pi \\ q(\omega) &= \sigma_Z^2\sin\omega d/\pi \\ \alpha_{XY}(\omega) &= \sigma_Z^2/\pi \\ \tan\phi_{XY}(\omega) &= -\tan\omega d. \end{aligned} \tag{8.17}$$

Then, as the two autospectra are given by

$$\begin{aligned} f_X(\omega) &= \sigma_Z^2/\pi \\ f_Y(\omega) &= 2\sigma_Z^2/\pi \end{aligned}$$

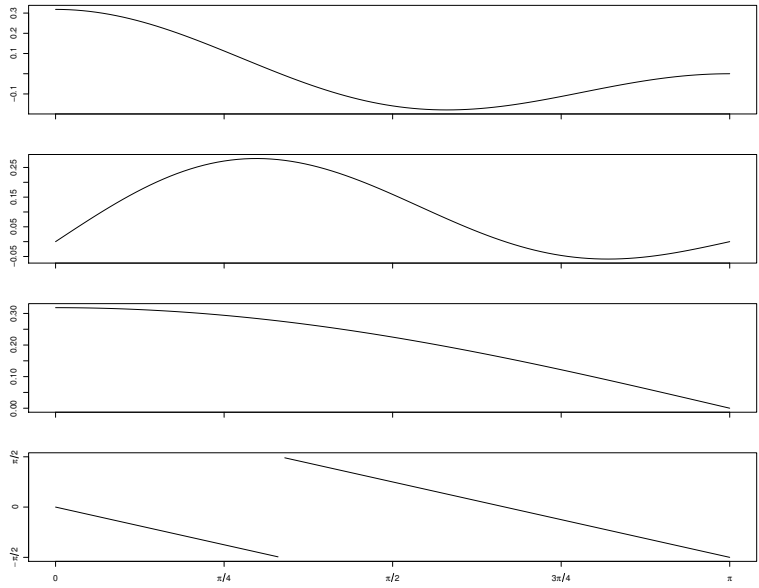

Figure 8.1 *The spectra $c(\omega)$, $q(\omega)$, $\alpha_{\mathrm{XY}}(\omega)$ and $\phi_{\mathrm{XY}}(\omega)$ for Example 8.3.*

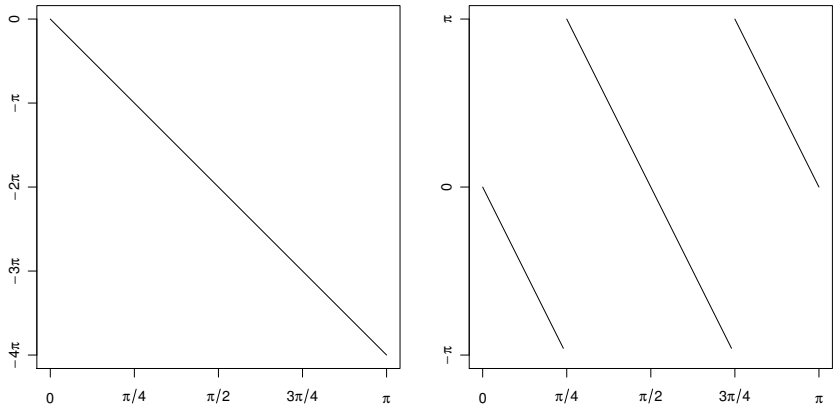

Figure 8.2 *The phase spectrum, $\phi_{\mathrm{XY}}(\omega)$, for Example 8.4 with $d = 4$, with phase unconstrained (left) and phase constrained (right).*

we find

$$C(\omega) = 1/2.$$

Note that all the above functions are defined on $(0, \pi)$. The function of particular interest in this example is the phase, which, from Equation (8.17), is a straight line with slope $-d$ when $\phi_{XY}(\omega)$ is unconstrained and is plotted against ω as a continuous function starting with zero phase at zero frequency (see the left panel of Figure 8.2). If, however, the phase is constrained to lie within the interval $(-\pi, \pi)$ then a graph like the right panel of Figure 8.2 will result, where the slope of each line is $-d$.

This result is often used to help in the identification of relationships between time series. If the estimated phase approximates a straight line through the origin, then this indicates a delay between the two series equal to the slope of the line. More generally, the time delay between two recording sites will change with frequency, due, for example, to varying speeds of propagation. This is called the **dispersive** case, and can be recognized by changes in the slope of the phase function.

8.2.2 Estimation

As in univariate spectral analysis, there are two basic approaches to estimating a cross-spectrum. First, we can take a Fourier transform of the truncated sample cross-covariance function (or of the cross-correlation function to get a normalized cross-spectrum). This is analogous to Equation (7.21) for the univariate case. The estimated co-spectrum is given by

$$\widehat{c}(\omega) = \frac{1}{\pi}\left[\sum_{k=-M}^{M} \lambda_k c_{XY}(k) \cos \omega k\right] \tag{8.18}$$

where M is the truncation point, and $\{\lambda_k\}$ is the lag window. The estimated quadrature spectrum is given by

$$\widehat{q}(\omega) = \frac{1}{\pi}\left[\sum_{k=-M}^{M} \lambda_k c_{XY}(k) \sin \omega k\right]. \tag{8.19}$$

Equations (8.18) and (8.19) are often used in the equivalent forms

$$\begin{aligned}
\widehat{c}(\omega) &= \frac{1}{\pi}\left\{\lambda_0 c_{XY}(0) + \sum_{k=1}^{M} \lambda_k [c_{XY}(k) + c_{XY}(-k)] \cos \omega k\right\} \\
\widehat{q}(\omega) &= \frac{1}{\pi}\left\{\sum_{k=1}^{M} \lambda_k [c_{XY}(k) - c_{XY}(-k)] \sin \omega k\right\}.
\end{aligned}$$

The truncation point M and the lag window $\{\lambda_k\}$ are chosen in a similar way to that used in spectral analysis for a single series, with the Tukey and Parzen windows being most popular.

Having estimated the co- and quadrature spectra, estimates of the cross-amplitude spectrum, phase and coherency follow, in an obvious way, from Equations (8.13), (8.14) and (8.15). Estimates of the power spectra of the two individual series, namely, $\widehat{f}_X$ and $\widehat{f}_Y$, are needed in the latter case. We find

$$\begin{aligned} \widehat{\alpha}_{XY}(\omega) &= \sqrt{\widehat{c}^2(\omega) + \widehat{q}^2(\omega)}, \\ \tan\widehat{\phi}_{XY}(\omega) &= -\widehat{q}(\omega)/\widehat{c}(\omega), \\ \widehat{C}(\omega) &= \widehat{\alpha}^2_{XY}(\omega)/\widehat{f}_X(\omega)\widehat{f}_Y(\omega). \end{aligned}$$

When plotting the estimated phase spectrum, similar remarks apply as to the (theoretical) phase. Phase estimates are undetermined by a multiple of 2π, but can usually be plotted as a continuous function, which is zero at zero frequency.

Before estimating the coherency, it may be advisable to **align** the two series. If this is not done, Jenkins and Watts (1968) have demonstrated that estimates of coherency will be biased if the phase changes rapidly. If the sample cross-correlation function has its largest value at lag s say, then the two series are aligned by translating one series a distance s so that the peak in the cross-correlation function of the aligned series is at zero lag.

The second approach to cross-spectral analysis is to smooth a function called the **cross-periodogram**. The univariate periodogram of a series $\{x_t\}$ can be written in the form

$$\begin{aligned} I(\omega_p) &= \left(\sum x_t \mathrm{e}^{\mathrm{i}\omega_p t}\right)\left(\sum x_t \mathrm{e}^{-\mathrm{i}\omega_p t}\right)/N\pi \\ &= N(a_p^2 + b_p^2)/4\pi \end{aligned} \tag{8.20}$$

using Equations (7.17) and (7.10), where $\{a_p\}$, $\{b_p\}$ are the coefficients in the Fourier series representation of $\{x_t\}$. By analogy with Equation (8.20), we may define the cross-periodogram of two series $\{x_t\}$ and $\{y_t\}$ as

$$I_{XY}(\omega_p) = \left(\sum x_t \mathrm{e}^{\mathrm{i}\omega_p t}\right)\left(\sum y_t \mathrm{e}^{-\mathrm{i}\omega_p t}\right)/N\pi. \tag{8.21}$$

After some algebra, it can be shown that the real and imaginary parts of $I_{XY}(\omega_p)$ are given by

$$N(a_{px}a_{py} + b_{px}b_{py})/4\pi \quad \text{and} \quad N(a_{px}b_{py} - a_{py}b_{px})/4\pi,$$

where (a_{px}, b_{px}), (a_{py}, b_{py}) are the Fourier coefficients of $\{x_t\}, \{y_t\}$, respectively at ω_p. These real and imaginary parts may then be smoothed to get consistent estimates of the co- and quadrature spectral density functions by

$$\begin{aligned} \widehat{c}(\omega_p) &= N \sum_{q=p-m^*}^{p+m^*} (a_{qx}a_{qy} + b_{qx}b_{qy})/4\pi m \\ \widehat{q}(\omega_p) &= N \sum_{q=p-m^*}^{p+m^*} (a_{qx}b_{qy} - a_{qy}b_{qx})/4\pi m, \end{aligned}$$

where $m = 2m^* + 1$ is chosen, as in the univariate case, so as to balance variance and resolution. These two equations are analogous to Equation (7.24) in the univariate case. The above estimates may then be used to estimate the cross-amplitude spectrum, phase and coherency as before.

The computational advantages of the second type of approach are clear. Once a periodogram analysis has been made of the two individual processes, nearly all the work has been done as the estimates of $c(\omega)$ and $q(\omega)$ only involve the Fourier coefficients of the two series. The disadvantage of the approach is that alignment is only possible if the cross-correlation function is calculated separately. This can be done directly or by the use of two (fast) Fourier transforms by an analogous procedure to that described in Section 7.4.5.

The properties of cross-spectral estimators are discussed, for example, by Jenkins and Watts (1968), Priestley (1981) and Bloomfield (2000). The following points are worth noting. Estimates of phase and cross-amplitude are imprecise when the coherency is relatively small. Estimates of coherency are constrained to lie between 0 and 1, and there may be a bias towards 1/2, which may be serious with short series. Finally, we note that rapid changes in phase may bias coherency estimates, which is another reason why alignment is generally a good idea.

8.2.3 Interpretation

Cross-spectral analysis is a technique for examining the relationship between two series in the frequency domain. The technique may be used for two time series that 'arise on a similar footing' and then the coherency spectrum is perhaps the most useful function. It measures the linear correlation between two series at each frequency and is analogous to the square of the ordinary product-moment correlation coefficient.

The other functions introduced in this chapter, such as the phase spectrum, are most readily understood in the context of linear systems, which will be discussed later in Chapter 9. We will therefore defer further discussion of how to interpret cross-spectral estimates until Section 9.3.

Example 8.5

In this example, we consider the total population (in millions) and birth rate (per 1,000) series for the United States from 1965 to 2015 (Figure 1.5, and explore the relationship between changes in these two series. Figure 8.3 shows the original and differenced values of the two series.

```
> pop<-read.csv("US_pop_birthrate.csv", header=T)
> ts.pop<-ts(pop[,2], start=1960)/10^6
> ts.birth<-ts(pop[,3], start=1960)
> ts.dpop<-diff(ts.pop)
> ts.dbirth<-diff(ts.birth)
```

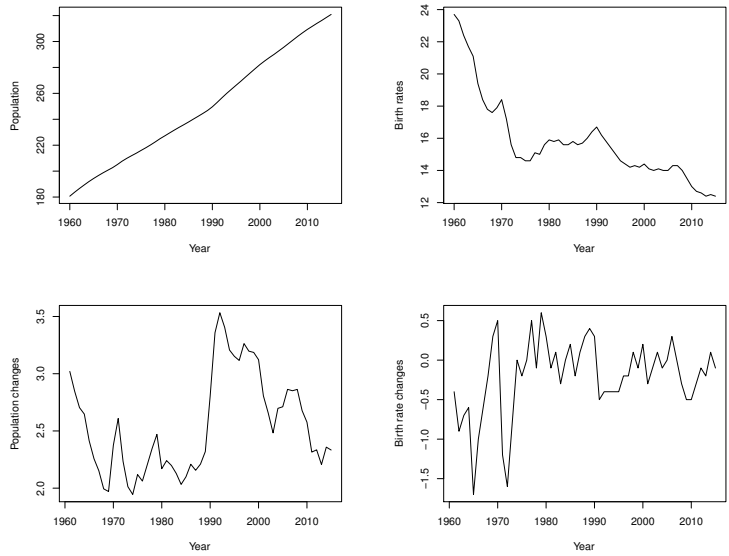

Figure 8.3 *The total population (Top left), the birth rates (Top right), the changes of population (Bottom left), and the changes of the birth rates (Bottom right) for the United States from 1965 to 2015.*

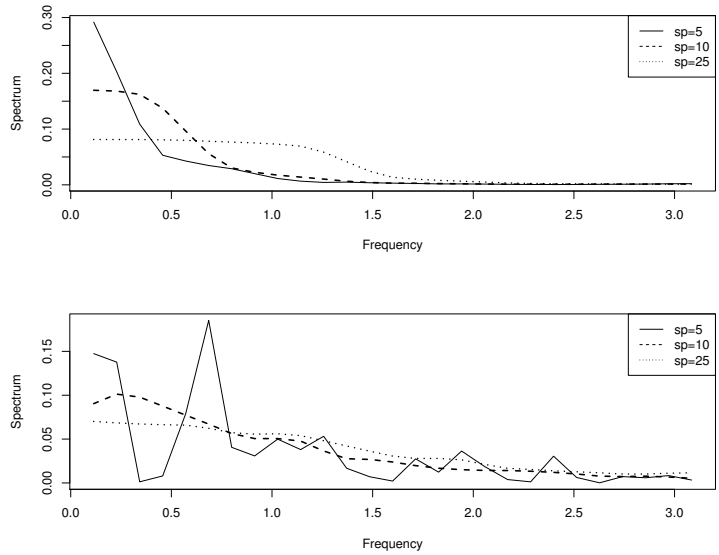

Figure 8.4 *Periodgrams of changes of population (Top) and changes of birth rates (Bottom) with different smoothing parameters.*

```
> par(mfrow=c(2,2), mar=c(4,4,4,4))
> plot(ts.pop, xlab="Year", ylab="Population")
> plot(ts.birth, xlab="Year", ylab="Birth rates")
> plot(ts.dpop, xlab="Year", ylab="Population changes")
> plot(ts.dbirth, xlab="Year", ylab="Birth rate changes")
```

We first show periodgrams of changes of population and changes of birth rates in Figure 8.4. The spectra of both series are large at low frequencies and small at middle and high frequencies. Besides, the shape of estimated spectra varies significantly when different smoothing parameters are used.

```
> # FIgure 8.4
> par(mfrow=c(2,1), mar=c(4,4,4,4))
> sp.dpop <- spec.pgram(ts.dpop,plot=F,sp=5)
> plot(sp.dpop$freq*2*pi,sp.dpop$spec/(2*pi),type="l",
    lty=1, xlab="Frequency", ylab="Spectrum")
> sp.dpop<-spec.pgram(ts.dpop,pl=F,sp=10)
> lines(sp.dpop$freq*2*pi, sp.dpop$spec/(2*pi),
    lty=2,lwd=2)
> sp.dpop <- spec.pgram(ts.dpop,pl=F,sp=25)
> lines(sp.dpop$freq*2*pi, sp.dpop$spec/(2*pi),
    lty=3,lwd=2)
> legend('topright',c("sp=5", "sp=10", "sp=25"),
    lty=c(1,2,3))

> sp.dbirth <- spec.pgram(ts.dbirth,plot=F)
> plot(sp.dbirth$freq*2*pi,sp.dbirth$spec/(2*pi),type="l",
    lty=1, xlab="Frequency", ylab="Spectrum")
> sp.dbirth<-spec.pgram(ts.dbirth, pl=F, sp=10)
> lines(sp.dbirth$freq*2*pi, sp.dbirth$spec/(2*pi),
    lty=2,lwd=2)
> sp.dbirth <- spec.pgram(ts.dbirth,pl=F,sp=25)
> lines(sp.dbirth$freq*2*pi, sp.dbirth$spec/(2*pi),
    lty=3,lwd=2)
> legend('topright',c("sp=5", "sp=10", "sp=25"),
    lty=c(1,2,3))
```

We then consider the cross-spectrum of the two series of changes. Figure 8.5 shows the squared coherency and the phase spectrum of the cross-spectrum for changes of populations and changes of birth rates. We find that the squared coherency is large at middle frequencies, suggesting that there is a significant linear relationship between two series at middle frequencies. We also note that the phase seems linear at both low and high fequencies, but with different slopes; on the other hand, the phase seems linear in the middle frequencies.

```
> # Figure 8.5
> par(mfrow=c(2,1), mar=c(4,4,4,4))
```

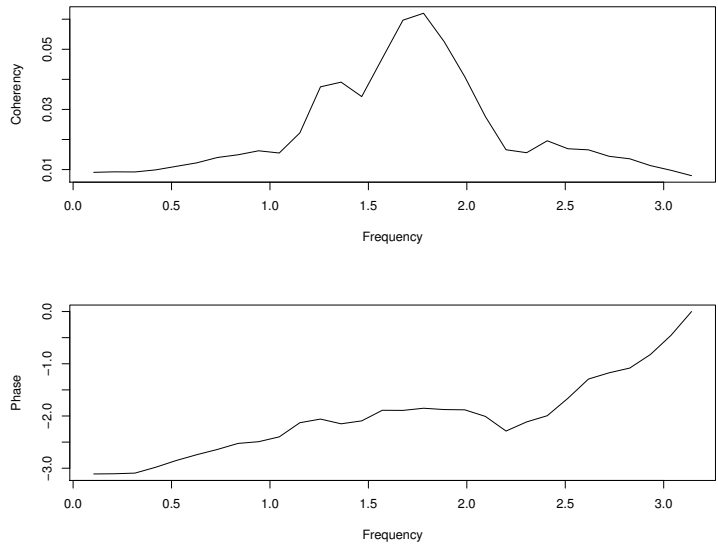

Figure 8.5 *The squared coherency (Top) and phase spectrum (Bottom) of the cross-spectrum.*

```
> p<-spec.pgram(xy,t=0,plot=F,sp=25)
> f<-p$freq*2*pi;
> plot(f,p$coh/(2*pi),type="l",xlab="Frequency",
    ylab="Coherency")
> plot(f,p$phase,type="l",xlab="Frequency", ylab="Phase")
```

Exercises

8.1 Show that the cross-covariance function of the stationary bivariate process $\{X_t, Y_t\}$ where

$$\begin{aligned} X_t &= Z_{1,t} + \beta_{11}\, Z_{1,t-1} + \beta_{12}\, Z_{2,t-1} \\ Y_t &= Z_{2,t} + \beta_{21}\, Z_{1,t-1} + \beta_{22}\, Z_{2,t-1} \end{aligned}$$

and $\{Z_{1,t}\}$, $\{Z_{2,t}\}$ are independent purely random processes with zero mean and variance σ_z^2, is given by

$$\gamma_{XY}(k) = \begin{cases} (\beta_{11}\beta_{21} + \beta_{12}\beta_{22})\sigma_z^2 & k = 0 \\ \beta_{21}\,\sigma_z^2 & k = 1 \\ \beta_{12}\,\sigma_z^2 & k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence evaluate the cross-spectrum.

8.2 Define the cross-correlation function $\rho_{XY}(\tau)$ of a bivariate stationary process and show that $|\,\rho_{XY}(\tau)\,| \leq 1$ for all τ.
Two first-order moving average processes

$$\begin{aligned} X_t &= Z_t + 0.4\,Z_{t-1} \\ Y_t &= Z_t - 0.4\,Z_{t-1} \end{aligned}$$

are formed from a purely random process $\{Z_t\}$, which has mean zero and variance σ_Z^2. Find the cross-covariance and cross-correlation functions of the bivariate process $\{X_t, Y_t\}$ and hence show that the cross-spectrum is given by

$$f_{XY}(\omega) = \sigma_Z^2\,(0.84 + 0.8\mathrm{i}\sin\omega)\,/\,\pi \qquad \text{for } 0 < \omega < \pi.$$

Evaluate the co-, quadrature, cross-amplitude, phase and coherency spectra.

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Chapter 9

Linear Systems

9.1 Introduction

An important problem in engineering and the physical sciences is that of identifying a model for a physical system (or process) given observations on the input and output to the system. For example, the yield from a chemical reactor (the output) depends in part on the temperature at which the reactor is kept (the input). Much of the literature assumes that the system can be adequately approximated over the range of interest by a linear model whose parameters do not change with time, although recently there has been increased interest in time-varying and non-linear systems. It turns out that the study of linear systems is useful, not only for examining the relationship between different time series, but also for examining the properties of linear filtering procedures such as many of the formulae for removing trend and seasonality.

This chapter confines attention to time-invariant linear systems. After defining such a system, Sections 9.2 and 9.3 look at their properties and show how to describe a linear system in the time and frequency domains, respectively. Then Section 9.4 discusses how to identify the structure of a linear system from observed data.

Much of the literature on linear systems (e.g. Bendat and Piersol, 2000) is written from an engineering viewpoint, and looks especially at such topics as control theory and digital communications (e.g. Glover and Grant, 1998), as well as the identification of input/output systems. We naturally concentrate on more statistical issues and acknowledge, in particular, the major contribution of Box et al. (1994).

We generally denote the input and output series by $\{x_t\}$, $\{y_t\}$, respectively in discrete time, and by $x(t)$, $y(t)$, respectively in continuous time, though we sometimes use the latter notation for either type of series, as in the following definition.

Definition of a Linear System. Suppose $y_1(t), y_2(t)$ are the outputs corresponding to inputs $x_1(t), x_2(t)$, respectively. Then the system is said to be **linear** if, and only if, any linear combination of the inputs, say $\lambda_1 x_1(t) + \lambda_2 x_2(t)$, produces the same linear combination of the outputs, namely, $\lambda_1 y_1(t) + \lambda_2 y_2(t)$, where λ_1, λ_2 are any constants[1].

[1] An alternative, neater way of writing this, is to denote the transformation effected by the system by $\mathcal{L}$ say. Mathematicians would call this an *operator*. Then if $\mathcal{L}x_1(t) = y_1(t)$

As two special cases of the above definition, we note that a linear system preserves addition and scalar multiplication, and some writers define linearity by specifying *two* conditions that correspond to these two special cases. For addition, we set $\lambda_1 = \lambda_2 = 1$, and get a condition sometimes called the **Principle of Superposition**, which says that the sum of the inputs gives rise to the sum of the outputs. For scalar multiplication, we set $\lambda_2 = 0$, and get a condition sometimes called the *homogeneity*, or *proportionality* or *scale-invariant* condition. Note that the latter condition means that if, for example, you double the input, then the output will also be doubled. We emphasize again that both these conditions are special cases of the more general definition given above.

We further confine attention to linear systems that are time-invariant. This term is defined as follows. If input $x(t)$ produces output $y(t)$, then the system is said to be **time-invariant** if a delay of time τ in the input produces the same delay in the output. In other words, $x(t-\tau)$ produces output $y(t-\tau)$, so that the input–output relation does not change with time. Generally speaking, any equation with constant coefficients defines a time-invariant system (though it need not be linear).

We only consider systems having one input and one output. The extension to several inputs and outputs is straightforward in principle, though more difficult in practice.

The reader may readily verify that the following equations both define a time-invariant linear system:

$$\begin{aligned} (1) \qquad & y_t = 0.5x_t + 0.25x_{t-1}, \\ (2) \qquad & y_t = 0.5y_{t-1} + 0.25x_t, \end{aligned}$$

where we note that (2) involves lagged values of the output. Differencing (or differentiation in continuous time) also gives a linear system. However, the reader should check that the equation

$$(3) y_t = 0.5x_t + 2,$$

which looks like a linear equation, does not in fact define a linear system[2], although it is time-invariant. Of course, an equation involving a non-linear function, such as $y_t = 0.5\,x_t^2$, is not linear either. Further note that an equation with time-varying coefficients, such as $y_t = 0.5\,t\,x_t$, will not be time-invariant, even if it is linear.

---

and $\mathcal{L}x_2(t) = y_2(t)$, the operator is linear if $\mathcal{L}[\lambda_1 x_1(t) + \lambda_2 x_2(t)] = \lambda_1 \mathcal{L}x_1(t) + \lambda_2 \mathcal{L}x_2(t) = \lambda_1 y_1(t) + \lambda_2 y_2(t)$.

[2] Hint: You can apply the definition of linearity to two carefully chosen known input series such as $x_{1,t} = K$ for all t, where K is a constant, and see what you get. Alternatively, write the equation in operator form as $\mathcal{L}x_t = [0.5x_t + 2]$, and then $\mathcal{L}[x_{1,t} + x_{2,t}] = [0.5(x_{1,t} + x_{2,t}) + 2] = \mathcal{L}x_{1,t} + \mathcal{L}x_{2,t} - 2$ and so does not satisfy the Principle of Superposition. Mathematicians would describe a non-linear transformation of this type as an *affine* transformation, which can readily be made linear by a suitable linear transformation, namely, $z_t = y_t - 2$.

9.2 Linear Systems in the Time Domain

A time-invariant linear system may generally be written in the form

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\ \mathrm{d}u \tag{9.1}$$

in continuous time, or

$$y_t = \sum_{k=-\infty}^{\infty} h_k x_{t-k} \tag{9.2}$$

in discrete time. The weight function, $h(u)$ in continuous time or $\{h_k\}$ in discrete time, provides a description of the system in the time domain. This function is called the **impulse response function** of the system, for reasons that will become clear in Section 9.2.2 below.

It is obvious that Equations (9.1) and (9.2) define a linear system. Moreover, the fact that the impulse response functions do not depend on t, ensures that the systems are time invariant. A linear system is said to be **physically realizable** or **causal** if

$$h(u) = 0 \qquad u < 0$$

in continuous time, or

$$h_k = 0 \qquad k < 0$$

in discrete time.

We further restrict attention to **stable** systems for which any bounded input produces a bounded output, although control engineers are often concerned with controlling unstable systems. In discrete time, a sufficient condition for stability is that the impulse response function should satisfy

$$\sum_k |h_k| < C,$$

where C is a finite constant. In continuous time, the above sum is replaced by an appropriate integral.

Engineers have been mainly concerned with continuous-time systems but are increasingly studying sampled-data control problems. Statisticians generally work with discrete data and so the subsequent discussion concentrates on the discrete-time case.

9.2.1 Some types of linear systems

The linear filters introduced in Section 2.5.2 are examples of linear systems. For example, the simple moving average given by

$$y_t = (x_{t-1} + x_t + x_{t+1})/3$$

has impulse response function

$$h_k = \begin{cases} 1/3 & k = -1, 0, +1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this filter is not 'physically realizable', as defined above, although it can of course be used in practice as a mathematical smoothing device. The filter can be made physically realizable by defining a new output variable, z_t say, where $z_t = y_{t-1}$.

Another general class of linear systems is those expressed as *linear differential equations* with constant coefficients in continuous time. For example, the equation

$$K\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t)$$

is a description of a linear system, where $K > 0$ for stability. In discrete time, the analogues of differential equations are *difference* equations given by

$$y_t + \alpha_1 \nabla y_t + \alpha_2 \nabla^2 y_t + \cdots = \beta_0 x_t + \beta_1 \nabla x_t + \beta_2 \nabla^2 x_t + \cdots \tag{9.3}$$

where $\nabla y_t = y_t - y_{t-1}$, and $\{\alpha_i\}, \{\beta_j\}$ are suitably chosen constants so as to make the system stable. Equation (9.3) can be rewritten as

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + b_0 x_t + b_1 x_{t-1} + \cdots. \tag{9.4}$$

It is clear that Equation (9.4) can be rewritten in the form (9.2) by successive substitution, or, more elegantly, by using polynomial equations in the backward shift operator B. For example, if

$$y_t = \frac{1}{2} y_{t-1} + x_t$$

then we can write the equation as $(1 - \frac{1}{2}B)y_t = x_t$, or as $y_t = (1 - \frac{1}{2}B)^{-1} x_t$ so that we find

$$y_t = x_t + \frac{1}{2} x_{t-1} + \frac{1}{4} x_{t-2} + \cdots.$$

Thus the impulse response function is given by

$$h_k = \begin{cases} (\frac{1}{2})^k & k = 0, 1, \ldots, \\ 0 & k < 0. \end{cases}$$

The reader will notice that h_k is defined for all positive k giving an impulse response function of infinite order. Engineers call such a system an **infinite impulse response** (IIR) system. More generally, whenever the equation for the system includes lagged values of the output, as in Equation (9.4), the system will generally be IIR. However, if there are no lagged values of the output and a finite sum of input values at different lags, then engineers call the system a **finite impulse response** (FIR) system.

Two very simple FIR linear systems are given by

$$y_t = x_{t-d} \tag{9.5}$$

called **simple delay**, where the integer d denotes the delay time, and

$$y_t = g\, x_t \tag{9.6}$$

called **simple gain**, where g is a constant called the gain. The impulse response functions of Equations (9.5) and (9.6) are

$$h_k = \begin{cases} 1 & k = d \\ 0 & \text{otherwise} \end{cases}$$

and

$$h_k = \begin{cases} g & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

respectively.

In continuous time, the impulse response functions of the corresponding equations for simple delay and simple gain, namely, $y(t) = x(t - \tau)$ and $y(t) = gx(t)$ can only be represented in terms of a function called the Dirac delta function, denoted by $\delta(u)$, which is mathematically tricky to handle – see Appendix B. The resulting impulse response functions are $\delta(u - \tau)$ and $g\,\delta(u)$, respectively.

An important class of impulse response functions, which often provides a reasonable approximation to physically realizable systems, is given by

$$h(u) = \begin{cases} [g\ \mathrm{e}^{-(u-\tau)/T}]/T & u > \tau \\ 0 & u < \tau. \end{cases}$$

A function of this type is called a **delayed exponential**, and depends on three constants, denoted by g, T and τ. The constant τ is called the **delay**. When $\tau = 0$, we have simple exponential response. The constant g is called the **gain**, and represents the eventual change in output when a step change of unit size is made to the input. The constant T governs the rate at which the output changes. The top panel of Figure 9.1 presents a graph of the impulse response function for a delayed exponential system together with an input showing a step change of unity at time zero and the corresponding output — see Section 9.2.3 below.

9.2.2 The impulse response function: An explanation

The impulse response function of a linear system describes how the output is related to the input as defined in Equations (9.1) or (9.2). The name 'impulse response' arises from the fact that the function describes the response of the system to an impulse input of unit size. For example, in discrete time, suppose

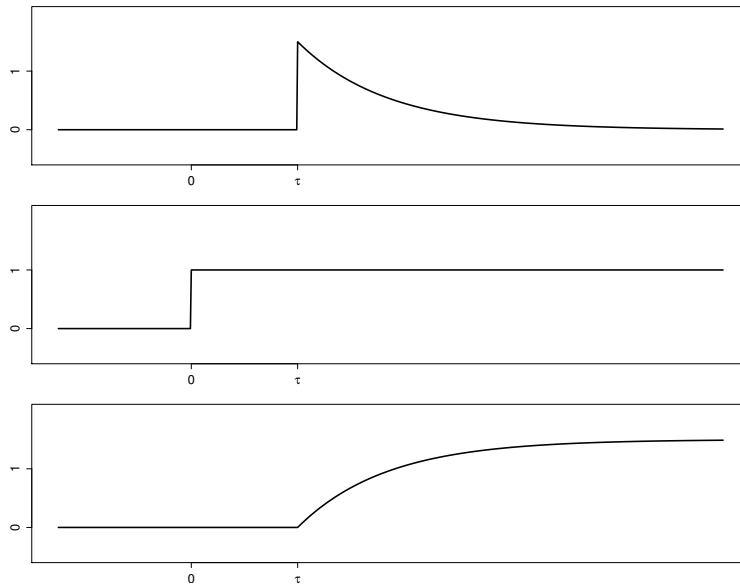

Figure 9.1 *A delayed exponential response system showing graphs of the impulse response function $h(u)$ (top), an input x_t with a unit step change at time zero (middle), and the corresponding output S_t (bottom).*

that the input x_t is zero for all t except at time zero when it takes the value unity. Thus

$$x_t = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0. \end{cases}$$

Then the output at time t is given by

$$\begin{aligned} y_t &= \sum h_k \, x_{t-k} \\ &= h_t. \end{aligned}$$

Thus the output resulting from the unit impulse input is the same as the impulse response function, and this explains why engineers often prefer the description 'unit impulse response function'.

9.2.3 The step response function

An alternative, equivalent, way of describing a linear system in the time domain is by means of a function called the step response function, which is defined by

$$S(t) = \int_{-\infty}^{t} h(u) \, \mathrm{d}u$$

in continuous time, and

$$S_t = \sum_{k \le t} h_k$$

in discrete time.

The name 'step response' arises from the fact that the function describes the response of the system to a unit step change in the input at time zero. For example, in discrete time, consider the input which is zero before time zero but unity thereafter. This can be represented mathematically by

$$x_t = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0. \end{cases}$$

The middle panel of Figure 9.1 shows such an input x_t. Then the corresponding output is given by

$$y_t = \sum_k h_k \, x_{t-k} = \sum_{k \le t} h_k = S_t,$$

so that the output is equal to the step response function.

Engineers sometimes use this relationship to measure the properties of a physically realizable system. The input is held steady for some time and then a unit step change is made to the input. The output is then observed and this provides an estimate of the step response function, and hence of its derivative, the impulse response function. A step change in the input may be easier to arrange in practice than an impulse.

The step response function for a delayed exponential system is given by

$$S(t) = g\,[1 - \mathrm{e}^{-(t-\tau)/T}] \qquad t > \tau$$

and the graph of $y(t)$ in the bottom panel of Figure 9.1 is also a graph of $S(t)$.

9.3 Linear Systems in the Frequency Domain

9.3.1 The frequency response function

An alternative way of describing a time-invariant linear system is by means of a function, called the **frequency response function**, which is the Fourier transform of the impulse response function. It is defined by

$$H(\omega) = \int_{-\infty}^{\infty} h(u) \mathrm{e}^{-\mathrm{i}\omega u} \, \mathrm{d}u \qquad 0 \le \omega < \infty \tag{9.7}$$

in continuous time, and

$$H(\omega) = \sum_k h_k \, \mathrm{e}^{-\mathrm{i}\omega k} \qquad 0 \le \omega \le \pi \tag{9.8}$$

in discrete time. The frequency response function is sometimes given the alternative description of **transfer function**, but the former term is arguably

more descriptive while the latter term is sometimes used in a different way – see below.

The frequency response and impulse response functions are equivalent ways of describing a linear system, in a somewhat similar way that the autocovariance and power spectral density functions are equivalent ways of describing a stationary stochastic process, one function being the Fourier transform of the other. We shall see that, for some purposes, $H(\omega)$ is much more useful than $h(u)$ or h_k. First, we prove the following theorem.

Theorem 9.1 A sinusoidal input to a linear system gives rise, in the steady state, to a sinusoidal output at the *same* frequency. The amplitude of the sinusoid may change and there may also be a phase shift.

Proof The proof is given for continuous time, the extension to discrete time being straightforward. Suppose that the input to a linear system, with impulse response function $h(u)$, is given by

$$x(t) = \cos \omega t \qquad \text{for all } t.$$

Then the output is given by

$$y(t) = \int_{-\infty}^{\infty} h(u) \cos \omega(t-u) \, \mathrm{d}u. \tag{9.9}$$

Now $\cos(A-B) = \cos A \cos B + \sin A \sin B$, so we may rewrite Equation (9.9) as

$$y(t) = \cos \omega t \int_{-\infty}^{\infty} h(u) \cos \omega u \, \mathrm{d}u + \sin \omega t \int_{-\infty}^{\infty} h(u) \sin \omega u \, \mathrm{d}u. \tag{9.10}$$

As the two integrals do not depend on t, it is now obvious that $y(t)$ is a mixture of sine and cosine terms at the same frequency ω. Thus the output is a sinusoidal perturbation at the same frequency ω as the input.

If we write

$$A(\omega) = \int_{-\infty}^{\infty} h(u) \cos \omega u \, \mathrm{d}u, \tag{9.11}$$

$$B(\omega) = \int_{-\infty}^{\infty} h(u) \sin \omega u \, \mathrm{d}u, \tag{9.12}$$

$$G(\omega) = \sqrt{A^2(\omega) + B^2(\omega)}, \tag{9.13}$$

$$\tan \phi(\omega) = -B(\omega)/A(\omega), \tag{9.14}$$

then Equation (9.10) may be rewritten as

$$\begin{aligned} y(t) &= A(\omega) \cos \omega t + B(\omega) \sin \omega t, \\ \text{or as} \qquad y(t) &= G(\omega) \cos [\omega t + \phi(\omega)]. \end{aligned} \tag{9.15}$$

Equation (9.15) shows that a cosine wave is amplified by a factor $G(\omega)$, which is called the **gain** of the system. The equation also shows that the cosine wave is shifted by an angle $\phi(\omega)$, which is called the **phase shift**. The above results have been derived for a particular frequency, ω, but hold true for any fixed ω such that $\omega > 0$. However, note that both the gain and phase shift may vary with frequency.

Looking at Equation (9.14), the reader may note that the phase shift is not uniquely determined, because $\tan x = \tan(x + 2\pi)$. If we take the positive square root in Equation (9.13), so that the gain is required to be positive, then the phase shift is undetermined by a multiple of 2π (see also Sections 8.2 and 9.3.2). This is not usually a problem in practice as the analyst may choose to constrain the phase to lie within a range, such as $(-\pi, \pi)$, or allow it to be a continuous function.

We have so far considered an input cosine wave. By a similar argument it can be shown that an input sine wave, $x(t) = \sin \omega t$, gives an output $y(t) = G(\omega) \sin[\omega t + \phi(\omega)]$, so that there is the same gain and phase shift. More generally if we consider an input given by

$$x(t) = \mathrm{e}^{\mathrm{i}\omega t} = \cos \omega t + \mathrm{i} \sin \omega t$$

then the output is given by

$$\begin{aligned} y(t) &= G(\omega)\{\cos[\omega t + \phi(\omega)] + \mathrm{i} \sin[\omega t + \phi(\omega)]\} \\ &= G(\omega)\, \mathrm{e}^{\mathrm{i}[\omega t + \phi(\omega)]} \\ &= G(\omega)\, \mathrm{e}^{\mathrm{i}\phi(\omega)} x(t). \end{aligned} \tag{9.16}$$

Thus we still have gain $G(\omega)$ and phase shift $\phi(\omega)$.

We can now link these results back to the frequency response function, $H(\omega)$. Equations (9.7), (9.11) and (9.12) give us

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(u) \mathrm{e}^{-\mathrm{i}\omega u}\, \mathrm{d}u \\ &= \int_{-\infty}^{\infty} h(u)(\cos \omega u - \mathrm{i} \sin \omega u)\, \mathrm{d}u \\ &= A(\omega) - \mathrm{i}B(\omega). \end{aligned}$$

This may be further rewritten, using Equations (9.13) and (9.14), in the form

$$H(\omega) = G(\omega)\, \mathrm{e}^{\mathrm{i}\phi(\omega)}. \tag{9.17}$$

Thus, when the input in Equation (9.16) is of the form $e^{\mathrm{i}\omega t}$, the output is obtained simply by multiplying by the frequency response function, and we have (in the steady-state situation) that

$$y(t) = H(\omega)x(t) = G(\omega)\, \mathrm{e}^{\mathrm{i}\phi(\omega)} x(t). \tag{9.18}$$

This completes the proof of Theorem 9.1. □

Transients The reader should note that Theorem 9.1 only applies in the **steady state** where it is assumed that the input sinusoid was applied at $t = -\infty$. If, in fact, the sinusoid is applied starting at say $t = 0$, then the output will take some time to settle to the steady-state form given by the theorem. The difference between the observed output and the steady-state output is called the **transient** component. If the system is stable, then the transient component tends to zero as $t \to \infty$. If the relationship between input and output is expressed as a differential equation (or a difference equation in discrete time), then the reader who knows how to solve such equations will recognize that the steady-state behaviour of the system corresponds to the particular integral of the equation, while the transient component corresponds to the complementary function.

It is easier to describe the transient behaviour of a linear system by using the **Laplace** transform[3] of the impulse response function. The Laplace transform also has the advantage of being defined for some unstable systems and so is sometimes preferred by engineers. However, statisticians customarily deal with steady-state behaviour for stable systems and so typically use Fourier transforms. We will continue this custom.

Discussion of Theorem 9.1 Theorem 9.1 helps to explain the importance of the frequency response function. For inputs consisting of an impulse or step change it is easy to calculate the output using the (time-domain) impulse response function. However, for a sinusoidal input, it is much easier to calculate the output using the frequency response function. (Compare the convolution in the time domain as in Equation (9.9) with the simple multiplication in the frequency domain using Equation (9.18).) More generally for an input containing several sinusoidal perturbations, namely

$$x(t) = \sum_j A_j(\omega_j) \mathrm{e}^{\mathrm{i}\omega_j t}$$

it is easy to calculate the output using the frequency response function as

$$y(t) = \sum_j A_j(\omega_j) H(\omega_j) \mathrm{e}^{\mathrm{i}\omega_j t}$$

and this is one type of situation where linear systems are easier to study in the frequency domain.

Further comments on the frequency response function Returning to the definition of the frequency response function as given by Equations (9.7) and (9.8), note that some authors define $H(\omega)$ for negative as well as positive frequencies. However, for real-valued processes we need only consider $H(\omega)$ for $\omega > 0$.

[3] The Laplace transform is defined in Appendix A.

Note that, in discrete time, $H(\omega)$ is only defined for frequencies up to the Nyquist frequency π (or $\pi/\Delta t$ if there is an interval Δt between successive observations). We have already introduced the Nyquist frequency in Section 7.2.1 and can apply similar ideas to a linear system measured at unit intervals of time. Given any sinusoidal input, with a frequency higher than π, we can find a corresponding sinusoid at a lower frequency in the range $(0, \pi)$, which gives identical readings at unit intervals of time. This alternative sinusoid will therefore give rise to an identical output when measured only at unit intervals of time.

We have already noted that $H(\omega)$ is sometimes called the frequency response function and sometimes the transfer function. We prefer the former term, as it is more descriptive, indicating that the function shows how a linear system responds to sinusoids at different frequencies. In any case the term 'transfer function' is used by some authors in a different way. Engineers sometimes use the term to denote the Laplace transform of the impulse response function – see, for example, Bendat and Piersol (2000, p. 30). For a physically realizable stable system, the Fourier transform of the impulse response function may be regarded as a special case of the Laplace transform. A necessary and sufficient condition for a linear system to be stable is that the Laplace transform of the impulse response function should have no poles in the right half-plane or on the imaginary axis. For an unstable system, the Fourier transform does not exist, but the Laplace transform does. However, we only consider stable systems, in which case the Fourier transform is adequate.

Further confusion can arise because Jenkins and Watts (1968) use the term 'transfer function' to denote the z-transform[4] of the impulse response function in the discrete-time case, while Box et al. (1994) use the term for a similar expression in connection with what they call transfer-function models – see Section 9.4.2. The z-transform is also used more generally by engineers for analysing discrete-time systems (e.g. Hayes, 1996, Chapter 2) but the Fourier transform is adequate for our purposes.

9.3.2 Gain and phase diagrams

The frequency response function $H(\omega)$ of a linear system is a complex function that may be written in the form

$$H(\omega) = G(\omega)\mathrm{e}^{\mathrm{i}\phi(\omega)}$$

where $G(\omega)$, $\phi(\omega)$ are the gain and phase, respectively – see Equation (9.17). In order to understand the properties of a linear system, it is helpful to plot $G(\omega)$ and $\phi(\omega)$ against ω to obtain what are called the **gain diagram** and the **phase diagram**.

The gain may be calculated via Equation (9.13) or equivalently as $|H(\omega)|$, which may be obtained via $G^2(\omega) = |H(\omega)|^2 = H(\omega)\overline{H(\omega)}$ where $\overline{H(\omega)}$

[4]The z-transform is defined in Appendix A.

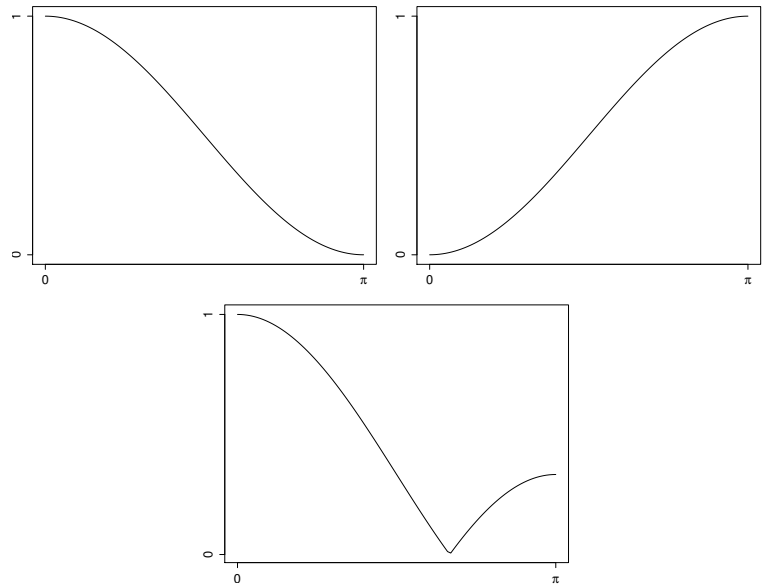

Figure 9.2 *Gain diagrams for, (a) a low-pass filter; (b) a high-pass filter; (c) a simple moving average of three successive observations.*

denotes the complex conjugate[5] of $H(\omega)$. If $G(\omega)$ is 'large' for low values of ω, but 'small' for high values of ω, as in Figure 9.2(a), then we have what is called a **low-pass** filter. This description is self-explanatory in that, if the input is a mixture of variation at several different frequencies, only those components with a low frequency will 'get through' the filter. Conversely, if $G(\omega)$ is 'small' for low values of ω, but 'large' for high values of ω, then we have a **high-pass** filter as in Figure 9.2(b).

The phase shift $\phi(\omega)$ may be calculated from the real and imaginary parts of $H(\omega)$ using Equation (9.14). Effectively this means plotting the value of $H(\omega)$ in the complex plane and examining the result to see which quadrant it is in. The phase can be written directly as $\tan^{-1}[-B(\omega)/A(\omega)]$ or equivalently as $\arctan[-B(\omega)/A(\omega)]$. However, as noted earlier, plotting the phase diagram is complicated by the fact that the phase is not uniquely determined. If the gain is always taken to be positive, then the phase is undetermined by a multiple of 2π and is often constrained to the range $(-\pi, \pi)$. Unfortunately this may result in spurious discontinuities when the phase reaches the upper or lower boundary and engineers often prefer to plot the phase as a continuous unconstrained function, using the fact that $\phi(0) = 0$ provided $G(0)$ is finite. Even then, the phase may have a discontinuity when the gain becomes zero as in Example 9.1 below, and this can only be avoided by allowing the gain to go negative.

[5] If $z = a + \mathrm{i}b$, then its complex conjugate is $\overline{z} = a - \mathrm{i}b$.

9.3.3 Some examples

Example 9.1 Consider the simple moving average

$$y_t = (x_{t-1} + x_t + x_{t+1})/3$$

which is a linear system with impulse response function

$$h_k = \begin{cases} 1/3 & k = -1, 0, +1 \\ 0 & \text{otherwise.} \end{cases}$$

The frequency response function of this filter is (using Equation (9.8))

$$\begin{aligned} H(\omega) &= \frac{1}{3}\mathrm{e}^{-\mathrm{i}\omega} + \frac{1}{3} + \frac{1}{3}\mathrm{e}^{\mathrm{i}\omega} \\ &= \frac{1}{3} + \frac{2}{3}\cos\omega \qquad 0 < \omega < \pi. \end{aligned}$$

This function happens to be real, not complex, and so the phase appears to be given by

$$\phi(\omega) = 0 \qquad 0 < \omega < \pi.$$

However, $H(\omega)$ is negative for $\omega > 2\pi/3$, and so if we adopt the convention that the gain should be positive, then we have

$$\begin{aligned} G(\omega) &= \left|\frac{1}{3} + \frac{2}{3}\cos\omega\right| \\ &= \begin{cases} \frac{1}{3} + \frac{2}{3}\cos\omega & 0 < \omega \leq 2\pi/3 \\ -\frac{1}{3} - \frac{2}{3}\cos\omega & 2\pi/3 < \omega < \pi \end{cases} \end{aligned}$$

and

$$\phi(\omega) = \begin{cases} 0 & 0 < \omega \leq 2\pi/3 \\ \pi & 2\pi/3 < \omega < \pi. \end{cases}$$

The gain is plotted in Figure 9.2(c) and is of low-pass type. This is to be expected as a moving average smooths out local fluctuations (high-frequency variation) and measures the trend (low-frequency variation). In fact it is arguably more sensible to allow the gain to go negative in $(2\pi/3, \pi)$ so that the phase is zero for all ω in $(0, \pi)$.

Example 9.2 A linear system showing simple exponential response has impulse response function

$$h(u) = [\, g\ \mathrm{e}^{-u/T}\,]/\,T \qquad u > 0.$$

Using Equation (9.7), the frequency response function is

$$H(\omega) = g(1 - \mathrm{i}\omega T)/(1 + \omega^2 T^2) \qquad \omega > 0.$$

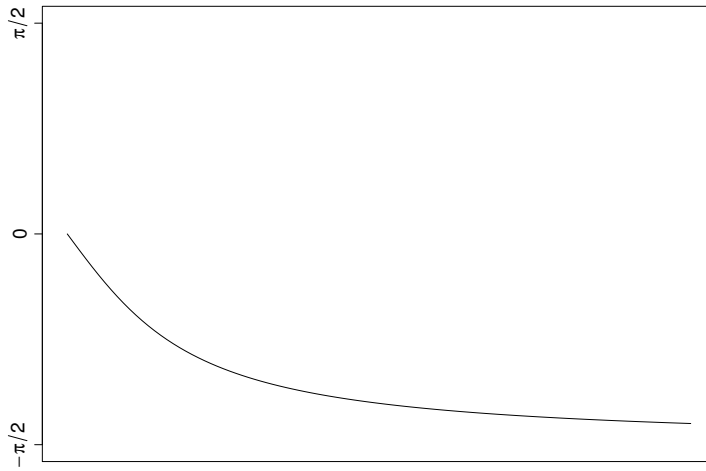

Figure 9.3 *Phase diagram for a simple exponential response system.*

Hence

$$\begin{aligned} G(\omega) &= g/\sqrt{(1+\omega^2 T^2)} \\ \tan\,\phi(\omega) &= -T\omega. \end{aligned}$$

As the frequency increases, $G(\omega)$ decreases so that the system is of low-pass type. As regards the phase, if we take $\phi(\omega)$ to be zero at zero frequency, then the phase becomes increasingly negative as ω increases until the output is completely out of phase with the input (see Figure 9.3).

Example 9.3 Consider the linear system consisting of pure delay, so that

$$y(t) = x(t-\tau)$$

where τ is a constant. The impulse response function is given by

$$h(u) = \delta(u-\tau)$$

where δ denotes the Dirac delta function — see Appendix B. Then the frequency response function is given by

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} \delta(u-\tau)\, \mathrm{e}^{-\mathrm{i}\omega u}\, \mathrm{d}u \\ &= \mathrm{e}^{-\mathrm{i}\omega\tau}. \end{aligned}$$

In fact $H(\omega)$ can be derived without using the rather difficult delta function by using Theorem 9.1. Suppose that input $x(t) = \mathrm{e}^{\mathrm{i}\omega t}$ is applied to the system. Then the output is $y(t) = \mathrm{e}^{\mathrm{i}\omega(t-\tau)} = \mathrm{e}^{-\mathrm{i}\omega\tau} \times$ input. Thus, by analogy with Equation (9.18), we have $H(\omega) = \mathrm{e}^{-\mathrm{i}\omega\tau}$. Then, using Equation (9.17), it is easy to see that this linear system has a constant gain equal to unity, namely, $G(\omega) = 1$, while the phase is given by $\phi(\omega) = -\omega\tau$.

9.3.4 General relation between input and output

So far, we have only considered sinusoidal inputs in the frequency domain. This section considers any type of input and shows that it is generally easier to work with linear systems in the frequency domain than in the time domain.

The general relation between input and output in continuous time is given by Equation (9.1), namely

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,\mathrm{d}u. \tag{9.19}$$

When $x(t)$ is not of a simple form, this integral may be hard to evaluate. Now consider the Fourier transform of the output, given by

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}u \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)x(t-u)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}u\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\mathrm{e}^{-\mathrm{i}\omega u}\,x(t-u)\,\mathrm{e}^{-\mathrm{i}\omega(t-u)}\,\mathrm{d}u\,\mathrm{d}t. \end{aligned}$$

However,

$$\int_{-\infty}^{\infty} x(t-u)\mathrm{e}^{-\mathrm{i}\omega(t-u)}\,\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

for all values of u, and is therefore the Fourier transform of $x(t)$, which we will denote by $X(\omega)$. Furthermore

$$\int_{-\infty}^{\infty} h(u)\,\mathrm{e}^{-\mathrm{i}\omega u}\,\mathrm{d}u = H(\omega)$$

so that

$$Y(\omega) = H(\omega)X(\omega). \tag{9.20}$$

Thus the integral in Equation (9.19) corresponds to a multiplication in the frequency domain provided that the Fourier transforms exist. A similar result holds in discrete time.

A more useful general relation between input and output, akin to Equation (9.20), can be obtained when the input $x(t)$ is a stationary process with a continuous power spectrum. This result will be given as Theorem 9.2.

Theorem 9.2 Consider a stable time-invariant linear system with gain function $G(\omega)$. Suppose that the input $X(t)$ is a stationary process with continuous power spectrum $f_X(\omega)$. Then the output $Y(t)$ is also a stationary process, whose power spectrum $f_Y(\omega)$ is given by

$$f_Y(\omega) = G^2(\omega) f_X(\omega). \tag{9.21}$$

Proof The proof will be given for continuous time, but a similar proof yields the same result in discrete time. It is easy to show that a stationary input to a stable linear system gives rise to a stationary output, and this will not be shown here.

We denote the impulse response and frequency response functions of the system by $h(u)$, $H(\omega)$, respectively. Thus $G(\omega) = |H(\omega)|$. By definition, the output is related to the input by

$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\ \mathrm{d}u.$$

For mathematical convenience, we assume that the input has mean zero, in which case the output also has mean zero. It is straightforward to extend the proof to an input with non-zero mean.

Denote the autocovariance functions (acv.f.s) of $X(t), Y(t)$ by $\gamma_X(\tau), \gamma_Y(\tau)$, respectively. Then

$$\begin{aligned}
\gamma_Y(\tau) &= E[Y(t)Y(t+\tau)] \qquad \text{since } E[Y(t)] = 0 \\
&= E\left[\int_{-\infty}^{\infty} h(u)X(t-u)\ \mathrm{d}u \int_{-\infty}^{\infty} h(u')X(t+\tau-u')\ \mathrm{d}u'\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(u')E[X(t-u)X(t+\tau-u')]\ \mathrm{d}u\ \mathrm{d}u'.
\end{aligned}$$

But

$$E[X(t-u)X(t+\tau-u')] = \gamma_X(\tau - u' + u).$$

Thus

$$\gamma_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(u')\gamma_X(\tau - u' + u)\ \mathrm{d}u\ \mathrm{d}u'. \tag{9.22}$$

The relationship between the acv.f.s of the input and the output in Equation (9.22) is not of a simple form. However, if we take Fourier transforms of both sides of Equation (9.22) by multiplying by $\mathrm{e}^{-\mathrm{i}\omega t}/\pi$ and integrating with respect to τ from $-\infty$ to $+\infty$, we find, using Equation (6.17), that the left-hand side is the spectrum of the output, namely

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \gamma_Y(\tau)\mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}\tau = f_Y(\omega).$$

The right-hand side of Equation (9.22) requires some work to simplify. We find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(u')\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma_X(\tau-u'+u)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right]\,\mathrm{d}u\,\mathrm{d}u'$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)\mathrm{e}^{\mathrm{i}\omega u}h(u')\mathrm{e}^{-\mathrm{i}\omega u'}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma_X(\tau-u'+u)\mathrm{e}^{-\mathrm{i}\omega(\tau-u'+u)}\,\mathrm{d}\tau\right]\,\mathrm{d}u\,\mathrm{d}u'$$

However,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma_X(\tau-u'+u)\mathrm{e}^{-\mathrm{i}\omega(t-u'+u)}\,\mathrm{d}\tau=\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma_X(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau=f_X(\omega)$$

for all u, u', and (denoting the complex conjugate of $H(\omega)$ by $\overline{H(\omega)}$)

$$\begin{aligned}\int_{-\infty}^{\infty} h(u)\mathrm{e}^{\mathrm{i}\omega u} &= \overline{H(\omega)}\\ &= G(\omega)\mathrm{e}^{-\mathrm{i}\phi(\omega)}.\end{aligned}$$

Thus

$$\begin{aligned}f_Y(\omega) &= \overline{H(\omega)}H(\omega)f_X(\omega)\\ &= G^2(\omega)f_X(\omega).\end{aligned}$$

This completes the proof of Theorem 9.2. □

The relationship between the spectra of the input and the output of a linear system is very important and yet of a surprisingly simple form. Once again a result in the frequency domain – Equation (9.21) – is much simpler than the corresponding result in the time domain – Equation (9.22).

Theorem 9.2 can be used in various ways and, in particular, can be used to evaluate the spectra of various classes of stationary processes in a simpler manner to that used in Chapter 6. There the procedure was to evaluate the acv.f. of the process and then find its Fourier transform. This can be algebraically tedious, and so we give three examples showing how to use Equation (9.21) instead.

(1) *Moving average (MA) processes*

An MA process of order q is given by

$$X_t=\beta_0 Z_t+\beta_1 Z_{t-1}+\cdots+\beta_q Z_{t-q},$$

where Z_t denotes a purely random process with variance σ_Z^2. Usually β_0 is one, but it simplifies the algebra to include this extra coefficient. Comparing with Equation (9.2), we see that the MA equation may be regarded as specifying a

linear system with $\{Z_t\}$ as input and $\{X_t\}$ as output. This system is stable and time-invariant and so, using Equation (9.8), has frequency response function

$$H(\omega) = \sum_{j=0}^{q} \beta_j \, \mathrm{e}^{-\mathrm{i}\omega j}.$$

As $\{Z_t\}$ is a purely random process, its spectrum is constant, namely,

$$f_Z(\omega) = \sigma_Z^2/\pi.$$

Thus, using Equation (9.21), the spectrum of $\{X_t\}$ is given by

$$f_X(\omega) = \left| \sum_{j=0}^{q} \beta_j \, \mathrm{e}^{-\mathrm{i}\omega j} \right|^2 \sigma_Z^2/\pi.$$

For example, for the first-order MA process

$$X_t = Z_t + \beta Z_{t-1}, \tag{9.23}$$

we have

$$H(\omega) = 1 + \beta \, \mathrm{e}^{-\mathrm{i}\omega}$$

and

$$\begin{aligned} G^2(\omega) &= |H(\omega)|^2 = (1 + \beta \cos \omega)^2 + \beta^2 \sin^2 \omega \\ &= 1 + 2\beta \cos \omega + \beta^2, \end{aligned}$$

so that $f_X(\omega) = (1 + 2\beta \cos \omega + \beta^2)\sigma_Z^2/\pi$ as already derived in Section 6.5.

This type of approach can also be used when $\{Z_t\}$ is not a purely random process. For example, suppose that the $\{Z_t\}$ process in Equation (9.23) is stationary with spectrum $f_Z(\omega)$. Then the spectrum of $\{X_t\}$ is given by

$$f_X(\omega) = (1 + 2\beta \cos \omega + \beta^2) f_Z(\omega).$$

(2) *Autoregressive (AR) processes*

The stationary first-order AR process

$$X_t = \alpha X_{t-1} + Z_t$$

with $|\alpha| < 1$, may be regarded as a linear system producing output X_t from input Z_t. It may also be regarded as a linear system 'the other way round', producing output Z_t from input X_t by

$$Z_t = X_t - \alpha X_{t-1}.$$

In the latter form, the frequency response function is

$$H(\omega) = 1 - \alpha \, \mathrm{e}^{-\mathrm{i}\omega}$$

and hence has (squared) gain given by

$$G^2(\omega) = 1 - 2\alpha\cos\omega + \alpha^2.$$

Thus, using Equation (9.21) with Z_t as output, we have

$$f_Z(\omega) = (1 - 2\alpha\cos\omega + \alpha^2) f_X(\omega). \tag{9.24}$$

However, if $\{Z_t\}$ denotes a purely random process with spectrum $f_Z(\omega) = \sigma_Z^2/\pi$, then we know the spectrum of the 'output' and may rewrite Equation (9.24) to get the spectrum of the 'input'. We find

$$f_X(\omega) = \sigma_Z^2/\pi(1 - 2\alpha\cos\omega + \alpha^2)$$

which has already been obtained as Equation (6.23) by our earlier method. I hope the reader agrees that this method is easier, and it can readily be extended to handle higher-order AR processes.

(3) *Differentiation*

Consider the process that converts a continuous input $X(t)$ into a continuous output $Y(t)$ by differentiation, namely

$$Y(t) = \frac{\mathrm{d}X(t)}{\mathrm{d}t}. \tag{9.25}$$

It can readily be shown that this is a time-invariant linear system. A differentiator is of considerable mathematical interest, although in practice only approximations to it are physically realizable.

If the input is sinusoidal, say $X(t) = \mathrm{e}^{\mathrm{i}\omega t}$, then, by differentiating, we find the output is given by

$$Y(t) = \mathrm{i}\omega\ \mathrm{e}^{\mathrm{i}\omega t},$$

so that, using Equation (9.18), the frequency response function is given by

$$H(\omega) = \mathrm{i}\omega.$$

If the input is a stationary process, with spectrum $f_X(\omega)$, then it appears that the output has spectrum

$$\begin{aligned} f_Y(\omega) &= |\mathrm{i}\omega|^2 f_X(\omega) \\ &= \omega^2 f_X(\omega) \end{aligned} \tag{9.26}$$

However, this result assumes that the linear system in Equation (9.25) is stable, when in fact it is only stable for certain types of input processes. For example, it can be shown that the response to a unit step change is an unbounded impulse, which means the system is not stable. In order for the system to be stable, the variance of the output must be finite. Now

$$\begin{aligned} \mathrm{Var}[Y(t)] &= \int_0^\infty f_Y(\omega)\mathrm{d}\omega \\ &= \int_0^\infty \omega^2 f_X(\omega)\mathrm{d}\omega. \end{aligned}$$

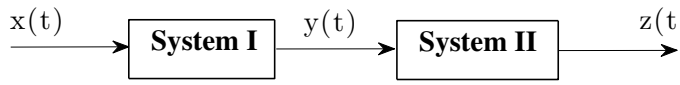

Figure 9.4 *Two linear systems in series.*

However, using equation (6.18), we have

$$\gamma_X(k) = \int_0^\infty f_X(\omega) \cos \omega k \ \mathrm{d}\omega$$

and

$$\frac{\mathrm{d}^2\gamma_X(k)}{\mathrm{d}k^2} = -\int_0^\infty \omega^2 f_X(\omega) \cos \omega k \ \mathrm{d}\omega$$

so that

$$\mathrm{Var}[Y(t)] = -\left[\frac{\mathrm{d}^2\gamma_X(k)}{\mathrm{d}k^2}\right]_{k=0}.$$

Thus $Y(t)$ has finite variance provided that $\gamma_X(k)$ can be differentiated twice at $k = 0$, and only then does Equation (9.26) hold.

9.3.5 Linear systems in series

The advantages of working in the frequency domain are also evident when we consider two or more *linear systems in series* (sometimes said to be *in cascade*). For example, Figure 9.4 shows two linear systems in series, where the input $x(t)$ to system I produces output $y(t)$, which in turn is the input to system II producing output $z(t)$. It is often of interest to evaluate the properties of the overall combined system, having $x(t)$ as input and $z(t)$ as output. It can readily be shown that the combined system is also linear, and we now find its properties. Denote the impulse response and frequency response functions of systems I and II by $h_1(\tau), h_2(\tau), H_1(\omega)$ and $H_2(\omega)$, respectively.

In the time domain, the relationship between $x(t)$ and $z(t)$ would be in the form of a double integral involving $h_1(u)$ and $h_2(u)$, which is rather complicated. However, in the frequency domain we can denote the Fourier transforms of $x(t), y(t), z(t)$ by $X(\omega), Y(\omega), Z(\omega)$, respectively, and use Equation (9.20). Then

$$Y(\omega) = H_1(\omega)X(\omega)$$

and

$$\begin{aligned} Z(\omega) &= H_2(\omega)Y(\omega) \\ &= H_2(\omega)H_1(\omega)X(\omega). \end{aligned}$$

Thus it is easy to see that the overall frequency response function of the combined system is

$$H(\omega) = H_1(\omega)H_2(\omega). \tag{9.27}$$

If

$$\begin{aligned} H_1(\omega) &= G_1(\omega)\,\mathrm{e}^{\mathrm{i}\phi_1(\omega)} \\ H_2(\omega) &= G_2(\omega)\,\mathrm{e}^{\mathrm{i}\phi_2(\omega)} \end{aligned}$$

then

$$H(\omega) = G_1(\omega)G_2(\omega)\,\mathrm{e}^{\mathrm{i}[\phi_1(\omega)+\phi_2(\omega)]}. \tag{9.28}$$

Thus the overall gain is the *product* of the component gains, while the overall phase is the *sum* of the component phases.

This result may be immediately applied to the case where the input $x(t)$ is a stationary process with power spectrum $f_X(\omega)$. Generalizing Equation (9.21), we find

$$f_Z(\omega) = G_1^2(\omega)G_2^2(\omega)f_X(\omega).$$

The above results are easily extended to the situation where there are m linear systems in series with respective frequency response functions $H_1(\omega), \ldots, H_m(\omega)$. The overall frequency response function is the product of the individual functions, namely

$$H(\omega) = H_1(\omega)H_2(\omega)\ldots H_m(\omega).$$

9.3.6 Design of filters

We are now in a position to reconsider in more depth the properties of the filters introduced in Sections 2.5.2 and 2.5.3. Given a time series $\{x_t\}$, the filters for estimating or removing trend are of the general form

$$y_t = \sum_k h_k x_{t-k}.$$

This equation clearly defines a time-invariant linear system and its frequency response function is given by

$$H(\omega) = \sum_k h_k \mathrm{e}^{-\mathrm{i}\omega k}$$

with gain function $G(\omega) = |H(\omega)|$.

How do we set about choosing an appropriate filter for a time series? The design of a filter involves a choice of $\{h_k\}$ and hence of $H(\omega)$ and $G(\omega)$. Two types of 'ideal' filters are shown in Figure 9.5. Both have sharp cut-offs, the low-pass filter completely eliminating high-frequency variation and the high-pass filter completely eliminating low-frequency variation.

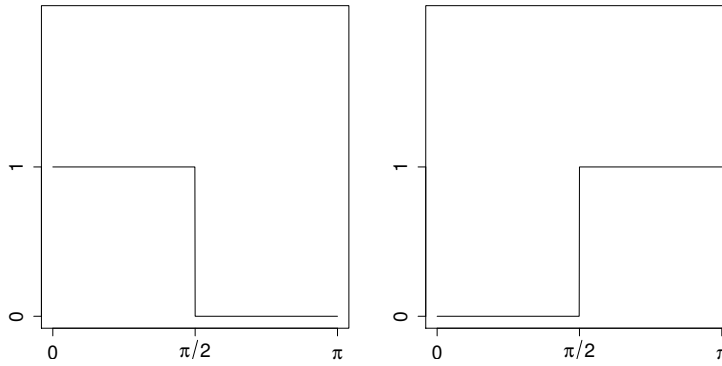

Figure 9.5 *Two types of ideal filters; (a) a low-pass filter or trend estimator; (b) a high-pass filter or trend eliminator.*

However, ideal filters of this type are impossible to achieve with a finite set of weights. Instead the smaller the number of weights used, the less sharp will generally be the cut-off property of the filter. For example, the gain diagram of a simple moving average of three successive observations is of low-pass type but has a much less sharp cut-off than the ideal low-pass filter (compare Figure 9.2(c) with Figure 9.5(a)). More sophisticated trend estimators, such as Spencer's 15-point moving average, have better cut-off properties.

As an example of a trend eliminator, consider first differencing, namely

$$y_t = x_t - x_{t-1}.$$

This has frequency response function

$$H(\omega) = 1 - \mathrm{e}^{-\mathrm{i}\omega}$$

and gain function

$$G(\omega) = \sqrt{2(1 - \cos\,\omega)}$$

which is plotted in Figure 9.6. This is indeed of high-pass type, as required, but the shape is a long way from the ideal filter in Figure 9.5(b). This should be borne in mind when working with first differences.

9.4 Identification of Linear Systems

We have so far assumed that the structure of the linear system under consideration is known. Given the impulse response function of a system, or equivalently the frequency response function, we can find the output corresponding to a given input. In particular, when considering the properties of filters for estimating or removing trend and seasonality, a formula for the 'system' is given when the filter is specified.

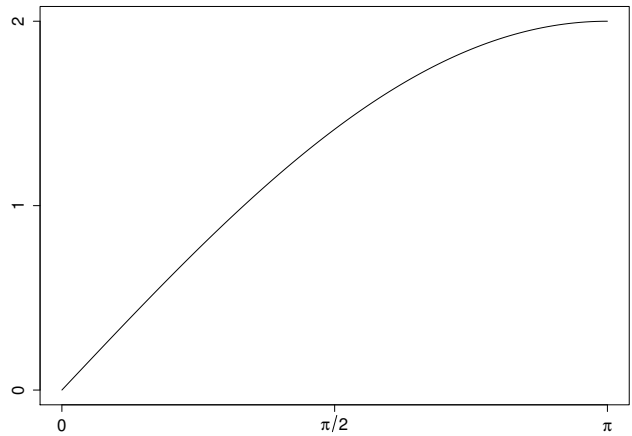

Figure 9.6 *The gain diagram $G(\omega)$ for the difference operator.*

However, many problems concerning linear systems are of a completely different type. The structure of the system is *not* known *a priori* and the problem is to examine the relationship between input and output so as to infer the properties of the system. This procedure is called the **system identification**. For example, suppose we are interested in the effect of temperature on the yield from a chemical process. Here we have a physical system, which we assume, initially at least, is approximately linear over the range of interest. By examining the relationship between observations on temperature (the input) and yield (the output) we can infer the properties of the chemical process.

The identification process is straightforward if the input to the system can be controlled and if the system is 'not contaminated by noise'. In this case, we can simply apply an impulse or step change input, observe the output, and hence calculate the impulse response or step response function. Alternatively, we can apply sinusoidal inputs at a range of different frequencies and observe the corresponding amplitude and phase shift of the output (which should be sinusoidal at the same frequency as the input if the system really is linear). This enables us to estimate the gain and phase diagrams.

However, many systems are contaminated by noise as illustrated in Figure 9.7, where $N(t)$ denotes a noise process. This noise process need not be white noise but it is usually assumed that it is uncorrelated with the input process $X(t)$.

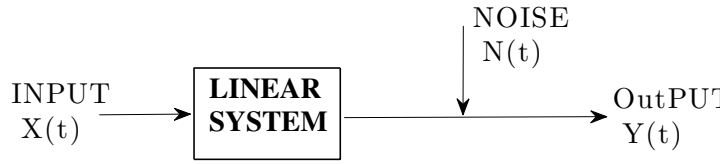

Figure 9.7 *A linear system with added noise.*

A further difficulty arises when the input can be observed but cannot be controlled. In other words one cannot make changes, such as a step change, to the input in order to see what happens to the output. For example, attempts have been made to treat the economy of a country as a linear system and to examine the relationship between observed variables like the retail price index (regarded as the input) and average wages (regarded as the output). However, price increases can only be controlled to a certain extent by governmental decisions, and this makes system identification difficult. Moreover, we will see later in Section 9.4.3 that there is an additional problem with this sort of data in that the output (e.g. wage increases) may in turn affect the input (e.g. price increases) and this is called a *feedback problem*.

When the system is affected by noise or the input is not controllable, more refined techniques are required to identify the system. We will describe two alternative approaches, one in the frequency domain and one in the time domain. Section 9.4.1 shows how cross-spectral analysis of input and output may be used to estimate the frequency response function of a linear system. Section 9.4.2 describes the Box–Jenkins approach to estimating the impulse response function of a linear system.

9.4.1 Estimating the frequency response function

Suppose that we have a system with added noise, as depicted in Figure 9.7. Although the structure of the system is unknown, it is often reasonable to assume that it is linear, time-invariant and stable, and that the noise is a stationary process that is uncorrelated with the input and has mean zero. Suppose that we have observations on the input and output over some time period. The input should be observations on a stationary process, in which case the output will also be stationary. Given these observations, we wish to estimate the frequency response function of the system. We will denote the (unknown) impulse response and frequency response functions of the system by $h(u)$, $H(\omega)$, respectively.

The reader may think that Equation (9.21), namely

$$f_Y(\omega) = G^2(\omega) f_X(\omega)$$

can be used to estimate the gain of the system, by estimating the spectra of input and output. However, this equation does not hold in the presence of noise and does not in any case give information about the phase of the system. Instead we derive a relationship involving the cross-spectrum of input and output.

In continuous time, the output $Y(t)$ is given by

$$Y(t) = \int_0^\infty h(u)X(t-u)\mathrm{d}u + N(t). \tag{9.29}$$

Note that we are only considering physically realizable systems, so that $h(u)$ is zero for $u < 0$. For mathematical convenience, we assume $E[X(t)] = 0$ so that $E[Y(t)] = 0$, but the following results also hold if $E[X(t)] \neq 0$. Multiplying through Equation (9.29) by $X(t-\tau)$ and taking expectations, we have

$$E[Y(t)X(t-\tau)] = \int_0^\infty h(u)E[X(t-u)X(t-\tau)]\mathrm{d}u + E[N(t)X(t-\tau)].$$

The last term on the right-hand side is zero as $N(t)$ is assumed to be uncorrelated with the input $X(t)$. Remembering that the processes have mean zero, the equation may be rewritten as

$$\gamma_{XY}(\tau) = \int_0^\infty h(u)\gamma_X(\tau-u)\mathrm{d}u, \tag{9.30}$$

where γ_{XY} is the cross-covariance function of $X(t)$ and $Y(t)$, and γ_X is the autocovariance function of $X(t)$. Equation (9.30) is called the Wiener–Hopf integral equation. Given γ_{XY} and γ_X, this equation can, in principle, be solved to give the impulse response function $h(u)$. However, it is often easier to work with the corresponding relationship in the frequency domain.

First, we revert to discrete time and note that the discrete-time analogue of Equation (9.30) is

$$\gamma_{XY}(\tau) = \sum_{k=0}^{\infty} h_k\gamma_X(\tau-k). \tag{9.31}$$

Take Fourier transforms of both sides of this equation by multiplying by $\mathrm{e}^{-\mathrm{i}\omega\tau}/\pi$ and summing from $\tau = -\infty$ to $+\infty$. Then we find

$$\begin{aligned} f_{XY}(\omega) &= \sum_{\tau=-\infty}^{\infty}\sum_{k=0}^{\infty} h_k\mathrm{e}^{-\mathrm{i}\omega k}\,\gamma_X(\tau-k)\,\mathrm{e}^{-\mathrm{i}\omega(\tau-k)}/\pi \\ &= \sum_{k=0}^{\infty} h_k\mathrm{e}^{-\mathrm{i}\omega k} f_X(\omega) \\ &= H(\omega)f_X(\omega), \end{aligned} \tag{9.32}$$

where f_{XY} is the cross-spectrum of input and output and f_X is the (auto)spectrum of the input. Thus, once again, a convolution in the time

domain corresponds to a multiplication in the frequency domain, and Equation (9.32) is much simpler than (9.30).

Estimates of $f_{XY}(\omega)$ and $f_X(\omega)$ can now be used to estimate $H(\omega)$ using Equation (9.32). Denote the estimated spectrum of the input by $\hat{f}_X(\omega)$, and the estimate cross-spectrum by $\hat{f}_{XY}(\omega)$. Then

$$\hat{H}(\omega) = \hat{f}_{XY}(\omega)/\hat{f}_X(\omega).$$

In practice, we normally use the equation

$$H(\omega) = G(\omega)\mathrm{e}^{\mathrm{i}\phi(\omega)}$$

and estimate the gain and phase separately. We have

$$\begin{aligned} \hat{G}(\omega) &= |\hat{H}(\omega)| = |\hat{f}_{XY}(\omega)/\hat{f}_X(\omega)| \\ &= |\hat{f}_{XY}(\omega)|/\hat{f}_X(\omega) \quad \text{since } \hat{f}_X(\omega) \text{ is real} \\ &= \hat{\alpha}_{XY}(\omega)/\hat{f}_X(\omega) \end{aligned} \tag{9.33}$$

where $\alpha_{XY}(\omega)$ is the cross-amplitude spectrum (see Equation (8.13)).

We also find

$$\tan\hat{\phi}(\omega) = -\hat{q}(\omega)/\hat{c}(\omega) \tag{9.34}$$

where $q(\omega)$, $c(\omega)$ are the quadrature and co-spectra, respectively (see Equation (8.14)). Thus, having estimated the cross-spectrum, Equations (9.33) and (9.34) enable us to estimate the gain and phase of the linear system, whether or not there is added noise.

We can also use cross-spectral analysis to estimate the properties of the noise process. The discrete-time version of Equation (9.29) is

$$Y_t = \sum_{k=0}^{\infty} h_k X_{t-k} \; + \; N_t. \tag{9.35}$$

For mathematical convenience, we again assume that $E(N_t) = E(X_t) = 0$ so that $E(Y_t) = 0$. If we multiply both sides of (9.35) by Y_{t-m}, we find

$$Y_t Y_{t-m} = \Big(\sum h_k X_{t-k} + N_t\Big)\Big(\sum h_k X_{t-m-k} + N_{t-m}\Big).$$

Taking expectations we find

$$\gamma_Y(m) = \sum_k \sum_j h_k h_j \gamma_X(m-k+j) \; + \; \gamma_N(m)$$

since $\{X_t\}$ and $\{N_t\}$ are assumed to be uncorrelated. Taking Fourier transforms of both sides of this equation, we find

$$f_Y(\omega) = H(\omega)\overline{H(\omega)}f_X(\omega) \; + \; f_N(\omega).$$

However,

$$\begin{aligned} H(\omega)\overline{H(\omega)} &= G^2(\omega) \\ &= C(\omega)f_Y(\omega)/f_X(\omega) \end{aligned}$$

where $C(\omega)$ denotes the *coherency* — see Equations (8.15) and (8.16). Thus

$$f_N(\omega) = f_Y(\omega)[1 - C(\omega)]. \tag{9.36}$$

Thus an estimate of $f_N(\omega)$ is given by

$$\hat{f}_N(\omega) = \hat{f}_Y(\omega)[1 - \hat{C}(\omega)]. \tag{9.37}$$

Equation (9.36) also enables us to see that if there is no noise, so that there is a pure linear relation between X_t and Y_t, then $f_N(\omega) = 0$ and $C(\omega) = 1$ for all ω. On the other hand if $C(\omega) = 0$ for all ω, then $f_Y(\omega) = f_N(\omega)$ and the output is not linearly related to the input. This confirms the point mentioned in Chapter 8 that the coherency $C(\omega)$ measures the linear correlation between input and output at frequency ω.

The results of this section not only show us how to identify a linear system by cross-spectral analysis but also give further guidance on the interpretation of functions derived from the cross-spectrum, particularly the gain, phase and coherency. Examples are given, for example, by Bloomfield (2000, Chapter 10) and Jenkins and Watts (1968).

In principle, estimates of the frequency response function of a linear system may be transformed to give estimates of the impulse response function (Jenkins and Watts, 1968, p. 444; Box et al. 1994, Appendix A11.1) but I do not recommend this. For instance, Example 8.4 appears to indicate that the sign of the phase may be used to indicate which series is 'leading' the other. However, for more complicated lagged models of the form given by Equation (9.35), it becomes increasingly difficult to make inferences from phase estimates. For such *time-domain* models, it is usually better to try to estimate the relationships directly (perhaps after pre-whitening the series), rather than via spectral estimates. One approach is the so-called Box–Jenkins approach, described in the next subsection.

9.4.2 The Box–Jenkins approach

This section gives a brief introduction to the method proposed by Box and Jenkins (1970, Chapters 10 and 11)[6] for identifying a physically realizable linear system, in the time domain, in the presence of added noise.

The input and output series are both differenced d times until both are stationary, and are also mean-corrected. The modified series will be denoted

[6] There is virtually no change in Chapters 10 and 11 in Box et al. (1994).

by $\{X_t\}$, $\{Y_t\}$, respectively. We want to find the impulse response function $\{h_k\}$ of the system, where

$$Y_t = \sum_{k=0}^{\infty} h_k X_{t-k} + N_t. \tag{9.38}$$

The 'obvious' way to estimate $\{h_k\}$ is to multiply through Equation (9.38) by X_{t-m} and take expectations to give

$$\gamma_{XY}(m) = h_0\gamma_X(m) + h_1\gamma_X(m-1) + \cdots \tag{9.39}$$

assuming that N_t is uncorrelated with the input. If we assume that the weights $\{h_k\}$ are effectively zero beyond say $k = K$, then the first $K+1$ equations of type (9.39) for $m = 0, 1, \ldots, K$, can be solved for the $K+1$ unknowns $h_0, h_1, \ldots, h_K$, on substituting estimates of γ_{XY} and γ_X. Unfortunately these equations do not, in general, provide good estimators for the $\{h_k\}$, and, in any case, assume knowledge of the truncation point K. The basic trouble, as already noted in Section 8.1.2, is that autocorrelation within the input and output series will increase the variance of cross-correlation estimates.

Box and Jenkins (1970) therefore propose two modifications to the above procedure. First, they suggest 'prewhitening' the input before calculating the sample cross-covariance function. Second, they propose an alternative form of Equation (9.35), which will in general require fewer parameters by allowing the inclusion of lagged values of the output[7]. They represent the linear system by the equation

$$\begin{aligned} Y_t - \delta_1 Y_{t-1} - \cdots - \delta_r Y_{t-r} \\ = \omega_0 X_{t-b} - \omega_1 X_{t-b-1} - \cdots - \omega_s X_{t-b-s}. \end{aligned} \tag{9.40}$$

This is rather like Equation (9.4), but is given in the notation used by Box and Jenkins (1970, Chapter 10) and involves an extra parameter b, which is called the **delay** of the system. The delay can be any non-negative integer. Using the backward shift operator B, Equation (9.40) may be written as

$$\delta(B)Y_t = \omega(B)X_{t-b} \tag{9.41}$$

where

$$\delta(B) = 1 - \delta_1 B - \cdots - \delta_r B^r$$

and

$$\omega(B) = \omega_0 - \omega_1 B - \cdots - \omega_s B^s.$$

Box and Jenkins (1970) describe Equation (9.41) as a **transfer function** model, which is a potentially misleading description in that the term 'transfer

[7] This is similar to the idea that the general linear process, represented by an MA model of possibly infinite order, can often be parsimoniously approximated by a mixed ARMA model of low order – see Section 3.4.5.

function' is sometimes used to describe some sort of transform of the impulse response function. Indeed Box and Jenkins (1970, Chapter 10) describe the generating function of the impulse response function, namely

$$h(B) = h_0 + h_1 B + h_2 B^2 + \cdots$$

as the **transfer function**[8].

The Box–Jenkins procedure begins by fitting an ARMA model to the (differenced) input. Suppose this model is of the form (see Section 3.4.5)

$$\phi(B)X_t = \theta(B)\alpha_t$$

where $\{\alpha_t\}$ denotes a purely random process, in the notation of Box and Jenkins (1970, Chapter 11). Thus we can transform the input to a white noise process by

$$\theta^{-1}(B)\phi(B)X_t = \alpha_t.$$

Suppose we apply the same transformation to the output, to give

$$\theta^{-1}(B)\phi(B)Y_t = \beta_t$$

and then calculate the cross-covariance function of the filtered input and output, namely, $\{\alpha_t\}$ and $\{\beta_t\}$. It turns out that this function gives a better estimate of the impulse response function, since if we write

$$h(B) = h_0 + h_1 B + h_2 B^2 + \cdots$$

so that

$$Y_t = h(B)X_t + N_t$$

then we find

$$\begin{aligned} \beta_t &= \theta^{-1}(B)\phi(B)Y_t \\ &= \theta^{-1}(B)\phi(B)[h(B)X_t + N_t] \\ &= h(B)\alpha_t + \theta^{-1}(B)\phi(B)N_t. \end{aligned}$$

If we now evaluate the cross-covariance function of the two derived series, namely, $\{\alpha_t\}$ and $\{\beta_t\}$, then we find

$$\gamma_{\alpha\beta}(m) = h_m \operatorname{Var}(\alpha_t) \tag{9.42}$$

since $\{\alpha_t\}$ is a purely random process, and N_t is uncorrelated with $\{\alpha_t\}$. Equation (9.42) is of a much simpler form than Equation (9.39) and will give more reliable estimates than those obtained by solving equations of type

[8]Note that Box and Jenkins (1970) use the notation $\{\nu_k\}$ for the impulse response function. Box and Jenkins also use x_t and y_t for the differenced input and output, respectively, but we retain the convention of using capital letters to denote random variables and lower case letters for observed values.

(9.39). This is partly because the sample cross-covariances of the derived series will have lower variances than those for the original series because there is less autocorrelation in the two series.

Observed values of the derived prewhitened series can be found by calculating $\hat{\alpha}_t = \hat{\phi}(B)\hat{\theta}^{-1}(B)x_t$ where $\hat{\phi}$, $\hat{\theta}$ denote estimates of ϕ and θ when fitting an ARMA model to the input, and x_t denotes the observed value of the input at time t. The same estimated transform is applied to the observed output values, and the sample cross-covariance function of $\hat{\alpha}_t$ and $\hat{\beta}_t$, namely, $c_{\alpha\beta}(k)$, is then computed. The observed variance of $\hat{\alpha}_t$, namely, s_α^2, is also computed. Then an estimate of h_k is given by

$$\hat{h}_k = c_{\alpha\beta}(k)/s_\alpha^2. \tag{9.43}$$

Box and Jenkins (1970) give the theoretical impulse response functions for various models given by Equation (9.40) and go on to show how the shape of the estimated impulse response function given by Equation (9.43) can be used to suggest appropriate values for the integers r, b and s in Equation (9.40). They then show how to obtain least squares estimates of $\delta_1, \delta_2, \ldots, \omega_0, \omega_1, \ldots$, given values of r, b and s. These estimates can in turn be used to obtain refined estimates of $\{h_k\}$ if desired.

Box and Jenkins go on to show how a transfer function model, with added noise, can be used for forecasting and control. Several successful case studies have been published (e.g. Jenkins, 1979; Jenkins and McLeod, 1982) and the method looks potentially useful. However, it should be noted that the main example discussed by Box and Jenkins (1970, Chapter 11), using some gas furnace data, has been criticized by Young (1984) and Chatfield (1977, p. 504) on a number of grounds, including the exceptionally high correlation between input and output, which means that virtually any identification procedure will give good results.

Finally, it is worth noting that a somewhat similar method to the Box–Jenkins approach has been independently developed in the control engineering literature (Astrom and Bohlin, 1966; Astrom, 1970) and is called the **Astrom–Bohlin approach**. This method also involves prewhitening and a model similar to Equation (9.40), but the control engineering literature does not discuss identification and estimation procedures in the same depth as the statistical literature. One difference in the Astrom–Bohlin approach is that non-stationary series may be converted to stationarity by high-pass filtering methods other than differencing.

The reader should note that we have implicitly assumed throughout this subsection that the output does not affect the input — in other words there is no feedback, as discussed below in Section 9.4.3. If there is a suspicion that feedback may be present, then it may be advisable to use alternative methods or try to fit the more general multivariate (vector) ARMA model as discussed later in Section 13.4. In fact we will see that a transfer function model can be regarded as a special case of the vector AR model.

At this point the reader may be wondering whether it is better to adopt a frequency-domain approach using cross-spectral analysis or to fit the sort of time-domain parametric model considered in this subsection. Our view is that it is unwise to attempt to make general pronouncements on the relative virtues of time-domain and frequency-domain methods. The two approaches are *complementary* rather than rivals, and it may be helpful to try both. It depends in part on the context, on the nature of the data and on what sort of model is desired.

9.4.3 Systems involving feedback

A system of the type illustrated in Figure 9.7 is called an **open-loop** system, and the procedures described in the previous two sections are appropriate for data collected under these conditions. However, data are often collected from systems where some form of **feedback control** is being applied, and then we have what is called a **closed-loop** system as illustrated in Figure 9.8. For example, when trying to identify a full-scale industrial process, it could be dangerous, or an unsatisfactory product could be produced, if some form of feedback control is not applied to keep the output somewhere near target. Similar problems arise in an economic context. For example, attempts to find a linear relationship showing the effect of price changes on wage changes are bedevilled by the fact that wage changes will in turn affect prices.

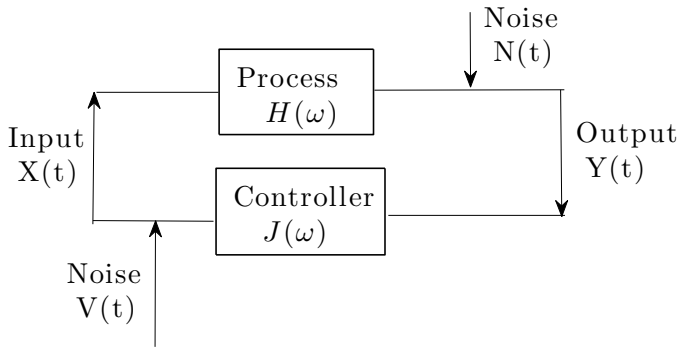

Figure 9.8 *A closed-loop system.*

The problem of identifying systems in the presence of feedback control is discussed, for example, by Gustavsson et al. (1977) and Priestley (1983). The comments by Granger and Newbold (1986, Section 7.3) on 'Causality and Feedback' are also relevant. The key message is that open-loop procedures may not be applicable to data collected in a closed-loop situation. The problem can be explained more clearly in the frequency domain. Assuming that all processes are stationary, let $f_{XY}(\omega)$ denote the cross-spectrum of $X(t)$ and

$Y(t)$ in Figure 9.8, and let $f_X(\omega), f_N(\omega), f_V(\omega)$ denote the spectra of $X(t)$, $N(t)$ and $V(t)$, respectively. Then if $H(\omega)$ and $J(\omega)$ denote the frequency response functions of the system and controller, respectively, it can be shown that

$$f_{XY}/f_X = (Hf_V + \bar{J}f_N)/(f_V + J\bar{J}f_N), \tag{9.44}$$

where all terms are functions of frequency, and $\bar{J}$ is the complex conjugate of J. Only if $f_N \equiv 0$ or $J \equiv 0$ is the ratio f_{XY}/f_X equal to H as is the case for an open-loop system (Equation (9.32)). Thus the estimate of H provided by $\hat{f}_{XY}/\hat{f}_X$ will be poor unless f_N/f_V is small. In particular, if $f_V \equiv 0, \hat{f}_{XY}/\hat{f}_X$ will provide an estimate of J^{-1} and *not* of H.

Similar remarks apply to an analysis in the time domain. The time-domain equivalent of Equation (9.44) is given by Box and MacGregor (1974).

The above problem is not specifically discussed by Box et al. (1994), although it is quite clear from the remarks in their Section 11.6 that their methods are only intended for use in open-loop systems. However, some confusion could be created by the fact that Box et al. (1994, Section 13.2) do discuss ways of choosing optimal feedback control, which is quite a different problem. Having identified a system in open loop, they show how to choose feedback control action so as to satisfy some chosen criterion.

Unfortunately, open-loop identification procedures have sometimes been used for a closed-loop system where they are not appropriate. Tee and Wu (1972) studied a paper machine while it was already operating under manual control and proposed a control procedure that has been shown to be worse than the existing form of control (Box and MacGregor, 1974). In marketing, several studies have looked at the relationship between advertising expenditure and sales of products such as dishwashing liquid and coffee. However, expenditure on advertising is often chosen as a result of changes in sales levels, so that any conclusions obtained by an open-loop analysis are open to doubt.

What then can be done if feedback is present? Box and MacGregor (1974) suggest one possible approach in which the analyst deliberately adds an independent noise sequence on top of the noise $V(t)$. Alternatively, one may have some knowledge of the noise structure or of the controller frequency response function. Akaike (1968) claims that it is possible to identify a system provided only that instantaneous transmission of information does not occur in both system and controller, and an example of his, rather complicated, procedure is given by Otomo et al. (1972).

We have left to last the final, and perhaps most important concern, namely, whether feedback is present within a particular system. Sometimes it is clear from the context whether feedback is present. For example, if one wanted to study the relationship between average (ambient) temperature and sales of a seasonal product, like ice cream, then it is clear that sales cannot possibly affect temperature. Thus one has an open-loop system and open-loop procedures, such as the Box–Jenkins approach in Section 9.4.2, can be used. However, it is not always clear from the context alone as to whether

feedback is present, particularly in economics and marketing. However, some contextual information may still be available. For example, economists know that prices typically affect wages, and wages in turn affect prices. If contextual information is not available, then the analyst may have to rely on data analysis to give some indication. For example, if significantly large cross-correlation coefficients between (prewhitened) input and output are observed at a zero or positive lag, then feedback may be present. However, this sort of inference is generally rather difficult to make in practice. The key point to remember is that, if feedback is present, then cross-correlation and cross-spectral analysis of the raw data may give misleading results if analysed in an open-loop way. Thus it is best to be conservative and allow for the possible presence of feedback if unsure about the true position.

Exercises

9.1 Which of the following equations define a time-invariant linear system?

(a) $y_t = 2x_t$
(b) $y_t = 0.7x_t - 3$
(c) $y_t = 0.5x_t + 0.3x_{t-1}$
(d) $y_t = 0.5y_{t-1} + 0.3x_t$
(e) $y(t) = 1.5\,t\,x_t$
(f) $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$
(g) $y(t) = 1/x(t)$
(h) $y(t) = x(t-\tau)$ for $\tau > 0$

9.2 Find the impulse response function, the step response function, the frequency response function, the gain and the phase shift for the following linear systems (or filters):

(a) $y_t = \frac{1}{2}x_{t-1} + x_t + \frac{1}{2}x_{t+1}$
(b) $y_t = \frac{1}{5}(x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2})$
(c) $y_t = \nabla x_t$
(d) $y_t = \nabla^2 x_t$

where in each case t is integer valued. Plot the gain and phase shift for filters (a) and (c). Which of the filters are low-pass and which high-pass?

If filters (a) and (b) are joined in series, find the frequency response function of the combined filter.

9.3 Find the frequency response functions of the following linear systems in continuous time:

(a) $y(t) = gx(t-\tau)$
(b) $y(t) = \frac{g}{T}\int_0^\infty \mathrm{e}^{-u/T}x(t-u)\,\mathrm{d}u$

where g, T and τ are positive constants.

9.4 Consider the AR(1) process, given by $X_t = \alpha X_{t-1} + Z_t$, where Z_t denotes a purely random process with zero mean and constant variance σ_z^2 and $|\alpha| < 1$ for stationarity. If we regard the model equation as a linear system with X_t denoting the input and Z_t denoting the output, find the frequency response function of the system. Hence find the power spectrum of X_t, knowing the spectrum of the 'output' is constant – see Section 9.3.4 if you get stuck.

Now consider the one-parameter second-order AR process

$$X_t = \alpha X_{t-2} + Z_t$$

where the lagged value of X is two steps before. Show that the process is second-order stationary if $|\alpha| < 1$ (as for the AR(1) process), and find the autocovariance and autocorrelation functions.

Show that the power spectral density function of the process is given by

$$f(\omega) = \sigma_z^2/\pi(1 - 2\alpha \cos 2\omega + \alpha^2) \qquad 0 < \omega < \pi$$

using two different methods: (a) by transforming the autocovariance function; (b) by using the approach of Section 9.3.4.

Suppose now that $\{Z_t\}$ is any stationary process with power spectrum $f_Z(\omega)$. What then is the power spectrum of $\{X_t\}$ as defined by the above AR(1) model?

9.5 If $\{X_t\}$ is a stationary time series in discrete time with power spectral density function $f(\omega)$, show that the smoothed time series

$$Y_t = \sum_{p=0}^{k} a_p X_{t-p}$$

where the a_p are real constants, is a stationary process with power spectral density function

$$\left[\sum_{q=0}^{k}\sum_{p=0}^{k} a_p a_q \cos{(p-q)\omega}\right] f(\omega).$$

In particular, if $a_p = 1/3$ for $p = 0, 1, 2$, show that the power spectrum of Y_t is

$$f(\omega)[1 - \cos 3\omega]/9(1 - \cos \omega).$$

(Hint: Use Equation (9.21) and the trigonometric relation $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$.)

9.6 Show that the power spectral density function of the ARMA(1, 1) process

$$X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}$$

is given by $f_X(\omega) = \sigma_z^2(1+2\beta \cos \omega+\beta^2)/\pi(1-2\alpha \cos \omega+\alpha^2)$ for $0 < \omega < \pi$, using the approach of Section 9.3.4. It may help to let $Y_t = Z_t + \beta Z_{t-1}$.

(This power spectrum may be shown to be equivalent to the normalized spectrum in Exercise 6.7 after some algebra.)

More generally, for the MA process $X_t = \theta(B)Z_t$, show that the frequency response function of the filter $Z_t \to X_t$ is $H(\omega) = \theta(\mathrm{e}^{-\mathrm{i}\omega})$, so that the spectrum of X_t is given by $\theta(\mathrm{e}^{-\mathrm{i}\omega})\theta(\mathrm{e}^{\mathrm{i}\omega})\sigma_Z^2/\pi$ for $0 < \omega < \pi$. Hence show that the spectrum of the general ARMA process $\phi(B)X_t = \theta(B)Z_t$ is given by $\theta(\mathrm{e}^{-\mathrm{i}\omega})\,\theta(\mathrm{e}^{\mathrm{i}\omega})\,\sigma_Z^2\,/\,\pi\,\phi(\mathrm{e}^{-\mathrm{i}\omega})\,\phi(\mathrm{e}^{\mathrm{i}\omega})$. Check this result on the above ARMA$(1, 1)$ process.

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Chapter 10

State-Space Models and the Kalman Filter

A general class of models, arousing much interest in many areas of application, is that of state-space models. They were originally developed by control engineers, particularly for applications requiring continuous updating of the current position. An example, from the field of navigation systems, is 'controlling the position of a space rocket'. However, state-space models have also found increasing use in many types of time-series problems, including parameter estimation, smoothing and prediction.

This chapter introduces state-space models for the time-series analyst, as well as describing the Kalman filter, which is an important general method of handling state-space models. Essentially, Kalman filtering is a method of signal processing, which provides optimal estimates of the current state of a dynamic system. It consists of a set of equations for recursively estimating the current state of a system and for finding variances of these estimates. More details may be found, for example, in Harvey (1989), Janacek and Swift (1993) and Durbin and Koopman (2001).

10.1 State-Space Models

When a scientist or engineer tries to measure any sort of signal, it will typically be contaminated by noise, so that the actual observation is given (in words) by

$$\text{OBSERVATION} \quad = \quad \text{SIGNAL} \quad + \quad \text{NOISE.} \tag{10.1}$$

The corresponding equation that is more familiar to statisticians is written as

$$\text{DATA} \quad = \quad \text{FIT} \quad + \quad \text{RESIDUAL.}$$

Both these equations can be intuitively helpful in understanding the key distinction between 'explained' and 'unexplained' variation. However, as is customary, we prefer to use Equation (10.1) here because it seems more natural for introducing state-space models.

The key step to defining the class of state-space models (in their basic linear form) is as follows. We assume that Equation (10.1) holds, but further assume that the signal is a linear combination of a set of variables, called **state variables**, which constitute what is called the **state vector** at time t. This vector describes the state of the system at time t, and is sometimes called

the 'state of nature'. The state variables can take many forms and the reader should wait to examine the various examples given below before expecting to fully understand this concept.

Although the above jargon derives from control engineering, it should be emphasized that the ideas are equally applicable in many other scientific areas. For example, in economics, the observation could be an economic variable, such as the unemployment rate, and the state variables could then include such (unobserved) quantities as the current true underlying level and the current seasonal factor (if any).

It is an unfortunate complication that there is no standard notation for state-space models and the Kalman filter. We confine attention to the case of a univariate observed time series and denote the observation at time t by X_t. We denote the $(m \times 1)$ state vector at time t by $\boldsymbol{\theta}_t$, and write Equation (10.1) as

$$X_t \;=\; \boldsymbol{h}_t^T \boldsymbol{\theta}_t \;+\; n_t, \tag{10.2}$$

where the $(m \times 1)$ column vector $\boldsymbol{h}_t$ is assumed to be a known vector[1] and n_t denotes the observation error.

The state vector $\boldsymbol{\theta}_t$ which is of prime importance, cannot usually be observed directly (i.e. is unobservable). The state variables are typically model parameters of some sort, such as regression coefficients in a regression model (see Section 10.1.6) or parameters describing the state of a system in a rather different way (see Sections 10.1.1–10.1.3). Thus, the analyst will typically want to use the observations on X_t to make inferences about $\boldsymbol{\theta}_t$. However, in some applications (see Section 10.1.4), the state vector will at least be partially known, but the state-space formulation and the Kalman filter may still be useful for making predictions and handling missing values, quite apart from estimating model parameters.

Although $\boldsymbol{\theta}_t$ may not be directly observable, it is often reasonable to assume that we know how it changes through time, and we denote the updating equation by

$$\boldsymbol{\theta}_t \;=\; G_t\, \boldsymbol{\theta}_{t-1} \;+\; \boldsymbol{w}_t, \tag{10.3}$$

where the $(m \times m)$ matrix G_t is assumed known, and $\boldsymbol{w}_t$ denotes a $(m \times 1)$ vector of deviations such that $\boldsymbol{w}_t^T = (w_{1,t}, w_{2,t}, \ldots, w_{m,t})$.

The pair of equations in (10.2) and (10.3) constitute the general form of the (univariate) **state-space model**. Equation (10.2) is called the **observation** (or **measurement**) equation, while (10.3) is called the **transition** (or **state** or **system**) equation.

The 'errors' in the observation and transition equations are generally assumed to be serially uncorrelated and also to be uncorrelated with each other at all time periods. We may further assume that n_t is $\mathrm{N}(0, \sigma_n^2)$ while $\boldsymbol{w}_t$ is multivariate normal with zero mean vector and a known variance–covariance

[1] We use the superscript ^{T} throughout to denote the transpose of a matrix or vector. Here $\boldsymbol{h}_t$ is a column vector, and so its transpose $\boldsymbol{h}_t^T$ is a row vector.

matrix[2] denoted by W_t. Note that if $\boldsymbol{w}_t$ is independent of $\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1}, \boldsymbol{\theta}_{t-2}, \ldots$, then the sequence $\{\boldsymbol{\theta}_t\}$ is said to have the Markov property in that $\boldsymbol{\theta}_t$ depends on $\boldsymbol{\theta}_{t-1}$ but not on earlier values.

The state-space model can readily be generalized to the case where X_t is a vector by making $\boldsymbol{h}_t$ a matrix of appropriate size and by making n_t a vector of appropriate length. It is also possible to add terms involving known linear combinations of explanatory (or exogenous) variables to the right-hand side of Equation (10.2).

The application of state-space models to engineering problems, such as controlling a dynamic system, is fairly clear. There the equations of motion of a system are often assumed to be known *a priori*, as are the properties of the system disturbances and measurement errors, although some model parameters may have to be estimated from data. Neither the equations nor the 'error' statistics need be constant as long as they are known functions of time. However, at first sight, state-space models may appear to have little connection with earlier time-series models. Nevertheless it can be shown, for example, that it is possible to put many types of time-series models into a state-space form. They include regression and autoregressive moving average (ARMA) models as well as various trend-and-seasonal models for which exponential smoothing methods are thought to be appropriate.

In fact, the class of state-space models actually covers a very wide collection of models, often appearing under different names in different parts of the literature. For example, the so-called **unobserved components models**, used widely by econometricians, are of a state-space form. Bayesian forecasting (see Section 10.1.5) relies on a class of models, called **dynamic linear models**, which are essentially a state-space representation, while some models with time-varying coefficients can also be represented in this way. Moreover, Harvey (1989) has described a general class of trend-and-seasonal models, called **structural models**, which involve the classical decomposition of a time series into trend, seasonality and irregular variation, but which can also be represented as state-space models. We pay particular attention to these important models.

Note that the model decomposition must be additive in order to get a linear state-space model. If, for example, the seasonal effect is thought to be multiplicative, then logarithms must be taken in order to fit a structural model, although this implicitly assumes that the 'error' terms are also multiplicative. A key feature of structural models (and more generally of linear state-space models) is that the observation equation involves a *linear* function of the state variables and yet does not restrict the model to be constant through time. Rather it allows local features, such as trend and seasonality, to be updated through time using the transition equation.

[2] This is a square symmetric matrix, of size $(m \times m)$, that specifies the variances of each element of $\boldsymbol{w}_t$ on its diagonal and the covariances between pairs of elements of $\boldsymbol{w}_t$ as the off diagonal terms. Thus the (i, j)th element of W_t is given by $\mathrm{Cov}\{w_{i,t}, w_{j,t}\}$.

Several examples of state-space models are presented in the following subsections, starting with the random walk plus noise model that involves just one state variable.

10.1.1 The random walk plus noise model

Suppose that the observation equation is known to be given by

$$X_t = \mu_t + n_t \tag{10.4}$$

where the unobservable local level μ_t is assumed to follow a random walk given by

$$\mu_t = \mu_{t-1} + w_t. \tag{10.5}$$

Here, Equation (10.5) is the transition equation, and the state vector $\boldsymbol{\theta}_t$ consists of a single state variable, namely, μ_t. Thus $\boldsymbol{\theta}_t$ is a scalar, rather than a vector, while $\boldsymbol{h}_t$ and G_t are also constant scalars, namely, unity. The model involves two error terms, namely, n_t and w_t, which are usually assumed to be independent and normally distributed with zero means and respective variances σ_n^2 and σ_w^2. The ratio of these two variances, namely, σ_w^2/σ_n^2, is called the **signal-to-noise ratio** and is an important quantity in determining the features of the model. In particular, if $\sigma_w^2 = 0$, then μ_t is a constant and the model reduces to a trivial, constant-mean model.

The state-space model defined by Equations (10.4) and (10.5) is usually called the **random walk plus noise model**, but has also been called the **local level** model and the **steady** model. Note that no trend term is included. While relatively simple, the model is very important since it can be shown that simple exponential smoothing produces optimal forecasts, not only for an ARIMA$(0, 1, 1)$ model (see Chapter 5) but also for the above model (see Section 10.2 and Exercise 10.1). The reader can readily explore the relation with the ARIMA$(0, 1, 1)$ model by taking first differences of X_t in Equation (10.4) and using Equation (10.5) to show that the first differences are stationary and have the same autocorrelation function as an MA(1) model. It can also be shown that the random walk plus noise model and the ARIMA$(0, 1, 1)$ model give rise to the same forecast function. Thus the random walk plus noise model is an alternative to the ARIMA$(0, 1, 1)$ model for describing data showing no long-term trend or seasonality but some short-term correlation.

10.1.2 The linear growth model

The linear growth model is specified by these three equations

$$\begin{aligned} X_t &= \mu_t + n_t, \\ \mu_t &= \mu_{t-1} + \beta_{t-1} + w_{1,t}, \\ \beta_t &= \beta_{t-1} + w_{2,t}. \end{aligned} \tag{10.6}$$

The first equation is the observation equation, while the next two are transition equations. The state vector $\boldsymbol{\theta}_t^T = (\mu_t, \beta_t)$ has two components, which can naturally be interpreted as the local level μ_t, and the local trend (or growth rate) β_t. Note that the latter state variable does not actually appear in the observation equation. Comparing with the general state-space form in Equations (10.2) and (10.3), the reader may readily verify that $\boldsymbol{h}_t^T = (1, 0)$ and $G_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are both constant through time.

The title 'linear growth model' is reasonably self-explanatory, although it might be clearer to add the adjective 'local'. If the trend term β_t is constant, then the current level μ_t changes linearly through time. However, the trend (or growth rate) may also evolve through time. Of course if $w_{1,t}$ and $w_{2,t}$ have zero variance, then the trend is constant (or deterministic) and we have what is called a **global** linear trend model. However, this situation is arguably unlikely to occur in practice and the modern preference is to use the above **local** linear trend model where the trend *is* allowed to change. In any case the global model is a special case of Equation (10.6) and so it seems more sensible to fit the latter, more general model.

The reader may easily verify that the second differences of X_t in Equation (10.6) are stationary and have the same autocorrelation function as an MA(2) model. In fact it can be shown that two-parameter exponential smoothing (where level and trend are updated) is optimal for an ARIMA$(0, 2, 2)$ model and also for the above linear growth model (e.g. see Abraham and Ledolter, 1986). It is arguably easier to get a variety of trend models from special cases of a general structural state-space model than from the Box–Jenkins ARIMA class of models.

10.1.3 The basic structural model

There are various ways of incorporating seasonality into a state-space model. An important example is the following model specified by four equations:

$$\begin{aligned} X_t &= \mu_t + i_t + n_t, \qquad (10.7) \\ \text{where } \mu_t &= \mu_{t-1} + \beta_{t-1} + w_{1,t} \\ \beta_t &= \beta_{t-1} + w_{2,t} \\ i_t &= -\sum_{j=1}^{s-1} i_{t-j} + w_{3,t} \end{aligned}$$

Here, μ_t denotes the local level, β_t denotes the local trend, i_t denotes the local seasonal index and s denotes the number of periods in 1 year (or season). The model also incorporates four separate 'error' terms that are all assumed to be additive and to have mean zero. Note that the fourth equation in (10.7) assumes that the expectation of the sum of the seasonal effects over 1 year is zero. In this model, the state vector has $s + 2$ components, namely,

$\mu_t, \beta_t, i_t, i_{t-1}, \ldots, i_{t-s+1}$. The model is similar in spirit to that implied by the additive Holt–Winters method (see Section 5.2.3). The latter depends on three smoothing parameters, which correspond in some sense to the three error variance ratios, namely, σ_1^2/σ_n^2, σ_2^2/σ_n^2 and σ_3^2/σ_n^2, where $\sigma_n^2 = \mathrm{Var}(n_t)$, and $\sigma_i^2 = \mathrm{Var}(w_{i,t})$ for $i = 1, 2, 3$.

The above model is called the **basic structural model** by Harvey (1989), who discusses the properties of the model in some detail. Harvey (1989) also discusses various extensions of the model, such as the incorporation of explanatory variables. Andrews (1994) gives some encouraging empirical results as regards the forecasting ability of structural models.

An alternative class of seasonal state-space models are described by Ord et al. (1997) and have the feature that they incorporate a single source of error. Some special cases of this class of models are found to be models for which exponential smoothing is optimal (Chatfield et al., 2001).

10.1.4 State-space representation of an AR(2) process

We illustrate the connection between state-space models and ARIMA models by considering the AR(2) model. The latter can be written as

$$X_t \;=\; \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t. \tag{10.8}$$

Consider the (rather artificial) state vector at time t defined by $\boldsymbol{\theta}_t^T = (X_t,\, \phi_2 X_{t-1})$. Then the observation equation may be written (trivially) as

$$X_t \;=\; (1, 0)\, \boldsymbol{\theta}_t$$

with $\boldsymbol{h}_t^T = (1, 0)$ and $\sigma_n^2 = 0$, while Equation (10.8) may be written as the first line of the transition equation

$$\boldsymbol{\theta}_t = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix} \boldsymbol{\theta}_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Z_t \tag{10.9}$$

since $\boldsymbol{\theta}_{t-1}^T = (X_{t-1},\ \phi_2 X_{t-2})$.

This looks (and is!) a rather contrived piece of mathematical trickery, and we normally prefer to use Equation (10.8), which appears more natural than Equation (10.9). (In contrast the state-space linear growth model in Equation (10.6) may well appear more natural than an ARIMA$(0, 2, 2)$ model.) However, Equation (10.9) does replace two-stage dependence with two equations involving one-stage dependence, and also allows us to use the general results relating to state-space models, such as the recursive estimation of parameters, should we need to do so. For example, the Kalman filter provides a general method of estimation for ARIMA models (e.g. Kohn and Ansley, 1986). However, it should also be said that the approach to identifying state-space models is generally quite different from that for ARIMA models in that more knowledge about model structure is typically assumed *a priori* (see Section 10.1.7).

Note that the state-space representation of an ARMA model is not unique and it may be possible to find many equivalent representations. For example, we could find alternative state-space representations of Equation (10.8) using the state vector $\boldsymbol{\theta}_t^T = (X_t, X_{t-1})$. In such case, the observation equation is still $X_t = (1, 0)\boldsymbol{\theta}_t$ with $\mathbf{h}_t = (1, 0)$ and $\sigma_n^2 = 0$, but the transition equation becomes

$$\boldsymbol{\theta}_t = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \boldsymbol{\theta}_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Z_t. \tag{10.10}$$

We may also consider using the (more useful?) state vector $\boldsymbol{\theta}_t^T = [X_t, \widehat{X}_t(1)]$, where $\widehat{X}_t(1)$ is the optimal one-step-ahead forecast at time t — see Exercise 10.4. Of course, in this case, the state vector *can* be observed directly, and the problem is no longer one of estimating $\boldsymbol{\theta}_t$. However, the state-space formulation may still be useful for other purposes, such as making predictions. For the other two forms of the state vector, the first component can be observed directly, but the second component contains unobserved parameters and so does need to be estimated.

10.1.5 Bayesian forecasting

Bayesian forecasting (West and Harrison, 1997) is a general approach to forecasting that includes a variety of methods, such as regression and exponential smoothing, as special cases. It relies on a model, called the **dynamic linear model**, which is closely related to the general class of state-space models. The Bayesian formulation means that the Kalman filter is regarded as a way of updating the (prior) probability distribution of $\boldsymbol{\theta}_t$ when a new observation becomes available to give a revised (posterior) distribution. The Bayesian approach also enables the analyst to consider the case where several different models are entertained and it is required to choose a single model to represent the process. Alternatively, when there are several plausible candidate models, the approach allows the analyst to compute some sort of combined forecast. For example, when the latest observation appears to be an outlier, one could entertain the possibility that this represents a step change in the process, or that it arises because of a single intervention, or that it is a 'simple' outlier with no change in the underlying model. The respective probabilities of each model being 'true' are updated after each new observation.

An expository introduction to the Bayesian approach, together with case studies and computer software, is given by Pole et al. (1994). Further developments are described by West and Harrison (1997). The approach has some staunch adherents, while others find the avowedly Bayesian approach rather intimidating. This author has no practical experience with the approach. Fildes (1983) suggested that the method is generally not worth the extra complexity compared with alternative, simpler methods. However, some of the examples in Pole et al. (1994) and in West and Harrison (1997) are persuasive, especially for short series with prior information.

10.1.6 A regression model with time-varying coefficients

This example demonstrates how state-space models can be used to generalize familiar constant-parameter models to the situation where the model parameters are allowed to change through time. Suppose that the observed variable X_t is known to be linearly related to a known explanatory variable u_t by

$$X_t \;=\; a_t \,+\, b_t\, u_t \,+\, n_t.$$

If the parameters a_t and b_t are constant, then we have the familiar linear regression model, but we suppose instead that the regression coefficients a_t and b_t are allowed to evolve through time according to a random walk. The two regression coefficients may be taken as the components of the (unobservable) state vector $\boldsymbol{\theta}_t$, while the values of the explanatory variable u_t are put into the known vector $\boldsymbol{h}_t$. Thus writing $\boldsymbol{\theta}_t^T = [a_t, b_t]$ and $\boldsymbol{h}_t^T = [1, u_t]$, the model may be written in in state-space form as

$$\begin{aligned} X_t &= \boldsymbol{h}_t^T\, \boldsymbol{\theta}_t \,+\, n_t, \\ \boldsymbol{\theta}_t &= \boldsymbol{\theta}_{t-1} \,+\, \boldsymbol{w}_t, \end{aligned} \tag{10.11}$$

where the transition matrix G_t is constant, namely, the constant (2×2) identity matrix given by $G_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Of course if the elements of $\boldsymbol{w}_t$ have zero variance, then $\boldsymbol{\theta}_t$ is constant, say $\boldsymbol{\theta}_t^T = (a, b)$, and we are back to the usual linear regression model with constant coefficients. In this case, the transition equation is a trivial identity and there is little point in using the state-space format. The advantage of Equation (10.11) is that it covers a much more general class of models, including ordinary linear regression as a special case, but also allowing the model parameters to change through time. As well as covering a wider class of possibilities, the state-space formulation means that we can apply the general theory relating to state-space models.

10.1.7 Model building

An important difference between state-space modelling in time-series applications and in some engineering problems is that the structure and properties of a time series will usually not be assumed known *a priori*. In order to apply state-space theory, we need to know $\boldsymbol{h}_t$ and G_t in the model equations and also to know the variances and covariances of the disturbance terms, namely, σ_n^2 and W_t. The choice of a suitable state-space model (i.e. the choice of suitable values for $\boldsymbol{h}_t$ and G_t) may be accomplished using a variety of aids including external knowledge and a preliminary examination of the data. For example, Harvey (1989) claims that the basic structural model (see Section 10.1.3) can describe many time series showing trend and seasonality, but, to use the standard basic structural model, the analyst must, for example,

check that the seasonal variation really is additive. If it is not additive, the analyst should consider transforming the data or trying an alternative model. In other words, the use of a state-space model does not take away the difficult problem of finding a suitable model for a given set of data. As usual, model fitting is easy, but model building can be hard.

Another problem in time-series applications is that the error variances are generally not known *a priori*. This can be dealt with by guesstimating them, and then updating them in an appropriate way, or, alternatively, by estimating them from a set of data over a suitable fit period.

10.2 The Kalman Filter

In state-space modelling, the prime objective is usually to estimate the signal in the presence of noise. In other words we want to estimate the $(m \times 1)$ state vector $\boldsymbol{\theta}_t$, which cannot usually be observed directly. The Kalman filter provides a general method for doing this. It consists of a set of equations that allow us to update the estimate of $\boldsymbol{\theta}_t$ when a new observation becomes available. We will see that this updating procedure has two stages, called the prediction stage and the updating stage.

Suppose we have observed a univariate time series up to time $(t-1)$, and that $\widehat{\boldsymbol{\theta}}_{t-1}$ is the 'best' estimator for $\boldsymbol{\theta}_{t-1}$ based on information up to this time. Here 'best' is defined as the minimum mean square error estimator. Further, suppose that we have evaluated the $(m \times m)$ variance–covariance matrix[3] of $\widehat{\boldsymbol{\theta}}_{t-1}$, which we denote by P_{t-1}. The first stage, called the **prediction stage**, is concerned with forecasting $\boldsymbol{\theta}_t$ from data up to time $(t-1)$, and we denote the resulting estimator in an obvious notation by $\widehat{\boldsymbol{\theta}}_{t|t-1}$. Considering Equation (10.3), where $\boldsymbol{w}_t$ is still unknown at time $t-1$, the obvious estimator for $\boldsymbol{\theta}_t$ is given by

$$\widehat{\boldsymbol{\theta}}_{t|t-1} \;=\; G_t\,\widehat{\boldsymbol{\theta}}_{t-1} \tag{10.12}$$

with variance–covariance matrix

$$P_{t|t-1} \;=\; G_t\,P_{t-1}\,G_t^T + W_t. \tag{10.13}$$

Equations (10.12) and (10.13) are called the **prediction equations**. Equation (10.13) follows from standard results on variance–covariance matrices for vector random variables (e.g. Chatfield and Collins, 1980, Equation (2.9)).

When the new observation at time t, namely, X_t, has been observed, the estimator for $\boldsymbol{\theta}_t$ can be modified to take account of this extra information. At time $(t-1)$, the best forecast of X_t is given by $\boldsymbol{h}_t^T\widehat{\boldsymbol{\theta}}_{t|t-1}$ so that the prediction error is given by

$$e_t \;=\; X_t - \boldsymbol{h}_t^T\,\widehat{\boldsymbol{\theta}}_{t|t-1}.$$

This quantity can be used to update the estimate of $\boldsymbol{\theta}_t$ and of its variance–covariance matrix. It can be shown that the best way to do this is by means

[3] See footnote on p. 204.

of the following equations:

$$\widehat{\boldsymbol{\theta}}_t \;=\; \widehat{\boldsymbol{\theta}}_{t|t-1} + K_t\, e_t \tag{10.14}$$

and

$$P_t \;=\; P_{t|t-1} - K_t\, \boldsymbol{h}_t^T P_{t|t-1} \tag{10.15}$$

where

$$K_t \;=\; P_{t|t-1}\boldsymbol{h}_t \,/\, [\boldsymbol{h}_t^T P_{t|t-1}\boldsymbol{h}_t + \sigma_n^2] \tag{10.16}$$

is called the Kalman gain matrix. In the univariate[4] case, K_t is just a vector of size $(m \times 1)$. Equations (10.14) and (10.15) constitute the second **updating stage** of the Kalman filter and are called the **updating equations**.

We will not attempt to derive the updating equations or to demonstrate the optimality of the Kalman filter. However, we note that the results may be found via least squares theory or using a Bayesian approach. A clear introduction to the Kalman filter is given by Meinhold and Singpurwalla (1983), while more detailed accounts are given by Harvey (1989; 1993, Chapter 4), Aoki (1990) and Durbin and Koopman (2001).

A major practical advantage of the Kalman filter is that the calculations are recursive, so that, although the current estimates are based on the whole past history of measurements, there is no need for an ever-expanding memory. Rather the new estimate of the signal is based solely on the previous estimate and the latest observation. A second advantage of the Kalman filter is that it converges fairly quickly when there is a constant underlying model, but can also follow the movement of a system where the underlying model is evolving through time.

The Kalman filter equations look rather complicated at first sight, but they may readily be programmed in their general form and reduce to much simpler equations in certain special cases. For example, consider the random walk plus noise model of Section 10.1.1 where the state vector $\boldsymbol{\theta}_t$ consists of just one state variable, the current level μ_t. After some algebra (e.g. Abraham and Ledolter, 1986), it can be shown that the Kalman filter for this model in the steady-state case (as $t \to \infty$) reduces to the simple recurrence relation

$$\widehat{\mu}_t \;=\; \widehat{\mu}_{t-1} \,+\, \alpha e_t, \tag{10.17}$$

where the smoothing constant α is a (complicated) function of the signal-to-noise ratio σ_w^2/σ_n^2 (see Exercise 10.1). Equation (10.17) is, of course, *simple exponential smoothing*. When σ_w^2 tends to zero, so that μ_t is a constant, we find that α tends to zero as would intuitively be expected, while as σ_w^2/σ_n^2 becomes large, then α approaches unity.

As a second example, consider the linear regression model with time-varying coefficients in Section 10.1.6. Abraham and Ledolter (1983, Section 8.3.3) show how to find the Kalman filter for this model. In particular, it is

[4]The observation X_t is univariate, but remember that the state vector $\boldsymbol{\theta}_t$ is $(m \times 1)$.

easy to demonstrate that, when W_t is the zero matrix, so that the regression coefficients are constant, then G_t is the identity matrix, while $P_{t|t-1} = P_{t-1}$. Then the Kalman filter reduces to the equations

$$\begin{aligned} \widehat{\boldsymbol{\theta}}_t &= \widehat{\boldsymbol{\theta}}_{t-1} + K_t\, e_t, \\ P_t &= P_{t-1} - K_t\, \boldsymbol{h}_t^T\, P_{t-1}, \end{aligned}$$

where

$$\begin{aligned} e_t &= X_t - \boldsymbol{h}_t^T\, \widehat{\boldsymbol{\theta}}_{t-1}, \\ K_t &= P_{t-1}\, \boldsymbol{h}_t\, [\boldsymbol{h}_t^T\, P_{t-1}\, \boldsymbol{h}_t + \sigma_n^2]^{-1}. \end{aligned}$$

Abraham and Ledolter (1983, Section 8.3.3) demonstrate that these equations are the same as the 'well-known' updating equations for recursive least squares provided that starting values are chosen in an appropriate way.

In order to initialize the Kalman filter, we need estimates of $\boldsymbol{\theta}_t$ and P_t at the start of the series. This can be done by *a priori* guesswork, relying on the fact that the Kalman filter will rapidly update these quantities so that the initial choices become dominated by the data. Alternatively, one may be able to estimate the $(m \times 1)$ vector $\boldsymbol{\theta}_t$ at time $t = m$ by least squares from the first m observations, since if we can write

$$\boldsymbol{X}_0 \;=\; M\, \boldsymbol{\theta}_m \;+\; \boldsymbol{e},$$

where $\boldsymbol{X}_0^T = (X_m, X_{m-1}, \ldots, X_1)$, M is a known non-singular $(m \times m)$ matrix and $\boldsymbol{e}$ is an m-vector of independent 'error' terms, then

$$\widehat{\boldsymbol{\theta}}_m \;=\; M^{-1} \boldsymbol{X}_0 \tag{10.18}$$

is the least squares estimate of $\boldsymbol{\theta}_m$ (since M is a square matrix). An example is given in Exercise 10.2.

Once a model has been put into state-space form, the Kalman filter can be used to provide recursive estimates of the signal, and they in turn lead to algorithms for various other calculations, such as making predictions and handling missing values. For example, *forecasts* may readily be obtained from the state-space model using the latest estimate of the state vector. Given data to time N, the best estimate of the state vector is written as $\widehat{\boldsymbol{\theta}}_N$ and the h-step-ahead forecast is given by

$$\begin{aligned} \widehat{X}_N(h) &= \boldsymbol{h}_{N+h}^T\, \widehat{\boldsymbol{\theta}}_{N+h} \\ &= \boldsymbol{h}_{N+h}^T\, G_{N+h}\, G_{N+h-1} \ldots G_{N+1}\, \widehat{\boldsymbol{\theta}}_N, \end{aligned}$$

where we assume $\boldsymbol{h}_{N+h}$ and future values of G_t are known. Of course if G_t is a constant, say G, then

$$\widehat{X}_N(h) \;=\; \boldsymbol{h}_{N+h}^T\, G^h\, \widehat{\boldsymbol{\theta}}_N. \tag{10.19}$$

If future values of $\boldsymbol{h}_t$ or G_t are *not* known, then they must themselves be forecasted or otherwise guesstimated.

The Kalman filter is applied to state-space models that are linear in the parameters. In practice many time-series models, such as multiplicative seasonal models, are non-linear. Then it may be possible to apply a filter, called the **extended Kalman filter**, by making a locally linear approximation to the model. Applications to data where the noise is not necessarily normally distributed are also possible but we will not pursue these more advanced topics here (see, for example, Durbin and Koopman, 2001).

As an example, we will evaluate the Kalman filter for the linear growth model of Section 10.1.2. Suppose that, from data up to time $(t-1)$, we have estimates $\widehat{\mu}_{t-1}$ and $\widehat{\beta}_{t-1}$ of the level and trend. At time $(t-1)$ the best forecasts of $w_{1,t}$ and $w_{2,t}$ are both zero so that the best forecasts of μ_t and β_t in Equation (10.6) are clearly given by

$$\widehat{\mu}_{t|t-1} \;=\; \widehat{\mu}_{t-1} + \widehat{\beta}_{t-1}$$

and

$$\widehat{\beta}_{t|t-1} \;=\; \widehat{\beta}_{t-1}.$$

These agree with Equation (10.12). When X_t becomes available, we can find the prediction error, namely, $e_t = X_t - \widehat{\mu}_{t|t-1}$, and this can then be inserted into Equation (10.14) to give the following pair of scalar equations, namely

$$\widehat{\mu}_t \;=\; \widehat{\mu}_{t|t-1} + k_{1,t}\, e_t \;=\; \widehat{\mu}_{t-1} + \widehat{\beta}_{t-1} + k_{1,t}\, e_t,$$

and

$$\widehat{\beta}_t \;=\; \widehat{\beta}_{t|t-1} + k_{2,t}\, e_t \;=\; \widehat{\beta}_{t-1} + k_{2,t}\, e_t,$$

where $k_{1,t}$, $k_{2,t}$ are the elements of the Kalman gain 'matrix' (here a 2×1 vector) K_t, which can be evaluated after some algebra. It is interesting to note that these two equations are of similar form to those in Holt's two-parameter (non-seasonal) version of exponential smoothing (see Section 5.2.3). There the level and trend are denoted by L_t, T_t, respectively, and we have, for example, that

$$\begin{aligned} L_t &= \alpha X_t + (1-\alpha)(L_{t-1} + T_{t-1}) \\ &= L_{t-1} + T_{t-1} + \alpha e_t, \end{aligned}$$

where $e_t = X_t - [L_{t-1} + T_{t-1}]$. In the steady state as $t \to \infty$, it can be shown that $k_{1,t}$ tends to a constant, which corresponds to the smoothing parameter α. This demonstrates that the forecasting method called Holt's (two-parameter) exponential smoothing is optimal for the linear growth model.

An intuitively reasonable way to initialize the two state variables from the first two observations is to take $\widehat{\mu}_2 = X_2$ and $\widehat{\beta}_2 = X_2 - X_1$ (see Exercise 10.2).

Exercises

10.1 Consider the random walk plus noise model in Section 10.1.1, and denote the signal-to-noise ratio σ_w^2/σ_n^2 by c. Show that the first-order autocorrelation coefficient of $(1-B)X_t$ is $-1/(2+c)$ and that higher-order autocorrelations are all zero.

For the ARIMA$(0,1,1)$ model

$$(1-B)X_t = Z_t + \theta Z_{t-1}$$

show that the first-order autocorrelation coefficient of $(1-B)X_t$ is $\theta/(1+\theta^2)$ and that higher-order autocorrelations are all zero. Thus the two models have equivalent autocorrelation properties when $\theta/(1+\theta^2) = -1/(2+c)$. Hence show that the invertible solution, with $|\theta| < 1$, is $\theta = \frac{1}{2}[(c^2+4c)^{1/2} - c] - 1$.

Applying the Kalman filter to the random walk plus noise model, we find (after some algebra) that, in the steady state (as $t \to \infty$ and $P_t \to$constant), we have

$$\widehat{\mu}_t \;=\; \widehat{\mu}_{t-1} + \alpha\, e_t$$

and

$$\alpha = 1 + \theta = \frac{1}{2}[(c^2+4c)^{1/2} - c]$$

and this is simple exponential smoothing. Now the ARIMA model is invertible provided that $-1 < \theta < 1$, suggesting that $0 < \alpha < 2$. However, the random walk plus noise model restricts α to the range $0 < \alpha < 1$ (and hence $-1 < \theta < 0$) and physical considerations suggest that this is generally a more sensible model. Do you agree?

10.2 Consider the following special case of the linear growth model:

$$\begin{aligned} X_t &= \mu_t + n_t \\ \mu_t &= \mu_{t-1} + \beta_{t-1} \\ \beta_t &= \beta_{t-1} + w_t \end{aligned}$$

where n_t, w_t are independent normal with zero means and respective variances σ_n^2, σ_w^2. Show that the initial least squares estimator of the state vector at time $t = 2$, in terms of the observations X_1 and X_2, is $[\widehat{\mu}_2, \widehat{\beta}_2] = [X_2,\ X_2 - X_1]$ with variance–covariance matrix

$$P_2 = \begin{bmatrix} \sigma_n^2 & \sigma_n^2 \\ \sigma_n^2 & 2\sigma_n^2 + \sigma_w^2 \end{bmatrix}.$$

If $\sigma_w^2 = 0$, so that we have ordinary linear regression with constant coefficients, and a third observation X_3 becomes available, apply the Kalman filter to show that the estimator of the state vector at time $t = 3$ is given by

$$[\widehat{\mu}_3, \widehat{\beta}_3] = \left[\frac{5}{6}X_3 + \frac{1}{3}X_2 - \frac{1}{6}X_1,\ (X_3 - X_1)/2\right].$$

Verify that these are the same results that would be obtained by ordinary least squares regression.

10.3 Find a state-space representation of (a) the MA(1) process $X_t = Z_t + \beta Z_{t-1}$, (b) the MA(2) process.
(Hint for (a): Try $\boldsymbol{\theta}_t^T = [X_t, \widehat{X}_t(1)] = [X_t, \beta Z_t]$.)

10.4 Find a state-space representation of the AR(2) process in Equation (10.8) based on the state vector $\boldsymbol{\theta}_t^T = [X_t, \widehat{X}_t(1)]$, where the optimal one-step-ahead predictor at time t is given by $\widehat{X}_t(1) = \phi_1 X_t + \phi_2 X_{t-1}$, and show that $G = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix}$ with $\boldsymbol{w}_t^T = (1, \phi_1) Z_t$.

10.5 Find a state-space representation of the AR(p) process

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t$$

based on the state vector $\boldsymbol{\theta}_t^T = (X_t, \ldots, X_{t-p})$ Note that the state vector is observable in this case – a rather unusual case. Show that

$$G = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Chapter 11

Non-Linear Models

Most of this book (like most of the time-series statistical literature) is concerned with *linear* methods and models. However, there is no reason why real-life generating processes should all be linear, and the assumption of linearity is often made more for mathematical and computational convenience rather than because it is really believed to be true. Nowadays, there is growing interest in non-linear models combined with a greater computational facility for fitting them, and so this chapter provides an introduction to various types of univariate non-linear models. A brief bibliography is given at the end of the chapter.

11.1 Introduction

This section motivates the need for non-linear models and attempts the difficult task of distinguishing between linear and non-linear models.

11.1.1 Why non-linearity?

Figure 11.1 displays a famous time series giving the average number of sunspots recorded in successive months. The data are listed, for example, by Andrews and Herzberg (1985), and the (updated) series may be obtained in R. Two representations of the data from 1749 to 1983 are given, with different vertical axes. Close inspection of either graph reveals that there is regular cyclic behaviour with a period of approximately 11 years. At first sight, the upper graph seems a more natural way to display the data, but the graph goes up and down so rapidly that it is really only possible to see where the maxima and minima occur. The lower graph uses a smaller height, and it[1] enables us to see that the series generally increases at a faster rate than it decreases. Behaviour like this cannot be explained by a linear model and is quite different from the asymmetry arising from a linear model with an asymmetric error distribution. Figure 11.1 can be reproduced by the following R script.

[1] This time plot provides an excellent example of the care that is needed in drawing graphs, particularly as regards the choice of scales (see Sections 2.3 and 14.4). One representation may be 'good' for one purpose (e.g. displaying the maxima) while a second representation may be 'good' for a different purpose (e.g. looking at the slope of increases and decreases). The plotting of the yearly version of the sunspot data is discussed by Cleveland (1994) together with additional relevant examples and advice.

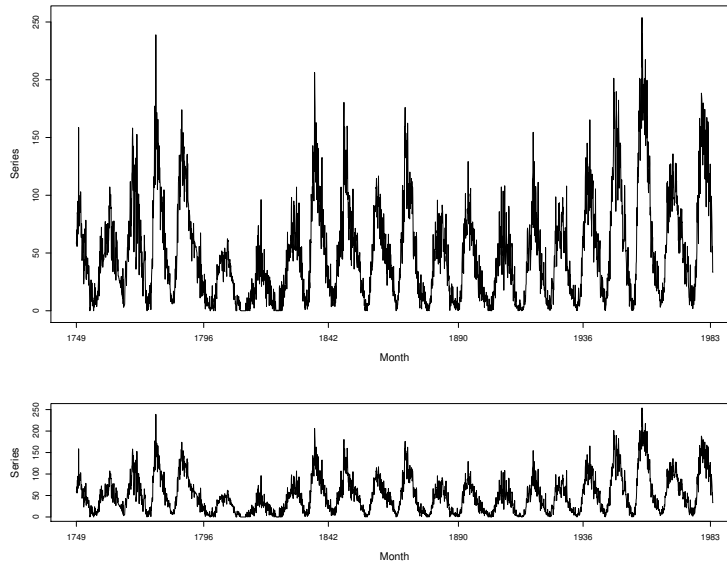

Figure 11.1 *Average monthly sunspot numbers from 1860 to 1983. The lower graph has a shortened vertical axis, which enables the rise and fall of the graph to be more easily assessed.*

```
> library(tseries)
> m<-rbind(1,1,2)
> layout(m)
> par(mar=c(4,4,4,4), cex.lab=1.2)
> plot(sunspots, xlab="Year", ylab="Series", xaxt="n")
> x.pos<-c(1749, 1796, 1842, 1890, 1936, 1983)
> axis(1, x.pos, x.pos)
> plot(sunspots, xlab="Year", ylab="Series", xaxt="n")
> x.pos<-c(1749, 1796, 1842, 1890, 1936, 1983)
> axis(1, x.pos, x.pos)
```

Another famous time series records the annual numbers of lynx trapped in the Mackenzie River district of Canada between 1822 and 1934 (see, for example, Hand et al., 1994, Data Set 109). These data are shown in Figure 11.2 and also show asymmetric cyclic behaviour but with the series *falling* faster than it rises. Asymmetric behaviour can also arise in studying the economy since the relationships between economic variables tend to be different when the economy is moving into a recession rather than when coming out of a recession – downturns are often steeper and more short-lived than upturns. Thus "it seems to be generally accepted that the economy is nonlinear" (Granger and Teräsvirta, 1993, p. 1). As yet another example, the amount

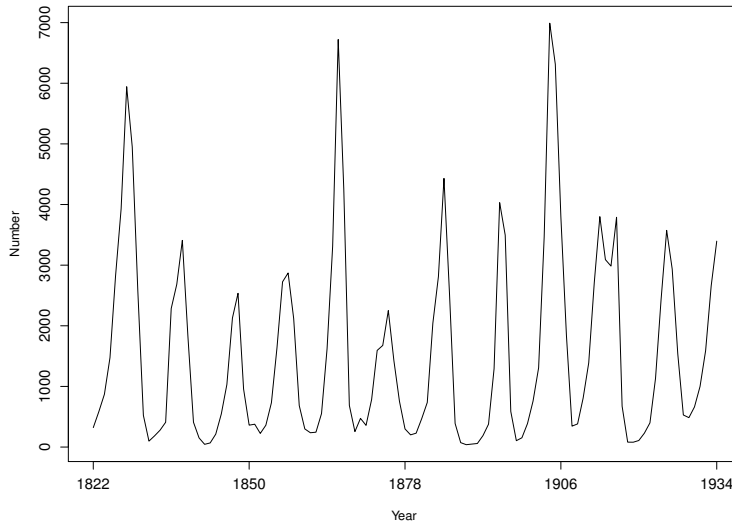

Figure 11.2 *Annual numbers of lynx trapped in the Mackenzie River district of Canada between 1821 and 1934.*

of water flowing down a river tends to increase sharply after a heavy storm and then to tail off gradually.

For seasonal series with a fixed cycle length, it may be possible to model asymmetric behaviour with a non-sinusoidal seasonal component, but when the cycle length is not fixed as in the above examples, a non-linear model is much more compelling for describing series with properties such as "going up faster than coming down". Non-linear models are also needed to describe data where the variance changes through time — see Section 11.3 and Figure 11.3.

Non-linear models can also be used to explain, and give forecasts for, data exhibiting regular cyclic behaviour. As such they provide an interesting alternative to the use of harmonic components, especially if the behaviour is asymmetric. For some non-linear models, if the noise process is 'switched off', then the process will converge asymptotically to a strictly periodic form called a **limit cycle** (Priestley, 1988; Tong, 1990).

Questions about non-linearity also arise when we consider transforming a variable using a non-linear transformation such as the Box–Cox transformation (see Section 2.4). Data may be transformed for a variety of reasons such as to make the data more normally distributed or to achieve constant variance. However, if we are able to fit a linear model to the transformed data, this will imply that a non-linear model is appropriate for the original data. In particular, a series that shows multiplicative seasonality can be transformed to additive seasonality by taking logs and can then be handled

using linear methods. However, the multiplicative model for the original data will be non-linear.

Before embarking on a non-linear analysis, it is sensible to check that the data really are non-linear (Darbellay and Slama, 2000). There are various technical procedures described in the literature for assessing and testing different aspects of non-linearity. A recent survey, with further references, is given by Tsay (2001), and the so-called BDS test, in particular, is briefly discussed later in this chapter. However, tests for non-linearity often have poor power, and the simplest, and arguably the most important, tool (as in the rest of time-series analysis) is a careful inspection of the time plot. Behaviour such as that seen in Figure 11.1 can be self-evident *provided the scales are chosen carefully*. It should also be noted that tests for non-linearity can have difficulty in distinguishing between data from a non-linear model and data from a linear model to which outliers have been added. While some non-linear models can give rise to occasional sharp spikes, the same is true of a linear model with occasional outliers. Here again, a careful inspection of the time plot can be both crucial and fruitful, especially when allied to expert contextual knowledge as to when, where and why unusual observations might occur. This highlights the close connection between non-linearity and non-normality. If, for example, a time series exhibits more 'spikes' up than down, then it is often not clear if this is due to non-linearity, non-normality, or both.

11.1.2 What is a linear model?

The first point to make is that there is no clear consensus as to exactly what is meant by a linear stochastic time-series model and hence no consensus as to what is meant by a *non-linear* model. In much of statistical methodology, the term **general linear model** is used to describe a model that is linear in the parameters but that could well involve non-linear functions of the explanatory variables in the so-called design matrix. In contrast a linear model (or linear system) in time series would certainly exclude non-linear functions of lagged or explanatory variables. To complicate matters further, an MA process, which is certainly a linear process (see Chapter 3) and which, at first sight, appears to be linear in the parameters, is actually regarded as non-linear in the parameters in that the one-step-ahead errors (on which least-squares estimation is based) are non-linear functions of the parameters. This means that explicit analytic formulae for estimators may not be available (see Section 4.3.1) and Box and Jenkins (1970) use the term **non-linear estimation** to describe procedures for minimizing a sum-of-squares function when numerical methods (such as hill-climbing) have to be used. This chapter only discusses the use of the term 'non-linear' as applied to models and forecasting methods.

The most obvious example of a linear model is the **general linear process** (see Section 3.11), which arises when the value of a time series, say X_t, can be expressed as a linear function of the present and past values of a purely random process, say Z_t. This class of models includes stationary autoregressive (AR),

moving average (MA), and ARMA models. The linearity of the process is clear when the model is viewed as a **linear system** (see Chapter 9) for converting the sequence of Z_ts into a sequence of X_ts. In addition the state-space model defined by Equations (10.2) and (10.3) is generally regarded as linear provided the disturbances are normally distributed, $\boldsymbol{h}_t$ is a constant known vector and G_t, W_t are constant known matrices (or at least are non-stochastic, so that if they change through time, they do so in a predetermined way).

A linear forecasting **method** is one where the h-steps-ahead forecast at time N can be expressed as a linear function of the observed values up to, and including, time N. This applies to exponential smoothing, and the additive (though not the multiplicative) version of Holt–Winters. It also applies to minimum mean square error (MMSE) forecasts derived from a stationary ARMA model with known parameters, as would be expected for a general linear process. However, note that when the model parameters have to be estimated from the data (as is normally the case), the MMSE forecasts will *not* be linear functions of past data.

The status of (non-stationary) ARIMA models is not so obvious. Apart from the non-stationarity (which means they can't be expressed as a general linear process), they look linear in other respects. An ARI model, for example, can be regarded as a linear system by treating $\{X_t\}$ and $\{Z_t\}$ as if they were the input and output, respectively, although it is really the other way round. Moreover MMSE forecasts from ARIMA models (assuming known model parameters) will be linear functions of past data. This suggests that it might be possible to define a linear model as any model for which MMSE forecasts are linear functions of observed data. However, while this is a necessary condition, it is not sufficient, because some models give linear prediction rules while exhibiting clear non-linear properties in other respects. A further complication is that it is possible to have models that are **locally linear**, but **globally non-linear** (see Sections 2.5 and 10.1.2). Thus it appears that it may not be feasible to define linearity precisely, but rather that it is possible to move gradually away from linearity towards non-linearity.

11.1.3 What is a non-linear model?

A non-linear model could be defined by exclusion as any model that is not linear. However, this is not helpful since (1) linear models have not been exactly defined, and (2) there is a blurred borderline between linearity and non-linearity. For example, some long-memory models (see Section 13.3) are more linear than non-linear, while for non-stationary models, the non-stationarity property is often more important than whether the model is linear. In any case, as noted earlier, a variable, which can be described by a linear model, becomes non-linear after applying a non-linear transformation.

This chapter therefore restricts attention to certain classes of models that are conventionally regarded as non-linear models, even though some of them

have some linear characteristics while some excluded models have non-linear characteristics.

11.1.4 What is white noise?

When examining the properties of non-linear models, it can be very important to distinguish between independent and uncorrelated random variables. In Section 3.4, **white noise** (or a **purely random process**) was defined to be a sequence of independent and identically distributed (i.i.d.) random variables. This is sometimes called **strict white noise** (SWN), and the phrase **uncorrelated white noise** (UWN) is used when successive values are merely uncorrelated, rather than independent. Of course if successive values follow a normal (Gaussian) distribution, then zero correlation implies independence so that Gaussian UWN is SWN. However, with non-linear models, distributions are generally non-normal and zero correlation need not imply independence.

In reading the literature, it is also helpful to understand the idea of a martingale difference. A series of random variables $\{X_t\}$ is called a **martingale** if $\mathrm{E}[X_{t+1}|\,\text{data to time}\,t\,]$ is equal to the observed value of X_t, say x_t. Then a series $\{Y_t\}$ is called a **martingale difference** (MD) if $\mathrm{E}[Y_{t+1}|\,\text{data to time}\,\,t\,] = 0$. This last result follows by letting $\{Y_t\}$ denote the first differences of a martingale, namely, $Y_t = X_t - X_{t-1}$. An MD is like UWN except that it does not need to have constant variance. Of course a Gaussian MD with constant variance is SWN.

UWN and MDs have known linear, second-order properties. For example, they have constant mean and zero autocorrelations. However, the definitions say nothing about the non-linear properties of such series. In particular, although $\{X_t\}$ may be UWN or an MD, the series of squared observations $\{X_t^2\}$ need not be. Only if $\{X_t\}$ is SWN, will $\{X_t^2\}$ be UWN. There are many tests for linearity whose power depends on the particular type of non-linearity envisaged for the alternative hypothesis (e.g. see Brock and Potter, 1993; Tsay, 2010, Section 4.2). These tests generally involve looking at the properties of moments of $\{X_t\}$, which are higher than second order, particularly at the autocorrelation function of $\{X_t^2\}$, which involves fourth-order moments.

Some analysts also like to look at **polyspectra**, which are the frequency domain equivalent of this and involve taking the Fourier transform of higher order moments of the process. The simplest example is the **bispectrum**, which is the transform of third-order terms of the general form $X_t X_{t-j} X_{t-k}$. For linear Gaussian processes, all polyspectra (including the bispectrum) are identically zero for order three or more and so bispectra have been used as part of test procedures for normality and for linearity. The sunspots data and the lynx data, for example, are both found to be non-linear and non-normal using tests based either on the bispectrum or on time-domain statistics. This author has little experience with polyspectra and will not discuss them here. It appears that they can be useful as a descriptive tool for the analyst interested

in frequency domain characteristics of data, but they can be difficult to interpret and may not be best suited for use in tests of linearity.

The rest of this chapater introduces some stochastic time-series models with a non-linear structure. Attention is restricted to some important classes of models that are of particular theoretical and practical importance.

11.2 Non-Linear Autoregressive Processes

An obvious way to generalize the (linear) AR model of order p is to assume that

$$X_t = f(X_{t-1}, X_{t-2}, \ldots, X_{t-p}) + Z_t, \tag{11.1}$$

where f is some non-linear function and Z_t denotes a purely random process. This is called a **non-linear autoregressive** (NLAR) model of order p. Note that the 'error' term is assumed to be additive. A more general error structure is possible, perhaps with Z_t incorporated inside the f function, but will not be considered here.

For simplicity consider the case $p = 1$. Then we can rewrite Equation (11.1) as

$$X_t = \phi(X_{t-1})X_{t-1} + Z_t \tag{11.2}$$

where ϕ is some non-constant function. It can be shown that a sufficient condition for Equation (11.2) to describe a stable model is that ϕ must satisfy the constraint that $|\phi(x)| < 1$, at least when $|x|$ is large. A model such as

$$X_t = \alpha X_{t-1}^2 + Z_t,$$

which does not satisfy this condition, will generally be explosive (unless Z_t is constrained to an appropriate finite interval), and thus will not be much use for describing real data, even though a comparable quadratic model, with X_{t-1}^2 replaced by the square of some explanatory variable, might seem plausible in a regression, rather than autoregression, context. Clearly non-linearity has to be introduced in a more subtle way in a time-series context.

Several variations of an AR model have been proposed, which allow the parameter(s) of the AR model to change through time, either deterministically (unusual) or stochastically or determined in some way by past data. Considering the stochastic option first, and taking the first-order case as an example, we could let

$$X_t = \alpha_t X_{t-1} + Z_t$$

where

$$\alpha_t = \gamma + \beta\alpha_{t-1} + \epsilon_t$$

and γ, β are constants with $\{\epsilon_t\}$ being an i.i.d. sequence independent of the $\{Z_t\}$ sequence. Thus the parameter of the AR(1) process for X_t itself follows an AR(1) process. Such models are called **time-varying parameter models** (e.g. see Nicholls and Pagan, 1985). In the case when $\beta = 0$ in Equation

(11.2), the model reduces to what is sometimes called a **random coefficient model**. While such models appear intuitively plausible at first sight – we all know the world is changing – they can be tricky to handle and it is hard to distinguish between the constant and time-varying parameter cases. It is also not immediately obvious if the model is linear or non-linear. The model may appear to be linear in Equation (11.2), but appears non-linear if Equations (11.2) and (11.2) are combined, with $\beta = 0$ for simplicity, as

$$X_t = \gamma X_{t-1} + Z_t + \epsilon_t X_{t-1}.$$

The last term is non-linear. The 'best' point forecast at time t is $\hat{\gamma} x_t$. This looks linear at first sight, but of course $\hat{\gamma}$ will involve past data leading to a non-linear function (though a similar situation arises for the constant-parameter AR model when the parameter(s) have to be estimated from the data). A more convincing argument for non-linearity is that, when Equation (11.2) is expressed as a state-space model (as in Equation 10.2), it is not a linear model since $\boldsymbol{h}_t = \alpha_t$ changes stochastically through time. Moreover it can be shown that the width of prediction intervals depends in a non-linear way on the latest value of the series.

Another general possibility arises if we assume that the function f in Equation (11.1) is piecewise linear. This means that it consists of two or more linear functions defined over different regions of the lagged values of X_t. Consider, for example, the NLAR model of order 1, namely

$$X_t = f(X_{t-1}) + Z_t. \tag{11.3}$$

If we assume that f is piecewise linear, then the parameters of f could, for example, depend on whether the value of X_{t-1} is larger or smaller than a critical value, customarily called a threshold. Thus we effectively allow the model parameters to be determined by past data. This leads to the idea of a threshold AR model, which will be considered in the next subsection.

11.3 Threshold Autoregressive Models

Following from Equation (11.3), a threshold AR model is a piecewise linear model where the parameters of an AR model are determined by the values taken by one or more of the lagged values of the time series. Consider, for example, the model

$$X_t = \begin{cases} \alpha^{(1)} X_{t-1} + Z_t & \text{if } X_{t-1} < \gamma, \\ \alpha^{(2)} X_{t-1} + Z_t & \text{if } X_{t-1} \geq \gamma, \end{cases} \tag{11.4}$$

where $\alpha_{(1)}$, $\alpha_{(2)}$, γ are constants and $\{Z_t\}$ denotes SWN. In one sense, this is an AR(1) model, but the AR parameter depends on whether X_{t-1} exceeds the value r called the **threshold**. Below r the AR parameter is $\alpha_{(1)}$, but above r it is $\alpha_{(2)}$. This feature makes the model non-linear and it is an example of a large class of models called **threshold autoregressive** (TAR) models.

Tong (1990) calls a TAR model **self-exciting** when the choice from the various sets of possible parameter values is determined by just one of the past values, say X_{t-d} where d is the *delay*. In Equation (11.4), the choice is determined by the value of X_{t-1} and so the model is indeed self-exciting. Model (11.4) can be readily be extended to higher order autoregressions and to more than two thresholds depending on the values of one or more past data values. In particular, a k-regime self-exciting TAR (SETAR) model with threshold variable X_{t-d} if it satisfies

$$X_t = \alpha_0^{(j)} + \alpha_1^{(j)} X_{t-1} + \ldots + \alpha_p^{(j)} X_{t-p} + Z_t^{(j)}, \quad \text{if } \gamma_{j-1} \leq X_{t-d} < \gamma_j, \tag{11.5}$$

in the jth regime, where k and d are positive integers, $j = 1, \ldots, k$, γ_j are real numbers such that $-\infty = \gamma_0 < \gamma_1 < \ldots < \gamma_k = \infty$, and $\{Z_t^{(j)}\}$ are i.i.d. sequences with mean 0 and variance σ_j^2 and are mutually independent for different j.

Some theory for threshold models is given by Tong (1990). Threshold models are piecewise linear in that they are linear in a particular subset of the sample space, and can be thought of as providing a piecewise linear approximation to some general non-linear function as in Equation (11.1). Threshold models sometimes give rise to periodic behaviour with a limit cycle and it can be fruitful to demonstrate such a property by plotting X_t against X_{t-1}, or more generally by plotting X_t against X_{t-k}. Such a graph is sometimes called a **phase diagram** (actually a discrete-time version of such a diagram). Such graphs are useful, not just for identifying TAR models, but more generally in assessing the general form of a lagged relationship, particularly whether it is linear or non-linear.

Estimation and forecasting for TAR models are, perhaps inevitably, rather more difficult than for linear AR models and will not be covered here. Estimation usually involves some sort of iterative procedure, while the difficulties involved in forecasting can be illustrated for the first-order model in Equation (11.4). Suppose we have data up to time N and that x_N happens to be larger than the threshold r. Then it is obvious from the model that $\hat{x}_N(1) = E[X_{N+1}|\text{data to time } N] = \alpha_{(2)} x_N$, and so the one-step-ahead forecast is easy to find. However, finding the two-steps-ahead forecast for the above model is much more complicated, as it will depend on whether the next observation happens to exceed the threshold. The expectation that arises is algebraically intractable and some sort of integration or approximation will need to be made to evaluate expectations over future error terms and the corresponding thresholds.

An interesting application of threshold models was made by Chappell et al. (1996) in regard to exchange rates within the European Union. These rates are supposed to stay within prescribed bounds. A threshold model led to improved forecasts as compared with a random walk model. A second example, using economic data, is given by Tiao and Tsay (1994). Although the latter authors

found little improvement in forecasts using a threshold model, the modelling process required to fit a non-linear model led to greater insight into economic relationships, particularly that the economy behaves in a different way when it is going into, or coming out of, recession. Tsay (1998) has extended threshold models to the multivariate case, and gives details on testing, estimation and modelling together with some examples.

Example 11.1

We consider an example of analyzing the average monthly sunspot numbers from 1860 to 1983 (see Figure 11.1) with SETAR models. Denote Y_t the sunspot number at year t. We first transform the data by taking logarithm,

$$\widetilde{Y}_t = \log(Y_t + 1).$$

The reason for adding 1 to Y_t is that the sunspot numbers at certain years are zero. We note that the solar cycle or solar magnetic actitivy cycle is nearly 11-years, so we consider the sunspot numbers are approximately periodic with 11-year cycles. Accordingly, we use the `R` function `decompose` to single out the seasonal effect in $\widetilde{Y}_t$. Figure 11.3 shows the estimated trend, seasonal effect S_t, and the remainder series. Figure 11.4 shows the deseasoned series $X_t := \widetilde{Y}_t - S_t$, and its sample ac.f.'s and partial sample ac.f.'s. Figures 11.3 and 11.4 can be reproduced by the following `R` script.

```
> # Decompose the transformed series with 11-year cycles
> ss<-ts(log(sunspots+1), frequency=12*11, start=c(1749,1))
> ss.de<-decompose(ss)
> ss.trend<-ts(ss.de$trend, frequency=12, start=c(1749,1))
> ss.random<-ts(ss.de$random, frequency=12, start=c(1749,1))
> ss.se<-ts(ss.de$seasonal, frequency=12, start=c(1749,1))
> x.pos<-c(1749, 1796, 1842, 1890, 1936, 1983)

> # Figure 11.3
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(ss.trend, xlab="Year", ylab="Trend", xaxt="n")
> axis(1, x.pos, x.pos)
> plot(ss.se, xlab="Year", ylab="Seasonal effect", xaxt="n")
> axis(1, x.pos, x.pos)
> plot(ss.random, xlab="Year", ylab="Remainder", xaxt="n")
> axis(1, x.pos, x.pos)

> # Figure 11.4
> ss.nose<-ts(ss-ss.de$seasonal, frequency=12, start=c(1749,1))
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(ss.nose, xlab="Year", ylab="Deseasoned series", xaxt="n")
> axis(1, x.pos, x.pos)
```

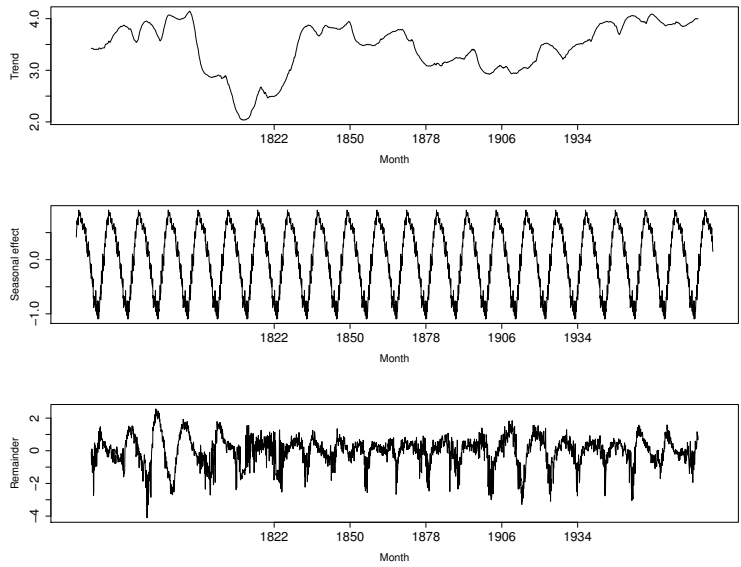

Figure 11.3 *The decomposition of the transformed sunspots series* $\widetilde{Y}_t$ *(Top: Estimated trend* m_t*; Middle: Estimated seasonal effect* S_t*; Bottom: The remainder series).*

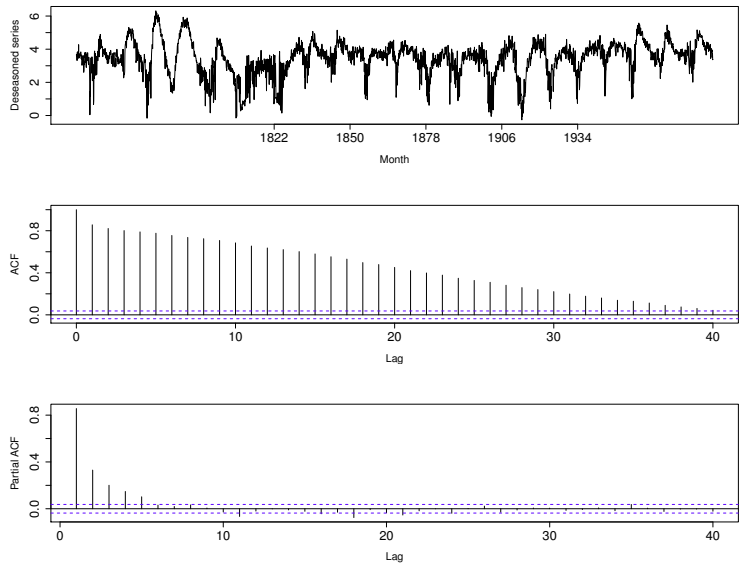

Figure 11.4 *The deseasoned series* $X_t = \widetilde{Y}_t - S_t$ *(top), and its sample ac.f. (middle) and sample partial ac.f. (bottom).*

```
> acf(as.numeric(ss.nose), ylab="ACF", main="", lag=40)
> pacf(as.numeric(ss.nose), ylab="Partial ACF", main="", lag=40)
```

The sample ac.f.'s and partial ac.f.'s in Figure 11.4 indicates that the series X_t can be approximately considered as an AR process. To fit nonlinear AR models to a time series, we may use the package `tsDyn` in `R`. For comparison purposes, we first use the model selection procedure in Section 4.8 to find a good ARMA model for X_t.

```
> ssnose.arma.aic<-matrix(0,10,10)
> for (i in 0:9) for (j in 0:9) {
  tmp.fit<-arima(ss.nose, order=c(i,0,j))
  ssnose.arma.aic[i+1,j+1]<-tmp.fit$aic
 }
> which(ssnose.arma.aic==min(ssnose.arma.aic), TRUE)
     row col
[1,]  10   7
```

The above code shows that an ARMA(9, 6) model gives the minimum AIC among ARMA(p, q) with $0 \leq p, q \leq 10$. Obviously, ARMA(9, 6) is not a good choice in practice as the model contains too many parameters. In such a case, it is worthwhile to try a nonlinear time series model, for example, a TAR model. The following `R` script shows how to fit a SETAR(2) model to X_t using function `setar` in `tsDyn`.

```
> library(tsDyn)
> fit.setar<-setar(ss.nose, m=2, mL=2, mH=2, thDelay=1)
> summary(fit.setar)

Non linear autoregressive model

SETAR model ( 2 regimes)
Coefficients:
Low regime:
  const.L    phiL.1    phiL.2
0.5325051 0.4889504 0.3329734

High regime:
  const.H    phiH.1    phiH.2
0.1210850 0.7154626 0.2441025

Threshold:
-Variable: Z(t) = + (0) X(t)+ (1)X(t-1)
-Value: 3.277
Proportion of points in low regime: 34.6%  High regime: 65.4%
```

```
Residuals:
      Min        1Q    Median        3Q       Max
-2.933373 -0.215514  0.024918  0.251950  2.421657

Fit:
residuals variance = 0.2403,  AIC = -4007, MAPE = 23.28%

Coefficient(s):

         Estimate  Std. Error  t value  Pr(>|t|)
const.L  0.532505    0.051617  10.3166 < 2.2e-16 ***
phiL.1   0.488950    0.021724  22.5078 < 2.2e-16 ***
phiL.2   0.332973    0.024634  13.5168 < 2.2e-16 ***
const.H  0.121085    0.085173   1.4216    0.1552
phiH.1   0.715463    0.031474  22.7317 < 2.2e-16 ***
phiH.2   0.244103    0.036852   6.6238 4.179e-11 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Threshold
Variable: Z(t) = + (0) X(t) + (1) X(t-1)

Value: 3.277
```

```
> # Figure 11.5
> fit.setar.resid<-ts(fit.setar$resid, frequency=12,
    start=c(1749,3))
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(fit.setar.resid, xlab="Year", ylab="Residual",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
> acf(fit.setar$resid, xlab="Lag",ylab="ACF", main="", lag=40)
> pacf(fit.setar$resid, xlab="Lag",ylab="Partial ACF",
    main="", lag=40)
```

The fitted SETAR(2) model is

$$X_t = \begin{cases} 0.5325_{(.0516)} + 0.4890_{(.0217)} X_{t-1} + 0.3330_{(.0246)} X_{t-2} + Z_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } X_{t-1} \le 3.277, \\ 0.1211_{(.0852)} + 0.7155_{(.0315)} X_{t-1} + 0.2441_{(.0369)} X_{t-2} + Z_t, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } X_{t-1} > 3.277, \end{cases}$$

where $Z_t \sim N(0, 0.2403)$. Figure 11.5 shows the residual series, and its sample ac.f.'s and partial ac.f.'s. Comparing the sample ac.f.'s and partial ac.f.'s of

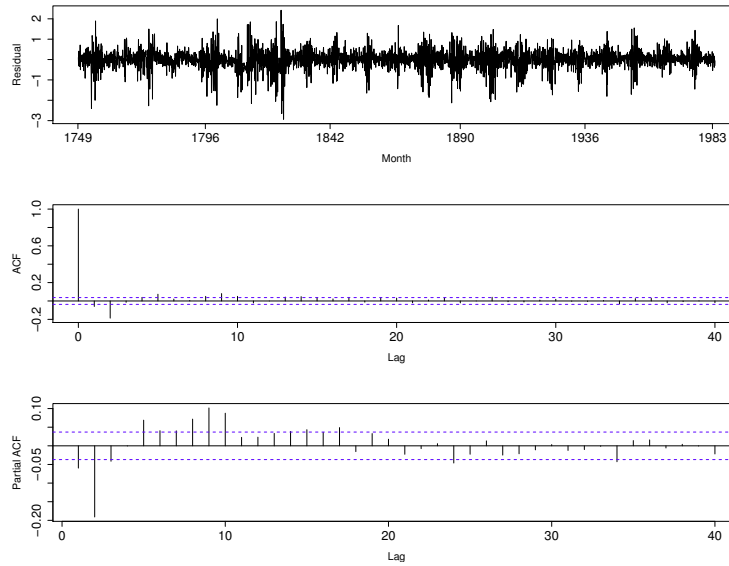

Figure 11.5 *The residual series in the fitted SETAR(2) model (top), and its their sample ac.f. (middle) and sample partial ac.f.(bottom).*

X_t and Z_t, we note that serial correlations of Z_t are significantly smaller than those of X_t.

11.4 Smooth Transition Autoregressive Models

A key feature of TAR models is the discontinuous nature of the AR relationship as the threshold is passed. Those who believe that nature is generally continuous may prefer an alternative model such as the **smooth transition autoregressive** (STAR) model where there is a smooth continuous transition from one linear AR model to another, rather than a sudden jump. For example, consider the model

$$X_t = a_0 + \sum_{j=1}^{p} a_j X_{t-j} + (b_0 + \sum_{j=1}^{p} b_j X_{t-j}) I(X_{t-d}) + Z_t, \tag{11.6}$$

where d is the delay parameter, and I is a smooth function with sigmoid characteristics. One example is the logistic function

$$I(x) = 1/[1 + \exp[\gamma(r - x)]]$$

which depends on two parameters, namely, r, which is comparable to the threshold, and $1/\gamma$, which controls the speed of the switch from one model to

the other. Of course, a STAR model reduces to a simple threshold model by choosing I to be an indicator function taking the value zero below a threshold and one above it. Note that the conditional mean of a STAR model is a weighted sum of two conditional means,

$$E(X_t|X_{t-1}, X_{t-2}, \ldots) = \mu_{t1} + I(X_{t-d})\mu_{t2},$$

where

$$\mu_{t1} = a_0 + \sum_{j=1}^{p} a_j X_{t-j}, \qquad \mu_{t2} = b_0 + \sum_{j=1}^{p} b_j X_{t-j}.$$

A favorable property of the STAR model versus the SETAR model is that the conditional mean function is differentiable.

A STAR model can be estimated using the `R` function `lstar` in package `tsDyn`; see details in the following example.

Example 11.2

Consider the average monthly sunspot numbers Y_t from 1860 to 1983 (see Figure 11.1). In Example 11.1 of Section 11.3, we first transformed the series, then removed the seasonal effect from the transformed series and obtained the series X_t (see the top panel of Figure 11.4). Instead of fitting a TAR model to the deseasoned series X_t, we can also try to fit a STAR model to X_t. In particular, the following `R` script fits a STAR(2) model to the deseasoned series X_t in Section 11.3. We shall note that the model representation in `lstar` is

$$X_t = \Big(\alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}\Big) I(X_{t-d}) + \Big(\beta_0 + \sum_{j=1}^{p} \beta_j X_{t-j}\Big)\big[1 - I(X_{t-d})\big] + Z_t,$$

which is different from the representation (11.6).

```
> library(tsDyn)
> fit.star<-lstar(ss.nose, m=2, d=1)
> summary(fit.star)

Non linear autoregressive model

LSTAR model
Coefficients:
Low regime:
  const.L    phiL.1    phiL.2
0.7016242 0.3207036 0.3535763

High regime:
    const.H      phiH.1      phiH.2
-0.58144876  0.35282417 -0.06858923
```

```
Smoothing parameter: gamma = 100

Threshold
Variable: Z(t) = + (1) X(t) + (0) X(t-1)

Value: 2.423

Residuals:
      Min        1Q    Median        3Q       Max
-2.960203 -0.215721  0.029487  0.254014  2.357019

Fit:
residuals variance = 0.2394,  AIC = -4015, MAPE = 23.27%

Coefficient(s):
          Estimate  Std. Error  t value  Pr(>|z|)
const.L   0.701624    0.071471   9.8168 < 2.2e-16 ***
phiL.1    0.320704    0.043619   7.3523 1.947e-13 ***
phiL.2    0.353576    0.033351  10.6018 < 2.2e-16 ***
const.H  -0.581449    0.088855  -6.5438 5.998e-11 ***
phiH.1    0.352824    0.051362   6.8693 6.451e-12 ***
phiH.2   -0.068589    0.041905  -1.6368    0.1017
gamma   100.000008  404.860071   0.2470    0.8049
th        2.422947    0.044916  53.9440 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Non-linearity test of full-order LSTAR model against
full-order AR model
 F = 25.696 ; p-value = 8.7349e-12

Threshold
Variable: Z(t) = + (1) X(t) + (0) X(t-1)

> # Figure 11.6
> fit.star.resid<-ts(fit.star$resid, frequency=12,
    start=c(1749,3))
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(fit.star.resid, xlab="Year", ylab="Residual",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
> acf(fit.star$resid, xlab="Lag",ylab="ACF", main="", lag=40)
> pacf(fit.star$resid, xlab="Lag",ylab="Partial ACF",
    main="", lag=40)
```

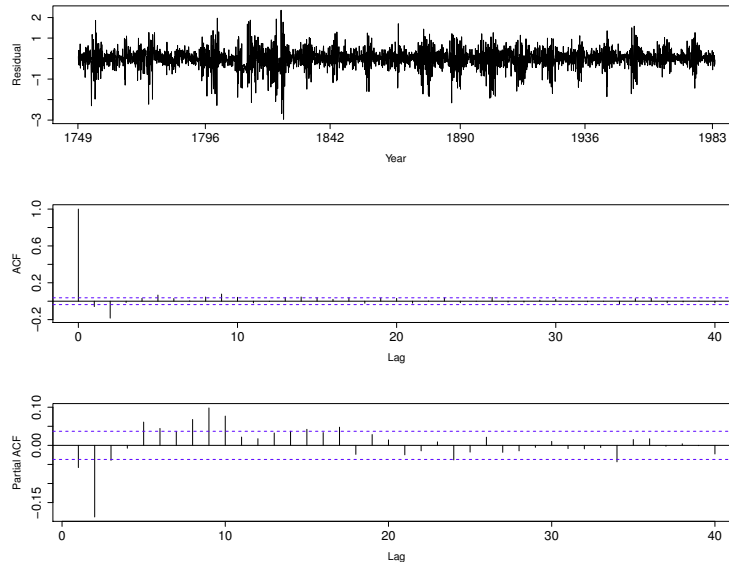

Figure 11.6 *The residual series in the fitted STAR(2) model (top), and its their sample ac.f. (middle) and sample partial ac.f. (bottom).*

The estimated model is given by

$$X_t = \Big(0.7016_{(.0715)} + 0.3207_{(.0436)}X_{t-1} + 0.3536_{(.0333)}X_{t-2}\Big)I(X_{t-1}) + \Big(-0.5814_{(.0889)} + 0.3528_{(.0513)}X_{t-1} - 0.0686_{(.0429)}X_{t-2}\Big)\big[1 - I(X_{t-1})\big] + Z_t,$$

where

$$I(x) = \Big[1 + \exp\Big(100.0_{(404.9)} \cdot (2.4230_{(.0449)} - x)\Big)\Big]^{-1}.$$

Note that the large standard error of γ is very large. Actually, other studies also show that the standard errors of γ and s are often very large; see Teräsvirta (1994). We also refer to it for further discussion. Figure 11.6 shows the residual series, and its sample ac.f.'s and partial ac.f.'s. Note that those plots are quite similar to those in Figures 11.5, as the difference between residual series in the fitted SETAR(2) and STAR(2) models is quite small. For example, the following R script shows that the mean and standard deviations of the difference between two residual series are quite small.

```
> resid.diff<-fit.star.resid-fit.setar.resid
> c(mean(resid.diff), sd(resid.diff))
[1] 2.106738e-14 5.829785e-02
```

11.5 Bilinear Models

A class of non-linear models, called the **bilinear** class, may be regarded as a plausible non-linear extension of the ARMA model, rather than of the AR model. Bilinear models incorporate cross-product terms involving lagged values of the time series and of the innovation process. The model may also incorporate ordinary AR and MA terms. Denoting the time series by $\{X_t\}$ and the innovation process by $\{Z_t\}$, a simple example could be

$$X_t = \alpha X_{t-1} + \beta Z_{t-1} X_{t-1} + Z_t, \tag{11.7}$$

where α and β are constants. As well as the innovation term Z_t, this model includes one AR term plus one cross-product term involving Z_{t-1} and X_{t-1}. It is this cross-product term that is the non-linear term and makes this a bilinear model.

One natural way for such a model to arise is to consider an AR(1) model where the AR parameter is not constant but is itself subject to innovations. If the perturbations in the AR parameter are such that the model for X_t is given by

$$X_t = (\alpha + \beta Z_{t-1}) X_{t-1} + Z_t \tag{11.8}$$

then this leads to Equation (11.7).

A bilinear model could appear to be uncorrelated white noise (UWN) when examined in the usual way by inspecting, and perhaps testing, the sample ac.f. For example, consider the model

$$X_t = \beta Z_{t-1} X_{t-2} + Z_t. \tag{11.9}$$

It can be shown (after some horrid algebra) that $\rho(k) = 0$ for all $k \neq 0$. Thus, the series has the second-order properties of UWN, even though it has a clear structure. If, instead, we examine the series $\{X_t^2\}$, then its ac.f. turns out to be of similar form to that of an ARMA$(2, 1)$ model. Thus Equation (11.9) is certainly not SWN.

This example re-emphasizes the earlier remarks that there is no point in looking at second-order properties of $\{X_t\}$ and hoping they will indicate any non-linearity, because they won't! Rather, the search for non-linearity must rely on specially tailored procedures. As noted earlier, one general approach is to look at the properties of the $\{X_t^2\}$ series. If both $\{X_t\}$ and $\{X_t^2\}$ appear to be UWN, then $\{X_t\}$ can reasonably be treated as SWN.

Bilinear models are interesting theoretically but are perhaps not particularly helpful in providing insight into the underlying generating mechanism; hence they have been little used in practice. They have, for example, been found to give a good fit to the famous sunspots data (see Figure 11.1), but subsequent analysis revealed that they gave poor long-term forecasts. The forecasts diverged and were not able to predict the periodic behaviour observed, because bilinear models are not designed to reflect such behaviour.

11.6 Regime-Switching Models

Several other classes of non-linear models have been introduced in the literature and we refer briefly to two of them. **State-dependent models** are described by Priestley (1988). They include threshold and bilinear models as special cases and can be thought of as a locally linear ARMA model. We say no more about them here.

A second class of non-linear models, called **regime-switching models**, has been widely applied in econometrics. The key feature is that the generating mechanism is different at different points in time and may be non-linear. When the model changes, it is said to *switch between regimes*. The time points at which the regime changes may be known in advance, or the regimes may change according to a Markov process. In particular, let s_t be a random variable that takes integer values $\{1, 2, \ldots, K\}$. Assume that the prior probability of s_0 taking value k is π_k, i.e.,

$$P(s_0 = k) = \pi_k, \qquad k \in \{1, \ldots, K\}.$$

Then obviously, $\pi_1 + \ldots + \pi_K = 1$. Suppose that

$$P(s_t = j \mid s_{t-1} = i, s_{t-2} = l, \ldots) = P(s_t = j \mid s_{t-1} = i) = p_{ij}; \qquad (11.10)$$

that is, the probability that s_t equals some value j depends on the past only through the most recent value s_{t-1}. Such a process is called an K-state **Markov chain** with transition probabilities $\{p_{ij}\}_{i,j=1,\ldots,K}$. Note that p_{ij} is the probability that regime i will be followed by regime j, and hence

$$p_{i1} + \ldots + p_{iK} = 1.$$

The transition probabilities $\{p_{ij}\}_{i,j=1,\ldots,K}$ are often summarized in a $K \times K$ matrix, called the **transition matrix**:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{pmatrix}.$$

Then a regime switching autoregressive model with order p can be written in the following form

$$X_t = \alpha_{s_t,1} X_{t-1} + \ldots + \alpha_{s_t,p} X_{t-p} + Z_t, \qquad (11.11)$$

where the regime s_t is the outcome of an unobserved K-state Markov chain with s_t independent of Z_t for all t. A simple illustration is provided by the following model for sales at time t in terms of demand and supply, namely

$$\text{sales}_t = \text{minimum } [\text{ demand}_t, \text{ supply}_t\,].$$

Here the two regimes are (1) demand exceeds supply; and (2) supply exceeds demand. While the above equation is deterministic, as well as non-linear, separate equations are needed to model demand and supply and these are likely to be stochastic. The reader is referred to Harvey (1993, Section 8.6) for an introduction to these models.

The estimation of regime-switching models is usually challenging. For the regime-switching AR model (11.10), one can still find an explicit expression for its likelihood which can be maximized numerically. Comparing to the estimation of model parameters, the more interesting quantities in regime-switching models is the conditional probability that the regime at t is i given the observed data, i.e.,

$$P(s_t = i \mid \{X_t\}).$$

The above probabilities, called *smoothed probabiities*, can be calculated based on conditional probabilities $P(s_t = i \mid s_{t-1} = j, \{X_t\})$; see details in Hamilton (1994, Section 22.4). The idea of regime-switching can also be extended to other time series models. For example, one may consider regime-switching ARCH or GARCH model; see ARCH and GARCH models in Sections 12.3 and 12.4. For those regime-switching models, the estimation becomes very difficult as the computation of the likelihood may involve path integrals. For illustration purpose, we provide an example of regime-switching AR models below.

Example 11.3

Consider again the average monthly sunspot numbers Y_t from 1860 to 1983 (see Figure 11.1). In Section 11.3, we first transformed the series, then removed the seasonal effect from the transformed series and obtained the series X_t (see the top panel of Figure 11.4). Note that the series seems to fluctuate between two regimes; this indicates that a two-state regime-switching model may be appropriate for interpreting the data. Specifically, we consider an AR(2) model with $K = 2$ regimes.

$$X_t = \begin{cases} \alpha_{1,0} + \alpha_{1,1}X_{t-1} + \alpha_{1,2}X_{t-2} + Z_{1,t}, & \text{if } s_t = 1, \\ \alpha_{2,0} + \alpha_{2,1}X_{t-1} + \alpha_{2,2}X_{t-2} + Z_{2,t}, & \text{if } s_t = 2, \end{cases}$$

where the noises $Z_{1,t} \sim N(0, \sigma_1^2)$ and $Z_{2,t} \sim N(0, \sigma_2^2)$. The two-state prior and transition probability matrix can be expressed as

$$P(s_0 = 1) = \pi \in [0, 1], \qquad P(s_0 = 2) = 1 - \pi.$$

$$P = \begin{pmatrix} \phi & 1 - \phi \\ 1 - \psi & \psi \end{pmatrix}.$$

This model can be estimated by using function `fit.MSAR` in the `R` package `NHMSAR`; see below.

```
> library("NHMSAR")
> data<-array(as.numeric(ss.nose),c(length(ss.nose),1,1))
> theta.init<-init.theta.MSAR(data,M=2,order=2,label="HH")
> fit.rs<-fit.MSAR(data, theta.init)
> fit.rs$theta   # estimated model parameters
$A
                A1        A2
Regime1 0.6406985 0.3069456
Regime2 0.4652951 0.2517079

$A0
              A01
Regime1 0.2022228
Regime2 0.7489775

$sigma

Regime1 0.07480023
Regime2 0.59640904

$prior

Regime1 1.000000e+00
Regime2 1.438698e-21

$transmat
           Regime1    Regime2
Regime1 0.98122641 0.01877359
Regime2 0.04352881 0.95647119

attr(,"NbComp")
[1] 1
attr(,"NbRegimes")
[1] 2
attr(,"order")
[1] 2
attr(,"label")
[1] "HH"
attr(,"n_par")
[1] 8
attr(,"emis.linear")
[1] FALSE
attr(,"class")
[1] "MSAR"
```

Therefore, the estimated model becomes

$$X_t = \begin{cases} 0.2022 + 0.6407X_{t-1} + 0.3069X_{t-2} + Z_{1,t}, & \text{if } s_t = 1, \\ 0.7490 + 0.4653X_{t-1} + 0.2517X_{t-2} + Z_{2,t}, & \text{if } s_t = 2, \end{cases}$$

$$Z_{1,t} \sim N(0, 0.0748^2), \qquad Z_{2,t} \sim N(0, 0.5964^2).$$

with the piror and transition probabilities

$$\pi_1 = 1, \qquad \pi_2 = 1 - \pi = 0,$$

$$P = \begin{pmatrix} 0.9812 & 0.0188 \\ 0.0435 & 0.9565 \end{pmatrix}.$$

The estimated conditional probabilities can be extracted from the fitted model. Figure 11.7 shows the estimated conditional probabilities for each regime at each time point, which can be reproduced using the following R script.

```
> fit.rs.smprob<-ts(fit.rs$smoothedprob, frequency=12,
    start=c(1749,2))
> x.pos<-c(1749, 1796, 1842, 1890, 1936, 1983)
> par(mfrow=c(2,1), mar=c(2,4,2,2), cex.axis=1.5,
    cex.lab=1.5)
> plot(fit.rs.smprob[,1], xlab="", ylab="", main="",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
> plot(fit.rs.smprob[,2], xlab="", ylab="", main="",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
```

After the model is estimated, we can compute the fitted values and residual as follows. Figure 11.8 shows the residual series, and its sample ac.f. and sample partial ac.f.

```
> # Compute the fitted values and residuals
> Xt<-as.numeric(ss.nose)
> n<-length(Xt)
> theta.regime1<-c(fit.rs$theta[2]$A0[1],fit.rs$theta[1]$A[1,])
> theta.regime2<-c(fit.rs$theta[2]$A0[2],fit.rs$theta[1]$A[2,])
> values.in.regime<-cbind(1, Xt[2:(n-1)], Xt[1:(n-2)]
      )%*%cbind(theta.regime1, theta.regime2)
> fit.rs.fitted<-rowSums(values.in.regime*fit.rs$smoothedprob)
> fit.rs.resid<- Xt[-c(1,2)]-fit.rs.fitted

> # Figure 11.8
> fit.rs.resid<-ts(fit.rs.resid, frequency=12, start=c(1749,3))
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
```

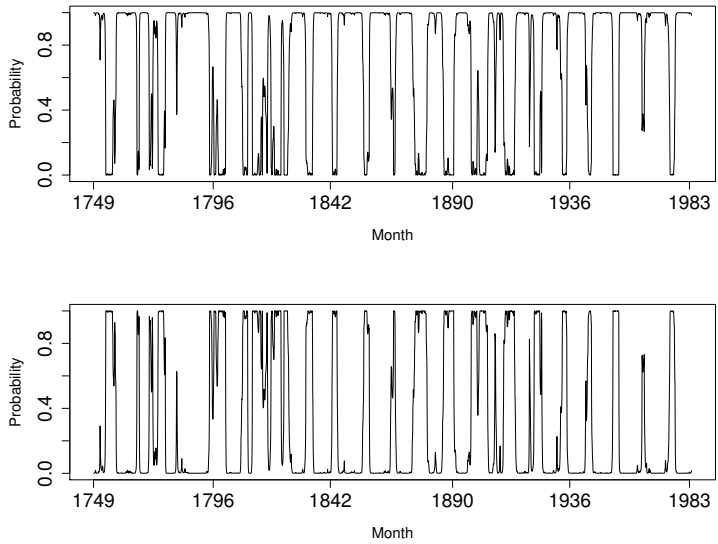

Figure 11.7 *Estimated conditional probabilities* $\widehat{P}(s_t = 1|\{X_t\})$ *(top) and* $\widehat{P}(s_t = 2|\{X_t\})$ *(bottom).*

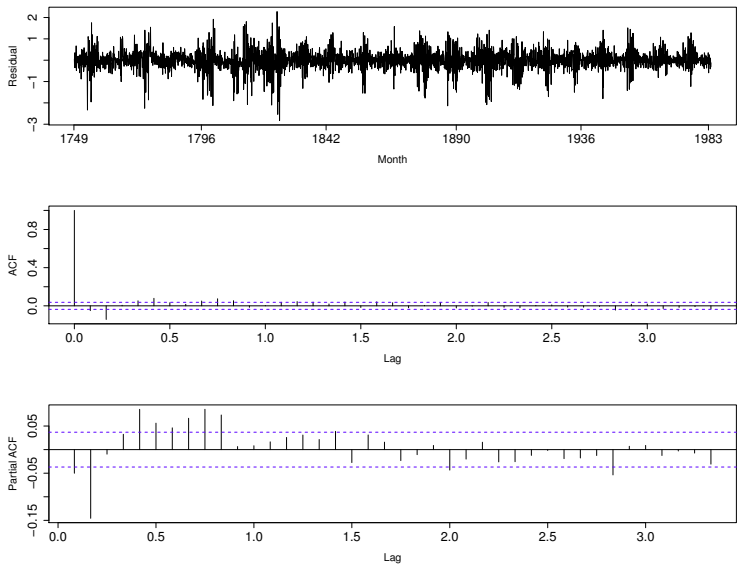

Figure 11.8 *The residual series in the fitted two-state regime switching model (top), and its sample ac.f. (middle) and sample partial ac.f. (bottom).*

```
> plot(fit.rs.resid, xlab="Year", ylab="Residual",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
> acf(fit.rs.resid, xlab="Lag", ylab="ACF", main="", lag=40)
> pacf(fit.rs.resid, xlab="Lag", ylab="Partial ACF",
    main="", lag=40)
```

11.7 Neural Networks

Neural networks (NNs) provide the basis for an entirely different non-linear approach to the analysis of time series. NNs originated in attempts at mathematical modelling of the way that the human brain works, but this connection is rather tenuous and probably not very helpful. Thus an NN is sometimes called an *artificial* NN (ANN) to emphasize that it is a mathematical model.

NNs have been applied to a wide variety of mathematical and statistical problems, many of which have little or no relation with time-series analysis. For example, NNs have been widely used in pattern recognition, where applications include the automatic reading of handwriting and the recognition of acoustic and visual facial features corresponding to speech sounds. Some of these applications have been very successful and the topic has become a rapidly expanding research area. In recent years, NNs have also been applied in time-series analysis and forecasting and we naturally concentrate on these applications. Recent reviews from a statistical perspective include Faraway and Chatfield (1998), Stern (1996), and Warner and Misra (1996), while the introductory chapter in Weigend and Gershenfeld (1994) presents a computer scientist's perspective on the use of NNs in time-series analysis.

A neural net can be thought of as a system connecting a set of inputs to a set of outputs in a possibly non-linear way. The connections between inputs and outputs are typically made via one or more hidden layers of **neurons**, sometimes alternatively called **processing units** or **nodes**. Figure 11.9 shows a simple example of an NN with three inputs, one hidden layer containing two nodes, and one output. The arrows indicate the direction of each relationship and the NN illustrated is typical in that there are no connections between units in the same layer and no feedback. This NN may therefore be described as being of a **feed-forward** design, and NNs are generally assumed to have this structure unless otherwise stated.

The structure, or **architecture**, of an NN has to be determined by the analyst. This includes determining the number of layers, the number of neurons in each layer and which variables to choose as inputs and outputs. In Figure 11.9, the architecture is chosen to forecast the value of a time series at time t (the output) using lagged values at time $(t-1)$ and $(t-4)$ together with a constant (the three inputs). The use of values at lags one and four would be natural when trying to forecast quarterly data. The number of layers is often

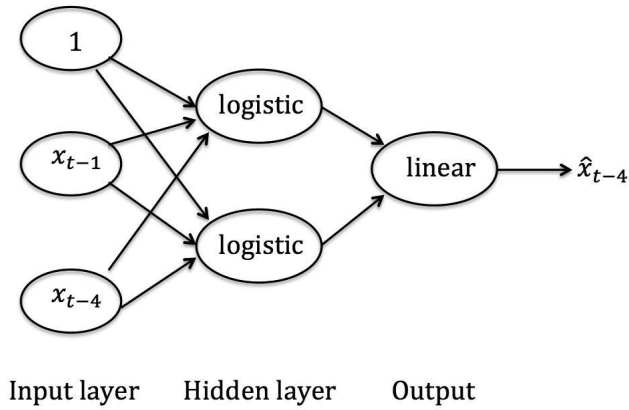

Figure 11.9 *An example of a neural network applied to time-series data with three inputs, one hidden layer of two neurons, and one output. The three inputs are a constant and lagged values at times* $(t-1)$ *and* $(t-4)$*. The output is the forecast at time* t*.*

taken to be one, while the number of hidden neurons is often found by trial and error using the data. Thus the architecture is chosen in general by using the context and the properties of the given data.

How then do we compute the output of an NN from the inputs, given the structure of the network? In general we denote the m input variables by $x_1, x_2, \ldots, x_m$, one of which will usually be a constant. We further assume there are H neurons in one hidden layer. We then attach the weight w_{ij} to the connection between input x_i and the jth neuron in the hidden level. These weights effectively measure the 'strength' of the different connections and are parameters that need to be estimated from the given data, as described below. Given values for the weights, the value to be attached to each neuron may then be found in two stages. First, a linear function of the inputs is found, say

$$v_j = \sum_i w_{ij} x_i$$

for $j = 1, 2, \ldots, H$. Second, the quantity v_j is converted to the final value for the jth neuron, say z_j, by applying a function, called an **activation function**, which has to be selected by the analyst. This function could be linear, but is more usually a non-linear sigmoid transformation such as the logistic function, $z_j = 1/(1+e^{-v_j})$, or the hyperbolic tangent, $z_j = \tanh(v_j)$. It is also possible to choose a discontinuous non-linear activation function such as the indicator function, which takes the value one when v_j exceeds a threshold value and zero otherwise.

Having calculated values for each neuron, a similar pair of operations can then be used to get the predicted value for the output using the values at the

H neurons. This requires a further set of weights, say w'_j, for $j = 1, 2, \ldots, H$, to be attached to the links between the neurons and the output, and also requires an appropriate activation function to be selected for this new stage. If there is a direct link between the constant input and the output, as in Figure 11.9, then we also need a weight for this connection, say w'_o. Overall the output, y say, is related to the inputs by the rather complicated-looking expression

$$y = \phi_o \left[\left(\sum_j w'_j \phi_h \left(\sum_i w_{ij} x_i \right) + w'_o \right) \right]$$

where ϕ_o and ϕ_h denote the activation functions at the output and hidden layers, respectively. In many applications of NNs, ϕ_o and ϕ_h are often chosen to have the same form, but in time-series forecasting this could be disastrous. The logistic function, for example, always gives a number between 0 and 1, and so this would only work for data that are scaled to lie between 0 and 1. Thus, in time-series forecasting, ϕ_o is often chosen to be the identity function so that the operation at the output stage remains linear. Of course, all the above operations have to be carried out for every time t, but we have simplified the presentation by omitting the subscript t that should really be applied to all values of the inputs x_i, to the neuron values ν_j and z_j, and to the output y.

The above exposition can be generalized in obvious ways to handle more than one hidden layer of neurons and more than one output. Overall, an NN can be likened to a sort of non-linear regression model.

Note that the introduction of a constant input 'variable', connected to every neuron in the hidden layer and also to the output as in Figure 11.9, avoids the necessity of separately introducing what computer scientists call a **bias**, and what statisticians would call an intercept term, for each relation. Essentially the 'biases' are replaced by the relevant weights, which become part of the overall set of weights (the model parameters) that can all be estimated in the same way.

How is this model fitting to be done? In time-series analysis, the weights are usually estimated from the data by minimizing the sum of squares of the within-sample one-step-ahead forecast errors, namely, $S = \sum_t (\hat{x}_{t-1}(1) - x_t)^2$. This is done over a suitable portion of the data, so as to get a good fit. Here we assume the output, y from the NN is the one-step-ahead forecast $\hat{x}_{t-1}(1)$.

Choosing the weights so as to minimize S is no easy task, and is a non-linear optimization problem. It is sound practice to divide the data into two sections, to fit the NN model to the first part of the data, called the **training set**, but to hold back the last part of the data, called the **test set**, so that genuine out-of-sample forecasts can be made and compared with the actual observations. This gives an independent check on the model's predictive ability.

Various fitting algorithms have been proposed for NN models, and many specialized packages are now available to implement them. A technique called **back-propagation** is commonly used, though other algorithms exist that may be more efficient. Details will not be given here – see, for example,

Bishop (1995). The NN literature typically describes the iterative estimation procedure as being a 'training' algorithm that 'learns by trial and error', and this is just one example of how the NN literature often uses different jargon from that used by statisticians. Unfortunately, some procedures may take several thousand iterations to converge, and yet may still converge to a local minimum. This is partly because of the non-linear nature of the objective function, and partly because there are typically much larger numbers of parameters to estimate than in traditional time-series models. For example, the relatively simple architecture in Figure 11.9 still involves nine connections and hence has nine parameters (weights). The large number of parameters means there is a real danger that model fitting will 'overtrain' the data and produce a spuriously good fit that does not lead to better forecasts. This motivates the use of model comparison criteria, such as Akaike's information criterion (AIC) (see Section 4.5), which penalizes the addition of extra parameters when comparing different NN architectures. It also motivates the use of an alternative fitting technique called **regularization** (e.g. Bishop, 1995, Section 9.2) wherein the 'error function' is modified to include a penalty term, which prefers 'small' parameter values. In order to start the iterative procedure, the analyst must select starting values for the weights, and this choice can be crucial. It is advisable to try several different sets of starting values to see if consistent results are obtained.

As well as the choice of software to fit an NN, we have seen that the analyst must also consider various other questions, such as how to choose the training set, what architecture to use and what activation function to apply. The approach is non-parametric in character in that little subject-domain knowledge is used in the modelling process (except in the choice of which input variables to include), and there is no attempt to model the 'error' component. When applied to forecasting, the whole process can be completely automated on a computer, which may be seen as an advantage or a disadvantage. The use of NNs then has the character of a *black-box approach* where a particular model is selected from a large class of models in a mechanistic way using little or no subjective skill and giving little understanding of the underlying mechanisms. A problem with black boxes is that they can sometimes give silly results and NNs are no exception. Faraway and Chatfield (1998) suggested that black-box modelling is generally unwise, but rather that a good NN model for time-series data must be selected by combining traditional modelling skills with knowledge of time-series analysis and of the particular problems involved in fitting NN models. Model building will, as always (see Section 4.10), take account of the context and the properties of the data.

The empirical evidence in regard to the forecasting ability of NNs is mixed (see Faraway and Chatfield, 1998; Zhang et al., 1998). There are, in fact, difficulties in making a fair comparison between the use of NNs and of alternative time-series forecasting methods. Moreover, measures of forecast accuracy, like MSE, are really intended for comparing linear models, while the importance of ensuring that forecasts are genuinely 'out-of-sample' is

not always appreciated. More complicated models usually do better within sample, but this proves nothing. Clearly there are exciting opportunities for collaborative work between statisticians and other scientists.

One important empirical study was the so-called Santa Fe competition where six series were analysed (Weigend and Gershenfeld, 1994). The series were very long compared with most time series that need to be forecasted (e.g. 34,000 observations) and five were clearly non-linear when their time plots were inspected. There was only one economic series. The organizers kept holdout samples for three of the series. The participants in the competition tried to produce the 'best' forecasts of the holdout samples using whichever method they preferred, though it is worth noting that little contextual information was provided for them. The results showed that the better NN forecasts did comparatively well for some series, but that some of the worst forecasts were also produced by NNs when applied in black-box mode without using some sort of initial data analysis before trying to fit an appropriate NN model. In particular, predictions "based solely on visually examining and extrapolating the training data did *much worse* than the best techniques, but also *much better* than the worst". The results also showed that there are "unprecedented opportunities to go astray". For the one economic series on exchange rates, there was a "crucial difference between training set and test set performance" and "out-of-sample predictions are on average worse than chance". In other words, better forecasts could have been obtained with the random walk. This is disappointing to say the least!

Other empirical evidence is less clear-cut. In subject areas where NNs have been applied successfully, there are often several thousand observations available for fitting, and the series may exhibit clear non-linear characteristics (as for some of the Santa Fe series). The evidence for shorter series is much less convincing, especially as researchers tend to publish results when new methods do better, but not otherwise. An exception is Racine (2001) who failed to replicate earlier results showing good NN forecasts for stock returns, but rather found linear regression more accurate. Recent applications in economic and sales forecasting have sometimes tried to use as few as 150 observations, and this now seems generally unwise. For many economic and financial series, a random walk forecast of no change is often better than an NN forecast. Simulations show linear methods do better than NNs for data generated by a linear mechanism, as one would intuitively expect.

While it may be too early to make a definitive assessment of the forecasting ability of NNs, it is clear that they are not the universal panacea that some advocates have suggested. Although they can be valuable for long series with clear non-linear characteristics, it appears that the analyst needs several hundred, and preferably several thousand, observations to be able to fit an NN with confidence. Even then, the resulting model is usually hard to interpret, and there is plenty of scope for going badly wrong during the modelling process. For the sort of short time series typically available in sales, economic

and financial forecasting, there is rarely enough data to reliably fit an NN and I cannot recommend their use.

Example 11.4

We show an example here to fit a *NN autoregessive* (NNAR) model to the deseasoned series X_t, which have been studied by other nonlinear time series models in Examples 11.1-11.3. Let d be the period of the seasonal effect in X_t. Denote NNAR$(p, P, k)_d$ as an NNAR model for X_t that uses inputs $(X_{t-1}, X_{t-2}, \ldots, X_{t-p}, X_{t-d}, X_{t-2d}, \ldots, X_{t-Pd})$ and k neurons in the hidden layer. The model can be expressed via the following equations,

$$\begin{aligned} v_j &= w_j^{(1)} + \sum_{i=1}^{p} w_{ij}^{(1)} X_{t-i} + \sum_{l=1}^{P} w_{ij}^{(2)} X_{t-dl}, \\ z_j &= \frac{1}{1+e^{-v_j}}, \\ X_t &= w_0^{(3)} + \sum_{j=1}^{k} w_j^{(3)} z_j. \end{aligned}$$

We can use the `R` function `nnetar` in the `R` package to estimate the model parameters. Specifically, the following `R` script shows how to fit a NNAR model to the deseasoned series X_t in Examples 11.1-11.3.

```
> library(forecast)
> fit.nnar<-nnetar(ss.nose)
> names(fit.nnar)
 [1] "x"       "m"       "p"       "P"       "scalex"    "size"
 [7] "subset" "model"   "nnetargs" "fitted" "residuals" "lags"
[13] "series" "method" "call"

> fit.nnar$model

Average of 20 networks, each of which is
a 34-18-1 network with 649 weights
options were - linear output units

> fit.nnar$method
[1] "NNAR(34,1,18)[12]"

> # Figure 11.10: Note that the first 33 values in the
> #   residual are NA
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(fit.nnar$resid, xlab="Month", ylab="Residual",
    type="l", xaxt="n")
> axis(1, x.pos, x.pos)
```

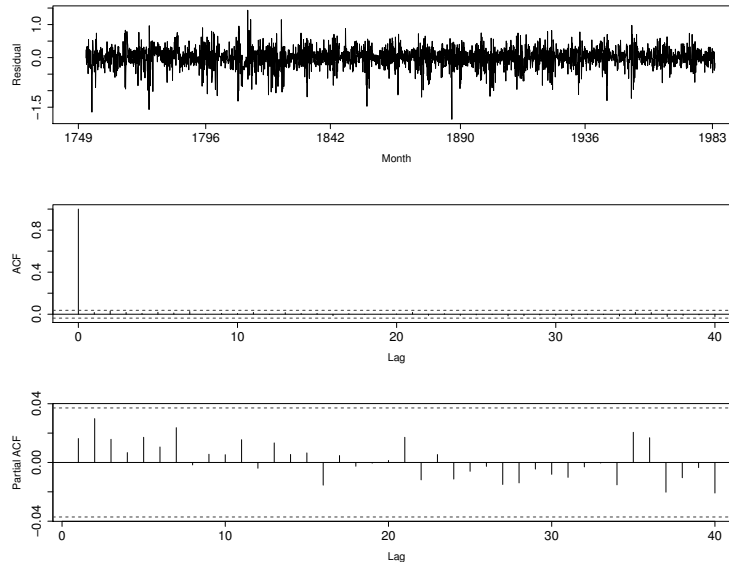

Figure 11.10 *The residual series in the fitted NNAR model (top), and its their sample ac.f. (middle) and sample partial ac.f. (bottom).*

```
> acf(fit.nnar$resid[-seq(1,34)], xlab="Lag",ylab="ACF",
     main="", lag=40)
> pacf(fit.nnar$resid[-seq(1,34)], xlab="Lag",
    ylab="Partial ACF", main="", lag=40)
```

The fitted model is $\text{NNAR}(34, 1, 18)_{12}$. Figure 11.10 shows the residual series, and its sample ac.f. and sample partial ac.f. Note that, comparing to sample ac.f. and sample partial ac.f. of residual series in Figures 11.5, 11.6, and 11.8, the fitted NNAR model seems to provide a very good fit, since the sample ac.f. and sample partial ac.f. of the residual series in the fitted NNAR model are not significant at all.

11.8 Chaos

The topic of **chaos** has attracted much attention in recent years, especially from applied mathematicians. Chaotic behaviour arises from certain types of non-linear models, and a loose definition is 'apparently random behaviour that is generated by a purely deterministic, non-linear system'. A non-technical overview is given by Gleick (1987) while Kantz and Schreiber (1997) provide a readable introduction from the perspective of mathematical physics. Chan and Tong (2001) and Isham (1993) provide a statistical perspective, while further helpful material is given by Tong (1990, Chapter 2).

If a chaotic deterministic system can appear to behave as if it were 'random', the statistician then has the task of deciding what is meant by random and further has to try to decide whether an apparently random time series has been generated by a stochastic model, by a chaotic (non-linear) deterministic model or by some combination of the two. It would be of great interest to scientists if fluctuations, previously thought to be random, turned out to have a deterministic explanation. Unfortunately, distinguishing between the different possibilities can be difficult in practice

The main idea is well illustrated by the famous example of chaos called the **logistic map**, sometimes alternatively called the **quadratic map**. This is an example of what mathematicians would call a *difference equation* or *map*, but which statisticians would probably regard as a deterministic time series.

Suppose a time series is generated by the (deterministic) equation

$$x_t = kx_{t-1}(1 - x_{t-1})$$

for $t = 1, 2, 3, \ldots$ with $x_0 \in (0, 1)$. Then, provided $0 < k \leq 4$, the series will stay within the range $(0, 1)$. For low values of k, the deterministic nature of the series will generally be self-evident. For $0 < k < 1$, it can easily be seen that the series will always decline to zero, whatever starting value is given. For $1 \leq k \leq 3$, it can be shown that the series will always converge to the value $x_t = (1 - 1/k)$ for large t, and this value is called a stable (or fixed) point, or, in the jargon of chaos theory, an **attractor** (although such a series is not in fact chaotic). Note that the stable point $(1 - 1/k)$ is the point of intersection of the 45° line and the quadratic curve, $x_t = kx_{t-1}(1 - x_{t-1})$ for $1 \leq k \leq 3$. For $3 < k < 3.57$, the series exhibits cyclic behaviour whose period depends on k but as k approaches 4, the series looks more and more chaotic. Indeed when $k = 4$, it can be shown that the series has a flat spectrum and has the second-order properties of UWN. The series 'jumps around' all over the interval $(0, 1)$ and the deterministic nature of the series may not be readily apparent in the time plot. This is an example of chaotic behaviour. Of course the deterministic nature of the series can readily be demonstrated in this case by plotting x_t versus x_{t-1} as the points will lie on a quadratic curve, as illustrated in Figure 11.11.

A chaotic system has the property that a small change in initial conditions will generally magnify through time rather than die out. This is exemplified in the so-called **butterfly effect**, whereby a butterfly flapping its wings could produce an effect that is eventually transformed into a tropical storm! The sensitivity to initial conditions (the rate at which a small perturbation is magnified) is measured by a quantity called the **Lyapunov exponent**. This will not be defined here, but values greater than zero indicate divergence. Moreover some starting values lead to greater instability than others (e.g. Yao and Tong, 1994) so that the width of 'error bounds' on predictions will depend on the latest value from which forecasts are to be made.

A chaotic series also has the property that a sequence of m values will generally lie in a restricted area of m-space. For example, for the logistic

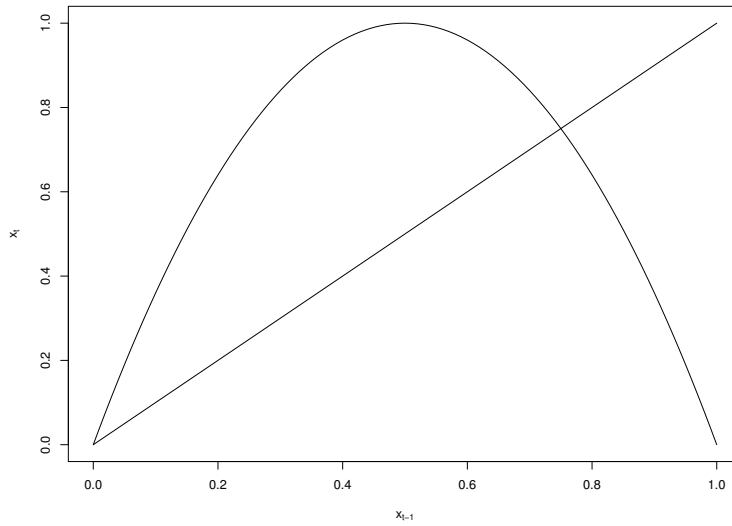

Figure 11.11 *A plot of x_t (y-axis) versus x_{t-1} (x-axis) for the logistic map with $k = 4$. The 45° line (where $x_t = x_{t-1}$) is also shown.*

map with $k = 4$, the values of paired observations of the form (x_t, x_{t-1}) lie on the quadratic curve illustrated in Figure 11.4 rather than anywhere in the unit square. This leads to the concept of the **dimension** of a chaotic system. Whereas random processes generally have infinite dimension, chaotic deterministic behaviour can take place in a much lower dimension. The dimension of a time series has been defined in several different ways, but none of the definitions is very accessible and so none will be given here. Note that non-integer values of dimension are possible as in the so-called **fractal** behaviour, which will also not be discussed here, except to note that fractals are **self-similar** in that they look to have the same properties regardless of the scale at which they are studied.

We have seen that a time series generated by the logistic map, with $1 \leq k \leq 3$, will always converge to the value $x_t = (1 - 1/k)$ for large t, and that this value can be called an attractor. However, for chaotic behaviour when $k = 4$, there is no convergence and the so-called domain of attraction is just the whole unit interval, meaning that values may be found anywhere in the range $(0, 1)$ for large t. However, as chaotic behaviour leads to successive values lying in a restricted subspace, it is sometimes found that the domain of attraction has a strange geometric form (e.g. fractals), and this is called a **strange attractor**.

As another example of the sort of strange behaviour that can result, Bartlett (1990) reproduces an example of a process that is random when

considered as going forward through time but that becomes a non-linear chaotic deterministic series *when time is reversed*. The fascinating topic of time reversibility is reviewed by Lawrance (1991). It can be shown that a stationary (Gaussian) ARMA model *is* reversible in that the properties of data generated by such a model look the same in either direction. In contrast a non-linear mechanism typically gives data, which are not time reversible. For example, the sunspots data in Figure 11.1 would look different if plotted backwards in time as the series would then decrease faster than it rises.

The study of chaos leads to much fascinating mathematics both with difference equations in discrete time (as considered here) or with non-linear differential equations in continuous time. However, are these results of any use to statisticians? One obvious question is whether it is possible to forecast a chaotic series. Because of sensitivity to initial conditions, long-term forecasting is not possible. However, for short-term forecasting of low-dimensional chaotic series, some progress is possible if we know the model (Berliner, 1991). Unfortunately, the analyst does *not* generally know the model *a priori* and so the next important question is whether it is possible to identify a chaotic model and, related to this, whether it is possible to distinguish between chaotic and stochastically random series. In economics, we would like to go further and disentangle the systematic component of a chaotic model from the "noise" that will inevitably affect the system, either in an additive way, so that $x_t = f(x_{t-1}) +$ noise, or as measurement error, so that we actually observe $y_t = x_t +$ noise even though the system equation for the $\{x_t\}$ series remains deterministic. Unfortunately it seems to be difficult to tell whether a series is (deterministically) chaotic or stochastic or some combination of the two, though some progress is being made. For long financial series, a test called the BDS test (Granger and Teräsvirta, 1993, Chapter 6; Brock and Potter, 1993) is often used to test the null hypothesis that a series is (linear) i.i.d. against a (non-linear) chaotic alternative, but it is not clear when this test is superior to other tests for non-linearity. More generally the extent to which a non-linear deterministic process retains its properties when corrupted by noise is also unclear. It is, however, worth noting that classical (linear) methods for smoothing series, which are designed to separate the signal from the noise, can actually make things worse for chaotic series (Kantz and Schreiber, 1997, Example 1.1). Work continues on these difficult questions. Sadly, it does appear to be the case that very large samples are needed to identify attractors in high-dimensional chaos.

A few years ago, there were high hopes that the use of chaotic models might lead to improved economic forecasts. Sadly, this has not yet occurred. For example, Granger (1992) says that it seems unlikely that the stock market could obey a simple deterministic model, while Granger and Teräsvirta (1993, p. 36) and Brock and Potter (1993) both say that there is strong evidence for nonlinearity in economic data but weak evidence that they are also chaotic. On the other hand, May (1987) has argued that chaos is likely to be pervasive in biology and genetics. In general, the analyst will find it difficult

to apply models with chaotic properties to real time-series data, but research is continuing (e.g. Tong, 1995) and the position may well change in the future. In any case, the study of chaotic models is fascinating and may contribute to our understanding of random behaviour in time-series modelling. It may even cause us to re-examine the meaning of the word 'random'. Even if a system under study is regarded in principle as deterministic, the presence of chaotic effects with unknown initial conditions (as will usually be the case in practice) means that prediction becomes difficult or impossible. Moreover a system may appear deterministic at the microscopic level, but appear stochastic at the macroscopic level. Putting this in a different way, it can be argued that whether a system is 'random' depends not on its intrinsic properties, but on how it appears to an observer in the given context with available knowledge. These are deep waters!!

11.9 Concluding Remarks

The real world generally changes through time and often behaves in a non-linear way. Thus alternatives to linear models with constant parameters should often be considered. There are several ways that the need for a non-linear model may be indicated, namely

- Looking at the time plot and noting asymmetry, changing variance, etc.
- Plotting x_t against x_{t-1}, or more generally against x_{t-k} for $k = 1, 2, \ldots$, and looking for strange attractors, limit cycles, etc.
- Looking at the properties of $\{x_t^2\}$ as well as at those of $\{x_t\}$
- Taking account of context, background knowledge, known theory, etc.

Non-linear models often behave quite differently from linear models. In general, they are harder to handle and require considerable technical and numerical skill. In particular, it is generally much more difficult to construct forecasts from a non-linear, rather than linear, model, for more than one step ahead. In the latter case, analytic formulae are generally available, but non-linear forecasting becomes increasingly difficult for longer lead times where some sort of numerical approach will generally be needed to calculate conditional expectations. Fortunately, specialist computer packages, such as STAR (see Tong, 1990, p. xv), are becoming available to do this. Two general references on non-linear forecasting are Lin and Granger (1994) and Chatfield (2001, Section 4.2.4).

An additional feature of non-linear models, which may be unexpected at first, is that the width of prediction intervals need not increase with the lead time. This may happen, for example, for data exhibiting multiplicative seasonality, where prediction intervals tend to be narrower near a seasonal trough rather than near a seasonal peak. Another peculiarity of non-linear models, even with normal errors, is that the distribution of the forecast error is *not* in general normal, and may even be bimodal or have some other unexpected shape. In the bimodal case, a sensible prediction interval may

comprise, not a single interval, but two disjoint intervals. This seems most peculiar at first sight, but second thoughts remind us of situations where we might expect a high or low outcome but not an intermediate result. Sales of a new fashion commodity could be like this. In such circumstances it could be particularly misleading to give a single point forecast by calculating the conditional expectation.

It is difficult to give advice on how to choose an appropriate non-linear model. As always contextual information and background knowledge are vital, but the only statistical advice I can give is that periodic behaviour suggests trying a threshold model, while, with a very long series, some analysts believe that it may be worth trying a neural net. It is also worth remembering that alternatives to linear models include, not only non-linear models but also models where the parameters change through time in a pre-determined way and models that allow a sudden change in structure. While the latter may be regarded as non-linear, it may be more helpful to think of them as non-stationary (see Section 13.2).

Non-linear models are mathematically interesting and sometimes work well in practice. The fitting procedure is more complicated than for linear models, but may lead to greater insight, and it should be remembered that the prime motivation for modelling is often to improve understanding of the underlying mechanism. This being so, it would be a bonus if non-linear models gave better forecasts as well. Sadly, it appears from the literature that gains in forecasting accuracy are often (usually?) modest if they exist at all, and may not by themselves compensate for the additional effort required to compute them. Thus even when the data appear to exhibit clear non-linear properties, it may still be safer to use a linear model for forecasting purposes.

11.10 Bibliography

The literature on non-linear models is growing rapidly and many references have been given throughout this chapter. Alternative introductions are given by Franses (1998, Chapter 8), Granger and Newbold (1986, Chapter 10) and Harvey (1993, Chapter 8). More detailed accounts are given by Priestley (1981, Chapter 11; 1988) and by Tong (1990). The advanced text by Granger and Teräsvirta (1993) is concerned with economic relationships and extends discussion to multivariate non-linear models. Fan and Yao (2003) introduced various parametric and nonparametric techniques for nonlinear time series analysis and prediction.

Exercise

It is difficult to set exercises on non-linear models that are mathematically and practically tractable. The reader may like to try the following exercise on the logistic map.

11.1 Consider the logistic map with $k = 4$, namely

$$x_t = 4x_{t-1}(1 - x_{t-1})$$

This series is non-linear, deterministic and chaotic. Given the starting value $x_0 = 0.1$, evaluate the first three terms of the series. Change x_0 to 0.11 and repeat the calculations. Show that x_3 changes from 0.289 to 0.179 (to 3 decimal places). Thus a change of 0.01 in the starting value leads to a change of 0.110 in the value of x_3 and this provides a simple demonstration that a chaotic series is very sensitive to initial conditions.

11.2 Use the following models to analyze the daily returns of the adjusted closing prices of the S&P500 index series, shown in Figure 1.2.

(a) Non-linear autoregressive models;
(b) Threshold autoregressive models;
(c) Smooth transition autoregressive models;
(d) Regime-switching models with two or three regimes;
(e) Neural network autoregressive models.

Chapter 12

Volatility Models

The non-linear models introduced in the previous chapter could be described as having structural non-linearity. They allow improved point forecasts of the observed variable to be made when the true model is known. This chapter considers various classes of non-linear models of a completely different type, which are primarily concerned with modelling *changes in variance* or *volatility*. They do not generally lead to better point forecasts of the measured variable, but may lead to better estimates of the (local) variance. This, in turn, allows more reliable prediction intervals to be computed and hence a better assessment of risk.

Volatility models have many applications in economics and finance. In options trading, volatility shows fluctuation levels of the market price of the underlying asset, and is one of key factors that determine option prices. In risk management, volatility models provide a simple approach to calculating the value at risk of a financial position. Volatility also plays an important role in asset allocation and portfolio optimization. Given the increasing interest in this topic, this chapter introduces various types of univariate volatility models. A brief bibliography is given at the end of the chapter.

12.1 Structure of a Model for Asset Returns

Suppose we have a time series from which any trend and seasonal effects have been removed and from which linear (short-term correlation) effects may also have been removed. We denote this derived series by $\{Y_t\}$, to distinguish it from the original observed series, $\{P_t\}$. Y_t might be the first differences (or the percentage changes) of a financial time series such as the natural log of a share price P_t, i.e.,

$$Y_t = \log P_t - \log P_{t-1}, \quad \text{or} \quad Y_t = (\frac{P_t}{P_{t-1}} - 1) \times 100\%$$

and in this context is often called the **return** or the **growth rate** of a series. For illustration purposes, let P_t be the adjusted closing prices of the Standard & Poor's 500 (S&P500) at the tth trading day, one can calculate its daily returns Y_t at each day; Figure 1.2 shows the daily return series from January 4, 1995 to December 30, 2016.

The basic idea in volatility modeling is that the return series $\{Y_t\}$ has very few serial correlations, but it is a dependent series. To see this, let Y_t

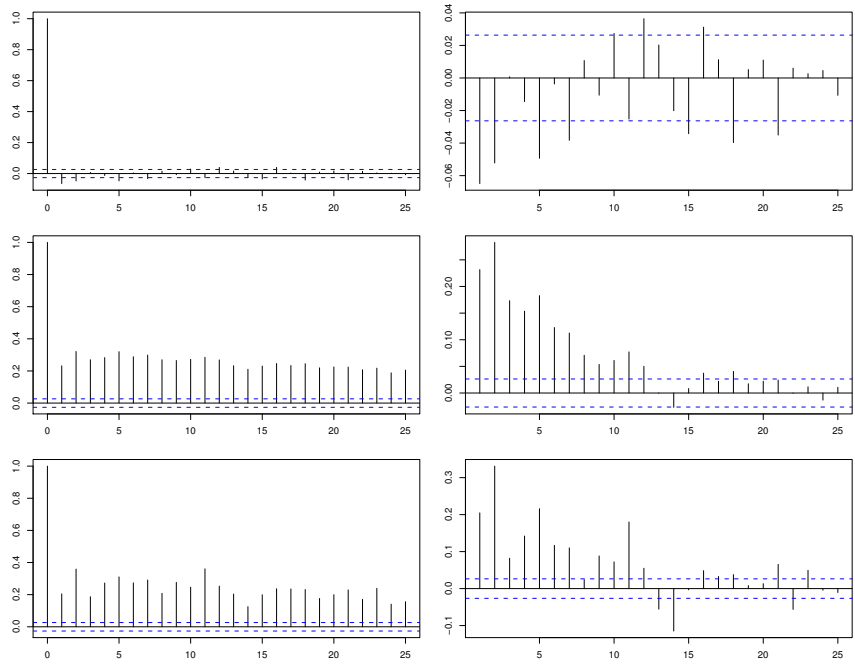

Figure 12.1 *Sample ac.f. (left) and sample partial ac.f. (right) of various functions of the daily returns, Y_t, of adjusted closing prices of S&P500 Index from January 4, 1995 to December 30, 2016. Top: Original series Y_t; Middle: Absoluate values of Y_t; Bottom: Squared values of Y_t.*

denote the daily returns Y_t of the S&P500 Index shown in Figure 1.2, we calculate sample ACFs and sample PACF of Y_t, $|Y_t|$, and Y_t^2, respectively, and show them in Figure 12.1. Note that the sample ACFs of the returns Y_t suggest no significant serial correlations except for small ones at lags 1, 2, and 5. However, the sample ACFs of the absolute and squared returns, $|Y_t|$ and Y_t^2, show strong dependence over all lags. From this example, we find an important feature of the return series; that is, the returns may seem serially uncorrelated, but it is dependent.

To focus our discussion on volatility of a return series, we may further assume that Y_t is the series of residuals from a regression or innovations in a linear time series model. In particular, suppose that X_t follow an ARMA(p, q) model described in Section 3.8,

$$\phi(B)X_t = \theta(B)Y_t,$$

where $\phi(B)$ and $\theta(B)$ are polynomials of B with orders p and q, respectively. Denote $\mathcal{F}_t$ the set of observed data up to time t, i.e., $\{X_1, \ldots, X_t\}$; the

observation X_t can be expressed as

$$X_t = \mu_t + Y_t, \tag{12.1}$$

where μ_t is the mean of X_t conditional on observed data $\mathcal{F}_{t-1}$ and given by

$$\mu_t = E(X_t|\mathcal{F}_{t-1}) = \phi(B)X_t - (\theta(B) - 1)Y_t, \tag{12.2}$$

and the innovation series Y_t has mean 0 and conditional variance

$$\sigma_t^2 = \text{Var}(X_t|\mathcal{F}_{t-1}) = \text{Var}(Y_t|\mathcal{F}_{t-1}). \tag{12.3}$$

Volatility models discussed in the rest of this chapter deal with the evolution of σ_t^2 over time.

12.2 Historic Volatility

For illustration purpose, we may assume the conditional mean of the process X_t is zero, i.e., $X_t = Y_t$. Let σ_t be the volatility of Y_t at time t; then motivated by Equation (12.3), σ_t^2 at time $t-1$ can be simply estimated by the sample variance based on the most recent k observations:

$$\widehat{\sigma}_t^2 = \frac{1}{k-1}\sum_{i=1}^{k}(Y_{t-i} - \bar{Y})^2, \tag{12.4}$$

where $\bar{Y} = \sum_{i=1}^{k} Y_{t-i}/k$. In finance, if σ_t in Equation (12.4) is in the daily basis, then it can be converted to the annual volatility by $\sqrt{A}\sigma$, where the annualizing factor A is the number of trading days, usually taken as around 252. The volatility estimate $\widehat{\sigma}_t$ given by (12.4) is called the *k-day historic volatility*. Figure 12.2 displays historic volatilities with two different window sizes k. Both estimates of σ_t capture important changes in the volatility process. In particular, they all show big volatilities during the 2008-2009 financial crisis. Note that for larger window size k, the time course of historic volatility is smoother.

Historic volatility uses equal weights for the observations in the moving window of returns. Since the most recent observations may have more contributions to levels of volatility, the conditional variance at time t may be approximated by

$$\widehat{\sigma}_t^2 = \sum_{i=1}^{k}\alpha_i(Y_{t-i} - \overline{Y})^2, \tag{12.5}$$

where $\sum_{i=1}^{k}\alpha_i = 1$. One variant of Equation (12.5) is to include a long-run variance rate V in the weighted sum, leading to

$$\widehat{\sigma}_t^2 = \gamma V + \sum_{i=1}^{k}\alpha_i(Y_{t-i} - \overline{Y})^2, \tag{12.6}$$

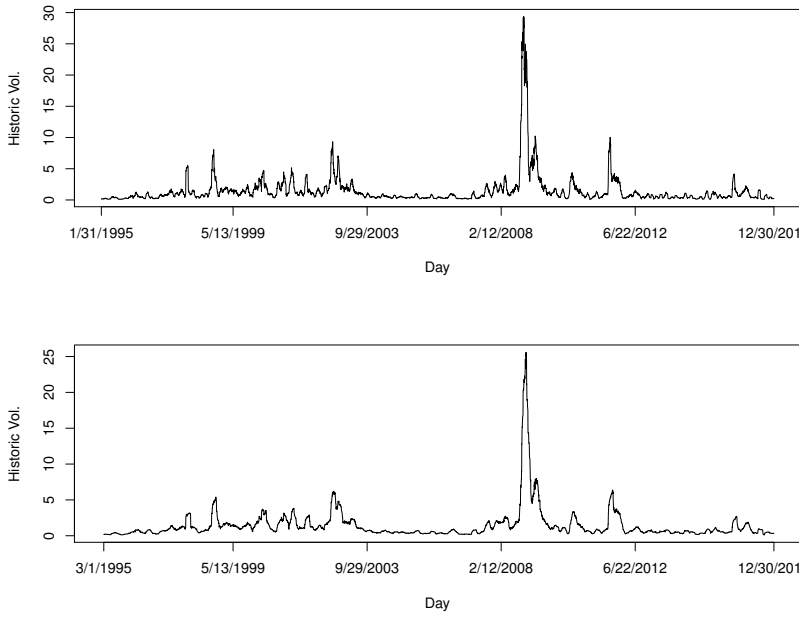

Figure 12.2 *Historic volatility estimated with different window sizes. Top:* $k = 20$*; bottom:* $k = 40$*.*

where $\gamma + \sum_{i=1}^{k} \alpha_i = 1$. Another variant of (12.5) is the *exponentially weighted moving average* (EWMA), which is a special case of Equation (12.5) and assumes that the weights α_i decrease exponentially fast. Specifically, let $k = \infty$, $u_j = 0 = \widehat{\sigma}_j$ for $j \leq 0$, and $\alpha_i = (1 - \lambda)\lambda^{i-1}$ for some $0 < \lambda < 1$; Equation (12.5) becomes

$$\widehat{\sigma}_t^2 = \gamma V + (1 - \lambda) \sum_{i=1}^{\infty} \lambda^{i-1} (Y_{t-i} - \overline{Y})^2, \tag{12.7}$$

which can be recursively represented as

$$\widehat{\sigma}_t^2 = (1 - \lambda)\gamma V + \lambda \widehat{\sigma}_{t-1}^2 + (1 - \lambda)(Y_{t-1} - \overline{Y})^2. \tag{12.8}$$

12.3 Autoregressive Conditional Heteroskedastic (ARCH) Models

To better describe the idea, we may represent Y_t having mean zero in the form

$$Y_t = \sigma_t \epsilon_t, \tag{12.9}$$

where $\{\epsilon_t\}$ denotes a sequence of i.i.d. random variables with zero mean and unit variance, and σ_t may be thought of as the local conditional standard deviation of the process. The ϵ_t may have a normal distribution but this assumption is not necessary for much of what follows. For example, one may assume that ϵ_t follow a standardized Student t-distribution. Let x_ν be a Student t-distribution with $\nu > 2$ degrees of freedom. Then $\epsilon_t = x_\nu / \sqrt{\nu/(\nu-2)}$ has a *standardized Student t-distribution* with variance 1 and probability density function

$$p(\epsilon) = \frac{\Gamma\big((\nu+1)/2\big)}{\Gamma(\nu/2)\sqrt{(\nu-2)\pi}} \left(1 + \frac{\epsilon^2}{\nu-2}\right)^{-(\nu+1)/2}.$$

In any case the unconditional distribution of Y_t generated by a non-linear model will not generally be normal but rather fat-tailed (or leptokurtic). Suppose we additionally assume that the square of σ_t depends on the most recent value of the derived series by

$$\sigma_t^2 = \omega + \sum_{j=1}^{k} \alpha_j Y_{t-j}^2, \tag{12.10}$$

where the parameters ω and $\alpha_1, \ldots, \alpha_k$ are nonnegative and satisfy

$$\alpha_1 + \ldots + \alpha_k < 1$$

to ensure that σ_t^2 is non-negative. A model for Y_t satisfying Equations (12.9) and (12.10) is called an **autoregressive conditionally heteroscedastic model** of order k (ARCH(k)). The adjective 'autoregressive' arises because the value of σ_t^2 depends on past values of the derived series, albeit in squared form. Note that Equation (12.10) does not include an 'error' term and so does not define a stochastic process.

Since (12.10) is similar to the autoregressive model in Chapter 3, for Y_t^2 to be covariance stationary, the roots of the equation $1 - \alpha_1 z - \cdots - \alpha_k z^k = 0$ are required to lie outside the unit circle. The unconditional variance V of Y_t is given by

$$V = \frac{\omega}{1 - \alpha_1 - \cdots - \alpha_k}. \tag{12.11}$$

ARCH(1) model

To understand the ARCH models, we examine the first-order case, i.e., $k = 1$. Then

$$Y_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2,$$

where $\omega > 0$ and $\alpha_1 > 0$. It is clear that the unconditional mean of Y_t is zero, i.e.,

$$E(Y_t) = E(E(Y_t|\mathcal{F}_{t-1})) = E(\sigma_t E(\epsilon_t)) = 0.$$

The unconditional variance of Y_t can be obtained as

$$\mathrm{Var}(Y_t) = E(Y_t^2) = E(\omega + \alpha_1 Y_{t-1}^2) = \omega + \alpha_1 E(Y_{t-1}^2).$$

Given that Y_t is stationary, we have $\mathrm{Var}(Y_t) = \mathrm{Var}(Y_{t-1}) = E(Y_{t-1}^2)$; hence

$$\mathrm{Var}(Y_t) = E(Y_t^2) = \omega/(1-\alpha_1).$$

One nice property of ARCH models is that the tail distribution of Y_t is heavier than that of a normal distribution. To see that, we need to first compute the unconditional kurtosis of Y_t. Assume that Z_t are i.i.d. standard normal random variables; we note that

$$\begin{aligned} E(Y_t^4) &= E[E(\sigma_t^4 \epsilon_t^4 | \mathcal{F}_{t-1})] = 3E[E(Y_t^2 | \mathcal{F}_{t-1})]^2 \\ &= 3E[\omega + \alpha_1 Y_{t-1}^2]^2 = 3E[\omega^2 + 2\omega\alpha_1 E(Y_{t-1}^2) + \alpha_1^2 E(Y_{t-1}^4)]. \end{aligned}$$

If Y_t is fourth-order stationary with $E(Y_t^4)$ being a constant, we can solve the above equation for $E(Y_t^4)$ and obtain that

$$E(Y_t^4) = \frac{3\omega(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

Therefore, the unconditional kurtosis of Y_t is given by

$$\kappa := \frac{E(Y_t^4)}{[\mathrm{Var}(Y_t)]^2} = 3\frac{1-\alpha_1^2}{1-3\alpha_1^2}.$$

When the fourth moment of Y_t is positive, we see that $1 - 3\alpha_1^2 > 0$ and then the unconditional kurtosis of Y_t is larger than 3, which is the unconditional kurtosis of a normal distribution. This shows that the tail distribution of Y_t is heavier than that of a normal distribution; hence the innovation process Y_t in a Gaussian ARCH(1) model tends to generate more 'outliers' than a Gaussian white noise process.

These properties also hold for general ARCH models with higher orders, but the argument become more complicated.

Estimation and prediction

Parameters in ARCH models can be estimated by maximzing their likelihood, which usually needs to be done numerically.

It can be shown that ARCH models are martingale differences (MDs), so that knowledge of the value of σ_t does not lead to improved point forecasts of Y_{t+h}. The value of modelling $\{\sigma_t\}$ lies in getting more reliable bounds for prediction intervals for Y_{t+h} and in assessing risk more generally. When the derived series $\{Y_t\}$ has mean zero, the point forecast of Y_{t+1} is zero and prediction intervals are typically calculated using the appropriate percentage point of the standard normal distribution, even when there are doubts about

the normality assumption (though alternative heavy-tailed distributions could be used).

Consider an ARCH(k) model. At the forecast origin N, the 1-step-ahead forecast for σ^2_{N+1} is

$$\widehat{\sigma}^2_N(1) = E(\sigma^2_{N+1}|\mathcal{F}_N) = \omega + \alpha_1 Y^2_N + \ldots + \alpha_k Y^2_{N+1-k},$$

and the l-step-ahead forecast for σ^2_{N+l} is

$$\widehat{\sigma}^2_N(l) = E(\sigma^2_{N+l}|\mathcal{F}_N) = \omega + \sum_{i=1}^{k} \alpha_i \widehat{\sigma}^2_N(l-k),$$

where $\widehat{\sigma}^2_N(l-k) = E(Y^2_{N+l-k}|\mathcal{F}_N)$ if $l > k$, and $\widehat{\sigma}^2_N(l-k) = Y^2_{N+l-k}$ if $l \leq k$. Thus, the one-step-ahead $100(1-\alpha')\%$ prediction interval for Y_{t+1} is typically taken to be of the form $0 \pm z_{\alpha'/2}\hat{\sigma}_t$, where $z_{\alpha'/2}$ denotes the value of the standard normal distribution for which the probability of being exceeded is $\alpha'/2$.

Example 12.1

We now apply the modeling procedure to build a simple ARCH model for the daily returns, Y_t, of adjusted closing prices of the S&P500 Index. Figure 1.2 shows the daily return series from January 4, 1995 to December 30, 2016. The sample ACF and PACF of the squared returns in the middle and bottom panels of Figure 12.1 show the existence of conditional heteroscedasticity. We consider an ARCH(2) model with the following specification for the daily return series

$$X_t = \mu + Y_t, \qquad Y_t = \sigma_t \epsilon_t, \qquad \sigma^2_t = \omega + \alpha_1 Y^2_{t-1} + \alpha_2 Y^2_{t-2}.$$

We use the `R` function `garchFit` in the `R` package `fGarch` to estimate the model. Assuming that ϵ_t are i.i.d. standard normal, we obtain the fitted model

$$\begin{aligned} X_t &= 0.0807_{(.0125)} + Y_t, \qquad Y_t = \sigma_t \epsilon_t, \\ \sigma^2_t &= 0.6580_{(.0216)} + 0.1965_{(.0202)} Y^2_{t-1} + 0.3523_{(.0246)} Y^2_{t-2}, \end{aligned}$$

where the standard errors of the parameters are given in the parenthses; see some of the `R` output below. Figure 12.3 shows the estimated volatility $\widehat{\sigma}_t$, the standardized residual $\widehat{\epsilon}_t$, and the same ac.f.'s and sample partial ac.f.'s of $\widehat{\epsilon}_t$ and $\widehat{\epsilon}^2_t$, respectively. Note that both the output and the plot show that the estimated residuals still have conditional heteroscedasticities, and hence indicates that the ARCH(2) model is not adequate.

```
> library("fGarch")
> fit1<-garchFit(~garch(2,0), data=sp500[,2])
> summary(fit1)
```

```
Title: GARCH Modelling

Call: garchFit(formula = ~garch(2, 0), data = sp500[, 2])

Conditional Distribution: norm

Coefficient(s):
     mu    omega   alpha1   alpha2
0.08072  0.65795  0.19649  0.35226

Std. Errors: based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu       0.08072     0.01252    6.447 1.14e-10 ***
omega    0.65795     0.02160   30.464  < 2e-16 ***
alpha1   0.19649     0.02021    9.720  < 2e-16 ***
alpha2   0.35226     0.02456   14.343  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood: -8184.593    normalized:  -1.47763

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  1909.616  0
 Shapiro-Wilk Test  R    W      NA        NA
 Ljung-Box Test     R    Q(10)  25.34687  0.004725394
 Ljung-Box Test     R    Q(15)  40.03293  0.0004483369
 Ljung-Box Test     R    Q(20)  43.53819  0.001734486
 Ljung-Box Test     R^2  Q(10)  363.8295  0
 Ljung-Box Test     R^2  Q(15)  563.6726  0
 Ljung-Box Test     R^2  Q(20)  730.354   0
 LM Arch Test       R    TR^2   430.027   0

Information Criterion Statistics:
     AIC      BIC      SIC     HQIC
2.956704 2.961485 2.956703 2.958371

> # Plot the result
> fit1.vol<-volatility(fit1)
> fit1.resid.st<-residuals(fit1)/volatility(fit1)
> x.pos<-c(seq(1,n,1400),n)
> par(mfrow=c(3,2), mar=c(2,2,2,2))
```

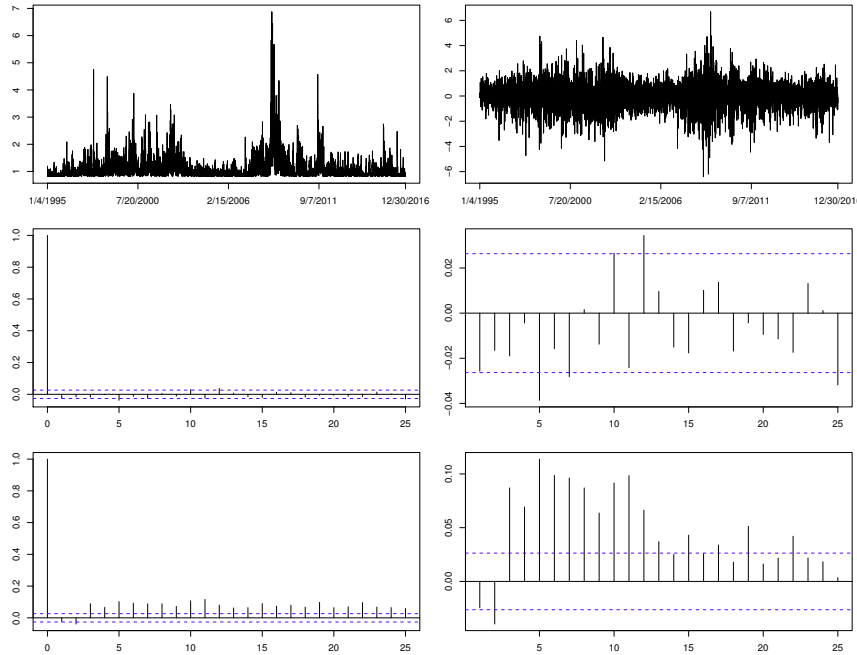

Figure 12.3 *Estimated volatility* $\widehat{\sigma}_t$ *(top left), the standardized residual* $\widehat{\epsilon}_t$ *(top right), and the same ac.f.'s and sample partial ac.f.'s of* $\widehat{\epsilon}_t$ *(middle) and* $\widehat{\epsilon}_t^2$ *(bottom), respectively, in the ARCH(2) model for daily returns of adjusted closing prices of the S&P500 Index from January 4, 1995 to December 30, 2016.*

```
> plot(fit1.vol, type="l", xlab="", ylab="", xaxt="n")
> axis(1, x.pos, sp500$Date[x.pos])
> plot(fit1.resid.st, type="l",xlab="", ylab="", xaxt="n")
> axis(1, x.pos, sp500$Date[x.pos])
> acf(fit1.resid.st, 25, ylab="", main="")
> acf(fit1.resid.st, 25, type="partial", ylab="", main="")
> acf(fit1.resid.st^2, 25, ylab="", main="")
> acf(fit1.resid.st^2, 25, type="partial", ylab="", main="")
```

12.4 Generalized ARCH Models

The ARCH model has been generalized to allow the variance to depend on past values of σ_t^2 as well as on past values of Y_t^2. A derived variable satisfying Equation (12.9) is said to follow a **generalised ARCH** (or GARCH) model

of order (h, k) when the local conditional variance is given by

$$\sigma_t^2 = \omega + \sum_{j=1}^{h} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{k} \alpha_i Y_{t-i}^2, \tag{12.12}$$

where $\omega \geq 0$ and $\alpha_i, \beta_j \geq 0$ for all i, j.

The GARCH(h, k) model (12.12) can be considered as an ARMA model of volatility with martingale difference innovations: Let $\beta_i = 0$ if $i > h$, $\alpha_j = 0$ if $j > k$, and $\eta_t = Y_t^2 - \sigma_t^2$. Note that

$$E(\eta_t) = E(\sigma_t^2 \epsilon_t^2) - E(\sigma_t^2) = E(\epsilon_t^2 E(\sigma_t^2 | \mathcal{F}_{t-1})) - E(\sigma_t^2) = 0.$$

Then η_t is a martingale difference. Moreover,

$$Y_t^2 = \omega + \sum_{j=1}^{\max(h,k)} (\alpha_j + \beta_j) Y_{t-j}^2 + \eta_t - \sum_{i=1}^{h} \beta_i \eta_{t-i}. \tag{12.13}$$

Hence the same invertibility and stationarity assumptions of ARMA models apply to GARCH models. For example, to ensure that Y_t is covariance stationary, it is required that all roots of $1 - \sum_{j=1}^{\max(h,k)} (\alpha_j + \beta_j) z^j = 0$ lie outside the unit circle. Since the α's and β's are usually assumed to be nonnegative, then we have

$$\sum_{j=1}^{k} \alpha_j + \sum_{i=1}^{h} \beta_i < 1. \tag{12.14}$$

Under (12.14), the unconditional variance of u_t is given by

$$E(u_t^2) = \frac{\omega}{1 - \sum_{i=1}^{h} \beta_i - \sum_{j=1}^{k} \alpha_j}. \tag{12.15}$$

GARCH(1,1) model

To understand properties of GARCH models, we examine the GARCH(1, 1) case,

$$Y_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2, \tag{12.16}$$

where $\omega > 0$ and $\alpha, \beta > 0$. Note that The GARCH(1, 1) model is closely related to the EWMA estimate (12.8).

The model implies that a large Y_{t-1}^2 or σ_{t-1}^2 will lead to a large σ_t^2, which in turn will give rise to a large $Y_t^2 = \sigma_t^2 \epsilon_t^2$. This is consistent with the volatility clustering observed in financial time series; hence, the GARCH(1, 1) model is often used to fit financial time series. Moreover, even when ϵ_t is standard normal, GARCH(1, 1) can still be highly leptokurtic since

$$\frac{E(Y_t^4)}{[\mathrm{Var}(Y_t)]^2} = \frac{3[1 - (\alpha + \beta)^2]}{1 - (\alpha + \beta)^2 - 2\alpha^2} > 3 \tag{12.17}$$

when $(\alpha+\beta)^2+2\alpha^2<1$.

Recall that $\eta_t = Y_t^2 - \sigma^2$ and let $\lambda = \alpha+\beta$; the GARCH(1, 1) model (12.16) can be expressed as

$$(1-\lambda B)\sigma_t^2 = \omega + \alpha\eta_{t-1},$$

where B is the backward shift operator. Therefore, when Y_t is covariance stationary, we have

$$\begin{aligned}\sigma_t^2 &= \frac{1}{1-\lambda B}(\omega+\alpha\eta_{t-1}) \\ &= \frac{\omega}{1-\lambda} + \alpha\big(1+\lambda B+\ldots+\lambda^i B^i+\ldots\big)\eta_{t-1} \\ &= \frac{\omega}{1-\lambda} + \alpha\big(\eta_{t-1}+\lambda\eta_{t-2}+\ldots+\lambda^i\eta_{t-1-i}\big).\end{aligned}$$

Note that as λ approaches 1, the weight of η_{t-i} in the above equation also approaches 1, indicating the persistent effect of events that occurred a long time ago on current volatility.

Prediction of volatilities

Similar to ARCH models, GARCH models do not affect point forecasts of the original observed variable, and it is therefore rather difficult to make a fair comparison of the forecasting abilities of different models for changing variance. Thus the modelling aspect (understanding the changing structure of a series), and the assessment of risk, are both more important than their ability to make point forecasts.

Replacing time index t by $t+1$ in (12.16) and taking expectations on both sides yields the one-step-ahead forecast of σ_{N+1}^2 at the forecast origin N,

$$\widehat{\sigma}_N^2(1) = \omega + \alpha Y_N^2 + \beta\sigma_N^2,$$

where σ_N can be estimated by fitting the model (12.16) to observations $Y_1,\ldots,Y_N$. To obtain a k-step-ahead forecast of σ_{N+k}^2, let $\sigma^2 = \omega/(1-\alpha-\beta)$. The GARCH(1, 1) model (12.16) can be written in the form

$$\sigma_t^2 - \sigma^2 = \alpha(Y_{t-1}^2-\sigma^2) + \beta(\sigma_{t-1}^2-\sigma^2). \tag{12.18}$$

Replacing time index t by future time $N+k$ in (12.18) yields

$$\sigma_{N+k}^2 - \sigma^2 = \alpha(Y_{N+k-1}^2-\sigma^2) + \beta(\sigma_{N+k-1}^2-\sigma^2).$$

This suggests the k-step-ahead forecast of σ_{N+k}^2 is

$$\widehat{\sigma}_N^2(k) := E(\sigma_{N+k}^2|\mathcal{F}_N) = \sigma^2 + \lambda^{k-1}\big\{\alpha(Y_N^2-\sigma^2)+\beta(\sigma_N^2-\sigma^2)\big\}. \tag{12.19}$$

Since $\sigma^2 = \omega/(1-\lambda)$, then

$$\widehat{\sigma}_N^2(k) = \omega(1-\lambda^k)/(1-\lambda) + (\alpha Y_N^2+\beta\sigma_N^2)\lambda^{k-1}. \tag{12.20}$$

Example 12.2

We now consider a GARCH(1, 1) model with the following specification for the daily return series of the S&P500 Index

$$X_t = \mu + Y_t, \qquad Y_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2.$$

We still use the `R` function `garchFit` to estimate the model. Assuming that ϵ_t are i.i.d. standard normal, we obtain the fitted model

$$\begin{aligned} X_t &= 0.0740_{(.0111)} + Y_t, \qquad Y_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.0189_{(.0029)} + 0.1005_{(.0085)} Y_{t-1}^2 + 0.8857_{(.0092)} \sigma_{t-1}^2, \end{aligned}$$

where the standard errors of the parameters are given in the parentheses; see the `R` output below. Figure 12.4 shows the estimated volatility $\widehat{\sigma}_t$, the standardized residual $\widehat{\epsilon}_t$, and the same ac.f.'s and sample partial ac.f.'s of $\widehat{\epsilon}_t$ and $\widehat{\epsilon}_t^2$, respectively. Comparing it to Figure 12.3, the standardized residuals in the fitted GARCH(1,1) model show little conditional heteroscedasticity, indicating the model is adequate.

```
> fit2<-garchFit(~garch(1,1), data=sp500[,2])
> summary(fit2)

Title: GARCH Modelling

Call: garchFit(formula = ~garch(1, 1), data = sp500[, 2])

Conditional Distribution: norm

Coefficient(s):
      mu     omega    alpha1     beta1
0.073951  0.018870  0.100541  0.885716

Std. Errors: based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.073951    0.011127    6.646 3.01e-11 ***
omega   0.018870    0.002920    6.464 1.02e-10 ***
alpha1  0.100541    0.008528   11.790  < 2e-16 ***
beta1   0.885716    0.009208   96.186  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood:
 -7703.91    normalized:  -1.390849
```

```
Description:
 Sun Nov 12 11:59:15 2017 by user:

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  692.0824  0
 Shapiro-Wilk Test  R    W      NA        NA
 Ljung-Box Test     R    Q(10)  25.4232   0.004598514
 Ljung-Box Test     R    Q(15)  34.91386  0.002529427
 Ljung-Box Test     R    Q(20)  39.16812  0.006353632
 Ljung-Box Test     R^2  Q(10)  14.55688  0.1490715
 Ljung-Box Test     R^2  Q(15)  21.04238  0.1354807
 Ljung-Box Test     R^2  Q(20)  22.33248  0.3227451
 LM Arch Test       R    TR^2   14.79585  0.2527914

Information Criterion Statistics:
     AIC      BIC      SIC     HQIC
2.783141 2.787922 2.783140 2.784808
```

12.5 The ARMA-GARCH Models

The linear time series models in Chapter 3 can be combined with GARCH to model the dynamics of asset returns and their volatilities. Assuming that X_t follows an ARMA model with GARCH innovations yields the following ARMA(p, q)-GARCH(h, k) model for (X_t, σ_t):

$$X_t = \mu + \sum_{i=1}^{p} \phi_i X_{t-i} + Y_t + \sum_{j=1}^{q} \psi_j Y_{t-j}, \qquad Y_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \omega + \sum_{j=1}^{k} \alpha_j Y_{t-j}^2 + \sum_{i=1}^{h} \beta_i \sigma_{t-i}^2. \tag{12.21}$$

The ϵ_t in (12.21) are i.i.d. standard normal or standardized Student-t random variables. The stationarity conditions for the ARMA part and the GARCH part are the same as those in Sections 3.8 and 12.4. The second equation in (12.21) can be replaced by other volatility models so that different aspects of volatilities can be characterized.

Given the model parameters, the two equations in (12.21) can be used to obtain one-step-ahead forecasts of the conditional mean and conditional variance of r_t. Specifically, from Sections 5.3 and 12.4, we have the one-step-ahead forecasts at the forecast origin N

$$\widehat{X}_N(1) = \mu + \sum_{i=1}^{p} \phi_i X_{N+1-i} + \sum_{j=1}^{q} \psi_j Y_{N+1-j}, \tag{12.22}$$

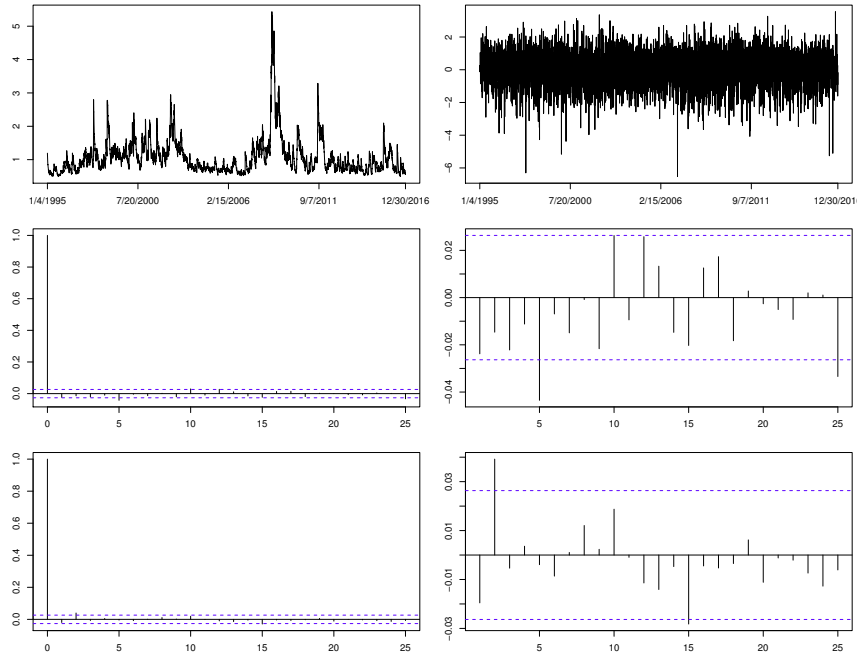

Figure 12.4 *Estimated volatility $\widehat{\sigma}_t$ (top left), the standardized residual $\widehat{\epsilon}_t$ (top right), and the same ac.f.'s and sample partial ac.f.'s of $\widehat{\epsilon}_t$ (middle) and $\widehat{\epsilon}_t^2$ (bottom), respectively, in the GARCH(1, 1) model for daily returns of adjusted closing prices of the S&P500 Index from January 4, 1995 to December 30, 2016.*

$$\widehat{\sigma}_N^2(1) = \omega + \sum_{j=1}^{k} \alpha_j Y_{N+1-j}^2 + \sum_{j=1}^{h} \beta_i \sigma_{N+1-i}^2. \tag{12.23}$$

The conditional distribution of X_{N+1} given the current and past observations up to time N is $N(\widehat{X}_N(1), \widehat{\sigma}_N^2(1))$.

The k-step-ahead forecast $\widehat{X}_N(k)$ can be evaluated by using the method for the ARMA model r_t described in Section 5.3.1. To evaluate the k-step-ahead forecast $\widehat{\sigma}_N^2(k)$, we first use the MA(∞) representation

$$X_t = \mu + \sum_{i=0}^{\infty} \psi_i Y_{t-i}$$

with $\psi_0 = 1$ (see Section 5.3.1). Therefore the k-step-ahead forecast of X_{N+k} can be expressed as

$$\widehat{X}_N(k) = \mu + \sum_{i=k}^{\infty} \psi_i Y_{t+k-i},$$

and the corresponding forecast error is

$$e_N(k) = X_{N+k} - \widehat{X}_N(k) = \sum_{i=1}^{k} \psi_{k-i} Y_{t+i}.$$

From this and the property $E(Y_{N+i}^2|\mathcal{F}_{N+i-1}) = \sigma_{N+i}^2$, which has been used to derive (12.19), it follows that the variance of the forecast error is

$$\text{Var}(e_N(k)) = \sum_{i=1}^{k} \psi_{k-i}^2 E(\sigma_{N+i}^2|\mathcal{F}_N) = \sum_{i=1}^{k} \psi_{k-i}^2 \sigma_N^2(i), \tag{12.24}$$

in which $\sigma_N^2(i)$ are given by (12.19).

Example 12.3

We now consider an ARMA(1, 1)-GARCH(1, 1) model with the following specification for the daily return series of the S&P500 Index

$$\begin{aligned} X_t &= \mu + \phi X_{t-1} + Y_t + \psi Y_{t-1}, \\ Y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

We still use the `R` function `garchFit` to estimate the model. Assuming that ϵ_t are i.i.d. standard normal, we obtain the fitted model

$$\begin{aligned} r_t &= 0.0066_{(.0042)} + 0.9116_{(.0540)} X_{t-1} + Y_t - 0.9373_{(.0456)}, \\ Y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.0185_{(.0029)} + 0.0994_{(.0084)} Y_{t-1}^2 + 0.8871_{(.0091)} \sigma_{t-1}^2, \end{aligned}$$

where the standard errors of the parameters are given in the parenthses; see the `R` output below. Figure 12.5 shows the estimated volatility $\widehat{\sigma}_t$, the standardized residual $\widehat{\epsilon}_t$, and the same ac.f.'s and sample partial ac.f.'s of $\widehat{\epsilon}_t$ and $\widehat{\epsilon}_t^2$, respectively.

```
> fit3<-garchFit(~arma(1,1)+garch(1,1), data=sp500[,2])
> fit3.vol<-volatility(fit3)
> fit3.resid.st<-residuals(fit3)/volatility(fit3)

> summary(fit3)

Title: GARCH Modelling

Call: garchFit(formula = ~arma(1, 1) + garch(1, 1),
           data = sp500[, 2])
```

```
Mean and Variance Equation: data ~ arma(1, 1) + garch(1, 1)

Conditional Distribution: norm

Coefficient(s):
       mu       ar1        ma1     omega    alpha1     beta1
0.0066356 0.9115801 -0.9372960 0.0184682 0.0993799 0.8871178

Std. Errors: based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.006636    0.004237    1.566    0.117
ar1     0.911580    0.054019   16.875  < 2e-16 ***
ma1    -0.937296    0.045582  -20.563  < 2e-16 ***
omega   0.018468    0.002859    6.461 1.04e-10 ***
alpha1  0.099380    0.008421   11.801  < 2e-16 ***
beta1   0.887118    0.009078   97.719  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood: -7693.559    normalized:  -1.38898

Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  750.2105  0
 Shapiro-Wilk Test  R    W      NA        NA
 Ljung-Box Test     R    Q(10)  15.42737  0.1172401
 Ljung-Box Test     R    Q(15)  26.54412  0.0326778
 Ljung-Box Test     R    Q(20)  32.5605   0.03767997
 Ljung-Box Test     R^2  Q(10)  16.5621   0.08463268
 Ljung-Box Test     R^2  Q(15)  22.95862  0.0850223
 Ljung-Box Test     R^2  Q(20)  24.22309  0.232783
 LM Arch Test       R    TR^2   16.26175  0.1795395

Information Criterion Statistics:
     AIC      BIC      SIC     HQIC
2.780126 2.787296 2.780124 2.782626
```

12.6 Other ARCH-Type Models

Many other types of ARCH models have been proposed. We introduce two of them in this section. We shall note that, it is difficult to choose between different models for changing variance using the data alone and so

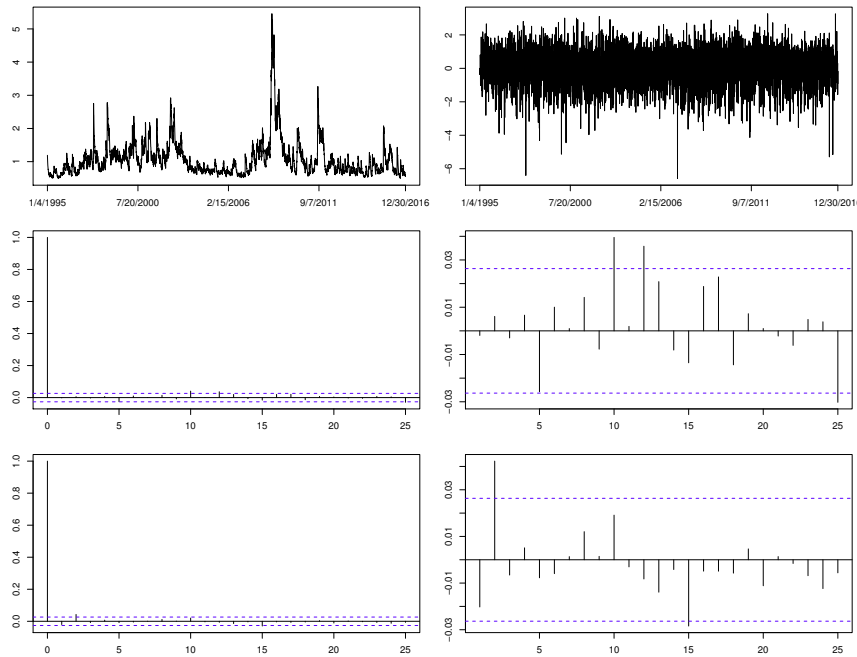

Figure 12.5 *Estimated volatility $\widehat{\sigma}_t$ (top left), the standardized residual $\widehat{\epsilon}_t$ (top right), and the same ac.f.'s and sample partial ac.f.'s of $\widehat{\epsilon}_t$ (middle) and $\widehat{\epsilon}_t^2$ (bottom), respectively, in the ARMA(1,1)-GARCH(1, 1) model for daily returns of adjusted closing prices of the S&P500 Index from January 4, 1995 to December 30, 2016.*

it is advisable, as usual, to use the context and any background theory, to supplement the results of the exploratory analysis of the data.

12.6.1 The integrated GARCH model

In many empirical analyses of financial time series, it is found that the volatilities of asset returns are highly persistent, since the $\alpha + \beta$ values in the fitted GARCH(1, 1) models are very close to 1. If $\alpha + \beta = 1$, the GARCH(1, 1) model becomes an **integrated GARCH** model

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + (1-\beta)u_{t-1}^2,$$

More generally, the integrated GARCH model, IGARCH(h, k), is of the form

$$\sigma_t^2 = \omega + \sum_{j=1}^{h} \beta_j \sigma_{t-j}^2 + \sum_{i=1}^{k} \alpha_i Y_{t-i}^2, \tag{12.25}$$

with $\sum_{j=1}^{k} \alpha_j + \sum_{i=1}^{h} \beta_i = 1$.

Note that the above condition implies that IGARCH(h, k) models have infinite unconditional variance, which can be seen by letting $\sum_{i=1}^{h} \beta_i + \sum_{j=1}^{k} \alpha_j$ approach 1 in (12.15), hence IGARCH models cannot be covariance stationary. However, IGARCH models are in fact strictly stationary. Another interesting property of IGARCH models is that, the k-step-ahead forecast of σ^2_{N+k} at forecast origin N can be very simple. In particular, letting $\lambda \to 1$ in (12.20) yields the following forecast of σ^2_{N+k}

$$\sigma^2_N(k) = E(\sigma^2_{t+k}|\mathcal{F}_t) = k\omega + \beta\sigma^2_t + (1-\beta)u^2_t. \tag{12.26}$$

12.6.2 The exponential GARCH model

As indicated in many empirical studies, a stylized fact of the volatility of asset returns is that the volatility response to a large positive return is considerably smaller than that of a negative return of the same magnitude. The GARCH model, which is defined by σ^2_t and Y^2_{t-j}, cannot incorporate this leverage effect. To accommodate the asymmetry, Nelson (1991) proposed the **exponential GARCH** model, EGARCH(h, k), that has the form

$$Y_t = \sigma_t\epsilon_t, \quad \log(\sigma^2_t) = \omega + \sum_{i=1}^{h} \beta_i \log(\sigma^2_{t-i}) + \sum_{j=1}^{k} f_j(\epsilon_{t-j}), \tag{12.27}$$

where the ϵ_t are i.i.d. with mean 0 and $f_j(\epsilon) = \alpha_j\epsilon + \gamma_j(|\epsilon| - E|\epsilon|)$. Note that the random variable $f_j(\epsilon_t)$ is the sum of two zero-mean random variables $\alpha_j\epsilon_t$ and $\gamma_j(|\epsilon_t| - E|\epsilon_t|)$. We can rewrite $f_j(\epsilon_t)$ as

$$f_j(\epsilon_t) = \begin{cases} (\alpha_j + \gamma_j)\epsilon_t - \gamma_j E|\epsilon_t|, & \text{if } \epsilon_t \geq 0, \\ (\alpha_j - \gamma_j)\epsilon_t - \gamma_j E|\epsilon_t|, & \text{if } \epsilon_t < 0, \end{cases}$$

which shows the asymmetry of the volatility response to positive and negative returns. Since (12.27) represents $\log(\sigma^2_t)$ in ARMA form with innovations $f_j(\epsilon_{t-j})$, σ^2_t and therefore Y_t also are stationary if the roots of $1 - \beta_1 z - \ldots - \beta_h z^h = 0$ lie outside the unit circle; see Section 3.7.2.

12.7 Stochastic Volatility Models

The reader will notice that the formulae for σ^2_t in all GARCH type models introduced in this chapter are essentially deterministic in that there is no 'error' term in either equation. An alternative to ARCH or GARCH models is to assume that σ_t follows a stochastic process. This is usually done by modelling the logarithm of σ^2_t or of σ_t to ensure that σ^2_t remains positive. A simple example is to assume that $\log(\sigma^2_t) = h_t$, say, follows an AR process with an 'error' component that is independent of the $\{\epsilon_t\}$ in the innovation series $\{Y_t\}$, that is,

$$\begin{aligned} Y_t &= \sigma_t\epsilon_t, \qquad \sigma^2_t = e^{h_t}, \\ h_t &= \phi_0 + \phi_1 h_{t-1} + \ldots + \phi_p h_{t-p} + \eta_t, \end{aligned} \tag{12.28}$$

which has AR(p) dynamics for $\log \sigma_t^2$. Models of this type are called **stochastic volatility** or **stochastic variance models**. The ϵ_t and η_t in (12.28) are assumed to be independent normal random variables with $\epsilon_t \sim N(0,1)$ and $\eta_t \sim N(0,\sigma^2)$. A complication of the SV model is that unlike in usual AR(p) models, the h_t in (12.28) is an unobserved state undergoing AR(p) dynamics, while the observations are u_t such that $u_t|h_t \sim N(0, e^{h_t})$. The likelihood function of $\boldsymbol{\theta} = (\sigma, \phi_0, \ldots, \phi_p)^T$, based on a sample of n observations $u_1, \ldots, u_n$, involves n-fold integrals, making it prohibitively difficult to compute the MLE by numerical integration for usual sample sizes.

Although the likelihood function is more difficult to handle, the model ties in more naturally with other finance models and is easier to generalize to the multivariate case. Moreover it seems intuitively more reasonable to assume that σ_t changes stochastically through time rather than deterministically, especially when one sees the sudden changes in volatility that can occur in the financial market as a result of a special event like a war involving oil-producing countries. More details may be found in Harvey (1993, Section 8.4). Taylor (1994) suggests that a judicious combination of both ARCH and stochastic volatility models may provide more satisfactory results than a single model.

12.8 Bibliography

Further information about ARCH and GARCH models is given by Bollerslev et al. (1992, 1994), Enders (1995, Chapter 3), Franses (1998, Chapter 7), Gourieroux (1997), Shephard (1996), and Lai and Xing (2008, Chapters 6 and 9).

Exercises

12.1 Consider an ARCH(1,1) model

$$Y_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = 0.1 + 0.5\sigma_{t-1}^2,$$

where ϵ_t are i.i.d. standard Gaussian random variables. What are the unconditional variance and unconditional kurtosis of Y_t?

12.2 Use the following `R` script to simulate an ARCH(1,1) process given in Exercise 12.1.

```
> set.seed(10)
> n<-500
> eps<-rnorm(n)
> sigma2<-rep(n)
> sigma2[1]<-0.1+0.5*1
> for (i in 2:n)        sigma2[i]<-0.1+0.5*sigma2[i-1]
> vol<-sqrt(sigma2)
> y<-vol*eps
```

(a) Plot the series Y_t and σ_t.

(b) Plot the sample ac.f. and sample ac.f. of Y_t, $|Y_t|$, Y_t^2, and σ_t^2.

12.3 Show that the kurtosis of the series generated by the GARCH(1, 1) model (12.16) is given by (12.17).

12.4 Consider an AR(1)-GARCH(1,1) model

$$X_t = \theta X_{t-1} + Y_t, \quad Y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha Y_{t-1}^2 + \beta \sigma_{t-1}^2.$$

(a) What are the 1-step and 2-step ahead forecasts of X_{n+1} at the forecast origin X_n?

(a) What are the 1-step and 2-step ahead forecasts of σ_{n+1}^2 at the forecast origin X_n?

12.5 Consider the daily returns, Y_t, of adjusted closing prices of the S&P500 Index, which has been analyzed in Examples 12.1-12.3. Fit a GARCH(2, 1) model with the following specification for Y_t

$$X_t = \mu + Y_t, \quad Y_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \omega + \alpha_1 Y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2.$$

(a) Compare the maximized log-likelihood and AIC of a GARCH(2,1) model with that of an ARCH(2) model (Example 12.1), a GARCH(1,1) model (Example 12.2), and an ARMA(1,1)-GARCH(1,1) model.

(b) What are the 1-step and 2-step ahead forecasts of σ_{n+1}^2 at the forecast origin X_n?

Chapter 13

Multivariate Time Series Modelling

13.1 Introduction

Observations are often taken simultaneously on two or more time series. For example, in meteorology we might observe temperature, air pressure and rainfall at the same site for the same sequence of time points. In economics, many different measures of economic activity are typically recorded at regular intervals. Examples include the retail price index, the gross domestic product and the level of unemployment. Given multivariate data like these, it may be helpful to develop a multivariate model to describe the interrelationships among the series.

Section 5.3.1 briefly discussed some multivariate forecasting procedures, focusing particularly on the use of multiple regression models. This chapter takes a more detailed look at some multivariate time-series models, giving particular attention to vector autoregressive (VAR) models.

The enormous improvement in computing capability over recent years has made it much easier to fit a given multivariate model from a computational point of view. However, the overall model-building process is still much more difficult for multivariate than univariate models. In particular, there are typically (far) more parameters to estimate than in the univariate case. Moreover, the pool of candidate models is much wider, and this can be seen as a strength – more possibilities – or as a weakness – it is harder to find the 'right' model. As regards the latter point, it appears that multivariate models are more vulnerable to misspecification than simpler univariate models, and this emphasizes the importance of getting sufficient background information so as to understand the context and identify all relevant explanatory variables before starting the modelling process. As always, it is vital to ask appropriate questions and formulate the problem carefully. An iterative approach to model building (see Section 4.8) is generally required, and the use to which the model will be put should be considered as well as the goodness of fit. There is always tension between seeking a parsimonious model (so that fewer parameters need to be estimated) while ensuring that important effects are not mistakenly omitted from the model.

With multivariate time-series data, the modelling process is complicated by the need to model the serial dependence *within* each series, as well as the interdependence *between* series. Information about the latter is provided by the cross-correlation function, which was introduced in Section 8.1. However, as noted in Section 8.1.3, the interpretation of cross-correlations is difficult

and choosing appropriate candidate models is not easy. There also tend to be more problems with the data in the multivariate case. Having more variables to measure means that there is more chance of mistakes in the data. Moreover, multivariate data are sometimes unsuitable for fitting multivariate models, perhaps because some explanatory variables have been held more or less constant in the past.

While univariate models can be very useful for describing short-term correlation effects, for forecasting large numbers of series, and as a benchmark in comparative forecasting studies, it is clear that multivariate models should also have much to offer in gaining a better understanding of the underlying structure of a given system and (hopefully) in getting better forecasts. Sadly, as noted in Section 5.4.2, the latter does not always happen. While multivariate models can usually be found that give a better *fit* than univariate models, there are a number of reasons why better forecasts need not necessarily result (though of course they sometimes do). We have seen that multivariate models, being more complicated, are generally more difficult to fit than univariate ones, while multivariate data may leave much to be desired. Furthermore, multivariate forecasts may require values of explanatory variables that are not yet available and so must themselves be forecast. If this cannot be done very accurately, then poor forecasts of the response variable may also result. Overall, the analyst should be prepared for the possibility that multivariate forecasts are not always as good as might be expected.

13.1.1 One equation or many?

One basic question is whether the model should involve a single equation or several equations. In multiple regression, for example, the model explains the variation in a single **response** variable, say y, in terms of the variation in one or more **predictor**, or **explanatory**, variables, say $x_1, x_2, \ldots$. This is done with the aid of a single equation. For a single-equation model to be appropriate, there must be only one response variable of interest (e.g. forecasts are only required for this one variable), and there should be no suggestion that the value of the response variable could itself affect the predictor variables. In other words, the regression equation assumes there is an **open-loop** system (see Figure 9.7). If the relationship between a single predictor variable and a single response variable can be modelled by a regression equation, some people would say that there is a **causal relationship** between the variables, though in practice it may be difficult to decide whether there is a direct link or if the link comes via relationships with a third, possibly unobserved, variable.

A completely different situation arises when the 'outputs' affect the 'inputs' so that there is **feedback** in a **closed-loop** system (see Figure 9.8). For example, in economics, we know that a rise in prices will generally lead to a rise in wages, which will in turn lead to a further rise in prices. Then a regression model is not appropriate. Instead, a model with more than one equation will be needed to satisfactorily model the system.

One type of model with more than one equation is the econometric simultaneous equation model (see Section 5.3.2). This model comprises a number of equations, which need not be linear, and which are generally constructed using economic theory. By including relevant policy variables, they may be used to evaluate alternative economic strategies, as well as improving understanding of the system (the economy) and producing forecasts (although the latter may not be the prime objective). Models of this type are typically constructed by econometricians and so we do not attempt to describe the modelling process in this time-series text, as the problems are usually more of an economic nature than statistical. We will, however, make the following general remarks.

We have already contrasted the differing viewpoints of econometricians and statisticians, while emphasizing the complementary nature of their skills. It is certainly true that economic models based on theory need to be validated with real data, but a statistical data-based approach to modelling the economy will not get very far by itself without some economic guidelines. This is because an unrestricted analysis will find it difficult to select a sensible model from a virtually infinite number of choices. However, we note the following three general points in regard to building models for economic data:

1. Economic data is naturally affected by feedback, and this always makes model building difficult. For a physical system, such as a chemical reactor, where the feedback is well-controlled, there may not be enough information in the available data to identify the structure of the system. However, this is not a major problem in that known perturbations can be superimposed on the system in order to see what effect they have. However, it is generally less easy to control an economy than something like a chemical reactor, and efforts to control the economy are often made in a fairly subjective way. As a result, the amount of information in economic data may be less than one would like. Furthermore, it is difficult to carry out experiments on the economy in the same sort of way that perturbations can be added to a physical system.
2. The economy has a complex, non-linear structure, which may well be changing through time, and yet data sets are often quite small.
3. Statistical inference is usually carried out conditionally on an assumed model and focuses on uncertainty due to sampling variation and having to estimate model parameters. However, specification errors, arising from choosing the wrong model, are often more serious, particularly in economics.

The main time-series alternative to econometric simultaneous equation models is VAR models and they will be covered in Section 13.3.

13.1.2 The cross-correlation function

A key tool in modelling multivariate time-series data is the cross-correlation function, which was defined in Section 8.1 in the context of a (bivariate) linear system. Before continuing, it may be helpful to redefine this function for an m-variate multivariate process, say $\{\boldsymbol{X}_t\}$, where $\boldsymbol{X}_t^T = (X_{1t}, X_{2t}, \ldots, X_{mt})$. Analogous to the univariate case, we begin by defining cross-covariances.

Let $\boldsymbol{\mu}_t$ denote the vector of **mean** values of $\boldsymbol{X}_t$ at time t, so that its ith component is $\mu_{it} = E(X_{it})$. Let $\Gamma(t, t+k)$ denote the **cross-covariance matrix** of $\boldsymbol{X}_t$ and $\boldsymbol{X}_{t+k}$, so that its (i,j)th element is the cross-covariance coefficient of X_{it} and $X_{j,t+k}$. A multivariate process is said to be **second-order stationary** if the mean and the cross-covariance matrices at different lags do not depend on time. Then $\boldsymbol{\mu}_t$ will be a constant, say $\boldsymbol{\mu}$, while $\Gamma(t, t+k)$ will be a function of the lag k only, say $\Gamma(k)$. Then the (i,j)th element of $\Gamma(k)$, say $\gamma_{ij}(k)$, is given by

$$\gamma_{ij}(k) = \mathrm{Cov}(X_{it}, X_{j,t+k}) = E[(X_{it} - \mu_i)(X_{j,t+k} - \mu_j)], \tag{13.1}$$

see Equation (8.1). In the stationary case, the set of cross-covariance matrices, $\Gamma(k)$ for $k = 0, \pm 1, \pm 2, \ldots$, is called the **covariance matrix function**.

It has rather different properties to the (auto)covariance function in univariate time series in that it is not an even function of lag. Actually, since

$$\gamma_{ij}(k) = \mathrm{Cov}(X_{it}, X_{j,t+k}) = \mathrm{Cov}(X_{j,t+k}, X_{it}) = \gamma_{ji}(-k),$$

we have

$$\Gamma(k) = \Gamma^T(-k), \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{13.2}$$

Note that the diagonal terms, $\gamma_{ii}(k)$, which are auto- rather than cross-covariances, still have the property of being an even function of lag.

Given the covariance matrix function, it is easy to standardize any particular element of any matrix (by dividing by the product of the standard deviations of the two relevant series) to find the corresponding cross-correlation and hence construct the set of $(m \times m)$ cross-correlation matrices, $R(k)$ for $k = 0, \pm 1, \pm 2, \ldots$, called the **correlation matrix function** of the process. Thus the (i,j)th element of $R(k)$ is given by

$$\rho_{ij}(k) = \mathrm{Corr}(X_{it}, X_{j,t+k}) = \gamma_{ij}(k)/\sigma_i\sigma_j \tag{13.3}$$

where σ_i, the standard deviation of X_{it}, can also be expressed as $\sqrt{\gamma_{ii}(0)}$. Note that $\rho_{ij}(k)$ is the correlation coefficient between X_{it} and $X_{j,t+k}$. When $k > 0$, this correlation coefficient measures the linear dependence of $X_{j,t+k}$ on X_{it}, which occurs after time t. Consequently, if $\rho_{ij}(k) \neq 0$ and $k > 0$, the series X_{it} leads the series X_{jt} at lag k. Furthermore, using (13.2), we obtain that

$$R(k) = R^T(-k), \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{13.4}$$

Due to the above property, it suffices in practice to consider the cross-correlation matrices $R(k)$ for $k \geq 0$.

Assuming that the same number of observations, say N, have been collected on the m variables over the same time period, the sample cross-covariances and cross-correlations may be calculated by a natural extension of the formulae given in Section 8.1.2. For example, the sample cross-covariance coefficient of X_i and X_j at lag k is given by

$$c_{ij}(k) = \begin{cases} \sum_{t=1}^{N-k} (x_{it} - \bar{x}_i)(x_{j,t+k} - \bar{x}_j)/N & k = 0, 1, \ldots, N-1 \\ \sum_{t=1-k}^{N} (x_{it} - \bar{x}_i)(x_{j,t+k} - \bar{x}_j)/N & k = -1, \ldots, -(N-1) \end{cases} \tag{13.5}$$

and the **sample cross-correlation coefficient** of X_i and X_j at lag k is given by

$$r_{ij}(k) = c_{ij}(k)/s_i s_j \tag{13.6}$$

where $s_i = \sqrt{c_{ii}(0)}$ denotes the sample standard deviation of observations on the ith variable.

13.1.3 Initial data analysis

In time-series analysis, the first step should normally be to plot the data. With multivariate data, this step is no less important. A time plot for each variable will indicate the presence of trend, seasonality, outliers and discontinuities. For stationary series, it will also be helpful to calculate the autocorrelation function (ac.f.) for each series in order to suggest an appropriate univariate model for each series. It may also be helpful to calculate the cross-correlation function for all meaningful pairs of variables, but the reader should refer back to Section 8.1.3, where the difficulties in interpreting cross-correlations are explained. In brief, cross-correlation variances are inflated by autocorrelation within the series and so some prefiltering is usually desirable to avoid spuriously large values.

With a good interactive graphics package, it can also be fruitful to scale all the series to have zero mean and unit variance, and then to plot pairs of variables on the same graph. One series can be moved backwards or forwards in time so as to make the visible characteristics of the series agree as closely as possible. If the cross-correlations between two series are generally negative around zero lag, then it may be necessary to turn one series 'upside-down' in order to get a good match. This type of approach may be helpful in alerting the analyst to the possible presence of linear relationships, perhaps with an in-built delay mechanism. However, the approach suffers from similar dangers as those involved in interpreting cross-correlations.

With a large number of variables, it can be very difficult to build a good multivariate time-series model, especially when many of the cross-correlations between predictor variables are 'large'. Then it may be fruitful to make some sort of multivariate transformation of the data (e.g. by using principal

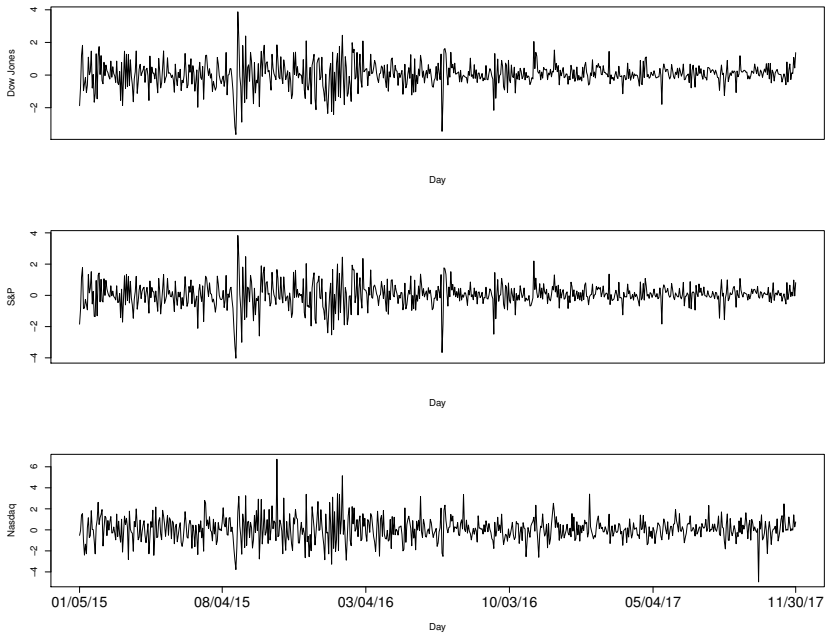

Figure 13.1 *Daily percentage log returns of adjusted closing prices of the Dow Jones Industrial Average (top), the S&P500 (middle), and the Nasdaq Composite indices from January 2, 2015 to November 30, 2017.*

component analysis) so as to reduce the effective dimensionality of the data. This type of approach will not be considered here (see, for example, Peña and Box, 1987).

Example 13.1. Analysis of daily returns

We show in Figure 13.1 daily percentage log returns of the Dow Jones Industrial Average, the S&P500, and the Nasdaq Composite indices from January 2, 2015 to November 30, 2017. These three market indices, labelled as X_1, X_2, and X_3, respectively, characterize the performance of the U.S. stock market from different perspectives; hence they should be highly correlated. To see their inter-dependence, we show in Figure 13.2 their sample cross-correlations at lag $k = 0, 1, \ldots, 25$. We find that the concurrent inter-dependence of X_1, X_2 and X_3 are very strong, while the lead-lag effect among these series is relatively weak. Figures 13.1 and 13.2 can be reproduced using the following R script.

```
> ret<-read.table("../data/indexret.txt", header=T)
```

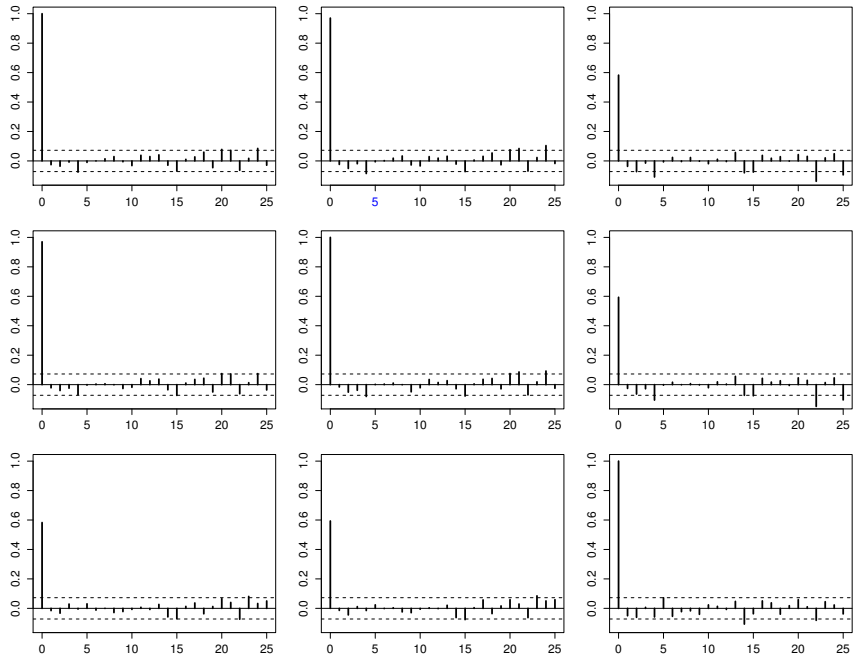

Figure 13.2 *Sample cross-correlations* $\widehat{\rho}_{ij}(k)$ $(i, j = 1, 2, 3)$*, of* X_i *and* X_j *at lag* $k = 0, 1, \ldots, 25$*.* X_i*,* $i = 1, 2, 3$*, are the daily percentage log returns of Adjusted closing prices of the Dow Jones Industrial Average, the S&P500 , and the Nasdaq Composite indices, respectively. The panel* (i, j) *shows the sample cross-correlation coefficients of* X_i *and* X_j*.*

```
> # Figure 13.1
> par(mfrow=c(3,1), mar=c(4,4,4,4))
> plot(ret$dji,type="l",xlab="Day",ylab="Dow Jones",xaxt="n")
> plot(ret$sp,type="l",xlab="Day",ylab="S&P",xaxt="n")
> plot(ret$nasdaq,type="l",xlab="Day",ylab="Nasdaq",xaxt="n")
> axis(1, x.pos, x.label, cex.axis=1.5)

> # a function of computing sample cross-correlation
> cross.corr<-function(dat1, dat2, k){
  dat.n<-length(dat1)
  cor(dat1[1:(dat.n-k)], dat2[(1+k):dat.n])
  }
> ret.ccor<-array(0, c(3,3,26))
> for (i in 1:3) for (j in 1:3) for (k in 1:26) {
     ret.ccor[i,j,k]<-cross.corr(ret[,i+1],ret[,j+1],k-1)
  }
```

```
> # Figure 13.2
> par(mfrow=c(3,3), mar=c(2,2,2,2))
> for (i in 1:3) for (j in 1:3) {
   plot(seq(0,25), ret.ccor[i,j,], type="h", xlab="",
     ylab="", ylim=c(-0.12, 1), lwd=2, cex.axis=1.3)
   abline(0,0)
   abline(1.96/sqrt(nrow(ret)), 0, lty=2)
   abline(-1.96/sqrt(nrow(ret)), 0, lty=2)
}
```

13.2 Single Equation Models

Section 13.1 said that it may be appropriate to model multivariate time-series data with a single equation when (1) there is a single response variable and several explanatory variables, and (2) there is no feedback from the response variable to the explanatory variables. The most obvious type of model to use is a multiple regression model, but the difficulties in fitting such models to time-series data have already been discussed in Section 5.3.1. We extend that discussion by considering the following simple model, namely

$$Y_t = a + bX_{t-d} + e_t \tag{13.7}$$

where a, b, d are constants and e_t denotes an error term (which may be autocorrelated).

If d is an integer greater than zero in Equation (12.5), then X_t is said to be a **leading indicator** for Y_t. Given data on X and Y until time t, this model enables forecasts of Y_t to be made directly for up to d steps ahead. However, to forecast more than d steps ahead, the required value of X_t will probably not be available and must itself be forecasted. In this sort of situation, a multivariate model is only able to give 'good' forecasts when forecasts of explanatory variables can be made (much) more accurately than those of the response variable.

Although Equation (13.7) appears to be a simple linear regression model at first sight, we said that the error terms may be autocorrelated and this makes the model non-standard. Furthermore, depending on the physical context, it may not be possible to control the values of X_t or there may even be feedback from Y_t to X_t, although this is not always immediately apparent. For all these reasons, it is often safer to fit an alternative class of models, which may include Equation (13.7) as a special case.

For an open-loop causal relationship between a single explanatory variable and a response variable, it is often worth trying to fit a member of the class of **transfer function models**. The latter were introduced in Section 9.4.2 and have the general form

$$Y_t = h(B)X_{t-d} + N_t, \tag{13.8}$$

where $h(B) = h_0 + h_1 B + h_2 B^2 + \cdots$ is a polynomial in the backward shift operator, B, d denotes a non-negative integer and N_t denotes noise (which may be autocorrelated). If $d > 0$, then X_t is said to be a **leading indicator** for Y_t.

As noted in Section 9.4.2, Equation (13.8) can sometimes be parsimoniously rewritten in the form

$$\delta(B)Y_t = \omega(B)X_t + e_t, \tag{13.9}$$

where $\delta(B), \omega(B)$ are low-order polynomials in B such that $\omega(B) = \delta(B)h(B)$, while $e_t = \delta(B)N_t$ is usually assumed to follow some sort of autoregressive moving average (ARMA) process. Note that Equation (13.9) may include lagged values of Y_t as well as of X_t. Models of this type can be fitted using the tools summarized in Section 9.4.2 and are fully described by Box et al. (1994, Chapter 11).

The **dynamic regression** models of Pankratz (1991) are of a somewhat similar type to those given above. Note that econometricians may refer to a model of the type described by Equation (13.8) as a **distributed lag model**, and when a polynomial lag function such as $h(B)$ is written as a *ratio* of polynomials as in the transfer function model of Equation (13.9), then econometricians may use the term **rational distributed lag**.

13.3 Vector Autoregressive Models

There are many situations where a single-equation model is inappropriate for multivariate time-series data. For example, there may be more than one response variable of interest, or the data may have been generated by a closed-loop system. In the latter case, it no longer makes sense to talk about an 'input' (an explanatory variable) and an 'output' (a response variable). More generally, there are many situations where there are two (or more) variables that, to a greater or lesser extent, 'arise on an equal footing', and which are all interrelated. Modelling such variables is often called **multiple time-series modelling**, and this section introduces arguably the most important class of models for this purpose.

13.3.1 VAR(1) models

With m variables, a natural way to represent them is by means of a $(m \times 1)$ vector $\boldsymbol{X}_t$ where $\boldsymbol{X}_t^T = (X_{1t}, \ldots, X_{mt})$. For simplicity, we initially restrict attention to the case $m = 2$. For stationary series, we may, without loss of generality, assume the variables have been scaled to have zero mean. In the latter case, a simple model would allow the values of X_{1t} and X_{2t} to depend linearly on the values of both series at time $(t-1)$. The resulting model for the two series would then consist of two equations, namely

$$\left.\begin{array}{rcl} X_{1t} & = & \phi_{11}X_{1,t-1} + \phi_{12}X_{2,t-1} + \varepsilon_{1t} \\ X_{2t} & = & \phi_{21}X_{1,t-1} + \phi_{22}X_{2,t-1} + \varepsilon_{2t}, \end{array}\right\} \tag{13.10}$$

where $\{\phi_{ij}\}$ are constants. The two 'error' terms ε_{1t} and ε_{2t} are usually both assumed to be white noise but are often allowed to be correlated contemporaneously. In other words, ε_{1t} could be correlated with ε_{2t} but not with past values of either ε_{1t} or ε_{2t}.

Note that if coefficients $\phi_{12} = \phi_{21} = 0$, then X_{1t} and X_{2t} are not dynamically correlated. In such case, both X_{1t} and X_{2t} follow a univariate AR(1) process, respectively, and hence can be analyzed separately. If one of ϕ_{12} and ϕ_{21} is not zero, say $\phi_{12} = 0$, but $\phi_{21} \neq 0$, then Equation (13.10) reduces to

$$\left.\begin{array}{rcl} X_{1t} & = & \phi_{11}X_{1,t-1} + \varepsilon_{1t} \\ X_{2t} & = & \phi_{21}X_{1,t-1} + \phi_{22}X_{2,t-1} + \varepsilon_{2t}. \end{array}\right\} \tag{13.11}$$

We notice that if ϕ_{12} is zero, then X_{1t} does not depend on the lagged value of X_{2t}. This means that, while X_{2t} depends on the lagged value of X_{1t}, there is no feedback from X_{2t} to X_{1t}. Put another way, this means any causality goes in one direction only and we can think of X_{1t} and X_{2t} as the input and output, respectively. The first equation in (13.10) is univariate while the model for X_{2t} could, in fact, be rewritten in the form of a transfer function model.

Equation (13.10) can be rewritten in vector form as

$$\boldsymbol{X}_t = \Phi \boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{13.12}$$

where $\boldsymbol{\varepsilon}_t^T = (\varepsilon_{1t}, \varepsilon_{2t})$ and

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Equation (13.12) looks like an AR(1) model except that $\boldsymbol{X}_t$ (and $\boldsymbol{\varepsilon}_t$) are now vectors instead of scalars. Since $\boldsymbol{X}_t$ depends on $\boldsymbol{X}_{t-1}$, it is natural to call this model a **vector autoregressive model** of order 1 (VAR(1)). Equation (13.12) can be further rewritten as

$$(I - \Phi B)\boldsymbol{X}_t = \boldsymbol{\varepsilon}_t \tag{13.13}$$

where B denotes the backward shift operator, I is the (2×2) identity matrix and ΦB represents the operator matrix

$$\begin{pmatrix} \phi_{11}B & \phi_{12}B \\ \phi_{21}B & \phi_{22}B \end{pmatrix}.$$

The stationarity of $\mathbf{X}_t$ can be extended from the argument for univariate X_t. In particular, the necessary and sufficient condition for the stationarity of (13.12) or (13.13) is that the roots of the determinant of $I - \Phi B$ lie outside the unit circle.

13.3.2 VAR(p) models

We can readily generalize the above model from two to m variables and from first-order autoregression to pth order. In general, a VAR model of order p

(VAR(p)) can be written in the form

$$\Phi(B)\boldsymbol{X}_t = \boldsymbol{\varepsilon}_t, \tag{13.14}$$

where $\boldsymbol{X}_t$ is a $(m \times 1)$ vector of observed variables, and Φ is a matrix polynomial of order p in the backward shift operator B such that

$$\Phi(B) = I - \Phi_1 B - \cdots - \Phi_p B^p,$$

where I is the $(m \times m)$ identity matrix and $\Phi_1, \Phi_2, \ldots, \Phi_p$ are $(m \times m)$ matrices of parameters.

Note that, if the variables can be ordered in such a way that each Φ_i matrix is lower triangular (meaning that all coefficients above the diagonal are zero), then a transfer-function model may be regarded as a special case of the VAR model in Equation (13.14). The last component of $\boldsymbol{X}_t$ is then the output (or response variable) in an open-loop system. In contrast, in a closed-loop system, the 'outputs' feed back to affect the 'inputs' and the general VAR model may then be appropriate to describe the behaviour of the mutually dependent variables.

We restrict attention to stationary processes, and hence, without loss of generality, we may assume the variables have been scaled to have zero mean. This explains why there is no constant on the right-hand side of Equation (13.14). The condition for stationarity is that the roots of the equation,

$$\text{determinant}\{\Phi(x)\} = |I - \Phi_1 x - \Phi_2 x^2 - \cdots - \Phi_p x^p| = 0,$$

should lie outside the unit circle. Note that this condition reduces to the familiar condition for stationarity in the univariate case when $m = 1$.

In Equation (13.14), we have used $\boldsymbol{\varepsilon}_t$ to denote m-dimensional white noise. Although we introduced bivariate white noise in the VAR(1) example above, we need to define white noise more generally in m dimensions. Let $\boldsymbol{\varepsilon}_t^T = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{mt})$ denote an $(m \times 1)$ vector of random variables. This multivariate time series will be called **multivariate white noise** if it is stationary with zero mean vector $\mathbf{0}$, and if the values of $\boldsymbol{\varepsilon}_t$ at different times are uncorrelated. Then the $(m \times m)$ matrix of the cross-covariances of the elements of $\boldsymbol{\varepsilon}_t$ with the elements of $\boldsymbol{\varepsilon}_{t+j}$ is given by

$$\text{Cov}(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t+j}) = \begin{cases} \Gamma_0 & j = 0, \\ 0_m & j \neq 0, \end{cases}$$

where Γ_0 denotes a $(m \times m)$ symmetric positive-definite matrix and 0_m denotes an $(m \times m)$ matrix of zeroes. This means that each component of $\boldsymbol{\varepsilon}_t$ behaves like univariate white noise. Notice that the covariance matrix at lag zero, namely, Γ_0, does not need to be diagonal, as an innovation at a particular time point could affect more than one measured variable at that time point. Thus we do allow the components of $\boldsymbol{\varepsilon}_t$ to be contemporaneously correlated.

The mathematics for a VAR model involves matrix polynomials that may look rather strange at first. To get a better feel for them, the reader is advised to look again at the first-order polynomial example included in Equation (13.13) and write out the scalar equations it represents.

Given the matrix Γ_0, describing the contemporaneous covariances of the white noise, it is possible in principle to evaluate the covariance matrix function, and hence the correlation matrix function of a VAR process. In practice, the algebra is usually horrid and it is only possible to find simple analytic functions in some simple cases. One gets a generalized matrix form of the Yule–Walker equations, which can be difficult to solve. Consider, for simplicity, the VAR(1) model in Equation (13.12). Multiply through on the right-hand side by $\boldsymbol{X}_{t-k}^T$ and take expectations. When $k > 0$, we get

$$\Gamma(k) = \Gamma(k-1)\Phi^T,$$

but when $k = 0$, we get

$$\Gamma(0) = \Gamma(-1)\Phi^T + \Sigma_0 = \Gamma(1)^T\Phi^T + \Sigma_0.$$

These equations are only easy to solve when Φ is diagonal, in which case one has m independent univariate AR(1) processes – see Exercise 13.3.

The definition of a VAR model in Equation (13.14) does not attempt to describe features such as trend and seasonality. It is possible to add deterministic terms to the right-hand side of Equation (13.14) to account for a non-zero mean, for trend and for seasonality (e.g. by including seasonal dummy variables). However, for seasonal data, it is usually easier to deseasonalize the data before attempting to model them, especially if the aim is to produce seasonally adjusted figures and forecasts. One simple possibility is to use seasonal differencing. As regards trend, non-seasonal (first) differencing may be employed to remove trend. However, the use of differencing is also not without problems, particularly if **co-integration** is present (see Section 13.6 below). Thus, fitting VAR models to real data is not easy, and will be discussed separately in Section 13.5 below.

13.4 Vector ARMA Models

As in the univariate case, the VAR model may be generalized to include moving average (MA) terms. Building on Equation (13.14), this is done in an 'obvious' way by writing

$$\Phi(B)\boldsymbol{X}_t = \Theta(B)\boldsymbol{\varepsilon}_t \tag{13.15}$$

where

$$\Theta(B) = I + \Theta_1 B + \cdots + \Theta_q B^q$$

is a matrix polynomial of order q in the backward shift operator B and $\Theta_1, \Theta_2, \ldots, \Theta_q$ are $(m \times m)$ matrices of parameters. Then $\boldsymbol{X}_t$ is said to follow a

vector ARMA (VARMA) model of order (p, q). Equation (13.15) is a natural generalization of the univariate ARMA model, and reduces to the familiar univariate ARMA model when $m = 1$. For stationary models, satisfying the stationarity condition given in Section 13.3, we can impose a condition for invertibility analogous to that in the univariate case. This requires that the roots of the equation,

$$\text{determinant}\{\Theta(x)\} = |I + \Theta_1 x + \Theta_2 x^2 + \cdots + \Theta_q x^q| = 0,$$

should lie outside the unit circle. This condition reduces to the usual univariate invertibility condition when $m = 1$.

If $\Phi(B)$ includes a factor of the form $I(1 - B)$, then the model is not stationary but rather acts on the first differences of the components of $\boldsymbol{X}_t$. By analogy with the univariate case, such a model is called a **vector ARIMA** (VARIMA) model. Note that it may not be optimal in practice to difference each component of $\boldsymbol{X}_t$ in the same way. Moreover the possible presence of co-integration (see Section 13.6 below) also needs to be considered before differencing multivariate data.

Forecasts can readily be computed for VAR, VARMA and VARIMA models by a natural extension of methods employed for univariate ARIMA models. Generally speaking, minimum mean square error (MMSE) forecasts can be obtained by replacing future values of white noise with zeroes while future values of $\boldsymbol{X}_t$ are replaced with MMSE forecasts. Present and past values of $\boldsymbol{X}_t$ and of $\boldsymbol{\varepsilon}_t$ are replaced by the observed values and the (one-step-ahead forecast) residuals, respectively. Details will not be given here, but see Exercise 13.5.

One problem with VARMA (or VARIMA) models is that there may be different, but equivalent (or exchangeable) ways of writing what is really the same model. There are various ways of imposing constraints on the parameters involved in Equation (13.15) to ensure that a model is **identifiable**, meaning that the model is unique, but the conditions are complicated and will not be given here. There are further problems involved in identifying and fitting a model with MA components – see Section 13.5 – and, analogous to the univariate case, VARMA models are generally (much) harder to handle than VAR models.

Finally, we note that VARMA models can be generalized by adding terms, involving additional exogenous variables, to the right-hand side of Equation (13.15) and such a model is sometimes abbreviated as a VARMAX model.

13.5 Fitting VAR and VARMA Models

There are various approaches to the identification of VARMA models. They involve assessing the orders p and q of the model, estimating the parameter matrices in Equation (13.15) and estimating the variance–covariance matrix of the 'noise' components. We do not give details here, but rather refer the reader, for example, to Priestley (1981), Lütkepohl (1993) and Reinsel (1997). A recent survey of VARMA models is given by Tiao (2001).

Identification of a VARMA model is inevitably a difficult and complicated process because of the large number of model parameters that may need to be estimated. The number of parameters increases quadratically with m and can become uncomfortably large when the lag length is more than one or two. This suggests that some constraints need to be placed on the model. One possibility is to use external knowledge or a preliminary analysis of the data to identify coefficient matrices where most of the parameters can *a priori* be taken to be zero. Such matrices are called *sparse* matrices. However, even with some sparse matrices included in the model, VARMA models are still difficult to fit, and so many analysts restrict attention to VAR models, which they hope will give an adequate approximation to VARMA models. Even then, there is still a danger of overfitting, and fitted VAR models do not always provide an approximation to real-life multivariate data that is as parsimonious and useful as AR models are for univariate data.

Because of the dangers of overfitting, a technique called **Bayesian vector autoregression** (BVAR) may be used to fit VAR models, in preference to using ordinary least squares. This approach can be used whether or not the analyst has a Bayesian philosophy. The technique essentially aims to prevent overfitting by shrinking parameters higher than first-order towards zero. The usual prior that is used for the parameters, called the **Minnesota prior**, has mean values, which assume *a priori* that every series is expected to be a random walk. Other priors have also been tried (e.g. Kadiyala and Karlsson, 1993). A tutorial paper showing how to select an appropriate BVAR model is given by Spencer (1993).

One important tool in VARMA model identification is the matrix of cross-correlation coefficients. The case of two-time series has already been considered in Section 8.1, and some difficulties involved in interpreting a sample cross-correlation function were noted. The author admits that he has typically found it difficult to interpret cross-correlation (and cross-spectral) estimates, even when there are only two variables. The analysis of three or more series is in theory a natural extension, but in practice is much more difficult and should only be attempted by analysts with substantial experience in univariate ARIMA model building. As previously noted, the interpretation of cross-correlations is complicated by the possible presence of autocorrelation within the individual series and by the possible presence of feedback between the series, and it is now generally recognized that series should be filtered or prewhitened before looking at cross-correlations.

A number of studies have been published, which suggest that, when carefully applied, the use of VARMA, and more especially of VAR models, can lead to improved forecasts as compared with univariate and other multivariate models. For example, the results in Boero (1990) suggest that a BVAR model is better than a large-scale econometric model for short-term forecasting, but not for long-term forecasts where the econometric model can benefit from judgemental interventions by the model user and may be able to pick up non-linearities not captured by (linear) VAR models. As is usually the case,

different approaches and models are complementary. As a second example, Bidarkota (1998) found that a bivariate ARMA model gave better out-of-sample forecasts of real U.S. interest rates than a univariate unobserved components model. There is evidence that unrestricted VAR models do not forecast as well as when using Bayesian vector autoregression (e.g. Kadiyala and Karlsson, 1993), presumably because unrestricted models may incorporate spuriously many parameters.

Of course, multivariate models are often constructed to help describe and understand the measured system, rather than (just) to produce forecasts. We provide below an example of analyzing the relationship among three macroeconomic variables.

Example 13.2. Analysis of macro-economic series

Figure 13.3 shows the time series plots of the U.S. *gross domestic product* (GDP), the civilian unemployment rate, and *consumer price index* (CPI) for all urban consumers from the first quarter of 1948 to the third quarter of 2017. Denote these series as W_{1t}, W_{2t}, and W_{3t}, respectively, and $\mathbf{W}_t = (W_{1t}, W_{2t}, W_{3t})^T$. Note that all W_i's displays some extest of nonstationarity; we transform the data by calculating the rate of changes or difference for each series. Specifically, we obtain the change rate of GDP, $X_{1t} = (W_{1t} - W_{1,t-1})/W_{1,t-1} \times 100\%$, the difference of the unemployment rate, i.e., $X_{2t} = W_{2t} - W_{2,t-1}$, and a measure of inflation, $X_{3t} = (W_{3t} - W_{3,t-1})/W_{3,t-1} \times 100\%$. The transformed series $\mathbf{X}_t = (X_{1t}, X_{2t}, X_{3t})^T$ are shown in Figure 13.4. Figures 13.3 and 13.4 can be reproduced by the following R command.

```
> data<-read.table("data/macrots.txt", header=T)
> x.pos<- c(1, 70, 140, 210, 279)
> x.label<-c("1948.Q1", "1965.Q2", "1982.Q4", "2000.Q2", "2017.Q3")

> # Figure 13.3
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(data$gdp,type="l",xlab="Quarter", ylab="GDP", xaxt="n")
> axis(1, x.pos, x.label)
> plot(data$unrate,type="l",xlab="Quarter", ylab="Unem. rate",
    xaxt="n")
> axis(1, x.pos, x.label)
> plot(data$cpi,type="l",xlab="Quarter", ylab="CPI", xaxt="n")
> axis(1, x.pos, x.label)

> gdp.rate<-diff(data$gdp)/data$gdp[1:(nrow(data)-1)]*100
> unrate.diff<-diff(data$unrate)
> cpi.rate<-diff(data$cpi)/data$cpi[1:(nrow(data)-1)]*100
> data2<-data.frame(data[-1,1], gdp.rate, unrate.diff, cpi.rate)
> colnames(data2)<-c("quarter", "gdprate", "unemdiff", "cpirate")
```

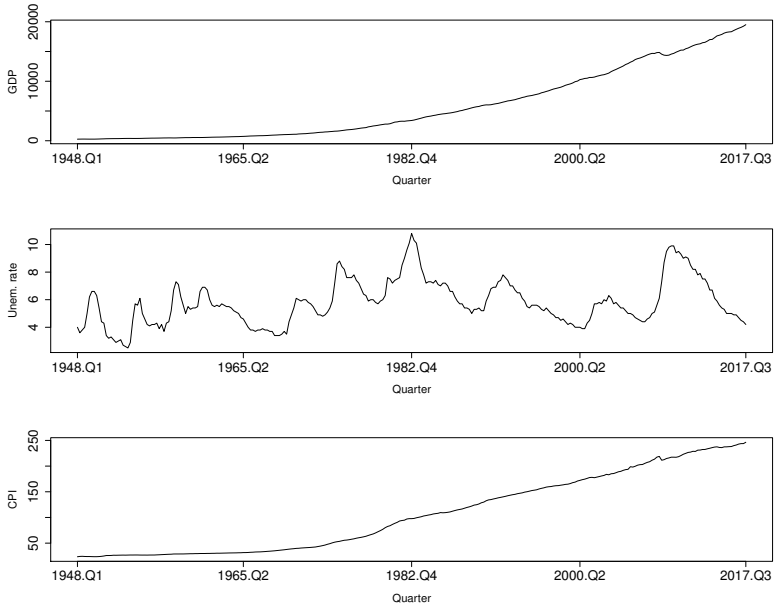

Figure 13.3 *Time series plot of GDP (top), unemployment rates (middle), and CPI (bottom).*

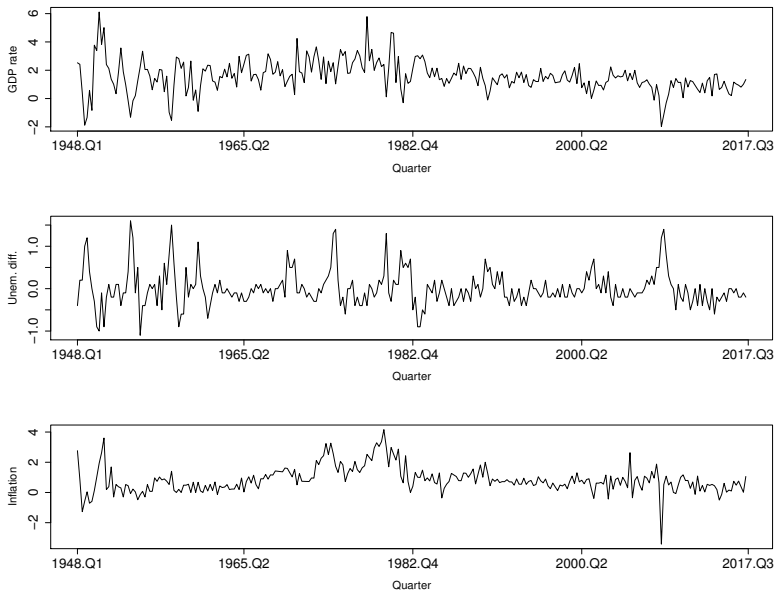

Figure 13.4 *Time series plot of the differenced series* $\mathbf{X}_t$.

```
> # Figure 13.4
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5, cex.lab=1.2)
> plot(data2$gdprate,type="l",xlab="Quarter",
    ylab="GDP rate", xaxt="n")
> axis(1, x.pos, x.label)
> plot(data2$unemdiff,type="l",xlab="Quarter",
    ylab="Unem. diff.", xaxt="n")
> axis(1, x.pos, x.label)
> plot(data2$cpirate,type="l",xlab="Quarter",
    ylab="Inflation",xaxt="n")
> axis(1, x.pos, x.label)
```

To see the cross-sectional dependence of these three series, we show in Figure 13.5 the scattor plot of X_i versus X_j, for $i, j = 1, 2, 3$. Note that the figure shows an obvious concurrent regression relationship between X_{1t} and X_{2t}, X_{1t} and X_{3t}, respectively. To see if there are any lead or lag effects among X_{it}'s, we show the sample cross-correlations in Figure 13.6. Figures 13.5 and 13.6 can be generated by the following R command.

```
> par(mar=c(2,2,2,2), cex.axis=1.5)
> plot(data2[,-1])

> par(mfrow=c(3,3), mar=c(2,2,2,2))
for (i in 1:3) for (j in 1:3) {
  plot(seq(0,25), data2.ccor[i,j,], type="h", xlab="", ylab="",
       ylim=c(-0.12, 1), lwd=2, cex.axis=1.3)
  abline(0,0)
  abline(1.96/sqrt(nrow(data2)), 0, lty=2)
  abline(-1.96/sqrt(nrow(data2)), 0, lty=2)
}
```

Given the concurrent and cross-sectional dependence of $\mathbf{X}_t$ in Figures 13.5 and 13.6, we may first try a single equation model for the series. For example, we consider the following equation

$$X_{1t} - \alpha X_{1,t-1} = \beta_0 + \beta_1 X_{2t} + \beta_2 X_{2,t-1} + Z_t,$$

where $Z_t \sim N(0, \sigma^2)$. This equation can be estimated as a linear regression via the following R script.

```
> n<-nrow(data2)
> attach(data2)
> fit1<-lm(gdprate[2:n]~gdprate[1:(n-1)]+unemdiff[2:n]
      +unemdiff[1:(n-1)])
> summary(fit1)
```

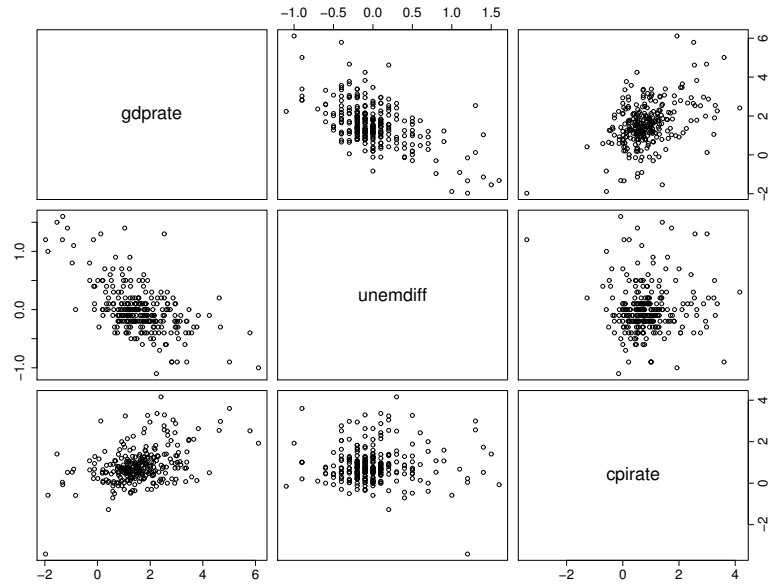

Figure 13.5 *Scattor plots of* X_i *and* X_j, $i, j = 1, 2, 3$.

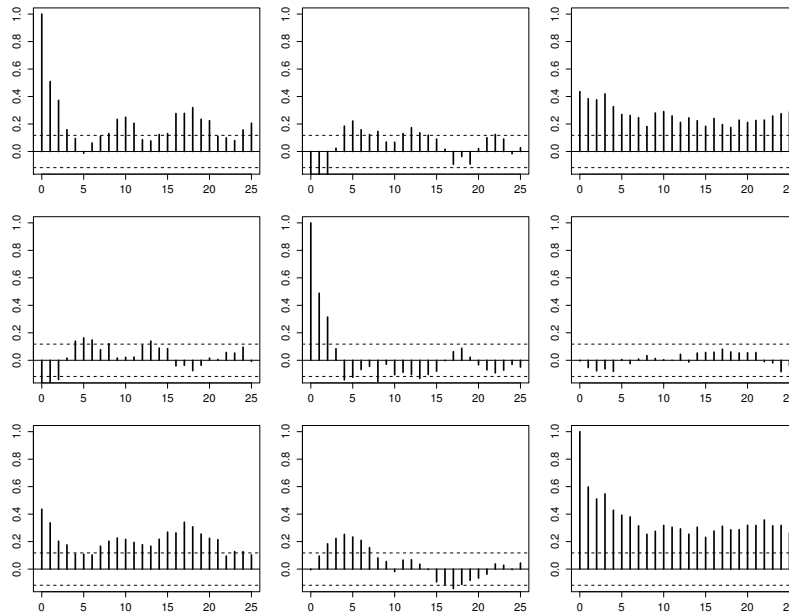

Figure 13.6 *Sample cross-correlations* $\widehat{\rho}_{ij}(k)$ $(i, j = 1, 2, 3)$, *of* $\mathbf{X}_t$.

```
Call:
lm(formula = gdprate[2:n] ~ gdprate[1:(n - 1)] + unemdiff[2:n] +
    unemdiff[1:(n - 1)])

Residuals:
    Min      1Q  Median      3Q     Max
-2.1327 -0.5507 -0.1327  0.4315  3.6300

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          0.95175    0.10117   9.408  < 2e-16 ***
gdprate[1:(n - 1)]   0.39013    0.05594   6.974 2.32e-11 ***
unemdiff[2:n]       -1.29118    0.14231  -9.073  < 2e-16 ***
unemdiff[1:(n - 1)]  0.27662    0.16110   1.717   0.0871 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8418 on 273 degrees of freedom
Multiple R-squared:  0.4357,    Adjusted R-squared:  0.4295
F-statistic: 70.26 on 3 and 273 DF,  p-value: < 2.2e-16
```

Note that, at the 95% significance level, the estimated model is

$$X_{1t} = 0.9518_{(.1012)} + 0.3901_{(.0560)} X_{1,t-1} - 1.2912_{(.1423)} X_{2,t} + Z_t,$$

where $\widehat{\sigma}^2 = 0.8418^2$. This suggests that the changes in unemployment rates depend on the past values of the GDP growth rate. Similarly, we may fit a single equation model for X_{1t} versus X_{3t}.

To get a full spectrum of relationships among the GDP growth rate, changes of unemployment rate, and the inflation rate, we consider a VAR model for $\mathbf{X}_t$. Specifically, we fit the following VAR(1) model to $\mathbf{X}_t$,

$$\mathbf{X}_t = \Phi \mathbf{X}_{t-1} + \boldsymbol{\varepsilon}_t,$$

where $\mathrm{Cov}(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t) = \Gamma_0$, and $\mathrm{Cov}(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_{t+j}) = 0$ for $j \neq 0$. This can be done by calling the function `VARMA` in the `R` package `MTS`. The following shows the `R` command and output of the estimation procedure.

```
> library("MTS")
> fit.var1<-VARMA(data2[,-1], p=1, q=0, include.mean=FALSE)

Number of parameters:  9
initial estimates:  0.7327 0.1979 0.251 -0.077 0.3654
      0.1182 0.2389 0.2681 0.5174
```

```
Par. lower-bounds:  0.6185 -0.1504 0.0784 -0.1156 0.2477
      0.0599 0.1651 0.0429 0.4058
Par. upper-bounds:  0.8469 0.5463 0.4237 -0.0384 0.483
      0.1765 0.3127 0.4933 0.629
Final Estimates:  0.711549 0.1507602 0.2785961 -0.06924344
      0.3811354 0.1073311 0.2308635 0.2543724 0.5149042

Coefficient(s):
          Estimate  Std. Error  t value Pr(>|t|)
gdprate    0.71155     0.05652   12.590  < 2e-16 ***
unemdiff   0.15076     0.17319    0.870 0.384027
cpirate    0.27860     0.08492    3.281 0.001035 **
gdprate   -0.06924     0.01926   -3.595 0.000324 ***
unemdiff   0.38114     0.05902    6.458 1.06e-10 ***
cpirate    0.10733     0.02894    3.709 0.000208 ***
gdprate    0.23086     0.03713    6.217 5.07e-10 ***
unemdiff   0.25437     0.11379    2.235 0.025393 *
cpirate    0.51490     0.05580    9.228  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
---
Estimates in matrix form:
AR coefficient matrix
AR( 1 )-matrix
        [,1]  [,2]  [,3]
[1,]  0.7115 0.151 0.279
[2,] -0.0692 0.381 0.107
[3,]  0.2309 0.254 0.515

Residuals cov-matrix:
           [,1]         [,2]        [,3]
[1,]  1.0499988 -0.161464568 0.209923395
[2,] -0.1614646  0.121935341 0.002619803
[3,]  0.2099234  0.002619803 0.453308001
----
aic=  -3.137723
bic=  -3.020283
```

The estimated VAR(1) model is given by

$$\begin{pmatrix} X_{1t} \\ X_{2t} \\ X_{3t} \end{pmatrix} = \begin{pmatrix} 0.7115 & 0.1508 & 0.2786 \\ -0.0692 & 0.3811 & 0.1073 \\ 0.2309 & 0.2544 & 0.5149 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{pmatrix}$$

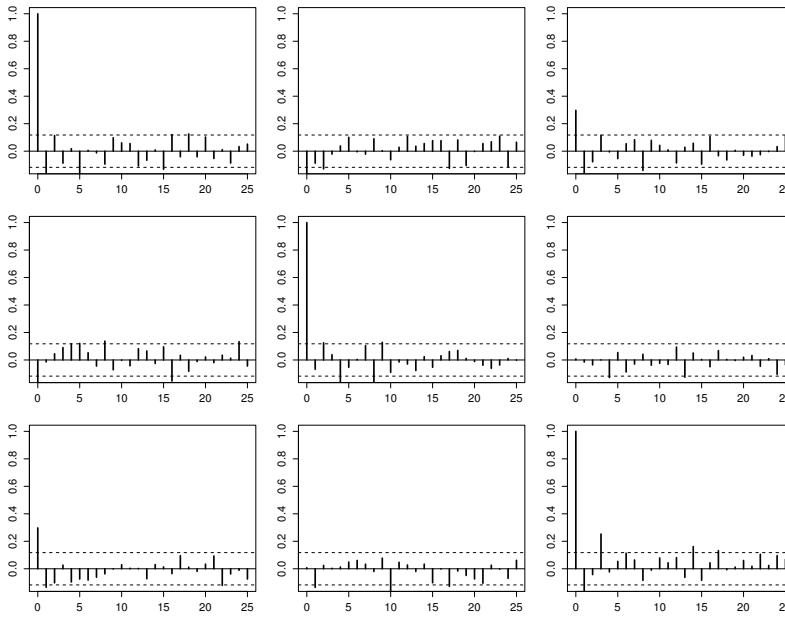

Figure 13.7 *Sample cross-correlations* $\widehat{\rho}_{ij}(k)$ $(i, j = 1, 2, 3)$*, of residual* $\boldsymbol{\varepsilon}_t$.

with

$$\Gamma_0 = \begin{pmatrix} 1.0500 & -0.1615 & 0.2099 \\ -0.1615 & 0.1219 & 0.0026 \\ 0.2099 & 0.0026 & 0.4533 \end{pmatrix}.$$

To check if cross-covariance of residuals are zeros or not, we show in Figure 13.7 the sample cross-correlations of $\boldsymbol{\varepsilon}_t$. Comparing the sample cross-correlations of $\mathbf{X}_t$ in Figure 13.6, the sample cross-correlations of $\boldsymbol{\varepsilon}_t$ at lag $k \neq 0$ are significantly smaller. Actually, the majority of them are within the 95% confidence bands, which indicates that most of serial correlation among $\boldsymbol{\varepsilon}_t$ is gone.

We may further consider fitting a VARMA model to the series $\mathbf{X}_t$, and it can also be done by calling the function `VARMA` in the `R` package `MTS`. Specifically, we can use the following command to fit a VARMA(1,1) model to $\mathbf{X}_t$.

```
fit.varma11<-VARMA(data2[,-1], p=1, q=1, include.mean=FALSE)
```

We omit the `R` output here and leave it to the interested readers for further exploration.

13.6 Co-Integration

Modelling multivariate time-series data is complicated by the presence of non-stationarity, particularly with economic data. One possible approach is to difference each series until it is stationary and then fit a VARMA model. However, this does not always lead to satisfactory results, particularly if different degrees of differencing are appropriate for different series or if the structure of the trend is of intrinsic interest in itself (and, in particular, assessing whether the trend is deterministic or stochastic). An alternative approach, much used in econometrics, is to look for what is called **co-integration**.

As a simple example, we might find that X_{1t} and X_{2t} are both non-stationary but that a particular linear combination of the two variables, say $(X_{1t} - kX_{2t})$ *is* stationary. Then the two variables are said to be co-integrated. If we now build a model for these two variables, there is no need to take first differences of both observed series, but rather the constraint implied by the stationary linear combination $(X_{1t} - kX_{2t})$ needs to be incorporated in the model.

A more general definition of co-integration is as follows. A series $\{X_t\}$ is said to be integrated of order d, written $I(d)$, if it needs to be differenced d times to make it stationary. If two series $\{X_{1t}\}$ and $\{X_{2t}\}$ are both $I(d)$, then any linear combination of the two series will usually be $I(d)$ as well. However, if a linear combination exists for which the order of integration is less than d, say $(d - b)$, then the two series are said to be co-integrated of order (d, b), written $CI(d, b)$. If this linear combination can be written in the form $\boldsymbol{\alpha}^T \boldsymbol{X}_t$, where $\boldsymbol{X}_t^T = (X_{1t}, X_{2t})$, then the vector $\boldsymbol{\alpha}$ is called a **co-integrating vector**.

Consider again the example given earlier in this section, where X_{1t} and X_{2t} are both non-stationary but $(X_{1t} - kX_{2t})$ is stationary. If X_{1t} and X_{2t} are both $I(1)$, then $d = b = 1$, $\boldsymbol{X}_t$ is $CI(1, 1)$ and a co-integrating vector is $\boldsymbol{\alpha}^T = (1, -k)$.

In a non-stationary vector ARIMA model, there is nothing to constrain the individual series to make them 'move together' in some sense, yet the laws of economics suggest that there are bound to be long-run equilibrium forces that will prevent some economic series from drifting too far apart. This is where the notion of co-integration comes in. The constraint(s) implied by co-integration enable the analyst to fit a more realistic multivariate model.

In an introductory text, it is not appropriate to give further details here. Some useful references on co-integration include Banerjee et al. (1993), Dhrymes (1997), Engle and Granger (1991) and Johansen (2001), but there have been many other contributions to the subject dealing with topics such as tests for co-integration, error-correction models and ways of describing 'common trends'. An amusing non-technical introduction to the concept of co-integration is given by Murray (1994). We recommend that co-integration should always be considered when attempting to model multivariate economic data.

13.7 Multivariate Volatility Models

All sections in this chapter so far focus on the multivariate linear time series, which can considered as an extension of univariate time series models in Chapter 3. We next consider an multivariate extension of the univariate volatility models in Chapter 12. Multivariate volatilities have many important applications in finance and other disciplines, and how to model and analyze multivariate volatilties has been an interesting topics during the last two decades.

Consider a multivariate time series $\mathbf{X}_t$. Similar to the univariate model (12.1), we write the series as

$$\mathbf{X}_t = \boldsymbol{\mu}_t + \mathbf{Y}_t, \tag{13.16}$$

where $\mathbf{Y}_t = (Y_{1t}, \ldots, Y_{mt})^T$ is the innovation of the series at time t with $E(\mathbf{Y}_t) = \mathbf{0}$, $\boldsymbol{\mu}_t$ is the mean of $\mathbf{X}_t$ conditional on observed data up to time $t-1$, $\boldsymbol{\mathcal{F}}_{t-1}$, and given by

$$\boldsymbol{\mu}_t = E(\mathbf{X}_t | \boldsymbol{\mathcal{F}}_{t-1}). \tag{13.17}$$

The process $\boldsymbol{\mu}_t$ is assumed to follow the conditional expectation of a multivariate time series model in Sections 13.4 and 13.5. For simplicity, we may assume that $\boldsymbol{\mu}_t$ follows

$$\boldsymbol{\mu}_t = \sum_{i=1}^{p} \Phi_i \mathbf{X}_{t-i} + \sum_{j=1}^{q} \Theta_j \mathbf{Y}_{t-j}. \tag{13.18}$$

The conditional covariance matrix of $\mathbf{Y}_t$ given $\boldsymbol{\mathcal{F}}_{t-1}$ is an $m \times m$ positive-definite matrix $\boldsymbol{\Sigma}_t$ defined as $\boldsymbol{\Sigma}_t = \text{Cov}(\mathbf{Y}_t | \boldsymbol{\mathcal{F}}_{t-1})$.

The major difficulty of extending univariate volatility models to their multivariate analog is the curse of dimensionality, because there are $m(m+1)/2$ quantities in $\boldsymbol{\Sigma}_t$ for an m-dimensional series. One way to overcome this difficulty is to consider some relatively simple multivariate volatility models. In this section, we introduce two kinds of volatility models. One is the exponentially weighted covariance matrix, and the other is called the BEKK model.

13.7.1 Exponentially weighted estimate

Given the information set $\boldsymbol{\mathcal{F}}_{t-1} = \{\mathbf{Y}_1, \ldots, \mathbf{Y}_{t-1}\}$, the covariance matrix of the innovation can be estimated by their sample covariance matrix, i.e.,

$$\widehat{\boldsymbol{\Sigma}}_t = \frac{1}{k-1} \sum_{i=1}^{k} \left(\mathbf{Y}_{t-i} - \bar{\mathbf{Y}}\right)^{\otimes 2}, \tag{13.19}$$

where $\mathbf{a}^{\otimes 2} := \mathbf{a}\mathbf{a}^T$. This estimate put equal weights for all observations in the moving window of $\mathbf{Y}_t$'s. To emphasize the contribution of most recent innovations, the conditional covariance matrix at time t can be approximated by

$$\widehat{\mathbf{\Sigma}}_t = \sum_{i=1}^{k} \alpha_i \big(\mathbf{Y}_{t-i} - \bar{\mathbf{Y}}\big)^{\otimes 2}, \tag{13.20}$$

where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k < 1$ and $\sum_{i=1}^{k} \alpha_i = 1$. One specification of (13.20) is the *exponentially weighted moving average* (EWMA) estimate of the covariance matrix. Specifically, let $k = \infty$ and $\alpha_i = (1-\lambda)\lambda^{i-1}$ for some $0 < \lambda < 1$; Equation (13.20) becomes

$$\widehat{\mathbf{\Sigma}}_t = (1-\lambda)\mathbf{Y}_{t-1}^{\otimes 2} + \lambda \widehat{\mathbf{\Sigma}}_{t-1}. \tag{13.21}$$

When $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ are assumed to follow a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}_t$, $\boldsymbol{\mu}_t$ in (13.18) is a function of parameters Φ_i's and Θ_j's; hence the log likelihood function of the data $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ can be derived, and maximized by numerical methods.

13.7.2 BEKK models

The BEKK model can be considered as a full extension of the univariate GARCH(1, 1) model. For an m-dimensional innovation process $\mathbf{Y}_t$, the BEKK(h, k) volatility model takes the form

$$\mathbf{\Sigma}_t = \mathbf{A}\mathbf{A}^T + \sum_{i=1}^{h} \mathbf{A}_i (\mathbf{Y}_{t-i}\mathbf{Y}_{t-i}^T)\mathbf{A}_i^T + \sum_{j=1}^{k} \mathbf{B}_j \mathbf{\Sigma}_{t-j} \mathbf{B}_j^T, \tag{13.22}$$

where $\mathbf{A}$ is a lower triangular matrix such that $\mathbf{A}\mathbf{A}^T$ is positive-definite, $\mathbf{A}_i$ and $\mathbf{B}_j$ are $m \times m$ matrices. The model contains $m^2(h+k) + m(m+1)/2$ parameters, which increases rapidly with h and k. It is difficult to study the property of a high-order BEKK model, so we focus only on an example for the BEKK(1,1) model.

Example 13.3

We consider the three macro-economic time series discussed in Section 13.5. In particular, we consider fitting a BEKK(1, 1) model to the transformed series $\mathbf{X}_t$; see Figure 13.4. For illustruation purposes, we study the volatilities of GDP growth rates and inflation. Denote $\mathbf{Y}_t = (Y_{1t}, Y_{2t}) := (X_{1t}, X_{3t})$; we fit the following 2-dimensional BEKK(1,1) to $\mathbf{Y}_t$:

$$\mathbf{\Sigma}_t = \mathbf{A}\mathbf{A}^T + \mathbf{A}_1(\mathbf{Y}_{t-1}\mathbf{Y}_{t-1}^T)\mathbf{A}_1^T + \mathbf{B}\mathbf{\Sigma}_{t-1}\mathbf{B}^T. \tag{13.23}$$

The estimation can be carried out by calling the function `BEKK11` in the `R` package `MTS`. The following script and output in `R` show how to estimate the model parameters.

```
> library("MTS")
> fit<-BEKK11(data2[,-c(1,3)], include.mean=F)

Initial estimates:  1.113999 0.3721313 0.7685585 0.1 0.02
    0.02 0.1 0.8 0.1 0.1 0.8
Lower limits:  0.2227999 0.07442627 0.1537117 1e-06 -0.5
    -0.5 1e-06 1e-06 -0.5 -0.5 1e-06
Upper limits:  1.225399 0.4093445 0.8454144 0.999999 0.5
    0.5 0.999999 0.999999 0.5 0.5 0.999999

Coefficient(s):
        Estimate  Std. Error  t value   Pr(>|t|)
A011  0.28744139  0.13290378  2.16278   0.030558 *
A021  0.12166368  0.11316513  1.07510   0.282331
A022  0.15371171  0.06343682  2.42307   0.015390 *
A11   0.43739948  0.08634034  5.06599 4.0628e-07 ***
A21   0.10413559  0.03614344  2.88118   0.003962 **
A12   0.00772987  0.18349600  0.04213   0.966399
A22   0.45216477  0.07440824  6.07681 1.2260e-09 ***
B11   0.89702033  0.06957063 12.89366 < 2.22e-16 ***
B21  -0.01202102  0.02645793 -0.45434   0.649581
B12  -0.03443632  0.15553538 -0.22141   0.824777
B22   0.79452952  0.06505837 12.21256 < 2.22e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The volatility equation of the fitted BEKK(1, 1) model is

$$\begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t} \end{bmatrix} = \begin{bmatrix} 0.287 & 0 \\ 0.122 & 0.154 \end{bmatrix} \begin{bmatrix} 0.287 & 0.122 \\ 0 & 0.154 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.437 & 0.008 \\ 0.104 & 0.452 \end{bmatrix} \begin{bmatrix} Y_{1,t-1}^2 & Y_{1,t-1}Y_{2,t-1} \\ Y_{1,t-1}Y_{2,t-1} & Y_{2,t-1}^2 \end{bmatrix} \begin{bmatrix} 0.437 & 0.104 \\ 0.008 & 0.452 \end{bmatrix}$$
$$+ \begin{bmatrix} 0.897 & -0.034 \\ -0.012 & 0.795 \end{bmatrix} \begin{bmatrix} \sigma_{11,t-1} & \sigma_{12,t-1} \\ \sigma_{21,t-1} & \sigma_{22,t-1} \end{bmatrix} \begin{bmatrix} 0.897 & -0.012 \\ -0.034 & 0.795 \end{bmatrix},$$

where seven elements are significant at 95% level. Figure 13.8 shows the estimated volatilities and the time-varying correlations of the BEKK(1,1) model. Note that the time-varying correlations of the BEKK(1, 1) model remain high during the most periods. Figure 13.8 can be reproduced by the following R command.

```
> x.pos<- c(1, 70, 140, 210, 279)
> x.label<-c("1948.Q1", "1965.Q2", "1982.Q4", "2000.Q2", "2017.Q3")
> est.corr<-fit$Sig[,2]/sqrt(fit$Sig[,1]*fit$Sig[,4])
```

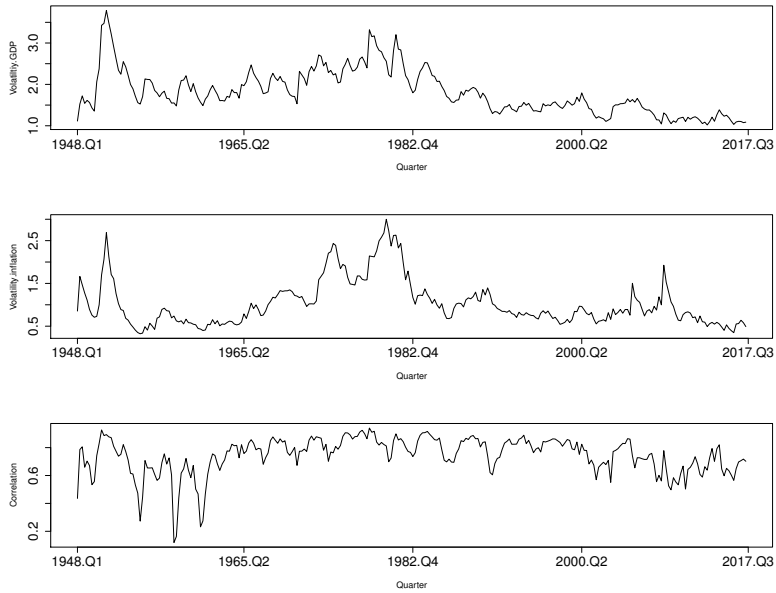

Figure 13.8 *Estimated volatilities (standard error) and time-varying correlations of a BEKK(1, 1) model for the quarterly GDP growth rate and the quarterly inflation rate from 1948 to 2017. Top: GDP growth rate volatility; Middle: Inflation rate volatility; Bottom: Time-varying correlations.*

```
> par(mfrow=c(3,1), mar=c(4,4,4,4), cex.axis=1.5)
> plot(sqrt(fit$Sig[,1]), type="l", xlab="Quarter",
    ylab="Volatiltiy.GDP", xaxt="n")
> axis(1, x.pos, x.label)
> plot(sqrt(fit$Sig[,4]), type="l", xlab="Quarter",
    ylab="Volatility.inflation", xaxt="n")
> axis(1, x.pos, x.label)
> plot(est.corr, type="l", xlab="Quarter",
    ylab="Correlation", xaxt="n")
> axis(1, x.pos, x.label)
```

13.8 Bibliography

The beginner may find it helpful to read Chapter 5 of Chatfield (2001), Chapters 7 and 8 of Granger and Newbold (1986) or Chapters 13 and 14 of Wei (1990). A thorough treatment of VAR and VARMA models is provided by Lütkepohl (1993), while Reinsel (1997) covers similar material in a somewhat

terser, more mathematical style. Multivariate versions of structural modelling, of Bayesian forecasting and of non-linear modelling have not been covered in this introductory chapter and the reader is referred to Harvey (1989), to West and Harrison (1997), to Granger and Teräsvirta (1993), and to Lai and Xing (2008, Chapter 9), respectively. Tsay (2015) gives a comprehensive summary of the basic concepts and ideas of analyzing multivariate dependent data and provides the `R` package `MTS` for all the examples of multivariate time series analysis used in the book.

Exercises

13.1 Express the following bivariate model in vector form and say what sort of model it is.

$$\begin{aligned} X_{1t} &= X_{1,t-1} + 0.5X_{2,t-1} + \varepsilon_{1t} \\ X_{2t} &= 0.2X_{1,t-1} + 0.7X_{2,t-1} + \varepsilon_{2t} \end{aligned}$$

Is the model stationary and invertible?

13.2 Consider the VAR(1) model for two variables as given by Equation (13.12). Determine whether the model is stationary for the following coefficient matrices at lag one;

(a) $\Phi = \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{pmatrix}$; (b) $\Phi = \begin{pmatrix} 0.7 & 0.7 \\ 0.7 & 0.7 \end{pmatrix}$;

(c) $\Phi = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix}$; (d) $\Phi = \begin{pmatrix} 0.9 & 0.5 \\ -0.1 & 0.3 \end{pmatrix}$.

13.3 Find the covariance matrix function $\Gamma(k)$, and the correlation matrix function $P(k)$ for model (c) in Exercise 13.2, assuming that $\boldsymbol{\varepsilon}_t$ denotes bivariate white noise with covariance matrix $\Gamma_0 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ at lag zero.

13.4 Consider the bivariate VARMA$(0, 1)$ model from Equation (13.15) with $\Phi(B) = I$ and $\Theta(B) = I + \Theta_1 B$ where $\Theta_1 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.4 \end{pmatrix}$. Is the model stationary and invertible?

13.5 Given data up to time N, find the one- and two-step-ahead forecasts for the bivariate VAR(1) model in Exercise 13.2 (d).

13.6 Consider the three dimensional series $\mathbf{X}_t = (X_{1t}, X_{2t}, X_{3t})$ in obtained in Example 13.2, where X_{1t} is the growth rate of GDP, X_{2t} is the change of the unemployment rate, and X_{3t} is the inflation rate. Fit a VARMA(1,1) model to the series $\mathbf{X}_t$.

13.7 Consider the three dimensional series $\mathbf{X}_t = (X_{1t}, X_{2t}, X_{3t})$ in obtained in Example 13.2, where X_{1t} is the growth rate of GDP, X_{2t} is the change of the unemployment rate, and X_{3t} is the inflation rate.

(a) Fit a BEKK(1,1) model to the series $\mathbf{Y}_t := (X_{1t}, X_{2t})$.

(b) Fit a BEKK(1,1) model to the series $\mathbf{Y}_t := (X_{2t}, X_{3t})$.

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Chapter 14

Some More Advanced Topics

This chapter provides a brief introduction to several more advanced topics that are unsuitable for detailed study in an introductory text. This gives a flavour of recent developments, and provides references to enable the reader to get further details if desired. Further reviews of many standard and non-standard aspects of time-series analysis, including research developments, are given, for example, by Newbold (1981, 1984, 1988) and in the collections of papers edited by Brillinger et al. (1992, 1993) and by Peña et al. (2001).

14.1 Modelling Non-Stationary Time Series

Much of the theory in the time-series literature is applicable to stationary processes. In practice most real time series have properties that do change with time, albeit slowly in many cases, and we have already met some methods for dealing with non-stationarity. For example, the use of differencing with ARIMA models allows stationary models to be fitted, while various explicit models for trend have already been introduced. More generally, even with a long, apparently stationary series, it is still a good idea to split the series into reasonably long, non-overlapping segments and compare the properties of the segments, particularly the general form of the ac.f. and spectrum.

This section discusses the problem of non-stationarity more generally, because it is important to understand the different types of non-stationarity that may arise and methods for dealing with them (see also Priestley, 1988, Chapter 6). For example, when the non-stationary features are not of primary concern (e.g. when instrument drift arises), then it is sensible to find a method that transforms the data to stationarity (e.g. differencing), as this enables us to fit stationary models and use the theory of stationary processes. Alternatively, when the non-stationary features of the data are of intrinsic interest in themselves, then it may be more rewarding to model them explicitly, rather than remove them and concentrate on modelling the stationary residuals. Thus there is, for example, a fundamental difference between fitting an ARIMA model, which describes non-stationary features implicitly, and fitting a state-space model, which describes them explicitly.

Slow changes in mean are one common source of non-stationarity. If a global (deterministic) function of time can be assumed, then such components can easily be fitted and removed (e.g. by fitting a polynomial). However, we have seen that it is now more common to assume that there are local changes

in the mean and perhaps fit a local linear trend that is updated through time. Some sort of filtering or differencing may then be employed. It helps if the filtered series has a natural contextual interpretation. Cyclical changes in mean (e.g. seasonality) can also be dealt with by filtering, by differencing or by fitting a global model perhaps using a few sine and cosine terms of appropriate frequency. Changes in variance may also be evident in the time plot and then the models of Chapter 12 should be considered.

Turning to parametric models, let us consider an AR process as an example of a linear model, and distinguish several different ways in which non-stationarity may arise for such a model. If the AR coefficients do not satisfy the stationarity conditions, because one or more roots of Equation (3.5) lie *on* the unit circle, then the series can be made stationary by differencing. Alternatively, if one or more roots lie *outside* the unit circle, then this leads to explosive behaviour, which cannot be made stationary by differencing. This sort of model[1] is much more difficult to handle. Another way that non-stationarity may arise is that the AR coefficients are changing through time, perhaps suddenly (e.g. Tyssedal and Tjostheim, 1988), or perhaps slowly (e.g. Swanson and White, 1997). The latter provides just one example of the many ways in which there can be a slow change in the underlying model structure. Changes in structure are generally more difficult to handle than something like a linear trend.

Changes in the model structure can be studied in the time domain, for example, by seeing how model parameters change through time, but can can also be studied in the frequency domain, and there are various ways of generalizing the spectrum to cope with non-stationary behaviour. The use of **evolutionary spectra** (Priestley, 1981, Chapter 11; 1988) is one possibility. This allows the spectrum to change slowly through time in a particular type of way. **Complex demodulation** is an alternative approach, which studies signals in a narrow frequency band to see how they change through time (e.g. Bloomfield, 2000, Chapter 7).

Perhaps the most difficult type of non-stationarity to handle is a sudden change in structure, due perhaps to the occurrence of a known external event such as an earthquake or a labour dispute. A **structural change** may produce a short-term transient effect or a long-term change in the model structure, such as a change in mean. With short-term effects, one or more outliers may be visible in the time plot and these can create problems with standard time-series methods, unless the outliers are modified in some way – see Section 14.4.5. The times at which sudden changes occur are called **change points** and the identification of such events is an important general problem in time-series analysis with a growing literature. Wherever possible, external knowledge of

[1] It is mathematically interesting to note, for example, that a non-stationary first-order AR process, with a lag-one coefficient greater than one, does have a stationary solution if time is reversed. However, this is usually regarded as unnatural, as the process is then no longer a **causal** process, by which is meant that the value of an observed time series is only allowed to depend on present and past data.

the given context should be used to decide where change points have occurred, though this may need to be substantiated by examination of the data; see analysis of change-points in a time series in Lai and Xing (2013).

Box et al. (1994, Chapter 12) show how to model sudden changes with a technique called **intervention analysis**, which is somewhat similar to the use of dummy variables in regression. Suppose, for example, that there is a sudden change in the level, of size K say, at time T. Then we can introduce a standardized **step change** variable, say S_t, of the form

$$S_t = \begin{cases} 0 & t < T \\ 1 & t \geq T \end{cases}$$

such that the quantity KS_t describes the resulting effect. The quantity KS_t can then be included in the model for the observed variable, and the constant K can be estimated along with all the other model parameters. It helps if the change-point time T is known from the context, though this parameter can also be estimated if necessary. The class of models could, for example, be an ARIMA or a transfer function model. An alternative approach is to use state-space models, or the dynamic linear models of **Bayesian forecasting** (see Chapter 10), as they can also be constructed so as to allow the user to deal with outliers and step changes in the mean and trend. An econometric approach to looking for change point dates is given by Bai and Perron (1998).

Despite the above remarks, in some situations it may be wiser to accept that there is no sensible way to model a non-stationary process. For example, Figure 5.1(a) showed data on the sales of insurance policies and this time plot shows there are some sudden changes in the underlying structure. When I was asked to produce forecasts for this series, I proceeded very cautiously. Rather than try to model these data in isolation, it is more important to ask questions to get appropriate background information as to why the time plot shows such unusual behaviour. In this case I found that the two large peaks corresponded to two recent sales drives. Thus the most important action is to find out whether there is going to be a third sales drive. Even if such information is forthcoming, it may still be difficult to incorporate it formally into a mathematical model. In the absence of such contextual information, it would be unwise to try to produce model-based forecasts.

14.2 Model Uncertainty

We have seen that model building is an important part of time-series analysis. Finding an appropriate model for a given set of data is generally an iterative, interactive process as described in Section 4.8. Typically the analyst will try many different models, choose the one that appears to be 'best' (perhaps using a criterion such as AIC described in Section 13.1) and then make inferences and predictions conditional on the selected model being true. Thus, while such inferences generally take account of uncertainty due to random variation

and to having estimates of model parameters, there is usually no allowance for the possibility that the wrong model may have been selected. This seems generally unwise but statisticians have only recently started to address the issues involved in **model uncertainty**. General reviews of the way that uncertainty about the model affects statistical problems are given by Chatfield (1995b) and Draper (1995), while more specialized reviews on the situation in time-series analysis and forecasting are given by Chatfield (1996; 2001, Chapter 8).

When searching for a model, it is common to try many different models. This latter activity is sometimes called **data dredging** or **data mining**. Although statistics, like the AIC and BIC, penalize more complex models, the reader should realize that there is still a danger that fitting many models to the same data may give a spuriously complex model that appears to give a good fit, but which nevertheless gives poor out-of-sample predictions.

When a model is selected using the data, rather than being specified *a priori*, the analyst needs to remember that (1) the true model may not have been selected, (2) the model may be changing through time or (3) there may not be a 'true' model anyway. It is indeed strange that we often implicitly admit that there is uncertainty about the underlying model by searching for a 'best-fit' model, but then ignore this uncertainty when making predictions. In fact it can readily be shown that, when the *same* data are used to formulate and fit a model, as is typically the case in time-series analysis, then least squares theory does not apply. Parameter estimates will typically be biased, often quite substantially. In other words, the properties of an estimator may depend, not only on the selected model *but also on the selection process*.

The effects of model uncertainty can be particularly alarming when fitting regression and time-series models, where data dredging is common. The analyst typically picks a 'best' model from a very wide range of choices. Thus an ARIMA(p, d, q) model may be chosen by considering all combinations of p, d and q such that $0 \leq p, q \leq 3$ and $d = 0$ or 1, so that a total of 32 $(= 4 \times 4 \times 2)$ models are entertained, quite apart from any additional models implied by considering whether to transform variables and whether to adjust or remove outliers. Clearly, there is a real danger that the analyst may go to excessive lengths to get a good fit to the observed data. Yet, having, for example, chosen suitable values of p, d and q for an ARIMA model, the analyst will typically make inferences and predictions as if the values of p, d and q were known *a priori*. This policy is logically unsound but hardly surprising given that most (all?) time-series texts, including this one, concentrate on describing methods that ignore model uncertainty. The only justification is that there is no simple alternative. However, this is not really satisfactory, given that empirical experience suggests that the accuracy of model predictions is generally not as good as would be expected from the goodness of fit.

As one important type of consequence, it has been found in practice that out-of-sample forecast accuracy is generally (much) worse than would be expected from the within-sample fit of time-series models. As a result,

prediction intervals tend to be too narrow in that 95% prediction intervals will typically contain fewer than 95% of future observations – see Chatfield (1993; 2001) for a review of the empirical evidence. Many analysts think that a narrow interval is somehow 'good', but theory suggests that it is actually safer to construct a wider interval that properly reflects model uncertainty as well as other sources of variation.

How then may we begin to cope with model uncertainty? Perhaps the most important step is to realise that any fitted model should be thought of as a useful approximation, and that our assessment of uncertainty is likely to be an underestimate. If we restrict ourselves to using a single best-fit model, then bear in mind that (1) a local model, which changes through time, may be preferred to a global model that has constant parameters — see, for example, the remarks on trend models in Section 2.5; (2) a simple model may be preferred to a complicated model, even if the latter appears to fit better (which is why model-selection criteria, like AIC, penalize the addition of extra parameters); (3) a robust model may be preferred to a model that is optimal for one set of conditions only.

However, instead of identifying a single model to utilize, it is often worth considering the use of more than one model, especially when several different models appear to fit a set of data about equally well. For example, if forecasts are required, it may be better to produce a range of forecasts based on different, clearly stated model assumptions. This is called **scenario analysis** – see Schoemaker (1991) – and is widely used in long-range forecasting. An alternative approach in forecasting is to **combine forecasts** from several different methods and models, perhaps by taking some sort of weighted average (e.g. see Diebold, 2001, Section 11.3). Clemen (1989) reviews the encouraging empirical evidence for this sort of strategy, but one drawback is that there is usually no resulting model. Yet another possibility is to mix several models as in an approach called **Bayesian model averaging** – see Draper (1995). It is, of course, also possible to use different models to describe different parts of the data, or to use different models for different purposes. The idea that there is a single 'true' model, which should always be used, is a fiction.

14.3 Control Theory

To many statisticians, the word 'control' implies statistical quality control using control charts, but in this section we use the word as in control engineering to denote the search for an automatic control procedure for a system whose structure may or may not be known. This is often the ultimate objective in identifying linear systems as in Chapter 9.

There are many different approaches to control, such as those based on linear parametric models of the form (cf. Equation (9.41))

$$\delta(B)Y_t = \omega(B)X_t + \theta(B)Z_t,$$

where $\{Y_t\}$ denotes an 'output', $\{X_t\}$ denotes an 'input', $\{Z_t\}$ denotes a purely random process, and δ, ω, θ are polynomials in the backward shift operator B. Many other approaches have been described, including the use of cross-spectral analysis and of instrumental variables but we do not attempt to discuss them here. The main contributions to control theory have naturally been made by control engineers, though mathematicians, statisticians and operational researchers have also played a part.

Control engineers were originally concerned mainly with deterministic control (and many of them, still are). Some of the reported research, such as the solution of non-linear differential equations subject to boundary conditions, has an obvious relevance to control theory but may equally be regarded as a branch of applied mathematics. In recent years attention has widened from deterministic problems to the consideration of **stochastic control**, where the system being controlled is subject to random disturbances. Some references covering the whole field include Davis and Vinter (1985), Jacobs (1993) and Priestley (1981).

In stochastic control, a basic problem is that of separating the signal from the noise (see Section 5.6), and there has been much work on the filtering problem starting with the work of Wiener and Kolmogorov. One major development has been the Kalman filter, which is a recursive method of estimating the state of a system in the presence of noise (see Chapter 10). Kalman filtering has been used in many applications including the control of a space rocket, where the system dynamics are well defined but the disturbances are unknown.

Although many statistical problems arise in control theory, statisticians have made a relatively small contribution to the subject. Box et al. (1994, Chapter 13) is one exception, which shows how to identify a linear parametric model for a system and hence find a good control procedure. The Box–Jenkins approach has been compared with the control engineer's state-space representation approach by Priestley (1981).

It is regrettable that there has not been more communication and cooperation between statisticians and control engineers, partly because of understandable differences in background and in the sort of problems typically tackled. There are also differences in terminology. One major practical difference is that the Box–Jenkins approach involves identifying a system using observed data, while much of the control literature appears to assume knowledge of the system structure. Perhaps the future will bring more collaboration to the benefit of all.

14.4 Miscellanea

This section gives an very brief introduction to a number of diverse topics, concentrating on giving appropriate references for the reader to follow up if desired.

14.4.1 Autoregressive spectrum estimation

Spectral analysis is concerned with estimating the spectrum of a stationary stochastic process. The approach described in Chapter 7, which is based on Fourier analysis, is essentially *non-parametric* in that no model is assumed *a priori*. A *parametric* approach called **autoregressive spectrum estimation** is a possible alternative method, and this will now be briefly described.

Many stationary stochastic processes can be adequately approximated by an autoregressive (AR) process of sufficiently high order, say

$$X_t = \alpha_0 + \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t. \tag{14.1}$$

The spectrum of (14.1) at frequency ω is inversely proportional to

$$\left| 1 - \sum_{k=1}^{p} \alpha_k e^{-ik\omega} \right|^2. \tag{14.2}$$

In order to estimate the spectrum using this class of models, the user has to assess p, the order of the process, perhaps using AIC. Then the AR parameters are estimated, as described in Section 4.2, and substituted into the reciprocal of Equation (14.2). The approach tends to give a smoother spectrum than that given by the non-parametric approach, though it can also pick out fairly narrow peaks in the spectrum. Further details and examples are given by Priestley (1981, Section 7.8–7.9), Percival and Walden (1993, Chapter 9), Hayes (1996, Section 8.5) and Broersen (2002), and the approach is certainly worth considering. Note that the selected order of the process is typically higher than when fitting models for forecasting purposes, as values around 10 are common and values as high as 28 have been reported.

An obvious development of this approach (Broersen, 2002) is to fit ARMA models rather than AR models to give what might be called ARMA spectrum estimation, but the simplicity of fitting AR models is likely to inhibit such a development. It is also worth noting that Percival and Walden (1993) suggest using the AR model as a pre-whitening device. Given that the AR model is likely to be approximation, rather than a true model, the residuals from the fitted AR model can themselves be the subject of a spectral analysis, from which the spectrum of the original series can be inferred by dividing by Equation (14.2). The reason for doing things this way is that it is generally easier to estimate a flattish spectrum than one with peaks, and we expect the residuals from the AR process to be close to white noise. Of course, if the AR model really is exactly appropriate, then the residuals will have a uniform spectrum.

14.4.2 Wavelets

Wavelet functions are an orthonormal set of functions, which can approximate a discontinuous function better than Fourier series. They can be used to

analyse non-stationary time series and give a distribution of power in two dimensions, namely, time and frequency, (rather than just one, namely, frequency, as in spectral analysis). One commonly-used wavelet family, with 'nice' properties, is the Daubechies family, but these functions have what appears, at first sight, to be a rather peculiar shape. Introductions to wavelets are given by Strang (1993) and Abramovich et al. (2000). The latter includes a section, with references, on applications to time-series analysis. Percival and Walden (2000) provide book-length coverage of the latter. Wavelets continue to be the subject of much current research.

14.4.3 'Crossing' problems

If one draws a horizontal line (an axis) through a stationary time series, then the time series will cross and recross the line in alternate directions. The times at which the series cuts the axis provide valuable information about the properties of the series. Indeed in some practical problems the crossing points are the only information available. The problems of inferring the properties of a time series from its 'level-crossing' properties are discussed by Cramér and Leadbetter (1967), Blake and Lindsey (1973), Kedem (1994) and Bendat and Piersol (2000, Section 5.5). The latter also discuss related topics such as when and where peak values occur.

14.4.4 Observations at unequal intervals, including missing values

This book has been mainly concerned with discrete time series measured at equal intervals of time. When observations are taken at unequal intervals, either by accident or design, there are additional complications in carrying out a time-series analysis.

There are various ways in which observations may arise at unequal time intervals. The most common situation is where observations are meant to be available at equal intervals but, for some reason, some are missing. In the latter case it is important to assess whether points are missing 'at random' (whatever that means), whether a group of consecutive observations is missing or there is some other systematic pattern, such as every fifth observation being absent. The reason why observations are missing must also be taken into account in the modelling process, and here the context is crucial. For example, special techniques are needed when the data are **censored** or **truncated**, as will arise, for example, when every observation that exceeds a certain threshold value is missing. In practice, it is common to find that a block of measurements is missing, perhaps because the measuring machine stopped working for a period, and then common sense may be used to impute the missing values. For example, given hourly temperature readings at a given site for 1 year, you may find that observations for two consecutive days are missing. Then it may be reasonable to insert appropriate hourly averages for that time of

year, perhaps adjusted by the difference between the previous day's average and the appropriate monthly average.

A rather different situation arises when observations are taken at random time points, as would arise, for example, if temperature measurements are only taken when it is raining. Here again the context is crucial in deciding what mechanism is appropriate for describing the pattern of time points, and hence deciding how to analyse the data.

When observations are missing at random, it may be desirable to estimate, or impute, the missing values so as to have a complete time series. Alternatively, some computations can be carried out directly on the 'gappy' data. For example, the autocovariance coefficient at lag k — see Equation (4.1) — may readily be adapted by summing over all pairs of observations where both values are present, and then dividing by the appropriate number of pairs. It is easy to go wrong when computing the latter number. If, for example, there is just one observation missing, say x_m, then *two* pairs could be absent, namely, (x_{m-k}, x_m) and (x_m, x_{m+k}), providing that x_m is not within k observations of the ends of the series. Thus the denominator in Equation (4.1) would then be $(N-2)$. When data are collected at unequal intervals, the computation of autocovariance and autocorrelation coefficients is no longer a viable possibility. An alternative is to compute a function, called the **(semi)variogram**, which is introduced later in Section 14.4.8. Diggle (1990, Section 2.5.2) shows how to apply this to unequally-spaced data.

In the frequency domain, the general definition of the periodogram – Equation (7.17) – can be adapted to cope with both missing observations and observations collected at unequal intervals, though, in the latter case, the values of t need no longer be integers.

A technique, called **spline regression**, can be used to compute a smoothed approximation to a time series, even when it is unequally spaced. A **spline** is a continuous piecewise polynomial function increasingly used in computational work. A brief introduction is given by Diggle (1990, Section 2.2.3), while further details in a regression context are given by Green and Silverman (1994).

Some further references on unequally spaced and missing observations include Jones (1985), Peña (2001) and the collection of papers edited by Parzen (1984), while additional references are listed by Priestley (1981, p. 586) and Hannan (1970, p. 48).

14.4.5 Outliers and robust methods

As in other areas of statistics, the presence of outliers can disrupt a time-series analysis. We have briefly referred to the problems raised by outliers in Sections 1.3, 2.7.2 and 14.1. Here we make some more general remarks and give a few more references.

Outliers, or aberrant observations, are often clearly visible in the time plot of the data. If they are obviously errors, then they need to be adjusted

or removed. Some measuring instruments record occasional values that are clearly anomalous and can be dealt with fairly easily. The situation is more difficult when it is not known whether the outlier is an error or a genuine extreme value. In the latter case, it could be misleading to remove the observation completely, but leaving the outlier in could 'mess up' the analysis. The context is crucial in deciding what to do. Most outliers that occur in practice are of the **additive** outlier type. They only affect the single observation where the outlier occurs, and can be thought of as the genuine observation plus or minus a one-off 'blip'. In contrast the **innovation outlier** occurs in the noise process and can affect all subsequent observations in the series. As a result, the innovation outlier is more difficult to deal with and may have a long-lasting effect.

A rather different type of effect arises when there is a sudden step change in the level or trend of a series. This may generate apparent outliers in the short term, but the problems are of a rather different kind. They may be best dealt with as a structural change, perhaps by the use of intervention analysis — see Section 14.1. Here again, the context is crucial in determining the times of any change points and in assessing the form of their effect.

Some useful references on outliers include the following: Tsay (1986) discusses model specification in the presence of outliers, while Chang et al. (1988) focus on parameter estimation. Abraham and Chuang (1989) discuss outlier detection for time series. Ledolter (1989) discusses the effect of outliers on forecasts, while Chen and Liu (1993) show that outliers are particularly dangerous when they are near the forecast origin at the end of the series. The latter paper considers the different types of outliers as well as some strategies for reducing potential difficulties. Luceño (1998) discusses how to detect multiple (possibly non-consecutive) outliers for industrial data assumed to be generated by an ARIMA process. Franses (1998, Chapter 6) provides a general introduction, while the new edition of Box et al. (1994, Section 12.2) also includes material on outlier analysis. Peña (2001) gives a recent review and, together with Chen and Liu (1993), provides a good source of further references about outliers. Work has started on outlier detection for vector time series (e.g. Tsay et al., 2000), but note that this can be much more difficult than in the univariate case. An outlier in one component of the series can cause 'smearing', not only in adjacent observations on the same component, but also in adjacent observations on other components.

Note that it can be difficult to tell the difference between outliers caused by errors, by non-linearity (see Chapter 11) and by having a non-normal 'error' distribution. It is worth bearing all three possibilities in mind when looking at apparently unusual values.

Instead of adjusting or removing outliers, an alternative approach is to use **robust** methods, which automatically downweight extreme observations. Three examples of this type of approach will be briefly mentioned. First, the use of **running median smoothers** (e.g. Velleman and Hoaglin, 1981) can be used to produce a smoothed version of a time series. Second, LOcally

WEighted regreSSion (LOWESS or LOESS) can also be used to smooth a time series, and can also be used iteratively to get smooth estimates of different components of a time series (Cleveland, 1993). Third, we note that the Kalman filter can also be robustified (Meinhold and Singpurwalla, 1989).

As a final comment, it is worth reiterating the general finding that local models seem to be more robust to departures from model assumptions than global models, and this explains, for example, why simple forecasting techniques like exponential smoothing often seem to compete empirically with more complicated alternative techniques, such as those based on ARIMA or vector autoregressive (VAR) models

14.4.6 Repeated measurements

In a **longitudinal study**, a series of measurements is taken through time on each of a sample of individuals or subjects. The data therefore comprise a collection of time series, which are often quite short. Special methods have been developed to tackle the analysis of such data. A modern introduction is given by Diggle et al. (2002, Chapter 5), while Jones (1993) gives a more advanced state-space approach.

14.4.7 Aggregation of time series

Many time series are produced by some sort of **aggregation**. For example, rainfall figures can be calculated by summing, or aggregating, over successive days or successive months. Aggregation over successive time periods is called *temporal* aggregation. If you find a model for a series at one level of aggregation (e.g. days), then it should be stressed that a completely different type of model may be appropriate for observations on the same variable over a different time period (e.g. months). Wei (1990, Chapter 16) discusses the effect of aggregation on the modelling process.

An alternative type of aggregation is *contemporaneous* aggregation, where aggregation is carried out by summing across series for the same time period. For example, sales of a particular product in a given time period could be aggregated over different regions, over different countries or over different brand sizes of the product. This raises rather different problems from temporal aggregation. For example, in inventory control, a typical question is to ask whether better forecasts can be developed by forecasting an aggregate series (e.g. total sales in a country) and then allocating them to the subseries (e.g. sales in regions) based on historical relative frequencies – called a *top-down* approach, or to forecast each component series and then add the forecasts in order to forecast the grand total – called the *bottom-up* approach. There is some empirical evidence that the latter approach is sometimes better (Dangerfield and Morris, 1992), but this question is very context dependent and it is not advisable to give general guidelines.

14.4.8 Spatial and spatio-temporal series

In some scientific areas, particularly in ecology and agriculture, data often arise that are ordered with respect to one or two spatial co-ordinates rather than with respect to time. Methods of analysing spatial data have been described, for example, by Ripley (1981) and Cressie (1993). Although there are many similarities between time-series and spatial analysis, there are also fundamental differences (Chatfield, 1977, Section 9). One tool that is commonly used in the analysis of spatial data is the **(semi)variogram**. For a one-dimensional series in space or time, this may be defined as

$$V(k) = \frac{1}{2}\text{Var}[X_{t+k} - X_t]. \tag{14.3}$$

Note the factor $\frac{1}{2}$ that is customarily introduced, and it is this that explains the adjective *semi*. For a stationary process, $V(k)$ is often written as $E[\{X_{t+k} - X_t\}^2]$, and, in this case, it can be shown that $V(k)$ is related to the ac.f., $\rho(k)$, of the process by

$$V(k) = \gamma(0)[1 - \rho(k)] \tag{14.4}$$

where $\gamma(0)$ denotes the variance of the process. The function is generally used to assess spatial autocorrelation and is covered in books on geostatistics and spatial statistics (e.g. Cressie, 1993), especially in relation to a technique called **kriging**, which is concerned with predicting a response variable from spatial data.

In recent years, several attempts have been made to apply the variogram to time-series data. For example Haslett (1997) shows that the variogram can be defined for some non-stationary series and used as a model-identification tool. Sadly, the estimation of $V(k)$ is rarely discussed in time-series books — an exception is Diggle (1990, Sections 2.5.2 and 5.4).

When data are recorded with their position in both space and time, we move into the difficult area of space-time modelling. One key question is whether the covariance function is separable, meaning that it can be expressed as a product of a purely temporal function and a purely spatial function.

14.4.9 Time series in finance

The application of time-series analysis in finance is a rapidly growing area. A modern overview is given in the article by Rydberg (2000) and (more extensively) in the book by Tsay (2010). Various types of data may arise and they may be measured at (very) different time intervals. Examples include yearly or quarterly financial results, daily share prices and high-frequency stock exchange data measured at intervals as short as 10 minutes[2]. High-frequency data are of particular interest and can generate very long series consisting of many thousands of observations.

[2]Data are recorded at even shorter intervals if they are 'tick-by-tick', meaning that every trade is recorded. Data like these are not standard time series.

Many financial time series, such as share prices, can be modelled (approximately) by a random walk. If we denote such a series by R_t, then it is common practice to analyse the first differences, $R_t - R_{t-1}$, or (more usually) the first differences of the logarithms, namely, $\log R_t - \log R_{t-1} = \log[R_t/R_{t-1}]$. The latter series is often called the (log) **returns**. Such series typically show little serial correlation and might be thought to be (approximately) a series of independent and identically distributed random variables. In fact, such series are usually not exactly independent because the ac.f. of the *squared* returns typically shows small but persistently positive values that decay slowly to zero. This long-memory effect may be due to non-linearity or to changing variance; see Chapter 10 for time series models on volatility.

It is also tempting to assume that such data are normally distributed, but they are typically found to have high kurtosis, in that they have 'heavy tails' (or 'fat tails'). This means that there are more extreme observations in both tails of the distribution, than would arise if data were normally distributed. There may also be evidence of asymmetry, as, for example, when returns show more large negative values than positive ones. Of course, the more aggregated the data, the better will be any normal approximation.

Generalized autoregressive conditionally heteroscedastic (GARCH) models are widely used for looking at finance series, as they allow for changing variance (or volatility clustering). As regards heavy tails, one useful family of distributions is the class of **stable** distributions (e.g. Brockwell and Davis, 1991, Section 13.3). Loosely speaking, a distribution is stable when the convolution of two such distributions leads to another distribution in the same family. The family includes the normal and Cauchy distributions, and a stable distribution may be symmetric or skew depending on the parameters. Using a stable distribution may lead to processes with infinite variance.

As many financial processes are close to a random walk, there is particular interest in finance in a variety of stochastic models that have similar properties. The famous **Black–Scholes model**, which should really be called the **Samuelson–Black–Scholes model**, is based on (geometric) Brownian motion and is popular because it leads to closed-form expressions for preference-independent derivative (or option) prices. However, the model assumes normally distributed increments and lacks some of the empirical features of real financial data. Thus, it is not a good model. The so-called **Levy process** is essentially a continuous-time random walk with a heavy-tailed step distribution and may be more realistic. However, these processes are defined in continuous time, and thus lead to stochastic differential equations rather than (discrete-time) difference equations. This means that the mathematics involved is much more difficult, and so these models will not be discussed here.

Rather than fit a univariate time-series model, multivariate time-series models are often applied to finance data. The reader is referred to Tsay (2010). Another topic of particular interest in finance is the study of **extreme values**,

which have obvious application in insurance and risk management – see Tsay (2002, Chapter 7).

Finally, we mention that computationally intensive statistical methods are increasingly used in finance. As in other application areas, the methods are used to solve problems that cannot be solved analytically. Tsay (2010, Chapter 10) gives a clear introduction to topics like **Markov Chain Monte Carlo (MCMC) methods** that can be used in simulation. However, note that **bootstrapping**, which is widely used to solve problems computationally by sampling from the observed empirical distribution, is particularly difficult to apply to time-series data because of the lack of independence – see Bühlmann (2002). Bootstrapping should be applied sparingly, or not at all, with time series!

14.4.10 Discrete-valued time series

This book has concentrated on discrete time series, meaning series measured at discrete intervals of time. The observed variable is typically continuous, but will, on occasion, be discrete. Some discrete variables, with large means, can be treated as if they are approximately normally distributed and modelled in much the same way as variables having a continuous distribution. However, data occasionally arise where a completely different approach is required, perhaps because zero counts are observed and the distribution is highly skewed. Examples can arise in measuring the abundance of animal populations or recorded cases of infectious diseases. As one example, the number of wading birds present at a particular site could be zero on many days, but then go up to several hundred (or even several thousand) on one particular day or for several days or weeks in a row. Such data raise special problems that will not be discussed here. A key issue is whether to invest in modelling the ecological/mechanistic background or simply fit a statistical model using a 'black-box' type of approach. Another issue is whether to model presence/absence rather than actual counts. A relevant reference is Macdonald and Zucchini (1997).

Appendix A Fourier, Laplace, and z-Transforms

This appendix provides a short introduction to the Fourier transform, which is a valuable mathematical tool in time-series analysis. The related Laplace and z-transforms are also briefly introduced.

Given a (possibly complex-valued) function $h(t)$ of a real variable t, the **Fourier transform** of $h(t)$ is usually defined as

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \tag{5}$$

provided the integral exists for every real ω. Note that $H(\omega)$ is in general complex. A sufficient condition for $H(\omega)$ to exist is that

$$\int_{-\infty}^{\infty} |h(t)|\, \mathrm{d}t < \infty.$$

If Equation (5) is regarded as an integral equation for $h(t)$ given $H(\omega)$, then a simple inversion formula exists of the form

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{6}$$

Then $h(t)$ is called the (inverse) Fourier transform of $H(\omega)$. The two functions $h(t)$ and $H(\omega)$ are commonly called a Fourier transform pair.

The reader is warned that many authors use a slightly different definition of a Fourier transform to Equation (5). For example, some authors put a constant $1/\sqrt{(2\pi)}$ outside the integral in Equation (5), and then the inversion formula for $h(t)$ has a symmetric form. In time-series analysis many authors (e.g. Cox and Miller, 1968, p. 315) find it convenient to put a constant $1/2\pi$ outside the integral in Equation (5). In the inversion formula, the constant outside the integral is then unity.

In time-series analysis, it is often convenient to work with the variable $f = \omega/2\pi$ rather than ω. The resulting Fourier transform pair is

$$G(f) = \int_{-\infty}^{\infty} h(t)\, \mathrm{e}^{-2\pi\mathrm{i}ft}\, \mathrm{d}t, \tag{7}$$

$$h(t) = \int_{-\infty}^{\infty} G(f)\, \mathrm{e}^{2\pi\mathrm{i}ft}\, \mathrm{d}f. \tag{8}$$

Note that the constant outside each integral is now unity.

When working with discrete-time series, we typically use the discrete form of the Fourier transform where $h(t)$ is only defined for integer values of t. Then

$$H(\omega) = \sum_{t=-\infty}^{\infty} h(t)\, \mathrm{e}^{-\mathrm{i}\omega t} \qquad \text{for } -\pi \leq \omega \leq \pi \tag{9}$$

is the discrete Fourier transform of $h(t)$. Note that $H(\omega)$ is only defined in the interval $[-\pi, \pi]$. The inverse transform is

$$h(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{10}$$

Fourier transforms have many useful properties, some of which are used during the later chapters of this book. However, we do not attempt to review them all here, as there are numerous mathematical books that cover the Fourier transform in varying depth.

One special type of Fourier transform arises when $h(t)$ is a real-valued even function such that $h(t) = h(-t)$. The autocorrelation function of a stationary time series has these properties. Then, using Equation (5) with a constant $1/\pi$ outside the integral, we find

$$\begin{aligned} H(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} h(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \\ &= \frac{2}{\pi} \int_{0}^{\infty} h(t)\, \cos \omega t\, \mathrm{d}t \end{aligned} \tag{11}$$

and it is easy to see that $H(\omega)$ is a real-valued even function. The inversion formula is then

$$\begin{aligned} h(t) &= \frac{1}{2} \int_{-\infty}^{\infty} H(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega \\ &= \int_{0}^{\infty} H(\omega)\, \cos \omega t\, \mathrm{d}\omega. \end{aligned} \tag{12}$$

Equations (11) and (12) are similar to a discrete Fourier transform pair and are useful when we only wish to define $H(\omega)$ for $\omega > 0$. We have generally adopted the latter convention in this book. When $h(t)$ is only defined for integer values of t, Equations (11) and (12) become

$$H(\omega) = \frac{1}{\pi} \left[h(0) + 2 \sum_{t=1}^{\infty} h(t)\, \cos \omega t \right] \tag{13}$$

$$h(t) = \int_{0}^{\pi} H(\omega)\, \cos\ \omega t\, \mathrm{d}\omega \tag{14}$$

and $H(\omega)$ is now only defined on $[0, \pi]$.

The **Laplace transform** of a function $h(t)$, which is defined for $t > 0$, is given by

$$H(s) = \int_0^\infty h(t)\, \mathrm{e}^{-st}\, \mathrm{d}t. \tag{15}$$

As compared with the Fourier transform, note that the lower limit of the integral is zero, and not $-\infty$, and that s is a complex variable, which may have real *and* imaginary parts. The integral converges when the real part of s exceeds some number called the 'abscissa of convergence'.

Given a function $h(t)$, such that

$$h(t) = 0 \qquad \text{for } t < 0 \tag{16}$$

then the Laplace and Fourier transforms of $h(t)$ are the same, provided that the real part of s is zero. More generally, the Laplace transform is a generalization of the Fourier transform for functions defined on the positive real line. Control engineers often prefer to use the Laplace transform when investigating the properties of a linear system, as this will cope with physically realizable systems which are stable *or* unstable. The impulse response function of a physically realizable linear system satisfies Equation (16) and so, for such functions, the Fourier transform is a special case of the Laplace transform. More details about the Laplace transform may be found in many mathematics books.

The **z-transform** of a function $h(t)$ defined on the non-negative integers is given by

$$H(z) = \sum_{t=0}^{\infty} h(t)\, z^{-t}, \tag{17}$$

where z is a complex variable. Comparing Equation (17) with (15) (or with (18) below), we see that the z-transform can be thought of as a discrete version of the Laplace transform, on replacing e^s by z. In discrete time, with a function satisfying Equation (16), some authors (e.g. Hayes, 1996, Section 2.2.5) prefer to use the z-transform rather than the discrete Fourier transform (i.e. Equation (9)) or the discrete form of the Laplace transform, namely

$$H(s) = \sum_{t=0}^{\infty} h(t)\, \mathrm{e}^{-st}. \tag{18}$$

All three transforms have somewhat similar properties, in that a convolution in the time domain corresponds to a multiplication in the frequency domain.

The more advanced reader will observe that, when $\{h(t)\}$ is a probability function such that $h(t)$ is the probability of observing the value t, for $t = 0, 1, \ldots$, then Equation (17) is related to the probability generating function of the distribution, while Equations (9) and (18) are related to the moment generating and characteristic functions of the distribution.

Exercises

A.1 If $h(t)$ is real, show that the real and imaginary parts of its Fourier transform, as defined by Equation (5), are even and odd functions, respectively.

A.2 If $h(t) = \mathrm{e}^{-a|t|}$ for all real t, where a is a positive real constant, show that its Fourier transform, as defined by Equation (5), is given by

$$H(\omega) = 2a/(a^2 + \omega^2) \qquad \text{for } -\infty < \omega < \infty$$

A.3 Show that the Laplace transform of $h(t) = \mathrm{e}^{-at}$ for $(t > 0)$, where a is a real constant, is given by

$$H(s) = 1/(s + a) \qquad \text{for } \mathrm{R}e(s) > a$$

where $\mathrm{R}e(s)$ denotes the real part of s.

Appendix B Dirac Delta Function

Suppose that $\phi(t)$ is any function which is continuous at $t = 0$. Then the **Dirac delta function** $\delta(t)$ is such that

$$\int_{-\infty}^{\infty} \delta(t)\,\phi(t)\,\mathrm{d}t = \phi(0). \tag{19}$$

Because it is defined in terms of its integral properties alone, it is sometimes called the 'spotting' function since it picks out one particular value of $\phi(t)$. It is also sometimes simply called the **delta function**.

Although it is called a 'function', it is important to realize that $\delta(t)$ is *not* a function in the usual mathematical sense. Rather it is what mathematicians call a generalized function, or distribution. This maps a function, $\phi(t)$ say, into the real line, by producing the value $\phi(0)$.

Some authors define the delta function by

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \tag{20}$$

such that

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1.$$

While this is often intuitively helpful, it is mathematically meaningless.

The Dirac delta function can also be regarded as the limit, as $\varepsilon \to 0$, of a pulse of width ε and height $1/\varepsilon$ (i.e. having unit area) defined by

$$u(t) = \begin{cases} 1/\varepsilon & 0 < t < \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

This definition is also not mathematically rigorous, but is heuristically useful. In particular, control engineers can approximate such an impulse by an impulse with unit area whose duration is short compared with the least significant time constant of the response to the linear system being studied.

Even though $\delta(t)$ is a generalized function, it can often be handled as if it were an ordinary function except that we will be interested in the values of integrals involving $\delta(t)$ and never in the value of $\delta(t)$ by itself.

The delta function has many useful properties and we have used $\delta(t)$ in Chapter 9 to analyse particular linear systems in continuous time.

Exercises

B.1 The function $\phi(t)$ is continuous at $t = t_0$. If $a < b$, show that

$$\int_a^b \delta(t - t_0)\, \phi(t)\, \mathrm{d}t = \begin{cases} \phi(t_0) & \text{for } a < t_0 < b \\ 0 & \text{for } t_0 < a \text{ or } t_0 > b. \end{cases}$$

B.2 The function $\phi(t)$ is continuous at $t = 0$. Show that $\phi(t)\, \delta(t) = \phi(0)\, \delta(t)$.

Appendix C Covariance and Correlation

This book assumes knowledge of basic statistical topics such as the laws of probability, probability distributions, expectation and basic statistical inference (including estimation, significance testing and linear regression). Any reader who is unfamiliar with these topics should consult one of the numerous elementary texts covering this material.

The topics of covariance and correlation are usually studied as part of elementary probability or linear regression. However, they are not always clearly understood at first, and are particularly important in the study of time series. Thus they will now be briefly reviewed.

Suppose two random variables X and Y have means $E(X) = \mu_X, E(Y) = \mu_Y$, respectively. Then the **covariance** of X and Y is defined to be

$$\mathrm{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] \tag{21}$$

and may be denoted γ_{XY}. If X and Y are independent, then

$$\begin{aligned} E[(X - \mu_X)(Y - \mu_Y)] &= E(X - \mu_X)E(Y - \mu_Y) \\ &= 0 \end{aligned}$$

so that the covariance is zero. If X and Y are *not* independent, then the covariance may be positive or negative depending on whether 'high' values of X tend to go with 'high' or 'low' values of Y. Here high means greater than the appropriate mean.

Covariance is a useful quantity for many mathematical purposes, but it is difficult to interpret, as it depends on the units in which X and Y are measured. Thus it is often useful to standardize the covariance between two random variables by dividing by the product of their respective standard deviations to give a quantity called the **correlation coefficient**. If we denote the standard deviations of X and Y by σ_X and σ_Y, respectively, then the correlation of X and Y is defined by

$$\mathrm{Corr}(X, Y) = \mathrm{Cov}(X, Y)/\sigma_X\sigma_Y \tag{22}$$

and is typically denoted by ρ_{XY}. It can readily be shown that a correlation coefficient must lie between ± 1 and is a useful measure of the linear association

between two variables. $\text{Corr}(X, Y) = 1$ if $Y = aX + b$ with $a > 0$, and $\text{Corr}(X, Y) = -1$ if $Y = aX + b$ with $a < 0$.

Given N pairs of observations, $\{(x_i, y_i);\ i = 1, \ldots, N\}$, the usual estimate of the covariance between two variables is given by

$$s_{xy} = \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})/(N - 1).$$

If we denote the sample variances of the two variables by s_x^2 and s_y^2, where $s_x^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2/(N - 1)$, then the usual estimate of the correlation coefficient is given by the sample correlation coefficient

$$r_{xy} = s_{xy}/s_x s_y. \tag{23}$$

This is the intuitive estimate of ρ_{XY} as defined above in Equation (22), and also agrees with Equation (2.3) in Chapter 2 on cancelling the denominators $(N - 1)$.

The above definitions are applied to time series as described in Chapters 2 and 3. For a stochastic process $\{X_t\}$ $\text{Cov}(X_t, X_{t+k})$ is a function of t and k, and a function of only k if these process is stationary. Then it is called the autocovariance coefficient function at lag k, denoted as $\gamma(k)$ with $\gamma(0) = \text{Cov}(X_t, X_t) = \text{Var}(X_t)$. Similarly for a stochastic process $\{X_t\}$,

$$\text{Corr}(X_t, X_{t+k}) = \frac{\text{Cov}(X_t, X_{t+k})}{\text{sd}(X_t)\text{sd}(X_{t+k})}$$

is a function of t and k. For a stationary process, it becomes a function of only k, called the autocorrelation coefficient function at lag k, equal to $\gamma(k)/\gamma(0)$ and denoted as $\rho(k)$. See Section 3.2.

Answers to Exercises

Chapter 2

2.1 There are various ways of assessing the trend and seasonal effects. A simple method is to calculate the four yearly averages in 1995, 1996, 1997 and 1998; and also the average sales in each of periods I, II, ..., XIII (i.e. calculate the row and column averages). The yearly averages provide a crude estimate of trend, while the differences between the period averages and the overall average estimate the seasonal effects. With such a small downward trend, this rather crude procedure may well be adequate for most purposes. It has the advantage of being easy to understand and compute. A more sophisticated approach would be to calculate a 13-month simple moving average, moving along one period at a time. This will give trend values for each period from period 7 to period 46. The end values for periods 1 to 6 and 47 to 52 need to be found by some sort of extrapolation or by using a non-centred moving average. The differences between each observation and the corresponding trend value provide individual estimates of the seasonal effects. The average value of these differences in each of the 13 periods can then be found to estimate the overall seasonal effect, assuming that it is constant over the 4-year period.

2.2 It is hard to guess autocorrelations even when plotting x_t against x_{t-1}. Any software package will readily give $r_1 = -0.55$.

2.4 The usual limits of 'significance' are at $\pm 2/\sqrt{N} = \pm 0.1$. Thus r_7 is just 'significant'. However, unless there is some contextual reason for an effect at lag 7, there is no real evidence of non-randomness, as one expects 1 in 20 values to be 'significant' when data really are random.

2.5 (b) Two seasonal differences, ∇_{12}^2, are needed to transform data to stationarity.

Chapter 3

3.1 For example, $\rho(1) = (0.7 - 0.7 \times 0.2)/(1 + 0.7^2 + 0.2^2) = 0.56/1.53$.

3.3 $\mathrm{Var}(X_t)$ is not finite. Then consider $Y_t = X_t - X_{t-1} = Z_t + (C-1)Z_{t-1}$. The latter expression denotes a stationary MA(1) process, with ac.f.

$$\rho_Y(k) = \begin{cases} 1 & k = 0, \\ (C-1)/[1 + (C-1)^2] & k = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.4 $\rho(k) = 0.7^{|k|}$ for $k = 0, \pm1, \pm2, \ldots$. Note this does not depend on μ.

3.6 Looking at the roots of the auxiliary equation (see Section 3.4.4, general-order process), the roots must satisfy $\left|[\lambda_1 \pm \sqrt{(\lambda_1^2 + 4\lambda_2)}]/2\right| < 1$. If $\lambda_1^2 + 4\lambda_2 > 0$, it can easily be shown that $\lambda_1 + \lambda_2 < 1$, and $\lambda_1 - \lambda_2 > -1$. If $\lambda_1^2 + 4\lambda_2 < 0$, the roots are complex and we find $\lambda_2 > -1$.
When $\lambda_1 = 1/3$ and $\lambda_2 = 2/9$, the Yule–Walker equations have auxiliary equation $y^2 - \frac{1}{3}y - \frac{2}{9} = 0$, which has roots $(\frac{2}{3})$ and $(-\frac{1}{3})$. Thus the general solution is $\rho(k) = A_1(\frac{2}{3})^{|k|} + A_2(-\frac{1}{3})^{|k|}$. Now use $\rho(0) = 1$ and $\rho(1) = [\frac{1}{3} + \frac{2}{9}\rho(1)]$, giving $\rho(1) = 3/7$, to evaluate $A_1 = 16/21$ and $A_2 = 5/21$.

3.8
$$\gamma_Y(k) = 2\gamma_X(k) - \gamma_X(k+1) - \gamma_X(k-1).$$

$$\gamma_Y(k) = \begin{cases} 2 - 2\lambda & k = 0 \\ -\lambda^{|k|-1}(1-\lambda)^2 & k = \pm1, \pm2, \ldots . \end{cases}$$

3.9 All three models are stationary and invertible. For model (a), we find $X_t = Z_t + 0.3Z_{t-1} + 0.3^2 Z_{t-2} + \cdots$.

3.12 (a) $p = d = q = 1$.

(b) Model is non-stationary, but is invertible.

(c) 0.7, 0.64, 0.628 —- decreasing very slowly as process is non-stationary.

(d) 0.7, 0.15, 0.075, 0.037 — decreasing quickly towards zero as invertible.

3.13 $\rho(k) = \frac{45}{38}(3/4)^{|k|} - \frac{7}{38}(1/4)^{|k|}$ $k = 0, \pm1, \pm2, \ldots$.
The AR(3) process is non-stationary, as the equation $(1 - B - cB^2 + cB^3) = 0$, has a root on the unit circle, namely, $B = 1$.

3.14 $\mathrm{Cov}[Ye^{\mathrm{i}\omega t}, \overline{Y}e^{-\mathrm{i}\omega(t+\tau)} = e^{-\mathrm{i}\omega\tau}\ \mathrm{Cov}[Y, \overline{Y}]$ does not depend on t.
$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$. When θ is uniformly distributed on $(0, 2\pi)$, then $\mathrm{E}(\cos\theta) = \mathrm{E}(\sin\theta) = 0$. Hence result.

Chapter 4

4.2 The least squares normal equations are the same as the sample Yule–Walker equations in Equation (4.12) except that the constant divisor $\sum(x_t - \bar{x})^2$ is omitted and the estimated autocovariance in Equation (4.1) is effectively estimated by summing over $(N - p)$, rather than $(N - k)$ cross-product terms.

4.3 When fitting an AR(2) process, π_2 is the coefficient α_2. For such a process, the first two Yule–Walker equations are: $\rho(2) = \alpha_1\rho(1) + \alpha_2$ and $\rho(1) = \alpha_1 + \alpha_2\rho(-1) = \alpha_1 + \alpha_2\rho(+1)$. Solve for $\pi_2 = \alpha_2$ by eliminating α_1.

4.4 π_j is zero for j larger than 2, the order of the process. π_2 equals the final coefficient in the model, namely, $\alpha_2 = 2/9$; and so we only have to find π_1, which equals $\rho(1)$. This is 9/21 — see Exercise 3.6.

4.5 The values come down quickly to zero, suggesting a stationary process. Values outside the range $\pm2/\sqrt{100} = \pm0.2$ are significantly different from zero, in this case just r_1, r_2 and r_5. An MA(2) process has an ac.f. of this

shape, meaning non-zero coefficients at lags 1 and 2. Given the small sample size, it is, however, possible that an AR(2) process could be appropriate, as the theoretical autocorrelations beyond lag 2 would be quite small, even if not exactly zero as for the MA(2) model. In practice, it is often difficult to distinguish between competitor models having similar ac.f.s and there may be no 'right answer'. All models are approximations anyway.

4.6 The ac.f. of the data indicates non-stationarity in the mean, as the values of r_k are only coming down to zero very slowly. So we need to take first differences (at least). The values of r_k for the first differences are all 'small' and are not significantly different from zero (all are less than $\pm 2/\sqrt{60}$). This suggests the series is now stationary and is just a purely random process. Thus a possible model for the original series is an ARIMA(0, 1, 0) model, otherwise known as a random walk. Of course, in practice, we would like to see a time plot of the data, check for outliers, etc. as well as getting more relevant background information about the economic variable concerned.

Chapter 5

5.4 Suppose we denote the model by

$$(1 - \alpha B^{12})W_t = (1 + \theta B)Z_t$$

$$\text{or} \qquad X_t = Z_t + \theta Z_{t-1} + (1 + \alpha)X_{t-12} - \alpha X_{t-24}.$$

Then $\hat{x}_N(1) = (1 + \hat{\alpha})x_{N-11} - \hat{\alpha}\, x_{N-23} + \hat{\theta}\, \hat{z}_N$ and
$\hat{x}_N(h) = (1 + \hat{\alpha})x_{N+h-12} - \hat{\alpha}\, x_{N+h-24}$ for $h = 2, 3, \ldots, 12$.

5.5
$$\hat{x}_N(1) = 1.2x_N - 0.2x_{N-1} - 0.5z_N$$

$$\hat{x}_N(2) = 1.2\hat{x}_N(1) - 0.2x_N$$

$\text{Var}[e_N(h)]$ equals σ_Z^2 when $h = 1$, $1.49\sigma_Z^2$ when $h = 2$, and $1.90\sigma_Z^2$ when $h = 3$.

Chapter 6

6.1 (b) $f^*(\omega) = (1 - \lambda^2)/\pi(1 - 2\lambda\cos\omega + \lambda^2)$ for $0 < \omega < \pi$.

6.2 (a) $f(\omega) = \sigma_Z^2[3 + 2(2\cos\omega + \cos 2\omega)]/\pi$ for $0 < \omega < \pi$.
(b) $f^*(\omega) = \sigma_Z^2[1.34 + 2(0.35\cos\omega - 0.3\cos 2\omega)]/\pi$ for $0 < \omega < \pi$.

6.3 The non-zero mean makes no difference to the acv.f., ac.f. or spectrum.

$$\gamma(k) = \begin{cases} 1.89\sigma_Z^2 & k = 0, \\ 1.2\sigma_Z^2 & k = \pm 1, \\ 0.5\sigma_Z^2 & k = \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho(k) = \begin{cases} 1 & k = 0, \\ 1.2/1.89 & k = \pm 1, \\ 0.5/1.89 & k = \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

6.4 This triangular spectrum gives rise to a rather unusual ac.f. Dividing through Equation (6.9) by σ_X^2 (or $\gamma(0)$), we get $\rho(k) = \int_0^\pi f^*(\omega)\cos\omega k\,\mathrm{d}\omega$ When $k = 0$, we find, as expected, that $\rho(0) = \int_0^\pi f^*(\omega)\,\mathrm{d}\omega = 1$. When k is even, $\cos\omega k$ completes an integer number of cycles between 0 and π so that $\int_0^\pi \cos\omega k\,\mathrm{d}\omega = 0$, and can further show that $\int_0^\pi \omega\cos\omega k\,\mathrm{d}\omega = 0$ after some messy algebra and integrating by parts (integrate $\cos\omega k$ and differentiate ω). When k is odd, $\int_0^\pi \cos\omega k\,\mathrm{d}\omega$ is still zero, but we find, on integrating by parts, that $\int_0^\pi \omega\cos\omega k\,\mathrm{d}\omega = 2/k^2$.

6.5 Clearly $E[X(t)] = 0$ and $\mathrm{Var}[X(t)] = 1$. Thus $\rho(u) = \gamma(u) = E[X(t)X(t+u)] = \mathrm{Prob}[X(t)$ and $X(t+u)$ have same sign$] - \mathrm{Prob}[X(t)$ and $X(t+u)$ have opposite sign$]$.
(Hint: Prob(observing an even number of changes in time u) $= \mathrm{e}^{-\lambda u}\left[1 + \frac{(\lambda u)^2}{2!} + \frac{(\lambda u)^4}{4!} + \cdots\right] = \mathrm{e}^{-\lambda u}(\mathrm{e}^{\lambda u} + \mathrm{e}^{-\lambda u})/2$.)
With a unit variance, $f(\omega) = f^*(\omega)$. As process is in continuous time, use Equation (6.17). Algebra is rather messy. It may help to write $\cos\omega u = [\mathrm{e}^{\mathrm{i}\omega u} + \mathrm{e}^{-\mathrm{i}\omega u}]/2$.

6.6 Here $f_Y = \sigma^2/\pi$ and $f_X = \sigma^2/\pi(1 - 2\alpha\cos\omega + \alpha^2)$ for $0 < \omega < \pi$.

6.7 Use Equation (6.16). As in the second part of the solution of Equation (6.5), algebra is a bit messy, and it may help to write $\cos\omega k$ as $[\mathrm{e}^{\mathrm{i}\omega k} + \mathrm{e}^{-\mathrm{i}\omega k}]/2$. Then you need to be able to sum a geometric progression (G.P.).

Chapter 8

8.1 $f_{XY}(\omega) = \sigma_Z^2\left[\beta_{11}\beta_{21} + \beta_{12}\beta_{22} + \beta_{21}\mathrm{e}^{-\mathrm{i}\omega} + \beta_{12}\mathrm{e}^{+\mathrm{i}\omega}\right]/\pi \quad$ for $0 < \omega < \pi$.

8.2 Use the fact that $\mathrm{Var}[\lambda_1 X(t) + \lambda_2 Y(t+\tau)] \geq 0$ for any constants λ_1, λ_2 (e.g. putting $\lambda_1 = \sigma_Y, \lambda_2 = \sigma_X$ gives $\rho_{XY} \geq -1$).
For the given bivariate process, we find

$$\gamma_{XY}(k) = \begin{cases} 0.84\sigma_Z^2 & k = 0, \\ -0.4\sigma_Z^2 & k = 1, \\ 0.4\sigma_Z^2 & k = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we find (for $0 < \omega < \pi$)
$c(\omega) = 0.84\sigma_Z^2/\pi, \qquad q(\omega) = -0.8\sigma_Z^2\sin\omega\,/\,\pi,$
$\alpha_{XY}(\omega) = \sigma_Z^2\sqrt{(0.84^2 + 0.8^2\sin^2\omega)\,/\,\pi}, \qquad \tan\phi_{XY}(\omega) = 0.8\sin\omega\,/\,0.84,$
$C(\omega) = 1.$

Chapter 9

9.1 They are all linear except (b) and (g). Note that (e) is linear but not time-invariant.

9.2 Selected answers only: For (a), h_k is $\frac{1}{2}$ for $k = \pm 1$, 1 for $k = 0$, and zero otherwise. Then we find

$$S_t = \begin{cases} 0 & t < -1, \\ \frac{1}{2} & t = -1, \\ 1\frac{1}{2} & t = 0, \\ 2 & t \geq 1. \end{cases}$$

Then $H(\omega) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega} + 1 + \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega} = 1 + \cos\omega$. As this is real and non-negative, $G(\omega) = H(\omega)$, and $\phi(\omega) = 0$.
(b) $H(\omega) = \frac{1}{5} + \frac{2}{5}\cos\omega + \frac{2}{5}\cos 2\omega$
(c)

$$h_k = \begin{cases} +1 & k = 0, \\ -1 & k = +1, \\ 0 & \text{otherwise.} \end{cases}$$

Filters (a) and (b) are low pass, while (c) and (d) are high pass. The combined filter from (a) and (b) has
$H(\omega) = (1 + \cos\omega)\big(\frac{1}{5} + \frac{2}{5}\cos\omega + \frac{2}{5}\cos 2\omega\big)$.

9.3 (a) $H(\omega) = g\mathrm{e}^{-\mathrm{i}\omega\tau}$ for $\omega > 0$.
(b) $H(\omega) = g(1 - \mathrm{i}\omega T) \,/\, (1 + \omega^2 T^2)$ for $\omega > 0$.

9.4 For the one-parameter AR(2) process, we find

$$\gamma(k) = \begin{cases} \alpha^{|k/2|}\sigma_Z^2 \,/\, (1 - \alpha^2) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$f_X(\omega) = f_Z(\omega) \,/\, (1 - 2\alpha\cos 2\omega + \alpha^2)$ for $0 < \omega < \pi$.

Chapter 10

10.1 It is generally sensible to restrict α to the range $(0, 1)$, although it is possible to devise a plausible ARIMA scenario with θ positive in the range $(0, 1)$, and this would extend the plausible range for α from $(0, 1)$ to $(0, 2)$.

10.2 $X_2 = \mu_2 + n_2$ and $X_1 = \mu_1 + n_1 = \mu_2 - \beta_1 + n_1 = \mu_2 - \beta_2 + n_1 + w_2$. Thus

$$\begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mu_2 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} n_2 \\ n_1 + w_2 \end{bmatrix}$$

So

$$\hat{\boldsymbol{\theta}}_2 = \begin{bmatrix} \hat{\mu}_2 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_2 - X_1 \end{bmatrix}$$

— see Equation (10.17).

To find P_2, $\text{Var}(\hat{\beta}_2)$, for example, is

$$E(X_2 - X_1 - \beta_2)^2 = E(n_2 - n_1 + \beta_2 - w_2 - \beta_2)^2 = 2\sigma_n^2 + \sigma_w^2.$$

Using the Kalman filter, we find $\hat{\boldsymbol{\theta}}_{3|2} = \begin{bmatrix} 2X_2 - X_1 \\ X_2 - X_1 \end{bmatrix}$, $P_{3|2} = \sigma_n^2 \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, and $K_3 = \begin{bmatrix} 5/6 \\ 1/2 \end{bmatrix}$. Hence $\hat{\boldsymbol{\theta}}_3$ using Equation (10.13).

10.3 (a) For $\boldsymbol{\theta}_t^T = [X_t, \beta Z_t]$, we find $\boldsymbol{h}^T = [1, 0]$; $G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $\boldsymbol{w}_t^T = [1, \beta]Z_t$. Alternatively, if we take $\boldsymbol{\theta}_t^T = [X_t, Z_t]$, for example, then we find $G = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$.

(b) Try $\boldsymbol{\theta}_t^T = [X_t, \hat{X}_t(1), \hat{X}_t(2)] = [X_t, \beta_1 Z_t + \beta_2 Z_{t-1}, \beta_2 Z_t]$ with $G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Or try $\boldsymbol{\theta}_t^T = [X_t, Z_t, Z_{t-1}]$.

Chapter 13

13.1 This is a VAR(1) model with a coefficient matrix at lag one equal to $\Phi = \begin{pmatrix} 1 & 0.5 \\ 0.2 & 0.7 \end{pmatrix}$. The roots of the equation: determinant $\{I - \Phi x\} = 0$ are $1/1.2$ and $1/0.5$. Thus one of the roots lies 'inside the unit circle'. (When, as here, the roots are real, then one only needs to look and see if the magnitude of the root exceeds unity.) This means the process is non-stationary. This could be guessed from the first equation of the model where the coefficient of $X_{1,t-1}$ is 'large' (unity) and so, in the absence of X_{2t} would give a (non-stationary) random walk for X_{1t} (this is not meant to be a proper mathematical argument).

13.2 The roots of the equation: determinant $\{I - \Phi x\} = 0$ are as follows: (a) a single root of $1/0.6$; (b) a single root of $1/1.4$; (c) two roots of $1/0.7$ and $1/0.3$; (d) two roots of $1/0.8$ and $1/0.4$. This means that models (a), (c) and (d) are stationary (giving roots exceeding unity in magnitude so they are 'outside the unit circle') but (b) is not.

13.3 Model (c) of Exercise 12.2 is stationary and has a diagonal coefficient matrix. This means that the model consists of two independent AR(1) processes. Any cross-covariances are zero and so the covariance and correlation matrix functions are diagonal. We find

$$\Gamma(k) = \begin{pmatrix} 0.7^{|k|}\sigma_1^2 & 0 \\ 0 & 0.3^{|k|}\sigma_2^2 \end{pmatrix} \text{ and } P(k) = \begin{pmatrix} 0.7^{|k|} & 0 \\ 0 & 0.3^{|k|} \end{pmatrix}.$$

13.4 All pure MA processes, whether univariate or multivariate, are stationary. The model is invertible if the roots of the equation, determinant $\{I + \Theta_1 x\} = 0$, are outside the unit circle. In this case, we find that the roots are $-1/0.8$ and $-1/0.2$, which are both smaller than -1 and so lie outside the unit circle. Thus the model *is* invertible.

13.5 Denote the forecast of $\boldsymbol{X}_{N+h}$ made at time N by $\hat{\boldsymbol{X}}_N(h)$. Then $\hat{\boldsymbol{X}}_N(1) = \Phi \boldsymbol{X}_N$ and $\hat{\boldsymbol{X}}_N(2) = \Phi \hat{\boldsymbol{X}}_N(1) = \Phi^2 \boldsymbol{X}_N$. Picking out the first component of $\boldsymbol{X}$, for example, we find $\hat{X}_{1,N}(1) = 0.9X_{1,N} + 0.5X_{2,N}$ while $\hat{X}_{1,N}(2) = 0.76X_{1,N} + 0.60X_{2,N}$.

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