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Nonparametric Statistical Inference

Fifth Edition

Jean Dickinson Gibbons
Subhabrata Chakraborti

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Nonparametric Statistical Inference

Fifth Edition

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To the memory of my parents,

John and Alice Dickinson,

And to my husband, John S. Fielden

Jean Dickinson Gibbons

To the memory of my father,

Himangshu,

To my mother, Pratima,

And to my wife, Anuradha, and son, Siddhartha Neil

Subhabrata Chakraborti

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Preface to the Fifth Edition

In the fifth edition of this book, we are going to digress from tradition and write a new preface that will replace all earlier prefaces.

This book began as a slim volume of only 306 pages written by Jean D. Gibbons and published by McGraw-Hill in 1971. The focus was on the theory of nonparametric tests of hypotheses in 14 chapters with some theoretical exercises at the end of each chapter. It was written while Jean was an associate professor of statistics at the Wharton School of the University of Pennsylvania, Philadelphia; it had a paper jacket with a picture of the author on the flyleaf.

The preface to this first edition stated that the organization of the material was primarily according to the type of statistical information collected and the type of questions to be answered by the inference procedures or according to the general type of mathematical derivation. This organization has proved highly satisfactory and hence has prevailed in each of the succeeding editions. For each statistic introduced, the null distribution theory is derived, or where this would be too tedious, the procedure one could follow is outlined, or when this would be too theoretical, the results are stated without proof. Generally, the other relevant mathematical details necessary for nonparametric inference are also included, for both hypothesis testing and confidence interval estimation. The purpose is to acquaint the reader with the mathematical logic on which a test is based, those test properties that are essential for understanding the procedures, and the basic tools necessary for comprehending the extensive literature published in the statistics journals. The book is not intended to be a user's manual for the application of nonparametric procedures, although we recognize that applications are very important and they are amply illustrated throughout the book by examples and exercises.

The reviews of this first edition were highly favorable and encouraging, labeling it ''the first nonparametrics theory book.'' It was widely adopted for classes taught to first- and second-year graduate students. The preface expressed the debt to the colleagues who had authored the only other nonparametrics books in existence at that time, namely, Gottfried E. Noether (1967), James V. Bradley (1968), and Maurice G. Kendall (1962), and also Jean's gratitude to her friend and mentor Herbert A. David. His training and encouragement helped make this first edition a reality.

The second edition grew to 424 pages and was published by Marcel Dekker in 1985, when Donald B. Owen was the editor of the very well respected *Statistics: Textbooks and Monographs Series*. This edition added a 15th chapter that gave data-oriented examples of applications of all of the methods described in the first edition plus extensive applied exercises. It also contained the many tables needed to work these exercises. At the time of publication,

the other nonparametrics books in print were Conover (1st edition, 1971), Lehmann (1975), Randles and Wolfe (1979), and Pratt and Gibbons (1981). Jean is very grateful to Don Owen for his encouragement and support in writing this edition.

In 1992, Subhabrata Chakraborti collaborated with Jean on the third edition and it grew to 544 pages. Subha had studied nonparametric statistics at SUNY Buffalo using the first edition before he joined the faculty at the University of Alabama, where Jean was now teaching. In this third edition, also published by Marcel Dekker, the 15th chapter was eliminated so that the methods and theory could be integrated in each chapter. The coverage was widely expanded to include many additional tests; confidence interval estimation was introduced, the references were increased, and some numerical examples with computer solutions were included.

The fourth edition, published in 2003 again by Marcel Dekker, encompassed 645 pages, over twice its original length from 1971. It contained extensive rewrites, updates, and expansions, including sample size determination, multiple comparisons, and power calculations. The chapter on order statistics was altered extensively in the hope of providing more motivation for the succeeding chapters covering procedures based on ranks. Perhaps the most significant changes to make this fourth edition more user-friendly and student-oriented were the inclusion of chapter summaries and answers to selected problems. Computer simulations for power calculations and a new chapter on analysis of count data were also added.

We feel that the topics covered in the fourth edition are more than enough for an instructor to pick and choose the material he wants to cover in a one-semester course. We have fine tuned this coverage and refined it in many places.

Specifically, for the fifth edition, we have

1. Entirely revised the textual material on at least 50% of the topics covered in the fourth edition to make it more readable and reader friendly
2. Updated many of the topics with material from more recent journal articles in the literature
3. Added a new section under goodness of fit tests in Chapter 4
4. Added a new Chapter 15 with practical guidance on how to choose among the various nonparametric procedures covered in the book
5. Reorganized some of the chapters, moving some old material into the new Chapter 14
6. Added some new problems and moved some problems from one chapter to another when appropriate
7. Redone a lot of the computer figures to make them more readable and hence user friendly

With a book of this length and detailed technical exposition, it is impossible to eliminate all errors. We have done our very best to give a careful job of proofreading, but are aware that an error-free book of this type and length is impossible. We would appreciate it if readers point out any errors, major or minor, by reporting them to us by e-mail. We should be able to incorporate these corrections in future printings of this edition and any future editions.

The tables required for solutions to data-oriented examples were first included in the second edition when methods were added to the text, and other tables have been added as additional topics were included in subsequent editions. We are very proud of our collection of tables and think one of this book's most significant features is providing tables specifically oriented toward (1) reporting *P* values, as opposed to using an arbitrary level of significance in hypothesis testing, and (2) obtaining confidence interval estimates of parameters. We are grateful to the many publishers who granted permission to reprint these tables, but would like to point out that the formats of these tables are original.

Over the years, the published book reviews and comments by users have been extremely helpful, and the authors have made every possible attempt to respond favorably to their suggestions. The fourth edition was reviewed by Ernst (2005), Jolliffe (2004), Ziegel (2004), and in Brief Reviews of Teaching Materials in *The American Statistician* (June 2004, p. 175). Ziegel (2004) stated in his review, ''There is no competition for this book and its comprehensive development and applications of nonparametric methods.'' A very special thank you goes to Professors Linda J. Davis and Clifton D. Sutton of George Mason University who offered extremely helpful comments and corrections in personal correspondence; David Buhner of the University of Texas Southwestern Medical School and Subhash Kochar of Portland State University also pointed out some typographical errors. We have tried to be responsive to all of these comments in this fifth edition.

Reviews of the third edition were published by Jones (1993), Prvan (1993), and Ziegel (1993). Clifton D. Sutton of George Mason University provided very useful comments on this third edition via personal correspondence. The second edition was reviewed by Moore (1986), Randles (1986), Sukhatme (1987), and Ziegel (1988). Finally, we are grateful for the comments of reviewers of the first edition, Dudewicz and Geller (1972), Johnson (1973), Klotz (1972), Govindarajulu (1972), and Noether (1972).

Ergmann, Ludbrook, and Spooren (2000) warn of possible meaningful differences in the outcomes of *P* values from different statistical packages. These differences can be due to the use of exact versus asymptotic distributions, use or nonuse of a continuity correction, or use or nonuse of a correction for ties. The computer output seldom gives such details of calculations, and even the ''Help'' facility and the manuals do not always give a clear description or documentation of the methods used to carry out the computations. Because this warning is quite valid, we have tried to explain to the

best of our ability any discrepancies between our hand calculations and the package results for each of our examples.

The book contains much-more-than-enough material for a one semester course. Even though Chapter 1 will be primarily review material for most students, we recommend that it be assigned at least for reading as homework to make sure everyone is using the same notation and terminology. Chapters 3, 4, and 14 can be omitted without interrupting any continuity, or they can be covered as outside readings. For those wishing to concentrate on procedures based on ranks, Chapters 2 and 6 through 13 are recommended. Many students will have already been exposed to the basic methods of the sign test, Wilcoxon signed rank test and Mann–Whitney–Wilcoxon two-sample test in other courses, but there is much additional coverage here on those topics, including treatment of ties and derivation of a correction for ties, recursive relations for generating null distributions, confidence interval estimates, power calculations, simulations, and sample size determination. Since the material on the various median tests in Chapters 6 and 10 have somewhat limited desirability in applications, they could be assigned for outside reading or even omitted if time is a problem, but these sections contain much interesting theory and discussion.

If professors want to supplement the course with an outside writing assignment, we suggest that they consider having the students go to the professional journals in a field of their interest and locate two articles that use nonparametric statistical methods, either correctly or incorrectly, and write a critique of these applications including the appropriate data analysis. If the data are not given in the article, the student can simulate data for their analyses.

As with any project of this magnitude, many persons have made contributions. We wish to thank David Grubbs of Taylor & Francis for encouraging us to prepare this new edition. We sincerely hope that this edition is close to error free, although we realize this is a faint hope. As always, readers' comments are welcome via e-mail and we will post them on the Web site. The e-mail addresses for the authors are jfielden@peoplepc.com and schakrab@cba.ua.edu, respectively. The book Web site is currently at http://bama.ua.edu/∼schakrab/NSI where information will be posted as necessary.

Jean Dickinson Gibbons
Subhabrata Chakraborti

1

Introduction and Fundamentals

1.1 Introduction

In many elementary statistics courses, the subject matter is somewhat arbitrarily divided into two categories, called descriptive and inductive statistics. *Descriptive statistics* usually relates only to the calculation or presentation of figures (visual or conceptual) to summarize or characterize a set of data. For such procedures, no assumptions are made or implied, and there is no question of legitimacy of techniques. The descriptive figures may be a mean, median, variance, range, histogram, etc. Each of these figures summarizes a set of numbers in its own unique way; each is a distinguishable and well-defined characterization of data. If such data constitute a random sample from a certain population, the sample represents the population in miniature and any set of descriptive statistics provides some information regarding this universe. The term *parameter* is generally employed to connote a characteristic of the population. A parameter is often an unspecified constant appearing in a family of probability distributions, but the word can also be interpreted in a broader sense to include almost all descriptions of population characteristics within a family.

When sample descriptions are used to infer some information about the population, the subject is called *inductive statistics* or *statistical inference*. The two types of problems most frequently encountered in this kind of statistics are estimation and tests of hypotheses. The factor that makes inference a scientific method, thereby differentiating it from mere guessing, is the ability to make evaluations or probability statements concerning the accuracy of an estimate or reliability of a decision. Unfortunately, such scientific evaluations cannot be made without some information regarding the probability distribution of the random variable relating to the sample description used in the inference procedure. This means that certain types of sample descriptions will be more popular than others, because of their distribution properties or mathematical tractability. The sample arithmetic mean is a popular figure for describing the characteristic of central tendency for many reasons but perhaps least of all because it is a mean. The unique position of the mean in inference stems largely from its ''almost normal'' distribution properties for

sample sizes larger than 30. If some other measure, say the sample median, had a property as useful as the central-limit theorem, surely it would share the spotlight as a favorite description of location.

The entire body of classical statistical-inference techniques is based on fairly specific assumptions regarding the nature of the underlying population distribution; usually its form and some parameter values must be stated. Given the right set of assumptions, certain test statistics can be developed using mathematical procedures that are frequently elegant and beautiful. The derived distribution theory is qualified by certain prerequisite conditions, and therefore all conclusions reached using these techniques are exactly valid only so long as the assumptions themselves can be substantiated. In textbook problems, the requisite postulates are frequently just stated and the student simply gets practice applying the appropriate technique. However, in a real-world problem, everything does not come packaged with labels of population of origin. A decision must be made as to what population properties may judiciously be assumed for the model. If the reasonable assumptions are not such that the traditional techniques are applicable, the classical methods may be used and inference conclusions stated only with the appropriate qualifiers (e.g., ''If the population is normal, then....'').

The mathematical statistician may claim that it is the users' problem to decide on the legitimacy of the postulates. Frequently in practice, the assumptions that are deemed reasonable by empirical evidence or past experience are not the desired ones, that is, those for which standard statistical techniques have been developed. Alternatively, the sample size may be too small or previous experience too limited to determine what is a reasonable assumption. Or, if the researchers are a product of the ''cookbook school'' of statistics, their particular expertise being in the area of application, they may not understand or even be aware of the preconditions implicit in the derivation of the statistical technique. In any of these three situations, the result often is a substitution of blind faith for scientific method, either because of ignorance or with the rationalization that an approximately accurate inference based on recognized and accepted scientific techniques is better than no answer at all or a conclusion based on common sense or intuition.

Alternative techniques are available. The mathematical bases for these procedures are the subject of this book. They may be classified as distribution-free and nonparametric procedures. In a *distribution-free* inference, whether for testing or estimation, the methods are based on functions of the sample observations whose corresponding random variable has a distribution that does not depend on the specific distribution function of the population from which the sample was drawn. Therefore, assumptions regarding the underlying population are not necessary. On the other hand, strictly speaking, the term *nonparametric test* implies a test for a hypothesis which is not a statement about parameter values. The type of statement permissible then depends on the definition accepted for the term parameter. If parameter is interpreted in the broader sense, the hypothesis can be concerned only with the form of the

population, as in goodness-of-fit tests, or with some characteristic of the probability distribution of the sample data, as in tests of randomness and trend. Needless to say, distribution-free tests and nonparametric tests are not synonymous labels or even in the same spirit, since one relates to the distribution of the test statistic and the other to the type of hypothesis to be tested. A distribution-free test may be for a hypothesis concerning the median, which is certainly a population parameter within our broad definition of the term.

In spite of the inconsistency in nomenclature, we will follow the customary practice and consider both types of tests as procedures in nonparametric inference, making no distinction between the two classifications. For the purpose of differentiation, the classical statistical techniques, whose justification in probability is based on specific assumptions about the population sampled, may be called *parametric methods*. This implies a definition of nonparametric statistics then as the treatment of either nonparametric types of inferences or analogies to standard statistical problems when specific distribution assumptions are replaced by very general assumptions and the analysis is based on some function of the sample observations whose sampling distribution can be determined without knowledge of the specific distribution function of the underlying population. The assumption most frequently required is simply that the population be continuous. More restrictive assumptions are sometimes made, for example, that the population is symmetrical, but not to the extent that the distribution is specifically postulated. The information used in making nonparametric inferences generally relates to some function of the actual magnitudes of the random variables in the sample. For example, if the actual observations are replaced by their relative rankings within the sample and the probability distribution of some function of these sample ranks can be determined by postulating only very general assumptions about the basic population sampled, this function will provide a distribution-free technique for estimation or hypothesis testing. Inferences based on descriptions of these derived sample data may relate to whatever parameters are relevant and adaptable, such as the median for a location parameter. The nonparametric and parametric hypotheses are analogous, both relating to location, and identical in the case of a continuous and symmetrical population.

Tests of hypotheses that are not statements about parameter values have no counterpart in parametric statistics and thus here nonparametric statistics provides techniques for solving new kinds of problems. On the other hand, a distribution-free test simply relates to a different approach to solving standard statistical problems, and therefore comparisons of the merits of the two types of techniques are relevant. Some of the more obvious general advantages of nonparametric-inference procedures can be appreciated even before our systematic study begins. Nonparametric methods generally are quick and easy to apply, since they involve extremely simple arithmetic. The theory of nonparametric inference relates to properties of the statistic used in the

inductive procedure. Discussion of these properties requires derivation of the random sampling distribution of the pertinent statistic, but this generally involves much less sophisticated mathematics than classical statistics. The test statistic in most cases is a discrete random variable with nonzero probabilities assigned to only a finite number of values, and its exact sampling distribution can often be determined by enumeration or simple combinatorial formulas. The asymptotic distributions are usually normal, chi-square, or other well-known functions. The derivations are easier to understand, especially for nonmathematically trained users of statistics. A cookbook approach to learning techniques is then not necessary, which reduces the danger of misuse of procedures. This advantage also minimizes the opportunities for inappropriate and indiscriminate applications, because the assumptions are so general. When no stringent postulations regarding the basic population are needed, there is little problem of violation of assumptions, with the result that conclusions reached in nonparametric methods usually need not be tempered by many qualifiers. The types of assumptions made in nonparametric statistics are generally easily satisfied, and decisions regarding their legitimacy almost obvious. Besides, in many cases, the assumptions are sufficient, but not necessary, for the test's validity. Assumptions regarding the sampling process, usually that it is a random sample, are not relaxed with nonparametric methods, but a careful experimenter can generally adopt sampling techniques which render this problem academic. With so-called ''dirty data,'' most nonparametric techniques are, relatively speaking, much more appropriate than parametric methods. The basic data available need not be actual measurements in many cases; if the test is to be based on ranks, for example, only the relative magnitudes are needed. The process of collecting and compiling sample data then may be less expensive and time consuming. Some new types of problems relating to sample-distribution characteristics are soluble with nonparametric tests. The scope of application is also wider because the techniques may be legitimately applied to phenomena for which it is impractical or impossible to obtain quantitative measurements. When information about actual observed sample magnitudes is provided but not used as such in drawing an inference, it might seem that some of the available information is being discarded, for which one usually pays a price in efficiency. This is really not true, however. The information embodied in these actual magnitudes, which is not directly employed in the inference procedure, really relates to the underlying distribution, which is a kind of information that is not relevant for distribution-free tests. On the other hand, if the underlying distribution is known, a classical approach to testing may legitimately be used and so this would not be a situation requiring nonparametric methods. The information of course may be consciously ignored, say for the purpose of speed or simplicity.

This discussion of relative merits has so far been concerned mainly with the application of nonparametric techniques. Performance is certainly a matter of concern to the experimenter, but generalizations about reliability are always

difficult because of varying factors like sample size, significance levels, or confidence coefficients, evaluation of the importance of speed, simplicity, and cost factors, and the nonexistence of a fixed and universally acceptable criterion of good performance. Box and Anderson (1955) state that ''to fulfill the needs of the experimenter, statistical criteria should (1) be sensitive to change in the specific factors tested, (2) be insensitive to changes, of a magnitude likely to occur in practice, in extraneous factors.'' These properties, usually called *power* and *robustness*, respectively, are generally agreed upon as the primary requirements of good performance in hypothesis testing. Parametric tests are often derived in such a way that the first requirement is satisfied for an assumed specific probability distribution, for example, using the likelihood-ratio technique of test construction. However, since such tests are, strictly speaking, not even valid unless the assumptions are met, robustness is of great concern in parametric statistics. On the other hand, nonparametric tests are inherently robust because their construction requires only very general assumptions. One would expect some sacrifice in power to result. It is therefore natural to look at robustness as a performance criterion for parametric tests and power for nonparametric tests. How then do we compare analogous tests of the two types?

Power calculations for any test require knowledge of the probability distribution of the test statistic under the alternative, but the alternatives in nonparametric problems are often extremely general. When the requisite assumptions are met, many of the classical parametric tests are known to be most powerful. In those cases where comparison studies have been made, however, nonparametric tests are frequently almost as powerful, especially for small sample sizes, and therefore may be considered more desirable whenever there is any doubt about assumptions. No generalizations can be made for moderate-sized samples. The criterion of asymptotic relative efficiency is theoretically relevant only for very large samples. When the classical tests are known to be robust, comparisons may also be desirable for distributions that deviate somewhat from the exact parametric assumptions. However, with inexact assumptions, calculation of power of classical tests is often difficult except by Monte Carlo techniques, and studies of power here have been less extensive. Either type of test may be more reliable, depending on the particular tests compared and type or degree of deviations assumed. The difficulty with all these comparisons is that they can be made only for specific nonnull distribution assumptions, which are closely related to the conditions under which the parametric test is exactly valid and optimal.

Perhaps the chief advantage of nonparametric tests lies in their very generality, and an assessment of their performance under conditions unrestricted by, and different from, the intrinsic postulates in classical tests seems more expedient. A comparison under more nonparametric conditions would seem especially desirable for two or more nonparametric tests, which are designed for the same general hypothesis testing situation. Unlike the body of classical techniques, nonparametric techniques frequently offer a selection

from interchangeable methods. With such a choice, some judgments of relative merit would be particularly useful. Power comparisons have been made, predominantly among the many tests designed to detect location differences, but again we must add that even with comparisons of nonparametric tests, power can be determined only with fairly specific distribution assumptions. The relative merits of the different tests depend on the conditions imposed. Comprehensive conclusions are thus still impossible for blanket comparisons of very general tests.

In conclusion, the extreme generality of nonparametric techniques and their wide scope of usefulness, while definite advantages in application, are factors that discourage objective criteria, particularly power, as assessments of performance, relative either to each other or to parametric techniques. The comparison studies so frequently published in the literature are certainly interesting, informative, and valuable, but they do not provide the sought-for comprehensive answers under more nonparametric conditions. Perhaps we can even say that specific comparisons are really contrary to the spirit of nonparametric methods. No definitive rules of choice will be provided in this book. The interested reader will find many pertinent articles in all the statistics journals. This book is a compendium of many of the large number of nonparametric techniques which have been proposed for various inference situations.

Before embarking on a systematic treatment of new concepts, some basic notation and definitions must be agreed upon and the groundwork prepared for development. Therefore, the remainder of this chapter will be devoted to an explanation of the notation adopted in this book and an abbreviated review of some of those definitions and terms from classical inference, which are also relevant to the special world of nonparametric inference. A few new concepts and terms will also be introduced that are uniquely useful in nonparametric theory. The general theory of order statistics will be the subject of Chapter 2, since they play a fundamental role in many nonparametric techniques. Quantiles, coverages, and tolerance limits are also introduced here. Starting with Chapter 3, the important nonparametric techniques will be discussed in turn, organized according to the type of inference problem (hypothesis to be tested) in the case of hypotheses not involving statements about parameters, or the type of sampling situation (one sample, two independent samples, etc.) in the case of distribution-free techniques, or whichever seems more pertinent. Chapters 3 and 4 will treat tests of randomness and goodness-of-fit, respectively, both nonparametric hypotheses which have no counterpart in classical statistics. Chapter 5 covers distribution-free tests of hypotheses and confidence interval estimates of the value of a population quantile in the case of one sample or paired samples. These procedures are based on order statistics, signs, and signed ranks. When the relevant quantile is the median, these procedures relate to the value of a location parameter and are analogies to the one-sample (paired-sample) tests for the population mean (mean difference) in classical statistics. Rank-order

statistics are also introduced here, and we investigate the relationship between ranks and variate values. Chapter 6 introduces the two-sample problem and covers some distribution-free tests for the hypothesis of identical distributions against general alternatives. Chapter 7 is an introduction to a particular form of nonparametric test statistic, called a linear rank statistic, which is especially useful for testing a hypothesis that two independent samples are drawn from identical populations. Those linear rank statistics, which are particularly sensitive to differences only in location and only in scale, are the subjects of Chapters 8 and 9, respectively. Chapter 10 extends this situation to the hypothesis that k-independent samples are drawn from identical populations. Chapters 11 and 12 are concerned with measures of association and tests of independence in bivariate and multivariate sample situations, respectively. For almost all tests, the discussion will center on logical justification, null distribution and moments of the test statistic, asymptotic distribution, and other relevant distribution properties. Whenever possible, related methods of interval estimation of parameters are also included. During the course of discussion, only the briefest attention will be paid to relative merits of comparable tests. Chapter 13 presents some theorems relating to calculation of asymptotic relative efficiency, a possible criterion for evaluating large sample performance of nonparametric tests relative to each other or to parametric tests when *certain* assumptions are met. These techniques are then used to evaluate the efficiency of some of the tests covered earlier. Chapter 14 covers some special tests based on count data.

Numerical examples of applications of the most commonly used nonparametric test and estimation procedures are included after the explanation of the theory. These illustrations will serve to solidify the reader's understanding of proper uses of nonparametric methods. All of the solutions show the calculations clearly. In addition, many of the solutions are then repeated using one or more statistical computer packages.

Problems are given at the end of each chapter. The theoretical problems serve to amplify or solidify the explanations of theory given in the text. The applied problems give the reader practice in applications of the methods. Answers to selected problems are given at the end of the book.

1.2 Fundamental Statistical Concepts

In this section, a few of the basic definitions and concepts of classical statistics are reviewed, but only very briefly since the main purpose is to explain notation and terms taken for granted later on. A few of the new fundamentals needed for the development of nonparametric inference will also be introduced here.

1.2.1 Basic Definitions

A *sample space* is the set of all possible outcomes of a random experiment.

A *random variable* is a set function whose domain is the elements of a sample space on which a probability function has been defined and whose range is the set of all real numbers. Alternatively, X is a random variable if for every real number x there exists a probability that the value assumed by the random variable does not exceed x, denoted by $P(X \leq x)$ or $F_X(x)$, and called the *cumulative distribution function* (cdf) of X.

The customary practice is to denote the random variable by a capital letter like X and the actual value assumed (value observed in the experiment) by the corresponding letter in lowercase, x. This practice will generally be adhered to in this book. However, it is not always possible, strictly speaking, to make such a distinction. Occasional inconsistencies will therefore be unavoidable, but the statistically sophisticated reader is no doubt already accustomed to this type of conventional confusion in notation.

The mathematical properties of any function $F_X(x)$, which is a cdf of a random variable X are as follows:

1. $F_X(x_1) \leq F_X(x_2)$ for all $x_1 \leq x_2$, so that F_X is nondecreasing.
2. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.
3. $F_X(x)$ is continuous from the right, or, symbolically, as $\varepsilon \to 0$ through positive values, $\lim_{\varepsilon \to 0} F_X(x + \varepsilon) = F_X(x)$.

A random variable X is called *continuous* if its cdf is continuous. Every continuous cdf in this book will be assumed differentiable everywhere with the possible exception of a finite number of points. The derivative of the cdf will be denoted by $f_X(x)$, a nonnegative function called the *probability density function* (pdf) of X. Thus when X is continuous,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \qquad f_X(x) = \frac{d}{dx} F_X(x) = F'_X(x) \geq 0$$

and

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

A random variable is called *discrete* if it can take on only a finite or a countably infinite number of values, called mass points. The *probability mass function* (pmf) of a discrete random variable X is defined as

$$f_X(x) = P(X = x) = F_X(x) - \lim_{\varepsilon \to 0} F_X(X - \varepsilon)$$

where $\varepsilon \to 0$ through positive values. For a discrete random variable $f_X(x) \geq 0$ and $\sum_{\text{all } x} f_X(x) = 1$, where the expression "all x" is to be interpreted as meaning all x at which $F_X(x)$ is not continuous; in other words, the summation is over all the mass points. Thus, for a discrete random variable, there is a nonzero probability for any mass point, whereas the probability that a continuous random variable takes on any specific value is zero.

The term *probability function* (pf) or *probability distribution* will be used to denote either a pdf or a pmf. For notation, capital letters will always be reserved for the cdf, while the corresponding lowercase letter denotes the pf.

The *expected value* of a function $g(X)$ of a random variable X, denoted by $E[g(X)]$, is

$$E[g(X)] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} g(x) f_X(x)\,dx & \text{if } X \text{ is continuous} \\ \displaystyle\sum_{\text{all } x} g(x) f_X(x) & \text{if } X \text{ is discrete} \end{cases}$$

Joint probability functions and expectations for functions of more than one random variable are similarly defined and denoted by replacing single symbols by vectors, sometimes abbreviated to

$$X_n = (X_1, X_2, \ldots, X_n)$$

A set of n random variables $(X_1, X_2, \ldots, X_n)$ is *independent* if and only if their joint probability function equals the product of the n individual marginal probability functions.

A set of n random variables $(X_1, X_2, \ldots, X_n)$ is called a *random sample* of the random variable X (or from the population F_X or f_X) if they are independent and identically distributed (i.i.d.) so that their joint probability density function is given by

$$f_{X_n}(x_1, x_2, \ldots, x_n) = f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

A *statistic* is any function of observable or sample random variables which does not involve (unknown) parameters.

A *moment* is a particular type of population parameter. The kth moment of X about the origin is $\mu_k' = E(X^k)$, where $\mu_1' = E(X) = \mu$, is the *mean* of X. The kth *central moment* about the mean is

$$\mu_k = E(X - \mu)^k$$

The second central moment about the mean μ_2 is the *variance* of X,

$$\mu_2 = \text{var}(X) = \sigma^2(X) = E(X^2) - \mu^2 = \mu_2' - (\mu_1')^2$$

The kth *factorial moment* is $E[X(X-1)\cdots(X-k+1)]$.

For two random variables, their *covariance* and *correlation*, respectively, are

$$\operatorname{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X \mu_Y$$
$$\operatorname{corr}(X, Y) = \rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sigma(X)\sigma(Y)}$$

The *moment-generating function* (mgf) of a function $g(X)$ of X is $M_{g(X)}(t) = E\{\exp[tg(X)]\}$.

The moments of X about the origin can be obtained by differentiating the mgf and evaluating at 0 as

$$\mu'_k = \frac{d^k}{dt^k} M_X(t)\bigg|_{t=0} = M_X^{(k)}(0), \quad k = 1, 2, \ldots$$

The mgf of a linear function $Y = a + bX$ is

$$M_{a+bX}(t) = e^{bt} M_X(at) \quad \text{for } a \text{ and } b \text{ constant}$$

The moments of Y can be obtained by differentiating its mgf and evaluating at 0.

1.2.2 Moments of Linear Combinations of Random Variables

Let $X_1, X_2, \ldots, X_n$ be n random variables and $a_i, b_i, i = 1, 2, \ldots, n$ be any constants. Then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$
$$\operatorname{var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + 2 \sum_{1 \le i < j \le n} \sum a_i a_j \operatorname{cov}(X_i, X_j)$$
$$\operatorname{cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{i=1}^{n} b_i X_i\right) = \sum_{i=1}^{n} a_i b_i \operatorname{var}(X_i) + \sum_{1 \le i < j \le n} \sum (a_i b_j + a_j b_i) \operatorname{cov}(X_i, X_j)$$

1.2.3 Probability Functions

Some special probability functions are shown in Table 1.2.1, along with the corresponding mean, variance, and mgf. Both discrete and continuous distributions are included; for a discrete distribution, the probability function means the pmf, whereas for a continuous distribution the probability

TABLE 1.2.1

Some Special Probability Functions

Name	Probability Function $f_X(x)$	mgf	$E(X)$	var(X)
Discrete Distributions				
Bernoulli	$\binom{n}{x}p^x(1-p)^{n-x}$ $x = 0, 1$ $0 \le p \le 1$	$pe^t + 1 - p$	P	$p(1-p)$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$ $x = 0, 1, \ldots, n$ $0 \le p \le 1$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson	$\dfrac{e^{-\mu}\mu^x}{x!}$ $x = 0, 1, 2, \ldots$ $\mu < 0$	$e^{\mu(e^t-1)}$	μ	μ
Multinomial	$\dfrac{N!}{x_1!x_2!\ldots x_k!}p_1^{x_1}p_2^{x_2}\ldots p_k^{x_k}$ $x_i = 0, 1, \ldots, N; \quad \sum x_i = N$ $0 \le p_i \le 1, \quad \sum p_i = 1$	$(p_1e^{t_1} + \cdots + p_{k-1}e^{t_{k-1}} + p_k)^N$	$E(x_i) = Np_i$	$\mathrm{Var}(X_i) = Np_i(1-p_i)$ $\mathrm{Cov}(X_i, X_j) = -Np_ip_j$

(*continued*)

TABLE 1.2.1 (continued)

Some Special Probability Functions

Name	Probability Function $f_X(x)$	mgf	$E(X)$	var(X)
Hypergeometric	$\dfrac{\binom{Np}{x}\binom{N-Np}{n-x}}{\binom{N}{x}}$ $x = 0, 1, \ldots, n$ $0 \le p \le 1$	*	Np	$np(1-p)\dfrac{(N-n)}{(N-1)}$
Geometric	$(1-p)^{x-1}p$ $x = 1, 2, \ldots$ $0 \le p \le 1$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{(1-p)}{p^2}$
Uniform on $1, 2, \ldots, N$	$\dfrac{1}{N}$ $x = 1, 2, \ldots, N$	$\sum_{x=1}^{N} \dfrac{e^{tx}}{N}$	$\dfrac{N+1}{2}$	$\dfrac{N^2-1}{12}$
Continuous Distributions				
Uniform on (a, b)	$\dfrac{1}{b-a} \quad a < x < b$	$\dfrac{e^{tb}-e^{ta}}{t(b-a)}$	$\dfrac{b+a}{2}$	$\dfrac{(b-a)^2}{12}$

Normal	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ $-\infty < x, \mu < \infty, \sigma > 0$	$e^{(t\mu + t^2\sigma^2/2)}$	μ	σ^2
Gamma	$\frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)} \quad 0 < x, a, b < \infty$	$(1-bt)^{-a}\ at < 1$	ab	ab^2
Exponential	Gamma with $a = 1$	$(1-bt)^{-1} bt < 1$	b	b^2
Chi-square (v)	Gamma with $a = v/2, \quad b = 2$	$(1-2t)^{-1} \quad 2t < 1$	v	$2v$
Weibull	$abx^{b-1}e^{-ax^b}$ $0 < a, b, x < \infty$	$a^{-t/b}\Gamma(1 + t/b)$	$a^{-1/b}\Gamma(1 + 1/b)$	$a^{-2/b}[\Gamma(1 + 2/b)$ $-\Gamma^2(1 + 1/b)]$
Beta	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$ $0 < x < 1, a, b > 0$	*	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$
Laplace (double exponential)	$\frac{e^{-\lvert x-a\rvert/b}}{2b}$ $-\infty < x, a < \infty, b > 0$	*	a	$2b^2$
Logistic	$\frac{e^{-(x-a)/b}}{b[1+e^{-(x-a)/b}]^2}$ $-\infty < x, a < \infty, b > 0$	*	a	$\frac{b^2\pi^2}{3}$

*The mgf is omitted here because the expression is too complicated.

function stands for the corresponding pdf. The term *standard normal* is used to designate the particular member of the normal family where $\mu = 0$ and $\sigma = 1$. The symbols $\phi(x)$ and $\Phi(x)$ will be reserved for the standard normal density and cumulative distribution functions, respectively.

Three other important distributions are:

Student's t_v: $f_X(x) = \dfrac{v^{-1/2}(1 + x^2/v)^{-(v+1)/2}}{B(v/2, 1/2)} \quad v > 0$

Snedecor's $F(v_1, v_2)$:

$$f_X(x) = \left(\frac{v_1}{v_2}\right)^{v_1/2} x^{v_1/2-1} \frac{(1 + v_1 x/v_2)^{-(v_1+v_2)/2}}{B(v_1/2, v_2/2)} \quad x > 0; \quad v_1, v_2 > 0$$

Fisher's $z(v_1, v_2)$:

$$f_X(x) = 2\left(\frac{v_1}{v_2}\right)^{v_1/2} e^{v_1 x} \frac{(1 + v_1 e^{2x}/v_2)^{-(v_1+v_2)/2}}{B(v_1/2, v_2/2)} \quad x > 1; \quad v_1,\ v_2 > 0$$

The gamma and beta distributions shown in Table 1.2.1 each contains a special constant, denoted by $\Gamma(a)$ and $B(a, b)$, respectively. The *gamma function*, denoted by $\Gamma(a)$, is defined as

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \quad \text{for } a > 0$$

and has the properties

$$\Gamma(a) = \begin{cases} (a-1)! & \text{for any positive integer } a \\ (a-1)\Gamma(a-1) & \text{for any } a > 1, \text{ not necessarily an integer} \\ \sqrt{\pi} & \text{for } a = 1/2 \end{cases}$$

For other fractional values of a, the gamma function can be found from special tables. For example, $\Gamma(1/4) = 3.6256, \Gamma(1/3) = 2.6789$, and $\Gamma(3/4) = 1.2254$ (Abramowitz and Stegun, 1972, p. 255). The gamma function can be conveniently evaluated in MINITAB.

The *beta function*, denoted by $B(a, b)$, is defined as

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx \quad \text{for } a > 0,\ b > 0$$

The beta and the gamma functions have the following relationship:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The gamma and the beta functions can be very helpful in evaluating some complicated integrals. For example, suppose we wish to evaluate the integral $I_1 = \int_0^\infty x^4 e^{-x}\,dx$. We identify I_1 as a gamma function $\Gamma(5)$ with $a = 5$ and then $I_1 = \Gamma(5) = 4! = 24$. Similarly, the integral $I_2 = \int_0^1 x^4(1-x)^7 dx$ is a beta function $B(5, 8)$ with $a = 5$ and $b = 8$, and thus using the relationship with the gamma function and simplifying, we easily get $I_2 = 4!7!/12! = 1/3960$.

1.2.4 Distributions of Functions of Random Variables Using the Method of Jacobians

Let $X_1, X_2, \ldots, X_n$ be n continuous random variables with joint pdf $f(x_1, x_2, \ldots, x_n)$, which is nonzero for an n-dimensional region S_x. Define the transformation

$$y_1 = u_1(x_1, x_2, \ldots, x_n),$$
$$y_2 = u_2(x_1, x_2, \ldots, x_n), \ldots, y_n = u_n(x_1, x_2, \ldots, x_n)$$

which maps S_x onto S_y, where S_x can be written as the union of a finite number m of disjoint spaces $S_1, S_2, \ldots, S_m$ such that the transformation from S_k onto S_y is one to one, for all $k = 1, 2, \ldots, m$. Then for each k, there exists a unique inverse transformation denoted by

$$x_1 = w_{1k}(y_1, y_2, \ldots, y_n),$$
$$x_2 = w_{2k}(y_1, y_2, \ldots, y_n), \ldots, x_n = w_{nk}(y_1, y_2, \ldots, y_n)$$

Assume that for each of these m sets of inverse transformations, the Jacobian

$$J_k(y_1, y_2, \ldots, y_n) = \frac{\partial(w_{1k}, w_{2k}, \ldots, w_{nk})}{\partial(y_1, y_2, \ldots, y_n)} = \det\left(\frac{\partial w_{ik}}{\partial y_i}\right)$$

exists and is continuous and nonzero in S_y, where $\det(a_{ij})$ denotes the determinant of the $n \times n$ matrix with entry a_{ij} in the ith row and jth column. Then the joint pdf of the n random variables $Y_1, Y_2, \ldots, Y_n$, where $Y_i = u_i(X_1, X_2, \ldots, X_n)$,

$$f(y_1, y_2, \ldots, y_n) = \sum_{k=1}^{m} |J_k(y_1, y_2, \ldots, y_n)| f[w_{1k}(y_1, y_2, \ldots, y_n), w_{2k}(y_1, y_2, \ldots, y_n), \ldots, w_{nk}(y_1, y_2, \ldots, y_n)]$$

for all $(y_1, y_2, \ldots, y_n) \in S_y$, and zero otherwise. The Jacobian of the inverse transformation is the reciprocal of the Jacobian of the direct transformation,

$$\frac{\partial(w_{1k}, w_{2k}, \ldots, w_{nk})}{\partial(y_1, y_2, \ldots, y_n)} = \left[\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}\right]^{-1}\Bigg|_{x_i = w_{ik}(y_1, y_2, \ldots, y_n)}$$

or

$$J_k(y_1, y_2, \ldots, y_n) = [J_k(x_1, x_2, \ldots, x_n)]^{-1}$$

Thus the pdf above can also be written as

$$f(y_1, y_2, \ldots, y_n) = \sum_{k=1}^{m} |[J_k(x_1, x_2, \ldots, x_n)]^{-1}| f(x_1, x_2, \ldots, x_n)$$

where the right-hand side is evaluated at $x_i = w_{ik}(y_1, y_2, \ldots, y_n)$ for $i = 1, 2, \ldots, n$. If $m = 1$ so that the transformation from S_x onto S_y is one to one, the subscript k and the summation sign may be dropped. In particular, when $m = 1$ and $n = 1$, the formula reduces to the familiar result for the pdf of a one-to-one function $Y = u(X)$

$$f_Y(y) = \left[f_X(x)\left|\frac{dy}{dx}\right|^{-1}\right]\Bigg|_{x = u^{-1}(y)}$$

1.2.5 Chebyshev's Inequality

Let X be any random variable with mean μ and a finite variance σ^2. Then for every $k > 0$, *Chebyshev's inequality* states that

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Note that the finite variance assumption guarantees the existence of the mean μ.

The following result, called the *central limit theorem* (CLT), is one of the most famous in statistics. We state it for the simplest i.i.d. situation.

1.2.6 Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean μ and variance $\sigma^2 > 0$ and let $\bar{X}_n$ be the sample mean. Then for $n \to \infty$, the random variable $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting distribution that is normal with mean 0 and variance 1.

The reader is referred to any standard graduate level book on mathematical statistics for a proof of this result, typically done via the mgf. In some of the non-i.i.d. situations, there are other types of CLTs available. For example, if the X's are independent but not identically distributed, there is a CLT generally attributed to Liapounov. We will not pursue these any further.

1.2.7 Point and Interval Estimation

A *point estimate* of a parameter is any single function of random variables whose observed value is used to estimate the true value. Let $\hat{\theta}_n = u(X_1, X_2, \ldots, X_n)$ be a point estimate of a parameter θ. Some desirable properties of $\hat{\theta}_n$ are defined as follows for all θ.

1. *Unbiasedness*: $E(\hat{\theta}_n) = \theta$ for all θ.
2. *Sufficiency*: $f_{X_1, X_2, \ldots, X_n | \hat{\theta}_n}(x_1, x_2, \ldots, x_n | \hat{\theta}_n)$ does not depend on θ, or, equivalently, $f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta) = g(\hat{\theta}_n; \theta) H(x_1, x_2, \ldots, x_n)$ where $H(x_1, x_2, \ldots, x_n)$ does not depend on θ.
3. *Consistency* (also called stochastic convergence and convergence in probability):

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| > \in) = 0 \quad \text{for every } \in > 0$$

   a. If $\hat{\theta}_n$ is an unbiased estimate of θ and $\lim_{n \to \infty} \text{var}(\hat{\theta}_n) = 0$, then $\hat{\theta}_n$ is a consistent estimate of θ, by Chebyshev's inequality.
   b. $\hat{\theta}_n$ is a consistent estimate of θ if the limiting distribution of $\hat{\theta}_n$ is a degenerate distribution with probability 1 at θ.
4. *Minimum mean squared error*: $E[(\hat{\theta}_n - \theta)^2] \leq E[(\hat{\theta}_n^* - \theta)^2]$, for any other estimate $\hat{\theta}_n^*$.
5. *Minimum variance unbiased*: $\text{var}(\hat{\theta}_n) \leq \text{var}(\hat{\theta}_n^*)$ for any other estimate $\hat{\theta}_n^*$ where both $\hat{\theta}_n$ and $\hat{\theta}_n^*$ are unbiased.

An *interval estimate* of a parameter θ with confidence coefficient $1 - \alpha$, or a $100(1 - \alpha)$ % *confidence interval* for θ, is a random interval whose end points U and V are functions of observable random variables (usually sample data) such that the probability statement $P(U < \theta < V) = 1 - \alpha$ is satisfied. The probability $P(U < \theta < V)$ should be interpreted as $P(U < \theta)$ and $P(V > \theta)$ since the confidence limits U and V are random variables (depending on the random sample) and θ is a fixed quantity. In many cases, the construction of a confidence interval is facilitated by choosing a pivotal statistic and obtaining the confidence limits via tabulated percentiles of standard probability distributions such as the standard normal or the chi-square. A *pivotal statistic* is a function of a statistic and the parameter of interest such that the distribution of the pivotal statistic is free from the parameter (and is often known

or at least derivable). For example, $t = \sqrt{n}(\bar{X} - \mu)/S$ is a pivotal statistic for setting up a confidence interval for the mean μ of a normal population with an unknown standard deviation. The random variable t follows a Student's t_{n-1} distribution and thus does not involve any unknown parameter. All standard books on mathematical statistics cover the topic of confidence interval estimation.

A useful technique for finding point estimates for parameters which appear as unspecified constants (or as functions of such constants) in a family of probability functions, say $f_X(x;\theta)$, is the method of *maximum likelihood*. The *likelihood function* of a random sample of size n from the population $f_X(x;\theta)$ is the joint probability function of the sample variables regarded as a function of θ, or

$$L(x_1, x_2, \ldots, x_n;\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

A *maximum-likelihood estimate* (MLE) of θ is a value $\hat{\theta}$ such that for all θ,

$$L(x_1, x_2, \ldots, x_n;\hat{\theta}) \geq L(x_1, x_2, \ldots, x_n;\theta)$$

Subject to certain regularity conditions, MLEs are sufficient and consistent and are asymptotically unbiased, minimum variance, and normally distributed. Here, as elsewhere in this book, the term *asymptotic* is interpreted as meaning large sample sizes. Another useful property of MLE is *invariance*, which says that if $g(\theta)$ is a *smooth* function of θ and $\hat{\theta}$ is the MLE of θ, then the MLE of $g(\theta)$ is $g(\hat{\theta})$. If more than one parameter is involved, θ above may be interpreted as a vector.

1.2.8 Hypothesis Testing

A *statistical hypothesis* is a claim or an assertion about the probability function of one or more random variables or a statement about the populations from which one or more samples are drawn, for example, its form, shape, or parameter values. A hypothesis is called simple if the statement completely specifies the population. Otherwise it is called *composite*. The *null hypothesis* H_0 is the hypothesis under test. The *alternative hypothesis*, H_1 or H_A, is the conclusion reached if the null hypothesis is rejected.

A *test* of a statistical hypothesis is a rule that enables one to make a decision whether or not H_0 should be rejected on the basis of the observed value of a *test statistic*, which is some function of a set of observable random variables. The probability distribution of the test statistic when H_0 holds is referred to as the *null distribution* of the test statistic.

A *critical region* or *rejection region* R for a test is that subset of values assumed by the test statistic which, in accordance with the test, leads to

rejection of H_0. The *critical values* of a test statistic are the bounds of R. For example, if a test statistic T prescribes rejection of H_0 for $T \leq t_\alpha$, then t_α is the critical value and R is written in symbols as $T \in R$ for $T \leq t_\alpha$.

A *type I error* is committed if the null hypothesis is rejected when it is true. A *type II error* is failure to reject a false H_0. For a test statistic T of H_0:$\theta \in \omega$ versus H_1:$\theta \in \Omega - \omega$, the probabilities of these errors are, respectively,

$$\alpha(\theta) = P(T \in R | \theta \in \omega) \quad \text{and} \quad \beta(\theta) = P(T \notin R | \theta \in \Omega - \omega)$$

The least upper bound value, or supremum, of $\alpha(\theta)$ for all $\theta \in \omega$ is often called the *size of the test. The significance level* is a preselected nominal bound for $\alpha(\theta)$, which may not be attained if the relevant probability function is discrete. Since this is usually the case in nonparametric hypothesis testing, some confusion might arise if these distinctions were adhered to here. So the symbol α will be used to denote either the size of the test or the significance level or the probability of a type I error, prefaced by the adjective ''exact'' whenever $\sup_{\theta \in \omega} \alpha(\theta) = \alpha$. A test is called *conservative* at level α if the probability of committing a Type I error is at most α, that is, $\sup_{\theta \in \omega} \alpha(\theta) \leq \alpha$.

The *power* of a test is the probability that the test statistic will lead to a rejection of H_0, denoted by $\text{Pw}(\theta) = P(T \in R)$. Power is of interest mainly as the probability of a correct decision, and so the power is typically calculated when H_0 if false, that is, when H_1 is true, and then $\text{Pw}(\theta) = P(T \in R | \theta \in \Omega - \omega) = 1 - \beta(\theta)$. The power depends on the following four variables:

1. The degree of falseness of H_0, that is, the amount of discrepancy between the assertions stated in H_0 and H_1
2. The size of the test α
3. The number of observable random variables involved in the test statistic, generally the sample size
4. The critical region or rejection region R

The *power function* of a test is the power when all but one of these variables are held constant, usually item 1. For example, we can study the power of a particular test as a function of the parameter θ, for a given sample size and α. Typically, the power function is displayed as a graph of the values of the parameter θ on the X-axis against the corresponding power values of the test on the Y-axis. To calculate the power of a test, we need the distribution of the test statistic under the alternative hypothesis. Sometimes, such a result is either unavailable or is much too complicated to be derived analytically; then *computer simulations* can be used to estimate the power of a test. To illustrate, suppose we would like to estimate the power of a test for the mean μ of a population with H_0:$\mu = 10$. We can generate on the computer a random sample from the normal distribution with mean 10 and variance equal to 1

and apply the test at a specified level α. If the null hypothesis is rejected, we call it a success. Now we repeat this process of generating a same size sample from the normal distribution with mean 10 and variance 1, say 1000 times. At the end of these 1000 simulations we find the proportion of successes, that is, the proportion of times when the test rejects the null hypothesis. This proportion is an empirical estimate of the nominal size of a test, which was set a priori. To estimate power over the alternative, for example, we repeat the same process but with samples from a normal distribution with, say, mean 10.5 and variance 1. The proportion of successes from these simulations gives an empirical estimate of the power (simulated power) of the test for the normal distribution when the mean is 10.5 and so on. The simulation technique is particularly useful when a new test is developed with an analytically complicated null and/or alternative distribution and we would like to learn about the test's performance. In the null case, the number of successes follows a binomial distribution with $n = 1000$ and $p = \alpha$ and this fact can be used to find the simulation error associated with the proportion of successes in terms of its standard error, which is $\sqrt{\alpha(1-\alpha)/1000}$.

A test is said to be *most powerful* for a specified alternative hypothesis if no other test of the same size has greater power against the same alternative.

A test is *uniformly most powerful* against a class of alternative hypotheses if it is most powerful with respect to each specific simple alternative hypothesis within the class of alternative hypotheses.

A ''good'' test statistic is one that is reasonably successful in distinguishing correctly between the conditions as stated in the null and alternative hypotheses. A method of constructing tests that often have good properties is the *likelihood-ratio principle*. A random sample of size n is drawn from the population $f_X(x;\theta)$ with likelihood function $L(x_1, x_2, \ldots, x_n;\theta)$, where θ is to be interpreted as a vector if more than one parameter is involved. Suppose that $f_X(x;\theta)$ is a specified family of functions for every $\theta \in \omega$ and ω is a subset of Ω. The *likelihood-ratio* test of

$$H_0{:}\theta \in \omega \quad \text{versus} \quad H_1{:}\theta \in \Omega - \omega$$

has the rejection region

$$T \in R \quad \text{for } T \leq c,\ 0 \leq c \leq 1$$

where T is the ratio

$$T = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

and $L(\hat{\omega})$ and $L(\hat{\Omega})$ are the maximums of the likelihood function with respect to θ for $\theta \in \omega$ and $\theta \in \Omega$, respectively. For an exact size α test of a simple H_0,

the number c which defines R is chosen such that $P(T \leq c|H_0) = \alpha$. Any monotonic function of T, say $g(T)$, can also be employed for the test statistic as long as the rejection region is stated in terms of the corresponding values of $g(T)$; the natural logarithm is one of the most commonly used $g(.)$ functions. The likelihood-ratio test is always a function of sufficient statistics, and the principle often produces a uniformly most powerful test when such exists. A particularly useful property of T for constructing tests based on large samples is that, subject to certain regularity conditions, the probability distribution of $-2 \ln T$ approaches the chi-square distribution with $k_1 - k_2$ degrees of freedom as $n \to \infty$, where k_1 and k_2 are, respectively, the dimensions of the spaces Ω and ω, $k_2 < k_1$.

All these concepts should be familiar to the reader, since they are an integral part of any standard introductory probability and inference course. We now turn to a few concepts that are especially important in nonparametric inference.

1.2.9 *P* Value

An alternative approach to hypothesis testing is provided by computing a quantity called the *P value*, sometimes called a *probability value* or the *associated probability* or the *significance probability*. A P value is defined as the probability, when the null hypothesis H_0 is true, of obtaining a sample result as extreme as, or more extreme than (in the direction of the alternative), the observed sample result. This probability can be computed for the observed value of the test statistic or some function of it like the sample estimate of the parameter in the null hypothesis. For example, suppose we are testing H_0:$\mu = 50$ versus H_1:$\mu > 50$ and we observe the sample result for $\bar{X}$ is 52. The P value is computed as $P(\bar{X} \geq 52|\mu = 50)$. The appropriate direction here is values of $\bar{X}$ that are greater than or equal to 52, since the alternative is μ greater than 50. It is frequently convenient to simply report the P value and go no further. If a P value is small, this is interpreted as meaning that our sample produced a result that is rather rare under the assumption of the null hypothesis. Since the sample result is a fact, it must be that the null hypothesis statement is inconsistent with the sample outcome. In other words, we should reject the null hypothesis. On the other hand, if a P value is large, the sample result is consistent with the null hypothesis and the null hypothesis is not rejected.

If we want to use the P value to reach a decision about whether H_0 should be rejected, we have to select a value for α. If the P value is less than or equal to α, the decision is to reject H_0; otherwise, the decision is not to reject H_0. The P value is therefore the smallest level of significance for which the null hypothesis would be rejected.

The P value provides not only a means of making a decision about the null hypothesis, but also some idea about how strong the evidence is against the null hypothesis. For example, suppose data set 1 with test T_1 results in a P

value of 0.012, while data set 2 with test T_2 (or T_1) has a P value of 0.045. The evidence against the null hypothesis is much stronger for data set 1 than for data set 2 because the observed sample outcome is much less likely in data set 1.

Most of the tables in the Appendix give exact P values for the nonparametric test statistics with small sample sizes. In some books, tables of critical values are given for selected α values. Since the usual α values, 0.01, 0.05, and the like, are seldom attainable exactly for nonparametric tests with small sample sizes, we prefer reporting P values to selecting a level α. If the asymptotic distribution of a test statistic is used to find a P value, this may be called an asymptotic or approximate P value.

If a test has a two-sided alternative, there is no specific direction for calculating the P value. One approach is simply to report the smaller of the two one-tailed P values, indicating that it is one-tailed. If the distribution is symmetric, it makes sense to double this one-tailed P value, and this is frequently done in practice. This procedure is sometimes used even if the distribution is not symmetric.

Finally, note that the P value can be viewed as a random variable. For example, suppose that the test statistic T has a cdf F under H_0 and a cdf G under a one-sided upper-tailed alternative H_1. The P value is the probability of observing a more extreme value than the present random T, so the P value is just the random variable $P = 1 - F(T)$. For a discussion of various properties and ramifications, the reader is referred to Sackrowitz and Samuel-Cahn (1999) and Donahue (1999).

1.2.10 Consistency

A test is *consistent* for a specified alternative if the power of the test, when that alternative is true, approaches 1 as the sample size approaches infinity. A test is consistent for a class (or subclass) of alternatives if the power of the test, when any member of the class (subclass) of alternatives is true, approaches 1 as the sample size approaches infinity.

Consistency is a ''good'' test criterion for parametric and nonparametric methods, and all of the standard test procedures clearly share this property. However, in nonparametric statistics the alternatives are often extremely general, and a wide selection of tests may be available for any one experimental situation. The consistency criterion provides an objective method of choosing among these tests (or at least eliminating some from consideration) when a less general subclass of alternatives is of major interest to the experimenter. A test which is known to be consistent against a specified subclass is said to be especially sensitive to that type of alternative and can generally be recommended for use when the experimenter wishes particularly to detect differences of the type expressed in that subclass.

Consistency of a test can often be shown by investigating whether or not the test statistic converges in probability to the parameter of interest. An

especially useful way to investigate consistency is described as follows. A random sample of size n is drawn from the population $f_X(x;\theta), \theta \in \Omega$. Let T be a test statistic for the general hypothesis $\theta \in \omega$ versus $\theta \in \Omega - \omega$, and let $g(\theta)$ be some function of θ such that

$$g(\theta) = \theta_0 \quad \text{if } \theta \in \omega$$

and

$$g(\theta) \neq \theta_0 \quad \text{if } \theta \in \Delta \text{ for } \Delta \subset \Omega - \omega$$

If for all θ we have

$$E(T) = g(\theta) \quad \text{and} \quad \lim_{n \to \infty} \text{var}(T) = 0$$

then the size α test with rejection region

$$T \in R \quad \text{for } |T - \theta| > c_\alpha$$

is consistent for the subclass Δ. Similarly, for a one-sided subclass of alternatives where

$$g(\theta) = \theta_0 \quad \text{if } \theta \in \omega$$

and

$$g(\theta) > \theta_0 \quad \text{if } \theta \in \Delta \text{ for } \Delta \subset \Omega - \omega$$

the consistent test of size α has rejection region

$$T \in R \quad \text{for } T - \theta_0 > c'_\alpha$$

The results follow directly from Chebyshev's inequality. (For a proof, see Fraser, 1957, pp. 267–268.) It may be noted that the unbiasedness condition may be relaxed to asymptotic ($n \to \infty$) unbiasedness.

1.2.11 Pitman Efficiency

Another type of objective criterion may be useful in choosing between two or more tests which are comparable in a well-defined way, namely, the concept of *Pitman efficiency*. In the theory of point estimation, the efficiency of two unbiased estimators for a parameter is defined as the ratio of their variances. In some situations, the limiting value of this ratio may be interpreted as the relative number of additional observations needed using the less efficient

estimator to obtain the same accuracy. The idea of efficiency is closely related when two test statistics are of interest if power is regarded as a measure of accuracy, but the tests must be compared under equivalent conditions (as both estimators were specified to be unbiased), and there are many variables in hypothesis testing. The most common way to compare two tests is to make all factors equivalent except sample size.

The *power efficiency* of a test A relative to a test B, where both tests are for the same simple null and alternative hypotheses, the same type of rejection region, and the same significance level, is the ratio n_b/n_a, where n_a is the number of observations required by test A for the power of test A to equal the power of test B when n_b observations are employed. Since power efficiency generally depends on the selected significance level, hypotheses, and n_b, it is difficult to calculate and interpret. The problem can be avoided in many cases by defining a type of limiting power efficiency.

Let A and B be two consistent tests of a null hypothesis H_0 and alternative hypothesis H_1, at significance level α. The *asymptotic relative efficiency* (ARE) of test A relative to test B is the limiting value of the ratio n_b/n_a, where n_a is the number of observations required by test A for the power of test A to equal the power of test B based on n_b observations while simultaneously $n_b \to \infty$ and $H_1 \to H_0$.

In many applications of this definition, the ratio is the same for all choices of α, so that the ARE is a single number with a well-defined interpretation for large samples. The requirement that both tests be consistent against H_1 is not a limitation in application, since most tests under consideration for a particular type of alternative will be consistent anyway. But with two consistent tests, their powers both approach 1 with increasing sample sizes. Therefore, we must let H_1 approach H_0 so that the power of each test lies on the open interval $(\alpha, 1)$ for finite sample sizes and the limiting ratio will generally be some number other than 1. The ARE is sometimes also called local asymptotic efficiency since it relates to large sample power in the vicinity of the null hypothesis. A few studies have been conducted that seem to indicate that in several important cases, the ARE is a reasonably close approximation to the exact efficiency for moderate-sized samples and alternatives not too far from the null case. Especially in the case of small sample sizes, however, the implications of the ARE value cannot be considered particularly meaningful. The methods of calculating the ARE for comparisons of particular tests will be treated fully in Chapter 13.

The problem of evaluating the relative merits of two or more comparable test statistics is by no means solved by introducing the criteria of consistency and asymptotic relative efficiency. Both are large-sample properties and may not have much import for small or even moderate-sized samples. Exact power calculations are tedious and often too specific to shed much light on the problem as it relates to nonparametric tests, which may explain the general acceptance of asymptotic criteria in the field of nonparametric inference.

The asymptotic relative efficiency of two tests is also defined as the ratio of the limits of the efficacies of the respective tests as the sample sizes approach infinity. The *efficacy* of a test for H_0:$\theta = \theta_0$ based on a sample size n is defined as the square of the derivative of the mean of the test statistic with respect to θ divided by the variance of the test statistic, both evaluated at the hypothesized value $\theta = \theta_0$. Thus, for large n, the efficacy measures the rate of change of the mean (expressed in standard units) of a test statistic at the null hypothesis values of θ. A test with a relatively large efficacy is especially sensitive to alternative values of θ close to θ_0 and therefore should have good power in the vicinity of θ_0. Details will be given in Chapter 13.

1.2.12 Randomized Tests

We now turn to a different problem, which, although not limited to nonparametric inference, is of particular concern in this area. For most classical test procedures, the experimenter chooses a ''reasonable'' significance level α in advance and determines the rejection-region boundary such that the probability of a type I error is exactly α for a simple hypothesis and does not exceed α for a composite hypothesis. When the null probability distribution of the test statistic is continuous, any real number between 0 and 1 may be chosen as the significance level. Let us call this preselected number the *nominal* α. If the test statistic T can take on only a countable number of values, that is, if the sampling distribution of T is discrete, the number of possible exact probabilities of a type I error is limited to the number of jump points in the cdf of the test statistic. These exact probabilities will be called *exact* α *values*, or *natural significance levels*. The region can then be chosen such that either (1) the exact α is the largest number, which does not exceed the nominal α or (2) the exact α is the number closest to the nominal α. Although most statisticians seem to prefer the first approach, as it is more consistent with classical test procedures for a composite H_0, this has not been universally agreed upon. As a result, two sets of tables of critical values of a test statistic may not be identical for the same nominal α; this can lead to confusion in reading tables. The entries in each table in our Appendix are constructed using the first approach for all critical values.

Disregarding that problem for now, suppose we want to compare the performance, as measured by power, of two different discrete test statistics. Their natural significance levels are unlikely to be the same, so identical nominal α values do not ensure identical exact probabilities of a type I error. Power is certainly affected by exact α, and power comparisons of tests may be quite misleading without identical exact α values. A method of equalizing exact α values is provided by *randomized test procedures*.

A *randomized decision rule* is one which prescribes rejection of H_0 always for a certain range of values of the test statistic, rejection sometimes for another nonoverlapping range, and no rejection otherwise. A typical rejection region of exact size as α might be written $T \in R$ with probability 1

if $T \geq t_2$, and with probability p if $t_1 \leq T < t_2$, where $t_1 < t_2$ and $0 < p < 1$ are chosen such that

$$P(T \geq t_2|H_0) + pP(t_1 \leq T < t_2|H_0) = \alpha$$

Some random device could be used to make the decision in practice, like drawing one card at random from 100, of which $100p$ are labeled reject. Such decision rules may seem an artificial device and are probably seldom employed by experimenters, but the technique is useful in discussions of theoretical properties of tests. The power of such a randomized test against an alternative H_1 is

$$\text{Pw}(\theta) = P(T \geq t_2|H_1) + pP(t_1 \leq T < t_2|H_1)$$

A simple example will suffice to explain the procedure. A random sample of size 5 is drawn from the Bernoulli population. We wish to test $H_0{:}\theta = 0.5$ versus $H_1{:}\theta > 0.5$ at significance level 0.05. The test statistic is X, the number of successes in the sample, which has the binomial distribution with parameter $\theta = 0.5$ and $n = 5$. A reasonable rejection region would be large values of X, and thus the six exact significance levels obtainable without using a randomized test are:

c	**5**	**4**	**3**	**2**	**1**	**0**
$P(X \geq c\|\theta = 0.5)$	1/32	6/32	16/32	26/32	31/32	1

A nonrandomized test procedure with rejection region $X \in R$ for $X = 5$ has nominal size 0.05 but exact size $\alpha = 1/32 = 0.03125$. The randomized test with exact $\alpha = 0.05$ is found with $t_1 = 4$ and $t_2 = 5$ as

$$P(X \geq 5|\theta = 0.5) + pP(4 \leq X < 5) = 1/32 + pP(X = 4) = 0.05$$
$$1/32 + 5p/32 = 0.05 \text{ and } p = 0.12$$

Thus, the rejection region is $X \in R$ with probability 1 if $X = 5$ and with probability 0.12 if $X = 4$. Using Table C, the power of this randomized test when $H_1{:}\theta = 0.6$ is

$$\begin{aligned}\text{Pw}(0.6) &= P(X = 5|\theta = 0.6) + 0.12\ P(X = 4|\theta = 0.6) \\ &= 0.0778 + 0.12(0.2592) = 0.3110\end{aligned}$$

1.2.13 Continuity Correction

The exact null distribution of most test statistics used in nonparametric inference is discrete. Tables of rejection regions or cumulative distributions

are often available for small sample sizes only. In many cases, some simple approximation to the null distribution is accurate enough for practical applications with moderate-sized samples. When these asymptotic distributions are continuous (like the normal or chi square), the approximation may be improved by introducing a *correction for continuity*. This is accomplished by regarding the value of the discrete test statistic as the midpoint of an interval. For example, if the domain of a test statistic T is only integer values, the observed value is considered to be $t \pm 0.5$. If the decision rule is to reject for $T \geq t_{\alpha/2}$ or $T \leq t'_{\alpha/2}$ and the large-sample approximation to the distribution of

$$\frac{T - E(T|H_0)}{\sigma(T|H_0)}$$

is the standard normal under H_0, the rejection region with *continuity correction* incorporated is determined by solving the equations

$$\frac{t_{\alpha/2} - 0.5 - E(T|H_0)}{\sigma(T|H_0)} = z_{\alpha/2} \quad \text{and} \quad \frac{t'_{\alpha/2} + 0.5 - E(T|H_0)}{\sigma(T|H_0)} = -z_{\alpha/2}$$

where $z_{\alpha/2}$ satisfies $\Phi(z_{\alpha/2}) = 1 - \alpha/2$. Thus, the continuity-corrected, two-sided, approximately size α rejection region is

$$T \geq E(T|H_0) + 0.5 + z_{\alpha/2}\sigma(T|H_0) \quad \text{or} \quad T \leq E(T|H_0) - 0.5 - z_{\alpha/2}\sigma(T|H_0)$$

One-sided rejection regions or critical ratios employing continuity corrections are found similarly. For example, in a one-sided test with rejection region $T \geq t_\alpha$, for a nominal size α, the approximation to the rejection region with a continuity correction is determined by solving for t_α in

$$\frac{t_\alpha - 0.5 - E(T|H_0)}{\sigma(T|H_0)} = z_\alpha$$

and thus the continuity-corrected, one-sided upper-tailed, approximately size α rejection region is

$$T \geq E(T|H_0) + 0.5 + z_\alpha\sigma(T|H_0)$$

Similarly, the continuity-corrected, one-sided lower-tailed, approximately size α rejection region is

$$T \leq E(T|H_0) - 0.5 - z_\alpha\sigma(T|H_0)$$

The P value for a one-sided test based on a statistic whose null distribution is discrete is often approximated by a continuous distribution, typically the

normal, for large sample sizes. Like the rejection regions above, this approximation to the P value can usually be improved by incorporating a correction for continuity. For example, if the alternative is in the upper tail, and the observed value of an integer-valued test statistic T is t_0, the exact P value $P(T \geq t_0|H_0)$ is approximated by $P(T \geq t_0 - 0.5|H_0)$. In the Bernoulli case with $n = 20$, $H_0{:}\theta = 0.5$ versus $H_1{:}\theta > 0.5$, suppose we observe $X = 13$ successes. The normal approximation to the P value with a continuity correction is

$$\begin{aligned} P(X \geq 13|H_0) = P(X > 12.5) &= P\left(\frac{X - 10}{\sqrt{5}} > \frac{12.5 - 10}{\sqrt{5}}\right) \\ &= P(Z > 1.12) \\ &= 1 - \Phi(1.12) = 0.1314 \end{aligned}$$

This approximation is very close to the exact P value of 0.1316 from Table C. The approximate P value without the continuity correction is 0.0901, and thus the continuity correction greatly improves the P value approximation. In general, let t_0 be the observed value of the test statistic T whose null distribution can be approximated by the normal distribution. When the alternative is in the upper tail, the approximate P value with a continuity correction is given by

$$1 - \Phi\left[\frac{t_0 - E(T|H_0) - 0.5}{\sigma(T|H_0)}\right]$$

In the lower tail, the continuity corrected approximate P value is given by

$$\Phi\left[\frac{t_0 - E(T|H_0) + 0.5}{\sigma(T|H_0)}\right]$$

When the alternative is two-sided, the continuity corrected approximate P value can be obtained using these two expressions and applying the recommendations given earlier under P value.

2

Order Statistics, Quantiles, and Coverages

2.1 Introduction

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a population with continuous cdf F_X, so that the probability is zero that any two or more of these random variables have equal magnitudes. In this situation, there exists a unique ordered arrangement within the sample. Suppose that $X_{(1)}$ denotes the smallest of $X_1, X_2, \ldots, X_n$; $X_{(2)}$ denotes the second smallest; ... and $X_{(n)}$ denotes the largest. Then

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

denotes the original random sample after arrangement in increasing order of magnitude, and these are collectively termed the *order statistics* of the random sample $X_1, X_2, \ldots, X_n$. The rth smallest, $1 \leq r \leq n$, $X_{(r)}$, is called the rth-order statistic. Some familiar applications of order statistics, which are obvious on reflection, are as follows:

1. $X_{(n)}$, the maximum (largest) value in the sample, is of interest in the study of floods and other extreme meteorological phenomena.
2. $X_{(1)}$, the minimum (smallest) value, is useful for phenomena where, for example, the strength of a chain depends on the weakest link.
3. The sample median, defined as $X_{[(n+1)/2]}$ for n odd and any number between $X_{(n/2)}$ and $X_{(n/2+1)}$ for n even, is a measure of location and an estimate of the population central tendency.
4. The sample midrange, defined as $(X_{(1)} + X_{(n)})/2$, is also a measure of central tendency.
5. The sample range $X_{(n)} - X_{(1)}$ is a measure of dispersion.
6. In some experiments, the sampling process ceases after collecting r of the observations. For example, in life-testing electric light bulbs, one may start with a group of n bulbs but stop taking observations after the rth bulb burns out. Then information is available only on the first

r ordered "lifetimes" $X_{(1)} < X_{(2)} < \cdots < X_{(r)}$, where $r \leq n$. Such data are often referred to as censored data.

7. Order statistics are used to study outliers or extreme observations, for example, when so-called dirty data are suspected.

Our study of order statistics in this chapter will be limited to their mathematical and statistical properties, including joint and marginal probability distributions, exact moments, asymptotic moments, and asymptotic marginal distributions. Two general uses of order statistics in distribution-free inference will be discussed in Chapter 5, namely, interval estimation and hypothesis testing of population percentiles. The topic of tolerance limits for distributions, including both one-sample and two-sample coverages, is discussed later in this chapter. But first, we must define another property of probability functions called the quantile function.

2.2 Quantile Function

We have already discussed how to use the mean, the variance, and other moments to describe a probability distribution. In some situations, we may be more interested in the percentiles of a distribution, like the 50th percentile (the median). For example, if X represents the breaking strength of an item, we might want to know the median strength, the 50th percentile, or the strength that is survived by 60% of the items, the 40th percentile point. Or we may want to know what percentage of the items will survive a pressure of say 3 lb. For questions like these, we need information about the quantiles of a distribution.

A *quantile* (or a percentile) of a distribution is that value of X such that a specific percentage of the probability is at or below it. Thus a quantile divides the area under the pdf into two parts of specific amounts. Only the area to the left of the number need be specified since the entire area is equal to one. Recall that the cdf $F_X(x)$ of a random variable X is the probability that X is less than or equal to some real value x. Thus the cdf and the quantile function are inversely related and a quantile can be conveniently defined in terms of the cdf. The pth quantile (or the $100p$th percentile) is that value of the random variable X, say X_p, such that $100p\%$ of the values of X in the population are less than or equal to X_p, for any positive fraction p $(0 < p < 1)$. If X (F_X) is continuous, X_p is a parameter that satisfies $P(X \leq X_p) = p$, or, in terms of the cdf, $F_X(X_p) = p$. Moreover if F_X is strictly increasing, the pth quantile is the unique solution to the equation $X_p = F_X^{-1}(p) = Q_X(p)$ say, for a given p and the inverse of the cdf $Q_X(p)$, $0 < p < 1$, is called the *quantile function* (qf) of the random variable X.

Consider, for example, a random variable from the exponential distribution with $b = 2$. Table 1.2.1 indicates that the cdf is should be

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases}$$

Since $1 - e^{-Xp/2} = p$ for $x > 0$, the inverse is $X_p = -2\ln(1-p)$ and hence the quantile function is $Q_X(p) = -2\ln(1-p)$ for $0 < p < 1$. The cdf and the quantile function for this exponential distribution are shown in Figures 2.2.1 and 2.2.2, respectively.

Suppose the distribution of the breaking strength random variable X is this exponential with $b = 2$. The reader can verify that the 0.5th quantile or the 50th percentile $Q_X(0.5)$ is 1.3863, and the 40th percentile $Q_X(0.4)$ is 1.0217. The proportion that exceeds a breaking strength of 3 pounds is 0.2231.

Thus the pth quantile is the solution to the equation $F_X(x) = p$. Since the cdf may not be increasing for all values, we define the pth quantile $Q_X(p)$ as the smallest X value at which the cdf is at least equal to p, or

$$Q_X(p) = F_X^{-1}(p) = \inf[x: F_X(x) \geq p] \quad 0 < p < 1$$

This definition gives a unique value for the quantile $Q_X(p)$ even when F_X is flat or is a step function, the latter being the case when X is discrete at or around the specified value p. For example, suppose X has a binomial distribution with parameters n and $p = 0.5$ and we want to find the median $X_{0.5}$. Using Table C, we see that there is no value of X at which the cdf is exactly equal to 0.5; however, applying the infimum definition, we find the median of X is 5, since 5 is the smallest value of X at which the cdf is at least 0.5. Similarly, the 25th and the 70th percentiles are found to be 4 and 6, respectively.

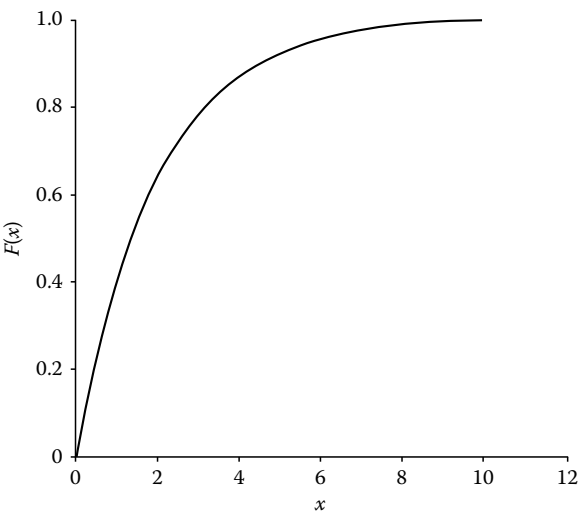


FIGURE 2.2.1
The exponential cdf with $b = 2$.

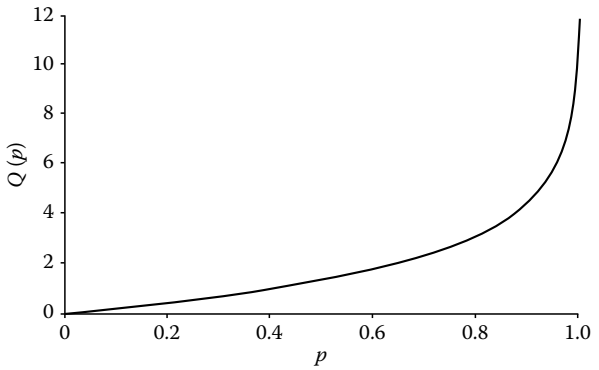


FIGURE 2.2.2
The exponential quantile function $b = 2$.

Some popular quantiles of a distribution are known as the *quartiles*. The first quartile is the 0.25th quantile, the second quartile is the 0.50th quantile (the median), and the third quartile is the 0.75th quantile. These are also referred to as the 25th, the 50th, and the 75th percentiles, respectively. Extreme quantiles (such as for $p = 0.95$, 0.99, or 0.995) of a distribution are important as critical values for many test statistics; calculating these is important in hypothesis testing.

The cdf and the qf provide similar information regarding the distribution; however, there are situations where one is more natural than the other. Note that formulas for the moments of X can also be expressed in terms of the quantile function. For example,

$$E(X) = \int_0^1 Q_X(p)dp \quad \text{and} \quad E(X^2) = \int_0^1 Q_X^2(p)dp \tag{2.2.1}$$

so that $\sigma^2 = \int_0^1 Q_X^2(p)dp - [\int_0^1 Q_X(p)dp]^2$.

The following result is useful when working with the qf. Let $f_X(x) = F'_X(x)$ denote the pdf of X.

THEOREM 2.2.1

Assuming that the necessary derivatives all exist, the first and the second derivatives of the quantile function $Q_X(p)$ are

$$Q'_X(p) = \frac{1}{f_X[Q_X(p)]} \quad \text{and} \quad Q''_X(p) = -\frac{f'_X[Q_X(p)]}{\{f_X[Q_X(p)]\}^3}$$

The proof of this result is straightforward and is left for the reader to solve.

It is clear that given some knowledge regarding the distribution of a random variable, one can try to use that information, perhaps along with some data, to aid in studying properties of such a distribution. For example, if we know that the distribution of X is exponential but we are not sure of its mean, typically, a simple random sample is taken and the population mean is estimated by the sample mean $\overline{X}$. This estimate can then be used to estimate properties of the distribution. For instance, the probability $P(X \leq 3.2)$ can be estimated by $1 - e^{-3.2/\overline{X}}$, which is the estimated cdf of X at 3.2. This is the approach of classical parametric analysis. In nonparametric analysis, we do not assume that the distribution is exponential (or anything else for that matter). The natural question then is how do we estimate the underlying cdf? This is where the *empirical distribution function* (edf) or the *empirical cumulative distribution function* (ecdf) plays a crucial role.

2.3 Empirical Distribution Function

For a random sample from the distribution with cdf F_X, the *empirical distribution function* or edf, denoted by $S_n(x)$, is simply the proportion of sample values less than or equal to the specified value x, that is,

$$S_n(x) = \frac{\text{number of sample values} \leq x}{n}$$

In the previous example, $S_n(3.2)$ can be used as a point estimate of $P(X \leq 3.2)$. The edf is most conveniently defined in terms of the order statistics of a sample as

$$S_n(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ i/n & \text{if } X_{(i)} \leq x < X_{(i+1)}, \quad i = 1, 2, \ldots, n-1 \\ 1 & \text{if } x \geq X_{(n)} \end{cases} \tag{2.3.1}$$

Suppose that a random sample of size $n = 5$ is given by 9.4, 11.2, 11.4, 12.0, and 12.6. The edf of this sample is shown in Figure 2.3.1. Clearly, $S_n(x)$ is a step (or a jump) function, with jumps occurring at the (distinct) ordered sample values, where the height of each jump is equal to the reciprocal of the sample size, namely, $1/5$ or 0.2.

When more than one observation has the same value, we say these observations are *tied*. In this case, the edf is still a step function but it jumps only at the distinct ordered sample values $X_{(j)}$ and the height of the jump is equal to k/n, where k is the number of values tied at $X_{(j)}$.

We now discuss some of the statistical properties of the edf $S_n(x)$. Let $T_n(x) = nS_n(x)$, so that $T_n(x)$ represents the total number of sample values that are less than or equal to the specified value x.

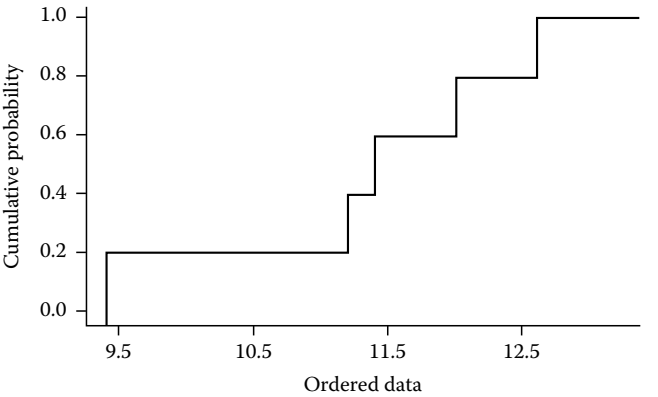


FIGURE 2.3.1
An empirical distribution function for $n = 5$.

THEOREM 2.3.1

For any fixed real value x, the random variable $T_n(x)$ has a binomial distribution with parameters n and $F_X(x)$.

Proof

For any fixed real constant x and $i = 1, 2, \ldots, n$, define the indicator random variables

$$\delta_i(x) = I_{[X_i \leq x]} = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$$

The $\delta_1(x)$, $\delta_2(x), \ldots, \delta_n(x)$ are independent and identically distributed, each with the Bernoulli distribution with parameter p, where $p = P[\delta_i(x) = 1] = P(X_i \leq x) = F_X(x)$. Now, since $T_n(x) = \sum_{i=1}^{n} \delta_i(x)$ is the sum of n independent and identically distributed Bernoulli random variables, it can be easily shown that $T_n(x)$ has a binomial distribution with parameters n and $p = F_X(x)$.

From Theorem 2.3.1, and using properties of the binomial distribution, we get the following results. The proofs are left for the reader.

COROLLARY 2.3.1.1

The mean and the variance of $S_n(x)$ are

(a) $E[S_n(x)] = F_X(x)$

(b) $\text{var}[S_n(x)] = F_X(x)[1 - F_X(x)]/n$

Part (a) of the corollary shows that $S_n(x)$, the proportion of sample values less than or equal to the specified value x, is an *unbiased* estimator of $F_X(x)$. Part (b) shows that the variance of $S_n(x)$ tends to zero as n tends to infinity. Thus, using Chebyshev's inequality, we can show that $S_n(x)$ is a consistent estimator of $F_X(x)$. Two useful results are stated below.

COROLLARY 2.3.1.2

For any fixed real value x, $S_n(x)$ is a consistent estimator of $F_X(x)$, or, in other words, $S_n(x)$ converges to $F_X(x)$ in probability.

COROLLARY 2.3.1.3

$E[T_n(x)\ T_n(y)] = n\ F_X(x) + n(n-1)F_X(x)F_X(y)$, *for* $x < y$.

The second corollary is useful in finding the covariance between $T_n(x)$ and $T_n(y)$, which is left as an exercise for the reader.

The convergence in Corollary 2.3.2 is for each value of x individually, whereas sometimes we are interested in all values of x, collectively. A probability statement can be made simultaneously for all x, as a result of the following important theorem [see Fisz (1963), for example, for a proof].

THEOREM 2.3.2 (Glivenko-Cantelli Theorem)

$S_n(x)$ converges uniformly to $F_X(x)$ with probability 1, *that is,*

$$P\left[\lim_{n\to\infty} \sup_{-\infty<x<\infty} [|S_n(x) - F_X(x)| = 0\right] = 1$$

Another useful property of the edf is its asymptotic normality, given in the following theorem.

THEOREM 2.3.3

As $n \to \infty$, the limiting probability distribution of the standardized $S_n(x)$ is standard normal, or

$$\lim_{n\to\infty} P\left\{\frac{\sqrt{n}[S_n(x) - F_X(x)]}{\sqrt{F_X(x)[1 - F_X(x)]}} \le t\right\} = \Phi(t)$$

Proof

Using Theorem 2.3.1, Corollary 2.3.1.1, and the central limit theorem, it follows that the distribution of

$$\frac{[nS_n(x) - nF_X(x)]}{\sqrt{nF_X(x)[1 - F_X(x)]}} = \frac{\sqrt{n}[S_n(x) - F_X(x)]}{\sqrt{F_X(x)[1 - F_X(x)]}}$$

approaches the standard normal as $n \to \infty$.

2.3.1 Empirical Quantile Function

Since the quantile function is the inverse of the cdf and the edf is an estimate of the cdf, it is natural to estimate the quantile function as the inverse of edf. This yields the *empirical quantile function* (eqf) $Q_n(u)$, $0 < u \leq 1$, defined below.

$$Q_n(u) = \begin{cases} X_{(1)} & \text{if } 0 < u \leq \dfrac{1}{n} \\ X_{(2)} & \text{if } \dfrac{1}{n} < u \leq \dfrac{2}{n} \\ \cdots & \\ X_{(n)} & \text{if } \dfrac{n-1}{n} < u \leq 1 \end{cases}$$

For the random sample of $n = 5$ discussed in Section 2.3 and presented as a graph in Figure 2.3.1, the edf and the eqf are given by

$$S_5(x) = \begin{cases} 0 & \text{if } x < 9.4 \\ 1/5 & \text{if } 9.4 \leq x < 11.2 \\ 2/5 & \text{if } 11.2 \leq x < 11.4 \\ 3/5 & \text{if } 11.4 \leq x < 12.0 \\ 4/5 & \text{if } 12.0 \leq x < 12.6 \\ 1 & \text{if } 12.6 \leq x \end{cases}$$

$$Q_5(u) = \begin{cases} 9.4 & \text{if } 0 < u \leq 1/5 \\ 11.2 & \text{if } 1/5 < u \leq 2/5 \\ 11.4 & \text{if } 2/5 < u \leq 3/5 \\ 12.0 & \text{if } 3/5 < u \leq 4/5 \\ 12.6 & \text{if } 4/5 < u \leq 1.0 \end{cases}$$

respectively.

Accordingly, the empirical quantiles are just the order statistics in a sample. For example, if $n = 10$, the 0.30th empirical quantile $Q_{10}(0.3)$ is simply $X_{(3)}$, since $2/10 < 0.3 \leq 3/10$. This is consistent with the usual definition of a sample quantile or a sample percentile since 30% of the data values are less than or equal to the third order statistic in a sample of size 10. However, note

that according to definition, the 0.25th quantile or the 25th percentile (or the 1st quartile) is also equal to $X_{(3)}$ since $2/10 < 0.25 \leq 3/10$.

Thus, the order statistics are natural point estimators of the corresponding population quantiles or percentiles. For this reason, a study of the properties of order statistics is as important in nonparametric analysis as the study of the properties of the sample mean in parametric analysis.

2.4 Statistical Properties of Order Statistics

In this section, we derive some of the statistical properties of order statistics.

2.4.1 Cumulative Distribution Function of $X_{(r)}$

THEOREM 2.4.1

For any fixed real t

$$P(X_{(r)} \leq t) = \sum_{i=r}^{n} P[nS_n(t) = i]$$
$$= \sum_{i=r}^{n} \binom{n}{i} [F_X(t)]^i [1 - F_X(t)]^{n-i} \quad -\infty < t < \infty \qquad (2.4.1)$$

This theorem can be proved in at least two ways. First, note that $X_{(r)} \leq t$ if and only if at least r of the X's are less than or equal to t, and Theorem 2.3.1 gives the exact probability of the number of X's less than or equal to t.

This result holds even if the underlying distribution is discrete and thus can be used to find probability distributions of discrete order statistics; an application is given as an exercise. Joint distributions of discrete order statistics are discussed in David and Nagaraja (2003). A second proof of the theorem, using mathematical properties of order statistics, is given later.

2.4.2 Probability Density Function of $X_{(r)}$

THEOREM 2.4.2

If the underlying cdf is continuous with $F'_X(x) = f_X(x)$, the pdf of the rth-order statistic is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F_X(x)]^{r-1} [1 - F_X(x)]^{n-r} f_X(x) \quad -\infty < x < \infty \qquad (2.4.2)$$

This can be proved from Theorem 2.4.1 by differentiation and some algebraic manipulations. A more direct derivation is provided later.

Theorems 2.4.1 and 2.4.2 clearly show that the sample quantiles are not distribution free. Therefore, these statistics are often not the most convenient to use as point estimators of the corresponding population quantiles. However, they frequently provide interesting starting points and in fact are the building blocks upon which many distribution-free procedures are based. The study of order statistics is thus vital to the understanding of distribution-free inference procedures.

Some important simplifications occur when we assume that the sample comes from the continuous uniform population on (0, 1). Note that for this distribution, $F_X(t) = t$ for $0 < t < 1$. Thus, from Theorem 2.4.1, the cdf of $X_{(r)}$ is

$$F_{X_{(r)}}(t) = P(X_{(r)} \leq t) = \sum_{i=r}^{n} P[nS_n(t) = i]$$

$$= \sum_{i=r}^{n} \binom{n}{i} t^i (1-t)^{n-i} \quad 0 < t < 1$$

and when F is continuous, from (2.4.2) the pdf of $X_{(r)}$ is a beta distribution given by

$$f_{X_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \quad 0 < t < 1 \tag{2.4.3}$$

This is summarized in Theorem 2.4.3.

THEOREM 2.4.3

For a random sample of size n from the uniform (0, 1) *distribution, the rth order statistic* $X_{(r)}$ *follows a beta* $(r, n-r+1)$ *distribution.*

The following result follows from Theorems 2.4.1 and 2.4.3.

COROLLARY 2.4.3.1

For the continuous uniform (0, 1) *distribution*

$$P(X_{(r)} \leq t) = \sum_{i=r}^{n} \binom{n}{i} t^i (1-t)^{n-i} = \frac{1}{B(r, n-r+1)} \int_0^t x^{r-1}(1-x)^{n-r} dx \tag{2.4.4}$$

The integral on the right is called an *incomplete beta integral* and is often written as $I_t(r, n-r+1)$. Thus the cumulative binomial probability can be calculated in terms of the *incomplete beta* integral. This integral has been tabulated by various authors and is now available in most modern statistical software. It can be verified that $1 - I_t(a, b) = I_{1-t}(b, a)$; we leave the verification as an exercise for the reader (Problem 2.3).

2.5 Probability-Integral Transformation

One key reason why the order statistics are so important in nonparametric statistics is that for any order statistic $X_{(r)}$ from a continuous cdf F, the transformed random variable $U_{(r)} = F(X_{(r)})$ has the same distribution as that of the rth-order statistic from the continuous uniform population on the interval (0, 1), regardless of the shape of F as long as it is continuous (normal, gamma, chi-square, etc.); in this sense, $F(X_{(r)})$ may be viewed as distribution free. This important property of continuous order statistics is called the *probability-integral transformation* (PIT), which is proved in the following theorem.

THEOREM 2.5.1 (Probability-Integral Transformation)

Let X be a random variable with cdf F_X. If F_X is continuous, the random variable Y produced by the transformation $Y = F_X(X)$ has the continuous uniform probability *distribution over the interval* (0, 1).

Proof

Since $0 \le F_X(x) \le 1$ for all x, letting F_Y denote the cdf of Y, we have $F_Y(y) = 0$ for $y \le 0$ and $F_Y(y) = 1$ for $y \ge 1$. For $0 < y < 1$, define u to be the largest number satisfying $F_X(u) = y$. Then $F_X(x) \le y$ if and only if $X \le u$, and it follows that

$$F_Y(y) = P[F(X) \le y] = P(X \le u) = F_X(u) = y$$

which is the cdf of the continuous uniform distribution defined over (0, 1).

This theorem can also be proved using moment-generating functions when they exist; this approach will be left as an exercise for the reader.

As a result of the PIT, we can conclude that if $X_1, X_2, \ldots, X_n$ is a random sample from any population with continuous cdf F_X, then $F_X(X_1), F_X(X_2), \ldots, F_X(X_n)$ is a random sample from the continuous uniform (0, 1) population.

Similarly, if $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics for the original sample, then

$$F_X(X_{(1)}) < F_X(X_{(2)}) < \cdots < F_X(X_{(n)})$$

are the order statistics in a sample from the continuous uniform (0, 1) distribution regardless of the original distribution F_X as long as it is continuous.

The PIT is a very important result in statistics, not only in the theoretical derivations of the properties of order statistics and the like, but also in practical applications such as generation of random numbers. The following two examples illustrate the utility of the PIT.

Example 2.5.1

Suppose we want to calculate the probability $P(2 < X \leq 3)$, where X follows a chi-square distribution with 3 degrees of freedom (df). Suppose $F_X(X)$ denotes the cdf of X. Since $F_X(X)$ has the uniform distribution on (0, 1) and F_X is non-decreasing, the probability in question is simply equal to $F_X(3) - F_X(2)$. Using the CHIDIST function with df = 3 in the software package EXCEL (note that EXCEL gives right-tail probabilities) we easily get $F_X(2) = 1 - 0.5724 = 0.4276$ and $F_X(3) = 1 - 0.3916 = 0.6084$, so that the required probability is simply $0.6084 - 0.4276 = 0.1808$. Thus, the computation is simplified by transforming the original probability into a probability with respect to the uniform distribution.

Example 2.5.2

An important practical application of the PIT is generating random samples from specified continuous probability distributions. For example, suppose we want to generate an observation X from an exponential distribution with mean 2. The cdf of X is $F_X(x) = 1 - e^{-X/2}$, and by the PIT, the transformed random variable $Y = 1 - e^{-X/2}$ is distributed as U, an observation from the uniform distribution over the interval (0, 1). Now set $1 - e^{-X/2} = U$ and solve for $X = -2\ln(1 - U)$. Using a uniform random number generator (most software packages and some pocket calculators provide one), we can obtain a uniform random number U and then the desired X from the transformation $X = -2\ln(1 - U)$. Thus, for example, if we get $u = 0.2346$ using a uniform random number generator, the corresponding value of X from the specified exponential distribution is 0.5347.

In summary, to generate a random sample of 2 or more from a specified continuous probability distribution, we may generate a random sample from the uniform (0, 1) distribution and apply the appropriate transformation to each observation in the sample. Several other applications of the probability-integral transformation are given in Problem 2.4.

2.6 Joint Distribution of Order Statistics

Since the observations $X_1, X_2, \ldots, X_n$ in a random sample from a continuous population with pdf f_X are independent and identically distributed, their joint pdf is

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

The joint pdf of the n order statistics $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ is not the same as the joint pdf of $X_1, X_2, \ldots, X_n$ since the order statistics are obviously neither independent nor identically distributed. However, the joint pdf is easily derived using the method of Jacobians for transformations.

The set of n order statistics is produced by the transformation

$$\begin{aligned}
Y_1 &= \text{smallest of } (X_1, X_2, \ldots, X_n) = X_{(1)} \\
Y_2 &= \text{second smallest of } (X_1, X_2, \ldots, X_n) = X_{(2)} \\
&\cdots \\
Y_r &= r\text{th smallest of } (X_1, X_2, \ldots, X_n) = X_{(r)} \\
&\cdots \\
Y_n &= \text{largest of } (X_1, X_2, \ldots, X_n) = X_{(n)}
\end{aligned}$$

This transformation is not one to one. In fact, since there are a total of $n!$ possible arrangements of the original random variables in increasing order of magnitude, there exist $n!$ inverses to the transformation.

One of these $n!$ permutations might be

$$X_5 < X_1 < X_{n-1} < \cdots < X_n < X_2$$

The corresponding inverse transformation is

$$\begin{aligned}
X_5 &= Y_1 \\
X_1 &= Y_2 \\
X_{n-1} &= Y_3 \\
&\cdots \\
X_n &= Y_{n-1} \\
X_2 &= Y_n
\end{aligned}$$

The Jacobian of this transformation is the determinant of an $n \times n$ identity matrix with rows rearranged, since each new Y_i is equal to one and only one of the original $X_1, X_2, \ldots, X_n$. The determinant therefore equals ± 1. The joint density function of the random variables in this particular transformation is thus

$$f_{X_1, X_2, \ldots, X_n}(y_2, y_n, \ldots, y_3, y_{n-1})|J| = \prod_{i=1}^{n} f_X(y_i) \quad \text{for } y_1 < y_2 < \cdots < y_n$$

It is easily seen that the same expression results for each of the $n!$ arrangements, since each Jacobian has absolute value 1 and multiplication is commutative. Therefore, applying the general Jacobian technique described in Chapter 1, the result is

$$\begin{aligned} f_{X_{(1)}, X_{(2)}, \ldots, X_{(n)}}(y_1, y_2, \ldots, y_n) &= \sum_{\substack{\text{over all } n! \text{ inverse} \\ \text{transformations}}} \prod_{i=1}^{n} f_X(y_i) \\ &= n! \prod_{i=1}^{n} f_X(y_i) \quad \text{for } y_1 < y_2 < \cdots < y_n \end{aligned} \tag{2.6.1}$$

In other words, the joint pdf of n order statistics is $n!$ times the joint pdf of the original sample. For example, for a random sample of size n from the normal distribution with mean μ and variance σ^2, we have

$$\begin{aligned} &f_{X_{(1)}, X_{(2)}, \ldots, X_{(n)}}(y_1, y_2, \ldots, y_n) \\ &\quad = \frac{n!}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \quad \text{for } -\infty < y_1 < y_2 < \cdots < y_n \end{aligned}$$

The usual method of finding the marginal pdf of any random variable can be applied to the rth order statistic by integrating out the remaining $(n-1)$ variables in the joint pdf in (2.6.1). For example, for the largest (maximum) element in the sample, $X_{(n)}$, we have

$$\begin{aligned} f_{X(n)}(y_n) &= n! f_X(y_n) \int_{-\infty}^{y_n} \int_{-\infty}^{y_{n-1}} \cdots \int_{-\infty}^{y_3} \int_{-\infty}^{y_2} \prod_{i=1}^{n-1} f_X(y_i) dy_i \\ &= n! f_X(y_n) \int_{-\infty}^{y_n} \int_{-\infty}^{y_{n-1}} \cdots \int_{-\infty}^{y_3} [F_X(y_2) f_X(y_2)] \prod_{i=3}^{n-1} f_X(y_i) dy_2 \cdots dy_{n-1} \\ &= n! f_X(y_n) \int_{-\infty}^{y_n} \int_{-\infty}^{y_{n-1}} \cdots \int_{-\infty}^{y_4} \frac{[F_X(y_3)]^2}{2(1)} f_X(y_3) \prod_{i=4}^{n-1} f_X(y_i) dy_3 \cdots dy_{n-1} \\ &\cdots \\ &= n! f_X(y_n) \frac{[F_X(y_n)]^{n-1}}{(n-1)!} \\ &= n[F_X(y_n)]^{n-1} f_X(y_n) \end{aligned} \tag{2.6.2}$$

Similarly, for the smallest (minimum) element, $X_{(1)}$, we have

$$
\begin{aligned}
f_{X_{(1)}}(y_1) &= n! f_X(y_1) \int_{y_1}^{\infty}\int_{y_2}^{\infty}\cdots\int_{y_{n-2}}^{\infty}\int_{y_{n-1}}^{\infty}\prod_{i=2}^{n} f_X(y_i)\,dy_n dy_{n-1}\cdots dy_3 dy_2 \\
&= n! f_X(y_1) \int_{y_1}^{\infty}\int_{y_2}^{\infty}\cdots\int_{y_{n-2}}^{\infty}[1-F_X(y_{n-1})]F_X(y_{n-1})\prod_{i=2}^{n-2} f_X(y_i)\,dy_{n-1}dy_{n-2}\cdots dy_2 \\
&= n! f_X(y_1) \int_{y_1}^{\infty}\int_{y_2}^{\infty}\cdots\int_{y_{n-3}}^{\infty}\frac{[1-F_X(y_{n-2})]^2}{2(1)} f_X(y_{n-2})\prod_{i=2}^{n-3} f_X(y_i)\,dy_{n-2}\cdots dy_2 \\
&\cdots \\
&= n! f_X(y_1)\frac{[1-F_X(y_1)]^{n-1}}{(n-1)!} \\
&= n[1-F_X(y_1)]^{n-1} f_X(y_1) \qquad (2.6.3)
\end{aligned}
$$

In general, for the rth-order statistic, the order of integration which is easiest to handle would be $\infty > y_n > y_{n-1} > \cdots > y_r$ followed by $-\infty < y_1 < y_2 < \cdots < y_r$, so that we have the following combination of techniques used for $X_{(n)}$ and $X_{(1)}$:

$$
\begin{aligned}
f_{X_{(r)}}(y_r) &= n! f_X(y_r) \int_{-\infty}^{y_r}\int_{-\infty}^{y_{r-1}}\cdots\int_{-\infty}^{y_2}\int_{y_r}^{\infty}\int_{y_{r+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty}\prod_{\substack{i=1\\ i\neq r}}^{n} f_X(y_i)\,dy_n\cdots dy_{r+2}dy_{r+1}dy_1\cdots dy_{r-1} \\
&= n! f_X(y_r)\frac{[1-F_X(y_r)]^{n-r}}{(n-r)!}\int_{-\infty}^{y_r}\int_{-\infty}^{y_{r-1}}\cdots\int_{-\infty}^{y_2}\prod_{i=1}^{r-1} f_X(y_i)\,dy_1\cdots dy_{r-2}dy_{r-1} \\
&\cdots \\
&= n! f_X(y_r)\frac{[1-F_X(y_r)]^{n-r}}{(n-r)!}\frac{[F_X(y_r)]^{r-1}}{(r-1)!} \\
&= \frac{n!}{(r-1)!(n-r)!}[F_X(y_r)]^{r-1}[1-F_X(y_r)]^{n-r} f_X(y_r) \qquad (2.6.4)
\end{aligned}
$$

It is clear that this method can be applied to find the marginal distribution of any subset of two or more order statistics and it is relatively easy to apply when finding the joint pdf of a set of successive order statistics, such as $X_{(1)}, X_{(2)}, \ldots, X_{(n-2)}$. In this case, we simply integrate out $X_{(n-1)}$ and $X_{(n)}$ as

$$\int_{x_{n-2}}^{\infty} \int_{x_{n-1}}^{\infty} f_{X_{(1)}, X_{(2)}, \ldots, X_{(n)}}(x_1, x_2, \ldots, x_n) dx_{(n)} dx_{(n-1)}$$

The approach, although direct, involves tiresome integration.

A much simpler method can be used that makes use of probability theory instead of pure mathematics. The technique will be illustrated first for the single order statistic $X_{(r)}$. Recall that by definition of a derivative, we have

$$\begin{aligned} f_{X_{(r)}}(x) &= \lim_{h \to 0} \frac{F_{X_{(r)}}(x+h) - F_{X_{(r)}}(x)}{h} \\ &= \lim_{h \to 0} \frac{P(x < X_{(r)} \leq x+h)}{h} \end{aligned} \tag{2.6.5}$$

Suppose that the x-axis is divided into the following three disjoint intervals:

$$\begin{aligned} I_1 &= (-\infty, x) \\ I_2 &= (x, x+h) \\ I_3 &= (x+h, \infty) \end{aligned}$$

The probabilities that X lies in each of these intervals are

$$\begin{aligned} p_1 &= P(X \in I_1) = F_X(x) \\ p_2 &= P(X \in I_2) = F_X(x+h) - F_X(x) \\ p_3 &= P(X \in I_3) = 1 - F_X(x+h) \end{aligned}$$

respectively. Now, $X_{(r)}$ is the rth-order statistic of the set $X_1, X_2, \ldots, X_n$ and lies in the interval I_2 if and only if exactly $r-1$ of the original X random variables lie in the interval I_1, exactly $n-r$ of the original X's lie in the interval I_3 and $X_{(r)}$ lies in the interval I_2. Since the original X values are independent and the intervals are disjoint, the multinomial probability distribution with parameters p_1, p_2, and p_3 can be used to evaluate the probability in (2.6.5). The result is

$$\begin{aligned} f_{X_{(r)}}(x) &= \lim_{h \to 0} \binom{n}{r-1, 1, n-r} p_1^{r-1} p_2 p_3^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} [F_X(x)]^{r-1} \\ &\quad \times \lim_{h \to 0} \left\{ \frac{F_X(x+h) - F_X(x)}{h} [1 - F_X(x+h)]^{n-r} \right\} \\ &= \frac{n!}{(r-1)!(n-r)!} [F_X(x)]^{r-1} f_X(x) [1 - F_X(x)]^{n-r} \end{aligned} \tag{2.6.6}$$

which agrees with the result previously obtained in (2.6.4) and Theorem 2.4.2.

For the joint distribution, let $X_{(r)}$ and $X_{(s)}$ be any two order statistics from the set $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. By the definition of partial derivatives, the joint pdf can be written as

$$\begin{aligned} f_{X_{(r)},X_{(s)}}(x,y) &= \lim_{\substack{h\to 0\\ t\to 0}} \frac{F_{X_{(r)},X_{(s)}}(x+h,y+t) - F_{X_{(r)},X_{(s)}}(x,y+t) - F_{X_{(r)},X_{(s)}}(x+h,y) + F_{X_{(r)},X_{(s)}}(x,y)}{ht} \\ &= \lim_{\substack{h\to 0\\ t\to 0}} \frac{P(x < X_{(r)} \le x+h, X_{(s)} \le y+t) - P(x < X_{(r)} \le x+h, X_{(s)} \le y)}{ht} \\ &= \lim_{\substack{h\to 0\\ t\to 0}} \frac{P(x < X_{(r)} \le x+h, y < X_{(s)} \le y+t)}{ht} \end{aligned} \tag{2.6.7}$$

For any $x < y$, the x-axis can be divided into the following five disjoint intervals with the corresponding probabilities that an original X observation lies in that interval:

Interval I	**$P(X \in I)$**
$I_1 = (-\infty, x]$	$p_1 = F_X(x)$
$I_2 = (x, x+h]$	$p_2 = F_X(x+h) - F_X(x)$
$I_3 = (x+h, y]$	$p_3 = F_X(y) - F_X(x+h)$
$I_4 = (y, y+t]$	$p_4 = F_X(y+t) - F_X(y)$
$I_5 = (y+t, \infty)$	$p_5 = 1 - F_X(y+t)$

With this interval separation and assuming without loss of generality that $r < s$, $X_{(r)}$ and $X_{(s)}$ are the rth- and sth-order statistics, respectively, *and* lie in the respective intervals I_2 and I_4 if and only if the n X values are distributed along the x-axis in such a way that exactly $r-1$ lie in I_1, 1 in I_2, 1 in I_4, and $n-s$ in I_5 since the one in I_4 is the sth in magnitude, and the remaining $s-r-1$ must therefore lie in I_3. Applying the multinomial probability distribution to these five types of outcomes with the corresponding probabilities, we obtain

$$\binom{n}{r-1, 1, s-r-1, 1, n-s} p_1^{r-1} p_2 p_3^{s-r-1} p_4 p_5^{n-s}$$

Substituting this for the probability in (2.6.7) gives

$$\begin{aligned} f_{X_{(r)},X_{(s)}}(x,y) &= \binom{n}{r-1, 1, s-r-1, 1, n-s} [F_X(x)]^{r-1} \\ &\quad \times \lim_{\substack{h\to 0\\ t\to 0}} \left\{ \frac{F_X(x+h) - F_X(x)}{h} [F_X(y) - F_X(x+h)]^{s-r-1} \right\} \\ &\quad \times \lim_{\substack{h\to 0\\ t\to 0}} \left\{ \frac{F_X(y+t) - F_X(y)}{t} [1 - F_X(y+t)]^{n-s} \right\} \\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} [F_X(x)]^{r-1} [F_X(y) - F_X(x)]^{s-r-1} \\ &\quad \times [1 - F_X(y)]^{n-s} f_X(x) f_X(y) \quad \text{for all } x < y \end{aligned} \tag{2.6.8}$$

This method could be extended in a similar manner to find the joint distribution of any subset of the n order statistics. In general, for any $k \leq n$, to find the joint distribution of k-order statistics, the x axis must be divided into $k+(k-1)+2=2k+1$ disjoint intervals and the multinomial probability law applied. For example, the joint pdf of $X_{(r_1)}, X_{(r_2)}, \ldots, X_{(r_k)}$, where $1 \leq r_1 < r_2 < \cdots < r_k \leq n$ and $1 \leq k \leq n$ is

$$f_{X_{(r_1)}, X_{(r_2)}, \ldots, X_{(r_k)}}(x_1, x_2, \ldots, x_k) = \frac{n!}{(r_1-1)!(r_2-r_1-1)!\cdots(n-r_k)} [F_X(x_1)]^{r_1-1} \times [F_X(x_2)-F_X(x_1)]^{r_2-r_1-1} \cdots [1-F_X(x_K)]^{n-r_k} \times f_X(x_1) f_X(x_2) \cdots f_X(x_k) \quad x_1 < x_2 < \cdots < x_k$$

In distribution-free techniques, we are often interested in the case where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are order statistics from the continuous uniform distribution over the interval (0, 1). Then, $F_X(x) = x$ and so the marginal pdf of $X_{(r)}$ and the joint pdf of $X_{(r)}$ and $X_{(s)}$ for $r < s$ are, respectively,

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} x^{r-1}(1-x)^{n-r} \quad 0 < x < 1 \tag{2.6.9}$$

$$f_{X_{(r)}, X_{(s)}}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} x^{r-1}(y-x)^{s-r-1}(1-y)^{n-s}, \quad 0 < x < y < 1 \tag{2.6.10}$$

from (2.6.4) and (2.6.8).

The density function in (2.6.9) will be recognized as the beta distribution with parameters r and $n-r+1$. Again, this agrees with the result of Theorem 2.4.3.

2.7 Distributions of the Median and Range

As indicated in Section 2.1, the median and range of a random sample are measured based on order statistics, which are descriptive of the central tendency and dispersion of the population, respectively. Their distributions are easily obtained now from the results in Section 2.6.

2.7.1 Distribution of the Median

For n odd, the median of a sample has the pdf of (2.6.4) with $r = (n+1)/2$. If n is even and a unique value is desired for the sample median U, the usual definition is

$$U = \frac{X_{(n/2)} + X_{[(n+2)/2]}}{2}$$

so that the distribution of U must be derived from the joint density function of these two-order statistics. Letting $n = 2m$, from (2.6.8) we have for $x < y$

$$f_{X_{(m)}, X_{(m+1)}}(x, y) = \frac{(2m)!}{[(m-1)!]^2}[F_X(x)]^{m-1}[1 - F_X(y)]^{m-1} f_X(x) f_X(y)$$

Making the transformation

$$u = \frac{x + y}{2}$$
$$v = y$$

and using the method of Jacobians, the pdf of the median U for $n = 2m$ is

$$f_U(u) = \frac{(2m)!2}{[(m-1)!]^2} \int_u^\infty [F_X(2u - v)]^{m-1}[1 - F_X(v)]^{m-1} \times f_X(2u - v) f_X(v) dv \tag{2.7.1}$$

As an example, consider the uniform distribution over (0, 1). The integrand in (2.7.1) is nonzero for the intersection of the regions

$$0 < 2u - v < 1 \quad \text{and} \quad 0 < v < 1$$

The region of integration then is the intersection of the three regions

$$u < v, \quad \frac{v}{2} < u < \frac{(v+1)}{2}, \quad \text{and} \quad 0 < v < 1$$

which is depicted in Figure 2.7.1. We see that the limits on the integral in (2.7.1) must be $u < v < 2u$ for $0 < u < 1/2$ and $u < v < 1$ for $1/2 < u < 1$. Thus, if $m = 2$, say, the pdf of the median of a sample of size 4 is

$$f_U(u) = \begin{cases} 8u^2(3 - 4u) & \text{for } 0 < u \le 1/2 \\ 8(4u^3 - 9u^2 + 6u - 1) & \text{for } 1/2 < u < 1 \end{cases} \tag{2.7.2}$$

In general, for any integer $m = 1, 2, \ldots$ one can obtain

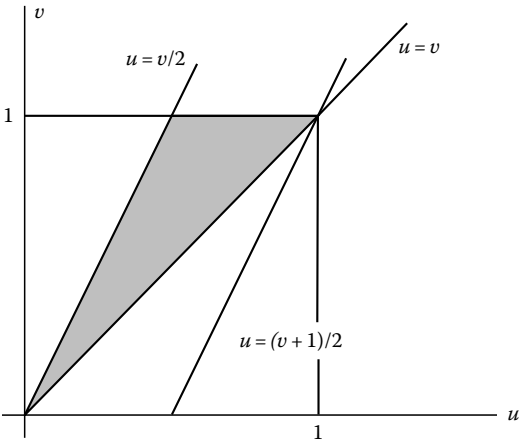


FIGURE 2.7.1
Region of integration is the shaded area.

$$f_U(u) = \begin{cases} \sum_{k=0}^{m-1} \dfrac{(2m)!2}{k!(m-1)!(m-k-1)!(k+m)} \\ \quad \times (2u-1)^{m-k-1}[(1-u)^{k+m} - (1-2u)^{k+m}] & \text{if } 0 < u \le 1/2 \\ \sum_{k=0}^{m-1} \dfrac{(2m)!2}{k!(m-1)!(m-k-1)!(k+m)} \\ \quad \times (2u-1)^{m-k-1}(1-u)^{k+m} & \text{if } 1/2 < u < 1 \end{cases}$$

Verification of these results is left for the reader.

2.7.2 Distribution of the Range

A similar procedure can be used to obtain the distribution of the range, defined as

$$R = X_{(n)} - X_{(1)}$$

The joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)[F_X(y) - F_X(x)]^{n-2} f_X(x) f_X(y) \quad \text{if } x < y$$

Now we make the transformation

$$u = y - x$$
$$v = y$$

and obtain, by integrating out v, the pdf of the range is, in general,

$$f_R(u) = \int_{-\infty}^{\infty} n(n-1)[F_X(v) - F_X(v-u)]^{n-2} f_X(v-u) f_X(v) dv \quad \text{for } u > 0 \quad (2.7.3)$$

For the uniform (0, 1) distribution, the integrand in (2.7.3) is nonzero for the intersection of the regions

$$0 < v - u < 1 \quad \text{and} \quad 0 < v < 1$$

but this region is simply $0 < u < v < 1$. Therefore, the pdf of the range is obtained from (2.7.3) by integrating v from u to 1 and we get

$$f_R(u) = n(n-1)u^{n-2}(1-u) \quad \text{for } 0 < u < 1 \tag{2.7.4}$$

which is the beta distribution with parameters $n-1$ and 2. This result for the uniform distribution is quite easy to handle. However, for a great many distributions, the integral in (2.7.3) is difficult to evaluate. For the case of a standard normal population, Hartley (1942) tabulated the cumulative distribution of the range for sample sizes not exceeding 20. The asymptotic distribution of the range is discussed in Gumbel (1944).

2.8 Exact Moments of Order Statistics

Expressions for any individual or joint moments of continuous order statistics can be found directly using the definition of moments and the specified pdf. The only practical limitation is the complexity of integration involved. Any distribution for which $F_X(x)$ is not easily expressible in a closed form is particularly difficult to handle. In some cases, a more convenient expression for the moments of $X_{(r)}$ can be found in terms of the quantile function $Q_X(u) = F_X^{-1}(u)$ defined in Section 2.2.

2.8.1 *k*th Moment about the Origin

The kth moment about the origin of the rth-order statistic from F_X is

$$\begin{aligned} E(X_{(r)}^k) &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} y^k [F_X(y)]^{r-1} [1 - F_X(y)]^{n-r} f_X(y) dy \\ &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} y^k [F_X(y)]^{r-1} [1 - F_X(y)]^{n-r} dF_X(y) \\ &= \frac{n!}{(r-1)!(n-r)!} \int_0^1 [Q_X(u)]^k u^{r-1} (1-u)^{n-r} du \\ &= E[Q_X(U)]^k \quad k = 1, 2, \ldots \end{aligned} \tag{2.8.1}$$

where the random variable U has a beta distribution with parameters r and $n-r+1$. This shows an important relationship between the moments of the order statistics from any arbitrary continuous distribution and the order statistics from the uniform (0, 1) distribution. In some cases, it may be more convenient to evaluate the integral in (2.8.1) by numerical methods, especially when a closed-form expression for the quantile function and/or the integral is not readily available.

As an example, consider the case of the uniform (0, 1) distribution so that $Q_X(u)=u$ identically on (0, 1) and hence the integral in (2.8.1) reduces to a beta integral with parameters $r+k$ and $n-r+1$. Thus, using the relationship between the beta and the gamma functions and factorials,

$$\begin{aligned} E(X_{(r)}^k) &= \frac{n!}{(r-1)!(n-r)!} B(r+k, n-r+1) \\ &= \frac{n!}{(r-1)!(n-r)!} \frac{(r+k-1)!(n-r)!}{(n+k)!} \\ &= \frac{n!(r+k-1)!}{(n+k)!(r-1)!} \end{aligned}$$

for any $1 \le r \le n$ and k. In particular, the mean is

$$E(X_{(r)}) = \frac{r}{n+1} \tag{2.8.2}$$

and the variance is

$$\operatorname{var}(X_{(r)}) = \frac{r(n-r+1)}{(n+1)^2(n+2)} \tag{2.8.3}$$

We immediately recognize (2.8.2) and (2.8.3) as the mean and the variance of a beta distribution with parameters r and $n-r+1$. This is of course true since as shown in Theorem 2.4.3, the distribution of $X_{(r)}$, the rth-order statistic of a random sample of n observations from the uniform (0, 1) distribution, is a beta distribution with parameters r and $n-r+1$.

2.8.2 Covariance between $X_{(r)}$ and $X_{(s)}$

Now consider the covariance between any two order statistics $X_{(r)}$ and $X_{(s)}$, $r<s$; $r, s=1, 2, \ldots, n$, from an arbitrary continuous distribution. From (2.6.8) we have

$$E(X_{(r)}X_{(s)}) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy[F_X(x)]^{r-1}[F_X(y) - F_X(x)]^{s-r-1} \times [1 - F_X(y)]^{n-s} f_X(x) f_X(y) dx\, dy$$

We now write $f_X(x)dx = dF_X(x)$, $f_X(y)dy = dF_X(y)$ and substitute $F_X(x) = u$ and $F_X(y) = v$, so that $x = F_X^{-1}(u) = Q_X(u)$ and $y = F_X^{-1}(v) = Q_X(v)$. Then the above expression reduces to

$$\frac{n!}{(r-1)!(s-r-1)!(n-s)!} \times \int_0^1 \int_0^{Q_X(v)} Q_X(u)Q_X(v)u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} du\, dv \tag{2.8.4}$$

As remarked before, (2.8.4) may be more convenient in practice for the actual evaluation of the expectation.

Specializing to the case of the uniform (0, 1) distribution so that $Q_X(u) = u$ and $Q_X(v) = v$, we obtain

$$E(X_{(r)}X_{(s)}) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_0^1 \int_0^v u^r v(v-u)^{s-r-1}(1-v)^{n-s} du\, dv$$

After substituting $z = u/v$ and simplifying, the inner integral reduces to a beta integral and the expectation simplifies to

$$\begin{aligned} &\frac{n!}{(r-1)!(s-r-1)!(n-s)!} B(r+1, s-r) \int_0^1 v^{s+1}(1-v)^{n-s} dv \\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} B(r+1, s-r)B(s+2, n-s+1) \\ &= \frac{n!r!(s-r-1)!(s+1)!(n-s)!}{(r-1)!(s-r-1)!(n-s)!s!(n+2)!} \\ &= \frac{r(s+1)}{(n+1)(n+2)} \end{aligned} \tag{2.8.5}$$

Now, the covariance is found using the formula

$$\operatorname{cov}(X_{(r)}, X_{(s)}) = E(X_{(r)}, X_{(s)}) - E(X_{(r)})E(X_{(s)})$$

which yields, for the uniform (0, 1) distribution

$$\begin{aligned}\operatorname{cov}(X_{(r)}, X_{(s)}) &= \frac{r(s+1)}{(n+1)(n+2)} - \frac{rs}{(n+1)^2} \\ &= \frac{r(n-s+1)}{(n+1)^2(n+2)} \quad \text{for } r < s \end{aligned} \tag{2.8.6}$$

Thus the correlation coefficient is

$$\operatorname{corr}(X_{(r)}, X_{(s)}) = \left[\frac{r(n-s+1)}{s(n-r+1)}\right]^{1/2} \quad \text{for } r < s \tag{2.8.7}$$

In particular then, the correlation between the minimum and maximum value in a sample of size n from the uniform (0, 1) distribution is

$$\operatorname{corr}(X_{(1)}, X_{(n)}) = \frac{1}{n}$$

which shows that the correlation is inversely proportional to the sample size.

We noted earlier that when the population is such that the cdf $F_X(x)$ or the quantile function $Q_X(u)$ cannot be expressed in a closed form, evaluation of the moments is often tedious or even impossible without the aid of a computer for numerical integration. Since the expected values of the order statistics from a normal probability distribution have especially useful practical applications, these results have been tabulated and are available, for example, in Harter (1961). For small n, these normal moments can be evaluated with appropriate techniques of integration. For example, if $n = 2$ and F_X is the standard normal, the mean of the first-order statistic is

$$\begin{aligned} E(X_{(1)}) &= 2 \int_{-\infty}^{\infty} x \left[1 - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{(-1/2)t^2} dt\right] \frac{1}{\sqrt{2\pi}} \mathrm{e}^{(-1/2)x^2} dx \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{x}^{\infty} x e^{(-1/2)(t^2+x^2)} dt\, dx \end{aligned}$$

Introducing a change to polar coordinates with

$$x = r \cos\theta \quad t = r \sin\theta$$

the integral above becomes

$$
\begin{aligned}
E(X_{(1)}) &= \frac{1}{\pi} \int_{\pi/4}^{5\pi/4} \int_{0}^{\infty} r^2(\cos\theta) e^{(-1/2)r^2} dr\, d\theta \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos\theta \left[\frac{1}{2} \int_{-\infty}^{\infty} \frac{r^2}{\sqrt{2\pi}} e^{(-1/2)r^2} dr \right] d\theta \\
&= \frac{1}{\sqrt{2\pi}} \int_{\pi/4}^{5\pi/4} \cos\theta\, d\theta \\
&= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{\pi}}
\end{aligned}
$$

Since $E(X_{(1)} + X_{(2)}) = 0$, we have $E(X_{(2)}) = 1/\sqrt{\pi}$.

Other examples of these techniques will be given in the problems.

2.9 Large-Sample Approximations to the Moments of Order Statistics

Evaluation of the exact moments of $X_{(r)}$ directly from the pdf requires numerical integration for many F_X of interest. Thus, for practical applications and in theoretical investigations, approximations to the moments of $X_{(r)}$ are needed. The PIT plays an important role here since the rth-order statistic from any continuous distribution is a function of the rth-order statistic from the uniform distribution. Letting $U_{(r)}$ denote the rth-order statistic from a uniform distribution over the interval (0, 1), this functional relationship can be expressed as

$$X_{(r)} = F_X^{-1}(U_{(r)}) = Q_X(U_{(r)}) \tag{2.9.1}$$

Now since the moments of $U_{(r)}$ are easily evaluated and $X_{(r)}$ is a function of $U_{(r)}$, the idea is to approximate the moments of $X_{(r)}$ in terms of some function of the moments of $U_{(r)}$. In other words, in studying the moments (or other statistical properties) of $X_{(r)}$ we try to take advantage of the facts that $X_{(r)}$ is a ''nice'' transformation (function) of the random variable $U_{(r)}$ and the properties of $U_{(r)}$ are easily found.

Consider first the general case of any random variable Z and any continuous function $g(Z)$ of Z. since the function $g(Z)$ is continuous, the Taylor series expansion of $g(Z)$ about a point μ is

$$g(Z) = g(\mu) + \sum_{i=1}^{\infty} \frac{(Z-\mu)^i}{i!} g^{(i)}(\mu) \tag{2.9.2}$$

where $g^{(i)}(\mu) = d^i g(Z)/dZ^i|_{z=\mu}$, and this series converges if

$$\lim_{n\to\infty} \frac{(Z-\mu)^n}{n!} g^{(n)}(z_1) = 0 \quad \text{for } \mu < z_1 < Z$$

Now if we let $E(Z) = \mu$ and $\text{var}(Z) = \sigma^2$ and take the expectation of both sides of (2.9.2), we obtain

$$E[g(Z)] = g(\mu) + \frac{\sigma^2}{2!} g^{(2)}(\mu) + \sum_{i=3}^{\infty} \frac{E[(Z-\mu)^i]}{i!} g^{(i)}(\mu) \tag{2.9.3}$$

From this, we immediately see that

1. A first approximation to $E[g(Z)]$ is $g(\mu)$.
2. A second approximation to $E[g(Z)]$ is $g(\mu) + \dfrac{\sigma^2}{2} g^{(2)}(\mu)$.

To find similar approximations to $\text{var}(Z)$, we form the difference between Equations 2.9.2 and 2.9.3 and square this difference, as follows:

$$\begin{aligned} g(Z) - E[g(Z)] &= (Z-\mu) g^{(1)}(\mu) + g^{(2)}(\mu) \frac{1}{2!} [(Z-\mu)^2 - \text{var}(Z)] \\ &\quad + \sum_{i=3}^{\infty} \frac{g^{(i)}(\mu)}{i!} \{(Z-\mu)^i - E[(Z-\mu)^i]\} \\ \{g(Z) - E[g(Z)]\}^2 &= (Z-\mu)^2 [g^{(1)}(\mu)]^2 + \frac{1}{4} [g^{(2)}(\mu)]^2 [\text{var}^2(Z) \\ &\quad - 2\,\text{var}(Z)(Z-\mu)^2] - g^{(1)}(\mu) g^{(2)}(\mu)\,\text{var}(Z)(Z-\mu) + h(Z) \end{aligned}$$

Then we take the expectation and get

$$\text{var}[g(Z)] = \sigma^2 [g^{(1)}(\mu)]^2 - \frac{1}{4} [g^{(2)}(\mu)]^2 \sigma^4 + E[h(Z)] \tag{2.9.4}$$

where $E[h(Z)]$ involves third or higher central moments of Z.

The first approximations to $E[g(Z)]$ and $\text{var}[g(Z)]$ are therefore

$$E[g(Z)] = g(\mu)$$

and

$$\text{var}[g(Z)] = [g^{(1)}(\mu)]^2 \sigma^2$$

The second approximations to $E[g(Z)]$ and $\text{var}[g(Z)]$ are

$$E[g(Z)] = g(\mu) + \frac{g^{(2)}(\mu)}{2}\sigma^2$$

and

$$\text{var}[g(Z)] = [g^{(1)}(\mu)]^2\sigma^2 - \left[\frac{g^{(2)}(\mu)\sigma^2}{2}\right]^2$$

respectively. The goodness of any of these approximations of course depends on the magnitude of the terms ignored, that is, the order of the higher central moments of Z.

In order to apply these generally useful results to the rth-order statistic of a sample of n from any continuous cdf F_X, we simply take $Z = U_{(r)}$ and note that the functional relationship $X_{(r)} = F_X^{-1}(U_{(r)})$ implies that our g function must be the quantile function, $g(.) = Q_X(.)$. Further, the moments of $U_{(r)}$ were found in (2.8.2) and (2.8.3) to be

$$\mu = E(U_{(r)}) = \frac{r}{n+1}$$

and

$$\sigma^2 = \text{var}(U_{(r)}) = \frac{r(n-r+1)}{(n+1)^2(n+2)}$$

Also, since the function g is the quantile function given in (2.9.1), the first two derivatives of the function g, $g^{(1)}$ and $g^{(2)}$, are obtained directly from Theorem 2.2.1. Evaluating these derivatives at $\mu = r/(n+1)$ we obtain

$$g^{(1)}(\mu) = \left\{f_X\left[F_X^{-1}\left(\frac{r}{n+1}\right)\right]\right\}^{-1}$$

$$g^{(2)}(\mu) = -f_X'\left[F_X^{-1}\left(\frac{r}{n+1}\right)\right]\left\{f_X\left[F_X^{-1}\left(\frac{r}{n+1}\right)\right]\right\}^{-3}$$

Substituting these results in the general result above, we can obtain the first and the second approximations to the mean and the variance of $X_{(r)}$. The first approximations are

$$E(X_{(r)}) = F_X^{-1}\left(\frac{r}{n+1}\right) \tag{2.9.5}$$

and

$$\text{var}(X_{(r)}) = \frac{r(n-r+1)}{(n+1)^2(n+2)} \left\{ f_X \left[F_X^{-1} \left(\frac{r}{n+1} \right) \right] \right\}^{-2} \tag{2.9.6}$$

Using (2.8.1), the third central moment of $U_{(r)}$ can be found to be

$$E[(U_{(r)} - \mu)^3] = \frac{r(2n^2 - 6nr + 4n + 4r^2 - 6r + 2)}{(n+1)^3(n+2)(n+3)} \tag{2.9.7}$$

so that for large n and finite r or r/n fixed, the terms from (2.9.3) and (2.9.4), which were ignored in reaching these approximations, are of small order. For greater accuracy, the second- or higher-order approximations can be found. This will be left as an exercise for the reader.

The use of (2.9.5) and (2.9.6) is particularly simple when f_X and F_X are tabulated. For example, to approximate the mean and variance of the fourth-order statistic of a sample of 19 from the standard normal population, we have

$$E(X_{(4)}) \approx \Phi^{-1}(0.20) = -0.84$$
$$\text{var}(X_{(4)}) \approx \frac{4(16)}{20^2(21)} [\phi(-0.84)]^{-2} = \frac{0.16}{21} 0.2803^{-2} = 0.097$$

The exact values of the means and variances of the normal order statistics are widely available, for example, in Ruben (1954) and Sarhan and Greenberg (1962). For comparison with the results in this example, the exact mean and variance of $X_{(4)}$ when $n = 19$ are -0.8859 and 0.107406, respectively.

2.10 Asymptotic Distribution of Order Statistics

As we found in the last section, evaluation of the exact probability density function of $X_{(r)}$ is sometimes rather complicated and it is useful to try to approximate its distribution. When the sample size n is large, such results can be obtained and they are generally called the asymptotic or the large sample distribution of $X_{(r)}$, as $n \to \infty$. Information concerning the form of the asymptotic distribution increases the usefulness of order statistics in applications, particularly for large sample sizes. In speaking of a general asymptotic distribution for any r, however, we must consider two distinct cases:

Case 1: As $n \to \infty, r/n \to p, 0 < p < 1$.
Case 2: As $n \to \infty, r$ or $n - r$ remains finite.

Case 1 would be of interest, for example, in the distribution of quantiles, whereas Case 2 would be appropriate mainly for the distribution of extreme values. Case 2 will not be considered here. The reader is referred to Wilks (1948) for a discussion of the asymptotic distribution of $X_{(r)}$ for fixed r under various conditions and to Gumbel (1958) for asymptotic distributions of extremes.

Under the assumptions of Case 1, we show in this section that the distribution of the standardized rth-order statistic from the uniform distribution approaches the standard normal distribution. This result can be shown in either of two ways. The most direct approach is to show that the probability density function of a standardized $U_{(r)}$ approaches the function $\varphi(u)$, the pdf of the standard normal distribution. To this end, in the density for $U_{(r)}$,

$$f_{U_{(r)}}(u) = \frac{n!}{(r-1)!(n-r)!} u^{r-1}(1-u)^{n-r} \quad 0 < u < 1$$

we make the transformation

$$z = \frac{u-\mu}{\sigma}$$

where μ and σ are, respectively, the mean and the variance of $U_{(r)}$, and obtain, for all z,

$$\begin{aligned} f_{Z_{(r)}}(z) &= \frac{n!}{(r-1)!(n-r)!}(\sigma z+\mu)^{r-1}(1-\sigma z-\mu)^{n-r}\sigma \\ &= n\binom{n-1}{r-1}\sigma\mu^{r-1}(1-\mu)^{n-r}\left(1+\frac{\sigma z}{\mu}\right)^{r-1}\left(1-\frac{\sigma z}{1-\mu}\right)^{n-r} \\ &= n\binom{n-1}{r-1}\sigma\mu^{r-1}(1-\mu)^{n-r}e^{v} \end{aligned} \tag{2.10.1}$$

where

$$v = (r-1)\ln\left(1+\frac{\sigma z}{\mu}\right) + (n-r)\ln\left(1-\frac{\sigma z}{1-\mu}\right) \tag{2.10.2}$$

Now using the Taylor series expansion

$$\ln(1+x) = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{x^i}{i} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

which converges for $-1 < x < 1$, and with the notation

$$\frac{\sigma}{\mu} = c_1 \qquad \frac{\sigma}{1-\mu} = c_2$$

we have

$$\begin{aligned} v &= (r-1)\left(c_1 z - c_1^2\frac{z^2}{2} + c_1^3\frac{z^3}{3} - \cdots\right) - (n-r)\left(c_2 z + c_2^2\frac{z^2}{2} + c_2^3\frac{z^3}{3} + \cdots\right) \\ &= z[c_1(r-1) - c_2(n-r)] - \frac{z^2}{2}\left[c_1^2(r-1) + c_2^2(n-r)\right] + \frac{z^3}{3}\left[c_1^3(r-1) - c_2^3(n-r)\right] - \cdots \end{aligned} \tag{2.10.3}$$

Since we are going to take the limit of v as $n \to \infty$, $r/n \to p$ fixed, $0 < p < 1$, c_1 and c_2 can be approximated as

$$c_1 = \left[\frac{(n-r+1)}{r(n+2)}\right]^{1/2} \approx \left(\frac{1-p}{pn}\right)^{1/2}$$

$$c_2 = \left[\frac{r}{(n-r+1)(n+2)}\right]^{1/2} \approx \left[\frac{p}{(1-p)n}\right]^{1/2}$$

respectively. Substitution of these values in (2.10.3) shows that as $n \to \infty$, the coefficient of z is

$$\frac{(r-1)\sqrt{1-p}}{\sqrt{np}} - \frac{(n-r)\sqrt{p}}{\sqrt{n(1-p)}} = \frac{r - np - (1-p)}{\sqrt{np(1-p)}} = -\frac{\sqrt{1-p}}{\sqrt{np}} \to 0$$

the coefficient of $-z^2/2$ is

$$\frac{(r-1)(1-p)}{np} + \frac{(n-r)p}{n(1-p)} = (1-p) - \frac{(1-p)}{np} + p = 1 - \frac{(1-p)}{np} \to 1$$

and the coefficient of $z^3/3$ is

$$\frac{(r-1)(1-p)^{3/2}}{(np)^{3/2}} - \frac{(n-r)p^{3/2}}{[n(1-p)]^{3/2}} = \frac{(np-1)}{n^{3/2}}\left(\frac{1-p}{p}\right)^{3/2} - \frac{p^{3/2}}{[n(1-p)]^{1/2}} \to 0$$

Substituting these results in (2.10.3) and ignoring terms of order $n^{-1/2}$ and higher, the limiting value is

$$\lim_{n\to\infty} v = \frac{-z^2}{2}$$

For the limiting value of the constant term in (2.10.1), we must use Stirling's formula

$$k! \approx \sqrt{2\pi}\mathrm{e}^{-k}k^{k+1/2}$$

for the factorials, which is to be multiplied by

$$\sigma\mu^{r-1}(1-\mu)^{n-r} = \frac{r^{r-1/2}(n-r+1)^{n-r+1/2}}{(n+1)^n(n+2)^{1/2}} \approx \frac{r^{r-1/2}(n-r+1)^{n-r+1/2}}{(n+1)^{n+1/2}}$$

So, as $n \to \infty$, the entire constant of (2.10.1) is written as

$$\begin{aligned} & n\binom{n-1}{r-1}\sigma\mu^{r-1}(1-\mu)^{n-r} \\ & = \frac{(n+1)!}{r!(n-r+1)!}\frac{r(n-r+1)}{n+1}\sigma\mu^{r-1}(1-\mu)^{n-r} \\ & \approx \frac{\sqrt{2\pi}\mathrm{e}^{-(n+1)}(n+1)^{n+3/2}}{2\pi\mathrm{e}^{-r}r^{r+1/2}\mathrm{e}^{-(n-r+1)}(n-r+1)^{n-r+3/2}}\frac{r^{r+1/2}(n-r+1)^{n-r+3/2}}{(n+1)^{n+3/2}} = \frac{1}{\sqrt{2\pi}} \end{aligned}$$

Thus we have the desired result

$$\lim_{n\to\infty} fz_{(r)}(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(1/2)z^2} = \phi(z)$$

Therefore, the *pdf* of the standardized $U_{(r)}$ approaches the *pdf* of the standard normal distribution. This can also be stated in terms of the cdf's

$$\lim_{n\to\infty} P\left(\frac{U_{(r)}-\mu}{\sigma} \le t\right) = \Phi(t)$$

To summarize, for large n, the distribution of $U_{(r)}$ can be approximated by a normal distribution with mean μ and variance σ^2.

For the rth-order statistic from any continuous distribution F_X, the relationship $X_{(r)} = F_X^{-1}(U_{(r)})$ allows us to conclude that the asymptotic distribution of $X_{(r)}$ is also approximately normal as long as the appropriate mean and variance are substituted. The key to this argument is the result that if a random variable is approximately normally distributed, then a smooth function of it (a transformation) is also approximately normally distributed with a certain mean and variance. Using the approximate mean and variance in (2.9.5) and (2.9.6) and using the fact that $r/n \to p$ as $n \to \infty$, we get

$$E(X_{(r)}) \to F_X^{-1}(p) \quad \text{and} \quad \mathrm{var}(X_{(r)}) \approx \frac{[p(1-p)]\{f_X[F_X^{-1}(p)]\}^{-2}}{n}$$

and state the following theorem.

THEOREM 2.10.1

*Let $X_{(r)}$ denote the r*th*-order statistic of a random sample of size n from any continuous cdf F_X. Then if $r/n \to p$ as $n \to \infty$, $0 < p < 1$, the distribution of*

$$\left[\frac{n}{p(1-p)}\right]^{1/2} f_X(\theta)[X_{(r)} - \theta]$$

tends to the standard normal, where $\theta = F_X^{-1}(p)$.

We can use this result to show that $X_{(r)}$ is a consistent estimator of $\theta = F_X^{-1}(p)$ if $r/n \to p$ as $n \to \infty$.

For the asymptotic joint distribution of any two-order statistics $X_{(r)}$ and $X_{(s)}$, $1 \le r < s \le n$, Smirnov (1935) obtained a similar result. Let $n \to \infty$ in such a way that $r/n \to p_1$ and $s/n \to p_2$, $0 < p_1 < p_2 < 1$, remain fixed. Then $X_{(r)}$ and $X_{(s)}$ are (jointly) asymptotically bivariate normally distributed with means μ_i, variances $p_i(1-p_i)[f_X(\mu_i)]^{-2}/n$, and covariance $p_1(1-p_2)[nf_X(\mu_1)f_X(\mu_2)]^{-2}$, where μ_i satisfies $F_X(\mu_i) = p_i$ for $i = 1, 2$.

2.11 Tolerance Limits for Distributions and Coverages

An important application of order statistics is in setting *tolerance limits* for distributions. The resulting procedure does not depend in any way on the underlying population as long as it is continuous. Such a procedure is therefore distribution free.

A *tolerance interval* for a continuous distribution with tolerance coefficient γ is a random interval (given by two endpoints that are random variables) such that the probability is γ that the area under the probability density function and between the endpoints of the interval is at least a certain preassigned value p. In other words, the probability is γ that this random interval covers or includes at least a specified percentage ($100p\%$) of the underlying distribution. If the endpoints of the tolerance interval are two order statistics $X_{(r)}$ and $X_{(s)}$, $r < s$, of a random sample of size n, the tolerance interval satisfies the condition

$$P[P(X_{(r)} < X < X_{(s)}) \ge p] = \gamma \tag{2.11.1}$$

The probability γ is called the *tolerance coefficient*. We need to find the two indices r and s, for a given tolerance coefficient, subject to the conditions

that $1 \le r < s \le n$. If the underlying distribution F_X is continuous, we can write

$$\begin{aligned} P[X_{(r)} < X < X_{(s)}] &= P(X < X_{(s)}) - P(X < X_{(r)}) \\ &= F_X(X_{(s)}) - F_X(X_{(r)}) \\ &= U_{(s)} - U_{(r)} \end{aligned}$$

according to the PIT. Substituting this result in (2.11.1), we find that the tolerance interval satisfies

$$P[U_{(s)} - U_{(r)} \ge p] = \gamma \tag{2.11.2}$$

Thus, the question of finding the indices r and s, for any arbitrary continuous distribution reduces to that of finding the indices for the uniform (0, 1) distribution. This is a matter of great simplicity, as we show in Theorem 2.11.1.

THEOREM 2.11.1

For a random sample of size n from the uniform (0, 1) distribution, *the difference* $U_{(s)} - U_{(r)}, 1 \le r < s \le n$, *is distributed as the* $(s-r)$th-*order statistic* $U_{(s-r)}$ *and thus has a beta distribution with parameters* $s - r$ *and* $n - s + r + 1$.

Proof

We begin with the joint distribution of $U_{(r)}$ and $U_{(s)}$ found in (2.6.8). To prove the theorem, we make the transformation

$$U = U_{(s)} - U_{(r)} \quad \text{and} \quad V = U_{(s)}$$

The joint distribution of U and V is then

$$f_{U,V}(u,v) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}(v-u)^{r-1}u^{s-r-1}(1-v)^{n-s} \quad 0 < u < v < 1$$

and so

$$f_U(u) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}u^{s-r-1}\int_u^1 (v-u)^{r-1}(1-v)^{n-s}dv$$

Under the integral sign, we make the change of variable $v - u = t(1-u)$ and obtain

$$\begin{aligned} f_U(u) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u^{s-r-1}(1-u)^{r-1} \\ &\quad \times \int_u^1 t^{r-1}[(1-u) - t(1-u)]^{n-s}(1-u)\, dt \\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u^{s-r-1}(1-u)^{n-s+r} B(r, n-s+1) \\ &= \frac{n!}{(s-r-1)!(n-s+r)!} u^{s-r-1}(1-u)^{n-s+r} \quad 0 < u < 1 \end{aligned} \tag{2.11.3}$$

This shows that U has a beta distribution with parameters $s-r$ and $n-s+r+1$, which is also the distribution of $U_{(s-r)}$ by Theorem 2.4.3. Thus, the required result in (2.11.2) can be written simply as

$$\gamma = P(U \geq p) = \int_p^1 \frac{n!}{(s-r-1)!(n-s+r)!} u^{s-r-1}(1-u)^{n-s+r} du$$

We can solve this for r and s for any given values of p, γ and n, or we can find the tolerance coefficient γ for given values of p, r, s and n. Note that all of the above results remain valid as long as the underlying *cdf* is continuous so that the PIT can be applied and hence the tolerance interval is distribution-free.

Corollary 2.11.1.1 *$U_{(r)} - U_{(r-1)}$ has a beta distribution with parameters* 1 *and* n.

2.11.1 One-Sample Coverages

The difference $F_X(X_{(s)}) - F_X(X_{(r)}) = U_{(s)} - U_{(r)}$ is called the *coverage* of the random interval $(X_{(r)}, X_{(s)})$, or simply an $s-r$ *cover*. The coverages are generally important in nonparametric statistics because of their distribution-free property. We define the set of successive elementary coverages as the differences.

$$C_i = F_X(X_{(i)}) - F_X(X_{(i-1)}) = U_{(i)} - U_{(i-1)} \quad i = 1, 2, \ldots, n+1$$

where we write $X_{(0)} = -\infty$, $X_{(n+1)} = \infty$, Thus,

$$\begin{aligned} C_1 &= F_X(X_{(1)}) = U_{(1)} \\ C_2 &= F_X(X_{(2)}) - F_X(X_{(1)}) = U_{(2)} - U_{(1)} \\ &\cdots \\ C_n &= F_X(X_{(n)}) - F_X(X_{(n-1)}) = U_{(n)} - U_{(n-1)} \\ C_{(n+1)} &= 1 - U_{(n)} \end{aligned} \tag{2.11.4}$$

Corollary 2.11.1.1 shows that the distribution of the ith elementary coverage C_i does not depend on the underlying cdf F_X, as long as F_X is continuous and thus the elementary coverages are distribution free. In fact, from Corollary 2.11.1.1 and properties of the beta distribution (or directly), it immediately follows that

$$E(C_i) = \frac{1}{n+1}$$

From this result, we can draw the interpretation that the n order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ partition the area under the pdf into $n+1$ parts, each of which has the same expected proportion of the total probability.

Since the Jacobian of the transformation defined in (2.11.4) mapping $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ onto $C_{(1)}, C_{(2)}, \ldots, C_{(n)}$ is equal to 1, the joint distribution of the n coverages is

$$f_{C_1,C_2,\ldots,C_n}(c_1, c_2, \ldots, c_n) = n! \quad \text{for } c_i \geq 0, \quad i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n+1} c_i = 1$$

A sum of any r successive elementary coverages is called an r *coverage*. We have the sum $C_i + C_{i+1} + \cdots + C_{i+r} = U_{(i+r)} - U_{(i)}, i + r \leq n$. Since the distribution of $C_1, C_2, \ldots, C_n$ is symmetric in $c_1, c_2, \ldots, c_n$, the marginal distribution of the sum of any r of the coverages must be the same for each fixed value of r, in particular equal to that of

$$C_1 + C_2 + \cdots + C_r = U_{(r)}$$

which is given in (2.6.9). The expected value of an r coverage then is $r/(n+1)$, with the same interpretation as before.

2.11.2 Two-Sample Coverages

Now suppose that a random sample of size m, $X_1, X_2, \ldots, X_m$ is available from a continuous cdf F_X and that a second independent random sample of

size n, $Y_1, Y_2, \ldots, Y_n$ is available from another continuous cdf F_Y. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the Y order statistics and let $I_1 = (-\infty, Y_{(1)}], I_2 = (Y_{(1)}, Y_{(2)}], \ldots, I_n = (Y_{(n-1)}, Y_{(n)}], I_{n+1} = (Y_{(n)}, \infty)$ denote the $(n+1)$ nonoverlapping *blocks* formed by the n Y-order statistics. The number of X observations belonging to the ith block, I_i, is called the ith *block frequency* and is denoted by B_i, say. Thus, there are $(n+1)$ block frequencies $B_1, B_2, \ldots, B_{n+1}$, where $B_{n+1} = m - B_1 - B_2 - \cdots - B_n$. A particularly appealing feature of the block frequencies is their distribution-free property, summarized in Theorem 2.11.2.

THEOREM 2.11.2

When $F_X = F_Y$ so that the underlying distributions are identical, the joint distribution of $B_1, B_2, \ldots, B_{n+1}$ is given by

$$P(B_1 = b_1, B_2 = b_2, \ldots, B_{n+1} = b_{n+1}) = \frac{1}{\binom{m+n}{n}}$$

$$\text{where } 0 \le b_j \le m \quad \text{and} \quad \sum_{j=1}^{n+1} b_j = m$$

In fact, one can show that when $F_X = F_Y$, the joint distribution of any t of the random variables $B_1, B_2, \ldots, B_{n+1}$, say $B_1^*, B_2^*, \ldots, B_t^*$ is given by

$$P(B_1^* = b_1^*, B_2^* = b_2^*, \ldots, B_t^* = b_t^*) = \frac{\binom{m+n-b_1^*-b_2^*-\cdots-b_t^*}{n-1}}{\binom{m+n}{n}}$$

$$\text{where } 0 \le b_j^* \le m$$

For proofs of these and other related results see Wilks (1962, pp. 442–446), for example.

We will later discuss a number of popular nonparametric tests based on the block frequencies. Some problems involving the block frequencies are given at the end of this chapter.

2.11.3 Ranks, Block Frequencies, and Placements

The ranks of observations play a crucial role in nonparametric statistics. The rank of the ith observation X_i, in a sample of m observations, is equal to the

number of observations that are less than or equal to X_i. In other words, using the indicator function,

$$\text{rank}(X_i) = \sum_{j=1}^{m} I(X_j \leq X_i) = mS_m(X_i)$$

where $S_m(X_i)$ is the edf of the sample. For the ordered observation $X_{(i)}$, the rank is simply equal to the index i, or

$$\text{rank}(X_{(i)}) = \sum_{j=1}^{m} I(X_j \leq X_{(i)}) = mS_m(X_{(i)}) = i$$

Thus, ranks of ordered observations in a single sample are similar to an empirical (data-based) version of the one-sample coverages studied earlier. We provide a functional definition of rank in Section 5.5 and study some of its statistical properties.

When there are two samples, say m X's and n Y's, the rank of an observation is often defined with respect to the combined sample of $(m+n)$ observations, say Z's. In this case, the rank of a particular observation can be defined again as the number of observations (X's and Y's) less than or equal to that particular observation. A functional definition of rank in the two-sample case is given in Section 7.2. However, to see the connection with two-sample coverages, let us examine, for example, the rank of $Y_{(j)}$ in the combined sample. Clearly, this is equal to the number of X's less than or equal to $Y_{(j)}$ plus j, the number of Y's less than or equal to $Y_{(j)}$, so that

$$\text{rank}(Y_{(j)}) = \sum_{i=1}^{m} I(X_i \leq Y_{(j)}) + j$$

However, $\sum_{i=1}^{m} I(X_i \leq Y_{(j)})$ is simply equal to $r_1 + r_2 + \cdots + r_j$ where r_i is the frequency of the ith block $(Y_{(i-1)}, Y_{(i)}]$, defined under two-sample coverages. Thus we have

$$\text{rank}(Y_{(j)}) = r_1 + r_2 + \cdots + r_j + j$$

and the rank of an ordered Y observation in the combined sample is a simple function of the block frequencies. Also let $P_{(j)} = mS_m(Y_{(j)})$ denote the number of X's that are less than or equal to $Y_{(j)}$. The quantity $P_{(j)}$ is called the *placement* of $Y_{(j)}$ among the X observations (Orban and Wolfe, 1982) and has been used in some nonparametric tests. Then, $P_{(j)} = r_j - r_{j-1}$ with $r_0 = 0$ and $\text{rank}(Y_{(j)}) = P_{(j)} + j$. This shows the connection between ranks and placements. More details regarding the properties of placements are given as problems. The reader is also referred to Fligner and Wolfe (1976) for related results.

2.12 Summary

In this chapter, we discussed some mathematical–statistical concepts and properties related to the empirical distribution function and the quantile function of a random variable. These include order statistics, which can be viewed as sample estimates of quantiles or percentiles of the underlying distribution. Other methods of estimating population quantiles have been considered in the literature, primarily based on linear functions of order statistics. Section 5.8 gives more details.

Problems

2.1 Let X be a discrete random variable taking on only positive integer values. Show that

$$E(X) = \sum_{i=1}^{\infty} P(X \geq i)$$

2.2 Let X be a nonnegative continuous random variable with cdf F_X. Show that

$$E(X) = \int_0^{\infty} [1 - F_X(x)]dx$$

(Hint: Use integration by parts on the definition of $E(X)$).

2.3 Show that

$$\sum_{x=a}^{n} \binom{n}{x} p^x (1-p)^{n-x} = \frac{1}{B(a, n-a+1)} \int_0^p y^{a-1}(1-y)^{n-a}\, dy$$

for any $0 < p < 1$. The integral on the right is called an incomplete beta integral and written as $I_p(a, n-a+1)$. Thus, if X is a binomial random variable with parameters n and p, the probability that X is less than or equal to a ($a = 0, 1, \ldots, n$) is

$$1 - I_p(a+1, n-a) = I_{1-p}(n-a, a+1)$$

2.4 Find the transformation to obtain, from an observation U following a continuous uniform (0, 1) distribution, an observation from each of the following continuous probability distributions:

(a) Exponential distribution with mean 1.
(b) Beta distribution with $a = 2$ and $b = 1$. The probability density function is given by

$$f(x) = 2x \quad \text{for } 0 < x < 1$$

(c) The logistic distribution defined by the probability density function

$$f(x) = \frac{e^{-(x-a)/b}}{[1 + e^{-(x-a)/b}]^2} \quad -\infty < x < \infty, -\infty < a < \infty, 0 < b < \infty$$

(d) The double exponential distribution defined by the probability density function

$$f(x) = \frac{1}{2b} e^{-(|x-a|)/b} \quad -\infty < x < \infty, -\infty < a < \infty, 0 < b < \infty$$

(e) The Cauchy distribution defined by the probability density function

$$f(x) = \frac{b}{\pi[b^2 + (x-a)^2]} \quad -\infty < x < \infty, -\infty < a < \infty, 0 < b < \infty$$

2.5 Prove the probability-integral transformation (Theorem 2.5.1) by finding the moment-generating function of the random variable $Y = F_X(X)$, where X is absolutely continuous and has cdf F_X.

2.6 If X is a continuous random variable with probability density function $f_X(x) = 2(1-x)$ for $0 < x < 1$, find the transformation $Y = g(X)$ such that the random variable Y has the uniform distribution over (0,2).

2.7 The order statistics for a random sample of size n from a discrete distribution are defined as in the continuous case except that now we have $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. Suppose a random sample of size 5 is taken with replacement from the discrete distribution $f_X(x) = 1/6$ for $x = 1, 2, \ldots, 6$. Find the probability mass function of $X_{(1)}$, the smallest order statistic.

2.8 A random sample of size 3 is drawn from the population $f_X(x) = \exp[-(x-\theta)]$ for $x > \theta$. We want to find a 95% confidence-interval estimate for the parameter θ. Since the maximum-likelihood estimate for θ is $X_{(1)}$, the smallest order statistic, a logical choice for the limits of the confidence interval would be some functions of $X_{(1)}$. If the upper limit is $X_{(1)}$, find the corresponding lower limit $g(X_{(1)})$ such that the confidence coefficient is 0.95.

2.9 For the n-order statistics of a sample from the uniform distribution over $(0, \theta)$, show that the interval $(X_{(n)}, X_{(n)}/\alpha^{1/n})$ is a $100(1-\alpha)\%$ confidence-interval estimate of the parameter θ.

2.10 Ten points are chosen randomly and independently on the interval (0, 1).
(a) Find the probability that the point nearest 1 exceeds 0.90.
(b) Find the number c such that the probability is 0.5 that the point nearest zero exceeds c.

2.11 Find the expected value of the largest order statistic in a random sample of size 3 from:
(a) The exponential distribution $f_X(x) = \exp(-x)$ for $x \geq 0$
(b) The standard normal distribution

2.12 Verify the result given in (2.7.1) for the distribution of the median of a sample of size $2m$ from the uniform (0, 1) distribution when $m = 2$. Show that this distribution is symmetric about 0.5 by writing (2.7.1) in the form

$$f_U(u) = 8(0.5 - |u - 0.5|)^2(1 + 4|u - 0.5|) \quad \text{for } 0 < u < 1$$

2.13 Find the mean and variance of the median of a random sample of n from the uniform (0.1) distribution:
(a) When n is odd
(b) When n is even and U is defined as in Section 2.7

2.14 Find the probability that the range of a random sample of size n from the population $f_X(x) = 2e^{-2x}$ for $x \geq 0$ does not exceed 4.

2.15 Find the distribution of the range of a random sample of size n from the exponential distribution $f_X(x) = 4\exp(-4x)$ for $x \geq 0$.

2.16 Give an expression similar to (2.7.3) for the probability density function of the midrange for any continuous distribution and use it to find the density function in the case of a uniform (0, 1) population.

2.17 By making the transformation $U = nF_X(X_{(1)})$, $V = n[1 - F_X(X_{(n)})]$ in (2.6.8) with $r = 1$, $s = n$, for any continuous F_X, show that U and V are independent random variables in the limiting case as $n \to \infty$, so that the two extreme values of a random sample are asymptotically independent.

2.18 Use (2.9.5) and (2.9.6) to approximate the mean and variance of:
(a) The median of a sample of size $2m + 1$ from a normal distribution with mean μ and variance σ^2.
(b) The fifth order statistic of a random sample of size 19 from the exponential distribution $f_X(x) = \exp(-x)$ for $x \geq 0$.

2.19 Let $X_{(n)}$ be the largest value in a sample of size n from the pdf f_X.
(a) Show that $\lim_{n\to\infty} P(n^{-1}X_{(n)} \leq x) = \exp(-a/\pi x)$ if $f_X(x) = a/[\pi(a^2+x^2)]$ (Cauchy).
(b) Show that $\lim_{n\to\infty} P(n^{-2}X_{(n)} \leq x) = \exp(-a\sqrt{2/\pi x})$ if $f_X(x) = (a/\sqrt{2\pi})x^{-3/2}\exp(-a^2/2x)$ for $x \geq 0$.

2.20 Let $X_{(r)}$ be the rth-order statistic of a random sample of size n from a cdf F_X.

(a) Verify that $P(X_{(r)} \le t) = \sum_{k=r}^{n} \binom{n}{k} [F_X(t)]^k [1 - F_X(t)]^{n-k}$.

(b) Verify the probability density function of $X_{(r)}$ given in (2.6.4) by differentiation of the result in (a).

(c) By considering $P(X_{(r)} > t/n)$ in the form of (a), find the asymptotic distribution of $X_{(r)}$ for r fixed and $n \to \infty$ if $F_X(x)$ is the uniform (0, 1) distribution.

2.21 Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be order statistics for a random sample from the exponential distribution $F_X(x) = 1 - \exp(-x)$ for $x \ge 0$.

(a) Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for any $s > r$.

(b) Find the distribution of $X_{(r+1)} - X_{(r)}$.

(c) Show that $E(X_{(i)}) = \sum_{j=1}^{i} 1/(n + 1 - j)$.

(d) Interpret the significance of these results if the sample arose from a life test on n light bulbs with exponential lifetimes.

2.22 Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the order statistics of a sample from a continuous unspecified distribution F_X. Define the n random variables

$$V_i = \frac{F_X(X_{(i)})}{F_X(X_{(i+1)})} \quad \text{for } 1 \le i \le n-1 \quad \text{and} \quad V_n = F_X(X_{(n)})$$

(a) Find the marginal distribution of V_r, $1 \le r \le n$.

(b) Find the joint distribution of V_r and $F_X(X_{(r-1)})$, $1 \le r \le n-1$, and show that they are independent.

(c) Find the joint distribution of $V_1, V_2, \ldots, V_n$.

(d) Show that $V_1, V_2, \ldots, V_n$ are independent.

(e) Show that $V_1, V_2^2, V_3^3, \ldots, V_n^n$ are independent and identically distributed with the uniform (0, 1) distribution.

2.23 Find the probability that the range of a random sample of size 3 from the uniform distribution is less than 0.8.

2.24 Find the expected value of the range of a random sample of size 3 from the uniform distribution.

2.25 Find the variance of the range of a random sample of size 3 from the uniform distribution.

2.26 Let the random variable U denote the proportion of the population lying between the two extreme values of a sample of n from some unspecified continuous population. Find the mean and variance of U.

2.27 Suppose that a random sample of size m, $X_1, X_2, \ldots, X_m$, is available from a continuous cdf F_X and a second independent random sample of

size n, $Y_1, Y_2, \ldots, Y_n$, is available from a continuous cdf F_Y. Let S_j be the random variable representing the number of Y blocks $I_1, I_2, \ldots, I_{n+1}$ (defined in Section 2.11.2) that contain exactly j observations from the X sample, $j = 0, 1, \ldots, m$.

(a) Verify that $S_0 + S_1 + \cdots + S_m = n + 1$ and $S_1 + 2S_2 + \cdots + mS_m = m$.

(b) If $F_X = F_Y$, show that the joint distribution of $S_0, S_1, \ldots, S_m$ is given by $\dfrac{(n+1)!}{s_0! s_1! \cdots s_m!} \binom{m+n}{n}^{-1}$.

(c) In particular show that, if $F_X = F_Y$, the marginal distribution of S_0 is given by $\binom{n+1}{s_0}\binom{m+1}{n-s_0} \Big/ \binom{m+n}{n}$ for $s_0 = n - m + 1, n - m + 2, \ldots, n$. A simple distribution-free test for the equality of F_X and F_Y can be based on S_0, the number of blocks that do not contain any X observation. This is the ''empty block'' test (Wilks, 1962, pp. 446–452).

2.28 *Exceedance statistics*. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from arbitrary continuous cdf's F_X and F_Y, respectively, and let $S_m(x)$ and $S_n(y)$ be the corresponding empirical cdf's. Consider, for example, the quantity $m[1 - S_m(Y_1)]$, which is simply the count of the total number of X's that exceed (or do not precede) Y_1 and may be called an exceedance statistic. Several nonparametric tests proposed in the literature are based on exceedance (or precedence) statistics and these are called exceedance (or precedence) tests. We will study some of these tests later.

Let $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ be the order statistics of the Y sample. Answer parts (a) through (h) assuming $F_X = F_Y$.

(a) Show that $S_m(Y_i)$, $i = 1, 2, \ldots, n$, is uniformly distributed over the set of points $(0, 1/m, 2/m, \ldots, 1)$.

(b) Show that the distribution of $S_m(Y_{(j)}) - S_m(Y_{(k)})$, $k < j$, is the same as the distribution of $S_m\,(Y_{(j-k)})$.

(Fligner and Wolfe, 1976)

(c) Show that the distribution of $P_{(i)} = mS_m(Y_{(i)})$ is given by

$$P[P_{(i)} = j] = \frac{\binom{m+n-i-j}{m-j}\binom{i+j-1}{j}}{\binom{m+n}{n}} \qquad j = 0, 1, \ldots, m$$

The quantity $P_{(i)}$ is the count of the number of X's that precede the ith order statistic in the Y sample and is called the ''placement'' of $Y_{(i)}$ among the observations in the X sample. Observe that $P_{(i)} = r_1 + \cdots + r_i$, where r_i is the ith block frequency and thus $r_i = P_{(i)} - P_{(i-1)}$.

(d) Show that

$$E(P_{(i)}) = \frac{mi}{n+1} \quad \text{and} \quad \text{var}(P_{(i)}) = \frac{i(n-i+1)m(m+n+1)}{(n+1)^2(n+2)}$$

(Orban and Wolfe, 1982)

(e) Let T_1 be the number of X observations exceeding the largest Y observation, that is, $T_1 = m[1 - S_m(Y_{(n)})] = m - P_{(n)}$. Show that

$$P(T_1 = t) = \frac{\binom{m+n-t-1}{m-t}}{\binom{m+n}{m}}$$

(f) Let T_2 be the number of X's preceding (not exceeding) the smallest Y observation; that is, $T_2 = mS_m(Y_{(1)}) = P_{(1)}$. Show that the distribution of $T_3 = T_1 + T_2$ is given by

$$P(T_3 = t) = (t+1)\frac{\binom{m+n-t-2}{m-t}}{\binom{m+n}{m}}$$

(Rosenbaum, 1954)

(g) Let T_4 be the number of X's in the interval $I = (Y_{(r)}, Y_{(n+1-r)}]$, where $Y_{(r)}$ is the pth sample quantile of the Y's. The interval I is called the interquartile range of the Y's. Note that $T_4 = m[S_m(Y_{(n+1-r)}) - S_m(Y_{(r)})]$. Show that the distribution of T_4 is given by

$$P(T_4 = t) = \frac{\binom{m+2r-i-1}{m-t}\binom{n+t-2r}{t}}{\binom{m+n}{m}} \quad t = 0, 1, \ldots, m$$

(h) Show that

$$E(T_4) = \frac{2m}{n+1} \quad \text{and} \quad \text{var}(T_4) = \frac{2m(n-1)(m+n+1)}{(n+1)^2(n+2)}$$

(Hackl and Katzenbeisser, 1984)

The statistics T_3 and T_4 have been proposed as tests for H_0: $F_X = F_Y$ against the alternative that the dispersion of F_X exceeds the dispersion of F_Y.

2.29 Let $S_m(x)$ be the edf of a random sample of size m from a continuous cdf F_X. Show that for $-\infty < x < y < \infty$.

$$\operatorname{cov}[S_m(x), S_m(y)] = \frac{F_X(x)[1 - F_X(y)]}{m}$$

2.30 Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution $f_X(x) = (2\theta)^{-1}e^{-x/2\theta}$, $x > 0$, $\theta > 0$, and let the ordered X's be denoted by $Y_1 \le Y_2 \le \cdots \le Y_n$. Assume that the underlying experiment is such that Y_1 becomes available first, then Y_2, and so on (for example, in a life-testing study) and that the experiment is terminated as soon as Y is observed for some specified r.

(a) Show that the joint pdf of $Y_1, Y_2, \ldots, Y_r$ is

$$(2\theta)^{-r}\frac{n!}{(n-r)!}\exp -\left[\frac{\sum_{i=1}^{r} y_i + (n-r)y_r}{2\theta}\right] \quad 0 \le y_1 \le \cdots \le y_r < \infty$$

(b) Show that $\theta^{-1}[\sum_{i=1}^{r} Y_i + (n-r)Y_r]$ has a chi-square distribution with $2r$ degrees of freedom.

2.31 A manufacturer wants to market a new brand of heat-resistant tiles which may be used on the space shuttle. A random sample of m of these tiles is put on a test and the heat resistance capacities of the tiles are measured. Let $X_{(1)}$ denote the smallest of these measurements. The manufacturer is interested in finding the probability that in a future test (performed by, say, an independent agency) of a random sample of n of these tiles, at least k ($k = 1, 2, \ldots, n$) will have a heat resistance capacity exceeding $X_{(1)}$ units. Assume that the heat resistance capacities of these tiles follows a continuous distribution with cdf F.

(a) Show that the probability of interest is given by

$$\sum_{r=k}^{n} P(r)$$

where

$$P(r) = \frac{mn!(r+m-1)!}{r!(n+m)!}$$

(b) Show that

$$\frac{P(r-1)}{P(r)} = \frac{r}{r+m-1}$$

a relationship that is useful in calculating $P(r)$.

(c) Show that the number of tiles n to be put on a future test such that all of the n measurements exceed $X_{(1)}$ with probability p is given by

$$n = \frac{m(1-p)}{p}$$

2.32 Define the indicator variable

$$\varepsilon(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Show that the random function defined by

$$F_n(x) = \sum_{i=1}^{n} \frac{\varepsilon(x - X_i)}{n}$$

is the empirical distribution function of a sample $X_1, X_2, \ldots, X_n$, by showing that

$$F_n(x) = S_n(x) \quad \text{for all } x$$

2.33 Prove that $\text{cov}[S_n(x), S_n(y)] = c[F_X(x), F_X(y)]/n$ where

$$c(s,t) = \min(s,t) - st = \begin{cases} s(1-t) & \text{if } s \leq t \\ t(1-s) & \text{if } s \geq t \end{cases}$$

and $S_n(.)$ is the empirical distribution function of a random sample of size n from the population F_X.

2.34 Let $S_n(x)$ be the empirical distribution function for a random sample of size n from the uniform (0, 1) distribution. Define

$$X_n(t) = \sqrt{n}|S_n(t) - t|$$
$$Z_n(t) = (t+1)X_n[t/(t+1)] \quad \text{for all } 0 \leq t \leq 1$$

Find $E[X_n(t)], E[Z_n(t)], \text{var}[X_n(t)]$ and $\text{var}[Z_n(t)]$, and conclude that $\text{var}[X_n(t)] \leq \text{var}[Z_n(t)]$ for all $0 \leq t \leq 1$ and all n.

3

Tests of Randomness

3.1 Introduction

Passing a line of 10 persons waiting to buy a ticket at a movie theater on a Saturday afternoon, suppose we observe the arrangement of 5 males and 5 females in the line to be M, F, M, F, M, F, M, F, M, F. Would this be considered a random arrangement by gender? Intuitively, the answer is no, since the alternation of the two types of symbols suggests intentional mixing by pairs. This arrangement is an extreme case, as is the configuration M, M, M, M, M, F, F, F, F, F, with intentional clustering. In less extreme situations, the randomness of an arrangement can be tested statistically using the theory of runs.

Given an ordered sequence of one or more types of symbols, a *run* is defined to be a succession of one or more types of symbols that are followed and preceded by a different symbol or no symbol at all. Clues to lack of randomness are provided by any tendency of the symbols to exhibit a definite pattern in the sequence. Both the number and the length of the runs, which are of course interrelated, should reflect the existence of some sort of pattern. Tests for randomness can therefore be based on either criterion or some combination thereof. Too few runs, too many runs, a run of excessive length, too many runs of excessive length, etc., can be used as statistical criteria for rejection of the null hypothesis of randomness, since these situations should occur rarely in a truly random sequence.

The alternative to randomness is often simply nonrandomness. In a test based on the total number of runs, both too few and too many runs suggest lack of randomness. A null hypothesis of randomness would consequently be rejected if the total number of runs is either too large or too small. However, the two situations may indicate different types of lack of randomness. In the movie theater example, a sequence with too many runs, tending toward the genders alternating, might suggest that the movie is popular with teenagers and young adults who attend as couples, whereas the other extreme arrangement may result if the movie is more popular with younger children.

Tests of randomness are an important addition to statistical theory, because the theoretical bases for almost all the classical techniques, as well as distribution-free procedures, begin with the assumption of a random sample. If the randomness of the observations is suspect, the information about order, which is almost always available, can be used to test a hypothesis of randomness. This kind of analysis is also useful in time-series and quality-control studies.

The symbols studied for pattern may arise naturally, as with gender in the theater example, or may be artificially imposed according to some dichotomizing criterion. Thus, the runs tests are applicable to either qualitative or quantitative data. In the latter case, the dichotomy is usually effected by comparing the magnitude of each number with a focal point, commonly the median or mean of the sample, and noting whether each observation exceeds or is exceeded by this value. When the data consist of actual numbers, two other types of runs analyses can be used to reach a conclusion about randomness. Both of these techniques use the information about relative magnitudes of adjacent numbers in the time-ordered sequence. These techniques, called the *runs up and down test* and the *rank von Neumann test*, use more of the available information and are especially effective when the alternative to randomness is either a trend or autocorrelation.

3.2 Tests Based on the Total Number of Runs

Assume an ordered sequence of n elements of two types, n_1 of the first type and n_2 of the second type, where $n_1 + n_2 = n$. If R_1 is the number of runs of type 1 elements and R_2 is the number of runs of type 2, the total number of runs in the sequence is the random variable $R = R_1 + R_2$. In order to derive a test for randomness based on R, we need the probability distribution of R when the null hypothesis of randomness is true.

3.2.1 Exact Null Distribution of *R*

The distribution of R will be found by first determining the joint probability distribution of R_1 and R_2 and then the distribution of their sum. Since under the null hypothesis every arrangement of the $n_1 + n_2$ objects is equiprobable, the probability that $R_1 = r_1$ and $R_2 = r_2$ is the number of distinguishable arrangements of $n_1 + n_2$ objects with r_1 runs of type 1 and r_2 runs of type 2 objects divided by the total number of distinguishable arrangements, which is $n!/(n_1!n_2!)$. For the numerator quantity, the following counting lemma can be used.

LEMMA 3.2.1

The number of distinguishable ways of distributing n-like objects into r distinguishable cells with no cell empty is $\binom{n-1}{r-1}, n \geq r, r \geq 1.$

Proof

Suppose that the n-like objects are all white balls. Place these n balls in a row and effect the division into r cells by inserting each of $r-1$ black balls between any two white balls in the line. Since there are $n-1$ positions in which each black ball can be placed, the total number of arrangements is $\binom{n-1}{r-1}$.

In order to obtain a sequence with r_1 runs of type 1, the n_1-like objects must be placed into r_1 cells, which can be done in $\binom{n_1-1}{r_1-1}$ different ways. The same reasoning applies to obtain r_2 runs for the other n_2 objects. The total number of distinguishable arrangements starting with a run of type 1 then is the product $\binom{n_1-1}{r_1-1}\binom{n_2-1}{r_2-1}$. Similarly, for a sequence starting with a run of type 2, the blocks of objects of type 1 and type 2 must alternate, and consequently either $r_1 = r_2 \pm 1$ or $r_1 = r_2$. If $r_1 = r_2 + 1$, the sequence must begin with run of type 1; if $r_1 = r_2 - 1$, a type 2 run must come first. But if $r_1 = r_2$, the sequence can begin with a run of either type, so the number of distinguishable arrangements must be doubled. We have thus proved the following result.

THEOREM 3.2.1

Let R_1 and R_2 denote the respective numbers of runs of n_1 objects of type 1 and n_2 objects of type 2 in a random sample of size $n = n_1 + n_2$. The joint probability distribution of R_1 and R_2 is

$$f_{R_1,R_2}(r_1,r_2) = \frac{c\binom{n_1-1}{r_1-1}\binom{n_2-1}{r_2-1}}{\binom{n_1+n_2}{n_1}} \quad \begin{array}{l} r_1 = 1,2,\ldots,n_1 \\ r_2 = 1,2,\ldots,n_2 \\ r_1 = r_2 \quad \text{or} \quad r_1 = r_2 \pm 1 \end{array} \tag{3.2.1}$$

where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.

COROLLARY 3.2.1

The marginal probability distribution of R_1 is

$$f_{R_1}(r_1) = \frac{\binom{n_1-1}{r_1-1}\binom{n_2+1}{r_1}}{\binom{n_1+n_2}{n_1}} \quad r_1 = 1, 2, \ldots, n_1 \tag{3.2.2}$$

Similarly for R_2 *with* n_1 *and* n_2 *interchanged.*

Proof

From (3.2.1), the only possible values of r_2 are $r_2 = r_1$, $r_1 - 1$, and $r_1 + 1$, for any r_1. Therefore, we have

$$f_{R_1}(r_1) = \sum_{r_2} f_{R_1, R_2}(r_1, r_2)$$

$$\begin{aligned}
\binom{n_1+n_2}{n_1} f_{R_1}(r_1) &= 2\binom{n_1-1}{r_1-1}\binom{n_2-1}{r_1-1} + \binom{n_1-1}{r_1-1}\binom{n_2-1}{r_1-2} + \binom{n_1-1}{r_1-1}\binom{n_2-1}{r_1} \\
&= \binom{n_1-1}{r_1-1}\left[\binom{n_2-1}{r_1-1} + \binom{n_2-1}{r_1-2} + \binom{n_2-1}{r_1-1} + \binom{n_2-1}{r_1}\right] \\
&= \binom{n_1-1}{r_1-1}\left[\binom{n_2}{r_1-1} + \binom{n_2}{r_1}\right] \\
&= \binom{n_1-1}{r_1-1}\binom{n_2+1}{r_1}
\end{aligned}$$

THEOREM 3.2.2

The probability distribution of R*, the total number of runs of* $n = n_1 + n_2$ *objects,* n_1 *of type* 1 *and* n_2 *of type* 2*, in a random sample is*

$$f_R(r) = \begin{cases} 2\binom{n_1-1}{r/2-1}\binom{n_2-1}{r/2-1} \Big/ \binom{n_1+n_2}{n_1} & \text{if } r \text{ is even} \\ \left[\binom{n_1-1}{(r-1)/2}\binom{n_2-1}{(r-3)/2} + \binom{n_1-1}{(r-3)/2}\binom{n_2-1}{(r-1)/2}\right] \Big/ \binom{n_1+n_2}{n_1} & \text{if } r \text{ is odd} \end{cases} \tag{3.2.3}$$

for $r = 2, 3, \ldots, n_1 + n_2$.

Proof

For r even, there must be the same number of runs of both types. Thus, the only possible values of r_1 and r_2 are $r_1 = r_2 = r/2$, and (3.2.1) is summed over this pair. If $r_1 = r_2 \pm 1$, r is odd. In this case, (3.2.1) is summed over the two pairs of values $r_1 = (r-1)/2$ and $r_2 = (r+1)/2$, $r_1 = (r+1)/2$ and $r_2 = (r-1)/2$, obtaining the given result. Note that the binomial coefficient $\binom{a}{b}$ is defined to be zero if $a < \text{b}$.

Using the result of Theorem 3.2.2, tables can be prepared for tests of significance of the null hypothesis of randomness. For example, if $n_1 = 5$ and $n_2 = 4$, we have

$$f_R(9) = \frac{\binom{4}{4}\binom{3}{3}}{\binom{9}{4}} = \frac{1}{126} = 0.008$$

$$f_R(8) = \frac{2\binom{4}{3}\binom{3}{3}}{\binom{9}{4}} = \frac{8}{126} = 0.063$$

$$f_R(2) = \frac{2\binom{4}{0}\binom{3}{0}}{\binom{9}{4}} = \frac{2}{126} = 0.016$$

$$f_R(3) = \frac{\binom{4}{1}\binom{3}{0} + \binom{4}{0}\binom{3}{1}}{\binom{9}{4}} = \frac{7}{126} = 0.056$$

For a two-sided test that rejects the null hypothesis for $R \leq 2$ or $R \geq 9$, the exact significance level α would be $3/126 = 0.024$. For the critical region defined by $R \leq 3$ or $R \geq 8$, $\alpha = 18/126 = 0.143$. As for all tests based on discrete probability distributions, there are a finite number of possible values of α. For a test with significance level at most 0.05, say, the first critical region above would be used even though the actual value of α is only 0.024. Tables which can be used to find rejection regions for the runs test are available in many sources. Swed and Eisenhart (1943) give the probability distribution of R for $n_1 \leq n_2 \leq 20$.

3.2.2 Moments of the Null Distribution of *R*

The kth moment of R is

$$\begin{aligned} E(R^k) &= \sum_r r^k f_R(r) \\ &= \left\{ \sum_{r \text{ even}} 2r^k \binom{n_1-1}{r/2-1}\binom{n_2-1}{r/2-1} \right. \\ &\quad + \sum_{r \text{ odd}} r^k \left[\binom{n_1-1}{(r-1)/2}\binom{n_2-1}{(r-3)/2} \right. \\ &\quad \left. \left. + \binom{n_1-1}{(r-3)/2}\binom{n_2-1}{(r-1)/2} \right] \right\} \Big/ \binom{n_1+n_2}{n_1} \end{aligned} \tag{3.2.4}$$

The smallest value of r is always 2. If $n_1 = n_2$, the largest number of runs occurs when the symbols alternate, in which case $r = 2n_1$. If $n_1 < n_2$, the maximum value of r is $2n_1 + 1$, since the sequence can both begin and end with a type 2 symbol. Assuming without loss of generality that $n_1 \leq n_2$ the range of summation for r is $2 \leq r \leq 2n_1 + 1$, letting $r = 2i$ for r even and $r = 2i + 1$ for r odd, the range of i is $1 \leq i \leq n_1$.

For example, the mean of R is expressed as follows using (3.2.4):

$$\begin{aligned} &\binom{n_1+n_2}{n_1} E(R) \\ &\quad = \sum_{i=1}^{n_1} 4i \binom{n_1-1}{i-1}\binom{n_2-1}{i-1} + \sum_{i=1}^{n_1} (2i+1)\binom{n_1-1}{i}\binom{n_2-1}{i-1} \\ &\qquad + \sum_{i=1}^{n_1} (2i+1)\binom{n_1-1}{i-1}\binom{n_2-1}{i} \end{aligned} \tag{3.2.5}$$

To evaluate these three sums, the following lemmas are useful.

LEMMA 3.2.2

$$\sum_{r=0}^{c} \binom{m}{r}\binom{n}{r} = \binom{m+n}{m} \quad \text{where } c = \min(m, n)$$

Proof

Since $(1+x)^{m+n} = (1+x)^m (1+x)^n$ for all x, we have

$$\sum_{i=0}^{m+n} \binom{m+n}{i} x^i = \sum_{j=0}^{m} \binom{m}{j} x^j \sum_{k=0}^{n} \binom{n}{k} x^k$$

Assuming without loss of generality that $c = m$ and equating the two coefficients of x^m on both sides of this equation, we obtain

$$\binom{m+n}{m} = \sum_{r=0}^{m} \binom{m}{m-r}\binom{n}{r}$$

LEMMA 3.2.3

$$\sum_{r=0}^{c} \binom{m}{r}\binom{n}{r+1} = \binom{m+n}{m+1} \quad \text{where } c = \min(m, n-1)$$

Proof

The proof follows as in Lemma 3.2, equating coefficients of x^{m+1}.

The algebraic process of obtaining $E(R)$ from (3.2.5) is tedious and will be left as an exercise for the reader. The variance of R can be found by evaluating the factorial moment $E[R(R-1)]$ in a similar manner.

A much simpler approach to finding the moments of R is provided by considering R as the sum of indicator variables as follows for $n = n_1 + n_2$. Let

$$R = 1 + I_2 + I_3 + \cdots + I_n$$

where in ordered sequences of the two types of symbols, we define

$$I_k = \begin{cases} 1 & \text{if the } k\text{th element } \neq \text{ the } (k-1)\text{th element} \\ 0 & \text{otherwise} \end{cases}$$

Then I_k is a Bernoulli random variable with parameter $p = n_1 n_2 \Big/ \binom{n}{2}$ so

$$E(I_k) = E(I_k^2) = \frac{2n_1 n_2}{n(n-1)}$$

Since R is a linear combination of these I_k, we have

$$E(R) = 1 + \sum_{k=2}^{n} E(I_k) = 1 + \frac{2n_1 n_2}{n_1 + n_2} \tag{3.2.6}$$

$$\begin{aligned} \operatorname{var}(R) &= \operatorname{var}\left(\sum_{k=2}^{n} I_k\right) = (n-1)\operatorname{var}(I_k) + \sum_{2 \le j \ne k \le n}\sum \operatorname{cov}(I_j, I_k) \\ &= (n-1)E(I_k^2) + \sum_{2 \le j \ne k \le n}\sum E(I_j I_k) - (n-1)^2[E(I_k)]^2 \end{aligned} \tag{3.2.7}$$

To evaluate the $(n-1)(n-2)$ joint moments of the type $E(I_j I_k)$ for $j \neq k$, the subscript choices can be classified as follows:

1. For the $2(n-2)$ selections where $j = k-1$ or $j = k+1$,

$$E(I_j I_k) = \frac{n_1 n_2 (n_1 - 1) + n_2 n_1 (n_2 - 1)}{n(n-1)(n-2)} = \frac{n_1 n_2}{n(n-1)}$$

2. For the remaining $(n-1)(n-2) - 2(n-2) = (n-2)(n-3)$ selections of $j \neq k$,

$$E(I_j I_k) = \frac{4 n_1 n_2 (n_1 - 1)(n_2 - 1)}{n(n-1)(n-2)(n-3)}$$

Substitution of these moments in the appropriate parts of (3.2.7) gives

$$\begin{aligned} \operatorname{var}(R) &= \frac{2n_1 n_2}{n} + \frac{2(n-2) n_1 n_2}{n(n-1)} + \frac{4 n_1 n_2 (n_1 - 1)(n_2 - 1)}{n(n-1)} - \frac{4 n_1^2 n_2^2}{n^2} \\ &= \frac{2 n_1 n_2 (2 n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)} \end{aligned} \tag{3.2.8}$$

3.2.3 Asymptotic Null Distribution

Although (3.2.3) can be used to find the exact distribution of R for any values of n_1 and n_2, the calculations are laborious unless n_1 and n_2 are both small. For large samples, an approximation to the null distribution, which gives reasonably good results as long as n_1 and n_2 are both larger than 10, can be used.

In order to find the asymptotic distribution, we assume that the total sample size $n \to \infty$ in such a way that $n_1/n \to \lambda$ and $n_2/n \to 1 - \lambda$, λ fixed, $0 < \lambda < 1$. For large samples then, the mean and variance of R from (3.2.6) and (3.2.8) are

$$\lim_{n \to \infty} E(R/n) = 2\lambda(1-\lambda) \quad \lim_{n \to \infty} \operatorname{var}(R/\sqrt{n}) = 4\lambda^2 (1-\lambda)^2$$

Forming the standardized random variable

$$Z = \frac{R - 2n\lambda(1-\lambda)}{2\sqrt{n}\lambda(1-\lambda)} \tag{3.2.9}$$

and substituting for R in terms of Z in (3.2.3), we obtain the standardized probability distribution of R, or $f_Z(z)$. If the factorials in the resulting expression are evaluated by Stirling's formula, the limit (Wald and Wolfowitz, 1940) is

$$\lim_{n\to\infty} \ln f_z(z) = -\ln\sqrt{2\pi} - \frac{1}{2}z^2$$

which shows that the limiting probability function of Z is the standard normal density.

For a two-sided test of size α using the normal approximation, the null hypothesis of randomness would be rejected when

$$\left|\frac{R - 2n\lambda(1-\lambda)}{2\sqrt{n}\,\lambda(1-\lambda)}\right| \geq z_{\alpha/2} \tag{3.2.10}$$

where z_γ is that number that satisfies $\Phi(z_\gamma) = 1 - \gamma$ or, equivalently, z_γ is the $(1-\gamma)$th quantile (or the upper γth quantile) point of the standard normal probability distribution. The exact mean and variance of R given in (3.2.6) and (3.2.8) can also be used in forming the standardized random variable, as the asymptotic distribution is unchanged. These approximations are generally improved by using a continuity correction of 0.5, as explained in Chapter 1.

3.2.4 Discussion

This runs test is one of the best known and easiest to apply among the tests for randomness in a sequence of observations. The data may be dichotomous as collected or may be classified as a dichotomous sequence according as each observation is above or below some fixed number, often the calculated sample median or mean. In the latter case, any observations equal to this fixed number are ignored in the analysis and n_1, n_2, and n are reduced accordingly. The runs test can be used with either one- or two-sided alternatives. If the alternative is simply nonrandomness, a two-sided test should be used. Since the presence of a trend would usually be indicated by a clustering of like objects, which is reflected by an unusually small number of runs, a one-sided test is more appropriate for trend alternatives.

Because of the generality of alternatives to randomness, no statement can be made concerning the overall performance of this runs test. However, its versatility should not be underrated. Other tests for randomness that are especially useful for trend alternatives have been proposed. The best known of these are tests based on the length of the longest run and tests based on the theory of runs up and down. These two types of test criteria will be discussed in the following sections.

3.2.5 Applications

Values of the null distribution of R computed from (3.2.3) are given in Table D for $n_1 \leq n_2 \leq 12$ as left-tail probabilities for R small and right-tail for R large. If the alternative is simply nonrandomness and the desired level

is α, we should reject for $R \leq r_{\alpha/2}$ or $R \geq r'_{\alpha/2}$, where $P(R \leq r_{\alpha/2}) \leq \alpha/2$ and $P(R \geq r'_{\alpha/2}) \leq \alpha/2$; the exact level of this two-tailed test is $P(R \leq r_{\alpha/2} + P(R \geq r'_{\alpha/2})$. If the desired alternative is a tendency for like elements to cluster, we should reject only for too few runs and therefore only for $R \leq r_\alpha$. On the other hand, if the appropriate alternative is a tendency for the elements to mix, we should reject only for too many runs and therefore only for $R \geq r'_\alpha$. Because Table D covers only $n_1 \leq n_2$, the type of element which occurs less frequently in the n observations should be called the type 1 element.

For $n_1 > 12$ and $n_2 > 12$, the critical values r and r' can be found from the normal approximation to the null distribution of the total number of runs. If we use the exact mean and variance of R given in (3.2.6) and (3.2.8) and a continuity correction of 0.5, the left-tail and right-tail critical regions are

$$\begin{aligned} \frac{R + 0.5 - 1 - 2n_1n_2/n}{\sqrt{2n_1n_2(2n_1n_2 - n)/[n^2(n-1)]}} &\leq -z_\alpha \\ \frac{R - 0.5 - 1 - 2n_1n_2/n}{\sqrt{2n_1n_2(2n_1n_2 - n)/[n^2(n-1)]}} &\geq z_\alpha \end{aligned} \tag{3.2.11}$$

The two-tailed critical region is a combination of the above with z_α replaced by $z_{\alpha/2}$.

Example 3.2.1

The recorded high temperature in a Florida resort town for each of 10 consecutive days during the month of January in this year is compared with the historical average high for the same day in previous years and noted as either above historical average (A) or below (B). For the data A A B A B B A A A B, test the null hypothesis of random direction of deviation from average high temperature against the alternative of nonrandomness, using level 0.05.

Solution

Since the data consist of six As and four Bs, B will be called the type 1 element to make $n_1 = 4$, $n_2 = 6$. The total number of runs observed is $R = 6$. Table D shows that $P(R \leq 2) = 0.010$ and $P(R \geq 9) = 0.024$, and these are the largest respective probabilities that do not exceed 0.025; the rejection region is then $R \leq 2$ or $R \geq 9$ with exact level 0.034. Our $R = 6$ does not fall into this region, so we do not reject the null hypothesis of randomness at the 0.05 level.

The STATXACT solution to Example 3.2.1 is shown below. The reader can verify using Table D that the exact right-tailed P value is $P(R \geq 6) = 0.595$ and the asymptotic P value from (3.2.11) with a continuity correction is $P(Z \geq -0.21) = 0.5832$.

```
*****************************************
STATXACT Solution to Example 3.2.1
*****************************************

ONE SAMPLE RUNS TEST

Summary of Exact distribution of ONE SAMPLE RUNS TEST statistic

      Value of Cutpoint =        10.00 ( User Defined )

Min          Max        Mean       Std-dev      Observed   Standardized
2.000       9.000       5.800        1.424         6.000       −0.2107

Asymptotic P-Value:
       Pr { Test Statistic .GE.        6.000 }          =       0.5835
       Two-sided: 2 * One-sided                         =       1.0000

Exact P-Value:
       Pr{ Test Statistic .GE.         6.000 }          =       0.5952
       Pr{ Test Statistic .EQ.         6.000 }          =       0.2857
       Pr{|Test Statistic-Mean|  .GE.  |Observed − Mean|} =      1.0000
```

3.3 Tests Based on the Length of the Longest Run

A test based on the total number of runs in an ordered sequence of n_1 objects of type 1 and n_2 objects of type 2 is only one way of using information about runs to detect patterns in the arrangement. Other statistics of interest are provided by considering the lengths of these runs. Since a run that is unusually long reflects a tendency for like objects to cluster and therefore possibly a trend, Mosteller (1941) suggested a test for randomness based on the length of the longest run. Exact and asymptotic probability distributions of the numbers of runs of given length are discussed in Mood (1940).

The joint probability distribution of R_1 and R_2 derived in Section 3.2, for the numbers of runs of the two types of objects, disregards the individual lengths and is therefore of no use in this problem. We now need the joint probability distribution for a total of $n_1 + n_2$ random variables, representing the numbers of runs of all possible lengths for each of the two types of elements in the dichotomy. Let the random variables R_{ij}, $i = 1, 2$; $j = 1, 2, \ldots, n_i$, denote, respectively, the numbers of

runs of objects of type i which are of length j. Then the following obvious relationships hold:

$$\sum_{j=1}^{n_i} jr_{ij} = n_i \quad \text{and} \quad \sum_{j=1}^{n_i} r_{ij} = r_i \quad \text{for } i = 1, 2$$

The total number of arrangements of the $n_1 + n_2$ symbols is still $\binom{n_1 + n_2}{n_1}$ and each is equally likely under the null hypothesis of randomness. We must compute the number of arrangements in which there are exactly r_{ij} runs of type i and length j, for all i and j. Assume that r_1 and r_2 are held fixed. The number of arrangements of the r_1 runs of type 1 which are composed of r_{1j} runs of length j for $j = 1, 2, \ldots, n_1$ is simply the number of permutations of the r_1 runs with r_{11} runs of length 1, r_{12} runs of length 2, $\ldots, r_{1n_1}$ of length n_1, where within each category the runs cannot be distinguished. This number is $r_1! \Big/ \prod_{j=1}^{n_1} r_{1j}!$. The number of arrangements for the r_2 runs of type 2 objects is similarly $r_2! \Big/ \prod_{j=1}^{n_2} r_{2j}!$. If $r_1 = r_2 \pm 1$, the total number of permutations of the runs of both types of objects is the product of these two expressions, but if $r_1 = r_2$, this number must be doubled since the sequence can begin with either types of object. Therefore, the following theorem is proved.

THEOREM 3.3.1

Under the null hypothesis of randomness, the probability that the r_1 runs of n_1 objects of type 1 and r_2 runs of n_2 objects of type 2 consist of exactly r_{1j}, $j = 1, 2, \ldots, n_1$, and r_{2j}, $j = 1, 2, \ldots, n_2$ runs of length j, respectively, is

$$f(r_{11}, \ldots, r_{1n_1}, r_{21}, \ldots, r_{2n_2}) = \frac{cr_1!r_2!}{\prod_{i=1}^{2} \prod_{j=1}^{n_i} r_{ij}! \binom{n_1 + n_2}{n_1}} \tag{3.3.1}$$

where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.

Combining the reasoning for Theorem 3.2.1 with that of Theorem 3.3.1, the joint distribution of the $n_1 + 1$ random variables R_2 and $R_{1j}, j = 1, 2, \ldots, n_1$, can be obtained as follows:

$$f(r_{11}, \ldots, r_{1n_1}, r_2) = \frac{cr_1! \binom{n_2 - 1}{r_2 - 1}}{\prod_{j=1}^{n_1} r_{1j}! \binom{n_1 + n_2}{n_1}} \tag{3.3.2}$$

The result is useful when only the total number, not the lengths, of runs of type 2 objects is of interest. This joint distribution, when summed over the values for r_2, gives the marginal probability distribution of the lengths of the r_1 runs of objects of type 1.

THEOREM 3.3.2

The probability that the r_1 runs of n_1 objects of type 1 consist of exactly r_{1j}, $j = 1, 2, \ldots, n_1$, runs of length j, respectively, is

$$f(r_{11}, \ldots, r_{1n_1}) = \frac{r_1! \binom{n_2 + 1}{r_1}}{\prod_{j=1}^{n_1} r_{1j}! \binom{n_1 + n_2}{n_1}} \tag{3.3.3}$$

The proof is exactly the same as for Corollary 3.2.1. The distribution of lengths of runs of type 2 objects is similar.

In the probability distributions given in Theorems 3.3.1 and 3.3.2, both r_1 and r_2, respectively, were assumed to be fixed numbers. Therefore, these are conditional probability distributions given the fixed values. If these are not to be considered as fixed, the conditional distributions are simply summed over the possible fixed values, since these are mutually exclusive.

Theorem 3.3.2 can be used to find the null probability distribution of a test for randomness based on K, the length of the longest run of type 1. For example, the probability that the longest in any number of runs of type 1 is of length k is

$$\sum_{r_1} \sum_{r_{11}, \ldots, r_{1k}} \frac{r_1! \binom{n_2 + 1}{r_1}}{\prod_{j=1}^{k} r_{1j}! \binom{n_1 + n_2}{n_1}} \tag{3.3.4}$$

where the sums are extended over all sets of nonnegative integers satisfying $\sum_{j=1}^{k} r_{1j} = r_1$, $\sum_{j=1}^{k} j r_{1j} = n_1, r_{1k} \geq 1, r_1 \leq n_1 - k + 1$, and $r_1 \leq n_2 + 1$. For example, if $n_1 = 5, n_2 = 6$, the longest possible run is of length 5. There can be no other runs, so that $r_1 = 1$ and $r_{11} = r_{12} = r_{13} = r_{14} = 0$, and

$$P(K = 5) = \frac{\binom{7}{1}}{\binom{11}{5}} = \frac{7}{462}$$

Similarly, we can obtain

$$P(K=4)=\frac{2!}{1!1!}\frac{\binom{7}{2}}{\binom{11}{5}}=\frac{42}{462}$$

$$P(K=3)=\frac{\frac{3!}{2!1!}\binom{7}{3}+\frac{2!}{1!1!}\binom{7}{2}}{\binom{11}{5}}=\frac{147}{462}$$

$$P(K=2)=\frac{\frac{4!}{3!1!}\binom{7}{4}+\frac{3!}{1!2!}\binom{7}{3}}{\binom{11}{5}}=\frac{245}{462}$$

$$P(K=1)=\frac{5!}{5!}\frac{\binom{7}{5}}{\binom{11}{5}}=\frac{21}{462}$$

For a test with significance level at most 0.05 when $n_1=5$, $n_2=6$, the null hypothesis of randomness is rejected when there is a run of type 1 elements of length 5. In general, the critical region would be the arrangements with at least one run of length t or more.

Theorem 3.3.1 must be used if the test is to be based on the length of the longest run of either type of element in the dichotomy. These two theorems are tedious to apply unless n_1 and n_2 are both quite small. Tables are available in Bateman (1948) and Mosteller (1941).

Tests based on the length of the longest run may or may not be more powerful than a test based on the total number of runs, depending on the basis for comparison. Both tests use only a portion of the information available, since the total number of runs, although affected by the lengths of the runs, does not directly make use of information regarding these lengths, and the length of the longest run only partially reflects both the lengths of other runs and the total number of runs. Power functions are discussed in Bateman (1948) and David (1947).

3.4 Runs Up and Down

When numerical observations are available and the sequence of numbers is analyzed for randomness according to the number or lengths of runs of elements above and below the median, some information is lost which

might be useful in identifying a pattern in the time-ordered observations. In this section, we consider runs up and down, where the magnitude of each element is compared with that of the immediately preceding element in the time sequence. If the next element is larger, a run up is started; if smaller, a run down is started. We can observe when the sequence increases, and for how long, when it decreases, and for how long. A decision concerning randomness then might be based on the number and lengths of these runs, whether up or down, since a large number of long runs should not occur in a truly random set of numerical observations. This type of analysis should be most sensitive to trend alternatives because an excessive number of long runs is usually indicative of a sequence with some sort of trend.

If the time-ordered observations are 8, 13, 1, 3, 4, 7, there is a run up of length 1, followed by a run down of length 1, followed by a run up of length 3. The sequence of six observations can be represented by five plus and minus signs, $+, -, +, +, +$, indicating their relative magnitudes. More generally, suppose there are n numbers, no two alike, say $a_1 < a_2 < \cdots < a_n$ when arranged in order of magnitude. The time-ordered sequence $S_n = (x_1, x_2, \ldots, x_n)$ represents some permutation of these n numbers. There are $n!$ permutations, each one representing a possible set of sample observations. Under the null hypothesis of randomness, each of these $n!$ arrangements is equally likely to occur. The test for randomness using runs up and down for the sequence S_n of dimension n is based on the derived sequence D_{n-1} of dimension $n-1$, whose ith element is the sign of the difference $x_{i+1} - x_i$, for $i = 1, 2, \ldots, n-1$. Let R_i denote the number of runs, either up or down, of length exactly i in the sequence $D_{n-1}n$. We have the obvious restrictions $1 \le i \le n-1$ and $\sum_{i=1}^{n-1} ir_i = n-1$. The test for randomness will reject the null hypothesis when there are at least r runs of length t or more, where r and t are determined by the desired significance level. Therefore, we must find the joint distribution of $R_1, R_2, \ldots, R_{n-1}$ under the null hypothesis when every arrangement S_n is equally likely. Let $f_n(r_{n-1}, r_{n-2}, \ldots, r_1)$ denote the probability that there are exactly r_{n-1} runs of length $n-1, \ldots, r_i$ runs of length $i, \ldots, r_1$ runs of length 1. If $u_n(r_{n-1}, \ldots, r_1)$ represents the corresponding frequency, then

$$f_n = \frac{u_n}{n!}$$

because there are $n!$ possible arrangements of S_n. The probability distribution will be derived as a recursive relation; we first consider the particular case where $n = 3$ and see how the distribution for $n = 4$ can be generated from it.

Given three numbers $a_1 < a_2 < a_3$, only runs of lengths 1 and 2 are possible. The $3! = 6$ arrangements and their corresponding values of r_2 and r_1 are given in Table 3.4.1. Since the probability of at least one run of length 2 or more is $2/6$, the significance level is 0.33 for this critical region. With this size sample, a smaller significance level cannot be obtained without resorting to a randomized decision rule.

TABLE 3.4.1

Listing of Arrangements When $n = 3$

S_3	D_2	r_2	r_1	Probability Distribution
(a_1, a_2, a_3)	$(+, +)$	1	0	$f_3(1,0) = 2/6$
(a_1, a_3, a_2)	$(+, -)$	0	2	
(a_2, a_1, a_3)	$(-, +)$	0	2	$f_3(0,2) = 4/6$
(a_2, a_3, a_1)	$(+, -)$	0	2	
(a_3, a_1, a_2)	$(-, +)$	0	2	$f_3(r_2, r_1) = 0$ otherwise
(a_3, a_2, a_1)	$(-, -)$	1	0	

Now consider the addition of a fourth number a_4, larger than all the others. For each of the arrangements in S_3, a_4 can be inserted in four different places. In the particular arrangement (a_1, a_2, a_3), for example, insertion of a_4 at the extreme left or between a_2 and a_3 would leave r_2 unchanged but add a run of length 1. If a_4 is placed at the extreme right, a run of length 2 is increased to a run of length 3. If a_4 is inserted between a_1 and a_2, the one run of length 2 is split into three runs, each of length 1.

Extending this analysis to the general case, the extra observation must either split an existing run, lengthen an existing run, or introduce a new run of length 1. The ways in which the run lengths in S_{n-1} are affected by the insertion of an additional observation a_n to make an arrangement S_n can be classified into the following four mutually exclusive and exhaustive cases:

1. An additional run of length 1 can be added in the arrangement S_n.
2. A run of length $i-1$ in S_{n-1} can be changed into a run of length i in S_n for $i = 2, 3, \ldots, n-1$.
3. A run of length $h = 2i$ in S_{n-1} can be split into a run of length i, followed by a run of length 1, followed by a run of length i, for $1 \leq i \leq [(n-2)/2]$, where $[x]$ denotes the largest integer not exceeding x.
4. A run of length $h = i + j$ in S_{n-1} can be split up into
   a. A run of length i, followed by a run of length 1, followed by a run of length j, or
   b. A run of length j, followed by a run of length 1, followed by a run of length i,

   where $h > i > j$, $3 \leq h \leq n-2$.

For $n = 4$, the arrangements can be enumerated systematically to show how these cases arise. Table 3.4.2 gives a partial listing. When the table is completed, the number of cases which result in any particular set (r_3, r_2, r_1)

TABLE 3.4.2

Partial Listing of Arrangements When $n = 4$

S_3	r_2	r_1	S_4	r_3	r_2	r_1	Case Illustrated
(a_1, a_2, a_3)	1	0	(a_4, a_1, a_2, a_3)	0	1	1	1
			(a_1, a_4, a_2, a_3)	0	0	3	3
			(a_1, a_2, a_4, a_3)	0	1	1	1
			(a_1, a_2, a_3, a_4)	1	0	0	2
(a_1, a_3, a_2)	0	2	(a_4, a_1, a_3, a_2)	0	0	3	1
			(a_1, a_4, a_3, a_2)	0	1	1	2
			(a_1, a_3, a_4, a_2)	0	1	1	2
			(a_1, a_3, a_2, a_4)	0	0	3	1

can be counted and divided by 24 to obtain the complete probability distribution. This will be left as an exercise for the reader. The results are

(r_3, r_2, r_1)	(1, 0, 0)	(0, 1, 1)	(0, 0, 3)	Other values
$f_4(r_3, r_2, r_1)$	2/24	12/24	10/24	0

There is no illustration for case 4 in the completed table of enumerated arrangements since here n is not large enough to permit $h \geq 3$. For $n = 5$, insertion of a_5 in the second position of (a_1, a_2, a_3, a_4) produces the sequence $(a_1, a_5, a_2, a_3, a_4)$. The one run of length 3 has been split into a run of length 1, followed by another run of length 1, followed by a run of length 2, illustrating case 4*b* with $h = 3$, $j = 1$, $i = 2$. Similarly, case 4*a* is illustrated by inserting a_5 in the third position. This also illustrates case 3.

More generally, the frequency u_n of cases in S_n having exactly r_1 runs of length 1, r_2 runs of length 2, ..., r_{n-1} runs of length $n-1$ can be generated from the frequencies for S_{n-1} by the following recursive relation:

$$\begin{aligned} u_n(r_{n-1}, r_{n-2}, \ldots, r_h, \ldots, r_i, \ldots, r_j, \ldots, r_1) &= 2u_{n-1}(r_{n-2}, \ldots, r_1 - 1) \\ &+ \sum_{i=2}^{n-1} (r_{i-1} + 1) u_{n-1}(r_{n-2}, \ldots, r_i - 1, r_{i_{-1}} + 1, \ldots, r_1) \\ &+ \sum_{\substack{i=1 \\ h=2i}}^{[(n-2)/2]} (r_h + 1) u_{n-1}(r_{n-2}, \ldots, r_h + 1, \ldots, r_i - 2, \ldots, r_1 - 1) \\ &+ 2 \sum_{\substack{i=2 \\ h=i+j \\ h \leq n-2}}^{n-3} \sum_{j=1}^{i-1} (r_h + 1) \\ &\times u_{n-1}(r_{n-2}, \ldots, r_h + 1, \ldots, r_i - 1, \ldots, r_j - 1, \ldots, r_1 - 1) \end{aligned} \tag{3.4.1}$$

The terms in this sum represent cases 1–4 in that order. For case 1, u_{n-1} is multiplied by 2 since for every arrangement in S_{n-1} there are exactly two places in which a_n can be inserted to add a run of length 1. These positions are always at an end or next to the end. If the first run is a run up (down), insertion of a_n at the extreme left (next to extreme left) position adds a run of length 1. A new run of length 1 is also created by inserting a_n at the extreme right (next to extreme right) in S_{n-1} if the last run in S_{n-1} was a run down (up). In case 4, we multiply by 2 because of the (a) and (b) possibilities.

The result is tricky and tedious to use but is much easier than enumeration. The process will be illustrated for $n = 5$, given

$$u_4(1,0,0) = 2 \quad u_4(0,1,1) = 12 \quad u_4(0,0,3) = 10 \quad u_4(r_3, r_2, r_1) = 0 \text{ otherwise}$$

Using (3.4.1), we have

$$\begin{aligned}
u_5(r_4, r_3, r_2, r_1) &= 2u_4(r_3, r_2, r_1 - 1) + [(r_1 + 1)u_4(r_3, r_2 - 1, r_1 + 1)] \\
&\quad + (r_2 + 1)u_4(r_3 - 1, r_2 + 1, r_1) \\
&\quad + (r_3 + 1)u_4(r_3 + 1, r_2, r_1) \\
&\quad + (r_2 + 1)u_4(r_3, r_2 + 1, r_1 - 3) \\
&\quad + 2(r_3 + 1)u_4(r_3 + 1, r_2 - 1, r_1 - 2) \\
u_5(1,0,0,0) &= 1u_4(1,0,0) = 2 \\
u_5(0,1,0,1) &= 2u_4(1,0,0) + 1u_4(0,1,1) \\
&\quad + 2u_4(2,0,1) = 4 + 12 + 0 = 16 \\
u_5(0,0,2,0) &= 1u_4(0,1,1) + 1u_4(1,2,0) = 12 + 0 = 12 \\
u_5(0,0,1,2) &= 2u_4(0,1,1) + 3u_4(0,0,3) + 1u_4(1,1,2) \\
&\quad + 2u_4(1,0,0) = 24 + 30 + 0 + 4 = 58 \\
u_5(0,0,0,4) &= 2u_4(0,0,3) + 1u_4(1,0,4) + 1u_4(0,1,1) \\
&= 20 + 0 + 12 = 32
\end{aligned}$$

The means, variances, and covariances of the numbers of runs of length t (or more) are found in Levene and Wolfowitz (1944). Tables of the exact probabilities of at least r runs of length t or more are given in Olmstead (1946) and Owen (1962) for $n \leq 14$ from which appropriate critical regions can be found. Olmstead gives approximate probabilities for larger sample sizes. See Wolfowitz (1944a,b) regarding the asymptotic distribution which is Poisson.

A test for randomness can also be based on the total number of runs, whether up or down, irrespective of their lengths. Since the total number of runs R is related to the R_i, the number of runs of length i, by

$$R = \sum_{i=1}^{n-1} R_i \tag{3.4.2}$$

the same recursive relation given in (3.4.1) can be used to find the probability distribution of R. Levene (1952) showed that the asymptotic null distribution of the standardized random variable R with mean $(2n-1)/3$ and variance $(16n-29)/90$ is the standard normal distribution.

3.4.1 Applications

For the test of randomness based on the total number of runs up and down, R, in an ordered sequence of n numerical observations, or equivalently in a sequence of $n-1$ plus and minus signs, the appropriate sign is determined by comparing the magnitude of each observation with the one immediately preceding it in the sequence. The appropriate rejection regions for each alternative are exactly the same as for the earlier test in Section 3.2, which was based on the total number of runs of two types of elements. Specifically, if the alternative is a tendency for like signs to cluster, the appropriate rejection region is small values of R. If the alternative is a tendency for like signs to mix with unlike signs, the appropriate rejection region is large values of R.

The exact distribution of R under the null hypothesis of randomness is given in Table E for $n \leq 25$ as left-tail probabilities for R small and right-tail for R large. For $n > 25$, the critical values of R can be found from the normal approximation to the null distribution of the number of runs up and down statistic. If we incorporate a continuity correction of 0.5, the left-tail and right-tail critical regions are

$$\frac{R + 0.5 - (2n-1)/3}{\sqrt{(16n-29)/90}} \leq -z_\alpha \quad \text{and} \quad \frac{R - 0.5 - (2n-1)/3}{\sqrt{(16n-29)/90}} \geq z_\alpha$$

The two-tailed critical region is a combination of the above with z_α replaced by $z_{\alpha/2}$.

One of the primary uses of the runs up and down test is for an analysis of time-series data. The null hypothesis of randomness is then interpreted as meaning the data can be regarded as independent and identically distributed. The alternative of a tendency to cluster is interpreted as an upward trend if the signs are predominantly plus, or a downward trend if the signs are predominantly minus, and the alternative of a tendency to mix is interpreted as cyclical variations. The total number of runs test could also be used with time series data if the data are first converted to two types of symbols by comparing the magnitude of each one to some standard for that period or to a single focal point, like the median of the data. This is frequently referred to as the runs above and below the median test. The test in the former case was actually illustrated by Example 3.2.1.

Example 3.4.1 illustrates an application of runs up and down in time series data.

Example 3.4.1

Tourism is regarded by all nations as big business because the industry brings needed foreign exchange and helps the balance of payments. The Travel Market Yearbook publishes extensive data on tourism. Analyze the annual data on total number of tourists to the United States for 1970–1982 to see if there is evidence of a trend, using the 0.01 level.

Year	Number of Tourists (Millions)
1970	12,362
1971	12,739
1972	13,057
1973	13,955
1974	14,123
1975	15,698
1976	17,523
1977	18,610
1978	19,842
1979	20,310
1980	22,500
1981	23,080
1982	21,916

SOLUTION

We use the runs up and down test of randomness for these $n = 13$ observations with the alternative of a trend. The sequence of 12 plus and minus signs is $+, +, +, +, +, +, +, +, +, +, +, -$ so that $R = 2$. The left-tail critical value from Table E is $R = 4$ at exact level 0.0026, the largest value that does not exceed 0.01. Since 2 is less than 4, we reject the null hypothesis and conclude there is a trend in number of tourists to the United States and it is positive because the signs are predominantly plus.

3.5 A Test Based on Ranks

Another way to test for randomness by comparing the magnitude of each element with that of the immediately preceding element in the time sequence is to compute the sum of the squares of the deviations of the pairs of successive elements. If the magnitudes of these elements are replaced by their respective ranks in the sequence before computing the sum of squares of successive deviations, we can obtain a nonparametric test.

Specifically, let the time-ordered sequence of observations be $S_n = (X_1, X_2, \ldots, X_n)$ as in Section 3.4. The test statistic

$$\text{NM} = \sum_{i=1}^{n-1} [\text{rank}(X_i) - \text{rank}(X_{i+1})]^2 \tag{3.5.1}$$

was proposed by Bartels (1982). A test based on a function of this statistic is the rank version of the ratio test for randomness developed by von Neumann using normal theory and is a linear transformation of the rank serial correlation coefficient introduced by Wald and Wolfowitz (1943).

It is easy to show that the test statistic NM ranges between $(n-1)$ and $(n-1)(n^2+n-3)/3$ if n is even and between $(n-1)$ and $[(n-1)(n^2+n-3)/3]-1$ if n is odd. The exact null distribution of NM can be found by enumeration and is given in Bartels (1982) for $4 \leq n \leq 10$. For larger sample sizes, the test statistic

$$\text{RVN} = \frac{\sum_{i=1}^{n-1} [\text{rank}(X_i) - \text{rank}(X_{i+1})]^2}{\sum_{i=1}^{n} [\text{rank}(X_i) - (n+1)/2]^2} \tag{3.5.2}$$

is asymptotically normally distributed with mean 2 and variance $4(n-2)(5n^2-2n-9)/5n(n+1)(n-1)^2$, which is approximately equal to $20/(5n+7)$. If there are no ties, the denominator of the rank von Neumann statistic (RVN) is equal to the constant $n(n^2-1)/12$.

Since a trend in either direction will be reflected by a small value of NM and therefore RVN, the appropriate rejection region to test randomness against a trend alternative is small values of NM or RVN. If the alternative is a tendency for the data to alternate small and large values, the appropriate rejection region is large values of the test statistic. Table S gives exact tail probabilities for selected values of NM for $n \leq 10$ and approximate left-tail critical values of RVN (based on the beta distribution) for larger sample sizes. Corresponding right-tail critical values are found using the fact that this beta approximation is symmetric about 2.0.

Bartels (1982) used simulation studies to show that this test is superior to the runs up and down test in many cases. Its asymptotic relative efficiency is 0.91 with respect to the ordinary serial correlation coefficient against the alternative of first-order *autocorrelation* under normality. Autocorrelation is defined as a measure of the dependence between observations that occur a fixed number of time units apart. Positive autocorrelation shows up in time-series data that exhibit a trend in either direction, while negative autocorrelation is indicated by fluctuations over time.

Example 3.5.1

We illustrate this RVN test using the data from Example 3.4.1 with $n = 3$ where the alternative is positive trend. We first rank the number of tourists from smallest to largest and obtain

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 11

The value of NM, the numerator of the RVN statistic, from (3.5.1) is then

$$\text{NM} = (1-2)^2 + (2-3)^2 + \cdots + (13-11)^2 = 18$$

And the denominator is $13(13^2-1)/12 = 182$. Thus, from (3.5.2), $\text{RVN} = 18/182 = 0.0989$, and Table S shows that the left-tail critical value based on the beta approximation is to reject the null hypothesis of randomness at the 0.005 level if $\text{RVN} \leq 0.74$. Therefore, we reject the null hypothesis and conclude that there is a significant positive trend. We also use these data to illustrate the test based on the normal approximation to the distribution of RVN. The mean is 2 and the variance for $n = 13$ is 0.2778. The standard normal test statistic is then $(0.0989 - 2)/\sqrt{0.2778} = -3.61$. At the 0.005 level, for example, the appropriate rejection region from Table A is $Z \leq -2.58$, so we again conclude that there is a significant positive trend.

3.6 Summary

In this chapter, we presented a number of tests that are appropriate for the null hypothesis of randomness in a sequence of observations whose order has some meaning. If the observations are simply two types of symbols, like M and F, or D and G, the total number of runs of symbols is the most appropriate test statistic. Tests based on the lengths of the runs are primarily of theoretical interest. If the observations are numerical measurements, the number of runs up and down or the RVN statistic provides the best test because too much information is lost by using the test based on runs above and below some fixed value like the median.

Usually, the alternative hypothesis is simply lack of randomness, and then these tests have no analog in parametric statistics. If the data represent a time series, the alternative to randomness can be exhibition of a trend. In this case, the RVN test is more powerful than the test based on runs up and down. Two additional tests for trend will be presented later in Chapter 11.

Problems

3.1 Prove Corollary 3.2.1 using a direct combinatorial argument based on Lemma 3.2.1.

3.2 Find the mean and variance of the number of runs R_1 of type 1 elements, using the probability distribution given in (3.2.2). Since $E(R) = E(R_1) + E(R_2)$, use your result to verify (3.2.6).

3.3 Use Lemmas 3.2.2 and 3.2.3 to evaluate the sums in (3.2.5), obtaining the result given in (3.2.6) for $E(R)$.

3.4 Show that the asymptotic distribution of the standardized random variable $[R_1 - E(R_1)]/\sigma(R_1)$ is the standard normal distribution, using the distribution of R_1 given in (3.2.2) and your answer to Problem 3.2.

3.5 Verify that the asymptotic distribution of the random variable given in (3.2.9) is the standard normal distribution.

3.6 By considering the ratios $f_R(r)/f_R(r-2)$ and $f_R(r+2)/f_R(r)$, where r is an even positive integer and $f_R(r)$ is given in (3.2.3), show that if the most probable number of runs is an even integer k, then k satisfies the inequality

$$\frac{2n_1n_2}{n} < k < \frac{2n_1n_2}{n} + 2$$

3.7 Show that the probability that a sequence of n_1 elements of type 1 and n_2 elements of type 2 begins with a type 1 run of length exactly k is

$$\frac{(n_1)_k n_2}{(n_1+n_2)_{k+1}} \quad \text{where } (n)_r = \frac{n!}{(n-r)!}$$

3.8 Find the rejection region with significance level not exceeding 0.10 for a test of randomness based on the length of the longest run when $n_1 = n_2 = 6$.

3.9 Find the complete probability distribution of the number of runs up and down of various lengths when $n = 6$ using (3.4.1) and the results given for $u_5(r_4, r_3, r_2, r_1)$.

3.10 Use your answers to Problem 3.9 to obtain the complete probability distribution of the total number of runs up and down when $n = 6$.

3.11 Verify the statement that the variance of the RVN test statistic is approximately equal to $20/(5n+7)$.

3.12 Analyze the data in Example 3.4.1 for evidence of trend using the total number of runs above and below
(a) The sample median
(b) The sample mean

3.13 A certain broker noted the following number of bonds sold each month for a 12 month period:

Jan. 19	July 22
Feb. 23	Aug. 24
Mar. 20	Sept. 25
Apr. 17	Oct. 28
May 18	Nov. 30
June 20	Dec. 21

(a) Use the runs up and down test to see if these data show a directional trend and make an appropriate conclusion at the 0.05 level.
(b) Use the runs above and below the sample median test to see if these data show a trend and make an appropriate conclusion at the 0.05 level.
(c) Use the RVN test to see if these data show a positive trend.
(d) Compare the conclusions reached in (a)–(c).

3.14 The following are 30 time lapses in minutes between eruptions of Old Faithful geyser in Yellowstone National Park, recorded between the hours of 8 a.m. and 10 p.m. on a certain day, and measured from the beginning of one eruption to the beginning of the next:

68, 63, 66, 63, 61, 44, 60, 62, 71, 62, 62, 55, 62, 67, 73,
72, 55, 67, 68, 65, 60, 61, 71, 60, 68, 67, 72, 69, 65, 66

A researcher wants to use these data for inference purposes, but is concerned about whether it is reasonable to treat such data as a random sample. What do you think? Justify your answer.

3.15 In a psychological experiment, the research question of interest is whether a rat ''learned'' its way through a maze during 64 trials. Suppose the time-ordered observations on number of correct choices by the rat on each trial are as follows:

0, 1, 2, 1, 1, 2, 3, 2, 2, 2, 1, 1, 3, 2, 1, 2, 1, 2, 2, 1, 1, 2, 2, 1, 4, 3, 1, 2, 2, 1, 2, 2,
2, 2, 3, 2, 2, 3, 4, 3, 2, 3, 3, 2, 3, 3, 2, 3, 3, 2, 3, 4, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 4, 4

(a) Test these data for randomness against the alternative of a tendency to cluster, using the dichotomizing criterion that 0, 1, or 2 correct choices indicate no learning, while 3 or 4 correct indicate some learning.
(b) Would the runs up and down test be appropriate for these data? Why or why not?

3.16 The data below represent departure of actual daily temperature in degrees Fahrenheit from the normal daily temperature at noon at a certain airport on 7 consecutive days.

Day	1	2	3	4	5	6	7
Departure	12	13	12	11	5	−1	−2

(a) Give an appropriate P value that reflects whether the pattern of positive and negative departures can be considered a random process against the alternative of a tendency to cluster.
(b) Give an appropriate P value that reflects whether the pattern of successive departures (from one day to the next) can be considered a random process against the alternative of a trend.

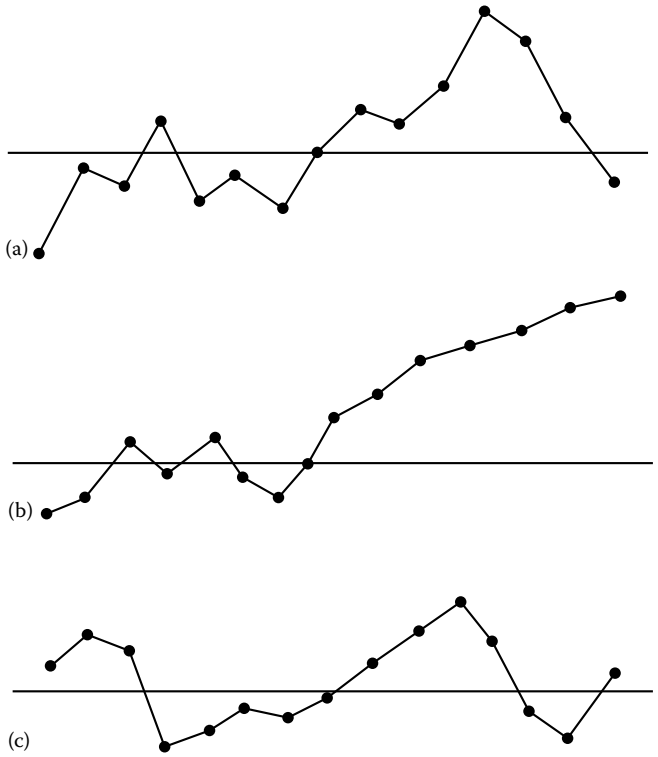


FIGURE P.3.17
Nonrandom patterns representing (a) cyclical movement, (b) trend movement, and (c) clustering.

3.17 The three graphs in Figure P.3.17 illustrate some kinds of nonrandom patterns. Time is on the horizontal axis. The data values are indicated by dots and the horizontal line denotes the median of the data. For each graph, compute the one-tailed P value for nonrandomness using two different nonparametric techniques.

3.18 Bartels (1982) illustrated the RVN test for randomness using data on annual changes in stock levels of corporate trading enterprises in Australia for 1968–1969 to 1977–1978. The values (in $A million) deflated by the Australian GDP are 528, 348, 264, −20, −167, 575, 410, −4, 430, −122. He tested randomness against the alternative of autocorrelation. Random stock-level changes occur when companies are well managed because future demands are accurately anticipated. ''Negative autocorrelation constitutes evidence for a tendency to overreact to shortfalls or excesses in stock levels, whereas positive autocorrelation suggests there is a long delay in reaching desired stock levels.'' The test statistic is NM = 169, which is not significant. Compare this result with that of (a) runs up and down, and (b) runs above and below the sample median.

4

Tests of Goodness of Fit

4.1 Introduction

An important problem in statistics relates to obtaining information about the form of the population from which a sample is drawn. The shape of this is distribution or some inference concerning a particular aspect of the population may be of primary interest. In this latter case, in classical statistics, information about the form generally must be postulated or incorporated in the null hypothesis to perform an exact parametric type of inference. For example, suppose we have a small number of observations from an unknown population with unknown variance and the hypothesis of interest concerns the value of the population mean. The traditional parametric test, based on Student's t distribution, is derived under the assumption of a normal population. The exact distribution theory and probabilities of both types of errors depend on this population form. Therefore, it might be desirable to check on the reasonableness of the normality assumption before forming any conclusions based on the t distribution. If the normality assumption appears not to be justified, some type of nonparametric inference for location might be more appropriate with a small sample size.

The compatibility of a set of observed sample values with a normal or any other distribution can be checked by a goodness-of-fit test. These tests are designed for a null hypothesis which is a statement about the form of the cumulative distribution or probability function of the parent population from which the sample is drawn. Ideally, the hypothesized distribution is completely specified, including all parameters. Since the alternative is necessarily quite broad, including differences only in location, scale, other parameters, form, or any combination thereof, rejection of the null hypothesis does not provide much specific information. Goodness-of-fit tests are customarily used when only the form of the population is in question, with the hope that the null hypothesis will be found acceptable.

In this chapter, we will consider two types of goodness-of-fit tests. The first type is designed for null hypotheses concerning a discrete distribution and compares the observed frequencies with the frequencies expected under the null hypothesis. This is the chi-square test proposed by Karl Pearson early in

the history of statistics. The second type of goodness-of-fit test is designed for null hypotheses concerning a continuous distribution and compares the observed cumulative relative frequencies with those expected under the null hypothesis. This group includes the Kolmogorov–Smirnov (K–S), Lilliefors tests, and Anderson–Darling (A–D) tests. The latter are designed for testing the assumption of a normal or an exponential distribution with unspecified parameters and are therefore important preliminary tests for justifying the use of parametric or classical statistical methods that require this assumption. Finally, we present some graphical approaches to assessing the form of a distribution.

4.2 The Chi-Square Goodness-of-Fit Test

A single random sample of size n is drawn from a population with unknown cdf F_X. We wish to test the null hypothesis

$$H_0 : F_X(x) = F_0(x) \quad \text{for all } x$$

where $F_0(x)$ is completely specified, against the general alternative

$$H_1 : F_X(x) \neq F_0(x) \quad \text{for some } x$$

In order to apply the chi-square test in this situation, the sample data must first be grouped according to some scheme in order to form a frequency distribution. In the case of count or qualitative data, where the hypothesized distribution would be discrete, the categories would be the relevant verbal or numerical classifications. For example, in tossing a die, the categories would be the numbers of spots; in tossing a coin, the categories would be the numbers of heads; in surveys of brand preferences, the categories would be the brand names considered. When the sample observations are quantitative, the categories would be numerical classes chosen by the experimenter. In this case, the frequency distribution is not unique and some information is necessarily lost. Even though the hypothesized distribution is most likely continuous with measurement data, the data must be categorized for analysis by the chi-square test.

When the population distribution is completely specified by the null hypothesis, one can calculate the probability that a random observation will be classified into each of the chosen or fixed categories. These probabilities multiplied by n give the frequencies that would be expected for each category if the null hypothesis were true. Except for sampling variation, there should be close agreement between these expected and observed frequencies if the sample data are compatible with the specified $F_0(x)$.

The corresponding observed and expected frequencies can be compared visually using a histogram, a frequency polygon, or a bar chart. The chi-square goodness-of-fit test provides a probability basis for making the comparison and deciding whether the lack of agreement is too great to have occurred by chance.

Assume that the n observations have been grouped into k mutually exclusive categories, and denote the observed and expected frequencies for the ith class by f_i and e_i, respectively, $i = 1, 2, \ldots, k$. The decision regarding fit is based on the deviations $f_i - e_i$. The sum of these k deviations is zero except for rounding. The test criterion suggested by Pearson (1900) is the sum of squares of these deviations, normalized by the expected frequency, or

$$Q = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i} \tag{4.2.1}$$

A large value of Q would reflect an incompatibility between the observed and expected frequencies, and therefore the null hypothesis used to calculate the e should be rejected for a large value of Q.

The exact probability distribution of the random variable Q is quite complicated, but for large samples, its distribution is approximately chi-square with $k - 1$ degrees of freedom, given here as Appendix Table B. The theoretical basis for this can be argued briefly as follows.

The only random variables of concern are the class frequencies $F_1, F_2, \ldots, F_k$, which constitute a set of random variables from the k-variate multinomial distribution with k possible outcomes, the ith outcome being the ith category in the classification system. With $\theta_1, \theta_2, \ldots, \theta_k$ denoting the probabilities of the respective outcomes and $f_1, f_2, \ldots, f_k$ denoting the observed values of the random variables, the likelihood function of the sample then is

$$L(\theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{k} \theta_i^{f_i} \quad f_i = 0, 1, \ldots, n; \quad \sum_{i=1}^{k} \theta_i = 1 \quad \sum_{i=1}^{k} f_i = n \tag{4.2.2}$$

The null hypothesis was assumed to specify the population distribution completely, from which the θ_i can be calculated. This hypothesis then is actually concerned only with the values of these parameters and can be equivalently stated as

$$H_0 : \theta_i = \theta_i^0 \quad \text{for } i = 1, 2, \ldots, k$$

It is easily shown that the maximum-likelihood estimates of the parameters in (4.2.2) are $\hat{\theta}_i = f_i/n$. The likelihood-ratio statistic for this hypothesis then is

$$T = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{L(\theta_1^0, \theta_2^0, \ldots, \theta_k^0)}{L(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k)} = \prod_{i=1}^{k} \left(\frac{\theta_i^0}{\hat{\theta}_i}\right)^{f_i}$$

As was stated in Chapter 1, the distribution of the random variable $-2 \ln T$ can be approximated by the chi-square distribution. The degrees of freedom are $k-1$, since the restriction $\sum_{i=1}^{k} \theta_i = 1$ leaves only $k-1$ parameters in Ω to be estimated independently. We have here

$$-2 \ln T = -2 \sum_{i=1}^{k} f_i \left(\ln \theta_i^0 - \ln \frac{f_i}{n} \right) \tag{4.2.3}$$

Some statisticians advocate using the expression in (4.2.3) as a test criterion for goodness of fit. We will now show that (4.2.3) is asymptotically equivalent to the expression for Q given in (4.2.1). The Taylor series expansion of $\ln \theta_i$ about $f_i/n = \hat{\theta}_i$ is

$$\ln \theta_i = \ln \hat{\theta}_i + (\theta_i - \hat{\theta}_i) \frac{1}{\hat{\theta}_i} + \frac{(\theta_i - \hat{\theta}_i)^2}{2!} \left(-\frac{1}{\hat{\theta}_i^2} \right) + \in$$

so that

$$\begin{aligned} \ln \theta_i^0 - \ln \frac{f_i}{n} &= \left(\theta_i^0 - \frac{f_i}{n} \right) \frac{n}{f_i} - \left(\theta_i^0 - \frac{f_i}{n} \right)^2 \frac{n^2}{2f_i^2} + \in \\ &= \frac{(n\theta_i^0 - f_i)}{f_i} - \frac{(n\theta_i^0 - f_i)^2}{2f_i^2} + \in \end{aligned} \tag{4.2.4}$$

where $\in$ represents the remainder term, the sum of terms alternating in sign

$$\sum_{j=3}^{\infty} (-1)^{j+1} \left(\theta_i^0 - \frac{f_i}{n} \right)^j \frac{n^j}{j! f_i^j}$$

Substituting (4.2.4) in (4.2.3), we have

$$\begin{aligned} -2 \ln T &= -2 \sum_{i=1}^{k} (n\theta_i^0 - f_i) + \sum_{i=1}^{k} \frac{(n\theta_i^0 - f_i)^2}{f_i} + \sum_{i=1}^{k} \in' \\ &= 0 + \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{f_i} + \in'' \end{aligned}$$

By the law of large numbers F_i/n is known to be a consistent estimator of θ_i, or

$$\lim_{n \to \infty} \left[P\left(\left| \frac{F_i}{n} - \theta_i \right| > \varepsilon \right) \right] = 0 \quad \text{for every } \varepsilon > 0$$

Thus, we see that the probability distribution of Q converges to that of $-2 \ln T$, which is chi square with $k-1$ degrees of freedom. An approximate α-level test then is obtained by rejecting H_0 when Q exceeds the $(1-\alpha)$th quantile point of the chi-square distribution, denoted by $\chi^2_{k-1,\alpha}$. This approximation can be used with confidence as long as every expected frequency is at least equal to 5. For any e_i smaller than 5, the usual procedure is to combine adjacent groups in the frequency distribution until this restriction is satisfied. The number of degrees of freedom then must be reduced to correspond to the actual number of categories used in the analysis. This rule of 5 should not be considered inflexible, however. It is conservative, and the chi-square approximation is often reasonably accurate for expected cell frequencies as small as 1.5.

Any case where the θ_i are completely specified by the null hypothesis is thus easily handled. The more typical situation, however, is where the null hypothesis is composite, that is, it states the form of the distribution but not all the relevant parameters. For example, when we wish to test whether a sample is drawn from some normal population, μ and σ would not be given. However, μ and σ must be known in order to calculate the probabilities θ_i, and hence the expected frequencies e_i, under H_0. If under H_0, the θ_i are estimated from the data as $\hat{\theta}_i^0$ and the expected frequencies are estimated as $n\hat{\theta}_i^0$, for $i = 1, 2, \ldots, k$, the goodness-of-fit test statistic in (4.2.1) becomes

$$Q = \sum_{i=1}^{k} \frac{\left(f_i - n\hat{\theta}_i^0\right)^2}{n\hat{\theta}_i^0} \tag{4.2.5}$$

The asymptotic distribution of Q then may depend on the method employed for estimation. When the estimates are found by the method of maximum likelihood for the grouped data, the $L(\hat{\omega})$ in the likelihood-ratio test statistic is $L\left(\hat{\theta}_1^0, \hat{\theta}_2^0, \ldots, \hat{\theta}_k^0\right)$, where the $\hat{\theta}_i^0$ are the MLEs of the θ_i^0 under H_0. The degrees of freedom for Q then are $k - 1 - s$, where s is the number of independent parameters in $F_0(x)$ which had to be estimated from the grouped data in order to estimate all the θ_i^0. In the normal goodness-of-fit test, for example, the μ and σ parameter estimates would be calculated from the grouped data and used with tables of the normal distribution to find the $n\hat{\theta}_i^0$, and the degrees of freedom for k categories would be $k-3$. When the original data are ungrouped and the MLEs are based or the likelihood function of all the observations, the theory is different. Chernoff and Lehmann (1954) have shown that the limiting distribution of Q is not the chi square in this case and that $P(Q > \chi^2_\alpha) > \alpha$. The test is then anticonservative. Their investigation shows that the error is considerably more serious for the normal distribution than the Poisson. A possible adjustment is discussed in their paper. In practice, the statistic in (4.2.5) is often treated as a chi-square variable anyway.

Example 4.2.1

A quality control engineer has taken 50 samples of size 13 each from a production process. The numbers of defectives for these samples are recorded below. Test the null hypothesis at level 0.05 that the number of defectives follows:

(a) The Poisson distribution
(b) The binomial distribution

Number of Defectives	Number of Samples
0	10
1	24
2	10
3	4
4	1
5	1
6 or more	0

Solution

Since the data are grouped and the hypothesized null distributions are discrete, the chi-square goodness-of-fit test is appropriate. Since no parameters are specified, they must be estimated from the data in order to carry out the test in both (a) and (b).

Solution to (a)

The Poisson distribution is $f(x) = e^{-\mu}\mu^x/x!$ for $x = 0, 1, 2, \ldots$, where μ here is the mean number of defectives in a sample of size 13. The maximum-likelihood estimate of μ is the mean number of defectives in the 50 samples, that is,

$$\hat{\mu} = \frac{0(10) + 1(24) + 2(10) + 3(4) + 4(1) + 5(1)}{50} = \frac{65}{50} = 1.3$$

We use this value in $f(x)$ to estimate the probabilities as $\hat{\theta}_i$ and to compute $\hat{e}_i = 50\hat{\theta}_i$. The calculations are shown in Table 4.2.1. Notice that the final $\hat{\theta}$ is not for exactly five defectives, but for five or more defectives; this is necessary to make $\sum\hat{\theta} = 1$. The final $\hat{e}$ is less than one, so it is combined with the previous category before calculating Q. The final result is $Q = 3.6010$ with 3 degrees of freedom; we start out with $k - 1 = 5$ degree of freedom and lose one for estimating θ and one more for combining the last two categories. Table B shows the 0.05 critical value for the chi-square distribution with 3 degrees of freedom is 7.81. Our $Q = 3.6010$ is smaller than this value, so we cannot reject the null hypothesis. The approximate P value is the right-tail probability $P(Q \geq 3.601)$ where Q follows a chi square distribution with 3 degrees of freedom. Using EXCEL, for example, the P value is found as 0.3078. Note that using Table B, we could say that the P value is between 0.25 and 0.50. Thus, our conclusion about the Poisson distribution is that we cannot reject the null hypothesis.

TABLE 4.2.1

Calculation of Q for Example 4.2.1(a)

Defectives	f	$\hat{\theta}$	$\hat{e}$		$\frac{(f-\hat{e})^2}{\hat{e}}$
0	10	0.2725	13.625		0.9644
1	24	0.3543	17.715		2.2298
2	10	0.2303	11.515		0.1993
3	4	0.0998	4.990		0.1964
4	1	0.324	1.620	2.155	0.0111
5 or more	1	0.0107	0.535		
					3.6010

TABLE 4.2.2

Calculation of Q for Example 4.2.1(b)

Defectives	f	$\hat{\theta}$		$\hat{e}$		$\frac{(f-\hat{e})^2}{\hat{e}}$
0	10	0.2542		12.710		0.5778
1	24	0.3671		18.355		1.7361
2	10	0.2448		12.240		0.4099
3	4	0.0997		4.986		0.1950
4	1	0.0277	0.0342	1.385	1.710	0.0492
5 or more	1	0.0065		0.385		
						2.9680

Solution to (b)

The null hypothesis is that the number of defectives in each sample of 13 follows the binomial distribution with $n = 13$ and p is the probability of a defective in any sample. The maximum-likelihood estimate of p is the total number of defectives, which we found in (a) to be 65, divided by the $50(13) = 650$ observations, or $\hat{p} = 65/650 = 0.1$. This is the value we use in the binomial distribution (or Table C) to find $\hat{\theta}$ and $\hat{e} = 50\,\hat{\theta}$ in Table 4.2.2. The final result is $Q = 2.9680$, again with 3 degrees of freedom, so the critical value at the 0.05 level is again 7.81. The approximate P value using EXCEL is 0.3966. Our conclusion about the binomial distribution is that we cannot reject the null hypothesis.

This example illustrates a common result with chi-square goodness-of-fit tests, that is, that each of two (or more) different null hypothesized distributions may be accepted for the same data set. Obviously, the true distribution cannot be both binomial and Poisson at the same time. Thus, the appropriate conclusion on the basis of a chi-square goodness-of-fit test is that we do not have enough information to distinguish between these two distributions.

The STATXACT solutions to Example 4.2.1 are shown below. Note that the numerical value of Q in each case agrees with the hand calculations. Each printout shows the degrees of freedom as 4 instead of 3 because the computer did not know that the expected frequencies entered were calculated by estimating one parameter from the data in each case and/or that two classes were combined. The P values do not agree because the degrees of freedom are different.

```
**********************************
STATXACT SOLUTION TO EXAMPLE 4.2.1(a)
**********************************

   CHI-SQUARE GOODNESS OF FIT TEST

  Statistic based on the observed five categories :
CH(X) = Pearson chi-square statistic =        3.601

Asymptotic P-value: (based on chi-square distribution with 4 df )
     Pr { CH(X)  .GE.       3.601 } =       0.4627

**********************************
STATXACT SOLUTION TO EXAMPLE 4.2.1(b)
**********************************

   CHI-SQUARE GOODNESS OF FIT TEST

  Statistic based on the observed five categories :
CH(X) = Pearson chi-square statistic =        2.968

Asymptotic P-value: (based on chi-square distribution with 4 df )
     Pr { Ch(X)  .GE.       2.968 } =       0.5633
```

4.3 The Kolmogorov–Smirnov (K–S) One-Sample Statistic

In the chi-square goodness-of-fit test, the comparison between observed and expected frequencies is made for a set of k groups. Only k comparisons are made even though there are n observations, where $k \leq n$. If the n sample observations are values of a continuous random variable, as opposed to strictly categorical data or values of a discrete variable, comparisons can be made between observed and expected cumulative relative frequencies for each of the different observed values. The *empirical cdf (edf)* defined in Section 2.3 is an estimate of the population cdf. Several goodness-of-fit test statistics are functions of the deviations between the empirical distribution function and the population cdf specified under the null hypothesis. The function of these deviations used to perform a goodness-of-fit test might be the sum of

squares, or absolute values, or the maximum deviation, to name only a few. The best-known test is the K–S one-sample statistic, which will be covered in this section.

The K–S one-sample statistic is based on the differences between the hypothesized cdf $F_0(x)$ and the *edf* of the sample defined in Chapter 2 as $S_n(x)$, the proportion of sample observations that are less than or equal to x for all real numbers x. We showed there that $S_n(x)$ provides a consistent point estimator for the true cdf $F_X(x)$. Further, by Theorem 2.3.2, we know that as n increases, the step function $S_n(x)$, with jumps occurring at the values of the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ for the sample, approaches the cdf $F_X(x)$ for all x. Therefore, for large n, the deviations between the true function and its statistical image, $|S_n(x) - F_X(x)|$, should be small for all values for x. This result suggests that, if H_0 is true, the statistic

$$D_n = \sup_x |S_n(x) - F_0(x)| \tag{4.3.1}$$

is, for any n, a reasonable measure of the accuracy of our estimate.

This D_n statistic, called the *K–S one-sample statistic,* is particularly useful in nonparametric statistical inference because the probability distribution of D_n does not depend on $F_0(x)$ as long as F_0 is continuous. Therefore, D_n is a distribution-free statistic.

The directional deviations defined as

$$D_n^+ = \sup_x [S_n(x) - F_0(x)] \quad D_n^- = \sup_x [F_0(x) - S_n(x)] \tag{4.3.2}$$

are called the *one-sided K–S statistics*. These measures are also distribution free, as proved in the following theorem.

THEOREM 4.3.1

The statistics D_n, D_n^+, and D_n^- are completely distribution-free for any specified continuous cdf F_0.

Proof

$D_n = \sup_x |S_n(x) - F_0(x)| = \max_x (D_n^+, D_n^-)$

Defining $X_{(0)} = -\infty$ and $X_{(n+1)} = \infty$, we can write

$$S_n(x) = \frac{i}{n} \quad \text{for } X_{(i)} \le x < X_{(i+1)}, i = 0, 1, \ldots, n$$

Therefore, we have

$$\begin{aligned} D_n^+ = \sup_x [S_n(x) - F_0(x)] &= \max_{0 \le i \le n} \sup_{X_{(i)} \le x < X_{(i+1)}} [S_n(x) - F_0(x)] \\ &= \max_{0 \le i \le n} \sup_{X_{(i)} \le x < X_{(i+1)}} \left[\frac{i}{n} - F_0(x)\right] \\ &= \max_{0 \le i \le n} \left[\frac{i}{n} - \inf_{X_{(i)} \le x < X_{(i+1)}} F_0(x)\right] \\ &= \max_{0 \le i \le n} \left[\frac{i}{n} - F_0(X_{(i)})\right] \\ &= \max \left\{ \max_{1 \le i \le n} \left[\frac{i}{n} - F_0(X_{(i)})\right], 0 \right\} \end{aligned} \tag{4.3.3}$$

Similarly

$$\begin{aligned} D_n^- &= \max \left\{ \max_{1 \le i \le n} \left[F_0(X_{(i)}) - \frac{i-1}{n}\right], 0 \right\} \\ D_n &= \max \left\{ \max_{1 \le i \le n} \left[\frac{i}{n} - F_0(X_{(i)})\right], \max_{1 \le i \le n} \left[F_0(X_{(i)}) - \frac{i-1}{n}\right], 0 \right\} \end{aligned} \tag{4.3.4}$$

The probability distributions of D_n, D_n^+, and D_n^- therefore depend only on the random variables $F_0(X_{(i)})$, $i = 1, 2, \ldots, n$; under H_0, these are the order statistics from the uniform distribution on (0, 1), regardless of the original F_0 as long as it is continuous and completely specified, because of the probability integral transformation discussed in Chapter 2. Thus D_n, D_n^+, and D_n^- have distributions which are independent of the particular F_0.

A simpler proof can be given by making the transformation $u = F_0(x)$ in D_n, D_n^+, or D_n^-. This will be left as an exercise for the reader. The above proof has the advantage of defining the K–S statistics in terms of order statistics.

In order to use the K–S statistics for inference, their sampling distributions must be known. Since the distributions are independent of F_X under H_0, we can assume without loss of generality that F_0 is the cdf of the uniform distribution on (0, 1). The derivation of the null distribution of D_n is rather tedious. However, the approach below illustrates a number of properties of order statistics and is therefore included here. For an interesting alternative derivation, see Massey (1950).

THEOREM 4.3.2

For $D_n = \sup_x |S_n(x) - F_0(x)|$, *where* $F_0(x)$ *is any specific continuous cdf, we have, under* H_0

$$P\left(D_n < \frac{1}{2n} + v\right) = \begin{cases} 0 & \text{for } v \le 0 \\ \int_{1/2n-v}^{1/2n+v} \int_{1/3n-v}^{1/3n+v} \cdots \int_{(2n-1)/2n-v}^{(2n-1)/2n+v} & \\ \quad \times f(u_1, u_2, \ldots, u_n) du_n \cdots du_1 & \text{for } 0 < v < \frac{2n-1}{2n} \\ 1 & \text{for } v \ge \frac{2n-1}{2n} \end{cases}$$

where

$$f(u_1, u_2, \ldots, u_n) = \begin{cases} n! & \text{for } 0 < u_1 < u_2 < \cdots < u_n < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof

As explained above, $F_0(x)$ can be assumed to be the cdf of the uniform (0, 1) distribution. We first determine the relevant domain of v. Since both $S_n(x)$ and $F_0(x)$ are between 0 and 1, $0 \le D_n \le 1$ always. Therefore, we must determine $P(D_n < c)$ only for $0 < c < 1$, which here requires

$$0 < \frac{1}{2n} + v < 1 \quad \text{or} \quad -\frac{1}{2n} < v < \frac{2n-1}{2n}$$

Now, for all $-1/2n < v < (2n-1)/2n$, where $X_{(0)} = 0$ and $X_{(n+1)} = 1$,

$$\begin{aligned} P\left(D_n < \frac{1}{2n} + v\right) &= P\left[\sup_x |S_n(x) - x| < \frac{1}{2n} + v\right] \\ &= P\left[|S_n(x) - x| < \frac{1}{2n} + v, \quad \text{for all } x\right] \\ &= P\left[\left|\frac{i}{n} - x\right| < \frac{1}{2n} + v \text{ for } X_{(i)} \le x < X_{(i+1)}, \quad \text{for all } i = 0, 1, \ldots, n\right] \\ &= P\left[\frac{i}{n} - \frac{1}{2n} - v < x < \frac{i}{n} + \frac{1}{2n} + v, \quad \text{for all } i = 0, 1, \ldots, n\right] \\ &= P\left[\frac{2i-1}{2n} - v < x < \frac{2i+1}{2n} + v \quad \text{for } X_{(i)} \le x < X_{(i+1)}, \quad \text{for all } i = 0, 1, \ldots, n\right] \end{aligned}$$

Consider any two consecutive values of i. We must have, for any $0 \le i \le n-1$, both

$$A_i: \left\{\frac{2i-1}{2n} - v < x < \frac{2i+1}{2n} + v \quad \text{for } X_{(i)} \le x \le X_{(i+1)}\right\}$$

and

$$A_{i+1}:\left\{\frac{2i+1}{2n}-v<x<\frac{2i+3}{2n}+v \quad \text{for } X_{(i+1)}\le x\le X_{(i+2)}\right\}$$

Since $X_{(i+1)}$ is the random variable common to both events and the common set of x is $(2i+1)/2n-v<x<(2i+1)/2n+v$ for $v\ge 0$, the event $A_i\cap A_{i+1}$ for any $0\le i\le n-1$ is

$$\frac{2i-1}{2n}-v<X_{(i+1)}<\frac{2i+1}{2n}+v \quad \text{for all } v\ge 0$$

In other words,

$$\frac{2i+1}{2n}-v<x<\frac{2i+1}{2n}+v \quad \begin{array}{l}\text{for } X_{(i)}\le x\le X_{(i+1)}\\ \text{for all } i=0,1,\ldots,n\end{array}$$

if and only if

$$\frac{2i+1}{2n}-v<X_{(i+1)}<\frac{2i+1}{2n}+v \quad \begin{array}{r}\text{for all } i=0,1,\ldots,n-1\\ v\ge 0\end{array}$$

The joint probability distribution of the uniform order statistics is

$$f_{X_{(1)},X_{(2)},\ldots,X_{(n)}}(x_1,x_2,\ldots,x_n)=n! \quad \text{for } 0<x_1<x_2<\cdots<x_n<1$$

Putting all this together now, we have

$$\begin{aligned}
&P\left(D_n<\frac{1}{2n}+v\right) \quad \text{for all } -\frac{1}{2n}<v<\frac{2n-1}{2n}\\
&\quad=P\left(\frac{2i+1}{2n}-v<X_{(i+1)}<\frac{2i+1}{2n}+v \quad \text{for all } i=0,1,\ldots,n-1\right) \quad \text{for all } 0\le v<\frac{2n-1}{2n}\\
&\quad=P\left[\left(\frac{1}{2n}-v<X_{(1)}<\frac{1}{2n}+v\right)\bigcap\left(\frac{3}{2n}-v<X_{(2)}<\frac{3}{2n}+v\right)\right.\\
&\qquad\left.\times\bigcap\cdots\bigcap\left(\frac{2n-1}{2n}-v<X_{(n)}<\frac{2n-1}{2n}+v\right)\right] \quad \text{for all } 0\le v<\frac{2n-1}{2n}
\end{aligned}$$

which is equivalent to the stated integral.

This result is tedious to evaluate. For the sake of illustration, consider $n = 2$. For all $0 \le v \le 3/4$,

$$P(D_2 < 1/4 + v) = 2! \int_{1/4-v}^{1/4+v} \int_{3/4-v}^{3/4+v} du_2 du_1 \quad 0 < u_1 < u_2 < 1$$

The limits overlap when $1/4 + v \ge 3/4 - v$, or $v \ge 1/4$. When $0 \le v < 1/4$, we have $u_1 < u_2$ automatically. Therefore, for $0 \le v < 1/4$,

$$P(D_2 < 1/4 + v) = 2 \int_{1/4-v}^{1/4+v} \int_{3/4-v}^{3/4+v} du_2 du_1 = 2(2v)^2$$

But for $1/4 \le v \le 3/4$, the region of integration is as illustrated in Figure 4.3.1. Dividing the integral into two pieces, we have for $1/4 \le v < 3/4$,

$$\begin{aligned} P(D_2 < 1/4 + v) &= 2\left[\int_{3/4-v}^{1/4+v} \int_{u_1}^{1} du_2 du_1 + \int_{0}^{3/4-v} \int_{3/4-v}^{1} du_2 du_1\right] \\ &= -2v^2 + 3v - 1/8 \end{aligned}$$

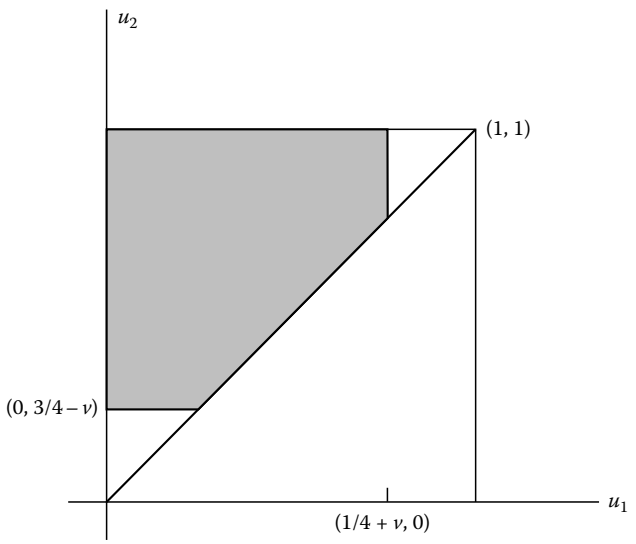


FIGURE 4.3.1
Shaded area is region of integration for $n = 2$.

Collecting the results for all v,

$$P(D_2 < 1/4 + v) = \begin{cases} 0 & \text{for } v \le 0 \\ 2(2v)^2 & \text{for } 0 < v < 1/4 \\ -2v^2 + 3v - 1/8 & \text{for } 1/4 \le v < 3/4 \\ 1 & \text{for } v \ge 3/4 \end{cases}$$

For any given v and n, we can evaluate $P(D_n < 1/2n + v)$ or use Table 1 of Birnbaum (1952). The inverse procedure is to find that number $D_{n,\alpha}$ such that $P(D_n > D_{n,\alpha}) = \alpha$. In our numerical example with $n = 2$, $\alpha = 0.05$, we find v such that

$$P(D_2 > 1/4 + v) = 0.05 \quad \text{or} \quad P(D_2 < 1/4 + v) = 0.95$$

and then set $D_{2,0.05} = 1/4 + v$. From the previous evaluation of the D_2 sampling distribution, either

$$2(2v)^2 = 0.95 \quad \text{and} \quad 0 < v < 1/4$$

or

$$-2v^2 + 3v - 0.125 = 0.95 \quad \text{and} \quad 1/4 \le v < 3/4$$

The first result has no solution, but the second yields the solution $v = 0.5919$. Therefore, $D_{2,0.05} = 0.8419$.

Numerical values of $D_{n,\alpha}$ are given in Table F for $n \le 40$ and selected tail probabilities α, and approximate values are given for larger n. More extensive tables are given in Dunstan et al. (1979, p. 42) for $n \le 100$.

For large sample sizes, Kolmogorov (1933) derived the following convenient approximation to the sampling distribution of D_n, and Smirnov (1939) gave a simpler proof. The result is given here without proof.

THEOREM 4.3.3

If F_X is any continuous distribution function, then for every $d > 0$,

$$\lim_{n\to\infty} P(D_n \le d/\sqrt{n}) = L(d)$$

where

$$L(d) = 1 - 2\sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2d^2}$$

TABLE 4.3.1

Exact and Asymptotic Values of $D_{n,\alpha}$ Such That $P(D_n > D_{n,\alpha}) = \alpha$ for $\alpha = 0.01, 0.05$

	Exact		Asymptotic		Ratio A/E	
n	**0.05**	**0.01**	**0.05**	**0.01**	**0.05**	**0.01**
2	0.8419	0.9293	0.9612	1.1509	1.142	1.238
3	0.7076	0.8290	0.7841	0.9397	1.108	1.134
4	0.6239	0.7341	0.6791	0.8138	1.088	1.109
5	0.5633	0.6685	0.6074	0.7279	1.078	1.089
10	0.4087	0.4864	0.4295	0.5147	1.051	1.058
20	0.2939	0.3524	0.3037	0.3639	1.033	1.033
30	0.2417	0.2898	0.2480	0.2972	1.026	1.025
40	0.2101	0.2521	0.2147	0.2574	1.022	1.021
50	0.1884	0.2260	0.1921	0.2302	1.019	1.018

Source: Birnbaum, Z.W., *J. Am. Statistic. Assoc.*, 47, 431, 1952, Table 2.

The function $L(d)$ has been tabulated in Smirnov (1948). Some of the results for the asymptotic approximation to $D_{n,\alpha} = d_\alpha/\sqrt{n}$ are given below.

$P(D_n > d_\alpha/\sqrt{n})$	0.20	0.15	0.10	0.05	0.01
d_α	1.07	1.14	1.22	1.36	1.63

This approximation has been found to be close enough for practical application as long as n exceeds 35. A comparison of exact and asymptotic values of $D_{n,\alpha}$ for $\alpha = 0.01$ and 0.05 is given in Table 4.3.1.

Since the one-sided K–S statistics are also distribution-free, knowledge of their sampling distribution would make them useful in nonparametric statistical inference as well. Their exact sampling distributions are considerably easier to derive than that for D_n. Only the statistic D_n^+ is considered in the following theorem, but D_n^+ and D_n^- have identical distributions because of symmetry.

THEOREM 4.3.4

For $D_n^+ = \sup_x [S_n(x) - F_0(x)]$ *where* $F_0(x)$ *is any continuous cdf, we have, under* H_0

$$P(D_n^+ < c) = \begin{cases} 0 & c \le 0 \\ \int_{1-c}^{1} \int_{(n-1)/n-c}^{u_n} \cdots \int_{2/n-c}^{u_3} \int_{1/n-c}^{u_2} \int (u_1, u_2, \ldots, u_2) du_1 \cdots du_n & 0 < c < 1 \\ 1 & c \ge 1 \end{cases}$$

where

$$f(u_1, u_2, \ldots, u_n) = \begin{cases} n! & \text{for } 0 < u_1 < u_2 < \cdots < u_n < 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof

As before, we first assume without loss of generality that F_0 is the cdf of the uniform (0, 1) distribution. Then we can write

$$D_n^+ = \max\left[\max_{1 \le i \le n}\left(\frac{i}{n} - X_{(i)}\right), 0\right]$$

the form found in (4.3.3). For all $0 < c < 1$, we have

$$\begin{aligned} P(D_n^+ < c) &= P\left[\max\left(\frac{i}{n} - X_{(i)}\right) < c\right] \\ &= P\left(\frac{i}{n} - X_{(i)} < c \quad \text{for all } i = 1, 2, \ldots, n\right) \\ &= P\left(X_{(i)} > \frac{i}{n} - c \quad \text{for all } i = 1, 2, \ldots, n\right) \\ &= \int_{1-c}^{\infty} \int_{(n-1)/n-c}^{\infty} \cdots \int_{2/n-c}^{\infty} \int_{1/n-c}^{\infty} f(x_1, x_2, \ldots, x_n) dx_1 \cdots dx_n \end{aligned}$$

where

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} n! & \text{for } 0 < x_1 < x_2 < \cdots < x_n < 1 \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to the stated integral.

Another form of this result, due to Birnbaum and Tingey (1951), which is more computationally tractable, is

$$P(D_n^+ > c) = (1 - c)^n + c \sum_{j=1}^{[n(1-c)]} \binom{n}{j} \left(1 - c - \frac{j}{n}\right)^{n-j} \left(c + \frac{j}{n}\right)^{j-1} \tag{4.3.5}$$

where $[x]$ denotes the greatest integer not exceeding x.

The equivalence of the two forms can be shown by induction. Birnbaum and Tingey give a table of those values $D_{n,\alpha}^+$, which satisfy $P(D_n > D_{n,\alpha}^+) = \alpha$ under H_0 for $\alpha = 0.01$, 0.05, 0.10 and selected values of n. For large sample sizes, we have the following theorem, which is given here without proof.

THEOREM 4.3.5

If F_0 is any specified continuous cdf, then under H_0 for every $d \geq 0$,

$$\lim_{n\to\infty} P(D_n^+ < d/\sqrt{n}) = 1 - e^{-2d^2}$$

As a result of this theorem, chi-square tables can be used for the null distribution of a function of D_n^+ as the following corollary shows.

COROLLARY 4.3.5.1 *If F_0 is any specified continuous cdf, then for every $d \geq 0$, the limiting null distribution of $V = 4nD_n^{+2}$, as $n \to \infty$, is the chi-square distribution with 2 degrees of freedom.*

Proof

We have $D_n^+ < d/\sqrt{n}$ if and only if $4nD_n^{+2} < 4d^2$ or $V < 4d^2$.

Therefore,

$$\text{under } H_0,\ \lim_{n\to\infty} P(V < 4d^2) = \lim_{n\to\infty} P\left(D_n^+ < d/\sqrt{n}\right) = 1 - e^{-2d^2} = 1 - e^{-4d^2/2}$$

$$\text{so that } \lim_{n\to\infty} P(V < c) = 1 - e^{-c/2} \quad \text{for all } c > 0$$

The right-hand side is the cdf of a chi-square distribution with 2 degrees of freedom.

As a numerical example of how this corollary enables us to approximate $D_{n,\alpha}^+$, let $\alpha = 0.05$. Table B shows that 5.99 is the 0.05 critical point of chi square with 2 degrees of freedom. Then we set $4nD_{n,0.05}^{+2} = 5.99$ and solve to obtain

$$D_{n,0.05}^+ = \sqrt{1.4975/n} = 1.22/\sqrt{n}$$

4.4 Applications of the Kolmogorov–Smirnov (K–S) One-Sample Statistics

The statistical use of the K–S statistic in a goodness-of-fit type of problem is obvious. Assume we have the random sample $X_1, X_2, \ldots, X_n$ and the hypothesis $H_0: F_X(x) = F_0(x)$ for all x, where $F_0(x)$ is a completely specified continuous cdf.

Since $S_n(x)$ is the statistical image of the population cdf, the differences between $S_n(x)$ and $F_0(x)$ should be small for all x except for sampling variation, if the null hypothesis is true. For the usual two-sided goodness-of-fit alternative.

$$H_1: F_X(x) \neq F_0(x) \quad \text{for some } x$$

large absolute values of these deviations tend to discredit the null hypothesis. Therefore, the K–S goodness-of-fit test with significance level α is to reject H_0 when $D_n > D_{n,\alpha}$. From Theorem 2.3.2, we know that $S_n(x)$ converges to $F_X(x)$ with probability 1, which implies consistency.

The value of the K–S goodness-of-fit statistic D_n in (4.3.1) can be calculated using (4.3.4) if all n observations have different numerical values (no ties). However, the following expression is considerably easier for algebraic calculation and applies when ties are present:

$$D_n = \sup_x |S_n(x) - F_0(x)| = \max_x [|S_n(x) - F_0(x)|, |S_n(x - \varepsilon) - F_0(x)|]$$

where ε denotes any small positive number. Example 4.4.1 will illustrate this easy algebraic method of calculating D_n. Quantiles of the exact null distribution of D_n are given in Table F for $n \leq 40$, along with approximate values for $n > 40$. The appropriate critical region is for D_n large.

Example 4.4.1

The 20 observations below were chosen randomly from the continuous uniform distribution over (0, 1), recorded to four significant figures, and rearranged in increasing order of magnitude. Determine the value of D_n, and test the null hypothesis that the square roots of these numbers also have the continuous uniform (0, 1) distribution.

0.0123	0.1039	0.1954	0.2621	0.2802
0.3217	0.3645	0.3919	0.4240	0.4814
0.5139	0.5846	0.6275	0.6541	0.6889
0.7621	0.8320	0.8871	0.9249	0.9634

Solution

The calculations needed to find D_n are shown in Table 4.4.1.

The entries in the first column, labeled x, are not the observations above, but their respective square roots, because the null hypothesis is concerned with the

TABLE 4.4.1

Calculation of D_n for Example 4.4.1

x	$S_n(x)$	$F_0(x)$	$S_n(x) - F_0(x)$	$S_n(x-\varepsilon) - F_0(x)$	$\lvert S_n(x) - F_0(x)\rvert$	$\lvert S_n(x-\varepsilon) - F_0(x)\rvert$
0.11	0.05	0.11	−0.06	−0.11	0.06	0.11
0.32	0.10	0.32	−0.22	−0.27	0.22	0.27
0.44	0.15	0.44	−0.29	−0.34	0.29	0.34
0.51	0.20	0.51	−0.31	−0.36	0.31	0.36
0.53	0.25	0.53	−0.28	−0.33	0.28	0.33
0.57	0.30	0.57	−0.27	−0.32	0.27	0.32
0.60	0.35	0.60	−0.25	−0.30	0.25	0.30
0.63	0.40	0.63	−0.23	−0.28	0.23	0.28
0.65	0.45	0.65	−0.20	−0.25	0.20	0.25
0.69	0.50	0.69	−0.19	−0.24	0.19	0.24
0.72	0.55	0.72	−0.17	−0.22	0.17	0.22
0.76	0.60	0.76	−0.16	−0.21	0.16	0.21
0.79	0.65	0.79	−0.14	−0.19	0.14	0.19
0.81	0.70	0.81	−0.11	−0.16	0.11	0.16
0.83	0.75	0.83	−0.08	−0.13	0.08	0.13
0.87	0.80	0.87	−0.07	−0.12	0.07	0.12
0.91	0.85	0.91	−0.06	−0.11	0.06	0.11
0.94	0.90	0.94	−0.04	−0.09	0.04	0.09
0.96	0.95	0.96	−0.01	−0.06	0.01	0.06
0.98	1.00	0.98	0.02	−0.03	0.02	0.03

distribution of these square roots. The $S_n(x)$ are the proportions of observed values less than or equal to each different observed x. The hypothesized distribution here is $F_0(x) = x$, so the third column is exactly the same as the first column. The fourth column is the difference $S_n(x) - F_0(x)$. The fifth column is the difference $S_n(x - \varepsilon) - F_0(x)$, that is, the difference between the S_n value for a number slightly smaller than an observed x and the F_0 value for that observed x. Finally, the sixth and seventh columns are the absolute values of the differences of the numbers in the fourth and fifth columns. The supremum is the largest entry in either of the last two columns; its value here is $D_n = 0.36$. Table F shows that the 0.01 level rejection region for $n = 20$ is $D_n \geq 0.352$, so we reject the null hypothesis that the square roots of these numbers are uniformly distributed.

The theoretical justification behind this example is as follows. Let Y have the continuous uniform (0, 1) distribution so that $f_Y(y) = 1$ for $0 \leq y \leq 1$. Then the pdf of $X = \sqrt{Y}$ can be shown to be (the reader should verify this) $f_X(x) = 2x$ for $0 \leq x \leq 1$, which is not uniform. In fact, this is a beta distribution with parameters $a = 2$ and $b = 1$.

4.4.1 One-Sided Tests

With the statistics D_n^+ and D_n^-, it is possible to use K–S statistics for a one-sided goodness-of-fit test, which would detect directional differences between $S_n(x)$ and $F_0(x)$. For the alternative

$$H_{1,+}: F_X(x) \geq F_0(x) \quad \text{for all } x$$

the appropriate rejection region is $D_n^+ > D_{n,\alpha}^+$, and for the alternative

$$H_{1,-}: F_X(x) \leq F_0(x) \quad \text{for all } x$$

H_0 is rejected when $D_n^- > D_{n,\alpha}^-$. Both of these tests are consistent against their respective alternatives.

Most tests of goodness of fit are two-sided and so applications of the one-sided K–S statistics will not be demonstrated here in detail. However, it is useful to know that the tail probabilities for the one-sided statistics are approximately one-half of the corresponding tail probabilities for the two-sided statistic. Therefore, approximate results can be obtained for the one-sided statistics by using the entries in Table F with each quantile being one-half of the value labeled. In our example, we find $D_n^+ = 0.02$ as the largest entry in the fourth column of Table 4.4.1. Now from Table F, we see that for $n = 20$, the smallest critical value corresponding to a two-tailed P value of 0.200 is 0.232. Since the observed value 0.02 is smaller than 0.232, we conclude that the approximate P value for testing H_0 against $H_{1,+}$: $F_X(x) \geq F_0(x)$ for all x is larger than 0.100 so we fail to reject H_0 in favor of $H_{1,+}$. For the alternative $H_{1,-}$: $F_X(x) \leq F_0(x)$ we find $D_n^- = 0.36$ from the fifth column of Table 4.4.1. The approximate P value from Table F is $P < 0.005$ and so we reject H_0 in favor of $H_{1,-}$ If, for example, we observed $D_n^+ = 0.30$, the approximate P value is between 0.01 and 0.025.

The STATXACT solution to Example 4.4.1 is shown below. The values of all three of the D_n statistics and P values agree with ours. The STATXACT package also provides K–S type goodness-of-fit tests for some specific distributions such as the binomial, Poisson, uniform, exponential, and normal. Some of these will be discussed and illustrated later in this chapter.

```
                ********************************
                STATXACT SOLUTION TO EXAMPLE 4.4.1
                ********************************

KOLMOGOROV-SMIRNOV ONE-SAMPLE TEST

Hypothesized distribution F(x) : Uniform(Cont.);
                                 Min =      0.0000 Max =   1.00

Let S(x) be the empirical distribution.
                        Sample size  :  20
```

```
Inference :
                                          Statistic
Item                    Sup{|S(x) − F(x)|}  Sup{S(x) − F(x)}  Sup{F(x) − S(x)}
Observed statistic             0.3600            0.02000           0.3600
Standard statistic              1.610            0.08944            1.610
Asymptotic P-value             0.0112             0.9841           0.0056
Exact P-value                  0.0079             0.9709           0.0040
Exact point prob.              0.0000             0.0000           0.0000
```

MINITAB offers the K–S test for the normal distribution and this will be discussed later in this chapter.

Two other useful applications of the K–S statistics relate to point and interval estimation of the unknown cdf F_X.

4.4.2 Confidence Bands

One important statistical use of the D_n statistic is in finding confidence bands on $F_X(x)$ for all x. From Table F, we can find the number $D_{n,\alpha}$ such that

$$P(D_n > D_{n,\alpha}) = \alpha$$

This is equivalent to the statement

$$P[\sup_x |S_n(x) - F_X(x)| < D_{n,\alpha}] = 1 - \alpha$$

which means that

$$P[S_n(x) - D_{n,\alpha} < F_X(x) < S_n(x) + D_{n,\alpha} \text{ for all } x] = 1 - \alpha$$

But we known that $0 \leq F_X(x) \leq 1$ for all x, whereas the inequality in this probability statement admits numbers outside this range. Thus we define

$$L_n(x) = \max[S_n(x) - D_{n,\alpha}, 0]$$

and

$$U_n(x) = \min[S_n(x) + D_{n,\alpha}, 1]$$

and call $L_n(x)$ a lower confidence band and $U_n(x)$ an upper confidence band for the cdf F_X, with associated confidence coefficient $1 - \alpha$.

The simplest procedure in application is to graph the observed $S_n(x)$ as a step function and draw parallel lines at a distance $D_{n,\alpha}$ in either direction, but always within the unit square. When $n > 40$, the value $D_{n,\alpha}$ can be determined from the asymptotic distribution. Of course, this confidence-band

procedure can be used to perform a test of the hypothesis $F_X(x) = F_0(x)$, since $F_0(x)$ lies wholly within the limits $L_n(x)$ and $U_n(x)$ if and only if the hypothesis cannot be rejected at significance level α. Similar applications of the D_n^- or D_n^+ statistics are obvious.

One criticism of confidence bands is that they are too wide, particularly in the tails of the distribution, where the order statistics have a lot of variation. Keeping this in mind, other approaches to constructing a confidence band on the cdf have been considered in the literature. One general idea is to base bands on $\sup_x \{\omega[F(x)]|S_n(x) - F_X(x)|\}$, where $\omega(x)$ is a suitable weight function. Some authors have also considered restricting the range of x to a finite interval. For example, Doksum (1977) used $\max_{a \le s_n(x) \le b} \{[F_X(x)(1 - F_X(x)]^{-1/2}|S_n(x) - F_X(x)|\}$, where a and b are constants with $0 \le a < b \le 1$, to set up a confidence band for $F_X(x)$. The resulting band is slightly wider in the middle but is much narrower in the tails.

SAS provides confidence bands for the cdf for various confidence levels. We present an example using the data in Example 4.4.1 and the module ''interactive data analysis.'' Figure 4.4.1 shows a screenshot, helping the reader find the procedure under SAS Version 9.1. The output is shown in Figure 4.4.2. The slider in the output allows one to change the confidence level interactively and examine the effect on the bands. Note also that the output includes the K–S test statistic and the associated P value, together with the estimates of the population mean and the standard deviation. The K–S test for the normal distribution with unknown mean and standard deviation is called Lilliefors's test and is discussed in Section 4.5. There we also show the MINITAB output using data from Example 4.5.1.

Next, we consider an application of the K–S test statistic to determine sample size.

4.4.3 Determination of Sample Size

In the case of a point estimate, the D_n statistic enables us to determine the minimum sample size required to guarantee, with a certain probability $1 - \alpha$, that the error in the estimate never exceeds a fixed positive value c. This allows us to formulate the sample size determination problem as follows. We want to find the minimum value of n that satisfies

$$P(D_n < c) = 1 - \alpha$$

This is equivalent to saying

$$1 - P(D_n < c) = P(D_n > c) = \alpha$$

and therefore c equals the value of $D_{n,\alpha}$ given in Table F. This means that the value of n can be read directly from Table F as that sample size

FIGURE 4.4.1
SAS interactive data analysis for Example 4.4.1.

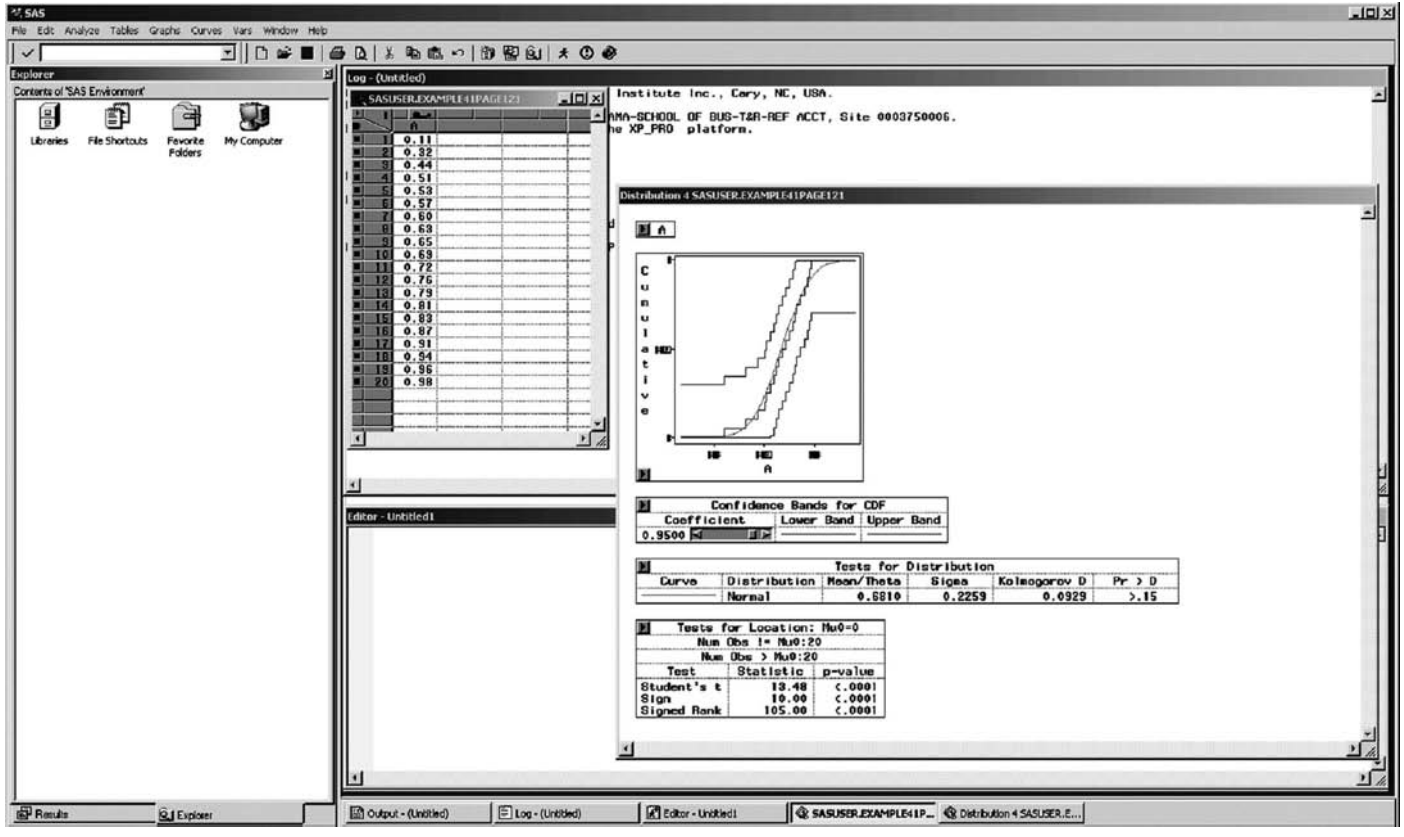


FIGURE 4.4.2
SAS interactive data analysis output for confidence band for Example 4.4.1.

corresponding to $D_{n,\alpha} = c$. If no $n \leq 40$ will meet the specified accuracy, the asymptotic distribution of Theorem 4.3.3 can be used by solving for n in $c = d/\sqrt{n}$, where $d/\sqrt{n}$ is given in the last row of Table F.

For example, suppose we want to take a sample of size n and use the resulting $S_n(x)$ as a point estimate of $F_X(x)$ for each x. We want the error in this estimate to be no more than 0.25 with probability 0.98. How large a sample should be taken? We look down the $0.02 = 1 - 0.98$ column in Table F until we find the largest c that is less than or equal to 0.25. This entry is 0.247, which corresponds to a sample size $n = 36$. If we want more precision in our point estimate and thereby specify a maximum error of 0.20 but keep the probability at 0.98, Table F shows that $n > 40$. The value is found by solving $1.52/\sqrt{n} = 0.20$ and we get $n = 57.76$, which is rounded up to require a sample size of 58 observations.

It should be noted that all of the theoretical properties of the K–S statistics require the assumption that F_X be continuous, since this is necessary to guarantee their distribution-free nature. The properties of the empirical distribution function given in Section 2.3, including the Glivenko–Cantelli theorem, do not require this continuity assumption. Furthermore, it is certainly desirable to have a goodness-of-fit test which can be used when the hypothesized distribution is discrete. Noether (1967, pp. 17–18) and others have shown that if the $D_{n,\alpha}$ values based on a continuous F_X are used in a discrete application, the significance level is at most α. However, Slakter (1965) used Monte Carlo techniques to show that this procedure is extremely conservative. Conover (1972) found recursive relationships for the exact distribution of D_n for F_0 discrete.

Pettitt and Stephens (1977) give tables of exact tail probabilities of nD_n that can be used with F_0 discrete and also for grouped data from a continuous distribution as long as the expected frequencies are equal. They show that in these two cases, the test statistic can be written as

$$nD_n = \max_{1 \leq j \leq k} \left| \sum_{i=1}^{j} (F_i - e_i) \right|$$

Because

$$S_n(x_j) = \sum_{i=1}^{j} \frac{F_i}{n} \quad \text{and} \quad F_0(x_j) = \sum_{i=1}^{j} \frac{e_i}{n}$$

for ordered $x_1 \leq x_2 \leq \cdots \leq x_n$. This expression shows that the distribution of D_n depends on the chosen grouping and also shows explicitly the relationship between D_n and the chi-square statistic Q.

This exact test has greater power than the chi-square test in the case of grouped data.

4.5 Lilliefors's Test for Normality

In this section, we consider the problem of a goodness-of-fit test for the normal distribution with no specified mean and variance. This problem is very important in practice because the assumption of a general normal distribution with unknown μ and σ is necessary to so many classical statistical inference procedures. In this case, note that the null hypothesis is composite because it states that the underlying distribution is some normal distribution. In general, K–S tests can be applied in the case of composite goodness-of-fit hypotheses after estimating the unknown parameters [$F_0(x)$ will then be replaced by $\hat{F}_0(x)$]. Unfortunately, the null distribution of the K–S test statistic with estimated parameter is far more complicated. In the absence of any additional information, one approach could be to use the tables of the K–S test to approximate the P value or to find the approximate critical value. For the normal distribution, Lilliefors (1967) showed that using the usual critical points developed for the K-S test gives extremely conservative results. He then used Monte Carlo simulations to develop a table for the K–S statistic that gives accurate critical values. As before, the K–S two-sided statistic is defined as

$$D_n = \sup_x |S_n(x) - \hat{F}_0(x)].$$

Here $\hat{F}_0(x)$ is computed as the cumulative standard normal distribution $\Phi(z)$ where $z = (x - \bar{x})/s$ for each observed x, $\bar{x}$ is the mean of the sample of n observations, and s^2 is the unbiased estimator of σ^2 (computed with $n - 1$ in the denominator). The appropriate rejection region is in the right tail and Table O gives the exact tail probabilities computed by Monte Carlo simulations. This table is taken from Edgeman and Scott (1987), in which more samples were used to improve the accuracy of the original results given by Lilliefors (1967).

Example 4.5.1

A random sample of 12 persons are interviewed to estimate median annual gross income in a certain economically depressed town. Use the most appropriate test for the null hypothesis that income data are normally distributed.

9,800	10,200	9,300	8,700	15,200	6,900
8,600	9,600	12,200	15,500	11,600	7,200

Solution

Since the mean and variance are not specified, the most appropriate test is the Lilliefors's test. The first step is to calculate $\bar{x}$ and s. From the data we get $\sum x = 124,800$ and $\sum (x - \bar{x})^2 = 84,600,000$ so that $\bar{x} = 10,400$ and

TABLE 4.5.1

Calculations for Lilliefors's Statistic in Example 4.5.1

x	z	$S_n(x)$	$\Phi(z)$	$\lvert S_n(x) - \Phi(z)\rvert$	$\lvert S_n(x-\varepsilon) - \Phi(z)\rvert$
6900	−1.26	0.0833	0.1038	0.0205	0.1038
7200	−1.15	0.1667	0.1251	0.0416	0.0418
8600	−0.65	0.2500	0.2578	0.0078	0.0911
8700	−0.61	0.3333	0.2709	0.0624	0.0209
9300	−0.40	0.4167	0.3446	0.0721	0.0113
9600	−0.29	0.5000	0.3859	0.1141	0.0308
9800	−0.22	0.5833	0.4129	0.1704	0.0871
10200	−0.07	0.6667	0.4721	0.1946	0.1112
11600	0.43	0.7500	0.6664	0.0836	0.0003
12200	0.65	0.8333	0.7422	0.0911	0.0078
15200	1.73	0.9167	0.9582	0.0415	0.1249
15500	1.84	1.0000	0.9671	0.0329	0.0504
			1.0000		0

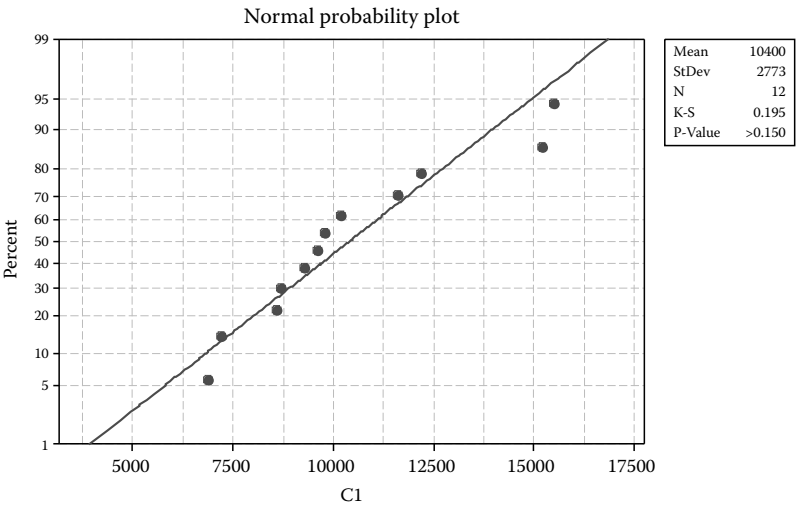
$s = \sqrt{84,600,000/11} = 2,773.25$. The corresponding standard normal variable is then $z = (x - 10{,}400)/2{,}773$. The calculations needed for D_n are shown in Table 4.5.1. We find $D_n = 0.1946$ and $P > 0.10$ for $n = 12$ from Table O. Thus, the null hypothesis that incomes are normally distributed is not rejected.

The following computer printouts illustrate the solution to Example 4.5.1 using the SAS and MINITAB packages.

```
**********************************
SAS/ANALYST SOLUTION TO EXAMPLE 4.5.1
**********************************

      The univariate procedure
      Fitted distribution for A

Parameters for Normal Distribution

   Parameter    Symbol    Estimate
   Mean         Mu           10400
   Std Dev      Sigma     2773.249

Goodness-of-Fit Tests for Normal Distribution

Test                   ---Statistic----    ----P Value-----
K-S                    D       0.19541250  Pr > D       >0.150
Cramer-von Mises       W-Sq    0.07074246  Pr > W-Sq    >0.250
Anderson-Darling (A-D) A-Sq    0.45877863  Pr > A-Sq     0.223
```

Note that both the MINITAB and SAS outputs refer to this as the K–S test and not the Lilliefors's test. Both calculate a modified K–S statistic using formulas given in D'Agostino and Stephens (1986); the results agree with ours to two decimal places. SAS also provides the results for two other tests, called the A–D and the Cramér–von Mises tests (see Problem 4.14). The A–D test is discussed in Section 4.7. In this particular example, each of the goodness-of-fit tests fails to reject the null hypothesis of normality. In MINITAB one can go from Stat to Basic Statistics to Normality Test and choose one of three tests: A–D, Ryan–Joiner (similar to Shapiro–Wilk), and K–S. The output is a graph, called normal probability plot along with some summary statistics, the value of the test statistic and a P value. The grid on this graph resembles that found on normal probability paper. The vertical axis is a probability scale and the horizontal axis is the usual data scale. The plotted points are the $S_n(x)$ values; the straight line is a least-squares line fitted to these points. If the points fall reasonably close to this straight line, the normal distribution hypothesis is supported. The P value given by both packages is an approximation based on linear interpolation in tables that are not the same as our Table O. See the documentation in the packages and D'Agostino and Stephens (1986) for additional details. Note that the P values obtained by MINITAB and SAS can be different. For example, for these data, MINITAB finds the P value for the A–D test to be 0.215 whereas the SAS P value shown on the output is 0.223.

```
**********************************
MINITAB SOLUTION TO EXAMPLE 4.5.1
**********************************
```



The SAS software package also provides a number of excellent (interactive) ways to study goodness-of-fit. We have given one illustration already for confidence bands using data from Example 4.4.1. Now we illustrate some of the other possibilities in Figures 4.5.1 through 4.5.5 using the data from

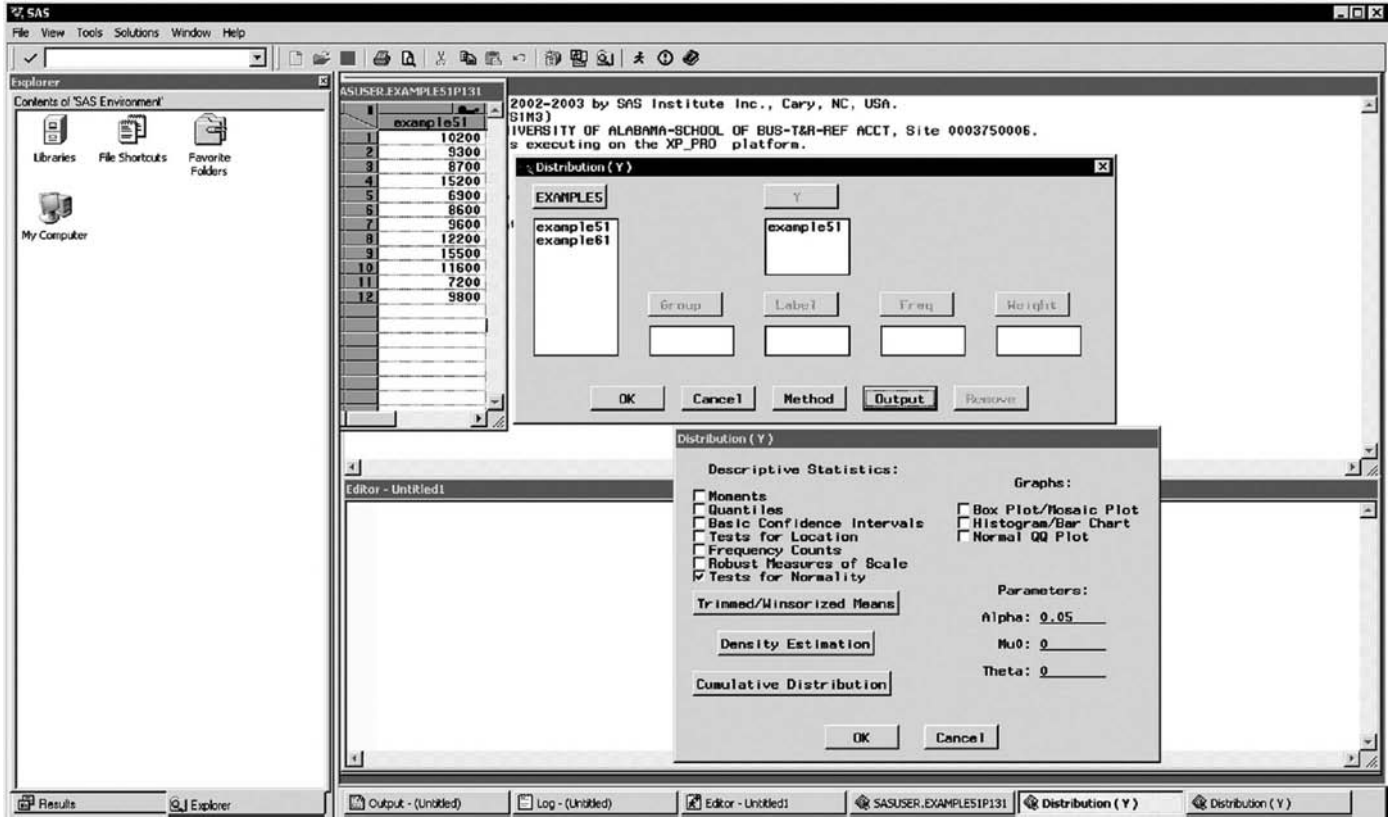
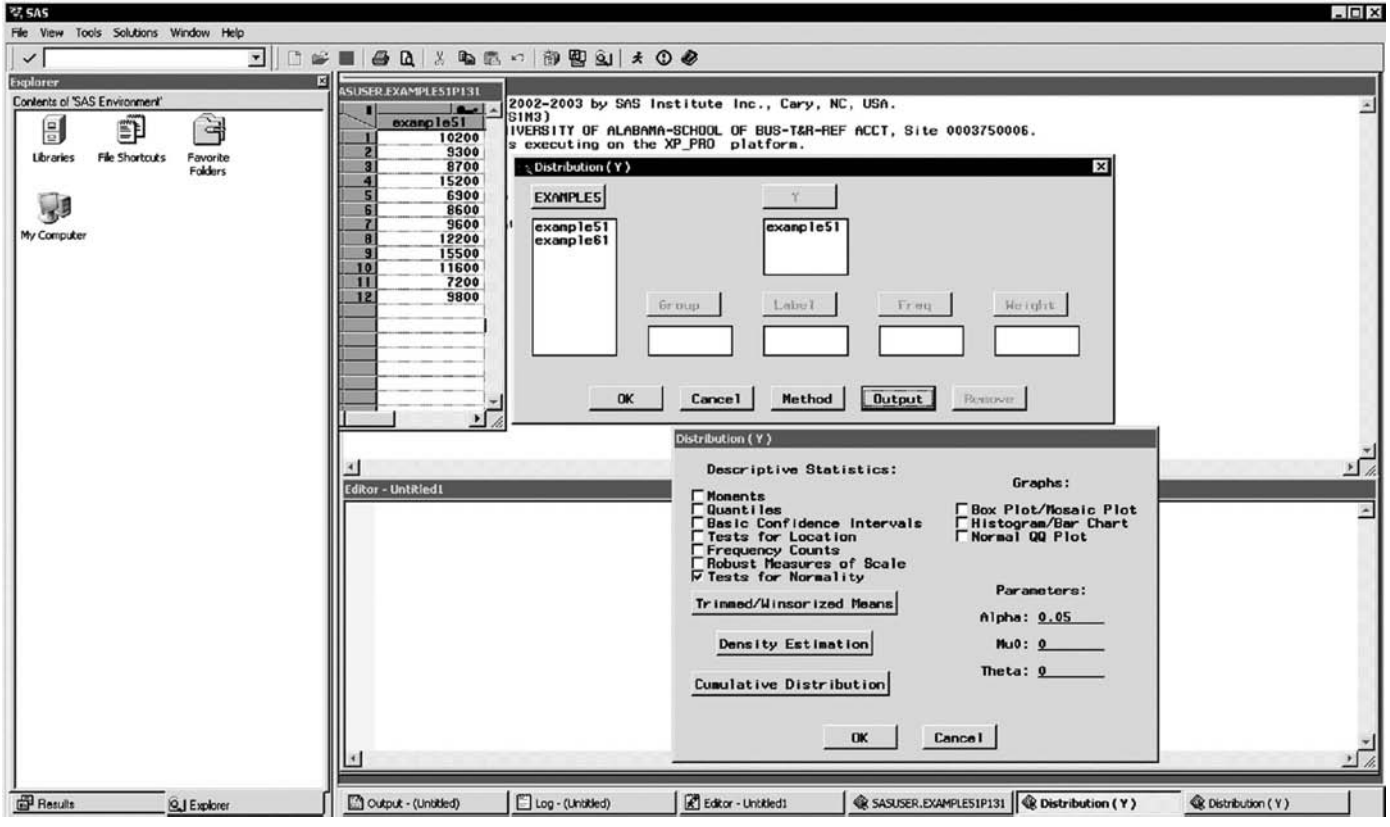


FIGURE 4.5.1
SAS/interactive data analysis of goodness of fit for Example 4.5.1.



	example51
1	10200
2	9300
3	8700
4	15200
5	6300
6	8600
7	9600
8	12200
9	15500
10	11600
11	7200
12	9800

FIGURE 4.5.2
SAS/interactive data analysis of goodness of fit for Example 4.5.1.

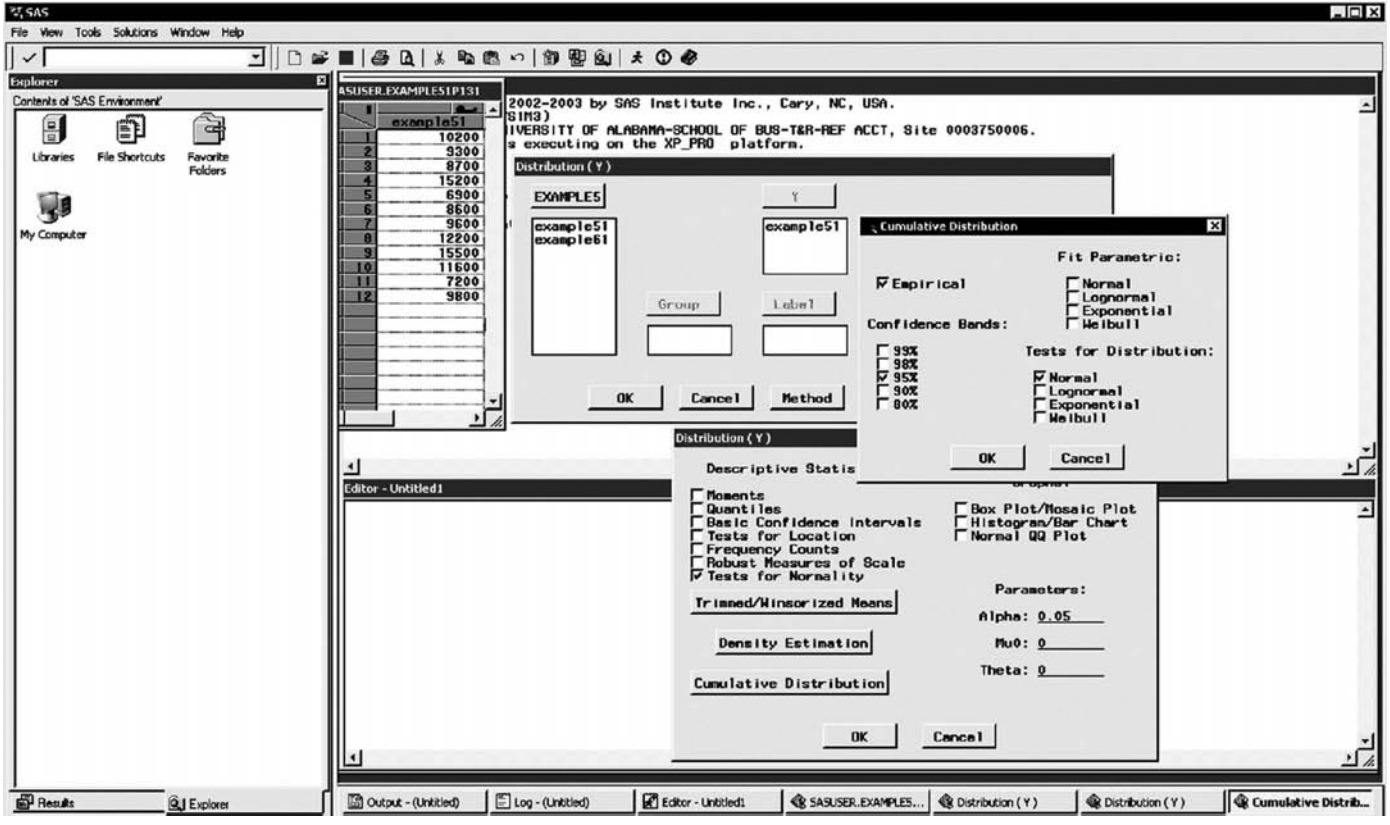


FIGURE 4.5.3
SAS/interactive data analysis of goodness of fit for Example 4.5.1.

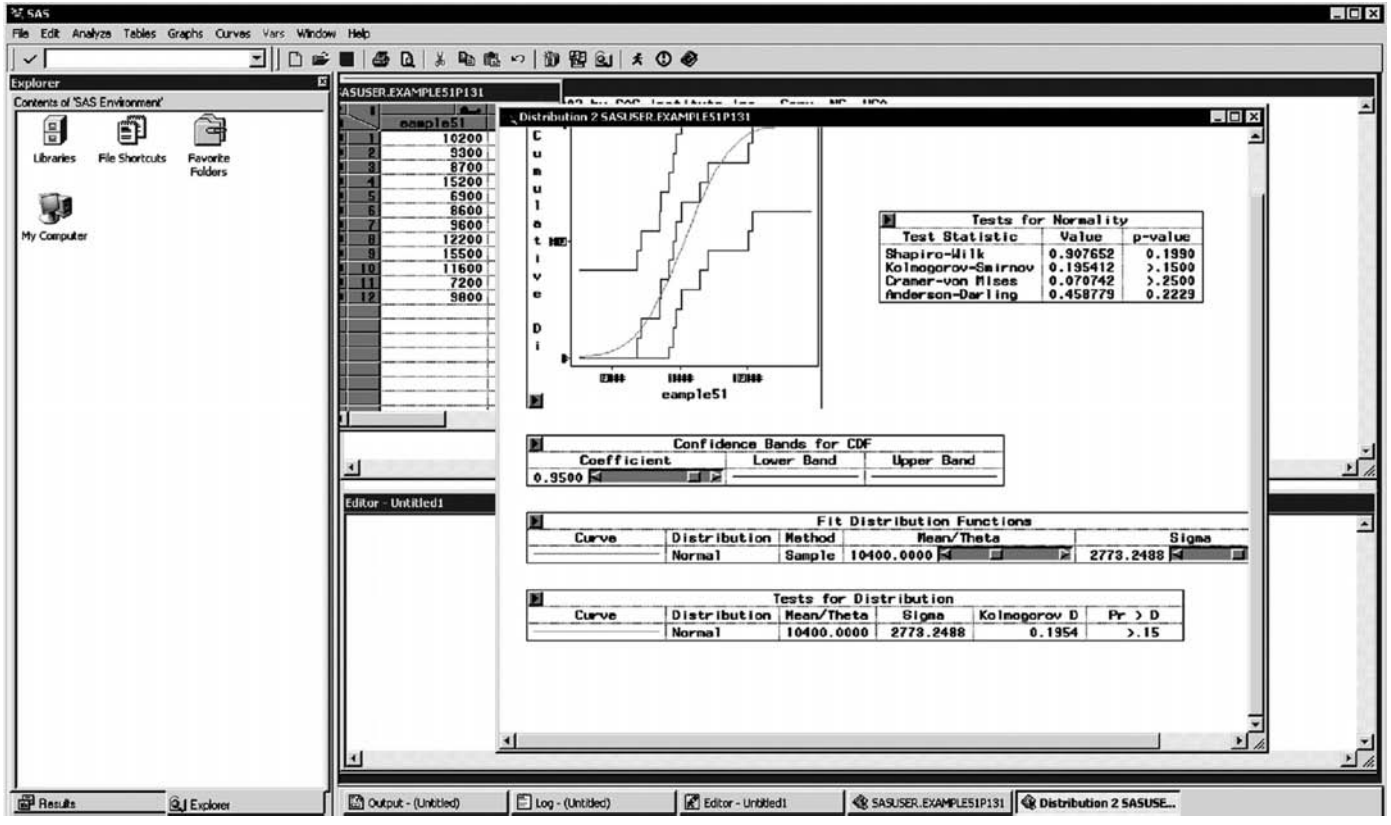


FIGURE 4.5.4
SAS/interactive data analysis of goodness of fit for Example 4.5.1.

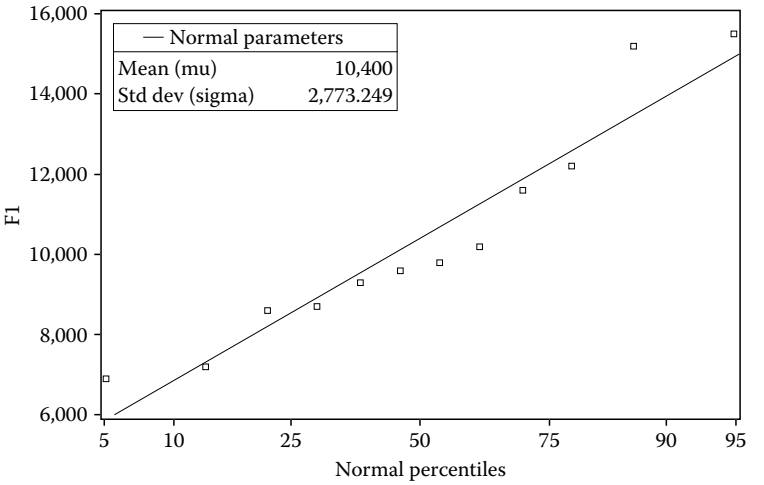


FIGURE 4.5.5
SAS/analyst *Q*–*Q* plot for Example 4.5.1.

Example 4.5.1 and the Interactive Data Analysis and the Analyst options, respectively, under SAS version 9.1. The highlights include a plot of the empirical cdf, a box plot, confidence bands, tests for a specific distribution the choices for which includes the normal, the lognormal, the exponential and the Weibull, a *Q*–*Q* plot (discussed in Section 4.8) together with a reference line, and other available options. Also interesting is a slider (the bottom panel in the output shown) where one can ''try out'' various means and standard deviations to be compatible with the data on the basis of the K–S test. For details on the features, the reader is referred to SAS version 9.1 online documentations.

4.6 Lilliefors's Test for the Exponential Distribution

Another important goodness-of-fit problem in practice is to test for the exponential distribution with no specified mean. The assumption of an exponential distribution with an unknown mean is made in many applications, particularly where the random variable under study is a waiting time, the time to the occurrence of an event. Lilliefors (1969) developed an analog of the K–S test in this situation and gave a table of critical values based on Monte Carlo simulations. As in the normal case with unknown parameters, the K–S two-sided statistic is defined as

$$D_n = \sup_x |S_n(x) - \hat{F}_0(x)|$$

Here, $\hat{F}_0(x)$ is computed as $1 - e^{-x/\bar{x}} = \hat{F}_0(z) = 1 - e^{-z}$, say, where $\bar{x}$ is the sample mean and $z = x/\bar{x}$ for each observed x. Thus, one forms the standardized variable $z_i = x_i/\bar{x}$, for each observed x_i, and calculates the usual K-S statistic between the empirical cdf and $\hat{F}_0(z_i)$. The appropriate rejection region is in the right tail and Table T gives the necessary critical values obtained by Monte Carlo simulations. This table is taken from Edgeman and Scott (1987) who used more samples to improve the accuracy of the original results given by Lilliefors (1969). Durbin (1975) provided exact quantiles.

Example 4.6.1

Test the null hypothesis that the data below arose from a one-parameter exponential distribution.

1.5, 2.3, 4.2, 7.1, 10.4, 8.4, 9.3, 6.5, 2.5, 4.6

Solution

Since the mean of the exponential distribution is not specified under the null hypothesis, we estimate it from the data by $\bar{x} = 5.68$. The standardized variable is $z = x/5.68$. The calculations for the Lilliefors's test are summarized in Table 4.6.1. We find $D_n = 0.233$ and using Table T with $n = 10$, the approximate P value is greater than 0.10.

MINITAB does not provide the Lilliefors's test for the exponential distribution, although a graphical output including the probability plot and the A–D test is available under Graph, Probability, Plot as well as under

TABLE 4.6.1

Calculations for Lilliefors's Test in Example 4.6.1

x	z	$S_n(x)$	$\hat{F}_0(x) = F_0(z)$	$\lvert S_n(x) - F_0(z)\rvert$	$\lvert S_n(x-\varepsilon) - F_0(z)\rvert$
1.5	0.26	0.1000	0.2321	0.1321	0.2321
2.3	0.40	0.2000	0.3330	0.1330	0.2330
2.5	0.44	0.3000	0.3561	0.0561	0.1561
4.2	0.74	0.4000	0.5226	0.1226	0.2226
4.6	0.81	0.5000	0.5551	0.0551	0.1551
6.5	1.14	0.6000	0.6816	0.1816	0.1816
7.1	1.25	0.7000	0.7135	0.0135	0.1135
8.4	1.48	0.8000	0.7721	0.0279	0.0721
9.3	1.64	0.9000	0.8055	0.0945	0.0055
10.4	1.83	1.0000	0.8397	0.1603	0.0603

Stat, Quality tools, and Individual Distribution Identification. The outputs from SAS/ANALYST and MINITAB for Example 4.6.1 are shown next. Note that the K–S test with the estimated mean is not referred to as Lilliefors's test but the numerical result agrees with ours. The approximate P value for Lilliefors's test (K–S test on the printout) is shown to be greater than 0.15. Thus, as with the hand calculations, we reach the same conclusion that there is not sufficient evidence to reject the null hypothesis of an exponential distribution. SAS uses internal tables that are similar to those given by D'Agostino and Stephens (1986) to calculate the P value. Linear interpolation is used in this table if necessary. As noted earlier, SAS also provides the values of two other test statistics, called the Cramér–von Mises and A–D tests; each fails to reject the null hypothesis and the P values are about the same. The A–D test is discussed in Section 4.7.

```
**********************************
SAS/ANALYST SOLUTION TO EXAMPLE 4.6.1
**********************************

      The univariate procedure
      Fitted distribution for A

Parameters for Exponential Distribution

  Parameter     Symbol     Estimate

  Threshold     Theta             0
  Scale         Sigma          5.68
  Mean                         5.68
  Std Dev                      5.68

   Goodness-of-Fit Tests for Exponential Distribution

Test              ----Statistic---   -----P Value----

K-S               D      0.23297622   Pr > D       >0.150
Cramer-von Mises  W-Sq   0.16302537   Pr > W-Sq     0.117
A-D               A-Sq   0.94547938   Pr > A-Sq     0.120
```

We now redo Example 4.6.1 for the null hypothesis of the exponential distribution with mean specified as $b = 5.0$. This is a simple null hypothesis for which the original K–S test of Section 4.5 is applicable. The calculations are shown in Table 4.6.2. The K–S test statistic is $D_n = 0.2687$ with $n = 10$ and we do not reject the null hypothesis since Table F gives $P > 0.200$.

The SAS solution in this case is shown below. Each of the tests fails to reject the null hypothesis and the P values are about the same.

TABLE 4.6.2

Calculations for the K–S Test with $b = 5.0$ for the Data in Example 4.6.1

x	$z = x/b$	$S_n(x)$	$F_0(z)$	$\lvert S_n(x) - F_0(z)\rvert$	$\lvert S_n(x-\varepsilon) - F_0(z)\rvert$
1.5	0.30	0.1	0.2592	0.1592	0.2592
2.3	0.46	0.2	0.3687	0.1687	0.2687
2.5	0.50	0.3	0.3935	0.0935	0.1935
4.2	0.84	0.4	0.5683	0.1683	0.2683
4.6	0.92	0.5	0.6015	0.1015	0.2015
6.5	1.30	0.6	0.7275	0.1275	0.2275
7.1	1.42	0.7	0.7583	0.0583	0.1583
8.4	1.68	0.8	0.8136	0.0136	0.1136
9.3	1.86	0.9	0.8443	0.0557	0.0443
10.4	2.08	1.0	0.8751	0.1249	0.0249

```
*********************************************
SAS/ANALYST SOLUTION TO EXAMPLE 4.6.1 WITH MEAN 5
*********************************************

          The UNIVARIATE Procedure
          Fitted distribution for A
    Parameters for Exponential Distribution

      Parameter    Symbol    Estimate

      Threshold    Theta            0
      Scale        Sigma            5
      Mean                          5
      Std Dev                       5

  Goodness-of-Fit Tests for Exponential Distribution

Test                ---Statistic---    ----P value-----
K-S                 D      0.26871635  Pr > D       >0.250
Cramer-von Mises    W-Sq   0.24402323  Pr > W-Sq     0.221
A-D                 A-Sq   1.29014013  Pr > A-Sq     0.238
```

Finally, suppose that we want to test the null hypothesis that the population is exponential with mean $b = 3.0$. Again, this is a simple null hypothesis for which SAS provides the three tests mentioned earlier and all of them reject the null hypothesis. However, note the difference in the magnitudes of the P values between the K–S test and the other two tests.

Thus the data are found to be consistent with an exponential distribution with mean 5 but not one with mean 3.

```
**********************************************
SAS/ANALYST SOLUTION TO EXAMPLE 4.6.1 WITH MEAN 3
**********************************************

The UNIVARIATE procedure
Fitted distribution for A

Parameters for Exponential Distribution

   Parameter     Symbol     Estimate

   Threshold     Theta             0
   Scale         Sigma             3
   Mean                            3
   Std Dev                         3

Goodness-of-Fit Tests for Exponential Distribution

Test                ---Statistic----     ----P Value----
K-S                 D       0.45340304   Pr > D       0.023
Cramer-von Mises    W-Sq    0.87408416   Pr > W-Sq    0.004
A-D                 A-Sq    4.77072898   Pr > A-Sq    0.004
```

4.7 Anderson–Darling (A–D) Test

Now we discuss another popular goodness-of-fit test that is available in several of the software packages, including MINITAB. The Anderson–Darling (A–D) (1954) test is based on the difference between the empirical distribution function and the hypothesized cdf $F_0(x)$, but here the difference here is measured in terms of the square instead of the absolute value used in the K–S and Lilliefors's tests. The A–D test is a special case of a general class of goodness-of-fit statistics proposed by Anderson and Darling (1952) as

$$W_n^2 = n \int_{-\infty}^{\infty} [S_n(x) - F_0(x)]^2 \psi[F_0(x)] dF_0(x)$$

where ψ is some non-negative weight function chosen by the experimenter to accentuate those values of $S_n(x) - F_0(x)$ that give the test the greatest sensitivity.

When $\psi(u) = 1$, W_n^2 is equal to n times the Cramer–von-Mises goodness-of-fit statistic (see Problem 4.14). The A–D test statistic uses $\psi(u) = [u(1-u)]^{-1}$, $0 < u < 1$. The motivation behind this choice is interesting. Since $nS_n(x)$ has a binomial distribution, the variance of $S_n(x)$ is $n^{-1}F_0(x)[1 - F_0(x)]$. Thus, the weight for the A–D test is the reciprocal of the variance under H_0, which

has the effect of emphasizing the differences between the empirical and the hypothesized distributions for more extreme values of x, both small and large. In other words, this choice gives more weight to the tails of the distribution than the middle. Since the difference between the empirical and the hypothesized distributions is likely to be more pronounced in the tails, this is a good choice and the resulting test has good power against a wide range of distributions as shown in Anderson and Darling (1954).

Under this choice of the weight function, the A–D test statistic can be shown to simplify to

$$W_n^2 = -n - \frac{1}{n}\sum_{j=1}^{n}(2j-1)[\ln Z_j + \ln(1 - Z_{n-j+1})] \tag{4.7.1}$$

where $Z_j = F_0(X_{(j)})$ and the $X_{(j)}$ are the order statistics of the random sample from F_0. A proof of this is left for the reader. Thus as for the K–S one-sample statistic discussed in Section 4.3, the null distribution of the A–D statistic depends only on the random variables Z_j, $j = 1, 2, \ldots, n$, which is a beta distribution with $a = j$, $b = n - j + 1$, and A–D test is therefore distribution-free. Also, as we argued for the K–S test, larger values of the test statistic W_n^2 tend to discredit the null hypothesis and hence the size α test rejects $H_0 : F_X(x) = F_0(x)$ against the two-sided alternative $H_1 : F_X(x) \neq F_0(x)$ when $W_n^2 > E_{n,\alpha}$, where $E_{n,\alpha}$ is the upper α percentile of the null distribution of the test statistic. The P value is calculated in the upper tail.

The exact sampling distribution of W_n^2 in (4.7.1) is rather complicated. When F_0 is a completely specified continuous distribution function, Anderson and Darling (1954) obtain the asymptotic distribution and provide a table of percentage points useful for hypothesis testing. Empirical studies suggest that the exact sampling distribution approaches the asymptotic distribution rather rapidly. Anderson and Darling (1954) claim that the asymptotic percentiles are safe to use for a sample size as small as 40. Stephens (1986) supports this view but claims that for practical purposes, the asymptotic distribution can be used for sample sizes as small as 5.

The theory of the A–D test applies exactly when F_0 is completely specified under H_0. But as we have already observed, this is frequently not the case, for example, when the null hypothesis is that the underlying distribution is of a specified form but some parameters are unknown or unspecified. As we did in the case of the Lilliefors's test for the normal and exponential distributions, we estimate the unknown parameters using maximum likelihood and calculate the goodness-of-fit statistic in (4.7.1) using these estimators. However, this changes the null distribution of the test statistic and, in fact, the resulting test is no longer distribution free. More advanced methods, including simulations, can be used to calculate or estimate the required percentage points to apply the modified test, as Lilliefors did.

A similar approach is adopted with the A–D test when some parameters are unknown in the underlying distribution under H_0. We estimate the

unknown parameters and calculate the ''estimated'' values of the Z statistics and thus calculate a modified W_n^2 statistic denoted by A^*. In general, the distribution of the A–D test statistic depends on the sample size n as well as the values of the unknown parameters. Stephens (1986) used various methods, including extensive Monte Carlo simulations, to study the distributions of various modifications of the original A–D statistic. The proposed modifications and the approximate upper tail percentage points are shown in Table 4.7.1 for each of three null hypothesis situations (a) completely specified distribution, (b) normal distribution with (1) unknown mean, (2) unknown variance, and (3) unknown mean and variance, and (c) exponential distribution. For situation (a), no modifications to W_n^2 have been suggested and the same is true for cases 1 and 2 of situation (b), the normal distribution. The given percentage points can be used with the unmodified W_n^2 for $n > 20$. Stephens (1986) also provides tables and expressions for the P values.

Example 4.7.1

We illustrate the A–D test using the data from Example 4.6.1 and first test the null hypothesis that the distribution is exponential with an unknown mean. As before, we use the maximum-likelihood estimator of the unknown mean, the sample mean 5.68. The next step is to calculate $Z_j = F_0(x;\hat{\theta})$ for the ordered sample values using the cdf of the exponential distribution, $1 - e^{-X_j/\bar{X}}$, as shown in Table 4.7.2. We find estimated $W_n^2 = 0.9455$ and hence $A^* = 0.9738$. When we compare 0.9738 with the entries in the last row of Table 4.7.1, we conclude that the null hypothesis cannot be rejected at any of the given significance levels. Thus, there is insufficient evidence to reject the null hypothesis that the data come from an exponential distribution. Note that the same conclusion was reached with the Lilliefors's test.

TABLE 4.7.1

A–D Test Statistic Modifications and Upper Tail Percentage Points

	Significance Level α				
	0.01	**0.025**	**0.05**	**0.10**	**0.15**
(a) F_X completely specified; use W_n^2 in (4.7.1) with no modification	3.857	3.070	2.492	1.933	1.610
(b) Normal distribution					
Case 1: Unknown mean,	1.551	1.285	1.087	0.894	0.782
Case 2: Unknown variance,	3.702	2.898	2.308	1.743	1.430
Case 3: Unknown mean and variance, Use $A^* = W_n^2(1 + 0.75/n + 2.25/n^2)$	1.035	0.873	0.752	0.631	0.561
(c) Exponential distribution					
Unknown mean, use $A^* = W_n^2(1 + 0.3/n)$	1.959	1.591	1.321	1.062	0.916

Source: Stephens, M.A., *Tests Based on EDF Statistics in Goodness-of-Fit Techniques*, Marcel Dekker, New York, 1986.

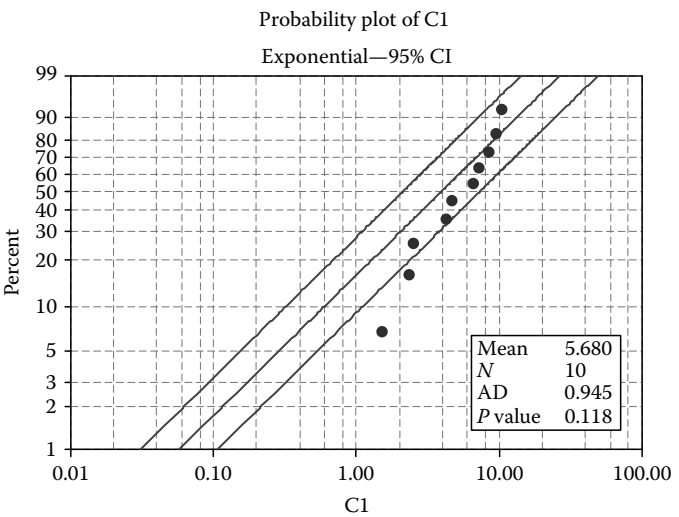
TABLE 4.7.2

Calculations for the A–D Test for the Data in Example 4.6.1

j	Ordered x_j	$x_j/\bar{x}$	$z_j = 1 - e^{-x_j/\bar{x}}$	$\ln z_j$	$1 - z_{11-j}$	$\ln(1 - z_{11-j})$	$\frac{-(2j-1)[\ln z_j + \ln(1 - z_{11-j})]}{10}$
1	1.5	0.2641	0.2321	−1.4606	0.1603	−1.8310	0.3292
2	2.3	0.4049	0.3330	−1.0997	0.1945	−1.6373	0.8211
3	2.5	0.4401	0.3561	−1.0327	0.2279	−1.4789	1.2558
4	4.2	0.7394	0.5226	−0.6489	0.2865	−1.2500	1.3292
5	4.6	0.8099	0.5551	−0.5886	0.3184	−1.1444	1.5597
6	6.5	1.1444	0.6816	−0.3833	0.4449	−0.8099	1.3125
7	7.1	1.2500	0.7135	−0.3376	0.4774	−0.7394	1.4001
8	8.4	1.4789	0.7721	−0.2586	0.6439	−0.4401	1.0482
9	9.3	1.6373	0.8055	−0.2163	0.6670	−0.4049	1.0561
10	10.4	1.8310	0.8397	−0.1747	0.7679	−0.2641	0.8336
						Sum = 10.9455	$W_n^2 = -10 + 10.9455$ $= 0.9455$

The MINITAB output for these data is shown below from the probability plot option. The calculated value of the A–D test statistic is equal to our hand-calculated value of the unmodified W_n^2 statistic from Table 4.7.2. MINITAB provides an approximate P value for W_n^2 but does not calculate the modified version of the test statistic, A^*, which is recommended when the parameter is unknown.

MINITAB Output for Example 4.7.1



Next, we use the same data from Example 4.6.1 to test the null hypothesis that the distribution is exponential with a specified mean of $b = 5.0$. This is a

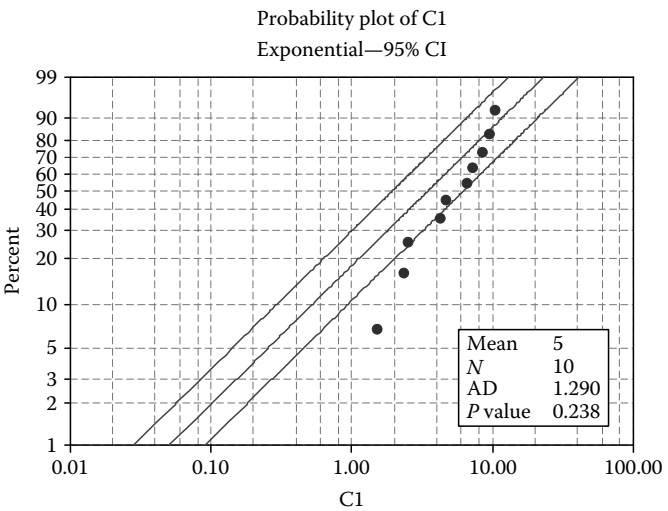
TABLE 4.7.3

Calculations for the A–D Test with $\mu = 5.0$ for the Data in Example 4.6.1

j	Ordered x_j	$x_j/\bar{x}$	$z_j = 1 - e^{-x_j/5}$	$\ln z_j$	$1 - z_{11-j}$	$\ln(1 - z_{11-j})$	$\frac{-(z_{j-1})[\ln z_j + \ln(1 - z_{11-j})]}{10}$
1	1.5	0.3000	0.2592	−1.3502	0.1249	−2.0800	0.3430
2	2.3	0.4600	0.3687	−0.9977	0.1557	−1.8600	0.8573
3	2.5	0.5000	0.3935	−0.9328	0.1864	−1.6800	1.3064
4	4.2	0.8400	0.5683	−0.5651	0.2417	−1.4200	1.3896
5	4.6	0.9200	0.6015	−0.5084	0.2725	−1.3000	1.6275
6	6.5	1.3000	0.7275	−0.3182	0.3985	−0.9200	1.3620
7	7.1	1.4200	0.7583	−0.2767	0.4317	−0.8400	1.4517
8	8.4	1.6800	0.8136	−0.2063	0.6065	−0.5000	1.0594
9	9.3	1.8600	0.8443	−0.1692	0.6313	−0.4600	1.0697
10	10.4	2.0800	0.8751	−0.1335	0.7408	−0.3000	0.8236
						Sum = 11.2902	$W_n^2 = -10 + 11.2902 = 1.2902$

completely specified null hypothesis for which the original A–D test is applicable with no modification. The calculations in Table 4.7.3 show that $W_n^2 = 1.2902$, which is found to be not significant (note that in this case the percentage points are read from row (a) of Table 4.7.1). Thus, there is insufficient evidence to reject the hypothesis that the data follow an exponential distribution with mean 5.0.

MINITAB Output for Example 4.6.1 with $b = 5.0$



The MINITAB output for this case is shown below again using the probability plot option and a specified mean of 5.0. Note that the calculated value of the test statistic is equal to our hand-calculated value of the W_n^2 statistic. This is the correct approach since the null hypothesis completely specifies the distribution. MINITAB also provides an approximate P value. The conclusion is the same.

The reader may verify that repeating these calculations for a specified mean of 4.0 yields $W_n^2 = 2.3618$, which does lead to the rejection of the null hypothesis at $\alpha = 0.10$ (but not at any other α).

4.8 Visual Analysis of Goodness of Fit

With the advent of easily available computer technology, visual approaches to statistical data analysis have become popular. The subject is sometimes referred to as exploratory data analysis (EDA), championed by statisticians like John Tukey. In the context of goodness-of-fit tests, the EDA tools include dot plots, histograms, probability plots, and quantile plots. The idea is to use some graphics to gain a quick insight into the underlying distribution and then, if desired, carry out a follow-up analysis with a formal confirmatory test using any of the tests in this chapter. Dot plots and histograms are valuable exploratory tools and are discussed in almost all statistics books but probability and quantile plots are seldom covered, even though one of the key papers on the subject was published in the 1960s (Wilk and Gnanadesikan, 1968). In this section we will present a brief discussion of these two topics. Fisher (1983) provides a good review of many graphical methods used in nonparametric statistics along with extensive references. Note that there are two-sample versions of each of these plots but we do not cover them here.

We first must distinguish between the theoretical and the empirical versions of a plot. The theoretical version is presented to understand the idea but the empirical version is the one that is implemented in practice. When there is no chance of confusion, the empirical plot is referred to as, simply, the plot.

Two types of plots are popular in practice. The first is the so-called probability plot, which is actually a probability versus probability plot, or a P–P plot. This plot is also called a percent-percent plot, for obvious reasons. In general terms, the theoretical P–P plot is the graph of a cdf $F(x)$ versus a cdf $G(x)$ for all values of x. Since the cdf's are probabilities, the P–P plot is confined to the unit square. If the two cdfs are identical, the theoretical P–P plot will be the main diagonal, the 45° line through the origin.

The second type of plot is the so-called quantile plot, which is actually a quantile versus quantile plot, or a Q–Q plot. The theoretical Q–Q plot is

the graph of the quantiles of a cdf F versus the corresponding quantiles of a cdf G, that is, the graph $[F^{-1}(p), G^{-1}(p)]$ for $0 < p < 1$. If the two cdf's are identical, the theoretical Q–Q plot will be the main diagonal, the 45° line through the origin. If $F(x) = G[(x - \mu)/\sigma]$, it is easily seen that $F^{-1}(p) = \mu + \sigma G^{-1}(p)$, so that the pth quantiles of F and G have a linear relationship. Thus, if two distributions differ only in location and/or scale, the theoretical Q–Q plot will be a straight line with slope σ and intercept μ.

In a goodness-of-fit problem, there is usually a specified target cdf, say F_0. Then the theoretical Q–Q plot is the plot $\left[F_0^{-1}(p), F_X^{-1}(p)\right]$, $0 < p < 1$. Since F_X is unknown, we can estimate it with the empirical cdf based on a random sample of size n, say S_n. Noting that the function S_n jumps only at the ordered values $X_{(i)}$, the empirical Q–Q plot is simply the plot of $F_0^{-1}(i/n)$ on the horizontal axis versus $S_n^{-1}(i/n) = X_{(i)}$ on the vertical axis, for $i = 1, 2, \ldots, n$. As noted before, F_0 is usually taken to be the standardized form of the hypothesized cdf, so that to establish the Q–Q plot, underlying parameters, (location and/or scale) do not need to be specified. This is one advantage of the Q–Q plot. The quantities $a_i = i/n$ are called plotting positions. At $i = n$, there is a problem since $a_n = F_0^{-1}(1) = \infty$; modified plotting positions have been considered, with various objectives. One simple choice is $a_i = (i - 0.5)/n$; other choices include $a_i = i/(n + 1)$ and $a_i = (i - 0.375)/(n + 0.25)$, the latter being highly recommended by Blom (1985). Many statistical software packages graph $\left\{F_0^{-1}[(i - 0.375)/(n + 0.25)], X_{(i)}\right\}$ as the empirical Q–Q plot. For a given standardized cdf F_0, the goodness-of-fit null hypothesis $F_X = F_0$ is not rejected if this plot is approximately a 45° straight line through the origin. Departures from this line suggest the types of differences that could exist between F_X and F_0. For example, if the plot resembles a straight line but with a nonzero intercept or with a slope other than 45°, a location-scale model is indicated. This means F_X belongs to the specified family of distributions but the location and the scale parameters, namely, μ and σ, are different from the standard values. When the empirical Q–Q plot is reasonably linear, the slope and the intercept of the plot can be used to estimate the scale and location parameters, respectively. When F_0 is taken to be the standard normal distribution, the Q–Q plot is called a normal probability plot. When F_0 is taken to be the standard exponential distribution (mean = 1), the Q–Q plot is called an exponential probability plot.

In summary, either the empirical P–P or Q–Q plot can be used as an informal tool for the goodness-of-fit problem but the Q–Q plot is more popular. If the plots appear to be close to the 45° straight line through the origin, the null hypothesis $F_X = F_0$ is tentatively accepted. If the Q–Q plot is close to some other straight line, then F_X is likely to be in the hypothesized location-scale family (as F_0) and the unknown parameters can be estimated from the plot. For example, if a straight line is fitted to the empirical Q–Q plot, the slope and the intercept of the line would estimate the unknown scale

and location parameters, respectively; then the estimated distribution is $\hat{F}_X = F_0[(x - \text{intercept})/\text{slope}]$. An advantage of the Q–Q plot is that the underlying parameters do not need to be specified since F_0 is usually taken to be the standard distribution in a family of distributions. By contrast, the construction of a P–P plot requires specification of the underlying parameters, so that the theoretical cdf can be evaluated at the ordered data values. The P–P plot is more sensitive to differences in the middle part of the data distribution and the hypothesized distribution, whereas the Q–Q plot is more sensitive to the difference in the tails of the two distributions.

One potential issue with using plots in goodness-of-fit problems is that the interpretation of a plot, with respect to linearity or near linearity, is bound to be somewhat subjective. Usually, a lot of experience is necessary to make the judgment with a reasonable degree of confidence. To make such an assessment more objective, several proposals have been made. One is based on the ''correlation coefficient'' between the x and y coordinates; see Ryan and Joiner (1976) for a test in the context of a normal probability plot. For more details, see D'Agostino and Stephens (1986, Chapter 5).

Example 4.8.1

For the sample data given in Example 4.6.1 using $a_i = (i - 0.375)/(n + 0.25)$, the calculations for normal and exponential Q–Q plots are shown in Table 4.8.1.

TABLE 4.8.1

Calculations for Normal and Exponential Q–Q Plot for Data in Example 4.6.1

Ordered Data	i	Plotpos $a_i = \frac{i - 0.375}{10.25}$	Standard Normal Quantiles $\Phi^{-1}(a_i)$	Standard Exponential Quantiles $-\ln(1 - a_i)$
1.5	1	0.060976	−1.54664	0.062914
2.3	2	0.158537	−1.00049	0.172613
2.5	3	0.256098	−0.65542	0.295845
4.2	4	0.353659	−0.37546	0.436427
4.6	5	0.451220	−0.12258	0.600057
6.5	6	0.548780	0.12258	0.795801
7.1	7	0.646341	0.37546	1.039423
8.4	8	0.743902	0.65542	1.362197
9.3	9	0.841463	1.00049	1.841770
10.4	10	0.939024	1.54664	2.797281

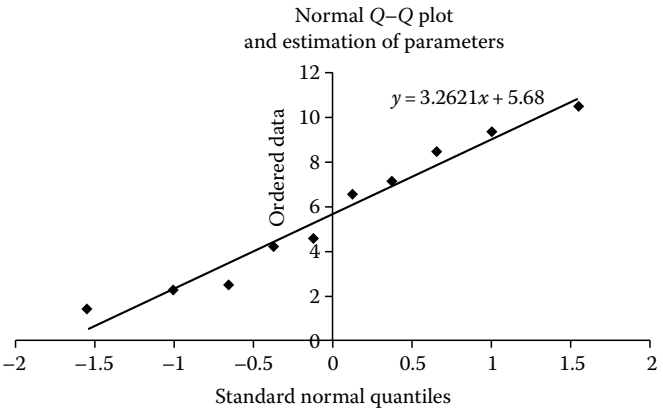


FIGURE 4.8.1
Normal *Q*–*Q* plot for Example 4.8.1.

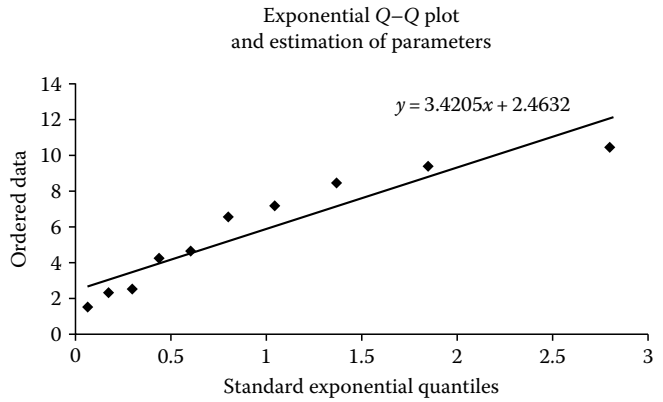


FIGURE 4.8.2
Exponential *Q*–*Q* plot for Example 4.8.1.

The two *Q*–*Q* plots are plotted in EXCEL and are shown in Figures 4.8.1 and 4.8.2. In each case, a least-squares line is fitted to the plot. The slope and intercept of the line are, respectively, the estimated scale and location parameters under the model. For these data, it appears that the normal distribution with mean 5.48 and standard deviation 3.3 provides a better fit than the exponential distribution with mean 3.52 because the points are closer to the line.

4.9 Summary

In this chapter, we presented procedures designed to help identify the population from which a random sample is drawn. The primary test criteria are the normalized sum of squares of deviations and the difference between the hypothesized and observed (sample) cumulative distributions. Chi-square tests are based on the first criterion and the K–S type tests are based on the second (including the Lilliefors's tests).

The chi-square test is specifically designed for use with categorical data, while the K–S statistics are for random samples from continuous populations. However, when the data are not categorical as collected, these two goodness-of-fit tests can be used interchangeably. The reader is referred to Goodman (1954), Birnbaum (1952), and Massey (1951c) for discussions of their relative merits. Only a brief comparison will be made here, which is relevant whenever raw ungrouped measurement data are available.

The basic difference between the two tests is that chi square is sensitive to vertical deviations between the observed and expected histograms, whereas K–S type procedures are based on vertical deviations between the observed and expected cumulative distribution functions. However, both types of deviations are useful in determining goodness-of-fit and probably are equally informative. Another obvious difference is that chi square requires grouped data whereas K–S does not. Therefore, when the hypothesized distribution is continuous, the K–S test allows us to examine the goodness-of-fit for each of the n observations, instead of only for the k classes, where $k \leq n$. In this sense, the K–S type procedures make more complete use of the available data. Further, the chi-square statistic is affected by the number of classes and their widths, which are often chosen arbitrarily by the experimenter.

One of the primary advantages of the K–S test is that the exact sampling distribution of D_n is known or can be simulated and tabulated when all parameters are specified, whereas the sampling distribution of Q is only approximately chi square for any finite n. The K–S test can be applied for any size sample, while the chi-square test should be used only for n large and each expected cell frequency not too small. When cells must be combined for chi-square application, the calculated value of Q is no longer unique, as it is affected by the scheme of combination. The K–S statistic is much more flexible than chi square, since it can be used in estimation to find minimum sample size and confidence bands. With the one-sided D_n^+ or D_n^- statistics, we can test for deviations in a particular direction, whereas the chi square is always concerned equally with differences in either direction. In most cases, K–S is easier to apply.

The chi-square test also has some advantages over K–S. A hypothesized distribution, which is discrete, presents no problems for the chi-square test, while the exact properties of D_n are violated by the lack of continuity.

As already stated, however, this is a minor problem with D_n, which can generally be eliminated by replacing equalities by inequalities in the P values. Perhaps the main advantage of the chi-square test is that by simply reducing the number of degrees of freedom and replacing unknown parameters by consistent estimators, a goodness-of-fit test can be performed in the usual manner even when the hypothesized distribution is not completely specified. If the hypothesized $F_0(x)$ in D_n contains unspecified parameters, which are estimated from the data, we obtain an estimate $\widehat{D}_n$ whose sampling distribution is different from that of D_n. The test is conservative when the D_n critical values are used. Lilliefors provided tables of the $\widehat{D}_n$ critical values for the null hypotheses of normal and exponential distributions.

As for relative performance, the power functions of the two statistics depend on different quantities. If $F_0(x)$ is the hypothesized cumulative distribution and $F_X(x)$ the true distribution, the power of K–S depends on

$$\sup_x |F_X(x) - F_0(x)|$$

while the power of chi-square test depends on

$$\sum_{i=0}^{k} \frac{\{[F_X(a_{i+1}) - F_X(a_i)] - [F_0(a_{i+1}) - F_0(a_i)]\}^2}{F_0(a_{i+1}) - F_0(a_i)}$$

where the a_i are the upper class limits in the numerical categories.

The power of the chi-square test can be improved by clever grouping in some situations. In particular, Cochran (1952) and others have shown that a choice of intervals which provide equal expected frequencies for all classes is a good procedure in this respect besides simplifying the computations. The number of classes k can be chosen such that the power is maximized in the vicinity of the point where power equals 0.5. This procedure also eliminates the arbitrariness of grouping. The expression for Q in (4.2.1) reduces to $(k \sum F_i^2 - n^2)/n$ when $e_i = n/k$ for $i = 1, 2, \ldots, k$.

Many studies of power comparisons have been reported in the literature over the years. Kac et al. (1955) showed that the K–S test is asymptotically more powerful than the chi-square test when testing for a completely specified normal distribution. Further, when the sample size is small, the K–S provides an exact test while the chi-square does not.

When the hypothesized distribution is normal or exponential and the parameters are specified (the null hypothesis is simple), the K–S test based on Table F gives an exact goodness-of-fit test. This test is conservative when parameters need to be estimated (the null hypothesis is composite). In these cases, the modified version of the D_n statistic sometimes known as the Lilliefors's test statistic should be used. The exact mathematical-statistical derivation of the distribution of this test statistic is often very complicated but Monte Carlo estimates of the percentiles of the null distribution can be

obtained. The tables given in Lilliefors (1967, 1969) were generated in this manner, as are our Tables O and T. Edgeman and Scott (1987) gave a step-by-step algorithm that included goodness-of-fit testing for the lognormal, the Rayleigh, Weibull, and the two-parameter exponential distribution.

Iman (1982) provided graphs for performing goodness-of-fit tests for the normal and the exponential distributions with unspecified parameters. These graphs are in the form of confidence bands based on Lilliefors's (1967) critical values for the normal distribution test and Durbin's (1975) critical values for the exponential distribution test. On a graph with these bands, the empirical distribution function of the standardized variable $S_n(z)$ is plotted on the vertical axis as a step function in order to carry out the test. If this plot lies entirely within the confidence bands, the null hypothesis is not rejected. These graphs can be obtained by running some MINITAB macro programs.

As we have noted, most statistics software packages including MINITAB and SAS provide a number of goodness-of-fit tests (including the K–S). Software packages such as Promodel can help a researcher fit a large collection of distributions in an automatic manner.

Goodness-of-fit continues to be an active area of research and development and not only in statistics. For example, Zhang (2002) proposed a general method based on the likelihood ratio that produces powerful goodness-of-fit tests. Cirrone et al. (2004) developed a statistical toolkit containing a large collection of goodness-of-fit tests that can be accessed on the internet. Yazici and Yolacan (2007) compared a number of goodness-of-fit tests for normality using a simulation study.

We also cover the Anderson–Darling (A–D) test which can be used for hypothesized normal exponential distributions with unknown parameters. Finally, we have the option of using *P–P* or *Q–Q* plots to glean information about the population distribution. The interpretation is usually subjective, however.

Problems

4.1 Two types of corn (golden and green-striped) carry recessive genes. When these were crossed, a first generation was obtained, which was consistently normal (neither golden nor green-striped). When this generation was allowed to self-fertilize, four distinct types of plants were produced: normal, golden, green-striped, and golden-green-striped. In 1200 plants, this process produced the following distribution:

Normal: 670
Golden: 230
Green-striped: 238
Golden-green-striped: 62

A monk named Mendel wrote an article theorizing that in a second generation of such hybrids, the distribution of plant types should be in a 9:3:3:1 ratio. Are the above data consistent with the good monk's theory?

4.2 A group of four coins is tossed 160 times, and the following data are obtained:

Number of heads	0	1	2	3	4
Frequency	16	48	55	33	8

Do you think the four coins are balanced?

4.3 A certain genetic model suggest that the probabilities for a particular trinomial distribution are, respectively, $\theta_1 = p^2, \theta_2 = 2p(1-p)$, and $\theta_3 = (1-p)^2, 0 < p < 1$. Assume that X_1, X_2, and X_3 represent the respective frequencies in a sample of n independent trials and that these numbers are known. Derive a chi-square goodness-of-fit test for this trinomial distribution if p is unknown.

4.4 According to a genetic model, the proportions of individuals having the four blood types should be related by

Type 0: q^2
Type A: $p^2 + 2pq$
Type B: $r^2 + 2qr$
Type AB: $2pr$

where $p + q + r = 1$. Given the blood types of 1000 individuals, how would you test the adequacy of the model?

4.5 If individuals are classified according to gender and color blindness, it is hypothesized that the distribution should be as follows:

	Male	**Female**
Normal	$p/2$	$p^2/2 + pq$
Color blind	$q/2$	$q^2/2$

for some $p + q = 1$, where p denotes the proportion of defective genes in the relevant population and therefore changes for each problem. How would the chi-square test be used to test the adequacy of the general model?

4.6 Show that in general, for Q defined as in (4.2.1),

$$E(Q) = E\left[\sum_{i=1}^{k} \frac{(F_i - e_i)^2}{e_i}\right] = \sum_{i=1}^{k} \left[\frac{n\theta_i(1-\theta_i)}{e_i} + \frac{(n\theta_i - e_i)^2}{e_i}\right]$$

From this we see that if the null hypothesis is true, $n\theta_i = e_i$ and $E(Q) = k - 1$, the mean of the chi-square distribution.

4.7 Show algebraically that when $e_i = n\theta_i$ and $k = 2$, we have

$$Q = \sum_{i=1}^{2} \frac{(F_i - e_i)^2}{e_i} = \frac{(F_1 - n\theta_1)^2}{n\theta_1(1-\theta_1)}$$

so that when $k = 2$, $\sqrt{Q}$ is the statistic commonly used for testing a hypothesis concerning the parameter of the binomial distribution for large samples. By the central-limit theorem, the distribution of $\sqrt{Q}$ approaches the standard normal distribution as $n \to \infty$ and the square of any standard normal variable is chi-square-distributed with 1 degree of freedom. Thus we have an entirely different argument for the distribution of Q when $k = 2$.

4.8 Give a simple proof that D_n, D_n^+, and D_n^- are completely distribution free for any continuous and completely specified F_X by appealing to the transformation $u = F_X(x)$ in the initial definitions of D_n, D_n^+, and D_n^-.

4.9 Prove that

$$D_n^- = \max\left\{\max_{1\le i\le n}\left[F_0(X_{(i)}) - \frac{i-1}{n}\right], 0\right\}$$

4.10 Prove that the null distribution of D_n^- is identical to the null distribution of D_n^+:

(a) Using a derivation analogous to Theorem 4.3.4
(b) Using a symmetry argument

4.11 Using Theorem 4.3.3, verify that

$$\lim_{n\to\infty} P\left(D_n > \frac{1.07}{\sqrt{n}}\right) = 0.20$$

4.12 Find the minimum sample size n required such that $P(D_n < 0.05) \ge 0.99$.

4.13 Use Theorem 4.3.4 to verify directly that $P\left(D_5^+ > 0.447\right) = 0.10$. Calculate this same probability using the expression given in (4.3.5).

4.14 Related goodness-of-fit test. The Cramér–von Mises type of statistic is defined for continuous $F_0(x)$ by

$$\omega_n^2 = \int_{-\infty}^{\infty} [S_n(x) - F_0(x)]^2 f_X(x)dx$$

(a) Prove that ω_n^2 is distribution free.
(b) Explain how ω_n^2 might be used for a goodness-of-fit test.
(c) Show that

$$n\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left[F_0\left(X_{(i)} - \frac{2i-1}{n} \right) \right]^2$$

This statistic is discussed in Cramér (1928), von Mises (1931), Smirnov (1936), and Darling (1957).

4.15 Suppose we want to estimate the cumulative distribution function of a continuous population using the empirical distribution function such that the probability is 0.90 that the error of the estimate does not exceed 0.25 anywhere. How large a sample size is needed?

4.16 If we wish to estimate a cumulative distribution within 0.20 units with probability 0.95, how large should n be?

4.17 A random sample of size 13 is drawn from an unknown continuous population $F_X(x)$, with the following results after array:

3.5, 4.1, 4.8, 5.0, 6.3, 7.1, 7.2, 7.8, 8.1, 8.4, 8.6, 9.0

A 90% confidence band is desired for $F_X(x)$. Plot a graph of the empirical distribution function $S_n(x)$ and resulting confidence bands.

4.18 In a vibration study, a random sample of 15 airplane components were subjected to server vibrations until they showed structural failures. The data given are failure times in minutes. Test the null hypothesis that these observations can be regarded as sample from the exponential population with density function $f(x) = e^{-x/10}/10$ for $x \geq 0$.

1.6, 10.3, 3.5, 13.5, 18.4, 7.7, 24.3, 10.7, 8.4, 4.9, 7.9, 12.0, 16.2, 6.8, 14.7

4.19 For the data given in Example 4.5.1, use the most appropriate test to see if the distribution can be assumed to be normal with mean 10,000 and standard deviation 2,000.

4.20 The data below represent earnings (in dollars) for a random sample of five common stocks listed on the New York Stock Exchange.

1.68, 3.35, 2.50, 6.23, 3.24

(a) Use the most appropriate test to see if these data can be regarded as a random sample from a normal distribution.
(b) Use the most appropriate test to see if these data can be regarded as a random sample from a normal distribution with $\mu = 3$, $\sigma = 1$.
(c) Determine the sample size required to use the empirical distribution function to estimate the unknown cumulative distribution function with 95% confidence such that the error in the estimate is (1) less than 0.25 and (2) less than 0.20.

4.21 It is claimed that the number of errors made by a typesetter is Poisson distributed with an average rate of 4 per 1000 words. One hundred random samples of 1000 words from this typesetter are examined and the number of errors are counted as shown below. Are these data consistent with the claim?

No. of errors	0	1	2	3	4	5
No. of samples	10	16	20	28	12	14

4.22 For the original data in Example 4.4.1 (not the square roots), test the null hypothesis that they come from the continuous uniform distribution, using level 0.01.

4.23 Use the D_n statistic to test the null hypothesis that the data in Example 4.2.1

(a) Come from the Poisson distribution with $\mu = 1.5$
(b) Come from the binomial distribution with $n = 13, p = 0.1$

These tests will be conservative because both hypothesized distribution are discrete.

4.24 Each student in a class of 18 is asked to list three people he likes and three he dislikes and label the people 0, 1, 2, 3, 4, 5 according to how much he likes them, with 0 denoting least liked and 5 denoting most liked. From this list, each student selects the number assigned to the person he thinks is the wealthiest of the six. The results in the form of an array are as follows:

0, 0, 0, 0, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 4, 5

Test the null hypothesis that the students are equally likely to select any of the number 0, 1, 2, 3, 4, 5, using the most appropriate test and the 0.05 level of significance.

4.25 During a 50 week period, demand for a certain kind of replacement part for TV sets is shown below. Find the theoretical distribution of weekly demands for a Poisson model with the same mean as the given data and perform an appropriate goodness-of-fit test.

Weekly Demand	**Number of Weeks**
0	28
1	15
2	6
3	1
More than 3	0
	50

4.26 Suppose that monthly collections for home delivery of the *New York Times* in a large suburb of New York City are approximately normally distributed with mean \$150 and standard deviation \$20. A random sample of 10 delivery persons in a nearby suburb is taken; the arrayed data for monthly collections in dollars are:

90, 106, 109, 117, 130, 145, 156, 170, 174, 190

Test the null hypothesis that the same normal distribution model applies to this suburb, using the most appropriate test.

4.27 A bank frequently makes large installment loans to builders. At any point in time, outstanding loans are classified in the following four repayment categories:

A: Current
B: Up to 30 days delinquent
C: 30–60 days delinquent
D: Over 60 days delinquent

The bank has established the internal standard that these loans are ''in control'' as long as the percentage in each category is as follows:

A: 80% B: 12% C: 7% D: 1%

They make frequent spot checks by drawing a random sample of loan files, noting their repayment status at that time and comparing the observed distribution with the standard for control. Suppose a sample of 500 files produces the following data on number of loans in each repayment category:

A: 358 B: 83 C: 44 D: 15

Does it appear that installment loan operations are under control at this time?

4.28 Durtco Incorporated designs and manufactures gears for heavy-duty construction equipment. One such gear, 9973, has the following specifications:

(a) Mean diameter 3.0 in.
(b) Standard deviation 0.001 in.
(c) Output normally distributed

The production control manager has selected a random sample of 500 gears from the inventory and measured the diameter of each. Nothing more has been done to the data. How would you determine statistically whether gear 9973 meets the specifications?

Be brief but specific about which statistical procedure to use and why and outline the steps in the procedure.

4.29 Compare and contrast the chi-square and K–S goodness-of-fit procedures.

4.30 For the data 1.0, 2.3, 4.2, 7.1, 10.4, use the most approximate procedure to test the null hypothesis that the distribution is

(a) Exponential $F_X(x) = 1 - e^{-x/b}$ (estimate b by $\bar{x}$)
(b) Normal

In each part, carry the parameter estimates to the nearest hundredth and the distribution estimates to the nearest ten thousandth.

4.31 A statistics professor claims that the distribution of final grades from A to F in a particular course invariably is in the ratio 1:3:4:1:1. The final grades this year are 26 A's, 50 B's, 80 C's, 35 D's, and 10 F's. Do these results refute the professor's claim?

4.32 The design department has proposed three different package designs for the company's product; the marketing manager claims that the first design will be twice as popular as the second design and that the second design will be three times as popular as the third design. In a market test with 213 persons, 111 preferred the first design, 62 preferred the second design, and the remainder preferred the third design. Are these results consistent with the marketing manager's claim?

4.33 A quality control engineer has taken 50 samples, each of size 13, from a production process. The numbers of defectives are recorded below.

Number of Defects	Sample Frequency
0	9
1	26
2	9
3	4
4	1
5	1
6 or more	0

(a) Test the null hypothesis that the number of defectives follows a Poisson distribution.
(b) Test the null hypothesis that the number of defectives follows a binomial distribution.
(c) Comment on your answers in (a) and (b).

4.34 Ten students take a test and their scores (out of 100) are as follows:

95, 80, 40, 52, 60, 80, 82, 58, 65, 50

Test the null hypothesis that the cumulative distribution function of the proportion of right answers a student gets on the test is

$$F_0(x) = \begin{cases} 0 & x < 0 \\ x^2(3 - 2x) & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

4.35 At the completion of a basketball training camp, participants are classified into categories, A (highest), B and C, according to their shooting ability. A sample of this year's participants yielded the following data:

A: 136 B: 38 C: 26

Historically, the percentages of participants falling into each category has been

A: 80% B: 12% C: 8%

Does it appear that this year's participants correspond to the historic percentages in their shooting ability?

4.36 Many genetic traits occur in large populations according to the Hardy–Weinberg Law, which is based on the binomial expansion

$$(p+q)^2 = p^2 + 2pq + q^2$$

The so-called M–N blood types of humans have three antigens, M only, N only, and both M and N, with respective probabilities p^2, q^2, and $2pq$. For the white population in the United States, studies have shown that $p = .54$ and $q = .46$. A sample of 200 white college students had 63 type M only, 32 type N only, and 105 type M and N. Do these data conform to the Hardy–Weinberg Law?

4.37 According to test theory, scores on a certain IQ test are normally distributed. This test was given to 18 girls of similar age and their scores were 114, 81, 87, 114, 113, 87, 111, 89, 93, 108, 99, 93, 100, 95, 93, 95, 106, 108. Test the null hypothesis that these scores are normally distributed with

(a) Unspecified mean and variance.
(b) Mean 100 and variance 100.

5

One-Sample and Paired-Sample Procedures

5.1 Introduction

In the general one-sample problem, the available data consist of a single set of observations, usually a random sample. The tests for randomness in Chapter 3 relate to inferences about a property of the joint probability distribution of a set of sample observations that are identically distributed but possibly dependent. The hypothesis in a goodness-of-fit study in Chapter 4 is concerned with the univariate population distribution from which the sample is drawn. These hypotheses are so general that no analogous counterparts exist in parametric statistics. Thus, these problems are viewed as nonparametric procedures. In a classical inference problem, the data from a single sample are used to obtain information about some particular aspect of the population distribution, usually one or more of its parameters. Nonparametric techniques are useful here too, particularly when a location parameter is of interest.

In this chapter, we will be concerned with the nonparametric analog of the normal-theory test (variance known) or Student's t test (variance unknown) for the hypotheses H_0: $\mu = \mu_0$ and H_0: $\mu_X - \mu_Y = \mu_D = \mu_0$ for the one-sample and paired-sample problems, respectively. These classical tests are derived under the assumption that the single population or the population of differences of pairs is normal. For the nonparametric tests, only certain continuity assumptions about the populations are needed to determine the sampling distributions of the test statistics. The hypotheses here are concerned with the median or some other quantile rather than the mean as the location parameter, but both the mean and the median are good indices of central tendency and they do coincide for symmetric populations. In any population, the median always exists (which is not true for the mean) and it is more robust as an estimate of location. The procedures covered here include confidence intervals and tests of hypotheses about any specified quantile. The case of the median is treated separately; the popular sign test and Wilcoxon signed-rank test are presented. The complete discussion in each case will be given only for the single-sample case, since with paired-sample data once the differences

are formed, we have essentially a single sample drawn from the population of differences and thus the methods of analysis are identical.

We also introduce rank-order statistics and present a measure of the relationship between ranks and variate values.

5.2 Confidence Interval for a Population Quantile

Recall from Chapter 2 that a quantile of a continuous random variable X is a real number that divides the area under the probability density function into two parts of specified amounts. Only the area to the left of the number need be specified since the entire area is equal to 1. Let F_X be the underlying cdf and define κ_p as any real number which is a solution to the equation $F_X(\kappa_p) = p$, or in terms of the *quantile function*, $\kappa_p = Q_X(p) = F_X^{-1}(p)$. We assume here that a unique solution (inverse) exists, as would be the case for a strictly increasing cdf F_X. Note that κ_p is a parameter of the population F_X; to emphasize this point we use the Greek letter κ_p instead of the Latin letter $Q_X(p)$ used in Chapter 2. For example, $\kappa_{0.50}$ is the median of the distribution, a measure of central tendency.

First, we consider the problem where a confidence-interval estimate of the parameter κ_p is desired for some specified value of p, given a random sample $X_1, X_2, \ldots, X_n$ from the cdf F_X. As discussed in Chapter 2, a natural point estimate of κ_p is the pth sample quantile, which is the (np)th-order statistic, provided of course that np is an integer. For example, since $100p\%$ of the population values are less than or equal to the pth population quantile, the estimate of κ_p is that value from a random sample such that $100p\%$ of the sample values are less than or equal to it. We define the order statistic $X_{(r)}$ to be the pth sample quantile where r is defined by

$$r = \begin{cases} np & \text{if } np \text{ is an integer} \\ [np + 1] & \text{if } np \text{ is not an integer} \end{cases}$$

and $[x]$ denotes the largest integer not exceeding x. This is just a convention adopted so that we can handle situations where np is not an integer. Other conventions are sometimes adopted. In our case, the pth sample quantile $Q_X(p)$ is equal to $X_{(np)}$ if np is an integer, and $X_{([np+1])}$ if np is not an integer.

A point estimate is not sufficient for inference purposes. We know from Theorem 2.10.1 that the rth-order statistic is a consistent estimator of the pth quantile of a distribution when $n \to \infty$ and $r/n \to p$. However, consistency is only a large-sample property. We would like a procedure for interval estimation of κ_p that will enable us to attach a confidence coefficient to our estimate for the given (finite) sample size. A logical choice for the confidence-interval endpoints are two-order statistics, say $X_{(r)}$ and $X_{(s)}$, $r < s$, from the

random sample drawn from the population F_X. To find the $100(1-\alpha)\%$ confidence interval, we must then find the two integers r and s, $1 \le r < s \le n$, such that

$$P(X_{(r)} < \kappa_p < X_{(s)}) = 1 - \alpha$$

for some given number $0 < \alpha < 1$. The quantity $1-\alpha$, which we frequently denote by γ, is called the *confidence level* or the *confidence coefficient*. Now, the event $X_{(r)} < \kappa_p$ occurs if and only if either $X_{(r)} < \kappa_p < X_{(s)}$ or $\kappa_p > X_{(s)}$, and these latter two events are clearly mutually exclusive. Therefore, for all $r < s$,

$$P(X_{(r)} < \kappa_p) = P(X_{(r)} < \kappa_p < X_{(s)}) + P(\kappa_p > X_{(s)})$$

or, equivalently,

$$P(X_{(r)} < \kappa_p < X_{(s)}) = P(X_{(r)} < \kappa_p) - P(X_{(s)} < \kappa_p) \tag{5.2.1}$$

Since we assumed that F_X is a strictly increasing function, $X_{(r)} < \kappa_p$ if and only if $F_X(X_{(r)}) < F_X(\kappa_p) = p$. But when F_X is continuous, the PIT implies that the probability distribution of the random variable $F_X(X_{(r)})$ is the same as that of $U_{(r)}$, the rth-order statistic from the uniform distribution over the interval (0, 1). Further, since $F_X(\kappa_p) = p$ by the definition of κ_p, we have

$$\begin{aligned} P(X_{(r)} < \kappa_p) &= P[F_X(X_{(r)}) < p] \\ &= P(U_{(r)} < p) \\ &= \int_0^p \frac{n!}{(r-1)!(n-r)!} x^{r-1}(1-x)^{n-r}dx \\ &= \int_0^p n\binom{n-1}{r-1} x^{r-1}(1-x)^{n-r}dx \end{aligned} \tag{5.2.2}$$

Thus, while the distribution of the rth-order statistic depends on the population distribution F_X, the probability in (5.2.2) does not. A confidence-interval procedure based on (5.2.1) is therefore distribution free.

In order to find the interval estimate of κ_p, we substitute (5.2.2) into (5.2.1) and find that r and s should be chosen such that

$$\int_0^p n\binom{n-1}{r-1} x^{r-1}(1-x)^{n-r}dx - \int_0^p n\binom{n-1}{s-1} x^{s-1}(1-x)^{n-s}dx = 1-\alpha \tag{5.2.3}$$

Clearly, this one equation will not give a unique solution for the two unknowns, r and s, and additional conditions are needed. For example, if

we want the narrowest possible interval for a fixed confidence coefficient, r and s would be chosen such that (5.2.3) is satisfied and $X_{(s)} - X_{(r)}$, or $E[X_{(s)} - X_{(r)}]$, is as small as possible. Alternatively, we could minimize $s - r$.

The integrals in (5.2.2) or (5.2.3) can be evaluated by integration by parts or by using tables of the incomplete beta function. However, (5.2.2) can be expressed in another form after integration by parts as follows:

$$
\begin{aligned}
P(X_{(r)} < \kappa_p) &= \int_0^p n \binom{n-1}{r-1} x^{r-1}(1-x)^{n-r}\,dx \\
&= n\binom{n-1}{r-1}\left[\frac{x^r}{r}(1-x)^{n-r}\Big|_0^p + \frac{n-r}{r}\int_0^p x^r(1-x)^{n-r-1}\,dx\right] \\
&= \binom{n}{r}p^r(1-p)^{n-r} + n\binom{n-1}{r}\left[\frac{x^{r+1}}{r+1}\right](1-x)^{n-r-1}\Big|_0^p \\
&\quad + \left(\frac{n-r-1}{r+1}\right)\int_0^p x^{r+1}(1-x)^{n-r-2}\,dx \\
&= \binom{n}{r}p^r(1-p)^{n-r} + \binom{n}{r+1}p^{r+1}(1-p)^{n-r-1} \\
&\quad + n\binom{n-1}{r+1}\int_0^p x^{r+1}(1-x)^{n-r-2}\,dx
\end{aligned}
$$

After repeating this integration by parts $n - r$ times, the result will be

$$
\begin{aligned}
&\binom{n}{r}p^r(1-p)^{n-r} + \binom{n}{r+1}p^{r+1}(1-p)^{n-r-1} + \cdots \\
&\quad + \binom{n}{n-1}p^{n-1}(1-p) + n\binom{n-1}{n-1}\int_0^p x^{n-1}(1-x)^0\,dx \\
&= \sum_{j=0}^{n-r}\binom{n}{r+j}p^{r+j}(1+p)^{n-r-j}
\end{aligned}
$$

or, after substituting $r + j = i$,

$$
P(X_{(r)} < \kappa_p) = \sum_{i=r}^{n}\binom{n}{i}p^i(1-p)^{n-i} \tag{5.2.4}
$$

In this final form, the integral in (5.2.2) is expressed as the sum of the last $n-r+1$ terms of the binomial distribution with parameters n and p. Thus, the probability in (5.2.1) can be expressed as

$$\begin{aligned} P(X_{(r)} < \kappa_p < X_{(s)}) &= \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=s}^{n} \binom{n}{i} p^i (1-p)^{n-i} \\ &= \sum_{i=r}^{s-1} \binom{n}{i} p^i (1-p)^{n-i} \\ &= P(r \le K \le s-1) \end{aligned} \tag{5.2.5}$$

where K has a binomial distribution with parameters n and p. This form is probably the easiest to use in choosing r and s such that $s-r$ is a minimum for fixed α. It is clear from (5.2.5) that this probability does not depend on the underlying cdf as long as it is continuous. The resulting confidence interval is therefore distribution–free.

In order to find the confidence interval for κ_p based on two order statistics, the right-hand side of (5.2.5) is set equal to $1-\alpha$ and the search for r and s is begun. Because of the discreteness of the binomial distribution, the exact nominal confidence level frequently cannot be achieved. In such cases, the confidence-level requirement can be changed from ''equal to'' to ''at least equal to'' $1-\alpha$. This is a conservative approach used for distribution-free confidence intervals; the actual probability is denoted by γ ($\ge 1-\alpha$) and is called the *exact confidence level*.

The result obtained in (5.2.4), found by integration of (5.2.2), can also be obtained by an argument based on simple counting, an argument used frequently in developing nonparametric procedures based on order statistics. Note that for any p, the event $X_{(r)} < \kappa_p$ occurs if and only if at least r of the n sample values, $X_1, X_2, \ldots, X_n$, are less than κ_p. Thus

$$\begin{aligned} P(X_{(r)} < \kappa_p) = {} & P(\text{exactly } r \text{ of the } n \text{ observations are } < \kappa_p) \\ & + P(\text{exactly}(r+1) \text{ of the } n \text{ observations} \\ & \quad \text{are } < \kappa_p) + \cdots \\ & + P(\text{exactly } n \text{ of the } n \text{ observations are } < \kappa_p) \end{aligned}$$

In other words,

$$P(X_{(r)} < \kappa_p) = \sum_{i=r}^{n} P(\text{exactly } i \text{ of the } n \text{ observations are } < \kappa_p)$$

This is a key observation. The probability that exactly i of the n observations are less than κ_p can be found as the probability of i successes in n independent Bernoulli trials, since the sample observations are all independent and

each observation can be classified as either a success or a failure, where a success is defined as an observation less than κ_p. In other words,

$$P(\text{exactly } i \text{ of the } n \text{ sample values are } < \kappa_p) = \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore

$$P(X_{(r)} < \kappa_p) = \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

This completes the proof.

In summary, the $(1-\alpha)100\%$ confidence interval for the pth quantile is given by $(X_{(r)}, X_{(s)})$, where r and s are integers such that $1 \leq r < s \leq n$ and

$$P(X_{(r)} < \kappa_p < X_{(s)}) = \sum_{i=r}^{s-1} \binom{n}{i} p^i (1-p)^{n-i} \geq 1 - \alpha \tag{5.2.6}$$

As indicated earlier, without a second condition, the confidence-interval endpoints will not be unique. One common approach in this case is to assign the probability $\alpha/2$ in each (right and left) tail. This yields the so-called ''equal-tails'' interval, where r and s are the *largest* and *smallest* integers $(1 \leq r < s \leq n)$ such that

$$\sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} \leq \frac{\alpha}{2} \quad \text{and} \quad \sum_{i=0}^{s-1} \binom{n}{i} p^i (1-p)^{n-1} \geq 1 - \frac{\alpha}{2} \tag{5.2.7}$$

respectively. These equations are easy to use in conjunction with Table C, which gives cumulative binomial probabilities. The exact confidence level is found from Table C as

$$\sum_{i=r}^{s-1} \binom{n}{r} p^i (1-p)^{n-i} = \sum_{i=0}^{s-1} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} = \gamma \tag{5.2.8}$$

We note that in some cases there may be no $r-1, r \geq 1$, that satisfies the first inequality in (5.2.7). In this case we take the left-hand confidence-interval endpoint $X_{(r)} = -\infty$. This means that for the given n, p and α, we obtain a one-sided (upper) confidence interval $(-\infty, X_{(s)})$ with exact confidence level $\sum_{i=0}^{s-1} \binom{n}{i} p^i (1-p)^{n-i}$ and we may want to choose s such that this level is at

least $1-\alpha$, rather than $1-\alpha/2$. Similarly, there may be no $s-1 \leq n$, which satisfies the second inequality in (5.2.7) and in that case we take the right-hand confidence-interval endpoint $X_{(s)} = \infty$, so that we obtain a one-sided (lower) confidence interval $(X_{(r)}, \infty)$ with exact confidence level $1 - \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} \geq 1-\alpha$. Thus, in practice there can be situations, depending on the n, p, and α, where a two-sided confidence interval cannot be achieved and we would present a one-sided confidence interval instead. These situations occur primarily when p is extreme, i.e., very close to 0 or 1, since the binomial distribution is highly skewed in these cases.

If the sample size is larger than 20 and therefore beyond the range of Table C, we can use the normal approximation to the binomial distribution with a continuity correction. The solutions are

$$\begin{aligned} r &= np + 0.5 - z_{\alpha/2}\sqrt{np(1-p)} \\ \text{and} \quad s &= np + 0.5 + z_{\alpha/2}\sqrt{np(1-p)} \end{aligned} \tag{5.2.9}$$

where $z_{\alpha/2}$ satisfies $\Phi(z_{\alpha/2}) = 1-\alpha/2$, as defined in Chapter 3. We round the result in (5.2.9) down to the nearest integer for r and up for s in order to be conservative (or to make the confidence level at least $1-\alpha$).

This equal-tails solution given by (5.2.7) and (5.2.9) is the most reasonable one and it generally gives the shortest (narrowest) interval when p is close to 0.5. Another possibility is to find the values of r and s that make $s-r$ as small as possible. This requires a trial-and-error solution in making (5.2.8) at least $1-\alpha$ and will be illustrated in Example 5.2.1.

Example 5.2.1

Suppose $n=10$, $p=0.35$, and $1-\alpha=0.95$. Using (5.2.7) with Table C, we find $r-1=0$ and $s-1=7$, making $r=1$ and $s=8$. The confidence interval for the 0.35th quantile is $(X_{(1)}, X_{(8)})$ with exact confidence level equal to $0.9952-0.0135=0.9817$ from (5.2.8). Note that this equal-tails solution yields $s-r=8-1=7$. Another possibility is to choose r and s such that $s-r$ is a minimum. The reader can verify that this solution yields $r=1$ and $s=7$, $s-r=6$, with exact confidence level $0.9740-0.0135=0.9605$. The normal approximation from (5.2.9) gives $r=1$ and $s=7$ with approximate confidence level 0.95.

Now suppose that $n=10$, $p=0.10$, and $1-\alpha=0.95$. Table C shows that no value of $r-1$ satisfies the left-hand condition of (5.2.7) so we take the lower confidence limit to be $-\infty$. Table C also shows that $s-1=3$ satisfies the right-hand condition of (5.2.7) with $\alpha/2$ replaced by α. So the 95% upper confidence interval for the 0.10th quantile is $(-\infty, X_{(4)})$ with exact confidence level $0.9872-0=0.9872$.

As another example, let $n = 11$, $p = 0.25$, $1 - \alpha = 0.95$. The equal-tails solution from (5.2.7) yields no value for r. We find $s - 1 = 5$ and the confidence interval is $(-\infty, X_{(6)})$ with exact level 0.9657.

As a final example, let $n = 10$, $p = 0.80$, $1 - \alpha = 0.95$. The solution from (5.2.7) gives $r - 1 = 4$ and no $s - 1 \leq 10$, so we take the upper confidence limit to be ∞. The resulting confidence interval $(X_{(5)}, \infty)$ has exact level $1 - 0.0328 = 0.9672$.

5.3 Hypothesis Testing for a Population Quantile

In a hypothesis testing type of inference concerned with quantiles, a distribution-free procedure is also possible. Given the order statistics $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ from any *unspecified* but continuous distribution F_X, a null hypothesis concerning the value of the pth quantile is written

$$H_0 : \kappa_p = \kappa_p^0$$

where κ_p^0 and p are both specified numbers. Under H_0, since κ_p^0 is the pth quantile of F_X, we have, by definition, $P(X \leq \kappa_p^0) = p$ and therefore we expect about np of the sample observations to be smaller than κ_p^0 if H_0 is true. If the actual number of sample observations smaller than κ_p^0 is considerably smaller than np, the data suggest that the true pth quantile is larger than κ_p^0 or there is evidence against H_0 in favor of the one-sided upper tailed alternative.

$$H_1 : \kappa_p > \kappa_p^0$$

This implies it is reasonable to reject H_0 in favor of H_1 if at most $r - 1$ sample observations are smaller than κ_p^0, for some r. Now, if at most $r - 1$ sample observations are smaller than κ_p^0, then it must be true that the rth-order statistic $X_{(r)}$ in the sample satisfies $X_{(r)} > \kappa_p^0$. Therefore, an appropriate rejection region R is

$$X_{(r)} \in R \quad \text{for } X_{(r)} > \kappa_p^0 \tag{5.3.1}$$

For a specified significance level α, the integer r should be chosen such that

$$P\left(X_{(r)} > \kappa_p^0 | H_0\right) = 1 - P\left(X_{(r)} \leq \kappa_p^0 | H_0\right) \leq \alpha$$

or, using (5.2.4), r is the largest integer such that

$$1 - \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i} \leq \alpha \tag{5.3.2}$$

Note that we find the critical region (or the constant r) so that the probability of a type I error is at most α and not necessarily exactly equal to α because the binomial distribution is discrete. As in the case of the confidence interval, where we set the probability to be at least $1-\alpha$, this provides a conservative solution.

We now express the rejection region in another form in order to be consistent with our later presentation in Section 5.4 for the sign test. Note that $X_{(r)} > \kappa_p^0$ if and only if at most $r-1$ of the observations are less than κ_p^0 so that at least $n-r+1$ are greater than κ_p^0. Define the random variable K as the total number of plus signs among the n differences $X_{(i)} - \kappa_p^0$ (the number of positive differences). Then the rejection region in (5.3.1) can be equivalently stated as

$$K \in R \quad \text{for } K \geq n-r+1$$

The differences $X_i - \kappa_p^0, i = 1, 2, \ldots, n$, are independent random variables, each having either a plus or a minus sign, and the probability of a plus sign under H_0 is

$$P\left(X_i - \kappa_p^0 > 0\right) = P\left(X_i > \kappa_p^0\right) = 1 - p$$

Hence, since K is the number of plus signs, we can write $K = \sum_{i=1}^{n} I(X_i > \kappa_p^0)$ where $I(A)$ equals 1 when the event A occurs and equals 0 otherwise. From the preceding discussion, the indicator variables $I(X_i > \kappa_p^0), i = 1, 2, \ldots, n$, are independent Bernoulli random variables with probability of success $1-p$ under H_0. Thus, under H_0, the distribution of K is binomial with parameters n and $1-p$ and so r is the largest integer that satisfies

$$P(K \geq n-r+1 | H_0) = \sum_{i=n-r+1}^{n} \binom{n}{i} (1-p)^i p^{n-i} \leq \alpha \tag{5.3.3}$$

which can be shown to agree with the statement in (5.3.2), by a change of summation index from i to $n-i$.

The advantage of using (5.3.2) is that the cumulative binomial probabilities given in Table C are directly involved.

On the other hand, if many more than np observations are smaller than κ_p^0, there is support against H_0 in favor of the one-sided lower-tailed alternative $H_1 : \kappa_p < \kappa_p^0$. Then we should reject H_0 if the number of sample observations smaller than κ_p^0 is at least, say s. This leads to the rejection region

$$X_{(s)} \in R \quad \text{for } X_{(s)} < \kappa_p^0$$

but this is equivalent to saying that the number of observations larger than κ_p^0 must be at most $n-s$. Thus, based on the statistic K defined before as the

number of positive differences, the appropriate region for the one-sided lower-tailed alternative $H_1 : \kappa_p < \kappa_p^0$ is

$$K \in R \quad \text{for } K \leq n - s$$

where $n - s$ is the largest integer such that

$$P(K \leq n - s | H_0) = \sum_{i=0}^{n-s} \binom{n}{i} (1-p)^i p^{n-i} \leq \alpha \tag{5.3.4}$$

For the two-sided alternative $H_1 : \kappa_p \neq \kappa_p^0$, the rejection region consists of the union of the two pieces specified above,

$$K \in R \quad \text{for } K \leq n - s \quad \text{or} \quad K \geq n - r + 1 \tag{5.3.5}$$

where r and $n - s$ are the largest integers such that each of (5.3.2) and (5.3.4) is less than or equal to $\alpha/2$.

Note that Table C can be used to find the exact critical values for $n \leq 20$, where $\theta = 1 - p$ in (5.3.4). For sample sizes larger than 20 the normal approximation to the binomial distribution with a continuity correction can be used. The rejection region for $H_1 : \kappa_p > \kappa_p^0$ is

$$K \geq 0.5 + n(1-p) + z_\alpha \sqrt{np(1-p)}$$

For $H_1 : \kappa_p < \kappa_p^0$, the rejection region is

$$K \leq -0.5 + n(1-p) - z_\alpha \sqrt{np(1-p)}$$

The rejection region for $H_1 : \kappa_p \neq \kappa_p^0$ is the combination of these two with z_α replaced by $z_{\alpha/2}$. Note that in all these formulas the standard normal deviate, say z_b, is such that the area to the right is b; in other words, z_b is the $100(1-b)$ th percentile [or the $(1-b)$th quantile] of the standard normal distribution.

Table 5.3.1 summarizes the appropriate rejection regions for the quantile test and the corresponding P values, both exact and approximate (asterisks are added to emphasize approximate), where K_0 is the observed value of the statistic K, the number of positive differences.

Example 5.3.1

The Educational Testing Service reports that the 75th percentile for scores on the quantitative portion of the Graduate Record Examination (GRE) is 693 in a certain year. A random sample of 15 first-year graduate students majoring in statistics report their GRE quantitative scores as 690, 750, 680, 700, 660, 710, 720, 730, 650, 670, 740, 730, 660, 750, and 690. Are the scores of students majoring in statistics consistent with the 75th percentile value for this year?

TABLE 5.3.1

Hypothesis Testing Guide for Quantiles

Alternative	Rejection Region	*P* Value
	Exact	*Exact*
$\kappa_p > \kappa_p^0$	$X_{(r)} > \kappa_p^0$ or $K \geq n - r + 1$, r from (5.3.2)	$P_U = \sum_{k=K_0}^{n} \binom{n}{k} (1-p)^k p^{n-k}$
	Approximate	*Approximate*
	$K \geq 0.5 + n(1-p) + z_\alpha \sqrt{n(1-p)p}$	$P_U^* = 1 - \Phi\left(\frac{K_0 - 0.5 - n(1-p)}{\sqrt{n(1-p)p}}\right)$
	Exact	*Exact*
$\kappa_p < \kappa_p^0$	$X_{(s)} < \kappa_p^0$ or $K \leq n - s$, s from (5.3.4)	$P_L = \sum_{k=0}^{K_0} \binom{n}{k} (1-p)^k p^{n-k}$
	Approximate	*Approximate*
	$K \leq -0.5 + n(1-p) - z_\alpha \sqrt{n(1-p)p}$	$P_L^* = \Phi\left(\frac{K_0 + 0.5 - n(1-p)}{\sqrt{n(1-p)p}}\right)$
	Exact	*Exact*
$\kappa_p \neq \kappa_p^0$	$X_{(r)} > \kappa_p^0$ or $X_{(s)} < \kappa_p^0$ or $K \geq n - r + 1$ or $K \leq n - s$, r and s from (5.3.5)	$2 \min(P_U, P_L)$
	Approximate	*Approximate*
	$K \geq 0.5 + n(1-p) + z_{\alpha/2} \sqrt{n(1-p)p}$ or $K \leq -0.5 + n(1-p) - z_{\alpha/2} \sqrt{n(1-p)p}$	$2 \min\left(P_U^*, P_L^*\right)$

SOLUTION

The question can be answered either by a hypothesis testing or a confidence-interval approach. We illustrate both at the 0.05 level. Here, we are interested in the 0.75th quantile (the third quartile) so that $p = 0.75$, and the hypothesized value of the 0.75th quantile, $\kappa_{0.75}^0$, is 693. Thus, the null hypothesis H_0:$\kappa_{0.75} = 693$ is to be tested against a two-sided alternative H_1:$\kappa_{0.75} \neq 693$. The value of the test statistic is $K = 8$, since there are eight positive differences among $X_i - 693$, and the two-sided rejection region is $K \in R$ for $K \leq n - s$ or $K \geq n - r + 1$, where r and $n - s$ are the largest integers that satisfy (5.3.2) and (5.3.4) with $\alpha/2 = 0.025$. For $n = 15$, $p = 0.75$, Table C shows that 0.0173 is the largest left-tail probability that does not exceed 0.025, so $r - 1 = 7$ and $r = 8$. Similarly, 0.0134 is the largest

left-tail probability that does not exceed 0.025 for $n = 15$ and $1 - p = 0.25$ (note the change in the success probability) so that $n - s = 0$ and $s = 15$. The two-sided critical region then is $K \leq 0$ or $K \geq 8$, and the exact significance level for this distribution-free test is $(0.0134 + 0.0173) = 0.0307$. Since the observed $K = 8$ falls in this rejection region, there is evidence that for this year the scores for the graduate majors in statistics are not consistent with the reported 75th percentile for all students in this year.

In order to find the P value, note that the alternative is two-sided and so we need to find the two one-tailed probabilities first. Using Table C with $n = 15$ and $\theta = 0.25$ we find $P(K \leq 8|H_0) = 0.9958$ and $P(K > 8|H_0) = 1 - 0.9827 = 0.0173$. Taking the smaller of these two values and multiplying by 2, the required P value is 0.0346, which also suggests rejecting the null hypothesis.

In order to find a 95% confidence interval for $\kappa_{0.75}$, we use (5.2.7). For the lower index r, the inequality on the left applies. From Table C with $n = 15$ and $\theta = 0.75$, the largest value of x such that the cumulative probability is less than or equal to 0.025 is 7, which yields $r = 8$ with corresponding probability 0.0173. For the upper index s, the inequality on the right in (5.2.7) applies, again with $n = 15$ and $\theta = 0.75$. From Table C, the smallest value of x such that the cumulative probability is greater than or equal to 0.975 is 14, so that $s = 15$ with corresponding probability 0.9866. The desired 95% confidence-interval endpoints are $X_{(8)}$ and $X_{(15)}$ which are 700 and 750, respectively. The exact confidence level using (5.2.8) is $\gamma = 0.9866 - 0.0173 = 0.9693$. Thus, we have at least 95% confidence, or exactly 96.93% confidence, that the 75th percentile (or the 0.75th quantile) score on the quantitative portion of the GRE exam of students majoring in statistics lies somewhere between 700 and 750. Note that, on the basis of this confidence interval, we would again reject $H_0 : \kappa_{0.75} = 693$ in favor of the alternative $H_1 : \kappa_{0.75} \neq 693$, since the hypothesized value lies outside of the confidence interval.

A very special quantile is the median (the 0.50th quantile or the 50th percentile). The median is an important and useful parameter, particularly when the underlying distribution is skewed, because the median is a far more robust estimate of the center of a distribution than the mean. The quantile tests and confidence intervals discussed here can both be applied to the case of the median using $p = 0.5$. However, because of its special importance, the case for the median is treated separately in the next section.

5.4 The Sign Test and Confidence Interval for the Median

Suppose that a random sample of N observations $X_1, X_2, \ldots, X_N$ is drawn from a population F_X with an unknown median M, where F_X is assumed to be continuous and strictly increasing, at least in the vicinity of M. In other words, the N

observations are independent and identically distributed, and $F_X^{-1}(0.5) = M$ uniquely. The total sample size notation is changed from n to N in this section in order to be consistent with the notation in the rest of this book.

The hypothesis to be tested concerns the value of the population median

$$H_0 : M = M_0$$

where M_0 is a specified value, against a corresponding one- or two-sided alternative. Since we assumed that F_X has a unique median, the null hypothesis states that M_0 is that value of X which divides the area under the pdf into two equal parts. In symbols, we have

$$H_0 : \theta = P(X > M_0) = P(X < M_0) = 0.50$$

Recalling the arguments used in developing a distribution-free test for an arbitrary quantile, we note that if the sample data are consistent with the hypothesized median value, on the average half of the sample observations will lie above M_0 and half below. Thus, the number of observations larger than M_0, denoted by K, can be used to test the validity of the null hypothesis. Also, when the sample observations are dichotomized in this way, they constitute a set of N independent random variables from the Bernoulli population with parameter $\theta = P(X > M_0)$, regardless of the population F_X. The sampling distribution of the random variable K then is the binomial with parameters N and θ, and θ equals 0.5 if the null hypothesis is true. Since K is the number of plus signs among the N differences $X_i - M_0, i = 1, 2, \ldots, N$, the nonparametric test based on K is called the *sign test*.

The rejection region for the upper-tailed alternative

$$H_1 : M > M_0 \quad \text{or} \quad \theta = P(X > M_0) > 0.5$$

is

$$K \in R \quad \text{for } K \geq k_\alpha$$

where k_α is chosen to be the smallest integer that satisfies

$$P(K \geq k_\alpha | H_0) = \sum_{i=k_\alpha}^{N} \binom{N}{i} (0.5)^N \leq \alpha \tag{5.4.1}$$

Any table of the binomial distribution, like Table C, can be used with $\theta = 0.5$ to find the particular value of k_α for the given N and α, but Table G is easier to use because it gives probabilities in both tails. Similarly, for a one-sided test with the lower-tailed alternative

$$H_1 : M < M_0 \quad \text{or} \quad \theta = P(X > M_0) < P(X < M_0)$$

the rejection region for an α-level test is

$$K \in R \quad \text{for } K \leq k'_\alpha$$

where k'_α is the largest integer satisfying

$$\sum_{i=0}^{k'_\alpha} \binom{N}{i} (0.5)^N \leq \alpha \tag{5.4.2}$$

If the alternative is two-sided,

$$H_1 : M \neq M_0 \quad \text{or} \quad \theta = P(X > M_0) \neq 0.5$$

the rejection region is $K \geq k_{\alpha/2}$ or $K \leq k'_{\alpha/2}$, where $k_{\alpha/2}$ and $k'_{\alpha/2}$ are respectively, the smallest and the largest integers such that

$$\sum_{i=k_{\alpha/2}}^{N} \binom{N}{i} (0.5)^N \leq \frac{\alpha}{2} \quad \text{and} \quad \sum_{i=0}^{k'_{\alpha/2}} \binom{N}{i} (0.5)^N \leq \frac{\alpha}{2} \tag{5.4.3}$$

Obviously, we have the relation $k_{\alpha/2} = N - k'_{\alpha/2}$ since the binomial distribution is symmetric when $\theta = 0.5$.

The sign test statistic with these rejection regions is consistent against the respective one- and two-sided alternatives. This is easy to show by applying the criterion of consistency given in Chapter 1. Since $E(K/N) = \theta$ and var $(K/N) = \theta(1-\theta)/N \to 0$ as $N \to \infty$, K provides a consistent test statistic.

5.4.1 *P* Value

The P value expressions for the sign test can be obtained as in the case of a general quantile test with $p = 0.5$. The reader is referred to Table 5.3.1, with n replaced by N throughout. For example, if the alternative is upper-tailed H_1: $M > M_0$, and K_0 is the observed value of the sign statistic, the P value for the sign test is given by the binomial probability in the upper tail

$$\sum_{i=K_0}^{N} \binom{N}{i} (0.5)^N$$

This value is easily read as a right-tail probability from Table G for the given N.

5.4.2 Normal Approximations

We could easily generate tables to apply the exact sign test for any sample size N. However, we know that the normal approximation to the binomial is

especially good when $\theta = 0.50$. Therefore, for moderate values of N (say at least 12), the normal approximation to the binomial can be used to determine the rejection regions. Since this is a continuous approximation to a discrete distribution, a continuity correction of 0.5 may be incorporated in the calculations. For example, for the alternative H_1: $M > M_0$, H_0 is rejected for $K \geq k_\alpha$, where k_α satisfies

$$k_\alpha = 0.5N + 0.5 + 0.5\sqrt{N}z_\alpha \tag{5.4.4}$$

Similarly, the approximate P value is

$$1 - \Phi\left(\frac{K_0 - 0.5 - 0.5N}{\sqrt{0.25N}}\right) \tag{5.4.5}$$

5.4.3 Zero Differences

A zero difference arises whenever $X_i = M_0$ for at least one i. Theoretically, zero differences do not cause a problem because the population was assumed to be continuous in the vicinity of the median. In reality, of course, zero differences can and do occur, either because the assumption of continuity is in error or because of imprecise measurements. Many zeros can be avoided by taking measurements to a larger number of significant figures.

The most common treatment of zeros is simply to ignore them and reduce N accordingly. The inferences are then conditional on the observed number of nonzero differences. An alternative approach is to treat half of the zeros as plus and half as minus. Another possibility is to assign to all zeros that sign which is least conducive to rejection of H_0; this is a strictly conservative approach. Finally, we could let chance determine the signs of the zeros by, say, flipping a balanced coin. These procedures are compared in Putter (1955) and Emerson and Simon (1979). A complete discussion, including more details on P values, is given in Pratt and Gibbons (1981). Randles (2001) proposed a more conservative method of handling zeros.

5.4.4 Power Function

In order to calculate the power of any test, the distribution of the test statistic under the alternative hypothesis should be available in a reasonably tractable form. In contrast to most nonparametric tests, the power function of the quantile tests is simple to determine since, in general, the random variable K follows the binomial probability distribution with parameters N and θ, where $\theta = P(X_i > \kappa_p)$ for the pth quantile. For the sign test the quantile of interest is the median and $\theta = P(X_i > M_0)$. For illustration, we will only consider the power of the sign test against the one-sided upper-tailed alternative H_1:$M > M_0$.

The power is a function of the unknown parameter θ, and the power curve or the power function is a graph of power versus various values of θ under the alternative. By definition, the power of the sign test against the alternative H_1 is the probability

$$\text{Pw}(\theta) = P(K \geq k_\alpha | H_1)$$

Under H_1, the distribution of K is binomial with parameters N and $\theta = P(X_i > M_0 \mid H_1)$ so the expression for power can be written as

$$\text{Pw}(\theta) = \sum_{i=k_\alpha}^{N} \binom{N}{i} \theta^i (1-\theta)^{N-i}$$

where k_α is the smallest integer such that

$$\sum_{i=k_\alpha}^{N} \binom{N}{i} (0.5)^N \leq \alpha$$

Thus, in order to evaluate the power function for the sign test, we first find the critical value k_α for a given significance level α, say 0.05. Then we need to calculate the probability $\theta = P(X_i > M_0 | H_1)$. If the power function is desired for a more parametric type of situation where the population distribution is fully specified then θ can be calculated. Such a power function would be desirable for comparisons between the sign test and some parametric test for location.

As an example, we calculate the power of the sign test of H_0:$M = 28$ versus H_1:$M > 28$ for $N = 16$ at significance level 0.05, under the assumption that the population is normally distributed with standard deviation 1 and median $M = 29.04$. Table G shows that the rejection region at $\alpha = 0.05$ is $K \geq 12$ so that $k_\alpha = 12$ and the exact size of this sign test is 0.0384. Now, under the assumption given, we can evaluate the underlying probability θ of a success as

$$\begin{aligned}
\theta &= P(X > 28 | H_1) \\
&= P\left(\frac{X - 29.04}{1} > \frac{28 - 29.04}{1}\right) \\
&= P(Z > -1.04) \\
&= 1 - \Phi(-1.04) \\
&= 0.8508 \\
&= 0.85, \text{say}
\end{aligned}$$

Note that the value of θ is larger than 0.5, which is in the legitimate region for the alternative H_1. Thus,

$$\begin{aligned}\text{Pw}(0.85) &= \sum_{i=12}^{16} \binom{16}{i} (0.85)^i (0.15)^{16-i} \\ &= 1 - \sum_{i=0}^{11} \binom{16}{i} (0.85)^i (0.15)^{16-i} = 0.9209\end{aligned}$$

This would be directly comparable with the normal theory test of $H_0: \mu = 28$ versus $H_1: \mu = 29.04$ with $\sigma = 1$, since the mean and median coincide for the normal distribution. The rejection region for this parametric test with $\alpha = 0.05$ is $\overline{X} > 28 + z_{0.05}/\sqrt{16} = 28.41$, and the power is

$$\begin{aligned}\text{Pw}(29.04) &= P[\overline{X} > 28.41 | X \sim \text{normal } (29.04, 1)] \\ &= P\left(\frac{\overline{X} - 29.04}{1/\sqrt{16}} > \frac{28.41 - 29.04}{0.25}\right) \\ &= P(Z > -2.52) \\ &= 0.9941\end{aligned}$$

Thus, the power of the normal theory test is larger than the power of the sign test, which is of course expected, since the normal theory test is known to be the best test when the population is normal. The problem with a direct comparison of the exact sign test with the normal theory test is that the powers of any two tests are directly comparable only when their sizes or significance levels are the same or nearly the same. In our case, the sign test has an exact size of 0.0384 whereas the normal theory test has exact size 0.05. This increase in the size of the test inherently biases the power comparison in favor of the normal theory test.

In order to ensure a more fair comparison, we might make the exact size of the sign test equal to 0.05 by using a randomized version of the sign test (as explained in Section 1.2.12). Alternatively, we might find the normal theory test of size $\alpha = 0.0384$ and compare the power of that test with the sign-test power of 0.9209. In this case, the rejection region is $\overline{X} > 28 + z_{0.0384}/\sqrt{16} = 28.44$ and the power is $\text{Pw}(29.04) = 0.9918$. This is still larger than the power of the sign test at $\alpha = 0.0384$ but two comments are in order. First and foremost, we have to assume that the underlying distribution is normal to justify using the normal theory test. No such assumption is necessary for the sign test. If the distribution is not normal and the sample size N is large, the calculated power is an approximation to the power of the normal theory test, by the central limit theorem. However, for the sign test, the size and the power calculations are exact for all sample sizes and no distribution assumptions are needed other than continuity. Further, the normal theory test is affected by the assumption about the population standard deviation σ, whereas the sign test calculations do not demand such knowledge. In order to obtain the power function, we can calculate the power at several values of M in the region of the alternative ($M > 28$) and

then plot the power versus the values of the median. This is easier under the normal approximation and is shown below.

Since under the alternative hypothesis H_1, the sign test statistic K has a binomial distribution with parameters N and $\theta = P(X > M_0|H_1)$, and the binomial distribution can be well approximated by the normal distribution, we can derive expressions to approximate the power of the sign test based on the normal approximation. These formulas are useful in practice for larger sample sizes and/or θ values for which exact tables are unavailable, although this appears to be much less of a problem with currently available software. We consider the one-sided upper-tailed case $H_1: M_1 > M_0$ for illustration; approximate power expressions for the other cases are left as exercises for the reader. The power for this alternative can be evaluated using the normal approximation with a continuity correction as

$$\begin{aligned} \text{Pw}(M_1) &= P(K \geq k_\alpha | H_1: M_1 > M_0) \\ &= P\left(Z > \frac{k_\alpha - N\theta - 0.5}{\sqrt{N\theta(1-\theta)}}\right) \\ &= 1 - \Phi\left(\frac{k_\alpha - N\theta - 0.5}{\sqrt{N\theta(1-\theta)}}\right) \end{aligned} \tag{5.4.6}$$

where $\theta = P(X > M_0|M_1 > M_0)$ and k_α is such that

$$\begin{aligned} \alpha &= P(K \geq k_\alpha | H_0) \\ &= P\left(Z > \frac{k_\alpha - N/2 - 0.5}{\sqrt{N/4}}\right) \\ &= 1 - \Phi\left(\frac{2k_\alpha - N - 1}{\sqrt{N}}\right) \end{aligned} \tag{5.4.7}$$

The equality in (5.4.7) implies that $k_\alpha = [N + 1 + \sqrt{N}\Phi^{-1}(1-\alpha)]/2$. Substituting this back into (5.4.6) and simplifying gives

$$\begin{aligned} \text{Pw}(M_1) &= P\left\{Z > \frac{0.5[N + 1 + \sqrt{N}\Phi^{-1}(1-\alpha)] - N\theta - 0.5}{\sqrt{N\theta(1-\theta)}}\right\} \\ &= 1 - \Phi\left[\frac{N(0.5-\theta) + 0.5\sqrt{N}z_\alpha}{\sqrt{N\theta(1-\theta)}}\right] \end{aligned} \tag{5.4.8}$$

where $z_\alpha = \Phi^{-1}(1-\alpha)$ is the $(1-\alpha)$th quantile of the standard normal distribution. For example, $z_{0.05} = 1.645$ and $z_{0.85} = -1.04$. Note that $z_\alpha = -z_{1-\alpha}$. The approximate power values are calculated and shown in Table 5.4.1 for $N = 16$ and $\alpha = 0.05$. A graph of the power function is shown in Figure 5.4.1.

TABLE 5.4.1

Normal Approximation to Power of the Sign Test for the Median When $N = 16$

θ	0.5	0.55	0.6	0.65	0.70	0.75	0.80	0.85	0.90
Power	0.0499	0.1054	0.1942	0.3204	0.4804	0.6591	0.8274	0.9471	0.9952

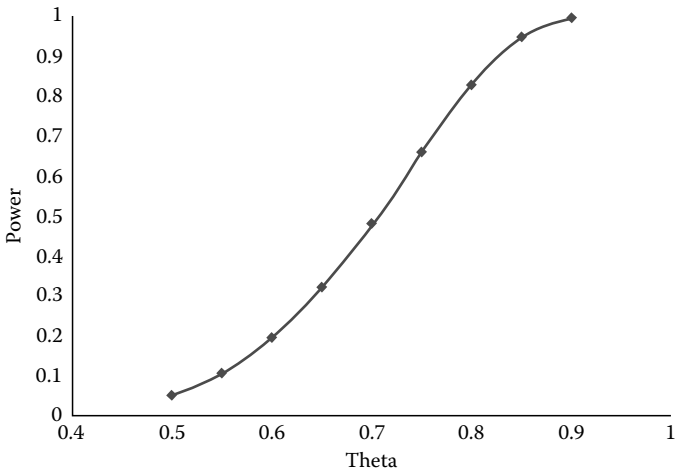


FIGURE 5.4.1
Normal approximation (with continuity correction) to the power function of the sign test for the median.

It should be noted that the power of the sign test depends on the alternative hypothesis through the probability $\theta = P(X > M_0|H_1{:}M_1 > M_0)$. Under H_0, we have $\theta = 0.5$, whereas $\theta > 0.5$ under H_1, since if $M_1 > M_0$, $P(X > M_0|H_1{:}M = M_1 > M_0) > P(X > M_1|H_1{:}M = M_1 > M_0)$ and therefore $\theta = P(X > M_0|H_1) > P(X > M_1|H_1) = 0.5$. Thus, the power of the sign test depends on the ''distance'' between the values of θ under the alternative and a value of $\theta > 0.5$ must be specified for the power calculation for $M_1 > M_0$. Noether (1987) suggested choosing a value of θ based on past information or a pilot study, or using an ''odds-ratio.'' In the normal theory test (such as the t test), however, the power depends directly on the ''distance'' $M_1 - M_0$, the difference between the values of the median under the null hypothesis and under the alternative. Note also that the approximate power is equal to the nominal size of the test when $\theta = 0.5$ (the null hypothesis is true). Expressions for approximate power against other alternatives are left as exercises for the reader.

5.4.5 Simulated Power

The power function for the sign test is easily found, particularly when the normal approximation is used for calculations. For many other nonparametric

tests, however, the power function can be quite difficult to calculate. In such cases, computer simulations can be used to estimate the power. We use a MINITAB Macro program to simulate the power of the sign test when the underlying distribution is normal with mean = median = M and variance σ^2.

The null hypothesis is H_0:$M = M_0$ and the alternative is H_0:$M = M_1 > M_0$. First we need to find the relationship between M_0, M_1, and θ where $\theta = P(X_i > M_0 | H_1)$. Assuming X is normally distributed with variance σ^2, we have

$$\theta = P\left(\frac{X - M_1}{\sigma} > \frac{M_0 - M_1}{\sigma}\right)$$
$$= \Phi\left(\frac{M_1 - M_0}{\sigma}\right)$$

This gives $(M_1 - M_0)/\sigma = \Phi^{-1}(\theta)$. Now assume arbitrarily that $M_0 = 0.5$ and $\sigma^2 = 1$. Then if $\theta = 0.55$, say, $\Phi^{-1}(0.55) = 0.1256$ and thus $M_1 = 0.6256$. Next we need to specify a sample size and the probability of a type I error. We arbitrarily choose $N = 13$ and $\alpha = 0.05$. From Table G, the rejection region for exact size 0.0461 is $K \geq 10$.

First, we generate 1000 random samples, each of size 13, from a normal distribution with $M = 0.6256$ and compute the value of the sign test statistic K for each sample generated, i.e., the number of X_i in that sample for which $X_i - M_0 = X_i - 0.5 > 0$. Then we note whether or not this K is in the rejection region. Then we count the number of times the value K is in the rejection region among the 1000 random samples generated. This count divided by 1000 is the simulated power at the point $\theta = 0.55$ (which corresponds to $M_1 = 0.6256$) in the case $N = 13$, $M_0 = 0.50$, $\sigma = 1$, and $\alpha = 0.0461$. Using a MINITAB Macro program, this value is found as 0.10. The program code is shown below. The Macro also calculates the simulated power of the signed-rank test to be discussed in Section 5.7.

```
Macro signrank
# simulates powers of sign and the signed-rank test
signrank
mcolumn c1 c2 c3 c4 pow1 pow2 theta phiinv mu
mconstant k1 k2 k3 k4 k5 cou mu1
let k5 = 1
set theta
(.5: .9/.05)1
end
print theta
invcdf theta phiinv;
normal 0.0 1.0.
```

```
let mu = 0.5 + phiinv
mlabel 4
let cou = 1
let pow1(k5) = 0
let pow2(k5) = 0
mlabel 1
let mu1 = mu(k5)
random 13 c1;
normal mu1 1.
Let c1 = c1 - 0.50
# calculate Sign and Signed Rank test statistics
let c2 = (c1 gt 0)
let k1 = sum(c2)
# k1 = value of Sign stat
let c3 = abs(c1)
let c3 = rank(c3)
let c4 = c2*c3
let k2 = sum(c4)
# k2 = value of Signed-Rank stat
# print c1 c2 c3 c4 k1 k2
# k3 is the critical value of sign test from Table G
# at n = 13, exact alpha = 0.0461
let k3 = 10
# k3 is the critical value for the signed
# rank test from Table H at n = 13, exact alpha = 0.47
let k4 = 70
if k1 ge 10
let pow1(k4) = pow1(k5) + 1
else
let pow1(k5) = pow1(k5) + 0
endif
if k2 ge 70
let pow2(k5) = pow2(k5) + 1
else
let pow2(k5) = pow2(k5) + 0
endif
let cou = cou + 1
if cou gt 1000
go to 2
else
go to 1
endif
mlabel 2
let pow1(k5) = pow1(k5)/1000
let pow2(k5) = pow2(k5)/1000
```

```
print k5 mu(k5) theta(k5) pow1(k5) pow2(k5)
let k5 = k5 + 1
if k5 gt 9
go to 3
else
go to 4
endif
mlabel 3
plot pow1*mu pow2*mu;
 connect;
 color 1 2;
axis 2;
 label 'simulated power';
overlay.
Endmacro
```

To run the Macro, type the statements into a plain text file using a text editor and save the file with a name say sign.mac (MINITAB uses the.mac extension for Macros). Then in MINITAB go to the command line editor and then type %a:\sign.mac and click on submit. The program will run and print the simulated power values and draw the power curves. The output from such a simulation is shown later in Section 5.7.3; the power curves are shown in Figure 5.7.1.

5.4.6 Sample Size Determination

In order to make an inference regarding the population median using the sign test, we need to have a random sample of observations. If we are allowed to choose the sample size, we might want to determine the value of N such that the test has size α and power $1-\beta$, given the null and the alternative hypotheses and other necessary assumptions. For the sign test against the one-sided upper-tailed alternative $H_1: M > M_0$, we need to find N such that

$$\sum_{i=k_\alpha}^{N} \binom{N}{i} (0.5)^N \leq \alpha \quad \text{and} \quad \sum_{i=k_\alpha}^{N} \binom{N}{i} \theta^i (1-\theta)^{N-i} \geq 1-\beta$$

where α, $1-\beta$ and $\theta = P(X > M_0 | H_1)$ are all specified. Note also that the size and power requirements have been modified to state ''at most'' α and ''at least'' $1-\beta$, in order to accommodate the fact that the binomial distribution is discrete. Tables are available to aid in solving these equations; see for example, Cohen (1977). We illustrate the process using the normal approximation to the power because the necessary equations are much easier to solve.

Under the normal approximation, the power of a size α sign test with $H_1: M > M_0$ is given in (5.4.8). Thus, we require that $1 - \Phi[N(0.5 - \theta) + 0.5\sqrt{N}z_\alpha / \sqrt{N\theta(1-\theta)}] = 1 - \beta$ and the solution for N is

$$N = \left[\frac{\sqrt{\theta(1-\theta)}\Phi^{-1}(\beta) - 0.5z_\alpha}{0.5 - \theta}\right]^2 = \left[\frac{\sqrt{\theta(1-\theta)}z_\beta + 0.5z_\alpha}{0.5 - \theta}\right]^2 \tag{5.4.9}$$

which should be rounded up to the next integer. The approximate sample size formula for the one-sided lower-tailed alternative $H_1: M < M_0$ is the same except that here $\theta = P(X > M_0 | H_1) < 0.5$. A sample size formula for the two-sided alternative is the same as (5.4.9) with α replaced by $\alpha/2$. The derivation is left as an exercise for the reader.

For example, suppose $\theta = 0.2$. If we take $\alpha = 0.05$ and $1 - \beta = 0.90$, then $z_\alpha = 1.645$ and $z_\beta = 1.282$. Then (5.4.9) yields $\sqrt{N} = 4.45$ and $N = 19.8$. Thus, we need at least 20 observations to meet the specifications.

5.4.7 Confidence Interval for the Median

A two-sided confidence-interval estimate for an unknown population median can be obtained from the acceptance region of the sign test against the two-sided alternative. The acceptance region for a two-tailed test of $H_0: M = M_0$, using (5.4.3), is

$$k'_{\alpha/2} + 1 \le K \le k_{\alpha/2} - 1 \tag{5.4.10}$$

where K is the number of positive differences among $X_i - M_0$, $i = 1, 2, \ldots, N$ and $k'_{\alpha/2}$ and $k_{\alpha/2}$ are integers such that

$$\sum_{i=k_{\alpha/2}}^{N} \binom{N}{i} (0.5)^N \le \frac{\alpha}{2} \quad \text{and} \quad \sum_{i=0}^{k'_{\alpha/2}} \binom{N}{i} (0.5)^N \le \frac{\alpha}{2}$$

or, equivalently,

$$P\left(k'_{\alpha/2} + 1 \le K \le k_{\alpha/2} - 1\right) \ge 1 - \alpha$$

Comparing these with Equations 5.2.6 and 5.2.7 for the quantile confidence interval, the equal-tailed confidence-interval endpoints for the unknown population median are the order statistics $X_{(r)}$ and $X_{(s)}$ where r and s are the largest and smallest integers respectively, such that

$$\sum_{i=0}^{r-1} \binom{N}{i} (0.5)^N \le \frac{\alpha}{2} \quad \text{and} \quad \sum_{i=s}^{N} \binom{N}{i} (0.5)^N \le \frac{\alpha}{2} \tag{5.4.11}$$

The solutions $r-1$ and s are easily found from Table G in the columns labeled left tail and right tail, respectively. However, because of the symmetry of the binomial distribution, only one of the constants, say r, needs to be determined, since $s = N - r + 1$.

For larger sample sizes, using (5.2.9)

$$r = k'_{\alpha/2} + 1 = 0.5N + 0.5 - 0.5\sqrt{N}z_{\alpha/2} \tag{5.4.12}$$

and

$$s = k_{\alpha/2} = 0.5N + 0.5 + 0.5\sqrt{N}z_{\alpha/2} \tag{5.4.13}$$

so that for the median, $s = N - r + 1$ or $k_{\alpha/2} = N - k'_{\alpha/2}$. We round down for r and round up for s for a conservative solution.

In order to compare the exact and approximate confidence-interval endpoints suppose $N = 15$ and $1 - \alpha = \gamma = 0.95$. Then, from Table G with $\theta = 0.5$, we find $r - 1 = 3$ so that $r = 4$ and $s = 12$ for $P = 0.0176$. Hence, the confidence-interval endpoints are $X_{(4)}$ and $X_{(12)}$ with exact confidence level $\gamma = 0.9648$. Note that the next value of P in Table G is 0.0592, which is greater than $\alpha/2 = 0.025$ and therefore $r = 5$ and $s = 11$ does not satisfy (5.4.11). For the approximate 95% confidence interval (5.4.12) gives $r = 0.5 + 7.5 - 0.5\sqrt{15}(1.96) = 4.20$, which we round down to 4. Hence, the confidence interval based on the normal approximation agrees with the one based on the exact binomial distribution.

5.4.8 Problem of Zeros

Zeros do not present a problem in finding a confidence-interval estimate of the median using this procedure. As a result, the sample size N is not reduced for zeros and zeros are counted as many times as they occur in determining confidence-interval endpoints. If the real interest is in hypothesis testing and there are many zeros, the power of the test will be greater if the test is carried out using a confidence-interval approach.

5.4.9 Paired-Sample Procedures

The one-sample sign-test procedures for hypothesis testing and confidence-interval estimation of M are equally applicable to paired-sample data. For a random sample of N pairs $(X_1, Y_1), \ldots, (X_N, Y_N)$, the N differences $D_i = X_i - Y_i$ are formed. If the population of differences is assumed continuous at its median M_D so that $P(D = M_D) = 0$, and θ is defined as $\theta = P(D > M_D)$, the same procedures are clearly valid here with X_i replaced everywhere by D_i.

It should be emphasized that this is a test for the median difference M_D, which is not necessarily the same as the difference of the two medians M_X and M_Y. The following simple example will illustrate this often misunderstood fact. Let X and Y have the joint distribution

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & \text{for } y-1 \le x \le y,\ -1 \le y \le 1 \\ & \quad \text{or } y+1 \le x \le 1,\ -1 \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Then X and Y are uniformly distributed over the shaded region in Figure 5.4.2. It can be seen that the marginal distributions of X and Y are identical, both being uniform on the interval $(-1, 1)$, so that $M_X = M_Y = 0$. It is clear that where X and Y have opposite signs, in quadrants II and IV,

$$P(X < Y) = P(X > Y)$$

while in quadrants I and III, $X < Y$ always. For all pairs, then, we have $P(X < Y) = 3/4$, while implies that the median of the population of differences is smaller than zero. It will be left as an exercise for the reader to show that the cdf of the difference random variable $D = X - Y$ is

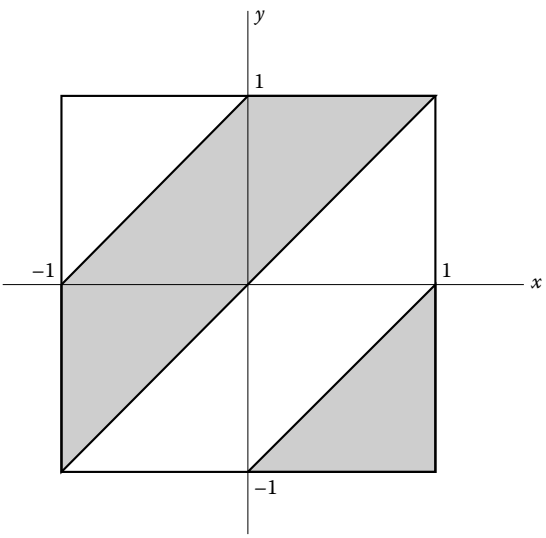


FIGURE 5.4.2
Region of integration is the shaded area.

$$F_D(d) = \begin{cases} 0 & \text{for } \leq -1 \\ (d+1)(d+3)/4 & \text{for } -1 < d \leq 0 \\ 3/4 & \text{for } 0 < d \leq 1 \\ d(4-d)/4 & \text{for } 1 < d \leq 2 \\ 1 & \text{for } d > 2 \end{cases} \tag{5.4.14}$$

The median difference is that value M_D, of the distribution of D, such that $F_D(M_D) = 1/2$. The reader can verify that this yields $M_D = -2 + \sqrt{3}$.

In general, then, it is not true that $M_D = M_X - M_Y$. On the other hand, it is true that a mean of differences equals the difference of means. Since the mean and median coincide for symmetric distributions, if the X and Y populations are both symmetric and $M_X = M_Y$, and if the difference population is also symmetric,* then $M_D = M_X - M_Y$ and $M_X = M_Y$ is a necessary and sufficient condition for $M_D = 0$. Note that for the case where X and Y are each normally distributed, the difference of their medians (or means) is equal to the median (or mean) of their differences $X - Y$, since $X - Y$ is also normally distributed with median (or mean) equal to the difference of the respective median (or means).

Earlier discussions of power and sample size also apply to the paired-sample data problems.

5.4.10 Applications

We note that the sign test is a special case of the quantile test with $p = 0.5$, since the quantile specified is the population median. This test is easier to apply than the general quantile test because the binomial distribution for $\theta = 0.5$ is symmetric for any N. We write the null hypothesis here as $H_0: M = M_0$. The appropriate rejection regions in terms of K, the number of plus signs among $X_1 - M_0$, $X_2 - M_0, \ldots, X_N - M_0$, and corresponding exact P values, are summarized as follows:

Alternative	Rejection Region	Exact P Value
$M > M_0$	$K \geq k_\alpha$	$\sum_{i=K_0}^{N} \binom{N}{i} (0.5)^N$
$M < M_0$	$K \leq k'_\alpha$	$\sum_{i=0}^{K_0} \binom{N}{i} (0.5)^N$
$M \neq M_0$	$K \leq k'_{\alpha/2}$ or $K \geq k_{\alpha/2}$	2 (smaller of the one-tailed P values)

Table C with $\theta = 0.5$ and n (representing N) can be used to determine the critical values. Table G is simpler to use because it gives both left-tail and right-tail binomial probabilities for $N \leq 20$ when $\theta = 0.5$.

* The difference population is symmetric if X and Y are symmetric and independent or if $f_{X,Y}(x, y) = f_{X,Y}(-x, -y)$.

For large sample sizes, the appropriate rejection regions and the P values, based on the normal approximations to the binomial distribution with a continuity correction, are as follows:

Alternative	Rejection Region	Approximate P Value
$M > M_0$	$K \geq 0.5N + 0.5 + 0.5z_\alpha\sqrt{N}$	$1 - \Phi\left(\frac{K_0 - 0.5N - 0.5}{0.5\sqrt{N}}\right)$
$M < M_0$	$K \leq 0.5N - 0.5 - 0.5z_\alpha\sqrt{N}$	$\Phi\left(\frac{K_0 - 0.5N + 0.5}{0.5\sqrt{N}}\right)$
$M \neq M_0$	Both above with $z_{\alpha/2}$	2 (smaller of the one-tailed P values)

If any zeros are present, we will ignore them and reduce N accordingly. As we have seen, a pre-specified significance level α often cannot be achieved with nonparametric statistical inference because most of the applicable sampling distributions are discrete. This problem is avoided if we determine the P value of a test result and use it to make our decision.

For a two-sided alternative, the common procedure is to define the P value as twice the smaller of the two one-sided P values, as described in the case for general quantiles. This ''doubling'' is particularly meaningful when the null distribution of the test statistic is symmetric, as is the case here. If instead we used the larger of the two one-tailed P values and doubled that, the final P value would be greater than 1, which is not acceptable. If the observed value of K is less than $N/2$, the upper-tail probability will be greater than 0.5 and the lower-tail probability will be less than 0.5 and so the P value is twice the lower- (left-) tail probability. If the observed value of K is greater than $N/2$, the upper-tail probability will be less than 0.5 so the P value is twice the upper- (right-) tail probability. If the observed value K is exactly equal to $N/2$, the two-tailed P value is taken to be 1.0. If we have a pre-specified α and wish to reach a decision, we should reject H_0 whenever the P value is less than or equal to α and accept H_0 otherwise.

For example, suppose we observe $K = 4$ plus signs among $N = 12$ nonzero sample differences. Table G shows that the left-tail P value is 0.1938; since there is no entry equal to 4 in the right-tail column, we know that the right-tail P-value exceeds 0.5. Thus, the two-sided P value is $2(0.1938) = 0.3876$. Note that for any $K < N/2 = 6$, the upper-tail probability is greater than 0.5 and the lower-tail probability is less than 0.5. Conversely, for any $K > N/2 = 6$, the upper-tail probability is less than 0.5 and the lower-tail probability is greater than 0.5. Also, by symmetry, the probability of say 4 or less is the same as the probability of 8 or more. Thus, to calculate the P value for the two-sided alternative, the convention is to take the smaller of the two one-tailed P values and double it. If instead we used the larger of the P values and doubled that, the final P value could possibly be more than 1.0, which is not acceptable. Note also that when the observed value of K is exactly equal to 6, the two-sided P value will be taken to be equal to 1.0.

In our example, the observed value 4 for $N = 12$ is less than 6, so the smaller one-tailed P value is in the lower tail and is equal to 0.1938 and this leads to a two-sided P value of 0.3876 as found earlier. If we have a prescribed α and wish to reach a decision, we should reject H_0 whenever the P value is less than or equal to α and accept H_0 otherwise.

The exact distribution-free confidence interval for the median can be found from Table C but is easier to find using Table G. The choice of exact confidence levels is limited to $1-2P$, where P is a tail probability in Table G for the appropriate value of N. From (5.4.10), the lower confidence limit is the $\left(k'_{\alpha/2}+1\right)$th $= r$th order statistic in the sample, where $k'_{\alpha/2}$ is the left-tail critical value of the sign test statistic K from Table G, for the given α and N such that P is less than or equal to $\alpha/2$. But since the critical values are all of the nonnegative integers, $\left(k'_{\alpha/2}+1\right)$ is simply the rank of $k'_{\alpha/2}$ among the entries in Table G for that N. The calculation of this rank will become clearer after we do Example 5.4.1.

For consistency with the results given later for confidence-interval endpoints based on other nonparametric test procedures, we note that r is the rank of the left-tail entry in Table G for this N, and we denote this rank by u. Further, by symmetry, we have $X_{(s)} = X_{(N-r+1)}$. The confidence-interval endpoints are the uth smallest and the uth largest order statistics, where u is the rank of the left-tail critical value of K from Table G that corresponds to $P \leq \alpha/2$. The corresponding exact confidence coefficient is then $\gamma = 1 - 2P$. For sample sizes outside the range of Table G we have

$$u = 0.5 + 0.5N - 0.5\sqrt{N}z_{\alpha/2} \tag{5.4.15}$$

from (5.4.4), and we always round the result of (5.4.15) down.

For example, for confidence level 0.95 and $N = 15$, $\alpha/2 = 0.025$, the P from Table G closest to 0.025 but not exceeding it is 0.0176. The corresponding left-tail critical value is 3, which has a rank of 4 among the left-tail critical values for this N. Thus, $u = 4$ and the 95% confidence interval for the median is the interval $(X_{(4)}, X_{(12)})$.The exact confidence level is $1 - 2P = 1 - 2(0.0176) = 0.9648$.

Note that unlike in the case of testing hypotheses, any zeros are counted as many times as they appear for determination of the confidence-interval endpoints.

Example 5.4.1

Suppose that each of 13 randomly chosen female registered voters was asked to indicate if she was going to vote for candidate A or candidate B in an upcoming election. The results show that 9 of the subjects preferred A. Is this sufficient evidence to conclude that candidate A is preferred to B by female voters?

Solution

With this kind of data, the sign test is one of the few statistical tests that is valid and can be applied. Let θ be the true probability that candidate A is preferred over candidate B. The null hypothesis is that the two candidates are equally preferred, that is, $H_0: \theta = 0.5$ and the one-sided upper-tailed alternative is that A is preferred over B, that is $H_1: \theta > 0.5$. The sign test can be applied here and the value of the test statistic is $K = 9$. Using Table G with $N = 13$, the exact P value in the right-tail is 0.1334; there is not sufficient evidence to conclude that the female voters prefer candidate A over B, at a commonly used significant level such as 0.05.

Example 5.4.2

Some researchers claim that susceptibility to hypnosis can be acquired or improved through training. To investigate this claim six subjects were rated on a scale of 1–20 accordingly to their initial susceptibility to hypnosis and then given 4 weeks of training. Each subject was rated again after the training period. In the ratings below, higher numbers represent greater susceptibility to hypnosis. Do these data support the claim?

Subject	Before	After
1	10	18
2	16	19
3	7	11
4	4	3
5	7	5
6	2	3

Solution

The null hypothesis is $H_0: M_D = 0$ and the appropriate alternative is $H_1: M_D > 0$ where M_D is the median of the differences, after training minus before training. The number of positive differences is $K_0 = 4$ and the right-tail P value for $N = 6$ from Table G is 0.3438. Hence, the data do not support the claim at any level smaller than 0.3438. This implies that 4 is not an extreme value of K under H_0 and rejection of the null hypothesis is not warranted. Also, from Table G, at $\alpha = 0.05$, the rejection region is $K \geq 6$, with exact size 0.0156. Since the observed value of $K = 4$, we again fail to reject H_0.

The following computer printouts illustrate the solution to Example 5.4.3 based on the STATXACT, MINITAB, and SAS packages. The STATXACT solution agrees with ours for the exact one-sided P value. Their asymptotic P value (0.2071) is based on the normal approximation without the continuity correction. The MINITAB solution agrees exactly with ours. The SAS solution shown in Figure 5.4.3 gives only the two-tailed P values. Their exact P value equals 2 times ours; they also give P values based on Student's t test and the signed-rank test discussed

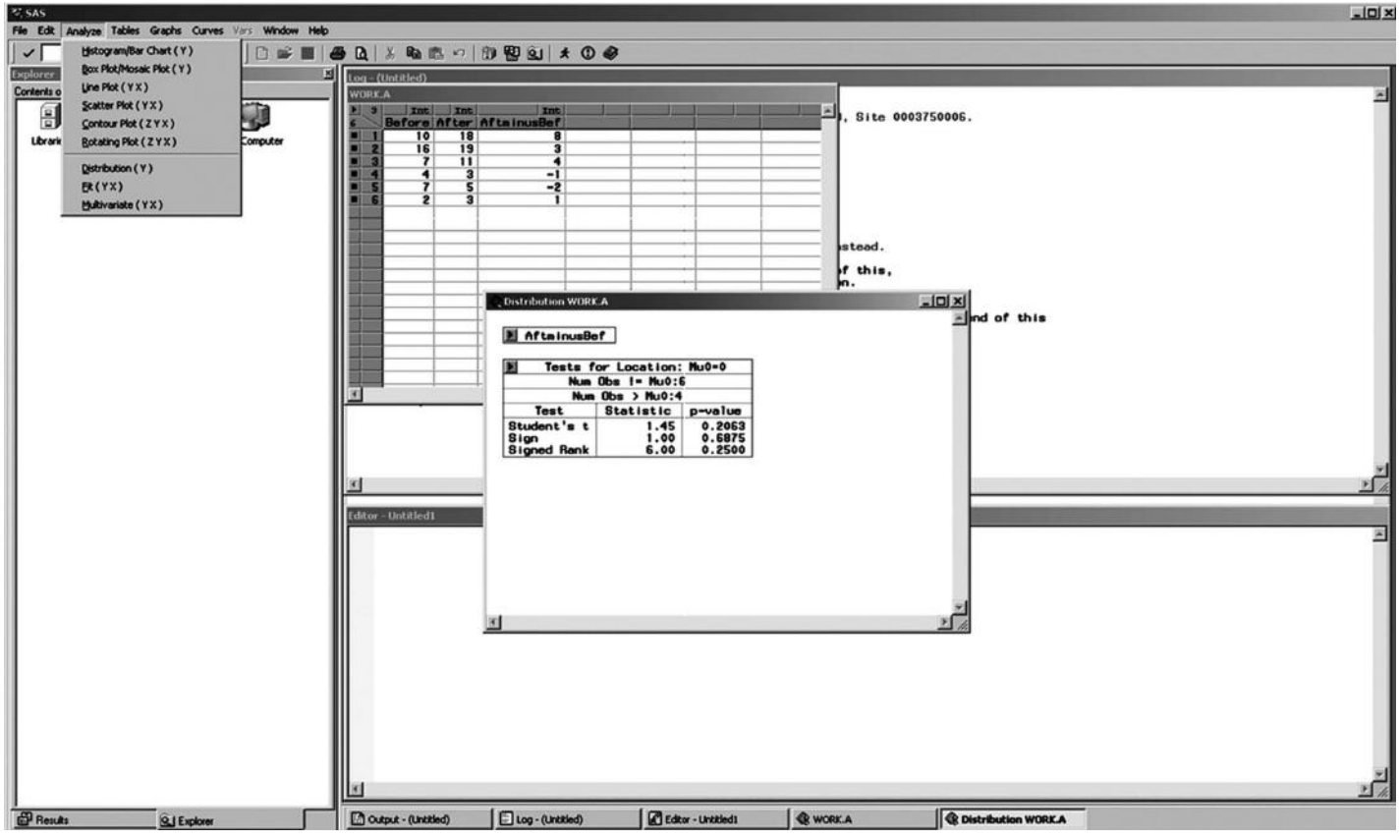


FIGURE 5.4.3
SAS Solution to Example 5.4.2.

later in this chapter. Note that SAS calculates the value of the sign test statistic as one-half of the difference between the number of positive and the number of negative differences.

```
                ****************************************
                STATXACT SOLUTION TO EXAMPLE 5.4.2
                ****************************************
SIGN TEST

Summary of Exact Distribution of SIGN Statistic:
Min          Max      Mean     Std-Dev     Observed     Standardized
0.0000     6.000     3.000       1.225        2.000          −0.8165

Asymptotic Inference:
One-sided P-value: Pr ( Test Statistic .LE. Observed ) =  0.2071
Two-sided P-value: 2 * One-sided                       =  0.4142

Exact Inference:
One-sided P-value:
Pr ( Test Statistic. LE. Observed )                    =  0.3438
Pr ( Test Statistic .EQ. Observed )                    =  0.2344
Two-sided P-value: 2*One-sided                         =  0.6875

                ****************************************
                MINITAB SOLUTION TO EXAMPLE 5.4.2
                ****************************************
Sign Test for Median: Af − Be

Sign test of median = 0.00000 versus  >  0.00000

          N     Below     Equal     Above          P     Median
Af-Be     6         2         0         4     0.3438      2.000
```

Now suppose we want to know, before an investigation starts, how many subjects should be included in the study when we plan to use the sign test for the median difference at a level of significance $\alpha = 0.05$, and we want to detect $P(D > 0) = 0.6$ with a power 0.85. Note that $P(D > 0) = 0.6$ means that the median difference, M_D, is greater than 0, the hypothesized value, and thus the test should have an upper-tailed alternative. With $\theta = 0.6$, $z_{0.05} = 1.645$, and $z_{0.15} = 1.0365$, Equation 5.4.9 gives $N = 176.96$, which we round up to 177. The MINITAB solution to this example is shown below. It also uses the normal approximation and the result 177 agrees with ours.

The solution also shows $N = 222$ observations will be required for a two-tailed test. The reader can verify this. The solution is labeled ''Test for One Proportion'' instead of ''Sign Test'' because it is applicable for a test for a quantile of any order p (as in Section 5.3).

```
************************************************
MINITAB SOLUTION TO POWER AND SAMPLE SIZE
************************************************
```

Power and Sample Size

Test for one proportion

Testing proportion = 0.5 (versus > 0.5)
Calculating power for proportion = 0.6
Alpha = 0.05 Difference = 0.1

Sample Size	Target Power	Actual Power
177	0.8500	0.8501

Power and Sample Size

Test for one proportion

Testing proportion = 0.5 (versus not = 0.5)
Calculating power for proportion = 0.6
Alpha = 0.05 Difference = 0.1

Sample Size	Target Power	Actual Power
222	0.8500	0.8511

Example 5.4.3

Nine pharmaceutical laboratories cooperated in a study to determine the median effective dose level of a certain drug. Each laboratory carried out experiments and reported its effective dose. For the results 0.41, 0.52, 0.91, 0.45, 1.06, 0.82, 0.78, 0.68, 0.75, estimate the interval of median effective dose with confidence level 0.95.

Solution

We go to Table G with $N = 9$ and find $P = 0.0195$ is the largest entry that does not exceed 0.025, and this entry has rank $u = 2$. Hence, the second smallest and second largest (or the $9 - 2 + 1 = 8$th smallest) order statistics, namely $X_{(2)}$ and $X_{(8)}$, provide the endpoints as $0.45 < M < 0.91$ with exact confidence coefficient $1 - 2(0.0195) = 0.961$. The MINITAB solution shown gives the two confidence intervals with the exact confidence coefficient on each side of 0.95, as well as an exact 95% interval labeled NLI, based on an interpolation scheme between the lower and upper endpoints. The nonlinear interpolation scheme is from Hettmansperger and Sheather (1986).

```
****************************************
MINITAB Solution to Example 5.4.3
****************************************
```

```
Sign CI: C1

Sign confidence interval for median
                                          Confidence
                           Achieved        Interval
        N    Median    Confidence      Lower      Upper    Position
CI      9    0.7500        0.8203     0.5200     0.8200           3
                           0.9500     0.4660     0.8895         NLI
                           0.9609     0.4500     0.9100           2
```

5.5 Rank-Order Statistics

The other one-sample procedure to be covered in this chapter is the Wilcoxon signed-rank test. This test is based on a special case of what are called rank-order statistics. The *rank-order statistics* for a random sample are any set of constants which indicate the order of the observations. The actual magnitude of any observation is used only in determining its relative position in the sample array and is thereafter ignored in any analysis based on rank-order statistics. Thus, statistical procedures based on rank-order statistics depend only on the relative magnitudes of the observations. Rank-order statistics might then be defined as the set of numbers that results when each original observation is replaced by the value of some order-preserving function. Suppose we have a random sample of N observations $X_1, X_2, \ldots, X_N$. Let the rank-order statistics be denoted by $r(X_1), r(X_2), \ldots, r(X_N)$ where r is any function such that $r(X_i) \leq r(X_j)$ whenever $X_i \leq X_j$. As with order statistics, rank-order statistics are invariant under monotone transformations, i.e., if $r(X_i) \leq r(X_j)$, then $r[F(X_i)] \leq r[F(X_j)]$, in addition to $F[r(X_i)] \leq F[r(X_j)]$, where F is any nondecreasing function.

For any set of N different sample observations, the simplest numbers to use to indicate relative position are the first N positive integers. In order to eliminate the possibility of confusion and to simplify and unify the theory of rank-order statistics, we will assume here that unless explicitly stated otherwise, the rank-order statistics are always a permutation of the first N integers. The ith rank-order statistic $r(X_i)$ then is called the rank of the ith observation in the original unordered sample. The value it assumes $r(x_i)$ is the number of observations x_j, $j = 1, 2, \ldots, N$, such that $x_j \leq x_i$. For example, the rank of the ith-order statistic is equal to i, or $r(x_{(i)}) = i$. A functional definition of the rank of any x_i in a set of N different observations is provided by

$$r(x_i) = \sum_{j=1}^{N} S(x_i - x_j) = 1 + \sum_{1 \le j \ne i \le N} S(x_i - x_j) \tag{5.5.1}$$

where

$$S(u) = \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } u \ge 0 \end{cases} \tag{5.5.2}$$

The rank-order statistic $r(X_i)$ is a discrete random variable and for a random sample from a continuous population it follows the discrete uniform distribution, or

$$P[r(X_i) = j] = 1/N \quad \text{for } j = 1, 2, \ldots, N$$

Although the terminology may seem confusing at the outset, a function of the rank-order statistics will be called a *rank statistic*. Rank statistics are particularly useful in nonparametric inference since they are usually distribution free. The methods are applicable to a wide variety of hypothesis-testing situations depending on the particular function used. The procedures are generally simple and quick to apply. Since rank statistics are functions only of the ranks of the observations, only this information is needed in the sample data. Actual measurements are often difficult, expensive, or even impossible to obtain. When actual measurements are not available for some reason but relative positions can be determined, rank-order statistics make use of all of the information available.

When the fundamental data consist of variate values and these actual magnitudes are ignored after obtaining the rank-order statistics, we may be concerned about loss of efficiency. One approach to a judgment concerning the potential loss of efficiency is to determine the correlation between the variate values and their assigned ranks. If the correlation is high, we would feel intuitively more justified in replacing actual values by ranks for the purpose of analysis. The hope is that inference procedures based on ranks alone will lead to conclusions that seldom differ from a corresponding inference based on actual variate values.

The ordinary product–moment correlation coefficient between two random variables X and Y is

$$\rho(X, Y) = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

Assume that for a continuous population with cdf F_X (pdf f_X) we would like to determine the correlation between the random variable X and its rank $r(X)$. Theoretically, a random variable from an infinite population cannot have a rank, since values on a continuous scale cannot be ordered. But an observation X_i, of a random sample of size N from this population, does have a

rank $r(X_i)$ as defined in (5.5.1). The distribution of X_i is the same as the distribution of X and the $r(X_i)$ are identically distributed though not independent. Therefore, it is reasonable to define the population correlation coefficient between ranks and variate values as the correlation between X_i and $Y_i = r(X_i)$, or

$$\rho[X, r(X)] = \frac{E(X_i Y_i) - E(X_i)E(Y_i)}{\sigma_X \sigma_{Y_i}} \tag{5.5.3}$$

The marginal distribution of Y_i for any i is the discrete uniform, so that

$$f_{Y_i}(j) = \frac{1}{N} \quad \text{for } j = 1, 2, \ldots, N \tag{5.5.4}$$

with moments

$$E(Y_i) = \sum_{j=1}^{N} \frac{j}{N} = \frac{N+1}{2} \tag{5.5.5}$$

$$E\left(Y_i^2\right) = \sum_{j=1}^{N} \frac{j^2}{N} = \frac{(N+1)(2N+1)}{6}$$

$$\text{var}(Y_i) = \frac{(N+1)(2N+1)}{6} - \frac{(N-1)^2}{4} = \frac{N^2-1}{12} \tag{5.5.6}$$

The joint pdf of X_i and its rank Y_i is

$$f_{X_i, Y_i}(x, j) = f_{X_i|Y_i=j}(x|j) f_{Y_i}(j) = \frac{f_{X_{(j)}}(x)}{N} \quad \text{for } j = 1, 2, \ldots, N$$

where $X_{(j)}$ denotes the jth-order statistic of a random sample of size N from the cdf F_X. From this expression we can write

$$E(X_i Y_i) = \frac{1}{N} \int_{-\infty}^{\infty} \sum_{j=1}^{N} j x f_{X_{(j)}}(x) dx = \sum_{j=1}^{N} \frac{j E(X_{(j)})}{N} \tag{5.5.7}$$

Substituting the results (5.5.5), (5.5.6), and (5.5.7) back into (5.5.3), we obtain

$$\rho[X, r(X)] = \left(\frac{12}{N^2-1}\right)^{1/2} \frac{\sum_{j=1}^{N} j E(X_{(j)}) - [N(N+1)/2]E(X)}{N\sigma_X} \tag{5.5.8}$$

Since the result here is independent of i, our definition in (5.5.3) may be considered a true correlation. The same result is obtained if the covariance between X and $r(X)$ is defined as the limit as $M \to \infty$ of the average of the M correlations that can be calculated between sample values and their ranks

where M samples of size N are drawn from this population. This method will be left as an exercise for the reader.

The expression given in (5.5.8) can be written in another useful form. If the variate values X are drawn from a continuous population with distribution F_X, the following sum can be evaluated:

$$\begin{aligned}
\sum_{i=1}^{N} iE(X_{(i)}) &= \sum_{i=1}^{N} \frac{iN!}{(i-1)!(N-i)!} \int_{-\infty}^{\infty} x[F_X(x)]^{i-1}[1-F_X(x)]^{N-i} f_X(x)dx \\
&= \sum_{j=0}^{N-1} \frac{(j+1)N!}{j!(N-j-1)!} \int_{-\infty}^{\infty} x[F_X(x)]^{j}[1-F_X(x)]^{N-j-1} f_X(x)dx \\
&= \sum_{j=1}^{N-1} \frac{N!}{(j-1)!(N-j-1)!} \\
&\quad \times \int_{-\infty}^{\infty} x[F_X(x)]^{j}[1-F_X(x)]^{N-j-1} f_X(x)dx \\
&\quad + \sum_{j=0}^{N-1} \frac{N!}{j!(N-j-1)!} \int_{-\infty}^{\infty} x[F_X(x)]^{j}[1-F_X(x)]^{N-j-1} f_X(x)dx \\
&= N(N-1) \int_{-\infty}^{\infty} xF_X(x) \sum_{j=1}^{N-1} \binom{N-2}{j-1} [F_X(x)]^{j-1}[1-F_X(x)]^{N-j-1} f_X(x)dx \\
&\quad + N \int_{-\infty}^{\infty} x \sum_{j=0}^{N-1} \binom{N-1}{j} [F_X(x)]^{j}[1-F_X(x)]^{N-j-1} f_X(x)dx \\
&= N(N-1) \int_{-\infty}^{\infty} xF_X(x) f_X(x)dx + N \int_{-\infty}^{\infty} xF_X(x)dx \\
&= N(N-1)E[XF_X(x)] + NE(X) \qquad (5.5.9)
\end{aligned}$$

If this quantity is now substituted in (5.5.8), the result is

$$\begin{aligned}
\rho[X, r(X)] &= \left(\frac{12}{N^2-1}\right)^{1/2} \frac{1}{\sigma_X} \left\{(N-1)E[XF_X(X)] + E(X) - \frac{N+1}{2}E(X)\right\} \\
&= \left(\frac{12}{N^2-1}\right)^{1/2} \frac{1}{\sigma_X} \left\{(N-1)E[XF_X(X)] - \frac{N-1}{2}E(X)\right\} \\
&= \left[\frac{12(N-1)}{N+1}\right]^{1/2} \frac{1}{\sigma_X} \left\{E[XF_X(X)] - \frac{1}{2}E(X)\right\} \qquad (5.5.10)
\end{aligned}$$

and

$$\lim_{N\to\infty} \rho[X, r(X)] = \frac{2\sqrt{3}}{\sigma_X}\left\{E[XF_X(X)] - \frac{1}{2}E(X)\right\} \tag{5.5.11}$$

Some particular evaluations of (5.5.11) are given in Stuart (1954).

5.6 Treatment of Ties in Rank Tests

In applying tests based on rank-order statistics, we usually assume that the population from which the sample was drawn is continuous. When this assumption is made, the probability of any two observations having identical magnitudes is equal to 0. The set of ranks as defined in (5.5.1) then will be N different integers. The exact properties of most rank statistics depend on this assumption. Two or more observations with the same magnitude are said to be *tied*. We may say only that *theoretically* no problem is presented by tied observations. In practice ties can certainly occur, either because the population is actually discrete or because of practical limitations on the precision of measurement. Some of the conventional approaches to dealing with ties in assigning ranks will be discussed generally in this section, so that the problem can be ignored in presenting the theory of some specific rank tests later.

In a set of N observations that are *not* all different, arrangement in order of magnitude produces a set of r groups of different numbers, the ith different value occurring with frequency t_i, where $\sum t_i = N$. Any group of numbers with $t_i \geq 2$ comprises a set of tied observations. The ranks are no longer well defined, and for any set of fixed ranks of N untied observations there are $\Pi t_i!$ possible assignments of ranks to the entire sample with ties, each assignment leading to its own value for a rank test statistic, although that value may be the same as for some other assignment. If a rank test is to be performed using a sample containing tied observations, we must have either a unique method of assigning ranks for ties so that the test statistic can be computed in the usual way or a method of combining the many possible values of the rank-test statistic to reach one decision. Several acceptable methods will be discussed briefly.

5.6.1 Randomization

In the method of randomization, one of the $\Pi t_i!$ possible assignments of ranks is selected by some random procedure. For example, in the set of observations

3.0, 4.1, 4.1, 5.2, 6.3, 6.3, 6.3, 9

there are 2!(3!) or 12 possible assignments of the integer ranks 1–8 which this sample could represent. One of these 12 assignments is selected by a

supplementary random experiment and used as the unique assignment of ranks. Using this method, some theoretical properties of the rank statistic are preserved, since each assignment occurs with equal probability. In particular, the null probability distribution of the rank-order statistic, and therefore of the rank statistic, is unchanged, so that the test can be performed in the usual way. However, an additional element of chance is artificially imposed, affecting the probability distribution under alternatives.

5.6.2 Midranks

The midrank method assigns to each member of a group of tied observations the simple average of the ranks they would have if distinguishable. Using this approach, tied observations are given tied ranks. The midrank method is perhaps the most frequently used, as it has much appeal experimentally. However, the null distribution of ranks is affected. The mean rank is unchanged, but the variance of the ranks is reduced. When the midrank method is used, a correction for ties can frequently be incorporated into the test statistic. We discuss these corrections when we present the respective tests. We note that it is not necessary to average all the tied ranks to find the value of the midrank. It is always equal to one-half of the sum of the smallest and largest of those ranks they would have if they were not tied.

5.6.3 Average Statistic

If one does not wish to choose a particular set of ranks as in the previous two methods, one may instead calculate the value of the test statistic for all the $\Pi t_i!$ assignments and use their simple average as the single sample value. Again, the test statistic would have the same mean but smaller variance.

5.6.4 Average Probability

Instead of averaging the test statistic for each possible assignment of ranks, one could find the probability of each resulting value of the test statistic and use the simple average of these probabilities for the overall probability. This requires availability of tables of the exact null probability distribution of the test statistic rather than simply a table of critical values.

5.6.5 Least Favorable Statistic

Having found all possible values of the test statistic, one might choose as a single value that one which minimizes the probability of rejection. This procedure leads to the most conservative test, i.e., the lowest probability of committing a type I error.

5.6.6 Range of Probability

Alternatively, one could compute two values of the test statistic: the one least favorable to rejection and the one most favorable. This method does not lead to a unique decision unless both values fall inside or both fall outside the rejection region.

5.6.7 Omission of Tied Observations

The final and most obvious possibility is to discard all tied observations and reduce the sample size accordingly. This method certainly leads to a loss of information, but if the number of observations to be omitted is small relative to the sample size, the loss may be minimal. This procedure generally introduces bias toward rejection of the null hypothesis.

The reader is referred to Savage's *Bibliography* (1962) for discussions of treatment of ties in relation to particular nonparametric rank test statistics. Pratt and Gibbons (1981) also give detailed discussions and many references. Randles (2001) gives a different approach to dealing with ties.

5.7 The Wilcoxon Signed-Rank Test and Confidence Interval

Since the one-sample sign test in Section 5.4 uses only the signs of the differences between each observation and the hypothesized median M_0, the magnitudes of these observations relative to M_0 are ignored. Assuming that such information is available, a test statistic which takes into account these individual relative magnitudes might be expected to give better performance. If we are willing to make the assumption that the parent population is symmetric, the Wilcoxon signed-rank test statistic provides an alternative test of location which is affected by both the magnitudes and signs of these differences. The rationale and properties of this test will be discussed in this section.

As with the one-sample situation of Section 5.4, we have a random sample of N observations $X_1, X_2, \ldots, X_N$ from a continuous cdf F with median M, but now we assume that F is symmetric about M. Under the null hypothesis

$$H_0: M = M_0$$

the differences $D_i = X_i - M_0$ are symmetrically distributed about zero. Thus, for any $c > 0$,

$$F_D(-c) = P(D_i \leq -c) = P(D_i \geq c) = 1 - P(D_i \leq c) = 1 - F_D(c)$$

With the assumption of a continuous population, we need not be concerned theoretically with zero or tied absolute differences $|D_i|$. Suppose we order these absolute differences $|D_1|, |D_2|, \ldots, |D_N|$ from smallest to largest and

assign them ranks 1, 2, ..., N, keeping track of the original signs of the differences D_i. If M_0 is the true median of the symmetrical population, the expected value of the sum of the ranks of the positive differences T^+ is equal to the expected value of the sum of the ranks of the negative differences T^-. Since the sum of all the ranks is a constant, that is, $T^+ + T^- = \sum_{i=1}^{N} i = N(N+1)/2$, test statistics based on T^+ only, T^- only, or $T^+ - T^-$ are linearly related and therefore equivalent criteria. In contrast to the ordinary one-sample sign test, the value of T^+, say, is influenced not only by the number of positive differences but also by their relative magnitudes. When the symmetry assumption can be justified, T^+ may provide a more efficient test of location for some distributions.

The derived sample data on which these test statistics are based consist of the set of N integer ranks (1, 2, ..., N) and a corresponding set of N plus and minus signs. The rank i is associated with a plus or minus sign according to the sign of $D_j = X_j - M_0$, where D_j occupies the ith position in the ordered array of absolute differences $|D_j|$. If we let $r(\cdot)$ denote the rank of a random variable, the *Wilcoxon signed-rank statistic* can be written in symbols as

$$T^+ = \sum_{i=1}^{N} Z_i r(|D_i|) \quad T^- = \sum_{i=1}^{N} (1 - Z_i) r(|D_i|) \tag{5.7.1}$$

where

$$Z_i = \begin{cases} 1 & \text{if } D_i > 0 \\ 0 & \text{if } D_i \leq 0 \end{cases}$$

Therefore,

$$T^+ - T^- = 2 \sum_{i=1}^{N} Z_i r|D_i| - \frac{N(N+1)}{2}$$

Under the null hypothesis, the Z_i are independent and identically distributed Bernoulli random variables with $P(Z_i = 1) = P(Z_i = 0) = 1/2$ so that $E(Z_i) = 1/2$ and $\text{var}(Z_i) = 1/4$. Using the fact that T^+ in (5.7.1) is a linear combination of these variables, we have

$$E(T^+ | H_0) = \sum_{i=1}^{N} \frac{r(|D_i|)}{2} = \frac{N(N+1)}{4}$$

Also, since Z_i is independent of $r(|D_i|)$ under H_0 (see Problem 5.25), we can show that

$$\text{var}(T^+ | H_0) = \sum_{i=1}^{N} \frac{[r|D_i|]^2}{4} = \frac{N(N+1)(2N+1)}{24} \tag{5.7.2}$$

In order to derive the mean and variance of T^+ in general we write

$$T^+ = \sum_{1 \le i \le j \le N} \sum T_{ij} \tag{5.7.3}$$

where

$$T_{ij} = \begin{cases} 1 & \text{if } D_i + D_j > 0 \\ 0 & \text{otherwise} \end{cases}$$

The D_i's are identically distributed under H_0. Now define for all distinct i, j, k the probabilities

$$\begin{aligned} p_1 &= P(D_i > 0) \\ p_2 &= P(D_i + D_j > 0) \\ p_3 &= P(D_i > 0 \text{ and } D_i + D_j > 0) \\ p_4 &= P(D_i + D_j > 0 \text{ and } D_i + D_k > 0) \end{aligned} \tag{5.7.4}$$

The moments of the indicator variables for all distinct i, j, k, h are then

$$\begin{aligned} &E(T_{ii}) = p_1 && E(T_{ij}) = p_2 \\ &\text{var}(T_{ii}) = p_1 - p_1^2 && \text{var}(T_{ij}) = p_2 - p_2^2 \\ &\text{cov}(T_{ii}, T_{ik}) = p_3 - p_1 p_2 && \text{cov}(T_{ij}, T_{ik}) = p_4 - p_2^2 \\ &\text{cov}(T_{ij}, T_{hk}) = 0 \end{aligned}$$

The mean and variance of the linear combination in (5.7.3) in terms of these moments are

$$E(T^+) = NE(T_{ii}) + \frac{N(N-1)E(T_{ij})}{2} = Np_1 + \frac{N(N-1)p_2}{2} \tag{5.7.5}$$

$$\begin{aligned} \text{var}(T^+) &= N \text{ var}(T_{ii}) + \binom{N}{2} \text{var}(T_{ij}) + 2N(N-1) \text{ cov}(T_{ii}, T_{ik}) \\ &\quad + 2N \binom{N-1}{2} \text{cov}(T_{ij}, T_{ik}) + \binom{N}{4} \text{cov}(T_{ij}, T_{hk}) \\ &= Np_1(1-p_1) + \frac{N(N-1)p_2(1-p_2)}{2} \\ &\quad + 2N(N-1)(p_3 - p_1 p_2) + N(N-1)(N-2)\left(p_4 - p_2^2\right) \\ &= Np_1(1-p_1) + N(N-1)(N-2)\left(p_4 - p_2^2\right) \\ &\quad + \frac{N(N-1)}{2}[p_2(1-p_2) + 4(p_3 - p_1 p_2)] \end{aligned} \tag{5.7.6}$$

The relevant probabilities from (5.7.4) are now evaluated under the assumption that the population is symmetric and the null hypothesis is true

$$
\begin{aligned}
p_1 &= P(D_i > 0) = 1/2 \\
p_2 &= P(D_i + D_j > 0) = \int_{-\infty}^{\infty} \int_{-v}^{\infty} f_D(u) f_D(v) du\, dv \\
&= \int_{-\infty}^{\infty} [1 - F_D(-v)] f_D(v) dv \\
&= \int_{-\infty}^{\infty} F_D(v) f_D(v) dv = \int_0^1 x\, dx = \frac{1}{2} \\
p_3 &= P(D_i > 0 \text{ and } D_i + D_j > 0) \\
&= \int_0^{\infty} \int_{-v}^{\infty} f_D(u) f_D(v) du\, dv = \int_0^{\infty} [1 - F_D(-v)] f_D(v) dv \\
&= \int_0^{\infty} F_D(v) f_D(v) dv = \int_{1/2}^1 x\, dx = \frac{3}{8} \\
p_4 &= P(D_i + D_j > 0 \text{ and } D_i + D_k > 0) \\
&= P(0 < D_i + D_j < D_i + D_k) + P(0 < D_i + D_k < D_i + D_j) \\
&= P(-D_i < D_j < D_k) + P(-D_i < D_k < D_j) \\
&= 2P(-D_i < D_j < D_k) \\
&= 2 \int_{-\infty}^{\infty} \int_{-w}^{\infty} \int_{-v}^{\infty} f_D(u) f_D(v) f_D(w) du\, dv\, dw \\
&= 2 \int_{-\infty}^{\infty} \int_{-w}^{\infty} [1 - F_D(-v)] f_D(v) f_D(w) dv\, dw \\
&= 2 \int_{-\infty}^{\infty} \int_{-w}^{\infty} f_D(v) f_D(w) dv\, dw - 2 \int_{-\infty}^{\infty} \int_{-w}^{\infty} F_D(v) f_D(v) f_D(w) dv\, dw \\
&= 2 \int_{-\infty}^{\infty} [1 - F_D(-w)] f_D(w) dw - \int_{-\infty}^{\infty} \{1 - [F_D(-w)]^2\} f_D(w) dw \\
&= 2 \int_{-\infty}^{\infty} F_D(w) dF_D(w) - 1 + \int_{-\infty}^{\infty} [1 - F_D(w)]^2 dF_D(w) \\
&= 2\frac{1}{2} - 1 + \frac{1}{3} = \frac{1}{3}
\end{aligned}
$$

The reader may verify that substitution of these results back in (5.7.5) and (5.7.6) gives the mean and variance already found in (5.7.2).

We use the method described in Chapter 1 to investigate the consistency of T^+. We can write

$$E\left[\frac{2T^+}{N(N+1)}\right] = \frac{2p_1}{N+1} + \frac{(N-1)p_2}{N+1}$$

which equals $1/2$ under H_0 and $\text{var}[2T^+/N(N+1)]$ clearly tends to zero as $N \to \infty$. Therefore, the test with rejection region

$$T^+ \in R \quad \text{for } \frac{2T^+}{N(N+1)} - \frac{1}{2} \geq k$$

is consistent against alternatives of the form $p_2 = P(D_1 + D_j > 0) > 0.5$. This result is reasonable since if the true population median exceeds M_0, the sample data would reflect this by having most of the larger ranks correspond to positive differences. A similar two-sided rejection region of T^+ centered on $N(N+1)/4$ is consistent against alternatives with $p_2 \neq 0.5$.

To determine the rejection regions precisely for this consistent test, the probability distribution of T^+ must be determined under the null hypothesis

$$H_0 : \theta = P(X > M_0) = 0.5$$

The extreme values of T^+ are zero and $N(N+1)/2$, occurring when all differences are of the same sign, negative or positive, respectively. The mean and variance were found in (5.7.2). Since T^+ is completely determined by the indicator variables Z_i in (5.7.1), the sample space can be considered to be the set of all possible N-tuples $\{z_1, z_2, \ldots, z_N\}$ with components either one or zero, of which there are 2^N. Each of these distinguishable arrangements is equally likely under H_0. Therefore, the null probability distribution of T^+ is given by

$$P(T^+ = t) = \frac{u(t)}{2^N} \tag{5.7.7}$$

where $u(t)$ is the number of ways to assign plus and minus signs to the first N integers such that the sum of the positive integers equals t. Every assignment has a conjugate assignment with plus and minus signs interchanged, and T^+ for this conjugate is

$$\sum_{i=1}^{N} i(1 - Z_i) = \frac{N(N+1)}{2} - \sum_{i=1}^{N} iZ_i$$

TABLE 5.7.1

Enumeration for the Distribution of T^+

Value of T^+	Ranks Associated with Positive Differences	Number of Sample Points $u(t)$
10	1, 2, 3, 4	1
9	2, 3, 4	1
8	1, 3, 4	1
7	1, 2, 4; 3, 4	2
6	1, 2, 3; 2, 4	2
5	1, 4; 2, 3	2

Since every assignment occurs with equal probability, this implies that the null distribution of T^+ is symmetric about its mean $N(N+1)/4$.

Because of the symmetry property, only one-half of the null distribution need be determined. A systematic method of generating the complete distribution of T^+ for $N=4$ is shown in Table 5.7.1.

$$f_{T^+}(t) = \begin{cases} 1/16 & t = 0, 1, 2, 8, 9, 10 \\ 2/16 & t = 3, 4, 5, 6, 7 \\ 0 & \text{otherwise} \end{cases}$$

Tables can be constructed in this way for all N.

To use the signed-rank statistics in hypothesis testing, the entire null distribution is not necessary. In fact, one set of critical values is sufficient for even a two-sided test, because of the relationship $T^+ + T^- = N(N+1)/2$ and the symmetry of T^+ about $N(N+1)/4$. Large values of T^+ correspond to small values of T^- and furthermore T^+ and T^- are identically distributed under H_0 since

$$\begin{aligned} P(T^+ \geq c) &= P\left[T^+ - \frac{N(N+1)}{4} \geq c - \frac{N(N+1)}{4}\right] \\ &= P\left[\frac{N(N+1)}{4} - T^+ \geq c - \frac{N(N+1)}{4}\right] \\ &= P\left[\frac{N(N+1)}{2} - T^+ \geq c\right] \\ &= P(T^- \geq c) \end{aligned}$$

Since it is more convenient to work with smaller sums, tables of the left-tail critical values are generally set up for the random variable T, which may denote either T^+ or T^-. If t_α is the number such that $P(T \leq t_\alpha) = \alpha$, the appropriate rejection regions for size α tests of $H_0: M = M_0$ are as follows:

$$\begin{array}{ll} T^- \le t_\alpha & \text{for } H_1: M > M_0 \\ T^+ \le t_\alpha & \text{for } H_1: M < M_0 \\ T^+ \le t_{\alpha/2} \text{ or } T^- \le t_{\alpha/2} & \text{for } H_1: M \ne M_0 \end{array}$$

Suppose that $N = 8$ and critical values are to be found for one- or two-sided tests at nominal $\alpha = 0.05$. Since $2^8 = 256$ and $256(0.05) = 12.80$, we need at least 13 cases of assignments of signs. We enumerate the small values of T^+ in Table 5.7.2. Since $P(T^+ \le 6) = 14/256 > 0.05$ and $P(T^+ \le 5) = 10/256 = 0.039$, $T_{0.05} = 5$; the exact probability of a type I error is 0.039. Similarly, we find $t_{0.025} = 3$ with exact $P(T^+ \le 3) = 0.0195$.

When the distribution is needed for several sample sizes, a simple recursive relation can be used to generate the probabilities. Let T_N^+ denote the sum of the ranks associated with positive differences D_i for a sample of N observations. Consider a set of $N-1$ ordered $|D_i|$, with ranks 1, 2, ..., $N-1$ assigned, for which the null distribution of T_{N-1}^+ is known. To obtain the distribution of T_N^+ from this, an extra observation D_N is added, and we can assume without loss of generality that $|D_N| > |D_i|$ for all $i \le N-1$. The rank of $|D_N|$ is then N. If $D_N > 0$, the value of T_N^+ will exceed that of T_{N-1}^+ by the amount N for every arrangement of the $N-1$ observations, but if $D_N < 0$, T_N^+ will be equal to T_{N-1}^+. Using the notation in (5.7.7), this can be stated as

$$\begin{aligned} P(T_N^+ = k) &= \frac{u_N(k)}{2^N} = \frac{u_{N-1}(k-N)P(D_N > 0) + u_{N-1}(k)P(D_N < 0)}{2^{N-1}} \\ &= \frac{u_{N-1}(k-N) + u_{N-1}(k)}{2^N} \end{aligned} \tag{5.7.8}$$

If N is moderate and systematic enumeration is desired, classification according to the number of positive differences D_i is often helpful. Define the random variable U as the number of positive differences; U follows the binomial distribution with parameter 0.5, so that

TABLE 5.7.2

Partial Distribution of T_N^+ for $N = 8$

Value of T^+	Ranks Associated with Positive Differences	Number of Sample Points
0		1
1	1	1
2	2	1
3	3; 1, 2	2
4	4; 1, 3	2
5	5; 1, 4; 2, 3	3
6	6; 1, 5; 2,4; 1,2,3	4

$$
\begin{aligned}
P(T^+ = t) &= \sum_{i=0}^{N} P(U = i \cap T^+ = t) \\
&= \sum_{i=0}^{N} P(U = i) P(T^+ = t | U = i) \\
&= \sum_{i=0}^{N} \binom{N}{i} (0.5)^N P(T^+ = t | U = i)
\end{aligned}
$$

A table of critical values and exact significance levels of the Wilcoxon signed-rank test is given in Dunstan et al. (1979) for $N \leq 50$, and the entire null distribution is given in Wilcoxon et al. (1972) for $N \leq 50$. Table H gives left-tail and right-tail probabilities of T^+ (or T^-) for $N \leq 15$. From a generalization of the central-limit theorem, the asymptotic distribution of T^+ is the normal. Therefore, using the moments given in (5.7.2), the null distribution of

$$
Z = \frac{4T^+ - N(N+1)}{\sqrt{2N(N+1)(2N+1)/3}} \tag{5.7.9}
$$

approaches the standard normal as $N \to \infty$. The test for, say, $H_1 : M > M_0$ can be performed for large N by computing (5.7.9) and rejecting H_0 for $Z \geq z_\alpha$. The approximation is generally adequate for $N \geq 15$. A continuity correction of 0.5 generally improves the approximation.

5.7.1 The Problem of Zero and Tied Differences

Since we assumed originally that the random sample was drawn from a continuous population, the problem of tied observations and zero differences could be ignored theoretically. In practice, generally any zero differences (observations equal to M_0) are ignored and N is reduced accordingly, although the other procedures described for the ordinary sign test in Section 5.4 can also be used here. In the case where two or more absolute values of differences are equal, that is, $|d_i| = |d_j|$ for at least one $i \neq j$, the observations are tied. The ties can be dealt with by any of the procedures described in Section 5.6. The midrank method is usually used, and the sign associated with the midrank of $|d_i|$ is determined by the original sign of d_i as before. The probability distribution of T is clearly not the same in the presence of tied ranks, but the effect is generally slight and no correction need be made unless the ties are quite extensive. A thorough comparison of the various methods of treating zeros and ties with this test is given in Pratt and Gibbons (1981).

With large sample sizes when the test is based on the standard normal statistic in (5.7.9), the variance can be corrected to account for the ties as long as the midrank method is used to resolve the ties. Suppose that t observations are tied for a given rank and that if they were not tied they would be given

the ranks $s+1, s+2, \ldots, s+t$. The midrank is then $s+(t+1)/2$ and the sum of squares of these ranks is

$$t\left[s+\frac{(t+1)}{2}\right]^2 = t\left[s^2+s(t+1)+\frac{(t+1)^2}{4}\right]$$

If these ranks had not been tied, their sum of squares would have been

$$\sum_{i=1}^{t}(s+i)^2 = ts^2+s(t+1)+\frac{t(t+1)(2t+1)}{6}$$

The presence of these t ties then decreases the sum of squares by

$$\frac{t(t+1)(2t+1)}{6}-\frac{t(t+1)^2}{4}=\frac{t(t+1)(t-1)}{12}=\frac{t(t^2-1)}{12} \tag{5.7.10}$$

Therefore, the reduced variance from (5.7.2) is

$$\text{var}(T^+|H_0)=\frac{N(N+1)(2N+1)}{24}-\frac{\sum t(t^2-1)}{48} \tag{5.7.11}$$

where the sum is extended over all sets of t ties. This is called the correction for ties.

5.7.2 Power Function

The distribution of T^+ is approximately normal for large sample sizes regardless of whether the null hypothesis is true. Therefore, a large sample approximation to the power can be calculated using the mean and variance given in (5.7.5) and (5.7.6). The distribution of $X-M_0$ under the alternative must be specified in order to calculate the probabilities in (5.7.4) to substitute in (5.7.5) and (5.7.6).

The asymptotic relative efficiency of the Wilcoxon signed-rank test relative to the t test is at least 0.864 for any distribution continuous and symmetric about zero, is 0.955 for the normal distribution, and is 1.5 for the double exponential distribution.

It should be noted that the probability distribution of T^+ is not symmetric when the null hypothesis is not true. Further, T^+ and T^- are not identically distributed when the null hypothesis is not true. We can still find the probability distribution of T^- from that of T^+, however, using the relationship

$$P(T^-=k)=P\left[\frac{N(N+1)}{2}-T^+=k\right] \tag{5.7.12}$$

5.7.3 Simulated Power

Calculating the power of the signed-rank test, even using the normal approximation, requires a considerable amount of work. It is much easier to simulate the power of the test, as we did for the sign test in Section 5.4. Again we use Macro program the MINITAB/shown in Section 5.4.5 for the calculations and compare the results with those obtained for the sign test when $N = 13$, $\alpha = 0.05$, $M_0 = 0.5$ and $M_1 = 0.6256$.

Simulating the power of the signed-rank test consists of the following steps similar to those for the sign test. First, we determine the rejection region of the signed-rank test from Table H as $T^+ \geq 70$ with exact $\alpha = 0.047$. We generate 1000 random samples each of size $N = 13$ from a normal distribution with mean 0.6256 and variance 1 and calculate the signed-rank statistic T^+ for each. For each of these statistics we check to see if it exceeds the critical value 70 or not. Finally, we count the number of times, out of 1000, that the signed-rank test rejects the null hypothesis and divide this number by 1000. This gives a simulated (estimated) value of the power of the signed-rank test with $N = 13$, $\alpha = 0.0461$, $M_0 = 0.50$, $M_1 = 0.6256$. The MINITAB program code is shown in Section 5.4. Note that the program also calculates the simulated power curves on the same graph, as shown in Figure 5.7.1.

The output from the Macro is shown in Table 5.7.3; pow1 and pow2 are the computed powers of the sign and the signed-rank test, respectively, based on 1000 simulations.

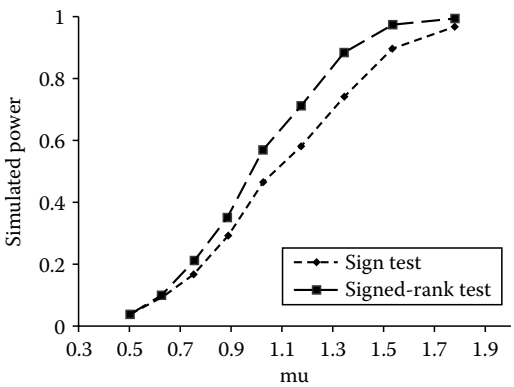


FIGURE 5.7.1
Simulated power of the sign and the signed-rank test for the normal distribution.

TABLE 5.7.3

Simulated Power of Sign and Signed-Rank Tests

Mu	Theta	pow1	pow2
0.5000	0.50	0.033	0.036
0.62566	0.55	0.095	0.100
0.75335	0.60	0.166	0.211
0.88532	0.65	0.292	0.352
1.02440	0.70	0.465	0.568
1.17449	0.75	0.582	0.710
1.34162	0.80	0.742	0.885
1.53643	0.85	0.897	0.974
1.78155	0.90	0.968	0.994

5.7.4 Sample Size Determination

In order to make an inference regarding the population median using the signed-rank test, we need to have a random sample of observations. If we are allowed to choose the sample size, we might want to determine the value of N such that the test has size α and power $1-\beta$, given the null and the alternative hypotheses and other necessary assumptions. Recall that for the sign test against the one-sided upper-tailed alternatives, we solved for N such that

$$\text{size} = \sum_{i=k_\alpha}^{N} \binom{N}{i} (0.5)^N \leq \alpha \quad \text{and} \quad \text{power} = \sum_{i=k_\alpha}^{N} \binom{N}{i} \theta^i (1-\theta)^{N-i} \geq 1-\beta$$

where α, $1-\beta$, and $\theta = P(X > M_0 | H_1)$ are all specified. We noted there that the solution is much easier to obtain using the normal approximation; the same is true for the Wilcoxon signed-rank test, as we now illustrate. The theory is presented here in terms of the signed-rank statistic T^+ but the same approach will hold for any test statistic whose distribution can be approximated by a normal distribution under both the null and the alternative hypotheses.

Under the normal approximation, the power of a size α signed-rank test against the alternative $H_1 : M > M_0$ is $P(T^+ \geq \mu_0 + z\sigma_0 | H_1)$, where μ_0 and σ_0 are respectively, the null mean and the null standard deviation of T^+. It can be easily shown (see Noether, 1987) that this power equals a specified $1-\beta$ if

$$\left(\frac{\mu_0 - \mu}{\sigma_0}\right)^2 = (z_\alpha + \rho z_\beta)^2 \tag{5.7.13}$$

where μ and σ are, respectively, the mean and the standard deviation of T^+ under the alternative hypothesis and $\rho = \sigma/\sigma_0$. Since σ is unknown and is

difficult to evaluate [see (5.7.6)], ρ is unknown. One possibility is to take $\rho = 1$; such an assumption is reasonable for alternative hypotheses that are not too different from the null hypothesis.

If we substitute the expressions for μ_0, σ_0, σ, and μ [see (5.7.5)] into (5.7.13), we need to solve for N in

$$\frac{\{N(p_1 - 0.5) + [N(N-1)(p_2 - 0.5)]/2\}^2}{N(N+1)(2N+1)/24} = (z_\alpha + z_\beta)^2 \tag{5.7.14}$$

Note that $p_1 = P(X_i > M_0)$ and $p_2 = P(X_i + X_j > 2M_0)$ under the alternative $H_1 : M > M_0$. The sample size calculations for N from (5.7.14), done in EXCEL using the solver application, are shown in Table 5.7.4 for $\alpha = 0.05$, $1 - \beta = 0.95$, assuming the underlying distribution is standard normal. The value of N, shown in the fifth column, needs to be rounded up to the next larger integer. For example, assuming normality and $M_0 = 0$, $M_1 = 0.5$, $\alpha = 0.05$, we need approximately 54 observations for a one-sided test with power 0.95.

A similar derivation can be used to find a sample size formula when the alternative is two-sided. The details are left as an exercise for the reader.

The sample size formula in (5.7.14) is not distribution-free since it depends on the underlying distribution through the parameters p_1 and p_2. Noether (1987) proposed approximating the left-hand side of (5.7.14) by $3N(p_2 - 0.5)^2$ and solving for N, which yields

$$N = \frac{(z_\alpha + z_\beta)^2}{3(p_2 - 0.5)^2} \tag{5.7.15}$$

TABLE 5.7.4

Calculations for Sample Size Determination in EXCEL

M_0	M_1	p_1	p_2	N	$p_1 - 0.5$	$0.5N(N-1)(p_2 - 0.5)$	$N(N+1)(2N+1)/24$
0	0.1	0.53983	0.55623	1141.7342	0.03983	36618.36868	124189277.00525
0	0.2	0.57926	0.61135	291.6422	0.07926	4719.26188	2077783.91705
0	0.3	0.61791	0.66431	134.2614	0.11791	1469.93485	203943.03149
0	0.4	0.65542	0.71420	79.2155	0.15542	663.56652	42211.37566
0	0.5	0.69146	0.76025	53.8075	0.19146	369.74250	13346.32363
0	0.6	0.72575	0.80193	40.0681	0.22575	236.31632	5562.95473
0	0.7	0.75804	0.83890	31.8487	0.25804	166.48314	2820.22992
0	0.8	0.78814	0.87105	26.5805	0.28814	126.14664	1654.40382
0	0.9	0.81594	0.89845	23.0355	0.31594	101.12744	1085.90868
0	1.0	0.84134	0.92135	20.5657	0.34134	84.77195	778.57707

This formula still depends on p_2; Noether (1987) suggested a choice for this parameter in terms of an ''odds-ratio,'' The reader is referred to his paper for details.

We illustrate the use of (5.7.15) for our previous example of a one-tailed test where $\alpha = 0.05$, $1 - \beta = 0.95$. If $M_1 = 0.1$ and $p_2 = 0.556$, we find $N = 1151$, whereas if $M_1 = 1.0$ and $p_2 = 0.921$, we find $N = 21$, both from (5.7.15). The corresponding values shown in Table 5.7.4 are $N = 1142$ and $N = 21$, respectively.

For a two-sided test, we can use (5.7.15) with α replaced by $\alpha/2$.

5.7.5 Confidence-Interval Procedures

As with the ordinary one-sample sign test, the Wilcoxon signed-rank procedure lends itself to confidence-interval estimation of the unknown population median M. The confidence limits are those values of M which do not lead to rejection of the null hypothesis. To find these limits for any sample size N, we first find the critical value $t_{\alpha/2}$ such that if the true population median is M and T is calculated for the derived sample values $X_i - M$, then

$$P(T^+ \leq t_{\alpha/2}) = \alpha/2 \quad \text{and} \quad P(T^- \leq t_{\alpha/2}) = \alpha/2$$

The null hypothesis will not be rejected for all numbers M which make $T^+ > t_{\alpha/2}$ and $T^- > t_{\alpha/2}$. The confidence-interval technique is to use trial and error to find those two numbers, say M_1 and M_2 where $M_1 < M_2$, such that when T is calculated for the two sets of differences $X_i - M_1$ and $X_i - M_2$, at significance level α, T^+ or T^-, whichever is smaller, is just short of significance, i.e., slightly larger than $t_{\alpha/2}$. This generally does not lead to a unique interval, and the manipulations can be tedious even for moderate sample sizes.

This technique is best illustrated by an example. The following eight observations are drawn from a continuous, symmetric population:

$$-1, 6, 13, 4, 2, 3, 5, 9 \tag{5.7.16}$$

For $N = 8$ the two-sided rejection region of nominal size 0.05 was found earlier by Table 5.7.2 to be $t_{\alpha/2} = 3$ with exact significance level

$$\alpha = P(T^+ \leq 3) + P(T^- \leq 3) = 10/256 = 0.039$$

In Table 5.7.5, we try six different values for M and calculate T^+ or T^-, whichever is smaller, for the differences $X_i - M$. The example illustrates a number of difficulties which arise. In the first trial choice of M, the number 4 was subtracted and the resulting differences contained three sets of tied pairs and one zero even though the original sample contained neither ties nor zeros. If the zero difference is ignored, N must be reduced to 7 and then the

TABLE 5.7.5

Trial-and-Error Determination of Endpoints

X_i	$X_i - 4$	$X_i - 1.1$	$X_i - 1.5$	$X_i - 9.1$	$X_i - 8.9$	$X_i - 8.95$
−1	−5	−2.1	−2.5	−10.1	−9.9	−9.95
6	2	4.9	4.5	−3.1	−2.9	−2.95
13	9	11.9	11.5	3.9	4.1	4.05
4	0	2.9	2.5	−5.1	−4.9	−4.95
2	−2	0.9	0.5	−7.1	−6.9	−6.95
3	−1	1.9	1.5	−6.1	−5.9	−5.95
5	1	3.9	3.5	−4.1	−3.9	−3.95
9	5	7.9	7.5	–0.1	0.1	0.05
T^+ or T^-		3	3.5	3	5	5

$t_{\alpha/2} = 3$ is no longer accurate for $\alpha = 0.039$. The midrank method could be used to handle the ties, but this also disturbs the accuracy of $t_{\alpha/2}$. Since there seems to be no real solution to these problems, we try to avoid zeros and ties by judicious choices for our M values for subtraction. These data are all integers, and hence a choice for M which is not an integer obviously reduces the likelihood of ties and makes zero values impossible. Since T^- for the differences $X_i - 1.5$ yields $T^- = 3.5$ using the midrank method, we will choose $M_1 = 1.5$. The next three columns represent an attempt to find an M which makes T^+ around 4. These calculations illustrate the fact that M_1 and M_2 are far from being unique. Clearly M_2 is in the vicinity of 9, but the differences $X_i - 9$ yield a zero. We conclude there is no need to go further. An approximate 96.1% confidence interval on M is given by $1.5 < M < 9$. The interpretation is that hypothesized values of M within this range will lead to acceptance of the null hypothesis for exact significance level 0.039.

This procedure is undoubtedly tedious, but the limits obtained are reasonably accurate. The numbers should be tried systematically to narrow down the range of possibilities. Thoughtful study of the intermediate results usually reduces the number of trials required.

A more systematic method of construction which leads to a unique interval and is much easier to apply is described in Noether (1967, pp. 57–58). The procedure is to convert the rejection region $T^+ > t_{\alpha/2}$ and $T^- > t_{\alpha/2}$ to an equivalent statement on M whose endpoints are functions of the observations X_i. For this purpose we must analyze the comparisons involved in determining the ranks of the differences $r(|X_i - M_0|)$ and the signs of the differences $X_i - M_0$ since T^+ and T^- are functions of these comparisons. Recall from (5.5.1) that the rank of any random variable in a set $\{V_1, V_2, \ldots, V_N\}$ can be written in symbols as

$$r(V_i) = \sum_{k=1}^{N} S(V_i - V_k) = \sum_{k \neq i} S(V_i - V_k) + 1$$

where

$$S(u) = \begin{cases} 1 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

To compute a rank, then we make $\binom{N}{2}$ comparisons of pairs of different numbers and one comparison of a number with itself. To compute the sets of all ranks, we make $\binom{N}{2}$ comparisons of pairs and N identity comparisons, a total of $\binom{N}{2} + N = N(N+1)/2$ comparisons. Substituting the rank function in (5.7.1), we obtain

$$\begin{aligned} T^+ &= \sum_{i=1}^{N} Z_i r(|X_i - M_0|) \\ &= \sum_{i=1}^{N} Z_i + \sum_{i=1}^{N} \sum_{k \neq i} Z_i S(|X_i - M_0| - |X_k - M_0|) \end{aligned} \tag{5.7.17}$$

Therefore, these comparisons affect T^+ as follows:

1. A comparison of $|X_i - M_0|$ with itself adds 1 to T^+ if $X_i - M_0 > 0$.
2. A comparison of $|X_i - M_0|$ with $|X_k - M_0|$ for any $i \neq k$ adds 1 to T^+ if $|X_i - M_0| > |X_k - M_0|$ and $X_i - M_0 > 0$, that is, $X_i - M_0 > |X_k - M_0|$. If $X_k - M_0 > 0$, this occurs when $X_i > X_k$, and if $X_k - M_0 < 0$, we have $X_i + X_k > 2M_0$ or $(X_i + X_k)/2 > M_0$. But when $X_i - M_0 > 0$ and $X_k - M_0 > 0$, we have $(X_i + X_k)/2 > M_0$ also.

Combining these two results, then, $(X_i + X_k)/2 > M_0$ is a necessary condition for adding 1 to T^+ for all i, k. Similarly, if $(X_i + X_k)/2 < M_0$, then this comparison adds 1 to T^-. The relative magnitudes of the $N(N+1)/2$ averages of pairs $(X_i + X_k)/2$ for all $i \leq k$, called the *Walsh averages*, then determine the range of values for hypothesized numbers M_0 which will not lead to rejection of H_0. If these $N(N+1)/2$ averages are arranged as order statistics, the two numbers which are in the $(t_{\alpha/2} + 1)$th position from either end are the endpoints of the $100(1-\alpha)\%$ confidence interval on M. Note that this procedure is exactly analogous to the ordinary sign-test confidence interval except that here the order statistics are for the averages of all pairs of observations instead of the original observations.

The Walsh averages are named after John Walsh, the author of the three-volume set *Handbook of Nonparametric Statistics*. It can be shown that the median of the Walsh averages is an unbiased estimator of the median of

the population and is called the *Hodges–Lehmann estimator* of the median. We note that zeros and ties do not present a problem with this confidence-interval procedure, since they are counted as many times as they occur. This was not the case for hypothesis testing. The value of the test statistic T^- for the hypothesized median M_0 is the number of Walsh averages that are less than M_0 plus one-half of the number that are equal to M_0. And the test statistic T^+ is the number of Walsh averages that are greater than M_0 plus one-half of the number that are equal to M_0.

The data in (5.7.16) for $N = 8$ arranged in order of magnitude are -1, 2, 3, 4, 5, 6, 9, 13, and the 36 Walsh averages are given in Table 5.7.5. For exact $\alpha = 0.039$, we found before that $t_{\alpha/2} = 3$. Since the fourth largest numbers from either end are 1.5 and 9.0, the confidence interval is $1.5 < M < 9$ with exact confidence coefficient $\gamma = 1 - 2\ (0.0195) = 0.961$. This result agrees exactly with that obtained by the previous method, but this will not always be the case since the trial-and-error procedure does not yield unique endpoints.

Now for these same data, suppose we want to test the null hypothesis $M = 4.5$ against a two-sided alternative at the 0.039 level. The reader may verify that the test statistics are $T^- = 17$ and $T^+ = 19$. The number of Walsh averages in Table 5.7.6 that are less than 4.5 is 16, the number equal to 4.5 is 2, and the number greater than 4.5 is 18. This illustrates the constant relationship between the number of negative, zero, and positive Walsh averages and the value of the Wilcoxon signed-rank statistics.

5.7.6 Paired-Sample Procedures

The Wilcoxon signed-rank test was actually proposed for use with paired-sample data in making inferences concerning the value of the median of the population of differences. Given a random sample of N pairs

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_N, Y_N)$$

TABLE 5.7.6

Walsh Averages for Data in (5.7.16)

−1.0	0.5	1.0	1.5	2.0	2.5	4.0	6.0
2.0	2.5	3.0	3.5	4.0	5.5	7.5	
3.0	3.5	4.0	4.5	6.0	8.0		
4.0	4.5	5.0	6.5	8.5			
5.0	5.5	7.0	9.0				
6.0	7.5	9.5					
9.0	11.0						
13.0							

their differences are

$$X_1 - Y_1, X_2 - Y_2, \ldots, X_N - Y_N$$

We assume these are independent observations from a population of differences which is continuous and symmetric with median M_D. In order to test the hypothesis

$$H_0 : M_D = M_0$$

form the N differences $D_i = X_i - Y_i - M_0$ and rank their absolute magnitudes from smallest to largest using integers $\{1, 2, \ldots, N\}$, keeping track of the original sign of each difference. Then the previous procedures for hypothesis testing and confidence intervals are equally applicable here with the same notation, except that the parameter M_D must be interpreted now as the median of the population of differences.

5.7.7 Use of Wilcoxon Statistics to Test for Symmetry

The Wilcoxon signed-rank statistics can also be used as tests for symmetry if the only assumption made is that the random sample is drawn from a continuous distribution. If the null hypothesis states that the population is symmetric with median M_0, the null distributions of T^+ and T^- are exactly the same as before. If the null hypothesis is accepted, we can conclude that the population is symmetric and has median M_0. But if the null hypothesis is rejected, we cannot tell which portion (or all) of the composite statement is not consistent with the sample outcome. With a two-sided alternative, for example, we must conclude that either the population is symmetric with median not equal to M_0, or the population is asymmetric with median equal to M_0, or the population is asymmetric with median not equal to M_0. Such a broad conclusion is generally not satisfactory, and this is why in most cases the assumptions that justify a test procedure are separated from the statement of the null hypothesis.

5.7.8 Applications

The appropriate rejection regions and P values for T^+, called the sum of the positive ranks, are given below. Note that t_0 is the observed value of T^+.

Alternative	Exact Rejection Region	Exact *P*-Value
$M > M_0$	$T^+ \geq t_\alpha$	$P\,(T^+ \geq t_0 \mid H_0)$
$M < M_0$	$T^+ \leq t'_\alpha$	$P\,(T^+ \leq t_0 \mid H_0)$
$M \neq M_0$	$T^+ \leq t'_{\alpha/2}$ or $T^+ \geq t_{\alpha/2}$	2 (smaller of the above)

Table H gives the distribution of T^+ for $N \leq 15$ as left-tail probabilities for $T^+ \leq N(N+1)/4$ and right-tail for $T^+ \geq N(N+1)/4$. This table can be used to find exact critical values for a given α or to find exact P values. For $N > 15$, the appropriate rejection regions and the P values based on the normal approximation with a continuity correction are as follows:

Alternative	Approximate Rejection Region	Approximate P Value
$M > M_0$	$T^+ \geq \frac{N(N+1)}{4} + 0.5 + z_\alpha \sqrt{\frac{N(N+1)(2N+1)}{24}}$	$1 - \Phi\left[\frac{t_0 - 0.5 - N(N+1)/4}{\sqrt{N(N+1)(2n+1)/24}}\right]$
$M < M_0$	$T^+ \leq \frac{N(N+1)}{4} - 0.5 - z_\alpha \sqrt{\frac{N(N+1)(2N+1)}{24}}$	$\Phi\left[\frac{t_0 + 0.5 - N(N+1)/4}{\sqrt{N(N+1)(2n+1)/24}}\right]$
$M \neq M_0$	Both above with $z_{\alpha/2}$	2 (smaller of the above)

If ties are present, the variance term in these rejection regions should be replaced by (5.7.11).

The corresponding confidence-interval estimate of the median has endpoints which are the $(t_{\alpha/2}+1)$st from the smallest and largest of the Walsh averages, where $t_{\alpha/2}$ is the left-tail critical value in Table H for the given N. The choice of exact confidence levels is limited to $1-2P$ where P is a tail probability in Table H. Therefore, the critical value $t_{\alpha/2}$ is the left-tail table entry corresponding to the chosen P. Since the entries are all of the nonnegative integers, $(t_{\alpha/2}+1)$ is the rank of $t_{\alpha/2}$ among the table entries for that N.

Thus, in practice, the confidence-interval endpoints are the uth smallest and uth largest of the $N(N+1)/2$ Walsh averages $W_{ik} = (X_i + X_k)/2$ for all $1 \leq i, k \leq N$, or

$$W_{(u)} \leq M \leq W_{[N(N+1)/2-u+1]}$$

The appropriate value of u for confidence $1-2P$ is the rank of that left-tail P among the entries in Table H for the given N. For $N > 15$, we find u from

$$u = \frac{N(N+1)}{4} + 0.5 - z_{\alpha/2}\sqrt{\frac{N(N+1)(2N+1)}{24}}$$

and round down to the next smaller integer if the result is not an integer. If zeros or ties occur in the averages, they should all be counted in determining the endpoints.

These Wilcoxon signed-rank test procedures are applicable to paired samples in exactly the same manner as long as X is replaced by the difference $D = X - Y$ and M is interpreted as the median M_D of the distribution of $X - Y$.

As in the case of the sign test, the confidence-interval estimate of the median or median difference can be based on all N observations even if

there are zeros and/or ties. Thus, a hypothesis test concerning a value for the median or median difference when the data contain zeros and/or ties will be more powerful if the decision is based on the confidence-interval estimate rather than on a hypothesis test procedure.

Example 5.7.1

A large company was disturbed about the number of person-hours lost per month due to plant accidents and instituted an extensive industrial safety program. The data below show the number of person-hours lost in a month at each of eight different plants before and after the safety program was established. Has the safety program been effective in reducing time lost from accidents? Assume the distribution of differences is symmetric.

Plant	Before	After
1	51.2	45.8
2	46.5	41.3
3	24.1	15.8
4	10.2	11.1
5	65.3	58.5
6	92.1	70.3
7	30.3	31.6
8	49.2	35.4

Solution

Because of the symmetry assumption, we can use the Wilcoxon signed-rank test instead of the sign test on these data. We take the differences $D =$ Before minus After and test $H_0 : M_D = 0$ versus $H_1 : M_D > 0$ since the program is effective if these differences are large positive numbers. Then we rank the absolute values and sum the positive ranks. The table below shows these calculations.

Plant	D	$\lvert D\rvert$	$r(\lvert D\rvert)$
1	5.4	5.4	4
2	5.2	5.2	3
3	8.3	8.3	6
4	−0.9	0.9	1
5	6.8	6.8	5
6	21.8	21.8	8
7	−1.3	1.3	2
8	13.8	13.8	7

We have $T^+ = 33$ and Table H for $N = 8$ gives the right-tail probability as 0.020. The program has been effective at the 0.05 level. The reader is asked to verify that

the number of positive Walsh averages also equals 33, which is the value of the signed rank statistic.

The following computer printouts illustrate the solution to Example 5.7.1 using the MINITAB, STATXACT, and SAS packages.

```
******************************************
MINITAB SOLUTION TO EXAMPLE 5.7.1
******************************************

Wilcoxon Signed Rank Test: B - A

Test of median = 0.000000 versus median  >  0.000000

                           Wilcoxon               Estimated
        N    N for Test   Statistic        P        Median
B - A   8             8        33.0    0.021         6.600

******************************************
STATXACT SOLUTION TO EXAMPLE 5.7.1
******************************************

Wilcoxon Signed Rank Test

Summary of Exact Distribution of WILCOXON SIGNED RANK Statistic:
   Min     Max     Mean    Std Dev   Observed   Standardized
0.0000   36.00    18.00      7.141      33.00          2.100

Asymptotic Inference:
  One-sided P value: Pr { Test Statistic .GE. Observed } = 0.0178
  Two-sided P value: 2 * One-sided                       = 0.0357

Exact Inference:
  One-sided P value: Pr { Test Statistic .GE. Observed } = 0.0195
                     Pr { Test Statistic .EQ. Observed } = 0.0078
  Two-sided P value: Pr { | Test Statistic - Mean |
                            GE. | Observed - Mean |
  Two-sided P value: 2*One-sided                         = 0.039

*************************************
SAS SOLUTION TO EXAMPLE 5.7.1
*************************************

Program:

DATA EX631;
    INPUT BEFORE AFTER;
   DIFF = BEFORE-AFTER;
```

```
DATALINES;
51.2  45.8
46.5  41.3
24.1  15.8
10.2  11.1
65.3  58.5
92.1  70.3
30.3  31.6
49.2  35.4
PROC UNIVARIATE DATA = EX631;
       VAR DIFF;
RUN;
```

Output:

Test	-Statistic-		----------*P* Value----------	
Student's t	*t*	2.754154	Pr > |*t*|	0.0283
Sign	*M*	2	Pr >= |*M*|	0.2891
Signed rank	*S*	15	Pr >= |*S*|	0.0391

The MINITAB solution uses the normal approximation with a continuity correction. The STATXACT solution gives the asymptotic results based on the normal approximation without a continuity correction. Only a portion of the output from SAS PROC UNIVARIATE is shown. This output provides a lot of information, including important descriptive statistics such as the sample mean, variance, interquartile range, etc., which are not shown. Note that the SAS signed-rank statistic is calculated as $T^+ - N(N+1)/4 = 33 - 18 = 15$ (labeled S) and the P value given is two-tailed. The required one-tailed P value can be found as $0.0391/2 = 0.0196$, which agrees with other calculations. It is interesting that for these data both the t-test and the signed-rank test clearly lead to a rejection of the null hypothesis at the 0.05 level of significance but the sign test does not.

Example 5.7.2

Assume the data in Example 5.4.2 come from a symmetric distribution and find a 90% confidence-interval estimate of the median difference, computed as After minus Before. We will not use these data to illustrate a hypothesis test because of the ties.

Solution

Table H for $N = 6$ shows that $P = 0.047$ for confidence $1 - 2(0.047) = 0.906$, and 0.047 has rank three in Table H so that $u = 3$. Thus, the 90.6% confidence-interval endpoints for the median difference are the third smallest and third largest Walsh averages.

The $6(7)/2 = 21$ Walsh averages of differences $(D_i + D_k)/2$ are shown in the table below.

−2.0	−1.0	1.0	3.0	4.0	8.0
−1.5	0.0	2.0	3.5	6.0	
−0.5	1.0	2.5	5.5		
0.5	1.5	4.5			
1.0	3.5				
3.0					

The third smallest and third largest Walsh averages are −1.0 and 5.5, respectively and the 90.6% confidence interval for the median difference is (−1.0, 5.5). Note that by listing the After minus Before data in an array across the top row of this table of Walsh averages, identification of the confidence-interval endpoints is greatly simplified.

The MINITAB and STATXACT solutions to this example are shown below. The MINITAB solution agrees exactly with our hand calculations. The ''Estimated Median'' 2.0 is the median of the 21 Walsh averages and is the Hodges–Lehmann estimator of the true median difference M_D. The STATXACT solution gives an asymptotic interval that agrees with our exact solution; the interval labeled exact uses the second smallest and the second largest Walsh averages, which provides the 93.8% confidence interval.

```
******************************************
MINITAB SOLUTION TO EXAMPLE 5.7.2
******************************************

Wilcoxon Signed Rank Cl: C1

                  Estimated    Achieved
           N        Median   Confidence    Confidence Interval
C1         6          2.00         90.7   (1 − 1.00,     5.50)

******************************************
STATXACT SOLUTION TO EXAMPLE 5.7.2
******************************************

HODGES-LEHMANN ESTIMATES OF MEDIAN DIFFERENCE
Summary of Exact distribution of WILCOXON SIGNED RANK

     Min      Max     Mean    Std-Dev    Observed  Standardized
  0.0000    21.00    10.50      4.757       16.50         1.261
```

```
Point Estimate of Median Difference : Lambda  =     2.000

90.00% Confidence Interval for Lambda :
          Asymptotic : (      −1.000,         5.500)
          Exact      : (      −1.500,         6.000)
```

5.8 Summary

In this chapter, we presented hypothesis testing and confidence-interval estimation for the pth quantile of any continuous distribution for any specified p, $0 < p < 1$, based on data from one sample or paired samples. These procedures are all based on using the pth sample quantile as a point estimate of the pth population quantile and use the binomial distribution; they have no parametric counter-parts. The sample quantiles are all order statistics of the sample. Other estimates of the population quantiles have been introduced in the literature; most of these are based on linear functions of order statistics, say $\sum a_i X_{(i)}$. The one proposed by Harrell and Davis (1982) has been shown to be better than ours for a wide variety of distributions. Dielman et al. (1994) present a Monte Carlo comparison of the performance of various sample quantile estimators for small sample sizes.

The pth quantile when $p = 0.5$ is the median of the distribution and we presented inference procedures based on the sign test in Section 5.4 and the Wilcoxon signed-rank test in Section 5.7. Both tests are generally useful in the same experimental situations regarding a single sample or paired samples. The assumptions required are minimal—independence of observations and a population which is continuous at M for the ordinary sign test and continuous everywhere and symmetric for the Wilcoxon signed-rank test. Experimentally, both tests have the problem of ties. Both tests are applicable when quantitative measurements are impossible or not feasible, as when rating scales or preferences are used. For the Wilcoxon test, information concerning relative magnitudes as well as directions of differences is required. Only the sign test can be used for strictly dichotomous data, like yes–no observations. Both are very flexible and simple to use for hypothesis testing or constructing confidence intervals. The null distribution of the sign test is easier to work with since binomial probabilities are easy to calculate and tables are readily available. The normal approximation is quite accurate for even moderate N in both cases, and neither is particularly hampered by the presence of a moderate number of zeros or ties.

For hypothesis testing, in the paired-sample case the hypothesis need not state an actual median difference but only a relation between medians if

both populations are assumed symmetric. For example, we might test the hypothesis that the X population values are on the average p percent larger than Y values. Assuming the medians are a reliable indication of size, we would write

$$H_0 : M_X = (1 + 0.01p)M_Y$$

and take differences $D_i = X_i - (1 + 0.01p)Y_i$ and perform either test on these derived data as before.

Both tests have a corresponding procedure for finding a confidence-interval estimate of the median of the population in the one-sample case and the median difference in the paired-sample case. Zeros and/or ties pose no problem with these confidence-interval procedures. We include expressions for sample size determination and power calculations.

Only the Wilcoxon signed-rank statistics are appropriate for tests of symmetry since the ordinary sign-test statistic is not at all related to the symmetry or asymmetry of the population. We have $P[(X_i - M) > 0] = 0.5$ always, and the sole criterion for determining K in the sign test is the number of positive signs, thus ignoring the magnitudes of the plus and minus differences. There are other extensions and modifications of the sign-test type of criteria [see, for example, Walsh (1949a,b)].

If the population is symmetric, both sign tests can be considered to be tests for location of the population mean and are therefore direct nonparametric counterparts to Student's test. As a result, comparisons of their performance are of interest. As explained in Chapter 1, one way to compare performance is by computing their asymptotic relative efficiency (ARE) under various distribution assumptions. The ARE of the ordinary sign test relative to the t test is $2/\pi = 0.637$, and the ARE of the Wilcoxon signed-rank test relative to the t test is $3/\pi = 0.955$, both calculated under the assumption of normal distributions. How these particular results were obtained will be discussed in Chapter 13. It is not surprising that both ARE values are less than 1 because the t test is the best test for normal distributions. It can be shown that the ARE of the Wilcoxon signed-rank test is always at least 0.864 for any continuous symmetric distribution, whereas the corresponding lower bound for the ordinary sign test is only 1/3. The ARE of the sign test relative to the Wilcoxon signed-rank test is 2/3 for the normal distribution and 1/3 for the uniform distribution. However, the result is 4/3 for the double exponential distribution; the fact that the ARE is greater than 1 means that the sign test performs better than the signed-rank test for this particular symmetric but heavy-tailed distribution. Similarly, the Wilcoxon signed-rank test performs better than the t test for some nonnormal distributions; for example, the ARE is 1.50 for the double exponential distribution and 1.09 for the logistic distribution, which are both heavy-tailed distributions.

Problems

5.1 Give a functional definition similar to (5.5.1) for the rank $r(X_i)$ of a random variable in any set of N independent observations where ties are dealt with by the midrank method. *Hint*: In place of $S(u)$ in (5.5.2), consider the function

$$c(u) = \begin{cases} 0 & \text{if } u > 0 \\ 1/2 & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases}$$

5.2 Find the correlation coefficient between variate values and ranks in a random sample of size N from
(a) The uniform distribution
(b) The standard normal distribution
(c) The exponential distribution

5.3 Verify the cdf of differences given in (5.4.14) and the result $M = -2 + \sqrt{3}$. Find and graph the corresponding probability function of differences.

5.4 Answer parts (a) through (e) using (1) the sign-test procedure and (2) the Wilcoxon signed-rank test procedure.
(a) Test at a significance level not exceeding 0.10 the null hypothesis H_0: $M = 2$ against the alternative H_1:$M > 2$, where M is the median of the continuous symmetric population from which the random sample $-3, -6, 1, 9, 4, 10, 12$ is drawn.
(b) Give the exact probability of a type I error in (a).
(c) On the basis of the following random sample of pairs:

X	126	131	153	125	119	102	116	163
Y	120	126	152	129	102	105	100	175

test at a significance level not exceeding 0.10 the null hypothesis $H_0: M = 2$ against the alternative H_1 $M \neq 2$, where M is the median of the continuous and symmetric population of differences $D = X - Y$.
(d) Give the exact probability of a type I error in (c).
(e) Give the confidence interval corresponding to the test in (c).

5.5 Generate the null sampling distributions of T^+ and T^- for a random sample of six unequal and nonzero observations.

5.6 Show by calculations from tables that the normal distribution provides reasonably accurate approximations to the critical values of one-sided tests for $\alpha = 0.01$, 0.05, and 0.10 when
(a) $N = 12$ for the sign test
(b) $N = 15$ for the signed-rank test

5.7 A random sample of 10 observations is drawn from a normal population with mean μ and variance 1. Instead of a normal-theory test, the ordinary sign test is used for $H_0 : \mu = 0, H_1 : \mu > 0$, with rejection region $K \in R$ for $K \geq 8$.
(a) Plot the power curve using the exact distribution of K.
(b) Plot the power curve using the normal approximation to the distribution of K.
(c) Discuss how the power functions might help in the choice of an appropriate sample size for an experiment.

5.8 Prove that the Wilcoxon signed-rank statistic $T^+ - T^-$ based on a set of nonzero observations $X_1, X_2, \ldots, X_N$ can be written in symbols as

$$\sum_{1 \leq i \leq j \leq N} \sum \operatorname{sgn}(X_i + X_j)$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

5.9 Let $D_1, D_2, \ldots, D_N$ be a random sample of N nonzero observations from some continuous population which is symmetric with median zero. Define

$$|D_i| = \begin{cases} X_i & \text{if } D_i > 0 \\ Y_i & \text{if } D_i < 0 \end{cases}$$

Assume there are m X values and n Y values, where $m + n = N$ and the X and Y values are independent. Show that the signed-rank test statistic T^+ calculated for these D_i is equal to the sum of the ranks of the X observations in the combined ordered sample of m X's and n Y's and also that $T^+ - T^-$ is the sum of the X ranks minus the sum of the Y ranks. This sum of the ranks of the X's is the test criterion for the Wilcoxon rank sum statistic in the two-sample problem to be discussed in Chapter 8. Show how T^+ might be used to test the hypothesis that the X and Y populations are identical.

5.10 Hoskin et al. (1986) investigated the change in fatal motor-vehicle accidents after the legal minimum drinking age was raised in 10 states. Their data were the ratios of the number of single-vehicle nighttime fatalities to the number of licensed drivers in the affected age group before and after the laws were changed to raise the drinking age, shown below. The researchers hypothesized that raising the minimum drinking age resulted in a reduced median fatality ratio. Investigate this hypothesis.

State	Affected Ages	Ratio Before	Ratio After
Florida	18	0.262	0.202
Georgia	18	0.295	0.227
Illinois	19–20	0.216	0.191
Iowa	18	0.287	0.209
Maine	18–19	0.277	0.299
Michigan	18–20	0.223	0.151
Montana	18	0.512	0.471
Nebraska	19	0.237	0.151
New Hampshire	18–19	0.348	0.336
Tennessee	18	0.342	0.307

5.11 The conclusion in Problem 5.10 was that the median difference (Before minus After) was positive for the affected age group, but this does not imply that the reduction was the result of raising the minimum legal drinking age. Other factors, countermeasures, or advertising campaigns [like MADD (Mothers Against Drunk Drivers)] may have affected the fatality ratios. In order to investigate further, these researchers compared the Before minus After ratios for the affected age group with the corresponding difference ratios for the 25–29 age group, who were not affected by the law change. Carry out an appropriate test and write a report of your conclusions.

State	Affected Age Group	25–29 Age Group
Florida	0.060	−0.025
Georgia	0.068	−0.023
Illinois	0.025	0.004
Iowa	0.078	−0.008
Maine	−0.022	0.061
Michigan	0.072	0.015
Montana	0.041	−0.035
Nebraska	0.086	−0.016
New Hampshire	0.012	−0.061
Tennessee	0.035	−0.051

5.12 Howard et al. (1986) reported a study designed to investigate whether computer anxiety changes between the beginning and end of a course on introduction to computers. The 14 student subjects were given a test to measure computer anxiety at the beginning of the term and then at the end of the 5 week summer course. High scores on this test indicate a high level of anxiety. For the data below, determine whether computer anxiety was reduced over the term.

Student	Before	After	Student	Before	After
A	20	20	H	34	19
B	21	18	I	28	13
C	23	10	J	20	21
D	26	16	K	29	12
E	32	11	L	22	15
F	27	20	M	30	14
G	38	20	N	25	17

5.13 Twenty-four students took both the midterm and the final exam in a writing course. Numerical grades were not given on the final, but each student was classified as either no change, improved, or reduced level of performance compared with the midterm. Six showed improvement, 5 showed no change, and 13 had a reduced level of performance. Find the P value for an appropriate one-sided test.

5.14 Reducing high blood pressure by diet requires reduction of sodium intake, which usually requires switching from processed foods to their natural counterparts. Listed below are the average sodium contents of five ordinary foods in processed form and natural form for equivalent quantities. Find a confidence-interval estimate of the median difference (processed minus natural) with confidence coefficient at least 0.87 using *two different* procedures.

Natural Food		Processed Food	
Corn of the cob	2	Canned corn	251
Chicken	63	Fried chicken	1220
Ground Sirloin	60	All-beef frankfurter	461
Beans	3	Canned beans	300
Fresh tuna	40	Canned tuna	409

5.15 For the data in Problem 4.20, use both the sign test and the signed-rank test to investigate the research hypothesis that median earnings exceed 2.0.

5.16 In an experiment to measure the effect of mild intoxication on coordination, nine subjects were each given ethyl alcohol in an amount equivalent to 15.7 ml/m^2 of body surface and then asked to write a certain phrase as many times as they could in 1 minute. The number of correctly written words was then counted and scaled such that a zero score represents the score a person not under the influence of alcohol would make, a positive score indicates increased writing speed and accuracy, and a

negative score indicates decreased writing speed and accuracy. For the data below, find a confidence-interval estimate of the median score at level nearest 0.95 using the procedure corresponding to the
(a) Sign test
(b) Wilcoxon signed-rank test where we assume symmetry

Subject	Score	Subject	Score
1	10	6	0
2	−8	7	−7
3	−6	8	5
4	−2	9	−8
5	15		

5.17 For the data in Example 5.4.3, test $H_0: M = 0.50$ against the alternative $H_1: M > 0.50$, using the
(a) Sign test
(b) Signed-rank test and assuming symmetry

5.18 For the data in Example 5.7.1, find a confidence-interval estimate of the median difference Before minus After using the level nearest 0.90.

5.19 In a trial of two types of rain gauge, 69 of type A and 12 of type B were distributed at random over a small area. In a certain period 14 storms occurred, and the average amounts of rain recorded for each storm by the two types of gauge are shown below. Another user claims to have found that the type B gauge gives consistently higher average readings than type A. Do these results substantiate such a conclusion? Investigate using two different nonparametric test procedures by finding the P value from
(a) Tables of the exact distribution
(b) Large sample approximations to the exact distributions

Storm	Type A	Type B	Storm	Type A	Type B
1	1.38	1.42	8	2.63	2.69
2	9.69	10.37	9	2.44	2.68
3	0.39	0.39	10	0.56	0.53
4	1.42	1.46	11	0.69	0.72
5	0.54	0.55	12	0.71	0.72
6	5.94	6.15	13	0.95	0.90
7	0.59	0.61	14	0.55	0.52

A total of four tests are to be performed. Discuss briefly the advisability of using nonparametric versus parametric procedures for such an

investigation and the relative merits of the two nonparametric tests used. Discuss assumptions in each case.

5.20 A manufacturer of suntan lotion is testing a new formula to see whether it provides more protection against sunburn than the old formula. The manufacturer chose 10 persons at random from among the company's employees, applied the two types of lotion to their backs, one type on each side, and exposed their backs to a controlled but intense amount of sun. Degree of sunburn was measured for each side of each subject, with the results shown below (higher numbers represent more severe sunburn).

(a) Test the null hypothesis that the difference (old–new) of degree of sunburn has median zero against the one-sided alternative that it is negative, assuming that the differences are symmetric. Does the new formula appear to be effective?

(b) Find a two-sided confidence interval for the median difference, assuming symmetry and with confidence coefficient near 0.90.

(c) Do (a) and (b) without assuming symmetry.

Subject	Old Formula	New Formula
1	41	37
2	42	39
3	48	31
4	38	39
5	38	34
6	45	47
7	21	19
8	28	30
9	29	25
10	14	8

5.21 Last year the elapsed time of long-distance telephone calls for a national retailer was skewed to the right with a median of 3 minutes 15 seconds. The recession has reduced sales, but the company's treasurer claims that the median length of long-distance calls now is even greater than last year. A random sample of 5625 calls is selected from recent records and 2890 of them are found to last more than 3 minutes 15 seconds. Is the treasurer's claim supported? Give the null and alternative hypotheses and the P value.

5.22 In order to test the effectiveness of a sales training program proposed by a firm of training specialists, a home furnishings company selects six sales representatives at random to take the course. The data are gross sales by these representatives before and after the course.

Representative	Sales Before	Sales After
1	90	97
2	83	80
3	105	110
4	97	93
5	110	123
6	78	84

(a) State the null and alternative hypotheses and use the sign test to find a P value relevant to the question of whether the course is effective.
(b) Use the sign-test procedure at level nearest 0.90 to find a two-sided confidence-interval estimate of the median difference in sales (after–before). Give the exact level.
(c) Use the signed-rank test to do (a). What assumptions must you make?
(d) Use the signed-rank test procedure to do (b).

5.23 In a marketing research test, 15 adult males were asked to shave one side of their face with a brand A razor blade and the other side with a brand B razor blade and state their preferred blade. Twelve men preferred brand A. Find the P value for the alternative that the probability of preferring brand A is greater than 0.5.

5.24 Let X be a continuous random variable symmetrically distributed about θ. Show that the random variables $|X - \theta|$ and Z are independent, where

$$Z = \begin{cases} 1 & \text{if } X > \theta \\ 0 & \text{if } X \leq \theta \end{cases}$$

5.25 Using the result in Problem 5.24, show that for the Wilcoxon signed-rank test statistic T^+, the $2N$ random variables $Z_1, r(|D_1|), Z_2, r(|D_2|), \ldots, Z_N, r(|D_N|)$ are mutually independent under H_0.

5.26 Show that the null distribution of the Wilcoxon signed-rank test statistic T^+ is the same as that of $W = \sum_{i=1}^{N} W_i$, where $W_1, W_2, \ldots, W_N$ are independent random variables with $P(W_i = 0) = P(W_i = i) = 0.5$, $i = 1, 2, \ldots, N$.

5.27 A study 5 years ago reported that the median amount of sleep by American adults is 7.5 hours out of 24 with a standard deviation of 1.5 h and that 5% of the population sleep 6 or less hours while another 5% sleep 9 or more hours. A current sample of eight adults reported their average amounts of sleep per 24 hours as 7.2, 8.3, 5.6, 7.4, 7.8, 5.2, 9.1, and 5.8 hours. Use the most appropriate statistical procedures to determine whether American adults sleep less today than they did

5 years ago and justify your choice. You should at least test hypotheses concerning the quantiles of order 0.05, 0.50, and 0.95.

5.28 Find a confidence-interval estimate of the median amount of sleep per 24 hours for the data in Problem 5.27 using confidence coefficient nearest 0.90.

5.29 Let $X_{(r)}$ denote the rth-order statistic of a random sample of size 5 from any continuous population and κ_p denote the pth quantile of this population. Find:
(a) $P(X_{(1)} < \kappa_{0.5} < X_{(5)})$
(b) $P(X_{(1)} < \kappa_{0.25} < X_{(3)})$
(c) $P(X_{(4)} < \kappa_{0.80} < X_{(5)})$

5.30 For order statistics of a random sample of size N from any continuous population F_X, show that the interval $(X_{(r)}, X_{(N-r+1)}, r < N/2)$, is a $100(1-\alpha)\%$ confidence-interval estimate for the median of F_X, where

$$1 - \alpha = 1 - 2N\binom{N-1}{r-1}\int_0^{0.5} x^{N-r}(1-x)^{r-1}dx$$

5.31 If $X_{(1)}$ and $X_{(N)}$ are the smallest and largest values, respectively, in a sample of size n from any continuous population F_X with median $\kappa_{0.50}$, find the smallest value of N such that:
(a) $P(X_{(1)} < \kappa_{0.50} < X_{(N)}) \geq 0.99$
(b) $P[F_X(X_{(N)}) - F_X(X_{(1)}) \geq 0.5] \geq 0.95$

5.32 Derive the sample size formula based on the normal approximation for the sign test against a two-sided alternative with approximate size α and power $1-\beta$.

5.33 Derive the sample size formula based on the normal approximation for the signed rank test against a two-sided alternative with approximate size α and power $1-\beta$.

6

The General Two-Sample Problem

6.1 Introduction

For the matched-pairs sign and signed-rank tests of Chapter 5, the data consisted of two samples, but each element in one sample was linked with a particular element of the other sample by some unit of association. This sampling situation can be described as two dependent samples or a single sample of pairs from a bivariate population. When the inferences to be drawn are related only to the population of differences of the paired observations, the first step in the analysis usually is to take the differences of the paired observations; this leaves only a single set of observations. Therefore, this type of data can be legitimately classified as a one-sample problem. In this chapter, we will be concerned with data consisting of two mutually independent random samples, that is, random samples drawn independently from each of two populations. Not only are the elements within each sample independent, but also every element in the first sample is independent of every element in the second sample.

The universe consists of two populations, which we call the X and Y populations, with cumulative distribution functions F_X and F_Y, respectively. We have a random sample of size m drawn from the X population and another random sample of size n drawn independently from the Y population,

$$X_1, X_2, \ldots, X_m \quad \text{and} \quad Y_1, Y_2, \ldots, Y_n$$

Usually the hypothesis of interest in the two-sample problem is that the two samples are drawn from identical populations, that is,

$$H_0 : F_Y(x) = F_X(x) \quad \text{for all } x$$

If we are willing to make assumptions concerning the forms of the underlying populations and assume that the differences between the two populations occur only with respect to some parameters, such as the means or the variances, the so-called best test can frequently be derived in a Neyman–Pearson framework. For example, if we assume that both populations are

normally distributed, it is well known that the two-sample Student's t test for equality of means and the F test for equality of variances are respectively the best tests. The performances of these two tests are well known. These and other classical tests may be sensitive to violations of the fundamental model assumptions inherent in the derivation and construction of these tests. Any conclusions reached using such tests are only as valid as the underlying assumptions. If there is reason to suspect a violation of any of these postulates, or if sufficient information to judge their validity is not available, or if a completely general test of equality for unspecified distributions is appropriate, some nonparametric procedure is desirable.

In practice, other assumptions are often made about the form of the underlying populations. One common assumption is called the *location model*, or the *shift model*. This model assumes that the X and Y populations are the same except possibly for an unknown shift value θ, or that

$$F_Y(x) = P(Y \leq x) = P(X \leq x - \theta) = F_X(x - \theta) \quad \text{for all } x \text{ and } \theta \neq 0$$

This means that $X + \theta$ and Y have the same distribution or that X is distributed as $Y - \theta$. The Y population is then the same as the X population if $\theta = 0$, is shifted to the right if $\theta > 0$, and is shifted to the left if $\theta < 0$. Under the shift assumption, the populations have the same shape and the same variance, and the amount of the shift θ must be equal to the difference between the population means, $\mu_Y - \mu_X$, the population medians, $M_Y - M_X$, and in fact the difference between any two respective location parameters or quantiles of the same order.

Another assumption about the form of the underlying population is called the *scale model*, which assumes that the X and Y populations are the same except possibly for a positive scale factor θ. The scale model can be written as

$$F_Y(x) = P(Y \leq x) = P(X \leq \theta x) = F_X(\theta x) \quad \text{for all } x \text{ and } \theta > 0,\ \theta \neq 1$$

This means that X/θ and Y have the same distribution for any positive θ or that X is distributed as θY. Also, the variance of X is θ^2 times the variance of Y and the mean of X is θ times the mean of Y.

A more general assumption about the form of the underlying populations is called a *location-scale model*. This model can be written as

$$P(Y - \mu_Y \leq x) = P(X - \mu_X \leq \theta x) \quad \text{for all } x \text{ and } \theta > 0,\ \theta \neq 1$$

which states that $(X - \mu_X)/\theta$ and $Y - \mu_Y$ are identically distributed (or similarly in terms of M_Y, M_X). Thus, the location-scale model incorporates properties of both the location and the scale models. Now the means of $X - \mu_X$ and $Y - \mu_Y$ are both zero and the variance of $X - \mu_X$ is θ^2 times the variance of $Y - \mu_Y$.

Regardless of the model assumed, the general two-sample problem is perhaps the most frequently discussed problem in nonparametric statistics. The null hypothesis is almost always formulated as identical populations with the common distribution completely unspecified except for the assumption that it is continuous. Thus, in the null case, the two random samples can be considered a single random sample of size $N = m + n$ drawn from the common, continuous, but unspecified population. Then the combined ordered configuration of the m X and n Y random variables in the sample is one of the $\binom{m+n}{m}$ possible equally likely arrangements. For example, suppose we have two independent random samples, $m = 3$ X's and $n = 2$ Y's. Under the null hypothesis that the X's and the Y's are identically distributed, each of the $\binom{5}{2} = 10$ possible arrangements of the combined sample shown below is equally likely.

1. $XXXYY$ 2. $XXYXY$ 3. $YXYXX$ 4. $XXYYX$ 5. $XYXXY$
6. $XYXYX$ 7. $YXXXY$ 8. $YXXYX$ 9. $XYYXX$ 10. $YYXXX$

In practice, the sample pattern of arrangement of X's and Y's provides information about the type of difference which may exist in the populations. For instance, if the observed arrangement is the one labeled either 1 or 10 in the above example, the X's and the Y's do not appear to be randomly mixed, suggesting a contradiction to the null hypothesis. Many statistical tests are based on some function of this combined arrangement. The type of function which is most appropriate depends on the type of difference one hopes to detect, which is indicated by the alternative hypothesis. An abundance of reasonable alternatives to H_0 may be considered, but the type easiest to analyze using distribution-free techniques states some functional relationship between the distributions. The most general two-sided alternative states simply

$$H_A : F_Y(x) \neq F_X(x) \quad \text{for some } x$$

and a corresponding general one-sided alternative is

$$\begin{aligned} H_1 : F_Y(x) &\geq F_X(x) \quad \text{for all } x \\ F_Y(x) &> F_X(x) \quad \text{for some } x \end{aligned}$$

In this latter case, we generally say that the random variable X is *stochastically larger* than the random variable Y. We can write this as $X \overset{\text{ST}}{>} Y$. Figures 6.1.1 and 6.1.2 are descriptive of the alternative that X is stochastically larger than Y, which includes as a subclass the more specific alternative $\mu_X > \mu_Y$. Some authors define $X \overset{\text{ST}}{>} Y$ to mean that $P(X > Y) > P(X < Y)$. (For the reverse inequality on F_X and F_Y, we say X is stochastically smaller than Y and write $X \overset{\text{ST}}{<} Y$).

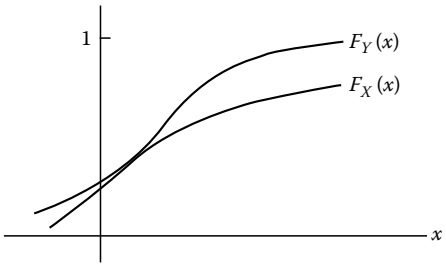


FIGURE 6.1.1
X is stochastically larger than Y.

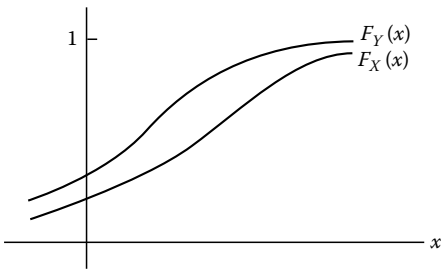


FIGURE 6.1.2
X is stochastically larger than Y.

If the particular alternative of interest is simply a difference in location, we use the location alternative or the location model

$$H_L : F_Y(x) = F_X(x - \theta) \quad \text{for all } x \text{ and some } \theta \neq 0$$

Under the location model, Y is distributed as $X + \theta$, so that Y is stochastically larger (smaller) than X if and only if $\theta > 0$ ($\theta < 0$). Similarly, if a difference in scale is of interest, we use the scale alternative

$$H_S : F_Y(x) = F_X(\theta x) \quad \text{for all } x \text{ and some } \theta \neq 1$$

Under the scale model, Y is distributed as X/θ so that Y is stochastically larger (smaller) than X if and only if $\theta < 1$ ($\theta > 1$).

Although the three special alternatives H_1, H_L, and H_S are the most frequently encountered in the general class H_A, other types of relations may be considered. For example, the alternative $H_{\text{LE}} : F_Y(x) = [F_X(x)]^k$, for some positive integer k and all x, called the *Lehmann* (1953) *alternative*, states that the Y random variables are distributed as the largest of k X variables. Under this alternative, Y is stochastically larger (smaller) than X if and only if $k > 1$ ($k < 1$).

The available statistical literature on the two-sample problem is quite extensive. A multitude of tests have been proposed for a wide variety of

functional alternatives, but only a few of the best-known tests are included in this book. The Wald–Wolfowitz runs test, the Kolmogorov–Smirnov (K–S) two-sample test, the median test, the control median test, and the Mann–Whitney U test will be covered in this chapter. Chapters 7 and 8 are concerned with a specific class of tests particularly useful for the location and scale alternatives, respectively.

6.2 The Wald–Wolfowitz Runs Test

Combine the two sets of independent random variables $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ into a single ordered sequence from smallest to largest, keeping track of which observations correspond to the X sample and which to the Y. Assuming that their probability distributions are continuous, a unique ordering is always possible, since theoretically ties do not exist. For example, with $m = 4$ and $n = 5$, the arrangement might be

$XYYXXYXYY$

which indicates that in the pooled sample the smallest element is an X, the second smallest a Y, etc., and largest a Y. Under the null hypothesis of identical distributions

$$H_0 : F_Y(x) = F_X(x) \text{ for all } x$$

we expect the X and Y random variables to be well mixed in the ordered configuration, since the $m + n = N$ random variables constitute a single random sample of size N from the common population. With a run defined as in Chapter 3 as a sequence of identical letters preceded and followed by a different letter or no letter, the total number of runs in the ordered pooled sample is indicative of the degree of mixing. In the arrangement $XYYXXYXYY$, the total number of runs is equal to 6 which shows a pretty good mixing of X's and Y's. A pattern of arrangement with too few runs would suggest that this group of N is not a single random sample but instead is composed of two samples from two distinguishable populations. For example, if the arrangement is $XXXXYYYYY$ so that all the elements in the X sample are smaller than all of the elements in the Y sample, there would be only two runs. This particular configuration might indicate not only that the populations are not identical, but also that the X's are stochastically smaller than the Y's. However, the reverse ordering also contains only two runs, and therefore a test criterion based solely on the total number of runs cannot distinguish these two cases.

The runs test is appropriate primarily when the alternative is completely general and two-sided, as in

$$H_A : F_Y(x) \neq F_X(x) \quad \text{for some } x$$

We define the random variable R as the total number of runs in the combined ordered arrangement of m X and n Y random variables. Since too few runs tend to discredit the null hypothesis when the alternative is H_A, the *Wald–Wolfowitz* (1940) *runs* test for significance level α generally has the rejection region in the lower tail as

$$R \leq c_\alpha$$

where c_α is chosen to be the largest integer satisfying

$$P(R \leq c_\alpha | H_0) \leq \alpha$$

The P value for the runs test is then given by

$$P(R \leq R_0 | H_0)$$

where R_0 is the observed value of the runs test statistic R.

Since the X and Y observations are two types of objects arranged in a completely random sequence if H_0 is true, the null probability distribution of R is exactly the same as we found for the runs test for randomness in Chapter 3. The distribution is given in Theorem 3.2.2 with n_1 and n_2 replaced by m and n, respectively, assuming the X are called type 1 objects and Y's are type 2 objects. The other properties of R discussed in Section 3.2, including the moments and asymptotic null distribution, are also unchanged. The only difference here is that the appropriate critical region for the alternative of different populations is too few runs. The null distribution of R is given in Table D with $n_1 = m$ and $n_2 = n$ for $m \leq n$. The normal approximation described in Section 3.2.3 is used for larger sample sizes. A numerical example of this test is given below.

Example 6.2.1

It is easy to show that the distribution of a standardized chi-square variable with large degrees of freedom can be approximated by the standard normal distribution. Here we investigate the agreement between these two distributions for moderate degrees of freedom. Two mutually independent random samples, each of size 8, were generated, one from the standard normal distribution and one from the chi-square distribution with $\nu = 18$ degrees of freedom. The resulting data are as follows:

Normal	−1.91	−1.22	−0.96	−0.72	0.14	0.82	1.45	1.86
Chi square	4.90	7.25	8.04	14.10	18.30	21.21	23.10	28.12

SOLUTION

Before testing the null hypothesis of equal distributions, the chi-square sample data must be standardized by subtracting the mean $\nu = 18$ and dividing by the standard deviation $\sqrt{2\nu} = \sqrt{36} = 6$. The transformed chi-square data are, respectively,

$$-2.18 \quad -1.79 \quad -1.66 \quad -0.65 \quad -0.05 \quad 0.54 \quad 0.85 \quad 1.69$$

We pool the normal data and these transformed data into a single array, ordering them from smallest to largest, underlining the transformed chi-square data, as

$$\underline{-2.18},\ -1.91,\ \underline{-1.79},\ \underline{-1.66},\ -1.22,\ -0.96,\ -0.72,\ \underline{-0.65},$$
$$\underline{-0.05},\ 0.14,\ \underline{0.54},\ 0.82,\ \underline{0.85},\ 1.45,\ \underline{1.69},\ 1.86$$

Let X and Y denote the standardized chi-square sample data and the normal sample data, respectively. To use the Wald–Wolfowitz runs test, we simply count the number of runs in the ordered combined configuration $X, Y, X, X, Y, Y, Y, X, X, Y, X, Y, X, Y, X, Y$ as $R = 12$. Table D shows that the P value, the left-tail probability with $R = 12$ for $m = 8$, $n = 8$, exceeds 0.5, and therefore we do not reject the null hypothesis of equal distributions. Using (3.2.11), we get $Z = 1.81$ and $P = 0.9649$ with a continuity correction, and $Z = 1.55$ and $P = 0.9394$ without a continuity correction.

The STATXACT solution to Example 6.2.1 using the runs test is shown below. The exact P value is 0.9683, which can be verified from Table D since $P(R \leq 12) = 1 - P(R \geq 13) = 0.968$. Note that their asymptotic P value is not the same as ours using (3.2.11).

```
*****************************************
STATXACT SOLUTION TO EXAMPLE 6.2.1
*****************************************
WALD-WOLFOWITZ RUNS TEST
Summary of Exact Distribution of Wald-Wolfowitz Runs Test Statistic

          Min           Max         Observed
        2.000         16.00            12.00

Asymptotic P value:
    Pr { Test Statistic .LE.           12.00 } =       0.9021
Exact P value:
    Pr { Test Statistic .LE.           12.00 } =       0.9683
    Pr { Test Statistic .EQ.           12.00 } =       0.0685
```

6.2.1 The Problem of Ties

Ideally, no ties should occur because of the assumption of continuous populations. Ties do not present a problem in counting the number of runs unless the tie is across samples; that is, two or more observations from different samples have exactly the same magnitude. For a conservative test, we can break all ties in all possible ways and compute the total number of runs for each resolution of all ties. The value of the test statistic R *is* the largest

computed value, since that is the one least likely to lead to rejection of H_0. For each group of ties across samples, where there are s x's and t y's of equal magnitude for some $s \geq 1, t \geq 1$, there are $\binom{s+t}{s}$ ways to break the ties. Thus, if there are k groups of ties, the total number of values of R to be computed is the product $\prod_{i=1}^{k} \binom{s_i + t_i}{s_i}$.

6.2.2 Discussion

The Wald–Wolfowitz runs test is extremely general and is consistent against all types of differences in populations (Wald and Wolfowitz, 1940). The very generality of the test weakens its performance against specific alternatives. Asymptotic power can be evaluated using the normal distribution with appropriate moments under the alternative, which are given in Wolfowitz (1949). Since power, whether exact or asymptotic, can be calculated only for completely specified alternatives, numerical power comparisons should not be the only criteria for this test. Its primary usefulness is in preliminary analyses of data when no particular form of alternative is yet formulated. Then, if the hypothesis is rejected, further studies can be made with other tests in an attempt to classify the type of difference between populations.

6.3 The Kolmogorov–Smirnov (K–S) Two-Sample Test

The K–S statistic is another one-sample test that can be adapted to the two-sample problem. Recall from Chapter 4 that as a goodness-of-fit criterion, this test compares the empirical distribution function of a random sample with a hypothesized cdf. In the two-sample case, the comparison is made between the empirical distribution functions of the two samples.

The order statistics corresponding to two random samples of size m and n from continuous populations F_X and F_Y are

$$X_{(1)}, X_{(2)}, \ldots, X_{(m)} \quad \text{and} \quad Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$$

Their respective empirical (sample) distribution functions, denoted by $S_m(x)$ and $S_n(x)$, are defined as before:

$$S_m(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ k/m & \text{if } X_{(k)} \leq x < X_{(k+1)} \quad \text{for } k = 1, 2, \ldots, m-1 \\ 1 & \text{if } x \geq X_{(m)} \end{cases}$$

and

$$S_n(x) = \begin{cases} 0 & \text{if } x < Y_{(1)} \\ k/n & \text{if } Y_{(k)} \le x < Y_{(k+1)} \\ 1 & \text{if } x \ge Y_{(n)} \end{cases} \quad \text{for } k = 1, 2, \ldots, n-1$$

In a combined ordered arrangement of the $m + n$ sample observations, $S_m(x)$ and $S_n(x)$ are the respective proportions of X and Y observations which do not exceed the specified value x.

If the null hypothesis

$$H_0 : F_Y(x) = F_X(x) \quad \text{for all } x$$

is true, the population distributions are identical and we have two samples from the same population. The empirical distribution functions for the X and Y samples are reasonable estimates of their respective population cdf's. Therefore, allowing for sampling variation, there should be reasonable agreement between the two empirical distributions if H_0 is true; otherwise the data suggest that H_0 is not true and therefore should be rejected. This is the intuitive logic behind most two-sample tests, and the problem is to define what is a reasonable agreement between the two empirical cdf's. In other words, how close do the two empirical cdf's have to be so that they could be viewed as not significantly different, taking account of the sampling variability. Note that this approach necessarily requires a definition of closeness. The two-sided *K–S two-sample test* criterion (sometimes called the *Smirnov* test), denoted by $D_{m,n}$, is based on the maximum absolute difference between the two empirical distributions

$$D_{m,n} = \max_x |S_m(x) - S_n(x)|$$

Since here only the magnitudes, and not the directions, of the deviations are considered, $D_{m,n}$ is appropriate for a general two-sided alternative

$$H_A : F_Y(x) \ne F_X(x) \quad \text{for some } x$$

and the rejection region is in the upper tail, defined by

$$D_{m,n} \ge c_\alpha$$

where

$$P(D_{m,n} \ge c_\alpha | H_0) \le \alpha$$

Because of the Glivenko–Cantelli theorem (Theorem 2.3.2), the test is consistent for this alternative. The P value is

$$P(D_{m,n} \geq D_0 | H_0)$$

where D_0 is the observed value of the two-sample K–S test statistic. As with the one-sample K–S statistic, $D_{m,n}$ is completely distribution-free for any continuous common population distribution since order is preserved under a monotone transformation. That is, if we let $z = F(x)$ for the common continuous cdf F, we have $S_m(z) = S_m(x)$ and $S_n(z) = S_n(x)$, where the random variable Z, corresponding to z, has the uniform distribution on the unit interval.

In order to implement the test, the exact cumulative null distribution of $mnD_{m,n}$ is given in Table I for $2 \leq m \leq n \leq 12$ or $m + n \leq 16$, whichever occurs first. Selected quantiles of $mnD_{m,n}$ are also given for $9 \leq m = n \leq 20$, along with the large sample approximation.

The derivation of the exact null distribution of $D_{m,n}$ is usually attributed to the Russian School, particularly Gnedenko (1954) and Korolyuk (1961), but the papers by Massey (1951b, 1952) are also important. Several methods of calculation are possible, generally involving recursive formulas. Drion (1952) derived a closed expression for exact probabilities for the case $m = n$ by applying random-walk techniques. Several approaches are summarized in Hodges (1958). One of these methods, which is particularly useful for small sample sizes, will be presented here as an aid to understanding.

To compute $P(D_{m,n} \geq d | H_0)$, where d is the observed value of $\max_x |S_m(x) - S_n(x)|$, we first arrange the combined sample of $m + n$ observations in increasing order of magnitude. The arrangement can be depicted graphically on a Cartesian coordinate system by a path which starts at the origin and moves one step to the right for an x observation and one step up for a y observation, ending at (m, n). For example, the sample arrangement $xyyxxyy$ is represented in Figure 6.3.1. The observed values of $mS_m(x)$ and $nS_n(x)$ are, respectively, the coordinates of all points (u, v) on the path where u and v are integers. The number d is the largest of the differences $|u/m - v/n| = |nu - mv|/mn$. If a line is drawn connecting the points (0, 0) and (m, n) on this graph, the equation of the line is $nx - my = 0$ and the vertical distance from any point (u, v) on the path to this line is $|v - nu/m|$. Therefore, nd for the observed sample is the distance from the diagonal line. In Figure 6.3.1 the farthest point is labeled Q, and the value of d is $2/4$.

The total number of arrangements of m X and n Y random variables is $\binom{m+n}{m}$, and under H_0 each of the corresponding paths is equally likely. The probability that an observed value of $D_{m,n}$ is not less than d then is the number of paths which have points at a distance not less than nd from the diagonal, divided by $\binom{m+n}{m}$.

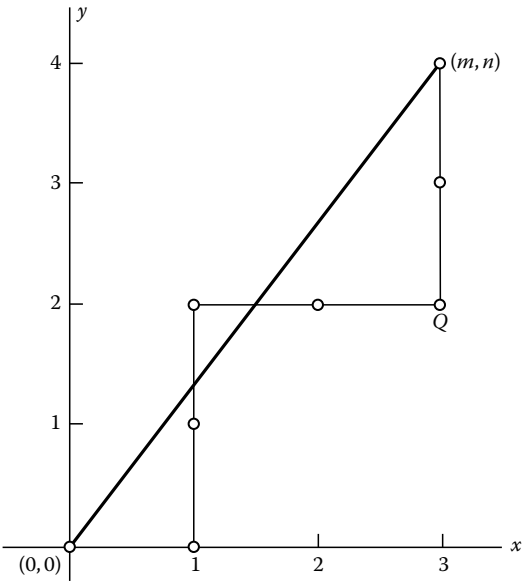


FIGURE 6.3.1
Path of *xyyxxyy*.

In order to count this number, we draw another figure of the same dimension as before and mark off two lines at vertical distance *nd* from the diagonal, as in Figure 6.3.2. Denote by $A(m, n)$ the number of paths from (0, 0) to (m, n) which lie entirely *within* (not on) these boundary lines. Then the desired probability is

$$P(D_{m,n} \geq d \mid H_0) = 1 - P(D_{m,n} < d \mid H_0) = 1 - \frac{A(m,n)}{\binom{m+n}{m}}$$

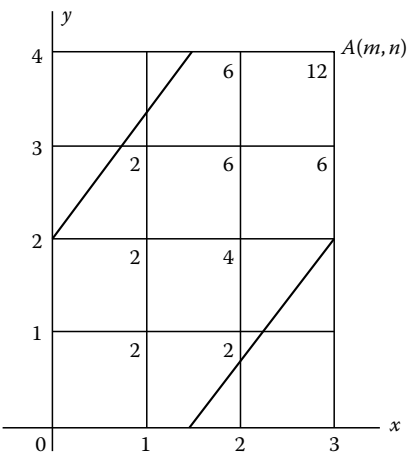


FIGURE 6.3.2
Evaluation of $A(u, v)$ for *xyyxxyy*.

$A(m, n)$ can easily be counted in the manner indicated in Figure 6.3.2. The number $A(u, v)$ at any intersection (u, v) clearly satisfies the recursion relation

$$A(u, v) = A(u - 1, v) + A(u, v - 1)$$

with boundary conditions

$$A(0, v) = A(u, 0) = 1$$

Thus, $A(u, v)$ is the sum of the numbers at the intersections where the previous point on the path could have been while still within the boundaries. This procedure is shown in Figure 6.3.2 for the arrangement $xyyxxyy$, where $nd = 2$. Since here $A(3, 4) = 12$, we have

$$P(D_{3,4} \geq 0.5) = 1 - \frac{12}{\binom{7}{4}} = \frac{23}{35} = 0.65714$$

For the asymptotic null distribution, that is, $m, n \to \infty$ in such a way that m/n remains constant, Smirnov (1939) proved the result

$$\lim_{m,n\to\infty} P\left(\sqrt{\frac{mn}{m+n}} D_{m,n} \leq d\right) = L(d)$$

where

$$L(d) = 1 - 2\sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2d^2}$$

Note that the asymptotic distribution of $\sqrt{mn/(m+n)}D_{m,n}$ is exactly the same as the asymptotic distribution of $\sqrt{N}D_N$ in Theorem 4.3.3. This is not surprising, since we know from the Glivenko–Cantelli theorem that as $n \to \infty, S_n(x)$ converges to $F_Y(x)$, which can be relabeled $F_X(x)$ as in the theorem. Then the only difference here is in the normalizing factor $\sqrt{mn/(m+n)}$, which replaces $\sqrt{N}$.

6.3.1 One-Sided Alternatives

A one-sided two-sample maximum-unidirectional-deviation test can also be defined, based on the statistic

$$D^+_{m,n} = \max_x [S_m(x) - S_n(x)]$$

For an alternative that the X random variables are stochastically smaller than the Y's,

$$\begin{aligned} H_1 : F_Y(x) &\leq F_X(x) \quad \text{for all } x \\ F_Y(x) &< F_X(x) \quad \text{for some } x \end{aligned}$$

the rejection region should be

$$D^+_{m,n} \geq c_\alpha$$

The one-sided test based on $D^+_{m,n}$ is also distribution-free and consistent against the alternative H_1. Since either sample may be labeled the X sample, it is not necessary to define another one-sided statistic for the alternative that X is stochastically larger than Y. The entries in Table I can also be used for a one-sided two-sample K–S statistic since the probabilities in the tails of this distribution are closely approximated using one-half of the corresponding tail probabilities on the two-sided, two-sample K–S statistic.

The graphic method described for $D_{m,n}$ can be applied here to calculate $P(D^+_{m,n} \geq d)$. The point Q^+, corresponding to Q, would be the point farthest *below* the diagonal line, and $A(m, n)$ is the number of paths lying entirely *above* the lower boundary line (see Problem 6.1). Tables of the null distribution of $D^+_{m,n}$ are available in Goodman (1954) for $m = n$.

As with the two-sided statistic, the asymptotic distribution of $\sqrt{mn/(m+n)}D^+_{m,n}$ is equivalent to the asymptotic distribution of $\sqrt{N}D^+_N$, which was given in Theorem 4.3.5 as

$$\lim_{m,n\to\infty} P\left(\sqrt{\frac{mn}{m+n}}D^+_{m,n} \leq d\right) = 1 - e^{-2d^2}$$

6.3.2 Ties

Ties within and across samples can be handled by considering only the r distinct ordered observations in the combined sample as values of x in computing $S_m(x)$ and $S_n(x)$ for $r \leq m$ and $r \leq n$. Then we find the empirical cdf for each different x and their differences at these observations and calculate the statistic in the usual way.

6.3.3 Discussion

The two-sample K–S tests are very easy to apply, using the exact distribution for any m and n within the range of the available tables and using the asymptotic distribution for larger sample sizes. They are useful mainly for the general alternatives H_A and H_1, since the test statistic is sensitive to all

types of differences between the cdf's. Their primary application then should be for preliminary studies of data, as was the runs test. Gideon and Mueller (1978) give a simple method for calculating $D_{m,n}$ and Pirie (1979) extends this method to samples with ties. The K–S tests are more powerful than the runs tests when compared against the Lehmann (1953) alternatives for large sample sizes. The large-sample performance of the K–S tests against specific location or scale alternatives varies considerably according to the population sampled. Capon (1965) made a study of these properties. Goodman (1954) showed that when applied to data from discrete distributions, these tests are conservative.

6.3.4 Applications

Example 6.3.1

We illustrate the K–S two-sample test against the two-sided alternative with the data from Example 6.2.1.The two empirical distribution functions and their differences are shown in Table 6.3.1. Note that the first column, labeled *t,* shows the combined (pooled) ordered sample. The maximum of the last column is $D_{m,n} = 2/8$ so that $mnD_{m,n} = 16$. Table I for $m = n = 8$, shows that $P(64D_{8,8} \geq 32|H_0) = 0.283$, so the required P value, $P(64D_{8,8} \geq 16|H_0)$, must be greater than 0.283. Thus, we do not reject the null hypothesis of identical distributions.

TABLE 6.3.1

Calculation of $D_{m,n}$ for Example 6.3.1

t	# $X \leq t$	$S_m(t)$	# $Y \leq t$	$S_n(t)$	$S_m(t) - S_n(t)$	$\lvert S_m(t) - S_n(t)\rvert$
−2.18	1	1/8	0	0	1/8	1/8
−1.91	1	1/8	1	1/8	0	0
−1.79	2	2/8	1	1/8	1/8	1/8
−1.66	3	3/8	1	1/8	2/8	2/8
−1.22	3	3/8	2	2/8	1/8	1/8
−0.96	3	3/8	3	3/8	0	0
−0.72	3	3/8	4	4/8	−1/8	1/8
−0.65	4	4/8	4	4/8	0	0
0.05	5	5/8	4	4/8	1/8	1/8
0.14	5	5/8	5	5/8	0	0
0.54	6	6/8	5	5/8	1/8	1/8
0.82	6	6/8	6	6/8	0	0
0.85	7	7/8	6	6/8	1/8	1/8
1.45	7	7/8	7	7/8	0	0
1.69	8	8/8	7	7/8	1/8	1/8
1.86	8	8/8	8	8/8	0	0

For the one-sided alternative, $D^+_{m,n} = 2/8$ and so the approximate P value is at least $(0.283)/2 = 0.142$. Thus, there is not sufficient evidence to reject H_0 against the one-sided alternative that the X's are stochastically smaller than the Y's.

The STATXACT solution to Example 6.3.1 using the K–S test is shown below. For the two-sided alternative, the exact and asymptotic P values are shown to be 0.9801 and 0.9639, respectively, both strongly suggesting that there is no significant evidence against the null hypothesis in these data. The exact two-sided P value is a Monte Carlo estimate; the algorithm is described in Hilton et al. (1994). The asymptotic two-sided P value is calculated using the Smirnov approximation, keeping only the first few terms. The exact one-sided P value is calculated from the permutation distribution of $D^+_{m,n}$. The reader is referred to the STATXACT user manual for details.

```
*****************************************************
STATXACT SOLUTION TO EXAMPLE 6.3.1: K-S TEST
*****************************************************

KOLMOGOROV-SMIRNOV TWO-SAMPLE TEST
POP_1 (F1)  :        1         POP_2 (F2) :        2

       Number of observations:
               POP_1 = 8
               POP_2 = 8
```

	|F1 − F2|	F1 − F2 (POP_1 is larger)	F2 − F1 (POP_2 is larger)
Observed statistic	0.2500	0.1250	0.2500
Asymptotic P value	0.9639	0.8825	0.6025
Exact P value	0.9801	0.8889	0.6222
Exact point prob.	0.3200	0.2667	0.2828

6.4 The Median Test

In order to test the null hypothesis of identical populations with two independent samples, the K–S two-sample test compares the proportions of observations from each sample which do not exceed x for all real numbers x. The test criterion is the maximum difference (absolute or unidirectional) between the two empirical distributions, which are defined for all x. Suppose that instead of using all possible differences, we choose some arbitrary but specific number δ and compare only the proportions of observations from each sample which are strictly less than δ. As before, the two independent samples are denoted by

$$X_1, X_2, \ldots, X_m \quad Y_1, Y_2, \ldots, Y_n$$

Each of the $m + n = N$ observations is to be classified according to whether it is less than δ or not. Let U and V denote the respective numbers of X and Y observations less than δ. Since the random variables in each sample have been dichotomized, U and V both follow the binomial probability distribution with parameters

$$p_X = P(X < \delta) \quad \text{and} \quad p_Y = P(Y < \delta)$$

and numbers of trials m and n, respectively. For two independent samples, the joint distribution of U and V then is

$$f_{U,V}(u, v) = \binom{m}{u}\binom{n}{v} p_X^u p_Y^v (1 - p_X)^{m-u}(1 - p_Y)^{n-v} \qquad (6.4.1)$$
$$u = 0, 1, \ldots, m \quad \text{and} \quad v = 0, 1, \ldots, n$$

The random variables U/m and V/n are unbiased point estimates of the parameters p_X and p_Y, respectively. The difference $U/m - V/n$ then is appropriate for testing the null hypothesis

$$H_0 : p_X - p_Y = 0$$

The exact null probability distribution of $U/m - V/n$ can easily be found from (6.4.1), and for m and n large its distribution can be approximated by the normal. The test statistic in either case depends on the common value $p = p_X = p_Y$, but the test can be performed by replacing p by its unbiased estimate $(u + v)/(m + n)$. Otherwise there is no difficulty in constructing a test (although approximate) based on the criterion of difference of proportions of observations less than δ. This is essentially a modified sign test for two independent samples, with the hypothesis that δ is the pth quantile point in each population, where p is unspecified but estimated from the data.

This test will not be pursued here since it is approximate and is not always appropriate to the general two-sample problem, where we are primarily interested in the hypothesis of identical populations. If the two populations are the same, the pth quantile points are equal for every value of p. However, two populations may be quite disparate even though some particular quantile points are equal. The value of δ, which is supposedly chosen without knowledge of the observations, then affects the sensitivity of the test criterion. If δ is chosen too small or too large, both U and V will have too small a range to be reliable. We cannot hope to have reasonable power for the general test without a judicious choice of δ. A test where the experimenter chooses a particular value of p (rather than δ), preferably a central value, would be more appropriate for our general hypothesis, especially if the type of difference one hopes to detect is primarily in location. In other words, we would rather control the *position* of δ, regardless of its actual value, but p and δ are hopelessly interrelated in the common population.

When the populations are assumed identical but unspecified, we cannot choose p and then determine the corresponding δ. Yet δ must be known at least positionally to classify each sample observation as less than δ or not. Therefore, suppose we decide to control the position of δ relative to the magnitudes of the *sample* observations. If the quantity $U+V$ is fixed by the experimenter prior to sampling, p is to some extent controlled since $(u+v)/(m+n)$ is an estimate of the common p. If p denotes the probability that any observation is less than δ, the probability distribution of $T=U+V$ is

$$f_T(t)=\binom{m+n}{t}p^t(1-p)^{m+n-t} \quad t=0,1,\ldots,m+n \tag{6.4.2}$$

The conditional distribution of U given $T=t$ is (6.4.1) divided by (6.4.2). In the null case where $p_X=p_Y=p$, the result is simply

$$f_{U|T}(u|t)=\frac{\binom{m}{u}\binom{n}{t-u}}{\binom{m+n}{t}} \quad u=\max(0,t-n),1,\ldots,\min(m,t) \tag{6.4.3}$$

which is the hypergeometric probability distribution. This result could also have been argued directly as follows. Each of the $m+n$ observations is dichotomized according to whether it is less than δ or not. Among all the observations, if $p_X=p_Y=p$, every one of the $\binom{m+n}{t}$ sets of t numbers is equally likely to comprise the less-than-δ group.

The number of sets that have exactly u from the X sample is $\binom{m}{u}\binom{n}{t-u}$. Since U/m is an estimate of p_X, u/m should be close to $t/(m+n)$ if the hypothesis $p_X=p_Y=p$ is true. A test criterion can then be found using the conditional distribution of U in (6.4.3) for any chosen t.

So far nothing has been said about the value of δ, since once t is chosen, δ really need not be specified to perform the test. Any number greater than the tth and not greater than the $(t+1)$st order statistic in the combined ordered sample will yield the same value of u. In practice, the experimenter would probably rather choose the fraction $t/(m+n)$ in order to control the value of p. Suppose we decide that if the populations differ at all, it is only in location. Then a reasonable choice of $t/(m+n)$ is 0.5. But $N=m+n$ may be odd or even, while t must be an integer. To eliminate inconsistencies in application, δ can be defined as the $[(N+1)/2]$nd order statistic if N is odd, and any number between the $(N/2)$nd and $[(N+2)/2]$nd order statistics for N even. Then a unique value of u is obtained for any set of N observations, and δ is actually defined to be the median of the combined samples. The probability distribution of U is given in (6.4.3), where $t=N/2$ for N even and

$t = (N-1)/2$ for N odd. The test based on U, the number of observations from the X sample which are less than the combined sample median, is called the *median test*. It is attributed mainly to Brown and Mood (1948, 1951), and Westenberg (1948) and is often referred to as Mood's median test or the joint median test.

The fact that δ cannot be determined before the samples are taken may be disturbing, since it implies that δ should be treated as a random variable. In deriving (6.4.3) we treated δ as a constant, but the same result is obtained for δ defined as the sample median value. Denote the combined sample median by the random variable Z and the cdf's of the X and Y populations by F_X and F_Y, respectively, and assume that N is odd. The median Z can be either an X or a Y random variable, and these possibilities are mutually exclusive. The joint density of U and Z for t observations less than the sample median where $t = (N-1)/2$ is the limit, as $\Delta z \to 0$, of the sum of the probabilities that (1) the X's are divided into three classifications, u less than z, one between z and $z + \Delta z$ and the remainder greater than $z + \Delta z$, and the Y's are divided such that $t - u$ are less than z, and (2) exactly u X's are less than z, and the Y's are divided such that $t - u$ are less than z, one is between z and $z + \Delta z$, and the remainder are greater than $z + \Delta z$. The result then is

$$\begin{aligned} f_{U,Z}(u,z) = {} & \binom{m}{u, 1, m-1-u} [F_X(z)]^u f_X(z) \\ & \times [1 - F_X(z)]^{m-1-u} \binom{n}{t-u} [F_Y(z)]^{t-u} [1 - F_Y(z)]^{n-t+u} \\ & + \binom{m}{u} [F_X(z)]^u [1 - F_X(z)]^{m-u} \binom{n}{t-u, 1, n-t+u-1} \\ & \times [F_Y(z)]^{t-u} f_Y(z) [1 - F_Y(z)]^{n-t+u-1} \end{aligned}$$

The marginal density of U is obtained by integrating the above expression over all z, and if $F_X(z) = F_Y(z)$ for all z, the result is

$$\begin{aligned} f_U(u) &= \left[m \binom{m-1}{u} \binom{n}{t-u} + n \binom{m}{u} \binom{n-1}{t-u} \right] \\ &\quad \times \int_{-\infty}^{\infty} [F(z)]^t [1 - F(z)]^{m+n-t-1} f(z) dz \\ &= \binom{m}{u} \binom{n}{t-u} [(m-u) + (n-t+u)] B(t+1, m+n-t) \\ &= \binom{m}{u} \binom{n}{t-u} \frac{t!(m+n-t)!}{(m+n)!} \end{aligned}$$

which agrees with the expression in (6.4.3).

Because of this result, we might say that before sampling, that is, before the value of δ is determined, the median test statistic is appropriate for the general hypothesis of identical populations, and after the samples are obtained, the hypothesis tested is that δ is the pth quantile value in both populations, where p is a number close to 0.5. The null distributions of the test statistic are the same for both hypotheses.

Even though the foregoing discussion may imply that the median test has some statistical and philosophical limitations in conception, it is well known and accepted within the context of the general two-sample problem. The procedure for two independent samples is to arrange the combined samples in increasing order of magnitude and determine the sample median δ, the observation with rank $(N+1)/2$ if N is odd and any number between the observations with rank $N/2$ and $(N+2)/2$ if N is even. A total of t observations are then less than δ, where $t = (N-1)/2$ or $N/2$ according as N is odd or even. Let U denote the number of X observations less than δ. If the two samples are drawn from identical continuous populations, the probability distribution of U for fixed t is

$$f_U(u) = \frac{\binom{m}{u}\binom{n}{t-u}}{\binom{m+n}{t}} \tag{6.4.4}$$

$$u = \max(0, t-n), \ldots, \min(m, t), \quad t = [N/2]$$

where $[x]$ denotes the largest integer not exceeding x. If the null hypothesis is true, then $P(X < \delta) = P(Y < \delta)$ for all δ, and in particular the two populations have a common median, which is estimated by δ.

Since U/m is an estimate of $P(X<\delta)$, which is approximately one-half under H_0, a test based on the value of U will be most sensitive to differences in location. If U is much larger than $m/2$, most of the X values are less than most of the Y values. This lends credence to the relation $P(X < \delta) > P(Y < \delta)$, that the X's are stochastically smaller than the Y's, so that the median of the X population is smaller than the median of the Y population, or that $\theta > 0$ in the location model. If U is too small relative to $m/2$, the opposite conclusion is implied. The appropriate rejection regions and P values for nominal significance level α then are as follows:

Alternative	**Rejection Region**	***P* Value**
$Y \overset{\text{ST}}{>} X, \theta > 0$ or $M_Y > M_X$	$U \geq c'_\alpha$	$P(U \geq U_0)$
$Y \overset{\text{ST}}{<} X, \theta < 0$ or $M_Y < M_X$	$U \leq c_\alpha$	$P(U \leq U_0)$
$\theta \neq 0$ or $M_X \neq M_Y$	$U \leq c$ or $U \geq c'$	2 (smaller of the above)

where c_α and c'_α are, respectively, the largest and smallest integers such that $P(U \le c_\alpha|H_0) \le \alpha$ and $P(U \ge c'_\alpha|H_0) \le \alpha, c$ and c' are two integers $c < c'$ such that

$$P(U \le c|H_0) + P(U \ge c'|H_0) \le \alpha$$

and U_0 is the observed value of the median test statistic U.

The critical values c and c' can easily be found from (6.4.4) or from tables of the hypergeometric distribution (Lieberman and Owen, 1961 or MINITAB or EXCEL) or tables of the binomial coefficients. If N is even, we choose $c'_\alpha = m - c_\alpha$. Since the distribution in (6.4.4) is not symmetric for $m \ne n$ if N is odd, the choice of an optimum rejection region for a two-sided test is not clear for this case. It could be chosen such that α is divided equally between the two tails or that the range of U is symmetric, or neither.

If m and n are so large that calculation or use of tables to find critical values is not feasible, a normal approximation to the hypergeometric distribution of U can be used. The mean and variance of U are easily found from Table 1.2.1 to be

$$E(U|t) = \frac{mt}{N} \quad \text{var}(U|t) = \frac{mnt(N-t)}{N^2(N-1)} \tag{6.4.5}$$

If $m, n \to \infty$ in such a way that m/n remains constant, this hypergeometric distribution approaches the binomial distribution for t trials with parameter m/N, which in turn approaches the normal distribution. For N large, the variance of U in (6.4.5) is approximately

$$\text{var}(U|t) = \frac{mnt(N-t)}{N^3}$$

and thus the asymptotic distribution of

$$Z = \frac{U - mt/N}{[mnt(N-t)/N^3]^{1/2}} \tag{6.4.6}$$

is approximately standard normal. A continuity correction of 0.5 can be used to improve the approximation. For example, when the alternative is $\theta < 0$ (or $M_Y < M_X$), the approximate P value with a continuity correction is given by

$$\Phi\left(\frac{U_0 + 0.5 - mt/N}{\sqrt{mnt(N-t)/N^3}}\right) \tag{6.4.7}$$

It is interesting to note that a test based on the statistic Z in (6.4.6) is equivalent to the usual normal-theory test for the difference between two independent proportions discussed here in Chapter 14.

This can be shown by algebraic manipulation of (6.4.6) with $t = u + v$ as follows

$$
\begin{aligned}
z &= \frac{Nu - mt}{\sqrt{mnt(N-t)/N}} = \frac{nu - m(t-u)}{\sqrt{mnN(t/N)(1-t/N)}} \\
&= \frac{u/m - v/n}{\sqrt{[(u+v)/N][1-(u+v)/N]N/mn}} \\
&= \frac{u/m - v/n}{\sqrt{\hat{p}(1-\hat{p})[(1/m)+(1/n)]}}
\end{aligned}
$$

If a success is defined as an observation being less than δ, u/m and v/n are the observed sample proportions of successes, and $\hat{p} = (u+v)/N$ is the best sample estimate of the common proportion. This then is the same approximate test statistic described at the beginning of this section for large samples, except that here $u + v = t$, a constant which is fixed by the choice of δ as the sample median.

The presence of ties either within or across samples presents no problem for the median test except in two particular cases. If N is odd and more than one observation is equal to the sample median, or if N is even and the $(N/2)$nd and $[(N+2)/2]$nd-order statistics are equal, t cannot be defined as before unless the ties are broken. The conservative approach is recommended, where the ties are broken in all possible ways and the value of u chosen for decision is the one which is least likely to lead to rejection of H_0.

6.4.1 Applications

Example 6.4.1

The production manager of a small company that manufactures a certain electronic component believes that playing some contemporary music in the production area will help reduce the number of nonconforming items produced. A group of workers with similar background (training, experience, etc.) are selected and five of them are assigned, at random, to work in the area while music is played. Then from the remaining group, four workers are randomly assigned to work in the usual way without music. The numbers of nonconforming items produced by the workers over a particular period of time are given below. Test to see if the median number of nonconforming items produced while music is played is less than that when no music is played.

Sample 1: Without Music	Sample 2: With Music
3, 4, 9, 10	1, 2, 5, 7, 8

SOLUTION

Let samples 1 and 2 denote the X and Y sample, respectively. Assume the shift model and test the null hypothesis $M_X = M_Y$ against the alternative $M_Y < M_X$. Then the P value for the median test is in the left tail. Since $N = 9$ is odd, $t = (9-1)/2 = 4$. The combined sample median is equal to 5 and thus $U = 2$. Using (6.4.4), the exact P value for the median test is

$$P(U \le 2|H_0) = \frac{\binom{4}{0}\binom{5}{4} + \binom{4}{1}\binom{5}{3} + \binom{4}{2}\binom{5}{2}}{\binom{9}{4}}$$

$$= 105/126 = 0.8333$$

There is not enough evidence in favor of the alternative H_1 and we do not reject H_0. The reader can verify using (6.4.7) that the normal approximation to the P value is $\Phi(0.975) = 0.8352$, leading to the same conclusion.

The MINITAB solution to Example 6.4.1 for the median test is shown below. For each group the MINITAB output gives the median and the interquartile range. Note that MINITAB does not calculate the P value for the exact median test but provides the chi-square approximation with df = 1. The chi-square test statistic is the square of the Z statistic in (6.4.6), based on the normal approximation without a continuity correction. Calculations yield $Z^2 = 0.09$ and from Table B, the critical value at $\alpha = 0.05$ is 3.84. Thus, the approximate test also fails to reject the null hypothesis. Using the chi-square approximation for these small sample sizes might not be advisable however. Besides, the test based on the chi-square approximation is always two-sided. Using MINITAB, the right-tail probability corresponding to the observed value of 0.09 under the chi-square distribution with df = 1 is 0.7642 and this is in fact the P value shown in the printout. MINITAB also provides a 95% confidence interval for each group median (based on the sign test and interpolation) and for the difference in the medians. The two individual confidence intervals overlap, and the interval for the difference of medians includes zero, suggesting that the corresponding population medians are the same. The MINITAB output does not show the result of a median test but it does show a confidence interval for the difference between the medians based on a median test; this will be discussed next. For these data, the confidence interval is $(-4.26, 8.26)$ which includes zero and this suggests, as before, that the population medians are not different. We note that Example 6.4.1 as stated calls for a one-sided alternative. Both of the MINITAB solutions, that is, the test based on the chi-square statistic and the confidence-interval estimate, are two-sided. A one-sided test cannot be performed using a chi-square statistic.

```
****************************************
MINITAB SOLUTION TO EXAMPLE 6.4.1
****************************************
Mood Median Test: C2 versus C1

Mood median test for C2
```

```
Chi-square = 0.09   df = 1   P = 0.764

                                          Individual 95.0% CIs
C1   N<=   N>   Median   Q3 − Q1   ---+-------+-------+-------+
1     2     2     6.50     6.50          (---------+---------)
2     3     2     5.00     6.00   (---------+-----------)
                                  -------+-------+-------+-------+
                                      2.5     5.0     7.5    10.0

Overall median = 5.00
* NOTE * Levels with < 6 observations have confidence < 95.0%

A 95.0% CI for median (1) − median (2): (−4.26, 8.26).
```

The output from STATXACT is shown next. It provides the approximate test based on the chi-square statistic as well as an exact *P* value, which is in the upper tail and does not agree with our hand calculations.

```
****************************************
STATXACT SOLUTION TO EXAMPLE 6.4.1
****************************************

Median test

Data file:

Row variable: Var2

Column variable: Var1

Summary of the test statistics:

Number of groups   2
Number of observ.  9
Overall median     5

Inference:

                                          P value
 Type          Statistic    DF   Tail    2-Sided    Point Prob.
Asymptotic       0.09       1     GE     0.7642
Exact            0.09             GE     1             0.4762
```

Finally, the SAS output is shown. SAS determines S, the number of observations from the sample with fewer observations that are larger than the combined sample median. In our case the X sample has fewer observations and $S = 2$. According to SAS documentation, a one-sided P value is calculated as $P_1 = P$(Test statistic $\geq S|H_0$) if $S >$ Mean, whereas if $S \leq$ Mean, the one-sided P value is calculated as $P_1 = P$(Test statistic $\leq S|H_0$). The mean of the median test statistic under H_0 is mt/N which equals $4(4)/9 = 1.78$ for our example. Thus,

SAS calculates the exact P value in the upper tail as $P(S \geq 2|H_0) = 81/126 = 0.6429$. This equals $1 - P(U \leq 1|H_0)$ and thus does not agree with our hand calculations. However, on the basis of S we reach the same conclusion of not rejecting H_0, made earlier on the basis of U. For the normal approximation to the P value, PROC NPAR1WAY calculates the Z statistic from $Z = (S - mt/N)/\sqrt{mnt(N-t)/N^2(N-1)}$ and incorporates a continuity correction unless one specifies otherwise. As with the exact P value, the SAS P value under the normal approximation also does not agree with our hand calculation based on U.

```
************************************************
SAS PROGRAM AND SOLUTION TO EXAMPLE 6.4.1
************************************************
```

Program:

```
data example;
      input sample number @@;
      datalines;
   1 3   1 4   1 9   1 10   2 1   2 2   2 5   2 7   2 8
   proc npar1way median data = example;
      class sample;
      var number;
      exact;
   run;
```

Output:

The NPAR1WAY procedure

Median Scores (Number of Points above Median) for Variable number Classified by Variable Sample

Sample	N	Sum of Scores	Expected Under H_0	Std Dev Under H_0	Mean Score
1	4	2.0	1.777778	0.785674	0.50
2	5	2.0	2.222222	0.785674	0.40

Median two-sample test

Statistic (S)	2.0000
Normal approximation	
Z	0.2828
One-sided Pr $> Z$	0.3886
Two-sided Pr $> \|Z\|$	0.7773

```
Exact test
One-sided Pr >= S            0.6429
Two-sided Pr >= |S - Mean|   1.0000

     Median one-way analysis

Chi-square        0.0800
DF                     1
Pr > chi-square   0.7773
```

6.4.2 Confidence-Interval Procedure

Assuming the shift model, the median-test procedure can be adapted to yield a confidence-interval estimate for the shift parameter in the following manner. Suppose that the two populations are identical in every way except for their medians. Denote these unknown parameters by M_X and M_Y, respectively, and the difference $M_Y - M_X$ by θ. In the shift model $F_Y(x) = F_X(x - \theta)$, the parameter θ represents the difference $F_Y^{-1}(p) - F_X^{-1}(p)$ between any two quantiles (of order p) of the two populations. In the present case, however, we assume that $\theta = M_Y - M_X$, the difference between the two medians ($p = 0.5$). From the original samples, if θ were known we could form the derived random variables

$$X_1, X_2, \ldots, X_m \quad \text{and} \quad Y_1 - \theta, Y_2 - \theta, \ldots, Y_n - \theta$$

and these would constitute samples from identical populations or, equivalently, a single sample of size $N = m + n$ from the common population. Then according to the median-test criterion with significance level α, the null hypothesis of identical distributions would be accepted for these derived samples if U, the number of X observations less than the median of the combined sample of derived observations, lies in the interval $c + 1 \le U \le c' - 1$. Recall that the rejection region against the two-sided alternative $\theta \neq 0$ is $U \le c$ or $U \ge c'$. The integers c and c' are chosen such that

$$\sum_{u=0}^{c} \frac{\binom{m}{u}\binom{n}{t-u}}{\binom{m+n}{t}} + \sum_{u=c'}^{t} \frac{\binom{m}{u}\binom{n}{t-u}}{\binom{m+n}{t}} \le \alpha \tag{6.4.8}$$

where $t = N/2$ or $(N-1)/2$ according as N is even or odd. Since H_0 is accepted for all U values in the interval $c + 1 \le U \le c' - 1$, and this acceptance region has probability $1 - \alpha$ under H_0, a $100(1 - \alpha)\%$ confidence-interval

estimate for θ consists of all values of θ for which the derived sample observations yield values of U which lie in the acceptance region. This process of obtaining a confidence interval for a parameter from the acceptance region (of a test of hypothesis) is called inverting the acceptance region (of the test), and the confidence interval is referred to as a *test-based confidence interval*.

To find the confidence interval, the range of θ corresponding to the acceptance region $c+1 \le U \le c'-1$, we first order the two derived samples separately from smallest to largest as

$$X_{(1)}, X_{(2)}, \ldots, X_{(m)} \quad \text{and} \quad Y_{(1)} - \theta, Y_{(2)} - \theta, \ldots, Y_{(n)} - \theta$$

The t smallest observations of the $N = m + n$ total are made up of exactly iX and $t - iY$ variables if each observation of the set

$$X_{(1)}, \ldots, X_{(i)}, Y_{(1)} - \theta, \ldots, Y_{(t-i)} - \theta$$

is less than each observation of the set

$$X_{(i+1)}, \ldots, X_{(m)}, Y_{(t-i+1)} - \theta, \ldots, Y_{(n)} - \theta$$

The value of i is at least $c+1$ if and only if for $i = c+1$, the largest X in the first set is less than the smallest Y in the second set, that is, $X_{(c+1)} < Y_{(t-c)} - \theta$. Arguing similarly, $X_{(c')} > Y_{(t-c'+1)} - \theta$ can be seen to be a necessary and sufficient condition for having at most $c'-1$ X observations among the t smallest of the total N (in this case the largest Y in the first set must be smaller than the smallest X in the second set). Therefore, the acceptance region for the median test corresponding to the null hypothesis of no difference between the two distributions (with respect to location) at significance level α can be equivalently written as

$$X_{(c+1)} < Y_{(t-c)} - \theta \quad \text{and} \quad X_{(c')} > Y_{(t-c'+1)} - \theta$$

or as

$$Y_{(t-c)} - X_{(c+1)} > \theta \quad \text{and} \quad Y_{(t-c'+1)} - X_{(c')} < \theta$$

The desired confidence interval $(Y_{(t-c'+1)} - X_{(c')}, Y_{(t-c)} - X_{(c+1)})$ follows from the last two inequalities. Now, using (6.4.8),

$$\begin{aligned} 1 - \alpha &= P(c+1 \le U \le c'-1 | H_0) \\ &= P(c+1 \le U \le c'-1 | \theta = 0) \\ &= P(Y_{(t-c'+1)} - X_{(c')} < \theta < Y_{(t-c)} - X_{(c+1)} | \theta = 0) \end{aligned}$$

Since the last equality is true for all values of θ, we can make the statement

$$P(Y_{(t-c'+1)} - X_{(c')} < \theta < Y_{(t-c)} - X_{(c+1)}) = 1 - \alpha$$

where c and c' are found from (6.4.8). Thus, the endpoints of the confidence-interval estimate for θ corresponding to Mood's median test are found simply from some order statistics of the respective random samples.

Example 6.4.2

We calculate the 95% confidence interval for the median difference for the data in Example 6.4.1. In order to find the constants c and c', we need to calculate the null distribution of U, using (6.4.3) for $m = 4, n = 5, t = 4$. The results are shown in Table 6.4.1. If we take $c = 0$ and $c' = 4$, then (6.4.8) equals 0.04762 so that the confidence interval for $\theta = M_Y - M_X$ is $(Y_{(1)} - X_{(4)}, Y_{(4)} - X_{(1)})$ with exact level 0.95238. Numerically, the interval is $(-9, 4)$. Also, the 95.238% confidence interval for $\theta = M_X - M_Y$ is $(-4, 9)$. Note that the MINITAB output given before states ''A 95.0% CI for median(1) – median(2): (−4.26, 8.26).'' This is based on the median test but c and c' are calculated using the normal approximation.

The median test is a member of a more general class of nonparametric two-sample tests, called *precedence tests*. Chakraborti and van der Laan (1996) provide an overview of the literature on these tests. A precedence test is based on a statistic W_r which denotes the number of Y observations that precede the rth-order statistic $X_{(r)}$ from the X sample (alternatively, one could use the number of X's that precede the sth-order statistic $Y_{(s)}$ from the Y sample). It can be seen, for example, that $W_r < w$ if and only if $X_{(r)} < Y_{(w)}$ so that a precedence test based on W_r can be interpreted in terms of two order statistics, one from each sample. To carry out the test, we first choose r, and then determine w such that the size of the test is α. It can be shown that the null distribution of W_r is distribution-free (see, for example, Problem 2.28), so that the precedence test is distribution-free. The median test a special case of a precedence test since as seen in the arguments for the confidence-interval procedure (see also, for example, Pratt, 1964), we have $U \leq u - 1$ if and only if $X_{(u)} < Y_{(t-u+1)}$. Several precedence tests have been proposed in the literature and we discuss some of them in this chapter.

6.4.3 Power of the Median Test

The power of the median test can be obtained as a special case of the power of a precedence test and we sketch the arguments for the more general case.

TABLE 6.4.1

Null Distribution of U for $m = 4$, $n = 5$, $t = 4$

u	0	1	2	3	4
$P(U = u)$	0.039683	0.31746	0.47619	0.15873	0.007937
	5/126	40/126	60/126	20/126	1/126

The distribution of the precedence statistic W_r for $i = 0, 1, \ldots, n$ under the alternative hypothesis is

$$P(W_r = i) = \frac{\binom{n}{i}}{B(r, m-r+1)} \times \int_0^1 \left\{F_Y\left[F_X^{-1}(u)\right]\right\}^i \left\{1 - F_Y\left[F_X^{-1}(u)\right]\right\}^{n-i} u^{r-1}(1-u)^{m-r} du \quad (6.4.9)$$

Thus, for a one-sided precedence test with rejection region $W_r < w_\alpha$, the power of the test is

$$\mathrm{Pw}(F_X, F_Y, m, n, \alpha) = \sum_{i=0}^{w_\alpha - 1} P(W_r = i) \quad (6.4.10)$$

where w_α is obtained from the size of the test; in other words, w_α is the largest integer such that

$$\sum_{s=0}^{w_\alpha - 1} P(W_r = s | H_0) \leq \alpha$$

To obtain the power of the median test, we just need to substitute suitable values in (6.4.10).

As an alternative development of power, note that under the assumption of the location model $F_Y(x) = F_X(x - \theta)$ the null hypothesis is that $\theta = 0$. From the confidence-interval procedure, we know that a necessary and sufficient condition for the median test to lead to accepting this hypothesis is that the number zero be included in the random interval

$$[(Y_{(t-c'+1)} - X_{(c')}), (Y_{(t-c)} - X_{(c+1)})]$$

Thus, the power of the median test in the location case is the probability that this interval does not include zero when $\theta \neq 0$, that is,

$$\mathrm{Pw}(\theta) = P(Y_{(t-c'+1)} - X_{(c')} > 0 \quad \text{or} \quad Y_{(t-c)} - X_{(c+1)} < 0 \text{ when } \theta \neq 0)$$

These two events, call them A and B, are mutually exclusive as we now show. For any $c' > c$, it is always true that $X_{(c')} \geq X_{(c+1)}$ and $Y_{(t-c'+1)} = Y_{(t-[c'-1])} \leq Y_{(t-c)}$. Thus, if A occurs, that is, $Y_{(t-c'+1)} > X_{(c')}$, we must also have $Y_{(t-c)} \geq Y_{(t-c'+1)} > X_{(c')} \geq X_{(c+1)}$ which makes $Y_{(t-c)} > X_{(c+1)}$,

a contradiction in B. As a result, the power function can be expressed as the sum of two probabilities involving order statistics:

$$\text{Pw}(\theta) = P(Y_{(t-c'+1)} > X_{(c')}) + P(Y_{(t-c)} < X_{(c+1)}).$$

Since the random variables X and Y are independent, the joint distribution of $X_{(r)}$ and $Y_{(s)}$ is the product of their marginal distributions, which can be easily found using the methods of Chapter 2 for completely specified populations F_X and F_Y or, equivalently, F_X and θ since $F_Y(x) = F_X(x - \theta)$. In order to calculate the power function then, we need only evaluate two double integrals of the following type:

$$P(Y_{(s)} < X_{(r)}) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{Y_{(s)}}(y) f_{X_{(r)}}(x) dy dx$$

The power function for a one-sided test is simply one integral of this type. For large sample sizes, since the marginal distribution of any order statistic approaches the normal distribution and the order statistics $X_{(r)}$ and $Y_{(s)}$ are independent here, the distribution of their difference $Y_{(s)} - X_{(r)}$ approaches the normal distribution with mean and variance

$$E(Y_{(s)}) - E(X_{(r)}) \quad \text{and} \quad \text{var}(Y_{(s)}) + \text{var}(X_{(r)})$$

Given the specified distribution function and the results in Chapter 2, we can approximate these quantities by

$$E(X_{(r)}) = F_X^{-1}\left(\frac{r}{m+1}\right) \quad E(Y_{(s)}) = F_X^{-1}\left(\frac{s}{n+1}\right)$$

$$\text{var}(X_{(r)}) = \frac{r(m-r+1)}{(m+1)^2(m+2)} \left\{ f_X\left[F_X^{-1}\left(\frac{r}{m+1}\right)\right]\right\}^{-2}$$

$$\text{var}(Y_{(s)}) = \frac{s(n-s+1)}{(n+1)^2(n+2)} \left\{ f_Y\left[F_Y^{-1}\left(\frac{s}{n+1}\right)\right]\right\}^{-2}$$

and an approximation to the power function can be found using normal probability tables.

It is clear that computing the exact or even the asymptotic power of the median test is involved. An easier approach might be to use computer simulations, as we did for the sign and the signed rank test in Chapter 5. We leave the details to the reader.

The asymptotic efficiency of the median test relative to Student's t test for normal populations is $2/\pi = 0.637$ (see Chapter 13). As a test for location, this is relatively poor performance. The Mann–Whitney test, discussed in Section 6.6, has greater efficiency for normal populations.

6.5 The Control Median Test

The median test, based on the number of X observations that precede the median of the combined samples, is a special case of a precedence test. A simple yet interesting alternative test is a *second precedence test*, based on the number of X (or Y) observations that precede the median of the Y (or X) sample. This is known as the *control median test* and is generally attributed to Mathisen (1943). The properties and various refinements of the test have been studied by Gart (1963), Gastwirth (1968), and Hettmansperger (1973), among others.

Without any loss of generality, suppose the Y sample is the control sample. The control median test is based on V, the number of X observations that precede the median of the Y observations. For simplicity let $n = 2r + 1$, so that the $(r + 1)$th-order statistic $Y_{(r+1)}$ is the median of the Y sample. Now $Y_{(r+1)}$ defines two non-overlapping blocks $(-\infty, Y_{(r+1)}]$ and $(Y_{(r+1)}, \infty)$ in the sample, and the control median test is based on V, the number of X observations in the first block. It may be noted that V is equal to $mS_m(Y_{(r+1)}) = P_{(r+1)}$, called the *placement* of $Y_{(r+1)}$, the median of the Y sample, among the X observations.

The control median test can be used to test the null hypothesis $H_0 : F_Y(x) = F_X(x)$ for all x against the general one-sided alternative that for example, the Y's are stochastically larger than the X's. In this case, the number of X's preceding $Y_{(r+1)}$ should be large and thus large values of V provide evidence against the null hypothesis. If we assume the shift model $F_Y(x) = F_X(x - \theta)$ then the problem reduces to testing the null hypothesis $H_0 : \theta = 0$ against the one-sided alternative $H_1 : \theta > 0$ and we reject for large values of V. In a similar manner it can be seen that for the one-sided alternative that the Y's are stochastically smaller than the X's or $H_1 : \theta < 0$, under the shift model we reject for small values of V.

Problem 2.28(c) (with $n = 2r + 1$ and $i = r + 1$) gives the null distribution of the test statistic V as:

$$P[V = j|H_0] = \frac{\binom{m+r-j}{m-j}\binom{j+r}{j}}{\binom{m+2r+1}{m}} \tag{6.5.1}$$

for $j = 0, 1, \ldots, m$. This can be easily evaluated to find the exact P value corresponding to an observed value V_0 or to calculate a critical value for a given level of significance α. Further, from Problem 2.28(d), the null mean and variance of V are

$$E(V) = \frac{m(r+1)}{(n+1)} = \frac{m}{2}$$

and

$$\begin{aligned}\text{var}(V) &= \frac{m(m+n+1)(r+1)(n-r)}{(n+1)^2(n+2)} \\ &= \frac{m(m+2r+2)}{4(2r+3)} = \frac{m(m+n+1)}{4(n+2)}\end{aligned}$$

In general, V/m is an estimator of $F_X(M_Y) = q$, say, and in fact is a consistent estimator. Now, $q = 0.5$ under H_0, $q < 0.5$ if and only if $M_Y < M_X$, and $q > 0.5$ if and only if $M_Y > M_X$. Hence, like the median test, a test based on V is especially sensitive to differences in the medians.

Gastwirth (1968) showed that when m, $n \to \infty$ such that $m/n \to \lambda$, and some regularity conditions are satisfied, $\sqrt{m}(V/m - q)$ is asymptotically normally distributed with mean zero and variance

$$q(1-q) + \lambda \frac{f_X^2(M_Y)}{4f_Y^2(M_Y)}$$

Under H_0 we have $q = 0.5, f_X = f_Y$, and $M_X = M_Y$, so that the asymptotic null mean and variance of V are $m/2$ and $m(m+n)/4n$, respectively. Thus, under H_0, the asymptotic distribution of

$$Z = \frac{V - m/2}{\sqrt{m(m+n)/4n}} = \frac{\sqrt{n}(2V-m)}{\sqrt{m(m+n)}}$$

is approximately standard normal.

Suppose we are interested in testing only the equality of the two medians (or some other quantiles) and not the entire distributions. In the context of location-scale models, the null hypothesis may concern only the equality of location parameters, without assuming that the scale parameters are equal. By analogy with a similar problem in the context of normal distributions, this is a nonparametric Behrens–Fisher problem. Note that under the current null hypothesis we still have $q = 0.5$ but the ratio $f_X(M)/f_Y(M)$, where M is the common value of the medians under the null hypothesis, does not necessarily equal one. This implies that in order to use the control median test for this problem we need to estimate this ratio of the two densities at M. Once a suitable estimate is found, the asymptotic normality of V can be used to construct an approximate test of significance. Several authors have studied this problem, including Pothoff (1963), Hettmansperger (1973), Hettmansperger and Malin (1975), Schlittgen (1979), Smit (1980), and Fligner and Rust (1982).

6.5.1 Curtailed Sampling

The control median test, or more generally any precedence test, is particularly useful in life-testing experiments, where observations are naturally time

ordered and collecting data is expensive. Precedence tests allow a decision to be made about the null hypothesis as soon as a preselected ordered observation becomes available. Thus, the experiment can be terminated (or the sampling can be curtailed) before all of the data have been collected, and precedence tests have the potential of saving time and resources. Note that the decision made on the basis of the curtailed sample is always the same as it would have been if all observations had been available.

As an illustration consider testing $H_0 : q = 0.5$ against the one-sided alternative $H_1 : q < 0.5$. Using the normal approximation, the control median test would reject H_0 in favor of H_1 if $V \leq d$, where

$$d = \frac{m}{2} - z_\alpha \left[\frac{m(m+n)}{4n}\right]^{1/2} \tag{6.5.2}$$

or equivalently if the median of the Y sample of size $2r+1$ satisfies

$$Y_{(r+1)} \leq X_{(d)}$$

where d is the solution from (6.5.2) after rounding down.

This restatement of the rejection region in terms of the X- and Y-order statistics clearly shows that a decision can be reached based on the control median test as soon as the median of the Y sample or the dth order statistic of the X sample is observed, whichever comes first. The index d is fixed by the size of the test and can be obtained exactly from the null distribution of V or the normal approximation as given in (6.5.2). The null hypothesis is rejected in favor of the alternative that $M_Y < M_X$ $(q < 0.5)$ if the median of the Y sample is observed before the dth-order statistic of the X sample; otherwise the null hypothesis is not rejected.

Gastwirth (1968) showed that in large samples the control median test always provides a decision earlier than the median test for both the one- and two-sided alternatives. For related discussions and other applications, see Pratt and Gibbons (1981) and Young (1973).

6.5.2 Power of the Control Median Test

Since the median test and the control median test are both precedence tests, the power of the control median test can be obtained in a manner similar to that for the median test. If the alternative is one-sided, say $q < 0.5$, the control median test rejects the null hypothesis $q = 0.5$ at significance level α if $V \leq d_\alpha$. Hence, the power of this control median test is $P(V \leq d_\alpha | q < 0.5)$. However, the event $V \leq d_\alpha$ is equivalent to $Y_{(r+1)} \leq X_{(d^*)}$, where $d^* = [d_\alpha]$ and therefore the power of the test is simply

$$\text{Pw}(\theta) = P[Y_{(r+1)} \leq X_{(d^*)} | q < 0.5]$$

where $q = F_X(M_Y)$ and d_α satisfies $P(V \leq d_\alpha | H_0) \leq \alpha$. Note that the power of the control median test depends on this composite function $q = F_X(M_Y)$, in addition to α, m and n, and q is not necessarily equal to the difference of medians even under the shift model. The quantity q can be thought of as a parameter that captures possible differences between the Y and the X distributions.

6.5.3 Discussion

The ARE of the control median test relative to the median test is equal to one regardless of the continuous parent distributions. In this sense the test is as efficient as the median test in large samples. In fact, the efficacy of the control median test is a symmetric function of the two sample sizes, which implies that ''the designation of the numbering of the samples is immaterial as far as asymptotic efficiency is concerned'' (Gart, 1963).

Because the ARE between the control median test and the median test is equal to one, this criterion suggests no preference. If some other measure of efficiency is used, interesting distinctions can be made between the two tests. For these and other related results see, for example, Killeen et al. (1972) and the references therein. As noted earlier, the control median test and the median tests are special cases of precedence tests. An overview of precedence tests for various problems can be found in Chakraborti and van der Laan (1996, 1997). Chakraborti (1990) and Chakraborti and Desu (1990) study generalizations of the control median test based on some quantile other than the median, called control quantile tests. Some of these results are discussed later in Chapter 10. Klotz (2001) provides the exact null distribution of the control quantile test in closed form in general and in the presence of ties. He finds the test more efficient than the Mann–Whitney test (covered in the next section) for the Cauchy distribution and suggests it is preferable for heavy tailed distributions.

6.5.4 Applications

The control median test is based on V, the number of X values that precede the median of the Y's. Writing $q = F_X(M_Y)$, the appropriate rejection regions for a nominal significance level α are shown in the following table along with expressions for P values where d_α and d'_α are, respectively, the largest and smallest integers such that $P(V \leq d_\alpha | H_0) \leq \alpha, P(V \geq d'_\alpha | H_0) \leq \alpha$ and d and d' $(d < d')$ are two positive integers such that

$$P(V \leq d | H_0) + P(V \geq d' | H_0) \leq \alpha$$

The exact critical values or P values can be easily found directly using (6.5.1) or from tables of the hypergeometric distribution (see Problem 6.10). In view

of the simple form of the null distribution, it might be easier to calculate a P value corresponding to an observed value of V, say V_0. Different from the median test, there is no difficulty here in assigning probability in the tails with two-tailed tests since the distribution of V is symmetric under H_0.

Alternative	**Rejection Region**	***P* Value**
$Y \overset{ST}{>} X$, $q > 0.5$ $\theta > 0$ or $M_Y > M_X$	$V \geq d'_\alpha$	$P(V \geq V_0 \mid H_0) = \sum_{j=V_0}^{m} P(V = j \mid H_0)$
$Y \overset{ST}{<} X$, $q < 0.5$ $\theta < 0$ or $M_Y < M_X$	$V \leq d_\alpha$	$P(V \leq V_0 \mid H_0) = \sum_{j=0}^{V_0} P(V = j \mid H_0)$
$\theta \neq 0$ or $M_Y \neq M_X$ $q \neq 0.5$	$V \leq d_\alpha$ or $V \geq d'_\alpha$	2 (smaller of above)

In practice, the asymptotic distribution is useful for finding the critical values or approximating the P value. For example, for the alternative $q < 0.5$, an approximation to the P value of the test with a continuity correction is

$$\Phi\left[\frac{\sqrt{n}(2v - m + 1)}{\sqrt{m(m+n)}}\right] \tag{6.5.3}$$

Example 6.5.1

We illustrate the control median test using the data in Example 6.4.1.

Solution

As before, let samples 1 and 2 denote the X and the Y sample, respectively and assume that the null hypothesis to be tested is $M_X = M_Y$ ($q = 0.5$) against the alternative $M_Y < M_X$ ($q < 0.5$) under the shift model. Then the P value for the control median test is also in the left tail. The median of the Y sample is 5 and thus $V = 2$. Using (6.5.1) with $m = 4$, $n = 5$, and $r = 2$, the exact P value for the control median test is

$$P(V \leq 2 \mid H_0) = \frac{\binom{6}{4}\binom{2}{0} + \binom{5}{3}\binom{3}{1} + \binom{4}{2}\binom{4}{2}}{\binom{9}{4}}$$
$$= 81/126 = 0.6429$$

Hence, there is not enough evidence against the null hypothesis in favor of the alternative. Also, from (6.5.3) the normal approximation to the P value is $\Phi(0.37) = 0.6443$ and the approximate test leads to the same decision as the exact test.

6.6 The Mann–Whitney *U* Test and Confidence Interval

Like the Wald–Wolfowitz runs test in Section 6.2, the Mann–Whitney U test (Mann and Whitney, 1947) is based on the idea that the particular pattern exhibited by the combined arrangement of the X and Y random variables in increasing order of magnitude provides information about the relationship between their populations. Instead of measuring the tendency to cluster by the total number of runs, the Mann–Whitney criterion is based on the magnitudes of, say, the Y's in relation to the X's, that is, the position of the Y's in the combined ordered sequence. A sample pattern of arrangement where most of the Y's are greater than most of the X's, or vice versa, would be evidence against a random mixing and thus tend to discredit the null hypothesis of identical distributions.

The *Mann–Whitney U test statistic* is defined as the number of times a Y precedes an X in the combined ordered arrangement of the two independent random samples

$$X_1, X_2, \ldots, X_m \quad \text{and} \quad Y_1, Y_2, \ldots, Y_n$$

into a single sequence of $m + n = N$ variables increasing in magnitude. We assume that the two samples are drawn from continuous distributions, so that the possibility $X_i = Y_j$ for some i and j need not be considered. If the mn indicator random variables are defined as

$$D_{ij} = \begin{cases} 1 & \text{if } Y_j < X_i \\ 0 & \text{if } Y_j > X_i \end{cases} \quad \text{for } i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \tag{6.6.1}$$

the Mann–Whitney U statistic can be written in symbols is

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij} \tag{6.6.2}$$

The logical rejection region for the one-sided alternative that the Y's are stochastically larger than the X's,

$$H_1 : F_Y(x) \leq F_X(x), \text{ with strict inequality for some } x$$

would clearly be small values of U. In order to show that U/mn is a consistent test criterion, we define

$$\begin{aligned} p = P(Y < X) &= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_Y(y) f_X(x) dy dx \\ &= \int_{-\infty}^{\infty} F_Y(x) dF_X(x) \end{aligned} \tag{6.6.3}$$

The hypothesis testing problem can now be redefined in terms of the parameter p. If $H_0 : F_Y(x) = F_X(x)$ for all x is true, then

$$p = \int_{-\infty}^{\infty} F_X(x) dF_X(x) = 0.5 \tag{6.6.4}$$

If, for example, the alternative hypothesis is $H_1 : F_Y(x) \leq F_X(x)$ that is $Y \overset{ST}{>} X$, then $H_1 : p \leq 0.5$. Thus, the null hypothesis of identical distributions can be parameterized as $H_0 : p = 0.5$ with the alternative $H_1 : p < 0.5$.

The mn random variables defined in (6.6.1) are Bernoulli variables, with moments

$$E(D_{ij}) = E(D_{ij}^2) = p \quad \text{var}(D_{ij}) = p(1-p) \tag{6.6.5}$$

For the joint moments we note that these random variables are not independent whenever the X subscripts or the Y subscripts are common, so that

$$\begin{aligned} &\text{cov}(D_{ij}, D_{hk}) = 0 \quad \text{for } i \neq h \text{ and } j \neq k \\ &\underset{j \neq k}{\text{cov}}(D_{ij}, D_{ik}) = p_1 - p^2 \qquad \underset{i \neq h}{\text{cov}}(D_{ij}, D_{hj}) = p_2 - p^2 \end{aligned} \tag{6.6.6}$$

where the additional parameters introduced are

$$\begin{aligned} p_1 &= P(Y_j < X_i \cap Y_k < X_i) \\ &= P(Y_j \text{ and } Y_k < X_i) \\ &= \int_{-\infty}^{\infty} [F_Y(x)]^2 dF_X(x) \end{aligned} \tag{6.6.7}$$

and

$$\begin{aligned} p_2 &= P(X_i > Y_j \cap X_h > Y_j) \\ &= P(X_i \text{ and } X_h > Y_j) \\ &= \int_{-\infty}^{\infty} [1 - F_X(y)]^2 dF_Y(y) \end{aligned} \tag{6.6.8}$$

Since U is defined in (6.6.2) as a linear combination of these mn random variables, the mean and variance of U are

$$E(U) = \sum_{i=1}^{m} \sum_{j=1}^{n} E(D_{ij}) = mnp \tag{6.6.9}$$

$$\begin{aligned}\operatorname{var}(U) = &\sum_{i=1}^{m}\sum_{j=1}^{n}\operatorname{var}(D_{ij}) + \sum_{i=1}^{m}\sum_{1\le j\ne k\le n}\sum\operatorname{cov}(D_{ij}, D_{ik}) \\ &+ \sum_{j=1}^{n}\sum_{1\le i\ne h\le m}\sum\operatorname{cov}(D_{ij}, D_{hj}) \\ &+ \sum_{1\le i\ne h\le m}\sum\sum_{1\le j\ne k\le n}\sum\operatorname{cov}(D_{ij}, D_{hk}) \end{aligned} \tag{6.6.10}$$

Now substituting (6.6.5) and (6.6.6) in (6.6.10), this variance is

$$\begin{aligned}\operatorname{var}(U) &= mnp(1-p) + mn(n-1)(p_1 - p^2) + nm(m-1)(p_2 - p^2) \\ &= mn[p - p^2(N-1) + (n-1)p_1 + (m-1)p_2] \end{aligned} \tag{6.6.11}$$

Since $E(U/mn) = p$ and $\operatorname{var}(U/mn) \to 0$ as $m, n \to \infty$, U/mn is a consistent estimator of p. Based on the method described in Chapter 1, the Mann–Whitney test can be shown to be consistent in the following cases:

Alternative		**Rejection Region**
$p < 0.5$	$F_Y(x) \le F_X(x)$	$U - mn/2 < k_1$
$p > 0.5$	$F_Y(x) \ge F_X(x)$	$U - mn/2 > k_2$
$p \ne 0.5$	$F_Y(x) \ne F_X(x)$	$\lvert U - mn/2\rvert > k_3$

(6.6.12)

In order to determine the size α critical regions of the Mann–Whitney test, we must now find the null probability distribution of U. Under H_0, each of the $\binom{m+n}{m}$ arrangements of the random variables into a combined sequence occurs with equal probability, so that

$$f_U(u) = P(U = u) = \frac{r_{m,n}(u)}{\binom{m+n}{m}} \tag{6.6.13}$$

where $r_{m,n}(u)$ is the number of distinguishable arrangements of the m X and n Y random variables such that in each sequence the number of times a Y precedes an X is exactly u. The values of u for which $f_U(u)$ is nonzero range between zero and mn, for the two most extreme orderings in which every x precedes every y and every y precedes every x, respectively. We first note that the probability distribution of U is symmetric about the mean $mn/2$ under the null hypothesis. This property may be argued as follows. For every particular arrangement z of the m x and n y letters, define the conjugate arrangement z' as the sequence z written backward. In other words, if z denotes a set of numbers written from smallest to largest, z' denotes the same numbers written from largest to smallest. Every y that precedes an x in z

then follows that x in z', so that if u is the value of the Mann–Whitney statistic for $z, mn - u$ is the value for z'. Therefore, under H_0, we have $r_{m,n}(u) = r_{m,n}(mn - u)$ or, equivalently,

$$P\left(U - \frac{mn}{2} = u\right) = P\left(U = \frac{mn}{2} + u\right)$$
$$= P\left[U = mn - \left(\frac{mn}{2} + u\right)\right] = P\left(U - \frac{mn}{2} = -u\right)$$

Because of this symmetry property, only lower tail critical values need be found for either a one- or two-sided test. We define the random variable U' as the number of times an X precedes a Y or, in the notation of (6.6.1),

$$U' = \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - D_{ij})$$

and redefine the rejection regions for size α tests corresponding to (6.6.12) as follows:

Alternative		Rejection Region
$p < 0.5$	$F_Y(x) \leq F_X(x)$	$U \leq c_\alpha$
$p > 0.5$	$F_Y(x) \geq F_X(x)$	$U' \leq c_\alpha$
$p \neq 0.5$	$F_Y(x) \neq F_X(x)$	$U \leq c_{\alpha/2}$ or $U' \leq c_{\alpha/2}$

To determine the critical value for any m and n, we enumerate the cases starting with $u = 0$ and work up until at least $\alpha\binom{m+n}{m}$ cases are counted. For example, for $m = 4, n = 5$, the arrangements with the smallest values of u, that is, where most of the X's are smaller than most of the $Y's$, are shown in Table 6.6.1. The rejection regions for this one-sided test for nominal significance levels of 0.01 and 0.05 would then be $U \leq 0$ and $U \leq 2$, respectively.

TABLE 6.6.1

Generation of the Left-Tail P Values of U for $m = 4, n = 5$

Ordering	u	
XXXXYYYYY	0	
XXXYXYYYY	1	$P(U \leq 0) = 1/126 = 0.008$
XXYXXYYYY	2	$P(U \leq 1) = 2/126 = 0.016$
XXXYYXYYY	2	$P(U \leq 2) = 4/126 = 0.032$
XYXXXYYYY	3	$P(U \leq 3) = 7/126 = 0.056$
XXYXYXYYY	3	
XXXYYYXYY	3	

Even though it is relatively easy to guess which orderings will lead to the smallest values of u, $\binom{m+n}{m}$ increases rapidly as m, n increase. Some more systematic method of generating critical values is needed to eliminate the possibility of overlooking some arrangements with u small and to increase the feasible range of sample sizes and significance levels for constructing tables. A particularly simple and useful recurrence relation can be derived for the Mann–Whitney statistic. Consider a sequence of $m+n$ letters being built up by adding a letter to the right of a sequence of $m+n-1$ letters. If the $m+n-1$ letters consist of m x and $n-1$ y letters, the extra letter must be a y. But if a y is added to the right, the number of times a y precedes an x is unchanged. If the additional letter is an x, which would be the case for $m-1$ x and n y letters in the original sequence, all of the y's will precede this new x and there are n of them, so that u is increased by n. These two possibilities are mutually exclusive. Using the notation (6.6.13) again, this recurrence relation can be expressed as

$$r_{m,n}(u) = r_{m,n-1}(u) + r_{m-1,n}(u-n)$$

and

$$\begin{aligned} f_U(u) = p_{m,n}(u) &= \frac{r_{m,n-1}(u) + r_{m-1,n}(u-n)}{\binom{m+n}{m}} \\ &= \frac{n}{m+n}\,\frac{r_{m,n-1}(u)}{\binom{m+n-1}{n-1}} + \frac{m}{m+n}\,\frac{r_{m-1,n}(u-n)}{\binom{m+n-1}{m-1}} \end{aligned}$$

or

$$(m+n)p_{m,n}(u) = np_{m,n-1}(u) + mp_{m-1,n}(u-n) \tag{6.6.14}$$

This recursive relation holds for all $u = 0, 1, 2, \ldots, mn$ and all integer-valued m and n if the following initial and boundary conditions are defined for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

$$\begin{aligned} &r_{i,j}(u) = 0 \quad \text{for all } u < 0 \\ &r_{i,0}(0) = 1 \quad r_{0,j}(0) = 1 \\ &r_{i,0}(u) = 0 \quad \text{for all } u \neq 0 \\ &r_{0,j}(u) = 0 \quad \text{for all } u \neq 0 \end{aligned}$$

If the sample with fewer observations is always labeled the X sample, tables are needed only for $m \leq n$ and left-tail critical points. Such tables are widely available, for example in Auble (1953) or Mann and Whitney (1947).

When m and n are too large for the existing tables, the asymptotic probability distribution can be used. Since U is the sum of identically distributed (though dependent) random variables, a generalization of the central-limit theorem allows us to conclude that the null distribution of the standardized U approaches the standard normal as $m, n \to \infty$ in such a way that m/n remains constant (Mann and Whitney, 1947). To make use of this approximation, the mean and variance of U under the null hypothesis must be determined. When $F_Y(x) = F_X(x)$, the integrals in (6.6.7) and (6.6.8) are equal to $p_1 = p_2 = 1/3$. Substituting these results in (6.6.9) and (6.6.11) along with the value $p = 1/2$ from (6.6.4) gives

$$E(U|H_0) = \frac{mn}{2} \quad \text{var}(U|H_0) = \frac{mn(N+1)}{12} \tag{6.6.15}$$

The large-sample test statistic then is

$$Z = \frac{U - mn/2}{\sqrt{mn(N+1)/12}}$$

whose distribution is approximately standard normal. This approximation has been found reasonably accurate for equal sample sizes as small as 6. Since U can assume only integer values, a continuity correction of 0.5 can be used.

6.6.1 The Problem of Ties

The definition of U in (6.6.2) was adopted for presentation here because most tables of critical values are designed for use in the manner described above. Since D_{ij} is not defined for $X_i = Y_j$, this expression does not allow for the possibility of ties across samples. If ties occur within one or both samples, a unique value of U is obtained. However, if one or more X is tied with one or more Y, our definition requires that the ties be broken in some way. The conservative approach may be adopted, which means that all ties are broken in all possible ways and the largest resulting value of U (or U') is used in reaching the decision. When there are many ties (as might be the case when each random variable can assume only a few ordered values such as very strong, strong, weak, very weak), an alternative approach may be preferable.

A common definition of the Mann–Whitney statistic which does allow for ties (see Problem 5.1) is

$$U_T = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij}$$

where

$$D_{ij} = \begin{cases} 1 & \text{if } X_i > Y_j \\ 0.5 & \text{if } X_i = Y_j \\ 0 & \text{if } X_i < Y_j \end{cases} \tag{6.6.16}$$

If the two parameters p^+ and p^- are defined as

$$p^+ = P(X > Y) \quad \text{and} \quad p^- = P(X < Y)$$

U_T may be considered as an estimate of its mean

$$E(U_T) = mn(p^+ - p^-)$$

A standardized U_T is asymptotically normally distributed. Since under the null hypothesis $p^+ = p^-$, we have $E(U_T|H_0) = 0$ whether ties occur or not. The presence of ties does affect the variance, however. Conditional upon the observed ties, the variance of U_T can be calculated in a manner similar to the steps leading to (6.6.11) by introducing some additional parameters. The result is

$$\text{var}(U_T|H_0) = \frac{mn(N+1)}{12}\left[1 - \frac{\sum t(t^2-1)}{N(N^2-1)}\right] \tag{6.6.17}$$

where t denotes the multiplicity of a tie and the sum is extended over all sets of t ties. The details will be left to the reader as an exercise (or see Noether, 1967, pp. 32–35). This correction for ties can be incorporated in the standardized variable used for the test statistic.

6.6.2 Confidence-Interval Procedure

If the populations from which the X and Y samples are drawn are identical in every respect except location, say $F_Y(x) = F_X(x - \theta)$ for all x and some θ, we say that the Y population is the same as the X population but shifted by an amount θ, which may be either positive or negative, the sign indicating the direction of the shift. We want to find a confidence interval for θ, the amount of shift, based on the Mann–Whitney test procedure. Under the assumption that $F_Y(x) = F_X(x - \theta)$ for all x and some θ, the sample observations $X_1, X_2, \ldots, X_m$ and $Y_1 - \theta, Y_2 - \theta, \ldots, Y_n - \theta$ come from identical populations. A confidence interval for θ with confidence coefficient $1 - \alpha$ consists of those values of θ for which the null hypothesis of identical populations will be accepted at significance level α.

To apply the Mann–Whitney procedure to this problem, the random variable U now denotes the number of times a $Y - \theta$ precedes an X, that is, the number of pairs $(X_i, Y_j - \theta), i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, for which $X_i > Y_j - \theta$, or equivalently, $Y_j - X_i < \theta$. If a table of critical values for a two-sided U test at level α gives a rejection region of $U \leq k$, say, we reject H_0 when no more than k differences $Y_j - X_i$ are less than the value θ, and accept H_0 when more than k differences are less than θ. The total number of differences $Y_j - X_i$ is mn. If these differences are ordered from smallest to largest according to actual (not absolute) magnitude, denoted by $D_{(1)}, D_{(2)}, \ldots, D_{(mn)}$, there are exactly k differences less than θ if θ is the $(k+1)$st $-$ ordered difference, $D_{(k+1)}$. Any number exceeding this $(k+1)$st difference will produce more than k differences less than θ. Therefore, the lower limit of the confidence interval for θ is $D_{(k+1)}$. Similarly, since the probability distribution of U is symmetric, an upper confidence limit is given by that difference which is $(k+1)$st from the largest, that is, $D_{(mn-k)}$. The confidence interval with coefficient $1 - \alpha$ then is

$$D_{(k+1)} < \theta < D_{(mn-k)}$$

This procedure is illustrated by the following numerical example. Suppose that $m = 3$, $n = 5$, $\alpha = 0.10$. By simple enumeration, we find $P(U \leq 1) = 2/56 = 0.036$ and $P(U \leq 2) = 4/56 = 0.071$, and so the critical value when $\alpha/2 = 0.05$ is 1, with the exact probability of a type I error 0.072. The confidence interval is then $D_{(2)} < \theta < D_{(14)}$.

Suppose that the sample data are X: 1, 6, 7; Y: 2, 4, 9, 10, 12. In order to find $D_{(2)}$ and $D_{(14)}$ systematically, we first order the x and y data separately, then subtract from each y, starting with the smallest y, the successive values of x as shown in Table 6.6.2, and order the differences. The interval here is $-4 < \theta < 9$ with an exact confidence coefficient of 0.928.

The median of the differences $Y_j - X_i$ is called the Hodges–Lehmann estimator of the shift parameter θ and this is an unbiased estimator of θ in the shift model. We are interpreting θ to be the difference of the medians $M_Y - M_X$ of the populations.

TABLE 6.6.2

Confidence-Interval Calculations

$y_j - 1$	$y_j - 6$	$y_j - 7$
1	−4	−5
3	−2	−3
8	3	2
9	4	3
11	6	5

6.6.3 Sample Size Determination

If we are in the process of designing an experiment and specify the size of the test as α and the power of the test as $1-\beta$, we can determine the sample size required to detect a difference between the populations measured by $p = P(Y > X)$. Noether (1987) showed that an approximate sample size for a one-sided test based on the Mann–Whitney statistic is

$$N = \frac{(z_\alpha + z_\beta)^2}{12c(1-c)(p-0.5)^2} \tag{6.6.18}$$

where $c = m/N$ and z_α, z_β are the upper α and β quantiles, respectively, of the standard normal distribution. The corresponding formula for a two-sided test is found by replacing α by $\alpha/2$ in (6.6.18). Verification of this is left to the reader.

These formulas are based on a normal approximation to the power of the Mann–Whitney test in the vicinity of the null hypothesis and can be calculated easily. Note that if we take $c = 0.5$, that is when $m = n$, the formula reduces to the sample size formula (5.7.15) for the Wilcoxon signed-rank test.

As an example, suppose we want to use a one-sided Mann–Whitney test at $\alpha = 0.05$ to detect a difference $p = P(Y > X) = 0.10$ with power 0.90. Suppose we want to take fewer X observations than Y, say $m = 0.8n$. This makes $c = 4/9$. With $z_{0.05} = 1.645$ and $z_{0.10} = 1.28$, we use (6.6.18) to find

$$N = \frac{(1.645 + 1.28)^2}{12\left(\frac{4}{9}\right)\left(\frac{5}{9}\right)(0.4)^2} = 18.05$$

Thus, we need a total of 19 observations, which translates to $m = 8$, $n = 11$.

Vollandt and Horn (1997) compare Noether's sample size formula to an alternative and find that Noether's approximation is sufficiently reliable with both small and large deviations from the null hypothesis. Chakraborti et al. (2006) consider the sample size determination problem in the context of the Mann–Whitney test and recommend using linearly smoothed versions of the empirical cdf's of some pilot samples along with Noether's formula.

6.6.4 Discussion

The Mann–Whitney U test is equivalent to another well-known test, called the Wilcoxon rank-sum test presented independently in Section 8.2. Because the Wilcoxon rank-sum test is easier to use in practice, we postpone giving numerical examples until then. The discussion here applies equally to both tests.

Only independence and continuous distributions need be assumed to test the null hypothesis of identical populations. The test is simple to use for any size samples, and tables of the exact null distribution are widely available. The large-sample approximation is quite adequate for most practical purposes, and corrections for ties can be incorporated in the test statistic. The test has been

found to perform particularly well as a test for equal means (or medians), since it is especially sensitive to differences in location. In order to reduce the generality of the null hypothesis in this way, however, we must feel that we can legitimately assume that the populations are identical except possibly for their locations. A particular advantage of the test procedure in this case is that it can be adapted to confidence-interval estimation of the difference in location.

When the populations are assumed to differ only in location, the Mann–Whitney test is directly comparable with the two-sample Student's t test for means. The asymptotic relative efficiency of U relative to t is *never* less than 0.864, and if the populations are normal, the ARE is quite high at $3/\pi = 0.9550$ (see Chapter 13). The Mann–Whitney test performs better than the t test for some nonnormal distributions. For example, the ARE is 1.50 for the double exponential distribution and 1.09 for the logistic distribution, which are both heavy-tailed distributions.

Many statisticians consider the Mann–Whitney (or equivalently the Wilcoxon rank-sum) test the best nonparametric test for location. Therefore, power functions for smaller sample sizes and/or other distributions are of interest. To calculate exact power, we sum the probabilities under the alternative for those arrangements of m X and n Y random variables which are in the rejection region. For any combined arrangement Z where the X random variables occur in the positions $r_1, r_2, \ldots, r_m$ and the Y's in positions $s_1, s_2, \ldots, s_n$, this probability is

$$P(Z) = m!n! \int_{-\infty}^{\infty} \int_{-\infty}^{u_N} \cdots \int_{-\infty}^{u_3} \int_{-\infty}^{u_2} \prod_{i=1}^{m} f_X(u_{r_i}) \prod_{j=1}^{n} f_Y(u_{s_j}) du_1 \ldots du_N \qquad (6.6.19)$$

which is generally extremely tedious to evaluate. The asymptotic normality of U holds even in the nonnull case, and the mean and variance of U in (6.6.9) and (6.6.11) depend only on the parameters p, p_1, and p_2 if the distributions are continuous. Thus, approximations to power can be found if the integrals in (6.6.3), (6.6.7), and (6.6.8) are evaluated. Unfortunately, even under the more specific alternative, $F_Y(x) = F_X(x - \theta)$ for some θ, these integrals depend on both θ and F_X, so that calculating even the approximation to power requires specification of the basic parent population.

6.7 Summary

In this chapter we covered several different inference procedures for the null hypothesis that two mutually independent random samples come from identical distributions against the general alternative that they differ in some way. The Wald–Wolfowitz runs test and K–S two-sample tests are both noted for their generality. Since they are sensitive to any type of difference between the

two distributions, they are not very powerful against any specific type of difference that could be stated in the alternative. Their efficiency is very low for location alternatives. They are really useful only in preliminary studies. The median test is a precedence test primarily sensitive to differences in location and it does have a corresponding confidence-interval procedure for estimating the difference in the medians. But it is not very powerful compared to other nonparametric tests for location. The same is true for the control median test, another precedence test, although it has some practical advantages in certain situations. Conover et al. (1978) examined the power of eight nonparametric tests, including the median test, compared to the *locally most powerful* (LMP) linear rank test when the distribution is exponential for small sample sizes. Even though the median test is asymptotically equivalent to the LMP test, it performed rather poorly. Friedlin and Gastwirth (2000) suggest that the median test be ''retired from routine use'' because their simulated power comparisons showed that other tests for location are more powerful for most distributions. Even the K–S two-sample test was more powerful for most of the cases they studied. Gibbons (1964) demonstrated the poor performance of the median test with exact power calculations for small sample sizes. Further, the hand calculations for an exact median test based on the hypergeometric distribution are quite tedious even for small sample sizes. The median test continues to be of theoretical interest, however, because it is valid under such weak conditions and has interesting theoretical properties. More recently, precedence type statistics have been used in statistical quality control to set up more robust nonparametric control charts. See for example, Chakraborti et al. (2004). For further details on precedence tests and precedence-type procedures see Balakrishnan and Ng (2006). Zhang (2006) studied powerful two-sample tests based on the likelihood ratio method.

The Mann–Whitney test is far preferable as a test of location for general use, as are the other rank tests for location to be covered later in Chapter 8. The Mann–Whitney test also has a corresponding procedure for confidence-interval estimation of the difference in population medians. And we can estimate the sample size needed to carry out a test at level α to detect a stated difference in locations with power $1 - \beta$.

Problems

6.1 Use the graphical method of Hodges described in Section 6.3 to find $P(D^+_{m,n} \geq d)$ under H_0, where d is the observed value of $D^+_{m,n} = \max_x [S_m(x) - S_n(x)]$ in the arrangement $xyyxyx$.

6.2 For the median-test statistic derive the complete null distribution of U for $m = 6, n = 7$, and set up one- and two-sided critical regions when $\alpha = 0.01, 0.05$, and 0.10.

6.3 Find the large-sample approximation to the power function of a two-sided median test for $m = 6, n = 7, \alpha = 0.10$, when F_X is the standard normal distribution.

6.4 Use the recursion relation for the Mann–Whitney test statistic given in (6.6.14) to generate the complete null probability distribution of U for all $m + n \leq 4$.

6.5 Verify the expressions given in (6.6.15) for the moments of U under H_0.

6.6 Answer parts (*a*) to (*c*) using (*i*) the median-test procedure and (ii) the Mann–Whitney test procedure (use tables) for the following two independent random samples drawn from continuous populations which have the same form but possibly a difference of θ in their locations:

X	79	13	138	129	59	76	75	53
Y	96	141	133	107	102	129	110	104

(a) Using the significance level 0.10, test

$$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta \neq 0$$

(b) Give the exact level of the test in (a).

(c) Give a confidence interval on θ, with an exact confidence coefficient corresponding to the exact level noted in (b).

6.7 Represent a sample of m X and n Y random variable by a path of $m + n$ steps, the ith step being one unit up or to the right according as the ith from the smallest observation in the combined sample is an X or a Y, respectively. What is the algebraic relation between the area under the path and the Mann–Whitney statistic?

6.8 Give some other functions of the difference $S_m(x) - S_n(x)$ (besides the maximum) which could be used for distribution-free tests of the equality of two population distributions.

6.9 The 2000 census statistics for Alabama give the percentage changes in population between 1990 and 2000 for each of the 67 counties. These counties were divided into two mutually independent groups, rural and nonrural, according to population size of less than 25,000 in 2,000 or not. Random samples of nine rural and seven nonrural counties gave the following data on percentage population change:

Rural	1.1, −21.7, −16.3, −11.3, −10.4, −7.0, −2.0, 1.9, 6.2
Nonrural	−2.4, 9.9, 14.2, 18.4, 20.1, 23.1, 70.4

Use all of the methods of this chapter to test the null hypothesis of equal distributions.

6.10 (a) Show that the distribution of the precedence statistic $P_{(i)}$ under the null hypothesis $F_X = F_Y$, given in Problem 2.28(c), can be expressed as

$$P(P_{(i)} = j|H_0) = \frac{n}{m+n} \frac{\binom{m}{j}\binom{n-1}{i-1}}{\binom{m+n-1}{j+i-1}}$$
$$= \frac{i}{j+1} \frac{\binom{m}{j}\binom{n}{i}}{\binom{m+n}{j+i}} \quad j = 0, 1, \ldots, m$$

These relationships are useful in calculating the null distribution of precedence statistics using tables of the hypergeometric distribution.

(b) Hence, show that the null distribution of the control median test statistic V, with $n = 2r + 1$, can be expressed as

$$\frac{2r+1}{m+2r+1} \frac{\binom{m}{j}\binom{2r}{r}}{\binom{m+2r}{j+r}} \quad j = 0, 1, \ldots, m$$

(c) Prepare a table of the cumulative probabilities of the null distribution of V for some suitable values of m and n (odd).

6.11 For the control median test statistic V, use Problem 2.28, or otherwise, to show that when $F_X = F_Y$,

$$E(V) = \frac{m}{2} \quad \text{and} \quad \text{var}(V) = \frac{2r+m+2}{4m(2r+3)}$$

[*Hint*: Use the fact that $E(X) = E_Y E(X|Y)$ and $\text{var}(X) = \text{var}_Y E(X|Y) + E_Y \text{var}(X|Y)$]

6.12 Show that when $m, n \to \infty$ such that $m/(m+n) \to \lambda, 0 < \lambda < 1$, the null distribution of the precedence statistic $P_{(i)}$ given in Problem 6.10 tends to the negative binomial distribution with parameters i and λ, or

$$\binom{j+i-1}{i-1} \lambda^j (1-\lambda)^j \quad j = 0, 1, \ldots, m \quad \text{(Sen, 1964)}$$

6.13 In some applications the quantity $\xi_p = F_X(\kappa_p)$, where κ_p is the pth quantile of F_Y, is of interest. Let $\lim_{n\to\infty}(m/n) = \lambda$, where λ is a fixed

quantity, and let $\{r_n\}$ be a sequence of positive integers such that $\lim_{n\to\infty}(r_n/n) = p$. Finally let $V_{m,n}$ be the number of X observations that do not exceed $Y_{(r_n)}$.

(a) Show that $m^{-1}V_{m,n}$ is a consistent estimator of ξ_p.

(b) Show that the random variable $m^{1/2}[m^{-1}V_{m,n} - \xi_p]$ is asymptotically normally distributed with mean zero and variance

$$\xi_p(1-\xi_p) + \lambda p(1-p)\,\frac{f_X^2(\kappa_p)}{f_Y^2(\kappa_p)}$$

where f_X and f_Y are the density functions corresponding to F_X and F_Y, respectively (Gastwirth, 1968; Chakraborti and Mukerjee, 1990).

6.14 A sample of three girls and five boys are given instructions on how to complete a certain task. Then they are asked to perform the task over and over until they complete it correctly. The numbers of repetitions necessary for correct completion are 1, 2, and 5 for the girls and 4, 8, 9, 10, and 12 for the boys. Find the P value for the alternative that on the average the girls learn the task faster than the boys, and find a confidence-interval estimate for the difference $\theta = M_Y - M_X$ with a confidence coefficient at least equal to 0.85, using the median test.

6.15 A researcher is interested in learning if a new drug is better than a placebo in treating a certain disease. Because of the nature of the disease, only a limited number of patients can be found. Out of these, five are randomly assigned to the placebo and five to the new drug. Suppose that the concentration of a certain chemical in blood is measured and smaller measurements are better and the data are:

Drug: 3.2, 2.1, 2.3, 1.2, 1.5 Placebo: 3.4, 3.5, 4.1, 1.7, 2.1

(a) For the median test and the control median test, give the null hypothesis, the alternative hypothesis, the value of the test statistic, the exact and the approximate P value and a conclusion. What assumptions are we making?

(b) Use the median test to calculate a confidence interval for the difference between the medians. What is the largest possible level of confidence? What assumptions are we making for this procedure?

7

Linear Rank Statistics and the General Two-Sample Problem

7.1 Introduction

We described the general two-sample problem in Chapter 6 and presented some tests based on various criteria related to the combined ordered arrangement of the sample observations. Many statistical procedures applicable to the two-sample problem are based on the rank-order statistics for the combined samples, since various functions of these rank-order statistics can provide information about the possible differences between populations. For example, if the X population has a larger mean than the Y population, the sample values will reflect this difference if most of the ranks of the X values exceed the ranks of the Y values.

Many commonly used two-sample rank tests can be classified as linear combinations of certain indicator variables for the combined ordered samples. Such functions are often called *linear rank statistics*. This unifying concept will be defined in the next section, and then some of the general theory of these linear rank statistics will be presented. Particular linear rank tests will then be treated in Chapters 8 and 9 for the location and scale problems, respectively.

7.2 Definition of Linear Rank Statistics

Assume we have two independent random samples, $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ drawn from populations with continuous cumulative distribution functions F_X and F_Y, respectively. Under the null hypothesis

$$H_0\colon F_X(x) = F_Y(x) = F(x) \quad \text{for all } x, F \text{ unspecified}$$

we then have a single set of $m+n=N$ random observations from the common but unknown population, to which the integer ranks 1, 2, ..., N can be assigned.

In accordance with the definition for the rank of an observation in a single sample given in (5.5.1), a functional definition of the rank of an observation in the combined sample with no ties can be given as

$$\begin{aligned} r_{XY}(x_i) &= \sum_{k=1}^{m} S(x_i - x_k) + \sum_{k=1}^{n} S(x_i - y_k) \\ r_{XY}(y_i) &= \sum_{k=1}^{m} S(y_i - x_k) + \sum_{k=1}^{n} S(y_i - y_k) \end{aligned} \tag{7.2.1}$$

where,

$$S(u) = \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } u \geq 0 \end{cases}$$

However, it is easier to denote the combined ordered sample by a vector of indicator random variables as follows. Let

$$Z = (Z_1, Z_2, \ldots, Z_N)$$

where $Z_i = 1$ if the ith random variable in the combined ordered sample is an X and $Z_i = 0$ if it is a Y, for 1, 2, ..., N, with $N = m+n$. The rank of the observation for which Z_i is an indicator is i, and therefore the vector Z indicates the rank-order statistics of the combined samples and in addition identifies the sample to which each observation belongs.

For example, given the observations $(X_1, X_2, X_3, X_4) = (2, 9, 3, 4)$ and $(Y_1, Y_2, Y_3) = (1, 6, 10)$, the combined ordered sample is $(1, 2, 3, 4, 6, 9, 10)$ or $(Y_1, X_1, X_3, X_4, Y_2, X_2, Y_3)$, and the corresponding Z vector is $(0, 1, 1, 1, 0, 1, 0)$. Since $Z_6 = 1$, for example, an X observation (in particular X_2) has rank 6 in the combined ordered array.

Many of the statistics based on rank-order statistics which are useful in the two-sample problem can be easily expressed in terms of this notation. An important class of statistics of this type is called a *linear rank statistic*, defined as a linear function of the indicator variables Z, as

$$T_N(Z) = \sum_{i=1}^{N} a_i Z_i \tag{7.2.2}$$

where the a_i are given constants called weights or scores. It should be noted that the statistic T_N is linear in the indicator variables and no similar restriction is implied for the constants.

7.3 Distribution Properties of Linear Rank Statistics

We will now prove some general properties of T_N in order to facilitate our study of particular tests based on linear-rank-statistics.

THEOREM 7.3.1

Under the null hypothesis $H_0: F_X(x) = F_Y(x) = F(x)$ *for all* x, *we have for* $i = 1, 2, \ldots, N$,

$$E(Z_i) = \frac{m}{N} \qquad \text{var}(Z_i) = \frac{mn}{N^2} \qquad \text{cov}(Z_i, Z_j) = \frac{-mn}{N^2(N-1)} \tag{7.3.1}$$

Proof

Since Z_i follows the Bernoulli distribution with

$$f_{Z_i}(z_i) = \begin{cases} m/N & \text{if } z_i = 1 \\ n/N & \text{if } z_i = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, 2, \ldots, N$$

its mean and variance are

$$E(Z_i) = \frac{m}{N} \quad \text{var}(Z_i) = \frac{mn}{N^2}$$

For the joint moments, we have for $i \neq j$,

$$E(Z_i Z_j) = P(Z_i = 1 \cap Z_j = 1) = \frac{\binom{m}{2}}{\binom{N}{2}} = \frac{m(m-1)}{N(N-1)}$$

so that

$$\text{cov}(Z_i, Z_j) = \frac{m(m-1)}{N(N-1)} - \left(\frac{m}{n}\right)^2 = \frac{-mn}{N^2(N-1)}$$

THEOREM 7.3.2

Under the null hypothesis $H_0: F_X(x) = F_Y(x) = F(x)$ *for all* x,

$$E(T_N) = m\sum_{i=1}^{N}\frac{a_i}{N}$$

$$\text{var}(T_N) = \frac{mn}{N^2(N-1)}\left[N\sum_{i=1}^{N}a_i^2 - \left(\sum_{i=1}^{N}a_i\right)^2\right] = \frac{mn}{N(N-1)}\sum_{i=1}^{N}(a_i-\bar{a})^2 \quad (7.3.2)$$

where $\bar{a} = \sum_{i=1}^{N} a_i/N$.

Proof

$$E(T_N) = \sum_{i=1}^{N} a_i E(Z_i) = m\sum_{i=1}^{N}\frac{a_i}{N}$$

$$\text{var}(T_N) = \sum_{i=1}^{N} a_i^2\ \text{var}(Z_i) + \sum\sum_{i\neq j} a_i a_j \,\text{cov}(Z_i, Z_j)$$

$$= \frac{mn\sum_{i=1}^{N}a_i^2}{N^2} - \frac{mn\sum_{i\neq j}\sum a_i a_j}{N^2(N-1)}$$

$$= \frac{mn}{N^2(N-1)}\left(N\sum_{i=1}^{N}a_i^2 - \sum_{i=1}^{N}a_i^2 - \sum\sum_{i\neq j} a_i a_j\right)$$

$$= \frac{mn}{N^2(N-1)}\left[N\sum_{i=1}^{N}a_i^2 - \left(\sum_{i=1}^{N}a_i\right)^2\right]$$

$$= \frac{mn}{N(N-1)}\left[\sum_{i=1}^{N}(a_i-\bar{a})^2\right]$$

THEOREM 7.3.3

If $B_N = \sum_{i=1}^{N} b_i Z_i$ and $T_N = \sum_{i=1}^{N} a_i Z_i$ are two linear rank statistics, under the null hypothesis H_0: $F_X(x) = F_Y(x) = F(x)$ for all x,

$$\text{cov}(B_N, T_N) = \frac{mn}{N^2(N-1)}\left(N\sum_{i=1}^{N}a_i b_i - \sum_{i=1}^{N}a_i\sum_{i=1}^{N}b_i\right)$$

Proof

$$\begin{aligned}\operatorname{cov}(B_N, T_N) &= \sum_{i=1}^{N} a_i b_i \operatorname{var}(Z_i) + \sum\sum_{i\neq j} a_i b_j \operatorname{cov}(Z_i Z_j) \\ &= \frac{mn}{N^2}\sum_{i=1}^{N} a_i b_i - \frac{mn}{N^2(N-1)}\sum\sum_{i\neq j} a_i b_j \\ &= \frac{mn}{N^2(N-1)}\left(N\sum_{i=1}^{N} a_i b_i - \sum_{i=1}^{N} a_i b_i - \sum\sum_{i\neq j} a_i b_j\right) \\ &= \frac{mn}{N^2(N-1)}\left(N\sum_{i=1}^{N} a_i b_i - \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i\right)\end{aligned}$$

Using these theorems, the exact moments under the null hypothesis can be found for any linear rank statistics. The exact null probability distribution of T_N depends on the probability distribution of the vector Z, which indicates the ranks of the X and Y random variables. This distribution was given in (6.6.19) for any distributions F_X and F_Y. In the null case, $F_X = F_Y = F$, say, and (6.6.19) reduces to

$$P(Z) = m!n! \int_{-\infty}^{\infty}\int_{-\infty}^{u_N}\cdots\int_{-\infty}^{u_2}\prod_{i=1}^{m} f(u_{r_i})\prod_{j=1}^{n} f(u_{s_j})\,du_1 \ldots du_N$$

where $r_1, r_2, \ldots, r_m$ and $s_1, s_2, \ldots, s_n$ are the ranks of the X and Y random variables, respectively, in the arrangement Z. Since the distributions are identical, the product in the integrand is the same for all subscripts, or

$$\begin{aligned}P(Z) &= m!n! \int_{-\infty}^{\infty}\int_{-\infty}^{u_N}\cdots\int_{-\infty}^{u_2}\prod_{i=1}^{N} f(u_i)\,du_1 \ldots du_n \\ &= \frac{m!n!}{N!}\end{aligned} \tag{7.3.3}$$

The final result follows from the fact that except for the terms $m!n!$, $P(Z)$ is the integral over the entire region of the density function of the N order statistics for a random sample from the population F. Since $\binom{m+n}{m} = \binom{N}{m}$ is the total number of distinguishable Z vectors, that is, distinguishable arrangements of m ones and n zeros, the result in (7.3.3) implies that all vectors Z are equally likely under H_0.

Since each Z occurs with probability $1\Big/\binom{N}{m}$, the exact probability distribution under the null hypothesis of any linear rank statistic can always be found by direct enumeration. The values of $T_N(Z)$ are calculated for each Z, and the probability of a particular value k is the number of Z vectors which lead to that number k divided by $\binom{N}{m}$. In other words, we have

$$P[T_N(Z) = k] = \frac{t(k)}{\binom{N}{m}} \tag{7.3.4}$$

where $t(k)$ is the number of arrangements of m X and n Y random variables such that $T_N(Z) = k$. Naturally, the tediousness of enumeration increases rapidly as m and n increase. For some statistics, recursive methods are possible. STATXACT calculates the exact P value for a linear rank test based on a complete enumeration of the values of the test statistic. Here, the data are permuted (rearranged) in all possible ways under the null hypothesis. The value of the test statistic is calculated for each permutation of the data; these values constitute the *permutation distribution* and allow calculation of the exact P value for any test based on ranks of any set of data.

When the null distribution of a linear rank statistic is symmetric, only one-half of the distribution needs to be generated. The statistic $T_N(Z)$ is symmetric about its mean μ if for every $k \neq 0$,

$$P[T_N(Z) - \mu = k] = P[T_N(Z) - \mu = -k]$$

Suppose that for every vector Z of m ones and n zeros, a conjugate vector Z' of m zeros and n ones exists such that whenever $T_N(Z) = \mu + k$, we have $T_N(Z') = \mu - k$. Then the frequency of the number $\mu + k$ is the same as that of $\mu - k$, and the distribution is symmetric. The condition for symmetry of a linear rank statistic then is that

$$T_N(Z) + T_N(Z') = 2\mu$$

The following theorem establishes a simple relation between the scores which will ensure the symmetry of $T_N(Z)$.

THEOREM 7.3.4

The null distribution of $T_N(Z)$ is symmetric about its mean $\mu = m\sum_{i=1}^{N} a_i/N$ whenever the weights satisfy the relation

$$a_i + a_{N-i+1} = c \quad c = \text{constant} \quad \text{for } i = 1, 2, \ldots, N$$

Proof

For any vector $Z = (Z_1, Z_2, \ldots, Z_N)$ of m ones and n zeros, define the conjugate vector $Z' = (Z'_1, \ldots, Z'_N)$, where $Z'_i = Z_{N-i+1}$. Then

$$\begin{aligned} T_N(Z) + T_N(Z') &= \sum_{i=1}^{N} a_i Z_i + \sum_{i=1}^{N} a_i Z_{N-i+1} \\ &= \sum_{i=1}^{N} a_i Z_i + \sum_{j=1}^{N} a_{N-j+1} Z_j \\ &= \sum_{i=1}^{N} (a_i + a_{N-i+1}) Z_i = c \sum_{i=1}^{N} Z_i = cm \end{aligned}$$

Since $E[T_N(Z)] = E[T_N(Z')]$, we must have $cm = 2\mu$, or $c = 2\mu/m = 2\sum_{i=1}^{N} a_i/N$.

The next theorem establishes the symmetry of the null disbtribution of *any* linear rank statistic when $m = n$.

THEOREM 7.3.5

The null distribution of $T_N(Z)$ is symmetric about its mean for any set of weights if $m = n = N/2$.

Proof

Since $m = n$, we can define our conjugate Z' with ith component $Z'_i = 1 - Z_i$. Then

$$T_N(Z) + T_N(Z') = \sum_{i=1}^{N} a_i Z_i + \sum_{i=1}^{N} a_i (1 - Z_i) = \sum_{i=1}^{N} a_i = 2\mu$$

A rather special but useful case of symmetry is given in the next theorem.

THEOREM 7.3.6

The null distribution of $T_N(Z)$ is symmetric about its mean μ if N is even and the weights are $a_i = i$ for $i \leq N/2$ and $a_i = N - i + 1$ for $i > N/2$.

Proof

The appropriate conjugate Z' has components $Z'_i = Z_{i+N/2}$ for $i \le N/2$ and $Z'_i = Z_{i-N/2}$ for $i > N/2$. Then

$$\begin{aligned} T_N(Z) + T_N(Z') &= \sum_{i=1}^{N/2} iZ_i + \sum_{i=N/2+1}^{N} (N-i+1)Z_i \\ &\quad + \sum_{i=1}^{N/2} iZ_{N/2+i} + \sum_{i=N/2+1}^{N} (N-i+1)Z_{i-N/2} \\ &= \sum_{i=1}^{N/2} iZ_i + \sum_{i=N/2+1}^{N} (N-i+1)Z_i \\ &\quad + \sum_{j=N/2+1}^{N} \left(j - \frac{N}{2}\right) Z_j + \sum_{j=1}^{N/2} \left(\frac{N}{2} - j + 1\right) Z_j \\ &= \sum_{i=1}^{N/2} \left(\frac{N}{2}+1\right) Z_i + \sum_{i=N/2+1}^{N} \left(\frac{N}{2}+1\right) Z_i \\ &= m\left(\frac{N}{2}+1\right) = 2\mu \end{aligned}$$

In determining the frequency $t(k)$ for any value k which is assumed by a linear-rank test statistic, the number of calculations required may be reduced considerably by the following properties of $T_N(Z)$, which are easily verified.

THEOREM 7.3.7

Property 1: *Let*

$$T = \sum_{i=1}^{N} a_i Z_i \quad \text{and} \quad T' = \sum_{i=1}^{N} a_i Z_{N-i+1}$$

Then

$$T = T' \quad \text{if } a_i = a_{N-i+1} \quad \text{for } i = 1, 2, \ldots, N$$

Property 2: *Let*

$$T = \sum_{i=1}^{N} a_i Z_i \quad \text{and} \quad T' = \sum_{i=1}^{N} a_i (1 - Z_i)$$

Then

$$T + T' = \sum_{i=1}^{N} a_i$$

Property 3: *Let*

$$T = \sum_{i=1}^{N} a_i Z_i \quad \text{and} \quad T' = \sum_{i=1}^{N} a_i (1 - Z_{N-i+1})$$

Then

$$T + T' = \sum_{i=1}^{N} a_i \quad \text{if } a_i = a_{N-i+1} \quad \text{for } i = 1, 2, \ldots, N$$

For large samples, that is, $m \to \infty$ and $n \to \infty$ in such a way that m/n remains constant, an approximation exists which is applicable to the distribution of almost all linear rank statistics. Since T_N is a linear combination of the Z_i, which are identically distributed (though dependent) random variables, a generalization of the central-limit theorem allows us to conclude that the probability distribution of a standardized linear rank statistic $[T_N - E(T_N)]/\sigma(T_N)$ approaches the standard normal probability distribution subject to certain regularity conditions.

The foregoing properties of linear rank statistics hold only in the hypothesized case of identical populations. Chernoff and Savage (1958) proved that the asymptotic normality property is valid also in the nonnull case, subject to certain regularity conditions relating mainly to the smoothness and size of the weights. The expressions for the mean and variance will be given here, since they are also useful in investigating the consistency and efficiency of most two-sample linear rank statistics.

A key feature in the Chernoff–Savage theory is that a linear rank statistic can be represented in the form of a Stieltjes integral. Thus, if the weights for a linear rank statistic are functions of the ranks, an equivalent representation of $T_N = \sum_{i=1}^{N} a_i Z_i$ is

$$T_N = m \int_{-\infty}^{\infty} J_N[H_N(x)]\, dS_m(x)$$

where the notation is defined as follows:

1. $S_m(x)$ and $S_n(x)$ are the empirical distribution functions of the X and Y samples, respectively.
2. $m/N \to \lambda_N, 0 < \lambda_N < 1$.
3. $H_N(x) = \lambda_N S_m(x) + (1 - \lambda_N) S_n(x)$, so that $H_N(x)$ is the proportion of observations from either sample which does not exceed the value x, or the empirical distribution function of the combined sample.
4. $J_N(i/N) = a_i$.

This Stieltjes integral form is given here because it appears frequently in the literature and is useful for proving theoretical properties. Since the following theorems are given here without proof anyway, the student not familiar with Stieltjes integrals can consider the following equivalent representation:

$$T'_N = m \sum_{\substack{\text{over all } x \text{ such} \\ \text{that } p(x)>0}} J_N[H_N(x)]p(x)$$

where

$$p(x) = \begin{cases} 1/m & \text{if } x \text{ is the observed value of an } X \text{ random variable} \\ 0 & \text{otherwise} \end{cases}$$

For example, in the simplest case where $a_i = i/N, J_N[H_N(x)] = H_N(x)$ and

$$\begin{aligned} T_N &= m \int_{-\infty}^{\infty} H_N(x)\, dS_m(x) = \frac{m}{N} \int_{-\infty}^{\infty} [mS_m(x) + nS_n(x)]\, dS_m(x) \\ &= \frac{m}{N} \int_{-\infty}^{\infty} (\text{number of observations in the combined sample} \le x) \\ &\quad \times (1/m \text{ if } x \text{ is the value of an } X \text{ random variable and } 0 \text{ otherwise}) \\ &= \frac{1}{N} \sum_{i=1}^{N} iZ_i \end{aligned}$$

Now when the X and Y samples are drawn from the continuous populations F_X and F_Y, respectively, we define the combined population cdf as

$$H(x) = \lambda_N F_X(x) + (1 - \lambda_N)F_Y(x)$$

The Chernoff and Savage theorem stated below is subject to certain regularity conditions not explicitly stated here, but given in Chernoff and Savage (1958).

THEOREM 7.3.8

Subject to certain regularity conditions, the most important of which are that $J(H) = \lim_{N \to \infty} J_N(H)$,

$$\left|J^{(r)}(H)\right| = |d^r J(H)/dH^r| \le K|H(1-H)|^{-r-1/2+\delta}$$

for $r = 0, 1, 2$, some $\delta > 0$ and any constant K which does not depend on m, n, N, F_X or F_Y, then for λ_N fixed,

$$\lim_{N\to\infty} P\left(\frac{T_N/m - \mu_N}{\sigma_N} \le t\right) = \Phi(t)$$

where

$$\mu_N = \int_{-\infty}^{\infty} J[H(x)]f_X(x)\,dx$$

$$N\sigma_N^2 = 2\left(\frac{1-\lambda_N}{\lambda_N}\right)\left\{\lambda_N \iint_{-\infty<x<y<\infty} F_Y(x)[1-F_Y(y)]J'[H(x)]J'[H(y)] \times f_X(x)f_Y(y)\,dx\,dy + (1-\lambda_N)\iint_{-\infty<x<y<\infty} F_X(x)[1-F_X(y)] \times J'[H(x)]J'[H(y)]f_X(x)f_Y(y)\,dx\,dy\right\}$$

provided $\sigma_N \neq 0$.

COROLLARY 7.3.1

If X and Y are identically distributed with common distribution $F(x) = F_X(x) = F_Y(x)$, we have

$$\mu_N = \int_0^1 J(u)\,du$$

$$N\lambda_N\sigma_N^2 = 2(1-\lambda_N)\iint_{0<x<y<1} x(1-y)J'(x)J'(y)\,dx\,dy$$

$$= 2(1-\lambda_N)\iiiint_{0<u<x<y<v<1} J'(x)J'(y)\,dx\,dy\,du\,dv$$

$$= 2(1-\lambda_N)\iint_{0<u<v<1}\int_u^v\int_x^v J'(x)J'(y)\,dy\,dx\,du\,dv$$

$$= 2(1-\lambda_N)\iint_{0<u<v<1}\int_u^v [J(v)-J(x)]J'(x)\,dx\,du\,dv$$

$$= 2(1-\lambda_N) \iint\limits_{0<u<v<1} \left[J(v)J(x) - \frac{J^2(x)}{2} \right]\Bigg|_u^v du\, dv$$

$$= (1-\lambda_N) \iint\limits_{0<u<v<1} [J^2(v) - 2J(v)J(u) + J^2(u)]\, du\, dv$$

$$= (1-\lambda_N) \left[\int_0^1 vJ^2(v)\, dv + \int_0^1 (1-u)J^2(u)\, du - \int_0^1 J(u)\, du \int_0^1 J(v)\, dv \right]$$

$$= (1-\lambda_N) \left\{ \int_0^1 J^2(u)\, du - \left[\int_0^1 J(u)\, du \right]^2 \right\}$$

These expressions are equivalent to those given in Theorem 7.3.2 for $a_i = J_N(i/N)$.

7.4 Usefulness in Inference

The general alternative to the null hypothesis in the two-sample problem is simply that the populations are not identical,

$$F_X(x) \neq F_Y(x) \quad \text{for some } x$$

or the analogous one-sided general alternative, which states a directional inequality such as

$$F_X(x) \leq F_Y(x) \quad \text{for all } x$$

with strict inequality for some x. The two-sample tests considered in Chapter 6, namely, the Kolmogorov–Smirnov, Wald–Wolfowitz runs, Mann–Whitney, and median tests, can be used for these alternatives. In most parametric two-sample situations, the alternatives are much more specific, as in the t and F tests for comparison of means and variances, respectively. Although all of the two-sample rank tests are for the same null hypothesis, particular test statistics may be especially sensitive to a particular form of alternative, thus increasing their power against that type of alternative.

Since any set of scores $a_1, a_2, \ldots, a_N$ can be used as the coefficients in a linear rank statistic, this form of test statistic lends itself particularly well to more specific types of alternatives. The appropriate choice depends on the type of difference between populations one hopes to detect. The simplest type of

situation to deal with is where the statistician has enough information about the populations to feel that if a difference exists, it is only in location or only in scale. These will be called, respectively, the *two-sample location problem* and the *two-sample scale problem*. In Chapters 8 and 9, we will discuss briefly some of the better-known and more widely accepted linear rank statistics useful in these problems. No attempt will be made to provide recommendations regarding which to use. The very generality of linear rank tests makes it difficult to make direct comparisons of power functions, since calculation of power requires more specification of the alternative probability distributions and moments. A particular test might have high power against normal alternatives but perform poorly for the gamma distribution. Furthermore, calculation of the exact power of rank tests is usually quite difficult. We must be able to determine the probability distribution of the statistic $T_N(Z)$ or the arrangement Z as in (6.6.19) under the specified alternative and sum these probabilities over those arrangements Z in the rejection region specified by the test. STATXACT can be useful in calculating the exact power of linear rank tests. Isolated and specific comparisons of power between nonparametric tests have received much attention in the literature. However, calculation of asymptotic relative efficiency of linear rank tests versus the t and F tests for normal alternatives is not particularly difficult, as we will show in Chapter 13.

Problems

7.1 One of the simplest linear rank statistics is defined as

$$W_N = \sum_{i=1}^{N} iZ_i$$

This is the Wilcoxon rank-sum statistic to be discussed in the next chapter. Use Theorem 7.3.2 to evaluate the mean and variance of W_N under H_0.

7.2 Express the two-sample median-test statistic U defined in Section 6.4 in the form of a linear rank statistic and use Theorem 7.3.2 to find its mean and variance. *Hint*: For the appropriate argument k, use the functions $S(k)$ defined as for (7.2.1).

7.3 Prove the three properties stated in Theorem 7.3.7.

7.4 For $m = n = 2$, derive the probability mass function of T_N, the sum of the X ranks, under H_0. Determine whether this distribution is symmetric and if so, identify the point of symmetry. Calculate the mean and variance of T_N.

8

Linear Rank Tests for the Location Problem

8.1 Introduction

Suppose that two independent samples of sizes m and n are drawn from two continuous populations so that we have a total of $N = m + n$ observations. We want to test the null hypothesis of identical distributions. The *location alternative* is that the populations have the same form but a different measure of central tendency. This can be expressed in symbols as follows:

$$H_0 : F_Y(x) = F_X(x) \quad \text{for all } x$$
$$H_L : F_Y(x) = F_X(x - \theta) \quad \text{for all } x \text{ and some } \theta \neq 0$$

The cumulative distribution of the Y population under H_L is functionally the same as that of the X population but shifted to the left if $\theta < 0$ and shifted to the right if $\theta > 0$, as shown in Figure 8.1.1. Therefore, the Y's are stochastically larger than the X's when $\theta > 0$ and the Y's are stochastically smaller than the X's when $\theta < 0$. For example, the median of the X population is larger than the median of the Y population when $\theta < 0$. (See also the discussion in Section 6.1.)

If F_X is the cumulative normal cdf, then the mean and median are equal and a one-sided normal-theory test with equal but unknown variances of the hypotheses

$$H_0 : \mu_Y - \mu_X = 0 \quad \text{versus} \quad H : \mu_Y - \mu_X < 0$$

is equivalent to the general location alternative with $H_0 : \theta = \mu_Y - \mu_X = 0$ versus $H : \theta = \mu_Y - \mu_X < 0$. The best parametric test against this alternative is based on Student's t statistic with $m + n - 2$ degrees of freedom:

$$t = \frac{\overline{X} - \overline{Y}}{\sqrt{\left[(m-1)s_X^2 + (n-1)s_Y^2\right]/(m+n-2)}\sqrt{(m+n)/mn}} \tag{8.1.1}$$

The t test has been shown to be robust under the assumptions of normality and equal variances. However, there are many good and simple nonparametric

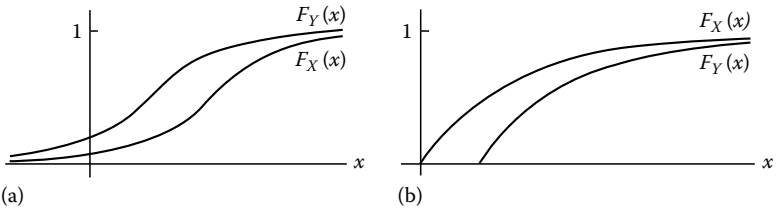


FIGURE 8.1.1
$F_Y(x) = F_X(x - \theta)$. (a) F_X normal, $\theta < 0$. (b) F_X exponential, $\theta > 0$.

tests for the location problem that do not require specification of the underlying distribution. Many of these are based on ranks since the ranks of the X's relative to the ranks of the Y's provide information about the relative size of the population medians. For a linear rank statistic, any set of scores that are nondecreasing or nonincreasing in magnitude would allow the statistic to reflect a combined ordered sample in which most of the X's are larger than the Y's, or vice versa. The Wilcoxon rank-sum test is one of the best known and easiest to use, since its scores are positive integers. The other tests that will be covered in this chapter are the Terry–Hoeffding-normal-scores test, inverse-normal-scores test, and percentile modified rank tests. Many other tests are discussed in the literature.

8.2 The Wilcoxon Rank-Sum Test and Confidence Interval

The ranks of the X's in the combined ordered arrangement of the two samples will generally be larger than the ranks of the Y's if the median of the X population exceeds the median of the Y population. Therefore, Wilcoxon (1945) proposed a test where we accept the one-sided location alternative $H_L : \theta < 0$ ($X \overset{\text{ST}}{>} Y$), if the sum of the ranks of the X's is too large, or $H_L : \theta > 0$ ($X \overset{\text{ST}}{<} Y$) if the sum of the ranks of the X's is too small, and the two-sided location alternative H_L: $\theta \neq 0$ if the sum of the ranks of the X's is either too large or too small. This function of the ranks expressed as a linear rank statistic has the simple weights $a_i = i$, $i = 1, 2, \ldots, N$. The *Wilcoxon rank-sum test* statistic is

$$W_N = \sum_{i=1}^{N} iZ_i \tag{8.2.1}$$

where the Z_i are the indicator random variables as defined for Equation 7.2.2.

If there are no ties, the exact mean and variance of W_N under the null hypothesis of equal distributions are easily found from Theorem 7.3.2 to be

$$E(W_N) = \frac{m(N+1)}{2} \quad \text{var}(W_N) = \frac{mn(N+1)}{12}$$

Verification is left for the reader. If $m \leq n$, W_N has a minimum value of $\sum_{i=1}^{m} i = m(m+1)/2$ and a maximum value of $\sum_{i=N-m+1}^{N} i = m(2N-m+1)/2$. Further, from Theorem 7.3.4, the statistic is symmetric about its mean under $H_0: \theta = 0$ since

$$a_i + a_{N-i+1} = N+1 \quad \text{for } i = 1, 2, \ldots, N.$$

The exact null probability distribution can be obtained systematically by enumeration using these properties. For example, suppose $m=3$, $n=4$. There are $\binom{7}{3} = 35$ possible distinguishable configurations of 1's and 0's in the vector Z, but these need not be enumerated individually. W_N will range between 6 and 18, symmetric about 12, the values occurring in conjunction with the ranks in Table 8.2.1, from which the complete null probability distribution is easily found. For example, Table 8.2.1 shows that $P(W_N \leq 17) = 2/35 = 0.0571$.

Several recursive schemes are also available for generation of the distribution. The simplest to understand is analogous to the recursion relations given in (5.7.8) for the Wilcoxon signed-rank statistic and (6.6.14) for the Mann–Whitney statistic. If $r_{m,n}(k)$ denotes the number of arrangements of m X and n Y random variables such that the sum of the X ranks is equal to k, it is evident that

$$r_{m,n}(k) = r_{m-1,n}(k-N) + r_{m,n-1}(k)$$

TABLE 8.2.1

Null Distribution of W_N

Value of W_N	Ranks of X's	Frequency
18	5, 6, 7	1
17	4, 6, 7	1
16	3, 6, 7; 4, 5, 7	2
15	2, 6, 7; 3, 5, 7;4, 5, 6	3
14	1, 6, 7;2, 5, 7; 3, 4, 7; 3, 5, 6	4
13	1, 5, 7; 2, 4, 7; 2, 5, 6; 3, 4, 6	4
12	1, 4, 7; 2, 3, 7; 1, 5, 6; 2, 4, 6; 3,4, 5	5

and

$$f_{W_N}(k) = p_{m,n}(k) = \frac{[r_{m-1,n}(k-N) + r_{m,n-1}(k)]}{\binom{m+n}{m}}$$

or

$$(m+n)p_{m,n}(k) = mp_{m-1,n}(k-N) + np_{m,n-1}(k) \tag{8.2.2}$$

Tail probabilities for the null distribution of the Wilcoxon rank-sum test statistic are given in Table J for $1 \leq m \leq n \leq 10$. More extensive tables are available in Wilcoxon et al. (1972).

For larger sample sizes, generation of the exact probability distribution is time consuming. However, the normal approximation to the distribution can be used because of the asymptotic normality of the general linear rank statistic (Theorem 7.3.8). The normal approximation for W_N has been shown to be accurate enough for most practical purposes for combined sample sizes N as small as 12.

The midrank method is easily applied to handle the problem of ties. The presence of a moderate number of tied observations seems to have little effect on the probability distribution. Corrections for ties have been thoroughly investigated (see, for example, Noether, 1967, pp. 32–35).

If the ties are handled by the midrank method, the variance of W_N in the normal approximation can be corrected to take the ties into account. As we found in (5.7.10), the presence of ties reduces the sum of squares of the ranks by $\sum t(t^2-1)/12$, where t is the number of X and Y observations that are tied at any given rank and the sum is over all sets of t tied ranks. Substituting this result in (7.3.2) gives the variance of W_N as

$$\begin{aligned}&\frac{mn}{N^2(N-1)}\left\{N\left[\frac{N(N+1)(2N+1)}{6} - \frac{\sum t(t^2-1)}{12}\right] - \left[\frac{N(N+1)}{2}\right]^2\right\} \\ &\quad = \frac{mn(N+1)}{12} - \frac{mn\sum t(t^2-1)}{12N(N-1)}\end{aligned} \tag{8.2.3}$$

The Wilcoxon rank-sum test is equivalent to the Mann–Whitney test discussed in Section 6.6, since a linear relationship exists between the two test statistics. With U defined as the number of times a Y precedes an X, as in (6.6.2), the Mann–Whitney U test statistic is

$$U = \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij} = \sum_{i=1}^{m}(D_{i1} + D_{i2} + \cdots + D_{in})$$

where

$$D_{ij} = \begin{cases} 1 & \text{if } Y_j < X_i \\ 0 & \text{if } Y_j > X_i \end{cases}$$

Then, $\sum_{j=1}^{n} D_{ij}$ is the number of values of j for which $Y_j < X_i$, or the rank of X_i reduced by t_i, the number of X's that are less than or equal to X_i. Thus, we can write

$$\begin{aligned} U &= \sum_{i=1}^{m} [r(X_i) - t_i] \\ &= \sum_{i=1}^{m} r(X_i) - (t_1 + t_2 + \cdots + t_m) \\ &= \sum_{i=1}^{m} iZ_i - (1 + 2 + \cdots + m) \\ &= W_N - \frac{m(m+1)}{2} \end{aligned} \tag{8.2.4}$$

The statistic U (or W_N) can be easily related to the placements introduced in Chapter 2. To see this, note that $\sum_{j=1}^{n} D_{ij}$, which counts the total number of Y's that are less than X_i, can be rewritten as $nG_n(X_i)$ where G_n is the empirical cdf of the Y sample. Now,

$$U = \sum_{i=1}^{m} nG_n(X_i) = \sum_{i=1}^{m} nG_n(X_{(i)}) = \sum_{i=1}^{m} \left[r(X_{(i)}) - i\right] \tag{8.2.5}$$

where $r(X_{(i)})$ is the rank of the ith-ordered X observation in the combined sample. The last equality in (8.2.5) also shows that the Mann–Whitney U statistic is a linear function of the Wilcoxon rank-sum test statistic. Thus, all the properties of the two tests are the same, including consistency and the minimum ARE of 0.864 relative to the t test. A confidence-interval procedure based on the Wilcoxon rank-sum test leads to the same result as the one based on the Mann–Whitney test.

As discussed in Chapter 10, the Wilcoxon rank-sum statistic is also equivalent to an ordinary analysis of variance of ranks for two groups (see Problem 10.5), a procedure that is easily extended to the case of more than two samples.

8.2.1 Applications

The appropriate rejection regions and P values for the Wilcoxon rank-sum test statistic W_N in terms of θ, the difference between the two location parameters, are as follows, where w_0 is the observed value of W_N.

Alternative	Rejection Region	*P* Value
$\theta < 0 (Y <^{ST} X)$	$W_N \geq w_\alpha$	$P(W_N \geq w_0)$
$\theta > 0 (Y >^{ST} X)$	$W_N \leq w'_\alpha$	$P(W_N \leq w_0)$
$\theta \neq 0$	$W_N \geq w_{a/2}$ or $W_N \leq w'_{\alpha/2}$	2 (smaller of above)

The exact cumulative null distribution of W_N is given in Table J for $m \leq n \leq 10$, as left-tail probabilities for $W_N \leq m(N+1)/2$ and right-tail for $W_N \geq m(N+1)/2$. For larger sample sizes, the appropriate rejection regions and P values based on the normal approximation with a continuity correction of 0.5 are as follows:

Alternative	Rejection Region	*P* Value
$\theta < 0$	$W_N \geq \frac{m(N+1)}{2} + 0.5 + z_\alpha \sqrt{\frac{mn(N+1)}{12}}$	$1 - \Phi\left(\frac{w_0 - 0.5 - m(N+1)/2}{\sqrt{mn(N+1)/12)}}\right)$
$\theta > 0$	$W_N \leq \frac{m(n+1)}{2} - 0.5 - z_\alpha \sqrt{\frac{mn(N+1)}{12}}$	$\Phi\left(\frac{w_0 + 0.5 - m(N+1)/2}{\sqrt{mn(N+1)/12)}}\right)$
$\theta \neq 0$	Both above with z_α replaced by $z_{\alpha/2}$	2 (smaller of above)

If ties are present, the correction for ties derived in (8.2.3) should be incorporated in the variance term of the rejection regions and P values.

Recall from Section 6.6 that the confidence-interval estimate of θ based on the Mann–Whitney test has endpoints that are the $(k+1)$st from the smallest and largest of the mn differences $y_j - x_i$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. The value of k is the left-tail rejection region cutoff point of $\alpha/2$ in the null distribution of the Mann–Whitney statistic. Let this $\alpha/2$ cutoff point be $c_{\alpha/2}$. The corresponding cutoff in terms of the Wilcoxon rank-sum statistic from the relationship in (8.2.3) is $w_{\alpha/2} = w'_{\alpha/2} - m(m+1)/2$. Thus, the value of k can be found by subtracting $m(m+1)/2$ from the left-tail critical value of W_N in Table J for the given m and n with $P = \alpha/2$.

For example, $m = 4$, $n = 5$, $P = 0.032$, $w'_{0.032} = 12$, and $c_{0.032} = 12 - 10 = 2$, so that $k + 1 = 3$ and the confidence level is 0.936. Notice that $k + 1$ is always equal to the rank of $w'_{\alpha/2}$ among the entries for the given m and n in Table J because $m(m+1)/2$ is the minimum value of the Wilcoxon rank-sum test statistic W_N.

In practice then, the corresponding confidence-interval endpoints to estimate θ are the uth smallest and uth largest of the mn Hodges–Lehmann estimators of the shift parameter, that is, the differences $Y_i - X_j$ for all i, j. The appropriate value for u is the rank of that left-tail P among the entries in Table J for the given m and n, for confidence level $1 - 2P$. For m and n outside the range of Table J, we use the normal approximation to find u from

$$u = \frac{mn}{2} + 0.5 - z_{\alpha/2} \sqrt{\frac{mn(N+1)}{12}} \tag{8.2.6}$$

and round down to the next smaller integer if the result is not an integer. Zeros and ties are counted as many times as they occur and therefore do not pose a problem for the confidence-interval estimate. If the ties are extensive, the variance under the radical sign in (8.2.6) should be replaced by (8.2.3).

Example 8.2.1

A time and motion study was made in the permanent mold department at Central Foundry to determine whether there was a pattern to the variation in the time required to pour the molten metal into the die and form a casting of a 6×4 in. Y-shaped branch. The metallurgical engineer suspected that pouring times before lunch were shorter than pouring times after lunch on a given day. Twelve independent observations were taken throughout the day, six before lunch and six after lunch. Find the P value for the alternative that mean pouring time before lunch is less than after lunch for the data below on pouring times in seconds.

Before Lunch		After Lunch	
12.6	11.2	16.4	15.4
11.4	9.4	14.1	14.0
13.2	12.0	13.4	11.3

Solution

With equal sample sizes $m = n = 6$, either period can be called the X sample. If X denotes pouring time before lunch, the desired alternative is $H_1{:}\theta = \mu_Y - \mu_X > 0$ and the appropriate P value is in the left tail for W_N. The pooled array with X values underlined is $\underline{9.4}$, $\underline{11.2}$, 11.3, $\underline{11.4}$, $\underline{12.0}$, $\underline{12.6}$, $\underline{13.2}$, 13.4, 14.0, 14.1, 15.4, 16.4, and $W_N = 1 + 2 + 4 + 5 + 6 + 7 = 25$. The P value is $P(W_N \leq 25) = 0.013$ from Table J for $m = 6$, $n = 6$. Thus, the null hypothesis $H_0{:}\theta = 0$ is rejected in favor of the alternative $H_1{:}\theta > 0$ at any significance level $\alpha \geq 0.013$.

The MINITAB, STATXACT, and SAS solutions to Example 8.2.1 are shown below. Note that both SAS and MINITAB compute the one-tailed P value as 0.0153 based on the normal approximation with a continuity correction. It is interesting to note that the MINITAB printout includes a confidence interval estimate of $\mu_X - \mu_Y$ that is based on the exact distribution of W_N and this agrees with what we would find (see Problem 8.16), while the test result is based on the normal approximation with a continuity correction. The STATXACT solution gives the exact P value, which agrees with ours, and the asymptotic P value based on the normal approximation without a continuity correction. The SAS solution gives the exact P value, which agrees with ours, and an asymptotic P value based on the normal approximation with a continuity correction (and it tells us so!). SAS also shows a t approximation based on what is called a *rank transformation*. The idea behind a rank transformation is to replace the original X

and Y data values by their ranks in the combined sample and calculate the usual t statistic from (8.1.1) using these ranks. The approximate P value is calculated from a t table with $N-2$ degrees of freedom. The rank transformation idea has been applied to various other classical parametric tests, thereby creating new nonparametric tests. The reader is referred to Conover and Iman (1981) for a good introduction to rank transformation and its applications. The SAS output also shows a two-sided result called the Kruskal–Wallis test, which we will cover in Chapter 10.

```
****************************************
MINITAB SOLUTION TO EXAMPLE 8.2.1
****************************************

Mann-Whitney test and CI: Before, after

Before      N =   6     Median =      11.700
After       N =   6     Median =      14.050
Point estimate for ETA1-ETA2 is      −2.400
95.5% CI for ETA1-ETA2 is (−4.600, −0.199)
W = 25.0
Test of ETA1 = ETA2 vs. ETA1 < ETA2 is significant at 0.0153.
```

```
****************************************
STATXACT SOLUTION TO EXAMPLE 8.2.1
****************************************

WILCOXON-MANN-WHITNEY TEST
[   Sum of scores from population <        1 > ]

Summary of exact distribution of WILCOXON-MANN-WHITNEY statistic:
Min     Max          Mean    Std.-dev.    Observed   Standardized
21.     57.00       39.00      6.245        25.00        −2.242
Mann-Whitney Statistic=        4.000

Asymptotic Inference:
One-sided P value: Pr {  Test-Statistic .LE. Observed } = 0.0125
Two-sided P value: 2 * one-sided                         = 0.0250

Exact Inference:
One-sided P value: Pr {  Test-Statistic.LE. Observed }  = 0.0130
                   Pr {  Test Statistic.EQ. Observed }  = 0.0054
Two-sided P value: Pr { | Test Statistic − Mean |
                          .GE.  | Observed − Mean |     = 0.0260
    Two-sided P value: 2*one-sided                       = 0.0260
```

```
************************************
SAS PROGRAM FOR EXAMPLE 8.2.1
************************************
DATE TIME;
  INPUT GROUP Time @@;
  DATALINES;
  1 12.6 1 11.2 1 11.4 1 9.4 1 13.2 1 12
  2 16.4 2 15.4 2 14.1 2 14  2 13.4 2 11.3
  ;
  PROC NPAR1WAY DATA=TIME WILCOXON;
  CLASS GROUP;
  VAR TIME;
  EXACT WILCOXON;
  RUN;
```

```
************************************
SAS SOLUTION TO EXAMPLE 8.2.1
************************************
```

The NPARIWAY procedure

Wilcoxon scores (rank sums) for variable time
Classified by Variable Group

Group	*N*	Sum of Scores	Expected Under H0	Std.-dev. Under H0	Mean Score
1	6	25.0	39.0	6.244998	4.166667
2	6	53.0	39.0	6.244998	8.833333

Wilcoxon two-sample test

Statistic (S)	25.000
Normal approximation	
Z	−2.1617
One-sided Pr < *Z*	0.0153
Two-sided Pr > |*Z*|	0.0306
t approximation	
One-sided Pr < *Z*	0.0268
Two-sided-Pr > |*Z*|	0.0535
Exact test	
One-sided Pr <= *S*	0.0130
Two-sided Pr >= |*S* − Mean|	0.0260

Z includes a continuity correction of 0.05.

Kruskal–Wallis test	
Chi-square	5.0256
DF	1
Pr > chi-square	0.0250

Example 8.2.2

In order to compare the relative effectiveness of a calorie-controlled diet and a carbohydrate-controlled diet, eight obese men were divided randomly into two independent groups. Three were placed on a strictly supervised calorie-controlled diet and their total weight losses in 2 weeks were 1, 6, and 7 lb; the others, on a carbohydrate-controlled diet, lost 2, 4, 9, 10, and 12 lb. Find a confidence-interval estimate for the difference in location between calorie diet and carbohydrate diet, with confidence coefficient near 0.90.

Solution

The X sample must be the calorie diet so that $m = 3 \leq n = 5$. The example requests a confidence interval for $\mu_X - \mu_Y$. We will proceed by finding a confidence interval on $\mu_Y - \mu_X$ and then take the negative of each endpoint. Table J shows that for $m = 3$, $n = 5$, the closest we can get to confidence 0.90 is $P = 0.036$ or exact confidence level 0.928; this entry has rank 2 so that $u = 2$.

The $3(5) = 15$ differences $Y - X$ are shown below. The second smallest difference is -4 and the second largest difference is 9, or $-4 \leq \mu_Y - \mu_X \leq 9$; the corresponding negative interval is $-9 \leq \mu_X - \mu_Y \leq 4$. Notice that by listing the Y values in an array and then subtracting successively larger X values, the smallest and largest differences are easy to find.

Y	$Y-1$	$Y-6$	$Y-7$
2	1	−4	−5
4	3	−2	−3
9	8	3	2
10	9	4	3
12	11	6	5

The MINITAB and STATXACT solutions to this example are shown below. Note that the MINITAB output shows a confidence level 0.926, which is almost equal to the exact level, 0.928. The MINITAB confidence limits also do not match exactly with ours but are very close. The STATXACT solution matches exactly with ours, although the output does not seem to indicate the exact confidence level. It is interesting to note that for this example, with such small sample sizes, the exact and the asymptotic methods produced identical results. Note also that STATXACT calls this procedure the Hodges–Lehmann estimate of the shift parameter, as we noted before.

```
****************************************
 MINITAB SOLUTION TO EXAMPLE 8.2.2
****************************************

MTB > Mann-Whitney 90.0 C1 C2;

SUBC >  Alternative 0.
```

```
Mann-Whitney test and CI: C1, C2

C1          N =   3     Median =      6.000
C2          N =   5     Median =      9.000
Point estimate for ETA1-ETA2 is     −3.000

92.6% CI for ETA1-ETA2-is (−8.998, 3.997)
W= 10.0
Test of ETA = ETA2 vs ETA1 not = ETA2 is significant at 0.3711

Cannot reject at alpha = 0.05


              ****************************************
               STATXACT SOLUTION TO EXAMPLE 8.2.2
              ****************************************

HODGES-LEHMANN ESTIMATES OF SHIFT PARAMETER

POP_1 :         1             POP_2 :         2

Summary of WILCOXON-MANN-WHITNEY statistic for pop_1

Min      Max       Mean    Std.-dev.   Observed   Standardized

6.000   21.00    13.50      3.354       10.00        −1.043
Mann-Whitney statistic =        4.000
Point estimate of shift : Theta = POP_1 - POP_2 =     −3.000

90.00% confidence interval for theta :
           Asymptotic : (      −9.000,      4.000)
           Exact      : (      −9.000,      4.000)
```

8.3 Other Location Tests

Generally, almost any set of monotone-increasing weights a_i used in a linear rank statistic will provide a consistent test for shift in location. Only a few of the better-known ones will be covered here.

8.3.1 Terry–Hoeffding (Normal Scores) Test

The *Terry* (1952) and *Hoeffding* (1951) or the *Fisher–Yates normal scores test* uses the weights $a_i = E(\xi_{(i)})$, where $\xi_{(i)}$ is the ith-order statistic from a standard normal population; the linear rank test statistic is

$$c_1 = \sum_{i=1}^{N} E(\xi_{(i)}) Z_i \tag{8.3.1}$$

These expected values of standard normal order statistics are tabulated for $N \leq 100$ and some larger sample sizes in Harter (1961), so that the exact null distribution can be found by enumeration. Tables of the distribution of the test statistic are given in Terry (1952) and Klotz (1964). The Terry test statistic is symmetric about the origin, and its variance is

$$\sigma^2 = mn \frac{\sum_{i=1}^{N} [E(\xi_{(i)})]^2}{N(N-1)} \tag{8.3.2}$$

The normal distribution provides a good approximation to the null distribution for larger sample sizes. An approximation based on the t distribution is even closer. This statistic is $t = r(N-2)^{1/2}/(1-r^2)^{1/2}$, where $r = c_1/[\sigma^2(N-1)]^{1/2}$ and the distribution is approximately Student's t with $N-2$ degrees of freedom.

The Terry test is asymptotically optimal against the alternative that the populations are both normal distributions with the same variance but different means. Under the classical assumptions for a test of location then, its ARE is 1, relative to Student's t test. For certain other families of continuous distributions, the Terry test is more efficient than Student's t test (ARE > 1) (Chernoff and Savage, 1958).

The weights used for the Terry test $E(\xi_{(i)})$ are often called *expected normal scores*, since the order statistics of a sample from the standard normal population are commonly referred to as normal scores. The idea of using expected normal scores instead of integer ranks as rank-order statistics is appealing generally, since for many populations the expected normal scores may be more ''representative'' of the raw data or variate values. This could be investigated by comparing the correlation coefficients between (1) variate values and expected normal scores and (2) variate values and integer ranks for particular families of distributions, as discussed in Section 5.5. For example, the limiting value of the correlation between variate values from a normal population and the expected normal scores is equal to 1.

Since the Terry test statistic is the sum of the expected normal scores of the variables in the X sample, it may be interpreted as identical to the Wicoxon rank-sum test of the previous section when the normal-scores transformation is used instead of the integer-rank transformation. Other linear rank statistics for location can be formed in the same way by using different sets of rank-order statistics for the combined samples. An obvious possibility suggested by the Terry test is to use the scores $\Phi^{-1}[i/(N+1)]$ where $\Phi(x)$ is the cdf of the standard normal distribution, since we showed in Chapter 2 that $\Phi^{-1}[i/(N+1)]$ is a first approximation to $E(\xi_{(i)})$. If κ_p is pth quantile of the standard normal distribution, $\kappa_p = \Phi^{-1}(p)$. Therefore, here the ith-order statistic in the combined ordered sample is replaced by the $[i/(N+1)]$st quantile of the standard normal. This is usually called the *inverse-normal-scores transformation* and forms the basis of the following test.

8.3.2 van der Waerden Test

The *van der Waerden* (1952, 1953) X_N *test* uses the inverse normal scores as weights in forming a linear rank statistic as

$$X_N = \sum_{i=1}^{N} \Phi^{-1}\left(\frac{i}{N+1}\right) Z_i \tag{8.3.3}$$

In other words, the weight a_i is the value on the abscissa of the graph of a standard normal density function such that the area to the left of a_i is equal to $i/(N+1)$. These weights a_i are easily found from tables of the cumulative normal distribution or from a software package such as EXCEL or MINITAB. Tables of critical values are given in van der Waerden and Nievergelt (1956) for $N \leq 50$. The X_N statistic is symmetric about zero and has variance

$$\sigma^2 = mn \frac{\sum_{i=1}^{N} \left[\Phi^{-1}(i/(N+1))\right]^2}{N(N-1)} \tag{8.3.4}$$

For larger sample sizes, the null distribution of the standardized X_N is well approximated by the standard normal distribution.

The X_N test is perhaps easier to use than the Terry test because the weights are easily found for any N. Otherwise, there is little basis for a choice between them. In fact, the van der Waerden test is asymptotically equivalent to the Terry test. Since $\text{var}(\xi_{(i)}) \to 0$ as $N \to \infty, \xi_{(i)}$ converges in probability to $E(\xi_{(i)})$, the weights for the Terry test statistic. However, by the probability-integral transformation, $\Phi(\xi_{(i)})$ is the ith-order statistic of a sample of size N from the uniform distribution. Therefore, from (2.8.2) and (2.8.3), $E[\Phi(\xi_{(i)})] = i/(N+1)$ and

$$\text{var}[\Phi(\xi_{(i)})] = \frac{i(N-i+1)}{(N+1)^2(N+2)} \to 0$$

as $N \to \infty$. This implies that $\Phi(\xi_{(i)})$ converges in probability to $i/(N+1)$ and $\xi_{(i)}$ converges to $\Phi^{-1}[i/(N+1)]$. We may conclude that the expected normal scores and the corresponding inverse normal scores are identical for all N as $N \to \infty$. As a result, the large sample properties, including the ARE, are the same for the Terry and the van der Waerden tests.

It should be noted that the expected normal scores and inverse normal scores may be useful in any procedures based on rank-order statistics. For example, in the one-sample and paired-sample Wilcoxon signed-rank test discussed in Section 5.7, the rank of the absolute value of the difference $|D_i|$

could be replaced by the corresponding expected value of the absolute value of the normal score $E(|\xi_{(i)}|)$ (which is not equal to the absolute value of the expected normal score). The sum of those ''ranks'' that correspond to positive differences D_i is then used as a test statistic. This statistic provides the asymptotically optimum test of location when the population of differences is normal and thus has an ARE of 1 relative to Student's t test. Expected normal scores are also useful in rank-correlation methods, which will be covered in Section 11.5.

Example 8.3.2

We illustrate the Terry and van der Waerden tests using data from Example 8.2.1 on pouring times with $m = 6$ and $n = 6$. The first six expected normal scores for the Terry test with $N = 12$ are -1.6292, -1.1157, -0.7928, -0.5368, -0.3122, and -0.1026; the other six are the same values but with positive signs by symmetry. For example, the seventh expected normal score is 0.1026, the eighth is 0.3122, and so on. We calculate $c_1 = -3.5939$ from (8.3.1) with variance $\sigma^2 = 2.6857$ from (8.3.2). The z statistic for the normal approximation is $z = -2.1930$ with a one-sided P value $P(Z \leq -2.1930) = 0.0142$. For the van der Waerden test, the first six inverse normal scores with $N = 12$ are -1.4261, -1.0201, -0.7363, -0.5024, -0.2934, and -0.0966; the remaining six are the same values but with positive signs by symmetry. The test statistic is $X_N = -3.242$ from (8.3.3), with variance $\sigma^2 = 2.1624$ from (8.3.4). The z statistic for the normal approximation is $z = -2.2047$ with a one-tailed P value $P(Z \leq -2.2047) = 0.0137$.

Note that we do not use a continuity correction in calculating either of these two z statistics, because the weights a_i for both of these test statistics are continuous variables and not integers.

The SAS solution for the data in Example 8.2.1 using the van der Waerden test is shown below. Note that it agrees exactly with ours. STATXACT has an option called the normal scores test, but it uses the inverse normal scores as weights, as opposed to the expected normal scores. In other words, it calculates the van der Waerden statistic. This solution is also shown below. Note also that both SAS and STATXACT provide exact P values corresponding to the test statistic -3.242 and this one-tailed P value is identical to the one found earlier in Example 8.1.1 for the Wilcoxon rank-sum statistic.

```
**********************************
SAS SOLUTION TO EXAMPLE 8.3.2
**********************************
```

```
Program:

DATA TIME;
INPUT GROUP TIME @@;
DATALINES;
1 12.6 11.2 1 11.4 1 9.4 1 13.2 1 12
2 16.4 2 15.4 2 14.1 2 14 2 13.4 2 11.3
;
```

```
PROC NPAR1WAY DATA=TIME VW;
CLASS GROUP;
VAR TIME;
EXACT VW;
RUN;
```

Output

The NPAR1WAY procedure

Van der Waerden Scores (Normal) for Variable Time
Classified by Variable GROUP

Group	*N*	Sum of Scores	Expected Under H0	Std.-dev. Under H0	Mean Score
1	6	−3.241937	0.0	1.470476	−0.540323
2	6	3.241937	0.0	1.470476	0.540323

Van der Waerden two-sample test

Statistic (*S*)	−3.2419
Normal approximation	
Z	−2.2047
One-sided Pr < *Z*	0.0137
Two-sided Pr > |*Z*|	0.0275
Exact test	
One-sided Pr <= *S*	0.0130
Two-sided Pr >= |*S* − Mean|	0.0260

Van der Waerden one-way analysis

Chi-square	4.8606
DF	1
Pr > chi-square	0.0275

```
****************************************
 STATXACT SOLUTION TO EXAMPLE 8.2.3
****************************************
```

NORMAL SCORES TEST

[Sum of scores from population < 1 >]

Summary of exact distribution of NORMAL SCORES statistic:

Min	Max	Mean	Std.-dev.	Observed	Standardized
−4.075	4.075	−7.772e−016	1.470	−3.242	−2.205

```
Asymptotic Inference:
One-sided P value: Pr { Test Statistic .LE. Observed } = 0.0137
Two-sided P value: 2 * one-sided                       = 0.0275

Exact Inference:
One-sided P value: Pr {  Test Statistic .LE. Observed } = 0.0130
                   Pr {  Test statistic .EQ. Observed } = 0.0011
Two-sided P value: Pr { | Test Statistic − Mean |
                        .GE.  | Observed − Mean |    = 0.0260
Two-sided P value: 2 * one-sided                       = 0.0260
```

8.3.3 Percentile Modified Rank Tests

Another interesting linear rank statistic for the two-sample location problem is a member of the class of *percentile modified linear rank tests* (Gastwirth, 1965). The idea is to select two number s and r, both between 0 and 1, and then score only the data in the upper sth and lower rth percentiles of the combined sample.

A linear rank statistic is formed in the usual way except that a score of zero is assigned to a group of observations in the middle of the combined array. In symbols, we let $S = [Ns] + 1$ and $R = [Nr] + 1$, where $[x]$ denotes the largest integer not exceeding the number x. Define B_r and T_s as

N odd:

$$B_r = \sum_{i=1}^{R} [R - i + 1] Z_i$$

and

$$T_s = \sum_{i=N-S+1}^{N} [i - (N - S)] Z_i \tag{8.3.5}$$

N even:

$$B_r = \sum_{i=1}^{R} \left[R - i + \frac{1}{2} \right] Z_i$$

and

$$T_s = \sum_{i=N-S+1}^{N} \left[i - (N - S) - \frac{1}{2} \right] Z_i$$

The combination $T_s - B_r$ provides a test for location and $T_s + B_r$ is a test for scale, which will be discussed in Chapter 9. It is easily seen that if N is even and $S = R = N/2$, so that no observations are assigned a score of zero, $T_s - B_r$ is equivalent to the Wilcoxon test. When N is odd and all the sample data are used, the tests differ slightly because of the different way of handing the middle observation $z_{(N+1)/2}$.

The mean and variance of the $T_s \pm B_r$ statistics can be calculated using Theorem 7.3.2 alone if $S + R \leq N$, remembering that $a_i = 0$ for $R + 1 \leq i \leq N - S$. Alternatively, Theorems 7.3.2 and 7.3.3 can be used on the pieces T_s and B_r along with the fact that

$$\text{var}(T_s \pm B_r) = \text{var}(T_s) + \text{var}(B_r) \pm 2\text{cov}(T_s, B_r)$$

The results for N even and $S = R$ are

$$E(T_s - B_r) = 0 \quad \text{var}(T_s - B_r) = \frac{mnS(4S^2 - 1)}{6N(N - 1)} \tag{8.3.6}$$

By Theorem 7.3.4, the null distribution of $T_s - B_r$ is symmetric about the origin for any m and n when $S = R$. Tables of the null distribution for $m = n \leq 6$ are given in Gibbons and Gastwirth (1966). It is also shown there empirically that for significance levels not too small, say at least 0.025, the normal distribution may be used to define critical regions with sufficient accuracy for most practical purposes when $m = n \geq 6$.

One of the main advantages of this test is that a judicious choice of s and r may lead to a test that attains larger power than the Wilcoxon rank-sum test without having to introduce complicated scoring systems. For example, any knowledge of asymmetry in the populations might be incorporated into the test statistic. The asymptotic relative efficiency of this test against normal alternatives reaches its maximum value of 0.968 when $s = r = 0.42$; when $s = r = 0.5$, the ARE is 0.955, as for the Wilcoxon rank-sum statistic.

8.4 Summary

In this chapter, we covered several additional tests for the two-sample problem; all of them are linear rank tests. The two-sample tests in Chapter 6 are appropriate for general alternatives that do not specify any particular kind of difference between the population distributions. The tests in this chapter are especially appropriate for the location alternative.

The Wilcoxon rank-sum test of Section 8.2 is by far the best known two-sample nonparametric test for location; it is equivalent to the Mann–Whitney

U test covered in Section 6.6. The discussions of power and sample size determination given there apply equally here. Other tests for location covered in this chapter are the Terry–Hoeffding expected normal scores test and the van der Waerden inverse normal scores test. These two tests are asymptotically equivalent and their asymptotic relative efficiency is one relative to the normal theory test for normal distributions. Thus, their ARE is somewhat larger than that of the Wilcoxon test for normal distributions, but they can have smaller power for other distributions. These other tests are not as convenient to use as the Wilcoxon test. Further, they do not have a convenient procedure for finding a corresponding confidence interval estimate for the difference in the medians. Finally, we cover the percentile modified rank tests for location, which are simply generalizations of the Wilcoxon rank-sum test.

Lehmann (2009) compares the Wilcoxon (Mann–Whitney) test, normal scores tests (Hoeffding or van der Waerden), permutation tests, and Student's t test on the basis of their ARE's for the uniform, normal, logistic, double exponential, and Cauchy distributions. He recommends the nonparametric tests be used in preference to Student's t test unless one is *firmly* convinced that the normal distribution applies, which is rarely the case. He notes that the simplicity and ease of interpretation of the Wilcoxon-type tests should be a strong factor in choosing an appropriate inference procedure.

Problems

8.1 Given independent samples of m X and n Y variables, define the following random variables for $i = 1, 2, \ldots, m$:

$K_i =$ rank of X_i among $X_1, X_2, \ldots, X_m$
$R_i =$ rank of X_i among $X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_n$

Use K_i and R_i to prove the linear relationship between the Mann–Whitney and Wilcoxon rank-sum statistics given in (8.2.4).

8.2 A single random sample $D_1, D_2, \ldots, D_N$ of size N is drawn from a population which is continuous and symmetric. Assume there are m positive values, n negative values, and no zero values. Define the $m + n = N$ random variables

$$X_i = D_i \quad \text{if } D_i > 0$$
$$Y_i = |D_i| \quad \text{if } D_i < 0$$

Then the $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ constitute two independent random samples of sizes m and n.

(a) Show that the two-sample Wilcoxon rank-sum statistic W_N of (8.2.1) for these two samples equals the Wilcoxon signed-rank statistic T^+ defined in (5.7.1).

(b) If these two samples are from identical populations, the median of the symmetric D population must be zero. Therefore the null distribution of W_N is identical to the null distribution of T^+ conditional upon the observed number of plus and minus signs. Explain fully how tables of the null distribution of W_N could be used to find the null distribution of T^+. Since for N large, m and n will both converge to the constant value $N/2$ in the null case, these two test statistics have equivalent asymptotic properties.

8.3 Generate by enumeration the exact null probability distribution of $T_s - B_r$ as defined in (8.3.5) for $m = n = 3$, all $S = R < 3$, and compare the rejection regions for $\alpha \leq 0.10$ with those for the Wilcoxon rank-sum test W_N when $m = n = 3$.

8.4 Verify the results given in (8.3.6) for the mean and variance of $T_s - B_r$ when $S = R$ and N is even and derive a similar result for $S = R$ when N is odd.

8.5 Show that the median test of Section 6.4 is a linear rank test.

8.6 Giambra and Quilter (1989) performed a study to investigate gender and age difference in ability to sustain attention when given Mackworth's clock-test. This clock is metal with a plain white face and a black pointer that moves around the face in 100 discrete steps of 36° each. During the test period, the pointer made 23 double jumps, defined as moving twice the normal distance or 7.2° in the same time period, at random and irregular intervals. Subjects were told that double jumps would occur and asked to signal their recognition of occurrence by pressing a button. Scores were the number of correct recognitions of the double jumps. The scores below are for 10 men aged 18–29 and 10 men aged 50–59. Determine whether median number of correct scores is larger for young men than for older men.

Age 18–29: 11, 13, 15, 15, 17, 19, 20, 21, 21, 22
Age 50–59: 8, 9, 10, 11, 12, 13, 5, 17, 19, 23

8.7 Elam (1988) conducted a double-blind study of 18 adult males to investigate the effects of physical resistance exercises and amino acid dietary supplements on body mass, body fat, and composite girth. Ten of the subjects received the diet supplement and eight received a placebo. All subjects participated in 15 resistance exercise workouts of 1 hour each spread over a 5 week period. Workloads were tailored to abilities of the individual subjects but escalated in intensity over the period. The data are the changes (after minus before) in body mass, body fat, and composite body girth for the amino acid (treatment) group and placebo

(control) groups of subjects. Were amino acid supplements effective in reducing the body mass (kg), fat (%), and girth (cm)?

Treatment Group				Control Group			
Subject	**Mass**	**Fat**	**Girth**	**Subject**	**Mass**	**Fat**	**Girth**
1	−2.00	1.14	−17.00	1	1.00	−0.56	11.00
2	0.00	−2.64	2.00	2	0.50	0.87	5.00
3	−1.00	−1.96	23.00	3	−0.75	−0.75	1.00
4	−4.00	0.86	13.00	4	−2.00	−0.60	35.00
5	−0.75	−2.35	2.00	5	−3.00	0.00	−5.00
6	−1.75	−2.51	5.00	6	−2.50	−2.54	2.00
7	−2.75	0.55	8.00	7	0.00	−3.10	3.00
8	0.00	3.40	3.00	8	0.25	3.48	−7.00
9	−1.75	0.00	7.00				
10	1.00	−4.94	10.00				

8.8 Howard et al. (1986) (see Problem 5.12) also investigated whether pretest anxiety scores differed for students enrolled in two different sections of the introduction to computer courses. Seven students were enrolled in each section, and the data are shown below. Is there a difference in median scores?

Section 1: 20, 32, 22, 21, 27, 26, 38
Section 2: 34, 20, 30, 28, 25, 23, 29

8.9 A travel agency wanted to compare the noncorporate prices charged by two major motel chains for a standard-quality single room at major airport locations around the country. A random sample of five Best Eastern motels and an independent sample of six Travelers' Inn motels, all located at major airports, gave the approximate current total costs of a standard single room as shown below. Find a 95% confidence interval estimate of the difference between median costs at Best Eastern and Travelers' Inn motels.

Best Eastern: $68, 75, 92, 79, 95
Travelers' Inn: $69, 76, 81, 72, 75, 80

8.10 Smokers are commonly thought of as nervous people whose emotionality is at least partly caused by the stimulating effect tobacco has on the nervous system. Nesbitt (1972) conducted a study with 300 college students and concluded that smokers are less emotional than nonsmokers, that smokers are better able to tolerate the physiological effects of anxiety, and that, over time, smokers become less emotional than nonsmokers. Subjects of both genders were drawn from three different colleges and classified as smokers if they smoked any number of cigarettes on a regular basis. In one aspect of the experiment, all subjects

were given the Activity Preference Questionnaire (APQ), a test designed to measure the emotionality of the subjects. The APQ is scored using an ordinal scale of 0–33, with lower scores indicating less emotionality, that is, greater sociopathy. The mean overall scores were 18.0 for smokers and 20.3 for nonsmokers. Suppose this experiment is repeated using a group of 8 randomly chosen smokers and 10 randomly chosen nonsmokers. Do these data support the same conclusion concerning emotionality as Dr. Nesbitt's data?

Smokers: 16, 18, 21, 14, 25, 24, 27, 12
Nonsmokers: 17, 15, 28, 31, 30, 26, 27, 20, 21, 19

8.11 A group of 20 mice are allocated to individual cages randomly. The cages are assigned in equal numbers, randomly, to two treatments, a control A and a certain drug B. All animals, in a random sequence, are infected with tuberculosis. The numbers of days until the mice die after infection are given as follows (one mouse in A got lost):

Control A: 5, 6, 7, 7, 8, 8, 8, 9, 12
Drug B: 7, 8, 8, 8, 9, 9, 12, 13, 14, 17

Since a preliminary experiment has established that the drug is not toxic, we can assume that the drug group cannot be worse (die sooner) than the control group under any reasonable conditions. Test the null hypothesis that the drug is without effect at a significance level of 0.05 and briefly justify your choice of test.

8.12 The following data represent two independent random samples drawn from continuous populations that are thought to have the same form but possibly different locations.

X: 79, 13, 138, 129, 59, 76, 75, 53
Y: 96, 141, 133, 107, 102, 129, 110, 104

Using a significance level not exceeding 0.10, test
(a) The null hypothesis that the two populations are identical and find the P value. (Do not use an approximate test.)
(b) The null hypothesis that the locations are the same and find the appropriate one-tailed P value.

8.13 A problem of considerable import to the small-scale farmer who purchases young pigs to fatten and sell for slaughter is whether there is any difference in weight gain for male and female pigs when the two genders are subjected to identical feeding treatments. If there is a difference, the farmer can optimize production by buying only one gender of pigs for fattening. As a public service, an agricultural experiment station decides to run a controlled experiment to determine whether gender is an important factor in weight gain. They placed eight young male pigs in one pen and eight young females in another pen and gave each pen identical feeding treatments for a fixed period of

time. The initial weights were all between 35 and 50 1b, and the amounts of weight gain in pounds for the two genders are recorded below. One of the female pigs died so there are only seven observations in that group. Analyze the data below using both a test and a confidence-interval approach with confidence coefficient near 0.90.

Female pigs: 9.31, 9.57, 10.21, 8.86, 8.52, 10.53, 9.21
Male pigs: 9.14, 9.98, 8.46, 8.93, 10.14, 10.17, 11.04, 9.43

8.14 How would you find the confidence-interval endpoints for the parameter of interest when the interval has confidence level nearest 0.90 and corresponds to

(a) The sign test with $n = 11$
(b) The Wilcoxon signed-rank test with $n = 11$
(c) The Wilcoxon rank-sum test with $m = 5$, $n = 6$

In each case define the function Z of the observations, give the numerical values of L and U for the order statistics $Z(L)$ and $Z(U)$, and give the exact confidence level.

8.15 A self-concept test was given to a random sample consisting of six normal subjects and three subjects under psychiatric care. Higher scores indicate more self-esteem. The data are as follows:

Normal: 62, 68, 78, 92, 53, 81
Psychiatric: 54, 70, 79

(a) Find a P value relevant to the alternative that psychiatric patients have lower self-esteem than normal patients.
(b) Find a confidence interval for the difference of the locations (level nearest 0.90).

8.16 Verify the confidence interval estimate of $\mu_X - \mu_Y$ with exact confidence coefficient at least 0.95 given in the MINITAB solution of Example 8.2.1.

9

Linear Rank Tests for the Scale Problem

9.1 Introduction

Consider again the null hypothesis that two independent samples are drawn from identical populations, but now suppose that we are interested in detecting differences in variability or dispersion. Some of the tests presented in Chapters 6 and 8, namely, the median, Mann–Whitney, Wilcoxon rank-sum, Terry, van der Waerden, and percentile-modified rank tests, were noted to be particularly sensitive to differences in location when the populations are identical otherwise, a situation described by the relation $F_Y(x) = F_X(x - \theta)$. These tests cannot be expected to perform especially well against other alternatives. The general two-sample tests of Chapter 6, like the Wald–Wolfowitz runs test or Kolmogorov–Smirnov tests, are affected by any type of difference in the populations and therefore cannot be expected to be very efficient for detecting differences in variability. Some other nonparametric tests are needed for the dispersion problem.

The classical test for which we are seeking an analog is the test for equality of variances, $H_0: \sigma_X^2 = \sigma_Y^2$, against one- or two-sided alternatives. If it is reasonable to assume that the two populations are both normal distributions with unknown means, the parametric test statistic is

$$F_{m-1,n-1} = \frac{\sum_{i=1}^{m} (X_1 - \overline{X})^2/(m-1)}{\sum_{i=1}^{n} (Y_i - \overline{Y})^2/(n-1)}$$

which has Snedecor's F distribution with $m-1$ and $n-1$ degrees of freedom. The F test is not particularly robust with respect to the normality assumption. If there is reason to question the assumptions inherent in the construction of the test, a nonparametric test of dispersion is appropriate.

The F test does not require any assumption regarding the locations of the two normal populations. The magnitudes of the two sample variances are directly comparable since they are each computed as measures of deviations around the respective sample means. The traditional concept of dispersion is a measure of spread around some population central value. The model for

the relationship between the two normal populations assumed for the F test might be written

$$F_{Y-\mu_Y}(x) = F_{X-\mu_X}\left(\frac{\sigma_X}{\sigma_Y}x\right) = F_{X-\mu_X}(\theta x) \quad \text{for all } x \text{ and some } \theta > 0 \qquad (9.1.1)$$

where $\theta = \sigma_X/\sigma_Y$ and $F_{(X-\mu_X)/\sigma_X}(x) = \Phi(x)$, and the null hypothesis to be tested is $H_0{:}\theta = 1$. We could say then that we assume that the distributions of $X - \mu_X$ and $Y - \mu_Y$ differ only by the scale factor θ for any μ_X and μ_Y, which need not be specified. The relationship between the respective moments is

$$E(X - \mu_X) = \theta E(Y - \mu_Y) \quad \text{and} \quad \text{var}(X) = \theta^2 \text{ var}(Y)$$

Since medians are the customary location parameters in distribution-free procedures, we might define nonparametric dispersion as spread around the respective medians. Then the nonparametric model corresponding to (9.1.1) is

$$F_{Y-M_Y}(x) = F_{X-M_X}(\theta x) \quad \text{for all } x \text{ and some } \theta > 0 \qquad (9.1.2)$$

Suppose that the test criterion for this model is to be based on the configuration of the X and Y random variables in the combined ordered sample, as in a linear rank test. The characteristics of respective locations and dispersions are inextricably mixed in the combined sample ordering, and possible location differences may mask dispersion differences. If the population medians are known, the model (9.1.2) suggests that the sample observations should be adjusted by

$$X_i' = X_i - M_X \quad \text{and} \quad Y_j' = Y_j - M_Y \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Then the X_i' and Y_j' populations both have zero medians, and the arrangements of X' and Y' random variables in the combined ordered sample should indicate dispersion differences as unaffected by location differences. The model is then H_S: $F_Y(x) = F_X(\theta x)$. In practice, M_X and M_Y would probably not be known, so that this is not a workable approach. If we simply assume $M_X = M_Y = M$ as unspecified, the combined sample arrangement of the unadjusted X and Y should still reflect dispersion differences. Since the X and Y populations differ only in scale, the logical model for this situation would seem to be the alternative

$$H_S{:}\ F_Y(x) = F_X(\theta x) \quad \text{for all } x \text{ and some } \theta > 0, \theta \neq 1 \qquad (9.1.3)$$

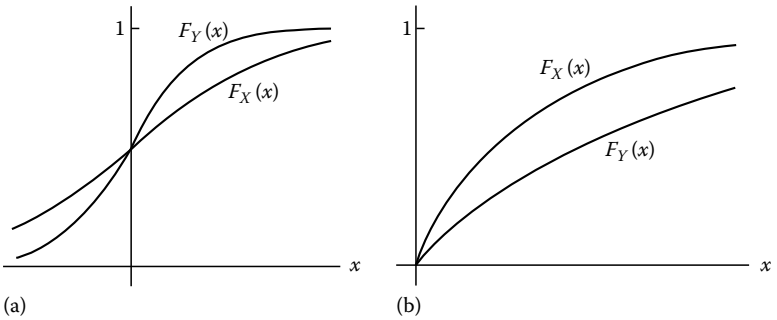


FIGURE 9.1.1
H_S: $F_Y(x) = F_X(\theta x)$. (a) F_X normal, $\theta > 1$. (b) F_X exponential, $\theta < 1$.

This is appropriately called the *scale alternative* because the cdf of the Y population is the same as that of the X population but with a compressed or enlarged scale according as $\theta > 1$ or $\theta < 1$, respectively.

In Figure 9.1.1a, the relation $H_S : F_Y(x) = F_X(\theta x)$ is shown for $F_X(x) = \Phi(x)$, the standard normal, and $\theta > 1$. Since $\mu_X = M_X = M_Y = \mu_Y = 0$ and $\theta = \sigma_X/\sigma_Y$, this model is a special case of (9.1.1) and (9.1.2).

Figure 9.1.1b illustrates the difficulty in thinking any arbitrary distribution may be taken for the scale alternative in (9.1.3) to be interpreted exclusively as a dispersion alternative. Here, we have a representation of the exponential distribution in H_S for $\theta < 1$, for example, $f_X(x) = e^{-x}, x > 0$, so that $f_Y(x) = \theta e^{-\theta x}, x > 0$, for some $\theta < 1$. Since $\text{var}(X) = 1$ and $\text{var}(Y) = 1/\theta^2$, it is true that $\sigma_Y > \sigma_X$. However, $E(X) = 1$ and $E(Y) = 1/\theta > E(X)$, and further $M_X = \ln 2$ while $M_Y = \ln 2/\theta > M_X$ for all $\theta < 1$. The combined ordered arrangement of samples from these exponential populations will be reflective of both the location and dispersion differences. The scale alternative in (9.1.3) can be interpreted as a dispersion alternative only if the population locations are equal or very close to being equal.

Actually, the scale model H_S: $F_Y(x) = F_X(\theta x)$ is not general enough even when the locations are the same. This relationship implies that $E(X) = \theta E(Y)$ and $M_X = \theta M_Y$, so that the locations are identical for all θ only if $\mu_X = \mu_Y = 0$ or $M_X = M_Y = 0$. A more general scale alternative can be written in the form

$$H_S : F_{Y-M}(x) = F_{X-M}(\theta x) \quad \text{for all } x \text{ and some } \theta > 0, \theta \neq 1 \tag{9.1.4}$$

where M is interpreted to be the common median. Both (9.1.3) and (9.1.4) are called the scale alternatives applicable to the two-sample scale problem, but in (9.1.3), we essentially assume without loss of generality that $M = 0$.

Many tests based on the ranks of the observations in a combined ordering of the two samples have been proposed for the scale problem. If they are to be useful for detecting differences in dispersion, we must assume either that the medians (or means) of the two populations are equal but unknown or that the sample observations can be adjusted to have equal locations by subtracting

the respective location parameters from one set. Under these assumptions, an appropriate set of weights for a linear rank-test statistic will provide information about the relative spread of the observations about their common central value. If the X population has a larger dispersion, the X values should be positioned approximately symmetrically at both extremes of the Y values. Therefore, the weights a_i should be symmetric, for example, small weights in the middle and large at the two extremes, or vice versa. We will consider several choices for simple sets of weights of this type that provide linear rank tests particularly sensitive to scale differences only. These are the Mood test, Freund–Ansari–Bradley–David–Barton tests, Siegel–Tukey test, Klotz normal-scores test, percentile modified rank tests, and Sukhatme test. Many other tests have also been proposed in the literature; some of these are covered in Section 9.9. Duran (1976) gives a survey of nonparametric scale tests. Procedures for finding confidence-interval estimates of relative scale are covered in Section 9.8. Examples and applications are given in Section 9.10.

9.2 The Mood Test

In the combined ordered sample of N variables with no ties, the average rank is the mean of the first N integers, $(N+1)/2$. The deviation of the rank of the ith ordered variable about its mean rank is $i-(N+1)/2$, and the amount of deviation is an indication of relative spread. However, as in the case of defining a measure of sample dispersion in classical descriptive statistics, the fact that the deviations are equally divided between positive and negative numbers presents a problem in using these actual deviations as weights in constructing a linear rank statistics. For example, if Z_i is the usual indicator variable for the X observations and $m=n=3$, the ordered arrangements

$$XYXYXY \quad \text{and} \quad XXYYYX$$

both have

$$\sum_{i=1}^{6}\left(i-\frac{N+1}{2}\right)Z_i=-1.5,$$

but the first arrangement suggests the variances are equal and the second suggests the X's are more dispersed than the Y's. The natural solution is to use either the absolute values or the squared values of the deviations to give equal weight to deviations on either side of the central value.

The Mood (1954) *test* is based on the sum of squares of the deviations of the X ranks from the average combined rank, or

$$M_N=\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^2 Z_i \tag{9.2.1}$$

A large value of M_N would imply that the X's are more widely dispersed, and a small M_N implies the opposite conclusion. Specifically, the weights are shown in Tables 9.2.1 and 9.2.2 for N even and N odd, respectively. The larger weights are in the tails of the arrangement. When N is odd, the median of the combined sample is assigned a weight of zero. In that case, therefore, the middle observation is essentially ignored, but this is necessary to achieve perfectly symmetric weights.

The moments of M_N under the null hypothesis are easily found from Theorem 7.3.2 as follows:

$$\begin{aligned} NE(M_N) &= m\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^2 \\ &= m\left[\sum i^2-(N+1)\sum i+\frac{N(N+1)^2}{4}\right] \\ &= m\left[\frac{N(N+1)(2N+1)}{6}-\frac{N(N+1)^2}{2}+\frac{N(N+1)^2}{4}\right] \end{aligned}$$

Then, $12NE(M_N) = mN(N+1)(N-1)$ and

$$E(M_N) = \frac{m(N^2-1)}{12} \tag{9.2.2}$$

TABLE 9.2.1

Mood Test Weights for N Even

i	1	2	3	$\cdots$	$\frac{N}{2}-1$	$\frac{N}{2}$
a_i	$\left(\frac{N-1}{2}\right)^2$	$\left(\frac{N-3}{2}\right)^2$	$\left(\frac{N-5}{2}\right)^2$	$\cdots$	$\left(\frac{3}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$
i	$\frac{N}{2}+1$	$\frac{N}{2}+2$	$\cdots$	$N-2$	$N-1$	N
a_i	$\left(\frac{1}{2}\right)^2$	$\left(\frac{3}{2}\right)^2$	$\cdots$	$\left(\frac{N-5}{2}\right)^2$	$\left(\frac{N-3}{2}\right)^2$	$\left(\frac{N-1}{2}\right)^2$

TABLE 9.2.2

Mood Test Weights for N Odd

i	1	2	3	$\cdots$	$\frac{N-1}{2}$	$\frac{N+1}{2}$
a_i	$\left(\frac{N-1}{2}\right)^2$	$\left(\frac{N-3}{2}\right)^2$	$\left(\frac{N-5}{2}\right)^2$	$\cdots$	$(1)^2$	0
i	$\frac{N+3}{2}$	$\cdots$	$N-2$	$N-1$	N	
a_i	$(1)^2$	$\cdots$	$\left(\frac{N-5}{2}\right)^2$	$\left(\frac{N-3}{2}\right)^2$	$\left(\frac{N-1}{2}\right)^2$	

Furthermore,

$$
\begin{aligned}
&N^2(N-1)\operatorname{var}(M_N) \\
&\quad = mn\left\{N\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^4-\left[\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^2\right]^2\right\} \\
&\quad = mn\left\{N\left[\sum i^4-4\frac{N+1}{2}\sum i^3+6\frac{(N+1)^2}{4}\sum i^2\right.\right. \\
&\qquad \left.\left.-4\frac{(N+1)^3}{8}\sum i+\frac{N(N+1)^4}{16}\right]-\left[\frac{N(N^2-1)}{2}\right]^2\right\}
\end{aligned}
$$

Using the following relations, which can be easily proved by induction,

$$
\sum_{i=1}^{N} i^3 = \left[\frac{N(N+1)}{2}\right]^2
$$

$$
\sum_{i=1}^{N} i^4 = \frac{N(N+1)(2N+1)(3N^2+3N-1)}{180}
$$

and simplifying, the desired result is

$$
\operatorname{var}(M_N) = \frac{mn(N+1)(N^2-4)}{180} \tag{9.2.3}
$$

The exact null probability distribution of M_N can be derived by enumeration in small samples. The labor is somewhat reduced by noting that since $a_i = a_{N-i+1}$, the properties of Theorem 7.3.7 apply. From Theorem 7.3.5 the distribution is symmetric about $N(N^2-1)/24$ when $m=n$, but the symmetry does not hold for unequal sample sizes. Exact critical values are tabled in Laubscher et al. (1968). For larger sample sizes, the normal approximation can be used with the moments in (9.2.2) and (9.2.3). Under the assumption of normal populations differing only in variance, the asymptotic relative efficiency of the Mood test relative to the F test is $15/2\pi^2 = 0.76$.

9.3 The Freund–Ansari–Bradley–David–Barton Tests

In the Mood test of the last section, the deviation of each rank from its average rank was squared to eliminate the problem of positive and negative deviations balancing out. If the absolute values of these deviations are

used instead to give equal weight to positive and negative deviations, the linear rank statistic is

$$A_N = \sum_{i=1}^{N} \left| i - \frac{N+1}{2} \right| Z_i = (N+1) \sum_{i=1}^{N} \left| \frac{i}{N+1} - \frac{1}{2} \right| Z_i \tag{9.3.1}$$

There are several variations of this test statistic in the literature, proposed mainly by Freund and Ansari (1957), Ansari and Bradley (1960), and David and Barton (1958). There seems to be some confusion over which test should be attributed to whom, but they are all essentially equivalent.

The *Freund–Ansari–Bradley test* can be written as a linear rank statistic in the form

$$F_N = \sum_{i=1}^{N} \left(\frac{N+1}{2} - \left| i - \frac{N+1}{2} \right| \right) Z_i = \frac{m(N+1)}{2} - A_N \tag{9.3.2}$$

or

$$F_N = \sum_{i=1}^{[(N+1)/2]} iZ_i + \sum_{i=[(N+1)/2]+1}^{N} (N - i + 1)Z_i \tag{9.3.3}$$

where $[x]$ denotes the largest integer not exceeding the value of x. Specifically, the weights assigned then are 1 to both the smallest and largest observations in the combined sample, 2 to the next smallest and next largest, etc., $N/2$ to the two middle observations if N is even, and $(N+1)/2$ to the one middle observation if N is odd. Since the smaller weights are at the two extremes here, which is the reverse of the assignment for the Mood statistic, a small value of F_N would suggest that the X population has larger dispersion. The appropriate rejection regions for the scale alternative

$$H_S: F_{Y-M}(x) = F_{X-M}(\theta x) \quad \text{for all } x \text{ and some } \theta > 0, \theta \neq 1$$

are then

Subclass of Alternatives	Rejection Region	*P* Value
$\theta > 1$	$F_N \leq k_1$	$P(F_N \leq f_0 \mid H_0)$
$\theta < 1$	$F_N \geq k_2$	$P(F_N \geq f_0 \mid H_0)$
$\theta \neq 1$	$F_N \leq k_3$ or $F_N \geq k_4$	2 (smaller of above)

where f_0 is the observed value of F_N. The fact that this test is consistent for these subclasses of alternatives will be shown in Section 9.7.

To determine the critical values for rejection, the exact null distribution of F_N can be found by enumeration. From Theorem 7.3.6, we note that the null distribution of F_N is symmetric about its mean if N is even. A recursion

relation may be used to generate the null distribution systematically. For a sequence of $m+n=N$ letters occurring in a particular order, let $r_{m,n}(f)$ denote the number of distinguishable arrangements of m X and n Y letters such that the value of F_N is the number f, and let $p_{m,n}(f)$ denote the corresponding probability. A sequence of N letters is formed by adding a letter to each sequence of $N-1$ letters. If $N-1$ is even (N odd), the extra score will be $(N+1)/2$, so that f will be increased by $(N+1)/2$ if the new letter is X and be unchanged if Y. If $N-1$ is odd, the extra score will be $N/2$. Therefore, we have the relations

$$N \text{ odd:} \quad r_{m,n}(f) = r_{m-1,n}\left(f - \frac{N+1}{2}\right) + r_{m,n-1}(f)$$

$$N \text{ even:} \quad r_{m,n}(f) = r_{m-1,n}\left(f - \frac{N}{2}\right) + r_{m,n-1}(f)$$

These can be combined in the single recurrence relation

$$r_{m,n}(f) = r_{m-1,n}(f-k) + r_{m,n-1}(f) \quad \text{for } k = \left[\frac{N+1}{2}\right]$$

In terms of the probabilities, the result is

$$p_{m,n}(f) = \frac{r_{m,n}(f)}{\binom{m+n}{m}}$$

$$(m+n)p_{m,n}(f) = mp_{m-1,n}(f-k) + np_{m,n-1}(f)$$

which is the same form as (6.6.14) and (8.2.2) for the Mann–Whitney and Wilcoxon rank-sum tests, respectively. Tables of the null probability distributions for $N \leq 20$ are available in Ansari and Bradley (1960).

For larger sample sizes, the normal approximation to the distribution of F_N can be used. The exact mean and variance are easily found by applying the result of Theorem 7.3.2 to F_N in the forms of (9.3.3) and (9.3.2) as follows, where $x=(N+1)/2$.

$$NE(F_N) = m\left[\sum_{i=1}^{[x]} i + \sum_{i=[x]+1}^{N} (N-i+1)\right] = m\left[\sum_{i=1}^{[x]} i + \sum_{j=1}^{N-[x]} j\right]$$

$$N \text{ even:} \quad E(F_N) = 2m\sum_{i=1}^{N/2} \frac{i}{N} = \frac{m(N+2)}{4}$$

$$N \text{ odd:} \quad E(F_N) = \frac{m\left[2\sum_{i=1}^{(N-1)/2} i + \frac{N+1}{2}\right]}{N} = \frac{m(N+1)^2}{4N}$$

$$\begin{aligned}
\operatorname{var}(F_N) &= \operatorname{var}(A_N) \\
&= \frac{mn}{N^2(N-1)}\left[N\sum_{i=1}^{N}\left(i-\frac{N+1}{2}\right)^2-\left(\sum_{i=1}^{N}\left|i-\frac{N+1}{2}\right|\right)^2\right] \\
&= \frac{mn}{N^2(N-1)}\left\{\frac{N^2(N^2-1)}{12}-\left[\frac{N}{m}E(A_N)\right]^2\right\} \\
&= \frac{mn}{N^2(N-1)}\left\{\frac{N^2(N^2-1)}{12}-\left[\frac{N(N+1)}{2}-\frac{N}{m}E(F_N)\right]^2\right\}
\end{aligned}$$

$$N \text{ even:} \quad \operatorname{var}(F_N) = \frac{mn}{N^2(N-1)}\left[\frac{N^2(N^2-1)}{12}-\left(\frac{N^2}{4}\right)^2\right]$$

$$= \frac{mn(N^2-4)}{48(N-1)}$$

$$N \text{ odd:} \quad \operatorname{var}(F_N) = \frac{mn}{N^2(N-1)}\left[\frac{N^2(N^2-1)}{12}-\left(\frac{N^2-1}{4}\right)^2\right]$$

$$= \frac{mn(N+1)(N^2+3)}{48N^2}$$

Collecting these results, we have

N Even	N Odd
$E(F_N) = m(N+2)/4$	$E(F_N) = m(N+1)^2/4N$
$\operatorname{var}(F_N) = \dfrac{mn(N^2-4)}{48(N-1)}$	$(N+1)\operatorname{var}(F_N) = \dfrac{mn(n+1)(N^2+3)}{48N^2}$

(9.3.4)

Another test which is almost identical is generally attributed to David and Barton (1958). This test also assigns symmetric integer weights but in the reverse order. That is, scores are given starting from the middle with 1 for N even, and 0 for N odd, and going out in both directions. The *David–Barton test* can be written as a linear rank statistic as

$$B_N = \sum_{i=1}^{[(N+1)/2]}\left(\left[\frac{N+2}{2}\right]-i\right)z_i + \sum_{i=[(N+1)/2]+1}^{N}\left(i-\left[\frac{N+1}{2}\right]\right)Z_i \qquad (9.3.5)$$

For N even, B_N and F_N have the exact same set of weights (but rearranged), and therefore, the means and variances are equal. But for N odd, this is not true because of the difference in relative assignment of the one ''odd'' weight,

that is, the middle observation. B_N assigns a weight of 0 to this observation, while F_N scores it as $(N+1)/2$. The following results are easily verified from Theorem 7.3.2:

N Even	N Odd
$E(B_N) = m(N+2)/4$	$E(B_N) = \dfrac{m(N^2-1)}{4N}$
$\text{var}(B_N) = \dfrac{mn(N^2-4)}{48(N-1)}$	$\text{var}(B_N) = \dfrac{mn(N+1)(N^2+3)}{48N^2}$

(9.3.6)

The exact relationship between B_N and F_N is

$$F_N + B_N = m\left[\frac{(N+2)}{2}\right] \tag{9.3.7}$$

Since this relation is linear, the tests have equivalent properties.

Tables of the null distribution of B_N are given in David and Barton (1958) for $m = n \leq 8$.

Since these three tests, F_N, B_N, and A_N, are all linearly related, they all have equivalent properties. All are consistent against the same alternatives. The asymptotic relative efficiency of each relative to the F test is $6/\pi^2 = 0.608$ for normal populations differing only in scale.

9.4 The Siegel–Tukey Test

Even simpler than the use of positive integer weights symmetric about the middle would be some arrangement of the first N positive integers. Since these are the weights used in the Wilcoxon rank-sum test W_N for location, tables of the probability distribution are then the same. Siegel and Tukey (1960) proposed a statistic that is sensitive to differences in scale using a rearrangement of the first N positive integers as weights. The weights for N even are

I	1	2	3	4	5	...	$N/2$[a]	...	$N-4$	$N-3$	$N-2$	$N-1$	N
a_i	1	4	5	8	9	...	N	...	10	7	6	3	2

[a] If $N/2$ is odd, $i = (N/2)+1$ here.

If N is odd, the middle observation in the array is thrown out and the same weights are used for the reduced N. This rearrangement achieves the desired symmetry in terms of sums of pairs of adjacent weights, although the weights themselves are not exactly symmetric. Since the weights are smaller

at the extremes, we should reject the null hypothesis in favor of an alternative that the X's have the greater variability when the test statistic is small.

The *Siegel–Tukey test statistic* is

$$S_N = \sum_{i=1}^{N} a_i Z_i$$

where

$$a_i = \begin{cases} 2i & \text{for } i \text{ even, } 1 < i \le N/2 \\ 2i-1 & \text{for } i \text{ odd, } 1 \le i \le N/2 \\ 2(N-i)+2 & \text{for } i \text{ even, } N/2 < i \le N \\ 2(N-i)+1 & \text{for } i \text{ odd, } N/2 < i \le N \end{cases} \tag{9.4.1}$$

Since the probability distribution of S_N is the same as that of the Wilcoxon rank-sum statistic W_N, the moments are also the same:

$$E(S_N) = \frac{m(N+1)}{2} \quad \text{var}(S_N) = \frac{mn(N+1)}{12} \tag{9.4.2}$$

To find critical values of S_N, tables of the distribution of W_N can be used, like Table J for $m \le n \le 10$.

The asymptotic relative efficiency and consistency properties of the Siegel–Tukey test are equivalent to those of the tests F_N, B_N, and A_N, because of the following relations. With N even, let S'_N be a test with weights constructed in the same manner as for S_N but starting at the right-hand end of the array, as displayed in Table 9.4.1 for $N/2$ even.

TABLE 9.4.1

Weights for Siegel–Tukey Test

		i						
Test	**Weights**	**1**	**2**	**3**	**4**	**5**	**…**	$N/2$
S_N	a_i	1	4	5	8	9	…	N
S'_N	a'_i	2	3	6	7	10	…	$N-1$
$S_N + S'_N$	$a_i + a'_i$	3	7	11	15	19	…	$2N-1$
S''_N	$(a_i + a'_i + 1)/4$	1	2	3	4	5	…	$N/2$

		i						
Test	**Weights**	$(N/2)+1$	**…**	$N-4$	$N-3$	$N-2$	$N-1$	N
S_N	a_i	$N-1$	…	10	7	6	3	2
S'_N	a'_i	N	…	9	8	5	4	1
$S_N + S'_N$	$a_i + a'_i$	$2N-1$	…	19	15	11	7	3
S''_N	$(a_i + a'_i + 1)/4$	$N/2$	…	5	4	3	2	1

If $N/2$ is odd, the weights $a_{N/2}$ and $a'_{N/2}$ are interchanged, as are $a_{(N/2)+1}$ and $a'_{(N/2)+1}$. In either case, the weights $(a_i + a'_i + 1)/4$ are equal to the set of weights for F_N when N is even, and therefore, the following complete cycle of relations to establish for N even:

$$S''_N = F_N = m\left(\frac{N}{2}+1\right) - B_N = \frac{m(N+1)}{2} - A_N \tag{9.4.3}$$

9.5 The Klotz Normal-Scores Test

The *Klotz* (1962) *normal-scores test* for scale is similar to the Mood test in that it uses the squares of the weights of the inverse-normal-scores test for location [van der Waerden test of (8.3.3)]. The test statistic is then

$$K_N = \sum_{i=1}^{N}\left[\Phi^{-1}\left(\frac{i}{N+1}\right)\right]^2 Z_i \tag{9.5.1}$$

where $\Phi(x)$ is the cumulative standard normal probability distribution. Since the larger weights are at the extremes, we again reject H_0 for large K_N for the alternative that the X population has the larger spread. Tables of critical values for $N \leq 20$ are given in Klotz (1962). The moments are

$$E(K_N) = \frac{m}{N}\sum_{i=1}^{N}\left[\Phi^{-1}\left(\frac{i}{N+1}\right)\right]^2$$

$$\text{var}(K_N) = \frac{mn}{N(N-1)}\sum_{i=1}^{N}\left[\Phi^{-1}\left(\frac{i}{N+1}\right)\right]^4 - \frac{n}{m(N-1)}[E(K_N)]^2$$

Since this is an asymptotically optimum test against the alternative of normal distributions differing only in variance, its ARE relative to the F test equals 1 when both populations are normal.

An asymptotically equivalent test proposed by Capon (1961) uses the expected values of the square of the normal order statistics as weights or

$$\sum_{i=1}^{N}\left[E\left(\xi_{(i)}^2\right)\right]Z_i \tag{9.5.2}$$

where $\xi_{(i)}$ is the ith-order statistic from a standard normal distribution. This test is the scale analog of the Terry test for location in (8.3.1). The weights are

tabled in Teichroew (1956), Sarhan and Greenberg (1962) for $N \leq 20$, and Tietjen et al. (1977) for $N \leq 50$.

9.6 The Percentile Modified Rank Tests for Scale

If the T_s and B_r statistics defined in (8.3.5) are added instead of subtracted, the symmetry of weights needed to detect scale differences is achieved. When N is even and $S = R = N/2$, $T_s + B_r$ is equivalent to the David–Barton type of test. The mean and variance of the statistic for N even and $S = R$ are

$$E(T_s + B_r) = \frac{mS^2}{N} \quad \text{var}(T_s + B_r) = \frac{mnS(4NS^2 - N - 6S^3)}{6N^2(N-1)}$$

The null distribution is symmetric for $S = R$ when $m = n$. Tables for $m = n \leq 6$ are given in Gibbons and Gastwirth (1966), and, as for the location problem, the normal approximation to critical values can be used for $m = n \geq 6$.

This scale test has a larger asymptotic relative efficiency than its full-sample counterparts for all choices of $s = r < 0.50$. The maximum ARE (with respect to s) is 0.850, which occurs for normal alternatives when $s = r = 1/8$. This result is well above the ARE of 0.76 for Mood's test and 0.608 for the tests of Sections 9.3 and 9.4. Thus, asymptotically at least, in the normal case, a test based on only 25% of the sample at each of the extremes is more efficient than a comparable test using the entire sample. The normal-scores tests of Section 9.5 have a larger ARE, but they are more difficult to use because of the complicated scores.

9.7 The Sukhatme Test

A number of other tests have been proposed for the scale problem. The only other one we discuss in detail here is the Sukhatme test statistic. Although it is less useful in applications than the others, this test has some nice theoretical properties. The test also has the advantage of being easily adapted to confidence-interval estimation of the ratio of the unknown scale parameters.

When the X and Y populations have or can be adjusted to have equal medians, we can assume without loss of generality that this common median is zero. If the Y's have a larger spread than the X's, those X observations which are negative should be larger than most of the negative Y observations, and the positive observations should be arranged so that most of the

Y's are larger than the X's. In other words, most of the negative Y's should precede negative X's, and most of the positive Y's should follow positive X's. Using the same type of indicator variables as for the Mann–Whitney statistic (6.6.2), we define

$$D_{ij} = \begin{cases} 1 & \text{If } Y_j < X_i < 0 \quad \text{or} \quad 0 < X_i < Y_j \\ 0 & \text{otherwise} \end{cases}$$

and the *Sukhatme test* statistic (Sukhatme, 1957) is

$$T = \sum_{i=1}^{m} \sum_{j=1}^{n} D_{ij} \tag{9.7.1}$$

The important parameter here is

$$\begin{aligned} p &= P(Y < X < 0 \quad \text{or} \quad 0 < X < Y) \\ &= \int_{-\infty}^{0} \int_{-\infty}^{x} f_Y(y) f_X(x)\, dy\, dx + \int_{0}^{\infty} \int_{x}^{\infty} f_Y(y) fx(x)\, dy\, dx \\ &= \int_{-\infty}^{0} F_Y(x)\, dF_X(x) + \int_{0}^{\infty} [1 - F_Y(x)] dF_X(x) \\ &= \int_{-\infty}^{0} [F_Y(x) - F_X(x)]\, dF_X(x) + \int_{0}^{\infty} [F_X(x) - F_Y(x)] dF_X(x) + 1/4 \end{aligned} \tag{9.7.2}$$

Then, the null hypothesis of identical populations has been parameterized to H_0:$p = 1/4$ and T/mn is an unbiased estimator of p since

$$E(T) = mnp$$

By redefining the parameters p, p_1, and p_2 of the Mann–Whitney statistic as appropriate for the present indicator variables D_{ij}, the variance of T can be expressed as in (6.6.10) and (6.6.11). The probabilities relevant here are

$$\begin{aligned} p_1 &= P[Y_j < X_i < 0 \quad \text{or} \quad 0 < X_i < Y_j) \cap (Y_k < X_i < 0 \quad \text{or} \quad 0 < X_i < Y_k)] \\ &= P[(Y_j < X_i < 0 \cap Y_k < X_i < 0)] + P[(Y_j > X_i > 0) \cap (Y_k > X_i > 0)] \\ &= \int_{-\infty}^{0} [F_X(x)]^2 dF_X(x) + \int_{0}^{\infty} [1 - F_Y(x)]^2 dF_X(x) \end{aligned} \tag{9.7.3}$$

$$
\begin{aligned}
p_2 &= P[(Y_j < X_i < 0 \quad \text{or} \quad 0 < X_i < Y_j) \cap (Y_j < X_h < 0 \quad \text{or} \quad 0 < X_h < Y_k)] \\
&= P[(Y_j < X_i < 0) \cap (Y_j < X_k < 0)] + P[(Y_j > X_i > 0) \cap (Y_j > X_k > 0)] \\
&= \int_{-\infty}^{0} [1/2 - F_X(y)]^2 dF_Y(y) + \int_{0}^{\infty} [F_X(y) - 1/2]^2 dF_Y(y)
\end{aligned} \tag{9.7.4}
$$

Then, from (6.6.11), the variance of T is

$$
\text{var}(T) = mn[p - p^2(N-1) + (n-1)p_1 + (m-1)p_2] \tag{9.7.5}
$$

Since $E(T/mn) = p$ and $\text{var}(T/mn) \to 0$ as $m, n \to \infty$, the Sukhatme statistic provides a consistent test for the following cases in terms of p in (9.7.2) and $\varepsilon = p - 1/4$ so that

$$
\varepsilon = \int_{-\infty}^{0} [F_Y(x) - F_X(x)]\, dF_X(x) + \int_{0}^{\infty} [F_X(x) - F_Y(x)]\, dF_X(x) \tag{9.7.6}
$$

The appropriate rejection regions and P values are

Subclass of Alternative	**Rejection Region**	***P* Value**
$p < 1/4$ $(\varepsilon < 0)$ $(\theta > 1)$	$T - mn/4 \le k_1$	$P(T \le t_0 \mid H_0)$
$p > 1/4$ $(\varepsilon > 0)$ $(\theta < 1)$	$T - mn/4 \ge k_2$	$P(T \ge t_0 \mid H_0)$
$p \ne 1/4$ $(\varepsilon \ne 0)$ $(\theta \ne 1)$	$\lvert T - mn/4 \rvert \ge k_3$	2 (smaller of above)

(9.7.7)

where t_0 is the observed value of T. It would be preferable to state these subclasses of alternatives as a simple relationship between $F_Y(x)$ and $F_X(x)$ instead of this integral expression for ε. Although (9.7.6) defines a large subclass, we are particularly interested now in the scale alternative model where $F_Y(x) = F_X(\theta x)$. Then,

1. If $\theta < 1, F_Y(x) > F_X(x)$ for $x < 0$ and $F_Y(x) < F_X(x)$ for $x > 0$.
2. If $\theta > 1, F_Y(x) < F_X(x)$ for $x < 0$ and $F_Y(x) > F_X(x)$ for $x > 0$.

In both cases, the two integrands in (9.7.6) have the same sign and can therefore be combined to write

$$
\varepsilon = \pm \int_{-\infty}^{\infty} |F_X(\theta x) - F_X(x)| dF_X(x) \tag{9.7.8}
$$

where the plus sign applies if $\theta < 1$ and the minus if $\theta > 1$. This explains the statements of subclasses in terms of θ given in (9.7.7).

The exact null distribution of T can be found by enumeration or a recursive method similar to that for the Mann–Whitney test. The null distribution of T is not symmetric for all m and n. The minimum value of T is zero and the maximum value is

$$S = UW + (m - U)(n - W) \tag{9.7.9}$$

where U and W denote the number of X and Y observations, respectively, which are negative. The minimum and maximum occur when the X or Y variables are all clustered. Tables of the exact distribution of T are given in Laubscher and Odeh (1976) and these should be used to find critical values for small sample sizes.

Another test statistic which could be used for this situation is

$$T' = \sum_{i=1}^{m} \sum_{j=1}^{n} D'_{ij} = S - T \quad \text{where } D'_{ij} = \begin{cases} 1 & \text{if } X_i < Y_j < 0 \\ & \text{or } 0 < Y_j < X_i \\ 0 & \text{otherwise} \end{cases} \tag{9.7.10}$$

where S is defined in (9.7.9). Then a two-sided critical region could be written as $T \leq t_{\alpha/2}$ or $T' \leq t'_{\alpha/2}$ where $t_{\alpha/2}$ and $t'_{\alpha/2}$ have respective left-tail probabilities equal to $\alpha/2$.

For larger sample sizes, U and W converge, respectively, to $m/2$ and $n/2$ and S converges to $mn/2$ while the distribution of T approaches symmetry and the normal distribution. Laubscher and Odeh (1976) showed that this approximation is quite good for m and n larger than 10. In the null case where $F_Y(x) = F_X(x)$ for all x, $p = 1/4$ and $p_1 = p_2 = 1/12$. Substituting these results in (9.7.5) gives the null mean and variance as

$$E(T) = \frac{mn}{4} \quad \text{and} \quad \text{var}(T) = \frac{mn(N + 7)}{48}$$

For moderate m and n, the distribution of

$$\frac{4\sqrt{3}(T - mn/4)}{\sqrt{mn(N + 7)}} \tag{9.7.11}$$

can be well approximated by the standard normal.

Problems arise for the T test statistic whenever $X_i = Y_j$, or $X_i = 0$, or $Y_j = 0$. The T statistic could be redefined in a manner similar to (6.6.16) so that a correction for ties can be incorporated into the expression for the null variance. Zeros also present a problem.

The Sukhatme test has a distinct disadvantage in application since it cannot be used without knowledge of both of the individual population medians M_X and M_Y. Even knowledge of the difference $M_Y - M_X$ is not sufficient information to adjust the observations so that both populations

have zero medians. Since the sample medians do converge to the respective population medians, the observations might be adjusted by subtracting the X and Y sample medians from each of the X and Y observations, respectively. The test statistic no longer has the same exact distribution, but for large sample sizes the error introduced by this estimating procedure should not be too large.

The Sukhatme test statistic can be written in the form of a linear rank statistic by a development similar to that used in Section 8.2 to show the relationship between the Wilcoxon and Mann–Whitney tests. Looking at (9.7.1) now, we know that for all values of i, $\sum_{j=1}^{n} D_{ij}$ is the sum of two quantities:

1. The number of values of j for which $Y_j < X_i < 0$, which is $r_{XY}(X_i) - U_i$
2. The number of values of j for which $Y_j > X_i > 0$, which is $N - r_{XY}(X_i) + 1 - V_i$

where

U_i is the number of X's less than or equal to X_i for all $X_i < 0$
V_i is the number of X's greater than or equal to X_i for all $X_i > 0$
$r_{XY}(x_i)$ is defined in (7.2.1)

Then, for $Z_i = 1$ if the ith variable in the combined array is an X and $Z_i = 0$ otherwise, we have

$$\begin{aligned} T &= \sum_{\substack{i=1 \\ X_i<0}}^{m} [r_{XY}(X_i) - U_i] + \sum_{\substack{i=1 \\ X_i>0}}^{m} [N - r_{XY}(X_i) + 1 - V_i] \\ &= \sum_{X<0} iZ_i + \sum_{X>0} (N - i + 1)Z_i - \sum U_i - \sum V_i \\ &= \sum_{X<0} iZ_i + \sum_{X>0} (N - i + 1)Z_i - \frac{U(U+1)}{2} - \frac{V(V+1)}{2} \end{aligned}$$

where $\sum_{X<0}$ indicates that the sum is extended over all values of i such that $X_i < 0$, U is the total number of X observations that are less than 0, and V is the number of X observations that are greater than 0. From this result, we can see that T is asymptotically equivalent to the Freund–Ansari–Bradley test, since as $N \to \infty$, the combined sample median will converge in probability to 0, the population median, and U and V will both converge to $m/2$, so that T converges to $F_N - m(m+2)/4$ with F_N defined as in (9.3.3). The test statistic is therefore asymptotically equivalent to all of the tests presented in Sections 9.3 and 9.4, and the large-sample properties are identical, including the ARE of $6/\pi^2$. Note that inasmuch as consistency is a large-sample property, the consistency of these other tests follows also from our analysis for T here.

9.8 Confidence-Interval Procedures

If the populations from which the X and Y samples are drawn are identical in every respect except scale, the nonparametric model of (9.1.2) with $M_X = M_Y = M$ is

$$F_{Y-M}(x) = F_{X-M}(\theta x) \quad \text{for all } x \text{ and some } \theta > 0$$

Since θ is the relevant scale parameter, a procedure for finding a confidence interval estimate of θ would be desirable. In the above model, we can assume without loss of generality that the common median M is zero. Then, for all $\theta > 0$, the random variable $Y' = Y\theta$ has the distribution

$$P(Y' \le y) = P\left(Y \le \frac{y}{\theta}\right) = F_Y\left(\frac{y}{\theta}\right) = F_X(y)$$

and Y' and X have identical distributions. The confidence-interval estimate of θ with confidence coefficient $1 - \alpha$ should consist of all values of θ for which the null hypothesis of identical populations will be accepted for the observations X_i and $Y_j\theta, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Using the Sukhatme test criterion of (9.7.1), where T denotes the number of pairs $(x_i, y_j\theta)$ for which either $y_j\theta < x_i < 0$ or $0 < x_i < y_j\theta$, or equivalently the number of positive pairs such that $x_i/y_j < \theta$. Suppose the rejection region for a two-sided test of size α based on the T criterion is to reject H_0 for $T \le k_1$ or $T \ge k_2$. The appropriate confidence interval with coefficient $1 - \alpha$ is then

$$\left(\frac{x_i}{y_j}\right)_{(k)} < \theta < \left(\frac{x_i}{y_j}\right)_{(k')} \tag{9.8.1}$$

where $(x_i/y_j)_{(k)}$ and $(x_i/y_j)_{(k')}$ denote the kth and k'th smallest in an array made from only those ratios x/y which are positive. For small sample sizes, k and k' are found from the tables in Laubscher and Odeh (1976). If m and n are larger than 10, the number k can be found using the normal approximation given in (9.7.9); the result with a continuity correction of 0.5 is

$$k = \frac{mn}{4} + 0.5 - z_{\alpha/2}\sqrt{\frac{mn(N+7)}{48}} \tag{9.8.2}$$

which should be rounded down to the next smaller integer. Then k' is found from $k' = mn/2 - k + 1$ since the approximation gives symmetric endpoints to the confidence-interval estimate.

One other approach to obtaining a confidence interval when there is no information about location is given later in Section 9.10.

9.9 Other Tests for the Scale Problem

All the tests for scale presented so far in this chapter are basically of the Mann–Whitney–Wilcoxon type, and except for the Mood and Klotz tests all are asymptotically equivalent. Other tests have been proposed—some are related to these while others incorporate essentially different ideas. A few will be summarized here even though they do not all fall within the category of linear rank statistics.

A test whose rationale is similar to the two-sample median test can be useful to detect scale differences. In two populations differing only in scale, the expected proportions of the two samples between two symmetric quantile points of the combined sample would not be equal. Since the total number of observations lying between the two quantiles is fixed by the order of the quantile, an appropriate test statistic could be the number of X observations lying between these two points.

If these quantiles are the first and third quartiles and the sample sizes are large so that the sample quartiles approach the corresponding population parameters in the null case, this might be considered asymptotically a test for equal population interquartile ranges. The null distribution of U, defined as the number of X observations within the Y-sample interquartile range, is the hypergeometric distribution, and the appropriate rejection region for the alternative that the X's are more widely dispersed is $U \leq u_\alpha$. If $m + n = N$ is divisible by 4, so that no observations equal the sample quartile values, the distribution is

$$f_U(u) = \binom{m}{u} \frac{\binom{n}{N/2 - u}}{\binom{N}{N/2}} \tag{9.9.1}$$

This test is usually attributed to Westenberg (1948).

Rosenbaum (1953) suggests that the number of observations R in the X sample which are either smaller than the smallest Y or larger than the largest Y is a reasonable test criterion for scale under the assumption that the population locations are the same. The null probability that exactly r X values lie outside the extreme values of the Y sample is

$$f_R(r) = n(n-1)\binom{m}{r} B(m + n - 1 - r, r + 2) \tag{9.9.2}$$

This result is easily verified by a combinational argument (Problem 9.9). Tables of critical values are given in Rosenbaum (1953).

Another criterion, suggested by Kamat, is based on the pooled sample ranks of the extreme X and Y observations. Let R_m and R_n denote the ranges

of the X ranks and Y ranks, respectively, in the combined sample ordering. If the locations are the same, a test statistic is provided by

$$D_{m,n} = R_m - R_n + n \tag{9.9.3}$$

Tables of critical values are given in Kamat (1956). It should be noted that when the X sample observations all lie outside the extremes of the Y sample, we have $D_{m,n} = R + n$, where R is Rosenbaum's statistic. The performance of these two tests is discussed in Rosenbaum (1965).

These three tests, as well as the others presented earlier in this chapter, are reasonable approaches to detecting dispersion differences only when the X and Y populations have the same location. If the populations do not have the same location but some measure of location is known for each population, say the medians M_X and M_Y, these values can be subtracted from the respective X and Y sample values to form samples from the $X' = X - M_X$ and $Y' = Y - M_Y$ populations which do have equal medians (in fact, zero). Then, any of the tests introduced earlier in this chapter can be performed on the X' and Y' variables. This is also true if the given data can be interpreted as deviations from some specified value or norm (as in Example 9.10.1). In this case, there is an alternative approach to testing the null hypothesis of equal scale. The absolute values of the deviations $X' = |X - M_X|$ and $Y' = |Y - M_Y|$ are themselves measures of spread for the respective populations. Each of the sample deviations x'_i and y'_j are estimates of the population deviation. If these sample deviations are arranged from smallest to largest in a single array, the arrangement of x' and y' is indicative of relative spread between the two populations. Thus, any of the two-sample location tests from Chapter 8 can be used on these absolute values to test for relative scale differences. This procedure will be illustrated in Example 9.10.1 using the Wilcoxon rank-sum test introduced in Section 8.2.

If the observations are adjusted before performing a test, say by subtracting the respective sample medians, the tests are no longer exact or even distribution-free. In fact, Moses (1963) shows that no test based on the ranks of the observations will be satisfactory for the dispersion problem without some sort of strong restriction, like equal or known medians, for the two populations. There is one type of approach to testing which avoids this problem. Although, strictly speaking, it does not qualify as a rank test, rank scores are used. The procedure is to divide each sample into small random subsets of equal size and calculate some measure of dispersion, for example, the variance, range, average deviation, for each subsample. The measures for both samples can be arranged in a single sequence in order of magnitude, keeping track of which of the X and Y samples produced the measure. A two-sample location test can then be performed on the result. For example, if m and n are both divisible by 2, random pairs could be formed and the Wilcoxon rank-sum test applied to the $N/2$ derived observations of

ranges of the form $|x_i - x_j|, |y_i - y_j|$. The test statistic then is an estimate of a linear function of $P(|X_i - X_j| > |Y_i - Y_j|)$. In general, for any sample dispersion measures denoted by U and V when computed for the X and Y subsamples, respectively, the Wilcoxon rank-sum test statistic estimates a linear function of $P(U > V)$. Questions such as the best subsample size and the best type of measure of dispersion remain to be answered generally. Tests of this kind are called *ranklike tests*. Their ARE's depend on the sizes of the random subsets, and range from 0.304 to a limiting value of 0.955 when the distributions are normal.

9.10 Applications

The Siegel–Tukey test for scale differences in Section 9.4 is the most frequently used procedure because it does not require a new set of tables. Table J for the distribution of the Wilcoxon rank-sum test can be used. The limitation of this test is that it can detect scale differences only when the locations are the same. The null hypothesis is $H_0 : \theta = \sigma_X/\sigma_Y = 1$, and the test statistic is S_N, the sum of the weights assigned to the X sample in the pooled array, where the method of assignment of all weights for $m + n = N$ even is spelled out in (9.4.1). The appropriate rejection regions and the P values for $m \le n \le 10$ are as follows, where s_0 denotes the observed value of the test statistic S_N.

Alternative	Rejection Region	*P* Value
$\theta = \sigma_X/\sigma_Y < 1$	$S_N \ge w_a$	$P(S_N \ge s_0 \mid H_0)$
$\theta = \sigma_X/\sigma_Y > 1$	$S_N \le w'_a$	$P(S_N \le s_0 \mid H_0)$
$\theta = \sigma_X/\sigma_Y \ne 1$	$S_N \ge w_{a/2}$ or $S_N \ge w'_{a/2}$	2 (smaller of above)

For larger sample sizes, the approximate rejection regions and P values based on the normal approximation with a continuity correction of 0.5 are as follows:

Alternative	Rejection Region	*P* Value
$\theta = \dfrac{\sigma_X}{\sigma_Y} < 1$	$S_N \ge \dfrac{m(N+1)}{2} + 0.5 + z_a\sqrt{\dfrac{mn(N+1)}{12}}$	$1 - \Phi\left[\dfrac{s_0 - 0.5 - m(N+1)/2}{\sqrt{mn(N+1)/12}}\right]$
$\theta = \dfrac{\sigma_X}{\sigma_Y} > 1$	$S_N \le \dfrac{m(N+1)}{2} - 0.5 - z_a\sqrt{\dfrac{mn(N+1)}{12}}$	$\Phi\left[\dfrac{s_0 + 0.5 - m(N+1)/2}{\sqrt{mn(N+1)/12}}\right]$
$\theta = \dfrac{\sigma_X}{\sigma_Y} \ne 1$	Both above with $z_{a/2}$	2 (smaller of above)

Example 9.10.1

An institute of microbiology is interested in purchasing microscope slides of uniform thickness and needs to choose between two different suppliers. Both have the same specifications for median thickness but they may differ in variability. The institute gauges the thickness of random samples of 10 slides from each supplier using a micrometer and reports the data shown below as the deviations from specified median thickness. Which supplier makes slides with a smaller variability in thickness?

Supplier X: 0.028, 0.029, 0.011, −0.030, 0.017, −0.012, −0.027, −0.018, 0.022, −0.023

Supplier Y: −0.002, 0.016, 0.005, −0.001, 0.000, 0.008, −0.005, −0.009, 0.001, −0.019

Solution

Since the given data represent differences from specified median thickness, the assumption of equal locations is tenable as long as both suppliers are meeting specifications.

First, we use the Siegel–Tukey test. The data arranged from smallest to largest, with X underlined, and the corresponding assignment of weights are shown in Table 9.10.1. The sum of the X weights is $S_N = 60$, and Table J gives the left-tail probability for $m = 10$, $n = 10$ as $P = 0.000$. Since this is a left-tail probability, the appropriate conclusion is to reject H_0 in favor of the alternative $H_1{:}\sigma_X/\sigma_Y > 1$ or $\sigma_X > \sigma_Y$. The data indicate that supplier Y has the smaller variability in thickness.

The STATXACT and SAS outputs for Example 10.1 are shown below. The answers and the conclusions are the same as ours.

Note that STATXACT provides both the exact and the asymptotic P values and so does SAS. The asymptotic P value using the STATXACT package is based on the value of the Z statistic without a continuity correction (−3.402) while the SAS package

TABLE 9.10.1

Array of Data and Weights

Data	Weight	Data	Weight
−0.030	1	0.000	19
−0.027	4	0.001	18
−0.023	5	0.005	15
−0.019	8	0.008	14
−0.018	9	0.011	11
−0.012	12	0.016	10
−0.009	13	0.017	7
−0.005	16	0.022	6
−0.002	17	0.028	3
−0.001	20	0.029	2

solution does use the continuity correction (−3.3639). It may be noted that at the time of this writing, MINITAB does not provide any nonparametric test for scale.

```
*****************************************
 STATXACT SOLUTION TO EXAMPLE 9.10.1
*****************************************

SIEGEL-TUKEY TEST
[  Sum of scores from population <        1 > ]

Min       Max     Mean      Std.-dev.   Observed     Standardized
55.00     155.0   105.0         13.23      60.00           −3.402


Asymptotic Inference:
   One-sided P value: Pr { Test Statistic.LE. Observed }  = 0.0003
   Two-sided P value: 2 * One-sided                       = 0.0007

Exact Inference:
   One-sided P value: Pr { Test Statistic .LE. Observed } = 0.0001
                      Pr {   Test Statistic .EQ. Observed } = 0.0000
   Two-sided P value: Pr { | Test Statistic − Mean |
                           .GE. | Observed  − Mean |     =    0.0002
   Two-sided P value: 2*One-sided                         =    0.0002
```

```
***********************************
SAS SOLUTION TO EXAMPLE 9.10.1
***********************************
```

Program:

```
DATA NAME;
INPUT GROUP Time @@;
  DATALINES;
1  0.028 1 0.029 1 0.011 1 −0.030 1 0.017 1 −0.012  1 −0.027 1 −0.018 1 0.022 1 −0.023
2 −0.002 2 0.016  2 0.005  2 −0.001 2 0.000 2  0.008  2 −0.005 2 −.009  2 0.001 2 −0.019
;
PROC NPAR1WAY ST DATA = TIME;
CLASS GROUP;
VAR TIME;
Exact;
RUN;
```

Output:

The NPAR1WAY Procedure

Siegel–Tukey Scores for Variable Time
Classified by Variable GROUP

GROUP	N	Sum of Scores	Expected Under H0	Std.-dev. Under H0	Mean Score
1	10	60.0	105.0	13.228757	6.0
2	10	150.0	105.0	13.228757	15.0

Siegel–Tukey two-sample test

Statistic (S)	60.0000
Normal approximation	
Z	−3.3639
One-sided Pr $< Z$	0.0004
Two-sided Pr $> \lvert Z \rvert$	0.0008
Exact test	
One-sided Pr $<= S$	1.028E-04
Two-sided Pr $>= \lvert S - \text{Mean} \rvert$	2.057E-04

Z includes a continuity correction of 0.5.

Siegel–Tukey one-way analysis

Chi-square	11.5714
DF	1
Pr > Chi-square	0.0007

Second, we use the Sukhatme test on these same data. The first step is to form separate arrays of the positive and negative deviations, with the X sample underlined. However, we have zeros here and the references do not recommend a procedure for their resolution. In order to be conservative, we will count zeros with the positive deviations and call them nonnegative deviations. This is consistent with the treatment of zeros in the procedure for a confidence-interval estimate based on the Mann–Whitney or Wilcoxon rank sum statistic.

Negatives: $-\underline{0.030}$, $-\underline{0.027}$, $-\underline{0.023}$, -0.019, $-\underline{0.018}$, $-\underline{0.012}$, -0.009, -0.005, -0.002, -0.001

Nonnegatives: 0.000, 0.001, 0.005, 0.008, $\underline{0.011}$, 0.016, $\underline{0.017}$, $\underline{0.022}$, $\underline{0.028}$ $\underline{0.029}$

We find $T = 2 + 1 = 3$ from (9.7.1) and $T' = 23 + 24 = 47$ from (9.7.10). The normal approximation is $z = -2.93$ without a continuity correction and a one-tailed P value $= 0.0017$. (The corrected value is $z = -2.87$ with P value $= 0.0021$.) The reader can verify the relation $T + T' = S$ where S is the maximum value of T. We note that the result is quite similar to the Siegel–Tukey test and the conclusion is the same. The Sukhatme test is not available in SAS or STATXACT at the time of this writing.

Third, we give another alternative for a test for equal scale. The data in this example represent deviations $X - M$ and $Y - M$ from some common median M. If the X and Y populations are both symmetric about M, each of the differences is equally likely to be positive and negative. If, further, the variables have the same scale, then the absolute values of these deviations $|X - M|$ and $|Y - M|$ should have the same median value of 0. Note that these absolute values are themselves measures of variability. Thus, we can use the Wilcoxon rank-sum test for location to measure the scale difference. Then the weights should be the ordinary ranks, that is, the integers 1–20 in their natural order, and the pooled ordered data and corresponding ranks are shown in Table 9.10.2. The sum of the X ranks here is $W_N = 149$ and the corresponding exact P value from Table J is 0.000, a right-tail probability, which makes us conclude that the median variability measure for X is larger than the median variability measure for Y. This result, while not the same as that obtained with the Siegel–Tukey or Sukhatme tests, is consistent with both previous conclusions. This will generally be true. We note, however, that the Wilcoxon test for location on the absolute value is consistent against scale alternatives only when the data are given in the form $X - M$ and $Y - M$ or can be written this way because M is known.

The advantage of this alternative test is that it has a corresponding confidence-interval procedure for estimation of the ratio $\theta = \sigma_X/\sigma_Y$ under the assumption of symmetry, the scale model relationship in (9.1.1), and the observations written in the form $X - M$ and $Y - M$ for equal medians or $X - M_X$ and $Y - M_Y$ for known M_X, M_Y in general. The procedure is to form the mn ratios $|X_i - M_X|/|Y_j - M_Y|$ for all i, j, and arrange them from smallest to largest. The confidence-interval end points are the uth smallest and uth largest among these ratios, where u is found in exactly the same manner as it was in Section 8.2 using Table J or the normal approximation.

TABLE 9.10.2

Array of Absolute Values of Data and Ranks

|Data|	Rank	|Data|	Rank
0.000	1	0.016	11
0.001	2.5	0.017	12
0.001	2.5	0.018	13
0.002	4	0.019	14
0.005	5.5	0.022	15
0.005	5.5	0.023	16
0.008	7	0.027	17
0.009	8	0.028	18
0.011	9	0.029	19
0.012	10	0.030	20

TABLE 9.10.3

Ratios of Absolute Values

	$\lvert X - M\rvert$									
$\lvert Y - M\rvert$	**0.011**	**0.012**	**0.017**	**0.018**	**0.022**	**0.023**	**0.027**	**0.028**	**0.029**	**0.030**
0.000	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.001	0.09	0.08	0.06	0.06	0.05	0.04	0.04	0.04	0.03	0.03
0.001	0.09	0.08	0.06	0.06	0.05	0.04	0.04	0.04	0.03	0.03
0.002	0.18	0.17	0.12	0.11	0.09	0.09	0.07	0.07	0.07	0.07
0.005	0.45	0.42	0.29	0.28	0.23	0.22	0.19	0.18	0.17	0.17
0.005	0.45	0.42	0.29	0.28	0.23	0.22	0.19	0.18	0.17	0.17
0.008	0.73	0.67	0.47	0.44	0.36	0.35	0.30	0.29	0.28	0.27
0.009	0.82	0.75	0.53	0.50	0.41	0.39	0.33	0.32	0.31	0.30
0.016	1.45	1.33	0.94	0.89	0.73	0.70	0.59	0.57	0.55	0.53
0.019	1.73	1.58	1.12	1.06	0.86	0.83	0.70	0.68	0.66	0.63

Confidence-interval estimate for Example 9.10.1: The $mn = 10(10) = 100$ ratios of absolute values $|Y - M|/|X - M|$ are shown in Table 9.10.3. Note that the ratios used are the reciprocals of the usual ratio and this will give an interval on σ_Y/σ_X; this is done in order to avoid division by zero. Note also that each set of sample data is written in increasing order of absolute magnitudes so that the uth smallest and uth largest can be easily identified. For $m = 10$, $n = 10$ and confidence coefficient nearest 0.95 say, Table J gives $P = 0.022$ with rank 24 so that $u = 24$. The interval estimate is $0.06 \leq \sigma_Y/\sigma_X \leq 0.53$ with

TABLE 9.10.4

Ratios $(Y - M)/(X - M)$

	$Y - M$				
$X - M$	**−0.019**	**−0.009**	**−0.005**	**−0.002**	**−0.001**
Negatives					
−0.030	0.6333	0.3000	0.1667	0.0667	0.0333
−0.027	0.7037	0.3333	0.1852	0.0741	0.0370
−0.023	0.8261	0.3913	0.2174	0.0870	0.0435
−0.018	1.0556	0.5000	0.2778	0.1111	0.0556
−0.012	1.5833	0.7500	0.4167	0.1667	0.0833
	0.000	**0.001**	**0.005**	**0.008**	**0.016**
Nonnegatives					
0.011	0	0.0909	0.4545	0.7273	1.4545
0.017	0	0.0588	0.2941	0.4706	0.9412
0.022	0	0.0455	0.2273	0.3636	0.7273
0.028	0	0.0357	0.1786	0.2857	0.5714
0.029	0	0.0345	0.1724	0.2759	0.5517

confidence coefficient $1 - 2(0.022) = 0.956$; taking the reciprocals we get $1.9 \leq \sigma_X/\sigma_Y \leq 16.7$.

We also use these data to illustrate the confidence-interval estimate of σ_X/σ_Y based on the Sukhatme test procedure. Here, we take only the nonnegative ratios $(X - M)/(Y - M)$ shown in Table 9.10.4 and there are 50 of them. As before, in order to avoid division by zero, we form the ratios $(Y - M)/(X - M)$ to find the interval on σ_Y/σ_X, and then take the reciprocal to obtain a confidence-interval estimate of σ_X/σ_Y. The value of k from (9.8.2) for 95% confidence is 10.80 and we round down and use $k = 10$. The confidence interval is $0.0435 \leq \sigma_Y/\sigma_X \leq 0.633$ and taking the reciprocals yields $1.58 \leq \sigma_X/\sigma_Y \leq 23.0$. Note that this interval is wider than the one based on the Wilcoxon rank-sum test. This will frequently be the case.

The confidence-interval procedure based on the Wilcoxon test for location can also be used when the data given are not variations from some central value and hence not measures of variability in themselves, but are from populations that can take on only positive values. Many variables fall into this category—for example, age, height, weight, income, GPA, test scores, survival times, relative efficiencies, and the like. For samples from such distributions, each of the mn ratios X_i/Y_j is itself a measure of the relative spread of the X and Y populations in the sense that it is an estimate of the range (measured from zero) of the X variable relative to the range of the Y variable if both are positive variables. In other words, we are here looking at scale as measured by total spread, as opposed to spread around a central value. Then, the confidence-interval endpoints are the uth smallest and the uth largest of the mn ratios X/Y, where u is found from Table J or from the normal approximation using (8.2.6). We call this the *method of positive variables* and illustrate it by Example 9.10.2.

Example 9.10.2

Two potential suppliers of streetlighting equipment, A and B, presented their bids to the city manager along with the following data as a random sample of life length in months.

A: 35, 66, 58, 83, 71
B: 46, 56, 60, 49

Test whether the life length of suppliers A and B have equal variability.

Solution

Before we can test for scale, we must determine whether we can assume that the locations can be regarded as equal. We use the Wilcoxon rank-sum test. Since supplier B has fewer observations, we label it the X sample so that $m = 4$ and $n = 5$. The pooled sample array with X underlined is 35, <u>46</u>, <u>49</u>, <u>56</u>, 58, <u>60</u>, 66, 71, 83. The test statistic is $W_N = 15$ and the one-tailed exact P value from Table J is $P = 0.143$. Thus, there is no reason not to assume that the locations are the same, and we use the Siegel–Tukey test for scale. The test statistic is $S_N = 24$

TABLE 9.10.5

Ratios B/A

	A				
B	**35**	**58**	**66**	**71**	**83**
46	1.314	0.793	0.697	0.648	0.554
49	1.400	0.845	0.742	0.690	0.590
56	1.600	0.966	0.848	0.789	0.675
60	1.714	1.034	0.909	0.845	0.723

with a one-sided exact P value of $P = 0.206$ from Table J. We conclude that there is no difference in the scales of the A and B populations. Note, however, that the test for scale was done after the test for location and that the two test statistics are not independent. Hence, one has to be careful about interpreting the results.

Now we will find a confidence-interval estimate of σ_B/σ_A using the method of positive variables with confidence coefficient near 0.95. From Table J with $m = 4$, $n = 5$, we find $u = 3$ for exact confidence level 0.936. The 20 ratios are shown in Table 9.10.5. The confidence-interval estimate is $0.648 \leq \sigma_B/\sigma_A \leq 1.400$. Note that this interval includes the ratio one, as was implied by our hypothesis test. These analyses imply that there is no basis for any preference between suppliers A and B.

9.11 Summary

In this chapter, we have covered many different tests for the null hypothesis $\sigma_X/\sigma_Y = 1$ that the scale parameters of two populations are identical. Each of these procedures (except the ranklike tests) required some assumption about the location of the two distributions. If we can assume that the locations are the same, then each of the procedures in Sections 9.2 through 9.6 can be carried out even when the common value is unknown or unspecified. When the locations are not the same but their difference $M_X - M_Y$ is known, we can form $X' = X - (M_X - M_Y)$ and carry out the same tests on X' and Y because X' and Y now have the same location (in fact, equal to M_Y). When the locations are not the same but are both known as, say, M_X and M_Y, these values can be subtracted to form $X' = X - M_X$ and $Y' = Y - M_Y$ and all of the tests in this chapter can be carried out because the locations of X' and Y' are now the same (in fact, equal to 0). If the medians are unknown and unequal, we can estimate them from the sample medians or use ranklike tests, but these are only ad hoc procedures whose performance is unknown.

Recall from Chapter 1 that the confidence-interval estimate of any parameter is the set of all values of the parameter, which, if stated in the null hypothesis, would be accepted by a hypothesis test at the α level that corresponds to one minus the confidence level. Therefore, in order to develop

a procedure for finding a confidence-interval estimate for $\theta = \sigma_X/\sigma_Y$, we must be able to generalize the test for $\theta = 1$ to a test for $\theta = \theta_0 \neq 1$.

1. First, assume that $M_X = M_Y$, unspecified. The natural approach would be to form $X' = X/\theta_0$ so that $\sigma_{X'}/\sigma_Y = 1$. But then $M_{X'} = M_X/\theta_0$ must be equal to M_Y which cannot be true unless $\theta_0 = 1$, a contradiction unless $M_X = M_Y = 0$.
2. Second, assume that $M_X - M_Y$ is known but not equal to 0. The natural approach would be to form $X' = [X - (M_X - M_Y)]/\theta_0$ so that $\sigma_{X'}/\sigma_Y = 1$. But then $M_{X'} = M_Y/\theta_0$ must be equal to M_Y, which cannot be true unless $\theta_0 = 1$ a contradiction unless $M_X = M_Y = 0$.
3. Third, assume that M_X and M_Y are known. The natural approach would be to form $X' = (X - M_X)/\theta_0$ and $Y' = Y - M_Y$ so that $\sigma_{X'}/\sigma_{Y'} = 1$. This makes $M_{X'} = M_{Y'} = 0$ and hence we can have a test of $\theta = \theta_0 \neq 1$.

This argument shows that M_X and M_Y must both be known in order to test the null hypothesis where $\theta_0 \neq 1$, and therefore, we can have a corresponding confidence-interval procedure only in this case. The simplest ones to use are those based on the Wilcoxon rank-sum test of the absolute values and the Sukhatme test, since tables are available in each case. The corresponding confidence-interval procedures were illustrated in Example 9.10.1.

If we assume only that X/θ and Y are identically distributed, we can test the null hypothesis $\theta = \theta_0 \neq 1$ and this gives us the confidence interval based on the method of positive variables. This procedure was illustrated by Example 9.10.2. But notice that this makes $M_X = \theta M_Y$ and hence the estimate of relative scale is based on spread about the origin and not spread about some measure of central tendency.

The asymptotic relative efficiency of each of the Freund–Ansari–Bradley–David–Barton tests of Section 9.3 is 0.608 relative to the F test for normal populations differing only in scale, 0.600 for the continuous uniform distribution, and 0.94 for the double-exponential distribution. The ARE for the Mood test of Section 9.2 is 0.76 for normal distributions differing only in scale. The Klotz and Capon tests of Section 9.5 have an ARE of 1.00 in this case. The ARE of the percentile modified rank tests for scale against the F test for normal alternatives differing only in scale reaches its maximum of 0.850 when $s = r = 1/8$.

Blair and Thompson (1992) study the properties of ranklike tests based on the absolute values of the differences between pairs of all possible subsamples of size two in the X and Y samples. Then they use the Wilcoxon rank sum test with Savage scores as a test statistic. Their ARE relative to the Siegel–Tukey test ranges from 0.222 for the uniform distribution to 3.55 for the exponential distribution. They note that Bickel and Lehmann (1979) make a distinction between the various tests for scale by claiming that dispersion is

measured relative to a location parameter and applies only to symmetric distributions while spread is a measure that is free of location.

In this sense, ranklike tests and the method of positive variables illustrated in Section 9.10 are tests for spread, and not tests for dispersion. The term tests for scale includes both.

Problems

9.1 Develop by enumeration for $m = n = 3$ the null probability distribution of Mood's statistic M_N of (9.2.1).

9.2 Develop by enumeration for $m = n = 3$ the null probability distribution of the Freund–Ansari–Bradley statistic of (9.3.3).

9.3 Verify the expression given in (9.2.3) for var(M_N).

9.4 Apply Theorem 7.3.2 to derive the mean and variance of the statistic A_N defined in (9.3.1).

9.5 Apply Theorem 7.3.2 to derive the mean and variance of the statistic B_N defined in (9.3.5).

9.6 Verify the relationship between A_N, B_N and F_N given in (9.4.3) for N even.

9.7 Use the relationship in (9.4.3) and the moments derived for F_N for N even in (9.3.4) to verify your answers to Problems 9.4 and 9.5 for N even.

9.8 Use Theorem 7.3.2 to derive the mean and variance of $T_s + B_r$ for N even, $S \neq R$, where $S + R \leq N$.

9.9 Verify the result given in (9.9.2) for the null probability distribution of Rosenbaum's R statistic.

9.10 Olejnik (1988) suggested that research studies in education and the social sciences should be concerned with differences in group variability as well as differences in group means. For example, a teacher can reduce variability in student achievement scores by focusing attention and classroom time on less able students. On the other hand, a teacher can increase variability in achievement by concentrating on the students with greatest ability and letting the less able students fall farther and farther behind. Previous research has indicated that mean student achievement for classes taught by teachers with a bachelor's degree is not different from that of classes taught by teachers with a master's degree. The present study was aimed at determining whether variability

in student achievement is the same for these two teacher groups. The data below are the achievement scores on an examination (10 = highest possible score) given to two classes of 10 students. Class 1 was taught by a teacher with a master's degree and class 2 by a teacher with a bachelor's degree. The mean score is 5 for each class. Is there a difference in the variability of scores?

Class 1	Class 2
7	3
4	6
4	7
5	9
4	3
6	2
6	4
4	8
3	2
7	6

9.11 The psychology departments of public universities in each of two different states accepted seven and nine applicants, respectively, for graduate study next fall. Their respective scores on the Graduate Record Examination are:

University X: 1200, 1220, 1300, 1170, 1080, 1110, 1130
University Y: 1210, 1180, 1000, 1010, 980, 1400, 1430, 1390, 970

The sample median and mean scores of applicants to the two universities are close to being equal, so an assumption of equal location may well be justified. Use the Siegel–Tukey test to see which university has the smaller variability in scores, if either.

9.12 In industrial production processes, each measurable characteristic of any raw material must have some specified average value, but the variability should also be small to keep the characteristics of the end product within specifications. Samples of lead ingots to be used as raw material are taken from two different distributors; each distributor has a specification of median weight equal to 16.0 kg. The data below represent actual weight in kilograms.

X: 15.7, 16.1, 15.9, 16.2, 15.9, 16.0, 15.8, 16.1, 16.3, 16.5, 15.5
Y: 15.4, 16.0, 15.6, 15.7, 16.6, 16.3, 16.4, 16.8, 15.2, 16.9, 15.1

(a) Use the deviations from specified median weight to find two different interval estimates of σ_X/σ_Y with confidence coefficient nearest 0.95.

(b) Use the method of positive variables to find a confidence-interval estimate of the ratio X/Y of scale measured relative to 0.

9.13 Data on weekly rate of item output from two different production lines for 7 weeks are as follows:

Line I: 36, 36, 38, 40, 41, 41, 42
Line II: 29, 34, 37, 39, 40, 43, 44

We want to investigate the relative variability between the two lines.

(a) Find a one-tailed P value using the Siegel–Tukey test and state all assumptions needed for an exact P.
(b) Find the one-tailed P value using the Wilcoxon procedure assuming the population medians are $M_{\mathrm{I}} = M_{\mathrm{II}} = 40$ and state all assumptions needed for an exact P.
(c) In (b), you should have found many ties. Is there another appropriate procedure for analyzing these data, one for which the ties present no problem? Explain fully and outline the procedure.
(d) Find a 95% confidence-interval estimate of the ratio of the scale of the output of line I relative to line II when spread is measured from zero.

10

Tests of the Equality of k Independent Samples

10.1 Introduction

The natural extension of the two-sample problem is the k-sample problem, where observations are taken under a variety of different and independent conditions. Assume that we have k independent sets of observations, one from each of k continuous cdfs $F_1(x), F_2(x), \ldots, F_k(x)$, where the ith random sample is of size $n_i, i = 1, 2, \ldots, k$ and there are a total of $\sum_{i=1}^{k} n_i = N$ observations. Note we are again assuming that the independence extends across samples in addition to within samples. The extension of the two-sample problem to the k-sample problem is that all k samples are drawn from identical populations given by the null hypothesis

$$H_0 : F_1(x) = F_2(x) = \cdots = F_k(x) \quad \text{for all } x$$

The general alternative is simply that the populations differ in some way.

The location model for the k-sample problem is that the cdf's are $F(x - \theta_1), F(x - \theta_2), \ldots, F(x - \theta_k)$, respectively, where θ_i denotes a location parameter of the ith population, frequently interpreted as the median or the treatment effect. Then the null hypothesis can be written as

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$$

and the general alternative is

$$H_1 : \theta_i \neq \theta_j \quad \text{for at least one } i \neq j$$

In classical statistics, the usual test for this problem is the analysis-of-variance (ANOVA) F test for a one-way classification. The underlying assumptions for this test are that the k populations are identical in shape, in fact normal and

with the same variance, and therefore may differ only in location. The test of equal means or equal treatment effects

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$$

is, within the context of this model, equivalent to the above hypothesis of k identical populations. Denoting the observations in the ith sample by $X_{i1}, X_{i2}, \ldots, X_{in_i}$, the ith sample mean by $\overline{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i$, and the grand mean by $\overline{X} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}/N$, the classical ANOVA F test statistic may be written as

$$F = \frac{\left(\sum_{i=1}^{k} n_i(\overline{X}_i - \overline{X})^2\right)/k - 1}{\left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2\right)/N - k} = \frac{\text{mean square between samples}}{\text{mean square within samples}}$$

This test statistic follows the F distribution with $k-1$ and $N-k$ degrees of freedom under the normality assumption when H_0 holds. The F test is robust for equal samples sizes, but it is known to be sensitive to the assumption of equality of variances when the sample sizes are unequal.

The nonparametric techniques for this k-sample problem require no assumptions beyond continuous populations and therefore are applicable more generally, and they involve only simple calculations. We will cover here the extensions of the two-sample median test and the control median test, the Kruskal–Wallis ANOVA test, some other extensions of rank tests from the two-sample problem, and tests against ordered alternatives including comparisons with a control or standard.

10.2 Extension of the Median Test

Under the hypothesis of identical populations, we have a single random sample of size $\sum_{i=1}^{k} n_i = N$ from the common population. The grand median δ of the pooled samples is an estimate of the median of this common population. Therefore, an observation from any of the k samples is as likely to be above δ as below it. The set of N observations will support the null hypothesis then if, for each of the k samples, about half of the observations are less than the grand sample median. A test based on this criterion is attributed to Mood (1950, pp. 398–406) and Brown and Mood (1948, 1951).

As in the two-sample case, the grand sample median δ will be defined as the observation in the pooled ordered sample which has rank $(N+1)/2$ if N is odd and any number between the two observations with ranks $N/2$ and

$(N+2)/2$ if N is even. Then, for each sample separately, the observations are dichotomized according as they are less then δ or not. Define the random variable U_i as the number of observations in sample number i, which are less than δ, and let t denote the total number of observations which are less then δ. Then, by the definition of δ,

$$t = \sum_{i=1}^{k} u_i = \begin{cases} N/2 & \text{if } N \text{ is even} \\ (N-1)/2 & \text{if } N \text{ is odd} \end{cases}$$

and we have the calculations below.

	Sample 1	**Sample 2**	$\cdots$	**Sample *k***	**Total**
$<\delta$	u_1	u_2	$\ldots$	u_k	t
$\geq\delta$	n_1-u_1	n_2-u_2	$\ldots$	n_k-u_k	$N-t$
Total	n_1	n_2	$\ldots$	n_k	N

Under the null hypothesis, each of the $\binom{N}{t}$ possible sets of t observations is equally likely to be in the less-than-δ category, and the number of dichotomizations with this particular sample outcome is $\prod_{i=1}^{k}\binom{n_i}{u_i}$. Therefore, given t, the null probability distribution of the U_i's is the multivariate extension of the hypergeometric distribution, or

$$f(u_1, u_2, \ldots, u_k|t) = \frac{\binom{n_1}{u_1}\binom{n_2}{u_2}\cdots\binom{n_k}{u_k}}{\binom{N}{t}} \tag{10.2.1}$$

Let θ denote the probability that an observation from the common population is less than δ. If any or many of the U_i's differ too much from their expected values $n_i\theta$, the null hypothesis should be rejected. Generally, it would be impractical to set up joint rejection regions for the statistics $U_1, U_2, \ldots, U_k$, because of the large variety of combinations of the sample sizes. Fortunately, we can use another test criterion, which, although an approximation, is reasonably accurate even for N as small as 25 if each sample consists of at least five observations. This test statistic can be derived by appealing to the analysis of goodness-of-fit tests in Chapter 4. Each of the N elements in the pooled sample is classified according to two criteria, sample number, and its magnitude relative to δ. Denote these $2k$ categories by (i, j), where $i = 1, 2, \ldots, k$ according to the sample number and $j = 1$ if the

observation is less than δ and $j = 2$ otherwise. Denote the observed and expected frequencies for the (i, j) category by f_{ij} and e_{ij}, respectively. Then

$$\begin{aligned} f_{i1} &= u_i \\ f_{i2} &= n_i - u_i \end{aligned} \quad \text{for } i = 1, 2, \ldots, k$$

and the expected frequencies under H_0 are estimated from the data by

$$\begin{aligned} e_{i1} &= \frac{n_i t}{N} \\ e_{i2} &= \frac{n_i(N-t)}{N} \end{aligned} \quad \text{for } i = 1, 2, \ldots, k$$

The goodness-of-fit test criterion for these $2k$ categories from (4.2.1) is then

$$\begin{aligned} Q &= \sum_{i=1}^{k} \sum_{j=1}^{2} \frac{(f_{ij} - e_{ij})^2}{e_{ij}} \\ &= \sum_{i=1}^{k} \frac{(u_i - n_i t/N)^2}{n_i t/N} + \sum_{i=1}^{k} \frac{[n_i - u_i - n_i(N-t)/N]^2}{n_i(N-t)/N} \\ &= N \sum_{i=1}^{k} \frac{(u_i - n_i t/N)^2}{n_i t} + N \sum_{i=1}^{k} \frac{(n_i t/N - u_i)^2}{n_i(N-t)} \\ &= N \sum_{i=1}^{k} \frac{(u_i - n_i t/N)^2}{n_i} \left(\frac{1}{t} + \frac{1}{N-t} \right) \\ &= \frac{N^2}{t(N-t)} \sum_{i=t}^{k} \frac{(u_i - u_i t/N)^2}{n_i} \end{aligned} \tag{10.2.2}$$

and Q has approximately the chi-square distribution under H_0. The parameters estimated from the data are the $2k$ probabilities that an observation is less than δ. But for each sample these probabilities sum to 1, and so only k independent parameters are estimated. The number of degrees of freedom for Q is then $2k - 1 - k$, or $k - 1$. The chi-square approximation to the distribution of Q is somewhat improved by multiplication of Q by the factor $(N-1)/N$. Then the rejection region is

$$Q \in R \quad \text{for } \frac{(N-1)Q}{N} \geq \chi^2_{k-1,\alpha}$$

As with the two-sample median test, tied observations do not present a problem unless more than one observation is equal to the median, which can occur only for N odd, or if N is even and the two middle observations are

equal. The conservative approach would base the decision on that resolution of ties which leads to the smallest value of Q. However, we could also leave the resolution to chance by specifying how to treat any ties before looking at the data. For example, we might specify that any observations equal to the median are to be counted as less than the median.

Example 10.2.1

A study has shown that 45% of normal sleepers snore occasionally while 25% snore almost all the time. More than 300 patents have been registered in the U.S. Patent Office for devices purported to stop snoring. Three of these devices are a squeaker sewn into the back of night clothes, a tie to secure the wrists to the sides of the bed, and a chin strap to keep the mouth shut. An experiment was conducted to determine which device is the most effective in stopping snoring or at least in reducing it. Fifteen men who are habitual snorers were divided randomly into three groups to test the devices. Each man's sleep was monitored for one night by a machine that measures amount of snoring on a 100-point scale while using a device. Analyze the results below to determine whether the three devices are equally effective or not.

Squeaker	Wrist Tie	Chin Strap
73	96	12
79	92	26
86	89	33
91	95	8
35	76	78

Solution

The overall sample median is 78. Since $N = 15$ is odd, we have $t = 7$ and the data are

Group	1	2	3
<78	2	1	4
≥ 78	3	4	1

We calculate $Q = 3.75$ from (10.2.2) and $(N-1)\,Q/N = 3.50$. With df $= 2$, we find $0.10 < P < 0.25$ from Table B. There is no significant evidence that the three medians differ.

The STATXACT solution to Example 10.2.1 is shown below. The results do not agree with ours because they define the U statistics as the number of sample observations that are less than or equal to delta, rather than the number strictly less than delta as we did. This means that they define $t = (N+1)/2$ when N is odd, while we define $t = (N-1)/2$. The U statistics with this definition of ≤ 78 are 2, 1, and 5, for groups 1, 2, and 3, respectively. The reader can verify that these values make $Q = 6.964$ as the printout shows. The difference in the answers is surprisingly large and the conclusions are not the same. The STATXACT printout also shows an exact P value and a point probability. We discuss these after Example 10.2.2.

```
************************************
STATXACT SOLUTION TO EXAMPLE 10.2.1
************************************

MEDIAN TEST

Statistics based on the observed one-way layout:
        Number of groups          = 3
        Number of observations    = 15
        The overall median        =          78.00
        Observed statistic        =           6.964

Asymptotic P-value: (based on chi-square distribution with 2 df)
     Pr { CH(X) .GE.     6.964 } =        0.0307

Exact P value and point probability:
     Pr { CH(X) .GE.     6.964 } =      0.0676
     Pr { CH(X) .EQ.     6.964 } =      0.0466
```

Example 10.2.2

The staff of a mental hospital is concerned with which kind of treatment is most effective for a particular type of mental disorder. A battery of tests administered to all patients delineated a group of 40 patients who were similar as regards diagnosis and also personality, intelligence, and projective and physiological factors. These people were randomly divided into four different groups of 10 each for treatment. For 6 months the respective groups received (1) electroshock, (2) psychotherapy, (3) electroshock plus psychotherapy, and (4) no type of treatment. At the end of this period, the battery of tests was repeated on each patient. The only type of measurement possible for these tests is a ranking of all 40 patients on the basis of their relative degree of improvement at the end of the treatment period; rank 1 indicates the highest level of improvement, rank 2 the second highest, etc. On the basis of the data in Table 10.2.1, does there seem to be any difference in effectiveness of the types of treatment?

Solution

We use the median test to see whether the four groups have the same location. The overall sample median is the observation with rank 20.5 since $N = 40$, and we have $t = 20$ and $n_i t/N = 5$. The results are

Group	1	2	3	4
<20.5	1	9	10	0
≥ 20.5	9	1	0	10

TABLE 10.2.1

Ranking of Patients

Groups			
1	**2**	**3**	**4**
19	14	12	38
22	21	1	39
25	2	5	40
24	6	8	30
29	10	4	31
26	16	13	32
37	17	9	33
23	11	15	36
27	18	3	34
28	7	20	35

We calculate $Q = 32.8$ from (10.2.2) and $(N-1)Q/N = 31.98$. From Table B with $df = 3$, we find $P < 0.001$ and we reject the null hypothesis of equal medians for the four groups.

The STATXACT solution for this example is shown below. The results agree with ours and always will when N is even. As with Example 10.2.1, the STATXACT solution shows the calculations of an exact P value and a point probability; this is exact in the sense that it is calculated using the multivariate hypergeometric distribution given in (10.2.1). STATXACT also provides a Monte Carlo estimate of the P value. The reader is referred to the STATXACT manual for more details.

```
**************************************
STATXACT SOLUTION TO EXAMPLE 10.2.2
**************************************

MEDIAN TEST

Statistics based on the observed one-way layout:
        Number of groups        = 4
        Number of observations = 40
        The overall median      = 20.50
        Observed statistic      = 32.80

Asymptotic P value: (based on chi-square distribution with 3 df )
     Pr { CH(X) .GE.       32.80 } =       0.0000

Exact P value and point probability:
     Pr { CH(X) .GE.       32.80 } =       0.0000
     Pr { CH(X) .EQ.       32.80 } =       0.0000
```

```
Monte Carlo estimate of P value:
      Pr { CH(X) .GE.       32.80 } =       0.0000
      99.00% Confidence interval    =  (0.0000, 0.0005)
```

10.3 Extension of the Control Median Test

The control median test was presented in Chapter 6 as an alternative to the median test for the two-sample problem. We showed that the control median test is simple to use, is as efficient as the median test in large samples, and is advantageous in certain experimental situations. We now present a generalization of the control median test (Sen, 1962) to the k-sample case, where the null hypothesis is that $k(\geq 2)$ populations are identical against the general alternative that the populations differ in some way.

Suppose that independent random samples of sizes $n_1, n_2, \ldots, n_k$ are taken from populations 1 through k. Without any loss of generality, let sample 1 be the control sample. First, we choose $q(\geq 1)$ fractions $0 < p_1 < p_2 < \cdots < p_q < 1$ and find the quantiles $X_1^{(1)} < X_1^{(2)} < \cdots < X_1^{(q)}$, corresponding to the fractions from the first sample. Thus, $X_1^{(i)}$ is the r_ith-order statistic of the first sample where $r_i = [n_1 p_i] + 1$ and $[x]$ denotes the largest integer not exceeding x.

The q quantiles define $(q+1)$ nonoverlapping and contiguous cells or blocks written as

$$I_j{:}(X_1^{(j)}, X_1^{(j+1)}] \quad \text{for } j = 0, 1, \ldots, q$$

where we take $X_1^{(0)} = -\infty$ and $X_1^{(q+1)} = \infty$. For the two-sample control median test, we have $k = 2, q = 1, r_1 = [n_1/2] + 1$, and the test is based on the number of observations in sample 2 that belong to I_0. For $k > 2$ samples and $q \geq 1$ quantiles, we count for the ith sample the number of observations V_{ij} that belong to block $I_j, j = 0, 1, \ldots, q, i = 1, 2, \ldots, k$, so that $n_i = \sum_{j=0}^{q} V_{ij}, V_{1j} = r_{j+1}$ for $j \geq 1$ and $V_{10} = r_1$. The generalization of the control median test is based on these counts. In the terminology introduced in Chapter 2, the count V_{ij} is the frequency of the jth block for the ith sample.

The derivation of the joint distribution of the counts $V_{ij}, i = 2, 3, \ldots, k, j = 0, 1, \ldots, q$, provides an interesting example of computing probabilities by conditioning. Given (conditional on) the q quantiles from the first sample, $X_1^{(1)} < X_1^{(2)} < \cdots < X_1^{(q)}$, the joint distribution of $V_{i0}, V_{i1}, \ldots, V_{iq}$, for any $i = 2, 3, \ldots, k$, is the multinomial

$$\frac{n_1!}{v_{i0}!v_{i1!}\ldots v_{iq!}}\left[F_i\left(X_1^{(1)}\right)\right]^{v_{i0}}\left[F_i\left(X_1^{(2)}\right)-F_i\left(X_1^{(1)}\right)\right]^{v_{i1}}$$
$$\cdots\left[F_i\left(X_1^{(q)}\right)-F_i\left(X_1^{(q-1)}\right)\right]^{v_{iq-1}}\left[1-F_i\left(X_1^{(q)}\right)\right]^{v_{iq}} \tag{10.3.1}$$

where $v_{iq}=n_i-(v_{i0}+v_{i1}+\cdots+v_{iq-1})$.

The desired (unconditional) joint probability distribution of $V_{i0}, V_{i1},\ldots,V_{iq}$ can be derived by calculating the expectation of the expression in (10.3.1) with respect to the chosen quantiles from the first sample. The joint distribution of the q quantiles from the first sample is

$$\frac{n_1!}{(v_{10}-1)!(v_{11}-1)!\ldots(v_{1q-1}-1)!v_{1q}!}[F_1(w_1)]^{v_{10}-1}$$
$$\times[F_1(w_2)-F_1(w_1)]^{v_{11}-1}\cdots[F_1(w_q)-F_1(w_{q-1})]^{v_{1q-1}-1}[1-F_1(w_q)]^{v_{1q}}$$

where $-\infty<w_1<w_2<\cdots<w_q<\infty$ and $v_{1q}=n_1-(v_{10}+v_{11}+\cdots+v_{1q-1})$. Given $X_1^{(1)}, X_1^{(2)},\ldots,X_1^{(q)}$, the distribution of $(V_{i0}, V_{i1},\ldots,V_{iq})$ is independent of that of $(V_{j0}, V_{1j},\ldots,V_{jq})$ for $i\neq j$. Thus we obtain the unconditional joint distribution of the V_{ij}'s as

$$P[V_{ij}=v_{ij}, i=2,3,\ldots,k, j=0.1,\ldots,q]$$
$$=\frac{n_1!}{v_{1q}!\prod_{j=0}^{q-1}v_{1j}-1!}\int_A^{\cdots}\int\prod_{j=0}^{q-1}[F_1(w_{j+1})-F_1(w_j)]^{v_{1j}-1}[1-F_1(w_q)]^{v_{1q}}$$
$$\times\prod_{i=2}^{k}\left[\frac{n_i!}{\prod_{j=0}^{q}v_{1q}!}\prod_{j=0}^{q}[F_i(w_{j+1}-F_i(w_j)]^{v_{ij}}\right]\prod_{j=1}^{q}dF_1(w_j) \tag{10.3.2}$$

where the region A is defined by $-\infty<w_0<w_1<\cdots<w_q<w_{q+1}<\infty$. Under H_0, the unconditional joint distribution of the V_{ij}'s reduces to

$$\frac{\prod_{i=1}^{k}n_i!}{N!}\frac{v_q!}{\prod_{i=1}^{k}v_{iq}!}\prod_{j=0}^{q-1}\left[\frac{(v_j-1)!}{v_{1j}-1!\prod_{i=2}^{k}v_{ij}!}\right] \tag{10.3.3}$$

where $N=\sum_{i=1}^{k}n_i$ and $v_j=\sum_{i=1}^{k}v_{ij}$ for $j=0,1,\ldots,q$.

As in the case of the median test, we reject the null hypothesis if any of the V_{ij} are too different from their expected values under the null hypothesis.

An exact P value can be calculated from (10.3.3) corresponding to the observed values of the counts and hence we can make a decision about rejecting H_0 for a given level of significance. In practice, however, such an implementation of the test is tedious, especially for large k, q, and/or sample sizes.

Alternatively, we can use a simplified test statistic defined as

$$Q^* = \sum_{j=0}^{q} \pi_j^{-1} \sum_{i=1}^{k} n_i \left[\frac{v_{ij}}{n_i} - \frac{v_j}{N}\right]^2$$

where $\pi_j = v_{1j}/(n_1 + 1)$ for $j = 0, 1, \ldots, q$. Massey (1951a) considered an extension of the median test based on a similar criterion. Under the null hypothesis, the distribution of Q^* can be approximated by a chi-square distribution with $(k-1)q$ degrees of freedom, provided that $N \to \infty$ with $n_i/N \to c_i$; $0 < c_1, \ldots, c_k < 1$ and $\pi_0, \pi_1, \ldots, \pi_q$ are all nonzero in the limit. Thus, an approximate size α test is to reject H_0 in favor of the general alternative if

$$Q^* \geq \chi^2_{(k-1)q,\alpha}$$

As with the median test, ties do not present any problems here except perhaps in the choice of the quantiles from the first sample. The test is consistent against the general alternative under some mild conditions on the cdf's. When the distributions belong to the location family, $F_i(x) = F(x - \theta_i)$ with $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k = 0$, the test is consistent against any deviations from H_0. However, when the distributions belong to the scale family, $F_i(x) = F(\theta_i x)$ with $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k = 1$, the test is consistent provided either $q \geq 2$, or $q \geq 1$ and $\xi_1 \neq 0$, where $F_1(\xi_1) = \pi_0$.

The asymptotic power and efficacy expressions for this test are given in Sen (1962). An important result is that when $q = 1$ and $\pi_0 = \pi_1 = 1/2$, the test is as efficient as the median test (ARE is 1). More generally, when the same set of quantiles (i.e., the same q and the same set of p's) is used, this test is as efficient as the generalization of the median test (based on q preselected quantiles of the pooled sample) studied by Massey (1951a). Hence, when the sample sizes are large, there is no reason, on the basis of efficiency alone, to prefer one test over the other. However, as a practical matter, finding a quantile or a set of quantiles is always easier in a single sample than in the combined samples, and thus the control median test would be preferred, especially in the absence of any knowledge about the performance of the tests when sample sizes are small. Finally, with regard to a choice of q, the number of quantiles on which the test is based, there is evidence (Sen, 1962) that even though the choice depends on the class of underlying alternative specifications, $q = 1$ or 2 is usually sufficient in practice.

10.4 The Kruskal–Wallis One-Way ANOVA Test and Multiple Comparisons

The median test for k samples uses information about the magnitude of each of the N observations relative to a single number, which is the median of the pooled samples. Many popular nonparametric k-sample tests use more of the available information by considering the relative magnitude of each observation when compared with every other observation. This comparison is effected in terms of ranks.

Since under H_0 we have essentially a single sample of size N from the common population, combine the N observations into a single ordered sequence from smallest to largest, keeping track of which observation is from which sample, and assign the ranks $1, 2, \ldots, N$ to the sequence. If adjacent ranks are well distributed among the k samples, which would be true for a random sample from a single population, the total sum of ranks, $\sum_{i=1}^{N} i = N(N+1)/2$, would be divided proportionally according to sample size among the k samples. For the ith sample, which contains n_i observations, the expected sum of ranks would be

$$\frac{n_i}{N}\frac{N(N+1)}{2} = \frac{n_i(N+1)}{2}$$

Equivalently, we can argue that since under H_0 the expected rank for any observation is the average rank $(N+1)/2$, the expected sum of ranks for n_i observations is $n_i(N+1)/2$. Denote the actual sum of ranks assigned to the elements in the ith sample by R_i. A reasonable test statistic could be based on a function of the deviations between these observed and expected rank sums. Since deviations in either direction indicate disparity between the samples and absolute values are not particularly tractable mathematically, the sum of squares of these deviations can be used as

$$S = \sum_{i=1}^{k}\left[R_i - \frac{n_i(N+1)}{2}\right]^2 \tag{10.4.1}$$

The null hypothesis is rejected for large values of S.

In order to determine the null distribution of S, consider the ranked sample data recorded in a table with k columns, where the entries in the ith column are the n_i ranks assigned to the elements in the ith sample. Then R_i is the ith-column sum. Under H_0, the integers $1, 2, \ldots, N$ are assigned at random to the k columns except for the restriction that there be n_i integers in column i. The total number of ways to make the assignment of ranks then is

the number of partitions of N distinct elements into k ordered sets, the ith of size n_i, and this is

$$\frac{N!}{\prod_{i=1}^{k} n_i!}$$

Each of these possibilities must be enumerated and the value of S calculated for each. If $t(s)$ denotes the number of assignments with the particular value s calculated from (10.4.1), then

$$f_S(s) = t(s) \prod_{i=1}^{k} \frac{n_i!}{N!}$$

Obviously, the calculations required are tedious and will not be illustrated here. Tables of exact probabilities for S are available in Rijkoort (1952) for $k = 3$, 4, and 5, but only for n_i equal and very small. Critical values for some larger equal sample sizes are also given there.

A somewhat more useful test criterion is a weighted sum of squares of deviations, with the reciprocals of the respective sample sizes used as weights. This test statistic, due to Kruskal and Wallis (1952), is defined as

$$H = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{1}{n_i} \left[R_i - \frac{n_i(N+1)}{2} \right]^2 \tag{10.4.2}$$

The consistency of H is investigated in Kruskal (1952). H and S are equivalent test criteria only for all n_i equal. Exact probabilities for H are given in Table K for $k = 3$, all $n_i \leq 5$. The tables in Iman et al. (1975) also cover $k = 4$, all $n_i \leq 4$ and $k = 5$, all $n_i \leq 3$ for the upper 10% of the exact distribution.

Since there are practical limitations on the range of tables that can be constructed, some reasonable approximation to the null distribution is required if a test based on S or H is to be useful in application.

Under the null hypothesis, the n_i entries in column i were randomly selected from the set $\{1, 2, \ldots, N\}$. They actually constitute a random sample of size n_i drawn without replacement from the finite population consisting of the first N integers. The mean and variance of this population are

$$\mu = \sum_{i=1}^{N} \frac{i}{N} = \frac{N+1}{2}$$

$$\sigma^2 = \sum_{i=1}^{N} \frac{[i - (N+1)/2]^2}{N} = \frac{N^2 - 1}{12}$$

The average rank sum for the ith column, $\overline{R}_i = R_i/n_i$, is the mean of this random sample, and as for any sample mean from a finite population,

$$E(\overline{R}_i) = \mu \quad \text{var}(\overline{R}_i) = \frac{\sigma^2(N - n_i)}{n_i(N - 1)}$$

Here then, under H_0, we have

$$E(\overline{R}_i) = \frac{N + 1}{2} \quad \text{var}(\overline{R}_i) = \frac{(N + 1)(N - n_i)}{12n_i}$$
$$\text{cov}(\overline{R}_i, \overline{R}_j) = -\frac{N + 1}{12}$$

Since $\overline{R}_i$ is a sample mean, if n_i is large, the central limit theorem allows us to approximate the distribution of

$$Z_i = \frac{\overline{R}_i - (N + 1)/2}{\sqrt{(N + 1)(N - n_i)/12n_i}} \tag{10.4.3}$$

by the standard normal. Consequently Z_i^2 is distributed approximately as chi square with one degree of freedom. This holds for $i = 1, 2, \ldots, k$, but the Z_i are clearly not independent random variables since $\sum_{i=1}^{k} n_i\overline{R}_i = N(N + 1)/2$, a constant. Kruskal (1952) showed that under H_0, if no n_i is very small, the random variable

$$\sum_{i=1}^{k} \frac{N - n_i}{N} Z_i^2 = \sum_{i=1}^{k} \frac{12n_i[\overline{R}_i - (N + 1)/2]^2}{N(N + 1)} = H \tag{10.4.4}$$

is distributed approximately as chi square with $k - 1$ degrees of freedom. The approximate size α rejection region is $H \geq \chi^2_{\alpha,k-1}$. Some other approximations to the null distribution of H are discussed in Alexander and Quade (1968) and Iman and Davenport (1976). Andrews (1954) discusses the power of this test.

Under the assumption that the populations are continuous, we do not have to deal with the problem of ties. However, ties can occur in practice. When two or more observations are tied within a column, the value of H is the same regardless of the method used to resolve the ties since the rank sum is not affected. When ties occur across columns, the midrank method is generally used. Alternatively, the ties can be broken in the way that is least conducive to rejection of H_0 for a conservative test.

If ties to the extent t are present and are handled by the midrank method, the variance of the finite population is

$$\sigma^2 = \frac{N^2 - 1}{12} - \frac{\sum t(t^2 - 1)}{12}$$

where the sum is over all sets of ties in the population, and this expression should be used in $\text{var}(\overline{R}_i)$ for the denominator of Z_i. In this case (10.4.4) becomes

$$\sum_{i=1}^{k} \frac{N - n_i}{N} \left\{ \frac{\left[\overline{R}_i - \frac{N(N+1)}{2}\right]^2}{\frac{(N+1)(N-n_i)}{12n_i} - \frac{N-n_i}{n_i(N-1)} \frac{\sum t(t^2-1)}{12}} \right\}$$

$$= \sum_{i=1}^{k} \frac{12n_i \left[\overline{R}_i - \frac{N(N+1)}{2}\right]^2}{N(N+1) - \frac{N \sum t(t^2-1)}{N-1}} = \frac{H}{1 - \frac{\sum t(t^2-1)}{N(N^2-1)}} \tag{10.4.5}$$

Hence the correction for ties is simply to divide H in (10.4.2) by the correction factor $1 - \sum t(t^2 - 1)/N(N^2 - 1)$ where the sum is over all sets of t tied ranks. The details are left as an exercise for the reader.

When the null hypothesis is rejected, we can compare any two groups, say i and j (with $1 \leq i < j \leq k$), by a *multiple comparisons procedure*. We calculate

$$Z_{ij} = \frac{|\overline{R}_i - \overline{R}_j|}{\sqrt{[N(N+1)/12](1/n_i + 1/n_j)}} \tag{10.4.6}$$

and compare it to the $[\alpha/k(k-1)]$st upper standard normal quantile $z^* = z_{\alpha/[k(k-1)]}$. If Z_{ij} exceeds z^*, the two groups are declared to be significantly different. The quantity α is called the *experimentwise error rate* or the *overall significance level*, which is the probability of at least one erroneous rejection among the $k(k-1)/2$ pairwise comparisons. Typically, one takes $\alpha \geq 0.20$ because we are making such a large number of statements. We note that $1 - \alpha$ is the probability that all of the statements are correct. It is not necessary to make all possible comparisons, although we usually do. For convenience, we give the z^* values to three decimal places for a total of $d = k(k-1)/2$ comparisons at $\alpha = 0.20$ as follows:

d	1	2	3	4	5	6	7	8	9	10
z^*	1.282	1.645	1.834	1.960	2.054	2.128	2.189	2.241	2.287	2.326

This multiple comparisons procedure is due to Dunn (1964).

10.4.1 Applications

The Kruskal–Wallis test is the natural extension of the two-sample Wilcoxon test for location to the case of k mutually independent samples from continuous populations. The null hypothesis is that the k populations are the same, but when we assume the location model, this hypothesis can be written in terms of the respective location parameters (or treatment effects) as

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$$
$$H_1 : \text{At least two } \theta\text{'s differ}$$

To perform the test, all $n_1 + n_2 + \cdots + n_k = N$ observations are pooled into a single array and ranked from 1 to N. The test statistic H in (10.4.2) is easier to calculate in the form

$$H = \frac{12}{N(N+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(N+1) \tag{10.4.7}$$

where R_i is the sum of the ranks from the ith sample. The appropriate rejection region is large values of H. The critical values or P values are found from Table K for $k = 3$, each $n_i \leq 5$. This statistic is asymptotically chi-square distributed with $k - 1$ degrees of freedom; the approximation is generally satisfactory except when $k = 3$ and the sample sizes are five or less. Therefore, Table B can be used when Table K cannot. When there are ties, we divide H by the correction factor, as shown in (10.4.5).

For multiple comparisons, using (10.4.6), we declare treatments i and j to be significantly different in effect if

$$|\overline{R}_i - \overline{R}_j| \geq z^* \sqrt{\frac{N(N+1)}{12}\left(\frac{1}{n_i} + \frac{1}{n_j}\right)} \tag{10.4.8}$$

If $n_i = n_j = N/k$ for all i and j, the right-hand side of (10.4.8) reduces to $z^*\sqrt{k(N+1)/6}$.

Example 10.4.1

For the experiment described in Example 10.2.2, use the Kruskal–Wallis test to see if there is any difference between the medians of the four groups.

SOLUTION

The data are already ranked from 1 to 40 in Table 10.2.1 so we need only calculate the rank sums as $R_1 = 260, R_2 = 122, R_3 = 90, R_4 = 348$. With $n_1 = n_2 = n_3 = n_4 = 10$, we get

$$H = \frac{12}{40(41)(10)}[260^2 + 122^2 + 90^2 + 348^2] - 3(41) = 31.89$$

with 3 degrees of freedom. The P value from Table B is $P < 0.001$, so we reject the null hypothesis that the four medians are the same and do a follow-up analysis by a multiple comparisons test using $\alpha = 0.20$. We have $\overline{R}_1 = 26.0, \overline{R}_2 = 12.2, \overline{R}_3 = 9.0$, and $\overline{R}_4 = 34.8$ and the right-hand side of (10.4.8) is 11.125. The treatments, which have significantly different medians, are 1 and 2, 1 and 3, 2 and 4, and 3 and 4.

The computer solutions to Example 10.4.1 are shown below using the MINITAB, SAS, and STATXACT packages. All of the results for H agree exactly.

```
**********************************
MINITAB SOLUTION TO EXAMPLE 10.4.1
**********************************

   Kruskal-Wallis test: C2 versus C1

      Kruskal-Wallis test: on C2

C1         N     Median     Ave Rank          Z
1         10     25.500         26.0       1.72
2         10     12.500         12.2      -2.59
3         10      8.500          9.0      -3.59
4         10     34.500         34.8       4.47
Overall   40                    20.5
H = 31.89  df = 3  P = 0.000
```

MINITAB shows the value of the test statistics as $H = 31.89$ and the asymptotic P value of 0.000 based on the chi-square approximation with 3 degree of freedom. If there had been ties in the data, MINITAB would have shown H(adjusted), which is calculated from (10.4.5). MINITAB also shows the median, average rank, and Z value for each group. The Z values given are calculated from (10.4.3). This is the standardized value of the deviation between the mean rank $\overline{R}_i$ for the ith group and its expected value $(N + 1)/2$ under the null hypothesis. The sign of the Z statistic indicates whether the mean rank is larger or smaller than expected, and the magnitude measures the relative deviation. The largest absolute Z value is 4.47, which indicates that the mean rank for group 4, which is 34.8, differs from the average rank of 20.5 more than that of any other group. And the smallest absolute Z value, 1.72, shows that the average for group 1, 26.0, differs from the average rank less than that of any other group.

Now we show the program code and the results for SAS and STATXACT.

```
                ******************************
                SAS SOLUTION TO EXAMPLE 10.4.1
                ******************************
Program:

data a;
input group N;
do i = 1 to N;
input battery @@;
output;
end;
cards;
1  10
19  22  25  24  29  26  37  23  27  28
2  10
14  21  2  6  10  16  17  11  18  7
3  10
12  1  5  8  4  13  9  15  3  20
4  10
39  39  40  30  31  32  33  36  34  35
;
proc npar1way wilcoxon;
class group;
run;

Output
```

The NPAR1WAY Procedure

Wilcoxon Scores (Rank Sums) for Variable Battery
Classified by Variable Group

Group	−*N*	Sum of Scores	Expected Under H0	Std.-dev. under H0	Mean Score
1	10	260.0	205.0	32.014119	26.00
2	10	122.0	205.0	32.014119	12.20
3	10	90.0	205.0	32.014119	9.00
4	10	348.0	205.0	32.014119	34.80

Average scores were used for ties.

Kruskal–Wallis test

Chi-square	31.8967
df	3
Pr > chi-square	< .0001

```
************************************
STATXACT SOLUTION TO EXAMPLE 10.4.1
************************************

KRUSKAL-WALLIS TEST [That the 4 populations are identically
distributed]

Statistic based on the observed data:

     T(X) = The observed test statistic =        31.89

Asymptotic P value: (based on chi-square distribution with 3 df)
     Pr  { T(X) .GE.       31.89 } =        0.0000

Monte Carlo estimate of P value :
     Pr  {  Statistic  .GE.    31.89  } =     0.0000
     99.00% Confidence interval = (      0.0000,      0.0005)
```

Example 10.4.2

For the experiment described in Example 10.2.1, use the Kruskal–Wallis test to see if there is any difference between the medians of the three groups.

Solution

The first step is to rank the data from 1 to 15, as shown below, where rank 1 is given to the smallest score, which indicates the most effective result.

	Squeaker	Wrist Tie	Chin Strap
	6	15	2
	9	13	3
	10	11	4
	12	14	1
	5	7	8
Sum	42	60	18

We calculate $\sum R^2/n = 5688/5 = 1137.6$ and $H = 12(1137.6)/15(16) - 3(16) = 8.88$. Table K for $k = 3, n_1 = n_2 = n_3 = 5$ shows that $0.001 < P$ value < 0.010, so the null hypothesis of equal treatment effects should be rejected. It appears that the chin strap is the most effective device in reducing snoring since it has the smallest sum of ranks. Since the null hypothesis was rejected, we carry out a multiple comparisons procedure at the 0.20 level. We have $z^* = 1.834$ for $d = 3$ and the right-hand side of (10.4.8) is 5.19. The sample mean ranks are $\overline{R}_1 = 8.4, \overline{R}_2 = 12, \overline{R}_3 = 3.6$. Our conclusion is that only groups 2 and 3 have significantly different treatment effects at the overall 0.20 significance level. Recall that our hand calculations did not lead to a rejection of the null hypothesis by the median test in Example 10.2.1.

The computer solutions to Example 10.4.2 are shown below using the SAS, STATXACT, and MINITAB packages. The results for the value of *H* agree exactly. The *P* value using the chi-square approximation is 0.012, which agrees with the outputs. Note that both STATXACT and SAS allow the user the option of computing what they call an exact *P* value based on the permutation distribution of the Kruskal–Wallis statistic. This can be very useful when the sample sizes are small so that the chi-square approximation could be suspect. However, the exact computation is time consuming even for moderate sample sizes, such as 10 as in Example 10.4.1. For this example, SAS finds this exact *P* value to be 0.0042, which agrees with our conclusion from Table K. MINITAB does not have an option to calculate an exact *P* value and it does not provide the correction for ties.

```
********************************
SAS SOLUTION TO EXAMPLE 10.4.2
********************************

            The NPAR1WAY Procedure

    Wilcoxon Scores (Rank Sums) for Variable Snore
           Classified by Variable Group

                 Sum of    Expected     Std.-dev.     Mean
Group      N     Scores    Under H0     Under H0      Score
1          5     42.0      40.0         8.164966       8.40
2          5     60.0      40.0         8.164966      12.00
3          5     18.0      40.0         8.164966       3.60

               Kruskal-Wallis test

     Chi-square                          8.8800
     df                                       2
     Asymptotic  Pr >    chi-square      0.0118
     Exact       Pr >=   chi-square      0.0042

  *************************************
  STATXACT SOLUTION TO EXAMPLE 10.4.2
  *************************************

KRUSKAL-WALLIS TEST [That the three populations are
identically distributed]

Statistic based on the observed data:

     T(X) = The Observed test Statistic =    8.880

Asymptotic P value: (based on chi-square distribution with 2 df )
     Pr { T(X) .GE. 8.880 } = 0.0118

Exact P value and point probability:
     Pr { Statistic .GE. 8.880 } = 0.0042
     Pr { Statistic .GE. 8.880 } = 0.0003
```

```
**********************************
MINITAB SOLUTION TO EXAMPLE 10.4.2
**********************************

Kruskal-Wallis test: C1 versus C2
Kruskal-Wallis test on C1

C2            N      Median      Ave Rank          Z
1             5       79.00           8.4       0.24
2             5       92.00          12.0       2.45
3             5       26.00           3.6      -2.69
Overall      15                       8.0

H = 8.88  DF = 2  P = 0.012
```

10.5 Other Rank-Test Statistics

A general form for any k-sample rank-test statistic, which follows the rationale of the Kruskal–Wallis statistic, can be developed as follows. Denote the $\sum_{i=1}^{k} n_i = N$ items in the pooled (not necessarily ordered) sample by $X_1, X_2, \ldots, X_N$, assign ranks and put a subscript on each rank to indicate which sample produced that observation. Thus $r_j(X_i)$ is the rank of X_i where X_i is from the jth sample, for some $j = 1, 2, \ldots, k$. The rank sum for the jth sample, previously denoted by R_j would now be denoted by $\sum_i r_j(X_i)$. Since the $r_j(X_i)$ for fixed j are a random sample of n_j numbers, the sum of any monotone increasing function g of $r_j(X_i)$ should, if the null hypothesis is true, on the average be approximately equal to the average of the function for all N observations multiplied by n_j for every j. The weighted sum of squares of these deviations provides a test criterion. Thus a general k-sample rank statistic can be written as

$$Q = \sum_{j=1}^{k} \frac{\left\{\sum_{i=1}^{n_j} g[r_j(X_i)] - n_j\left(\sum_{j=1}^{k}\sum_{i=1}^{n_j} g[r_j(X_i)]\right)\Big/N\right\}^2}{n_j} \tag{10.5.1}$$

For simplicity, we now denote the set of all N values of the function $g[r_j(x_i)]$ by $a_1, a_2, \ldots, a_N$ and their mean by

$$\bar{a} = \sum_{i=1}^{N} \frac{a_i}{N} = \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{g[r_j(x_i)]}{N}$$

It can be shown (see Hajék and Sidak, 1967, pp. 170–172) that as minimum $(n_1, n_2, \ldots, n_k) \to \infty$, under certain regularity conditions the probability distribution of

$$\frac{(N-1)Q}{\sum_{i=1}^{N}(a_i - \bar{a})^2}$$

approaches the chi-square distribution with $k-1$ degrees of freedom.

Two obvious possibilities for our function g are suggested by the scores in the two-sample location problem for the Terry (normal scores) and the van der Waerden (inverse normal scores) test statistics. Since in both these cases the scores are symmetric about zero, $\bar{a}$ is zero and the k-sample analogs are

$$T = \frac{N-1}{\sum_{i=1}^{N}[E(\xi_{(i)})]^2} \sum_{j=1}^{k} \frac{\left[\sum_{i=1}^{n_j} E(\xi_{(i)})_j\right]^2}{n_j}$$

$$X = \frac{N-1}{\sum_{i=1}^{N}\left[\Phi^{-1}\left(\frac{i}{N+1}\right)\right]^2} \sum_{j=1}^{k} \frac{\left[\sum_{i=1}^{n_j} \Phi^{-1}\left(\frac{i}{N+1}\right)_j\right]^2}{n_j}$$

The T and X tests are asymptotically equivalent as before.

Example 10.5.1

For the data in Example 10.2.2, the normal scores test is illustrated using STATXACT to see if there are any differences in the medians of the four groups.

```
************************************
STATXACT SOLUTION TO EXAMPLE 10.5.1
************************************

NORMAL SCORES TEST [That the four populations are identically
distributed]

Statistic based on the observed data:
     The observed statistic  =  29.27

Asymptotic P value: (based on chi-square distribution with 3 df )
     Pr  {  Statistic  .GE.   29.27  }  =  0.0000.

Monte Carlo estimate of P value:
     Pr  {  Statistic  .GE.   29.27  }  =  0.0000
     99.00% Confidence interval      =  (  0.0000, 0.0005)
```

The value of the T statistic is found to be 29.27 with an approximate P value close to 0 and this leads to rejection of the null hypothesis. Recall that for these data, both the median test and the Kruskal–Wallis test also led to rejection of the null hypothesis.

So far we have discussed the problem of testing the null hypothesis that k continuous populations are identical against the general (omnibus) alternative that they differ in some way. In practice, the experimenter may expect, in advance, specific kinds of departures from the null hypothesis, say in a particular direction. For example, it might be of interest to test that a group of treatments have an increasing (or decreasing) effect on some response variable. Conceptually, some of these problems can be viewed as generalizations of one-sided alternatives to the case of more than two samples. It seems reasonable to expect that we will be able to construct tests that are more sensitive (powerful) in detecting the specific departures from the null hypothesis than an omnibus test, like the Kruskal–Wallis test, since the latter does not use the prior information in a postulated ordering.

The problem of testing the null hypothesis of homogeneity against alternative hypotheses that are more specific or restricted in some manner than a global alternative (of nonhomogeneity) has been an area of active research. The seminal work of Barlow et al. (1972) and the book by Robertson et al. (1988) are excellent references on this subject. We will discuss some of these problems in the following sections.

10.6 Tests against Ordered Alternatives

Suppose we are concerned with testing the null hypothesis H_0 that the populations are identical against the alternative hypothesis that the location parameters are in an increasing order,

$$H_1 : \theta_1 \leq \theta_2 \leq \cdots \leq \theta_k$$

where at least one of the inequalities is strict. This alternative would be of interest if, for example, it is expected that increasing the value of a factor would result in an increase in the value of a response. If the expected direction in H_1 is not the natural order, we simply relabel the treatments so that the postulated order agrees with the natural order. This alternative is known as the ''simple order.'' Under the location model when $\theta_i < \theta_j$, the population corresponding to F_j is stochastically larger than the population corresponding to F_i.

A number of distribution-free tests for the problem of simple order are available in the literature. One way to motivate some of these tests is to note that the alternative hypothesis H_1 may be written in an expanded form as

$$\begin{aligned}
&\theta_1 \le \theta_2, \theta_1 \le \theta_3, \ldots, \theta_1 \le \theta_k \\
&\qquad \theta_2 \le \theta_3, \theta_2, \le \theta_4, \ldots, \theta_2 \le \theta_k \\
&\qquad\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \theta_{k-2} \le \theta_{k-1}, \theta_{k-2} \le \theta_k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \theta_{k-1} \le \theta_k
\end{aligned} \tag{10.6.1}$$

where at least one of the $k(k-1)/2$ inequalities is strict. Hence the problem of testing H_0 against H_1 may be viewed as a collection of $k(k-1)/2$ test problems, each of which is a two-sample problem. This observation allows us to contemplate several different tests based on those already available for the two-sample problem.

It is clear that all of the $k(k-1)/2$ two-sample test statistics must be of the same type, for example, Mann–Whitney, median, or control median, but which of the available two-sample tests should be used? We can use either the optimal test (from the ARE point of view) or some test that is more attractive from a practical point of view. After a two-sample test is chosen, and $k(k-1)/2$ of these tests are performed, the next question is to decide how to combine these tests into a single final test with desirable properties.

A popular test for the ordered alternatives problem was proposed by Terpstra (1952) and Jonckheere (1954) independently, hereafter called the JT test. The JT test uses a Mann–Whitney statistic U_{ij} for the two-sample problem of comparing samples i and j, where $i, j = 1, 2, \ldots, k$ with $i < j$, and an overall test statistic is constructed simply by adding all the U_{ij}. Thus the test statistic is

$$\begin{aligned}
B &= U_{12} + U_{13} + \cdots + U_{1k} + U_{23} + U_{24} + \cdots + U_{2k} + \cdots + U_{k-1,k} \\
&= \sum_{1 \le i < j \le k} \sum U_{ij} = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} I(X_{ir} < X_{js})
\end{aligned}$$

where

X_{ir} is the rth observation in sample i
X_{js} is the sth observation in sample j
I is the usual indicator function

The appropriate rejection region is large values of B because if the alternative H_1 is true, observations from the jth sample will tend to be larger than

observations from the ith sample. Thus the appropriate rejection region at level α is

$$B \geq B(\alpha, k, n_1, n_2, \ldots, n_k)$$

where $P[B \geq B(\alpha, k, n_1, n_2, \ldots, n_k)] \leq \alpha$ is satisfied under H_0.

The JT test is distribution-free if the cdf's are all continuous. Since all $N!/\left(\prod_{i=1}^{k} n_i!\right)$ rank assignments are equally likely under H_0, the null distribution of the test statistic B can be obtained by computing the value of B for each possible ranking and enumerating. The required calculations are tedious, especially for large n_i, and will not be illustrated here. Some exact critical values have been tabulated for $k = 3, 2 \leq n_1 \leq n_2 \leq n_3 \leq 8$; $k = 4, 5, 6, n_1 = n_2 = \cdots = n_k = 2(1)(6)$ and are given in Table R for selected α.

In practice, for larger sample sizes, it is more convenient to use an approximate test. If n_i/N tends to some constant between 0 and 1, the distribution of the random vector $(U_{12}, U_{13}, \ldots, U_{k-1,k})$ under H_0 can be approximated by a $k(k-1)/2$-dimensional normal distribution. From the results in Section 6.6 for the Mann–Whitney test we have $E(U_{ij}) = n_i n_j/2$ under H_0 so that

$$E_0(B) = \sum_{1 \leq i < j \leq k} \sum \frac{n_i n_j}{2} = \frac{N^2 - \sum_{i=1}^{k} n_i^2}{4} \tag{10.6.2}$$

The derivation of the variance of B is more involved and is left as an exercise for the reader. The result under H_0 is

$$\text{var}_0(B) = \frac{N^2(2N+3) - \sum_{i=1}^{k} n_i^2(2n_i+3)}{72} \tag{10.6.3}$$

In view of these results, an approximate level α test based on the JT statistic is to reject H_0 in favor of H_1 if

$$B \geq E_0(B) + z_\alpha [\text{var}_0(B)]^{1/2}$$

where z_α is the $(1-\alpha)$th quantile of the standard normal distribution.

Because of our continuity assumption, theoretically there can be no tied observations within or between samples. However, ties do occur in practice, and when the number of ties is large the test should be modified in a suitable manner. When observation(s) from sample i are tied with observation(s) from sample j, we replace U_{ij} by U_{ij}^* defined as

$$U_{ij}^* = \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} D_{rs}$$

where

$$D_{rs} = \begin{cases} 1 & \text{if } X_{ir} < X_{js} \\ 0.5 & \text{if } X_{ir} = X_{js} \\ 0 & \text{if } X_{ir} > X_{js} \end{cases}$$

This is equivalent to replacing the ranks of the tied observations by their midranks in the combined samples i and j. The JT test statistic in the case of ties is then

$$B^* = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} U_{ij}^* \tag{10.6.4}$$

Under the null hypothesis, the expectation of B^* is the same as that of B, given in (10.6.2). The variance of B^* under H_0 is

$$\begin{aligned} \operatorname{var}_0(B^*) = (72)^{-1} &\left[N(N-1)(2N+5) - \sum_{i=1}^{k} n_i(n_i-1)(2n_i+5) - \sum\nolimits_1 t(t-1)(2t+5) \right] \\ &+ [36N(N-1)(N-2)]^{-1} \left[\sum_{i=1}^{k} n_i(n_i-1)(n_i-2) \right] \left[\sum\nolimits_1 t(t-1)(t-2) \right] \\ &+ [8N(N-1)]^{-1} \left[\sum_{i=1}^{k} n_i(n_i-1) \right] \left[\sum\nolimits_1 t(t-1) \right] \end{aligned} \tag{10.6.5}$$

where $\sum_1$ denotes the summation over all groups of t ties among the N observations, and t denotes the number of observations tied at any one value.

For a proof of this result see, for example, Kendall and Gibbons (1990, pp. 95–96). When there are no ties, $t = 1$ for all N observations and (10.6.5) reduces to (10.6.3). Although tables are apparently not yet available for the exact null distribution of B^*, an approximate test for large sample sizes can be based on the statistic $[B^* - E_0(B^*)]/[\operatorname{var}_0(B^*)]^{1/2}$ and the standard normal distribution. Thus, in the case of ties, an approximately size α JT test is to reject H_0 in favor of H_1 if

$$B^* \geq E_0(B^*) + z_\alpha [\operatorname{var}_0(B^*)]^{1/2} \tag{10.6.6}$$

The JT test is consistent against ordered alternatives under the assumption that n_i/N tends to some constant between 0 and 1 as $N \to \infty$. The asymptotic relative efficiency and some asymptotic power comparisons with competing tests are discussed in Puri (1965).

A vast body of literature is available on related and alternative procedures for this problem. Chacko (1963), Hogg (1965), Puri (1965), Conover (1967),

Shorack (1967), Johnson and Mehrotra (1971), Tryon and Hettmansperger (1973), Shirahata (1980), Fairley and Fligner (1987), and Hettmansperger and Norton (1987), among others, consider various extensions. Among these, Puri (1965) studies a generalization of the JT test by introducing a class of linear rank statistics using an arbitrary score function and any two-sample Chernoff–Savage statistic. When the expected normal scores are used, Puri's procedure is highly recommended (Berenson, 1982) for samples from (1) normal populations; (2) light-tailed populations of the beta family, regardless of symmetry; (3) heavy-tailed and moderately skewed populations; and (4) heavy-tailed and very skewed populations. Tryon and Hettmansperger generalize Puri's class of tests by including weighting coefficients to form linear combinations of two-sample Chernoff–Savage statistics and provide some interesting results about how to determine the optimal weighting coefficients. Chakraborti and Desu (1988a) adopt a similar approach using the two-sample control quantile (median) test statistics and show that their optimal test has higher ARE in certain situations.

10.6.1 Applications

The JT test rejects H_0 against the ordered alternative H_1 when B is significantly large. The exact P value is

$$P(B \geq b_0 | H_0)$$

where b_0 is the observed value of the test statistic B. When sample sizes are moderately large, the normal approximation to the P value is given by

$$1 - \Phi\left[\frac{B - E_0(B)}{\sqrt{\text{var}_0(B)}}\right]$$

where $E_0(B)$ and $\text{var}_0(B)$ are given in (10.6.2) and (10.6.3).

Example 10.6.1

Experts have long claimed that speakers who use some sort of audiovisual aids in their presentations are much more effective in communicating with their audience. A consulting agency would like to test this claim; however, it is very difficult to find many speakers who can be regarded as (virtually) identical in their speaking capabilities. The agency is successful in locating only 15 such speakers from a nationwide search. The speakers are randomly assigned to one of three groups. The first group of speakers were not allowed to use any audiovisual aids, the second group were allowed to use a regular overhead projector and a microphone, and the third group could use a 35 mm color slide projector together with a microphone and a tape recorder (which played prerecorded audio messages). After a certain period of time, each speaker made a presentation in an auditorium, on a certain issue, in front of a live audience and a selected panel of judges. The contents of their presentations were virtually the

same, so that any differences in effectiveness could be attributed only to the audiovisual aids used by the speakers. The judges scored each presentation on a scale of 30–100, depending on their own judgment and the reaction of the audience, with larger scores denoting greater effectiveness; the scores are given below. It seems reasonable to expect that the use of audiovisual aids would have some beneficial effect and hence that the median score for group 1 will be the lowest, that for group 3 the highest, and the median score for group 2 somewhere in between.

Group 1	Group 2	Group 3
74, 58, 68, 60, 69	70, 72, 75, 80, 71	73, 78, 88, 85, 76

SOLUTION

The null hypothesis to be tested is $H_0: \theta_1 = \theta_2 = \theta_3$, where θ_i is the median of the ith group, against $H_1: \theta_1 \leq \theta_2 \leq \theta_3$, where at least one of the inequalities is strict. Here $k = 3$ and the three two-sample Mann–Whitney statistics U_{12}, U_{13}, and U_{23} are needed in order to apply the JT test. We find $U_{12} = 22, U_{13} = 24$, and $U_{23} = 21$ and hence $B = 67$. The exact P value for the JT test from Table R is $P(B \geq 67|H_0) < 0.0044$. Thus H_0 is rejected in favor of H_1 at any commonly used value of α and we conclude that audiovisual aids do help in making a presentation more effective. In fact, when all other factors are equal, there is evidence that the more audiovisual aids are used, the more effective the presentation. Also, we have $E_0(B) = 37.5$ and $\text{var}_0(B) = 89.5833$ from (10.6.2) and (10.6.3), respectively, so that using the normal approximation, we find $z = 3.1168$ (without a continuity correction) and the approximate P value from Table A is $1 - \Phi(3.12) = 0.0009$; the approximate JT test leads to the same conclusion. The SAS and STATXACT computer solutions shown below agree exactly with ours. Although MINITAB provides an option for the Mann–Whitney statistic, it actually calculates the Wilcoxon rank sum statistics discussed in Section 8.2. The equivalent U_{ij} can be found from these.

```
*********************************
 SAS SOLUTION TO EXAMPLE 10.6.1
*********************************
```

```
Program:

data example;
input group score @@;
datalines;
1 74 1 58 1 68 1 60 1 69 2 70 2 72 2 75 2 80 2 71
3 73 3 78 3 88 3 85 3 76
;
proc freq data=example;
tables group*score/noprint;
exact JT;
run;
```

Output

```
                    The FREQ Procedure

          Statistics for Table of Group by Score

          Jonckheere-Terpstra test

        Statistic (JT)                  67.0000
        Z                                3.1168

      Asymptotic test
      One-sided Pr > Z                  0.0009
      Two-sided Pr > |Z|                0.0018

      Exact test
      One-sided Pr >= JT             5.259E-04
      Two-sided Pr >= | JT - Mean |     0.0011

           Sample size = 15


          ************************************
          STATXACT SOLUTION TO EXAMPLE 10.6.1
          ************************************

JONCKHEERE-TERPSTRA TEST [That the three populations are iden-
tically distributed]
Statistic Based on the 15 Observations:

Mean    Std.-dev.    Observed (JT (x))    Standardized (JT* (x))
37.50    9.465             67.00                3.117

Asymptotic P value:
  One-sided:   Pr { JT* (X)  .GE.    3.117 }   =   0.0009
  Two-sided:   2  *  One-sided                 =   0.0018

Exact P values:
  One-sided:   Pr { JT* (X)  .GE.    3.117 }   =   0.0005
               Pr { JT* (X)  .EQ.    3.117 }   =   0.0002
  Two-sided:   Pr { |JT* (X)| .GE.   3.117 }   =   0.0011
```

10.7 Comparisons with a Control

The case of ordered alternatives is one example of a situation where the experimenter wishes to detect, a priori, not just any differences among a group of populations, but differences in some specific direction. We now consider another example where the alternative to the null hypothesis of homogeneity is restricted in a specific direction.

Suppose we want to test only a partial ordering of the $(k-1)$ distributions with respect to a common distribution. This will be the situation where the only comparison of interest is between a control distribution and each other distribution; what happens among the $(k-1)$ other distributions is irrelevant. For example, in a drug screening study, we may want to compare a group of treatments under development to what is currently in use (which may be nothing or a placebo), called the control, and then subject those treatments that are ''better'' than the control to more elaborate studies. In a business environment, people might consider changing their current investment policy to one of a number of newly available comparable policies provided the payoff is higher.

The alternative of interest here is another generalization of the one-sided alternatives problem and constitutes a subset of the ordered alternatives problem discussed in the last section. One would expect, at least intuitively, to be able to use the available pertinent information to construct a test that is more powerful than the test for either the general or the ordered alternatives problem.

Without any loss of generality let F_1 be the cdf of the control population and let F_i be the cdf of the ith treatment population, $i = 2, 3, \ldots, k$, where $F_i = F(x - \theta_i)$, with $F(0) = p$, so that θ_i is the pth $(0 < p < 1)$ quantile of F_i. Our problem is to test, for a specified p, the null hypothesis that the pth treatment quantiles are equal to each other and equal to the pth quantile in the control population,

$$H_0 : \theta_2 = \theta_3 = \cdots = \theta_k = \theta_1$$

against the one-sided alternative

$$H_1 : \theta_2 \geq \theta_1, \theta_3 \geq \theta_1, \ldots, \theta_k \geq \theta_1$$

where at least one of the inequalities is strict. As pointed out earlier, when $\theta_i > \theta_1$, F_i is stochastically larger than F_1. In the literature on hypothesis testing with restricted alternatives, this alternative is known as the simple-tree alternative. Miller (1981), among others, calls this a ''many-one'' problem.

In some situations, it might be of interest to test the alternative in the opposite direction, $\theta_2 \leq \theta_1, \theta_3 \leq \theta_1, \ldots, \theta_k \leq \theta_1$ where at least one of the inequalities is strict. The tests we discuss can easily be adapted for this case.

Our approach in the many-one problem is similar to that in the ordered alternatives problem in that we view the test as a collection of $(k-1)$ subtest problems $H_{0i}: \theta_i = \theta_1$ against $H_{1i}: \theta_i \geq \theta_1$ for $i = 2, 3, \ldots, k$. Thus, a distribution-free test is obtained in two steps. First an appropriate one-sample test for the ith subtest problem is selected and then $(k-1)$ of these tests are combined in a suitable manner to produce an overall test.

Before considering specific tests, we need to make a distinction between Case I where sufficient prior knowledge about the control is at hand so that the control population may be assumed to be known (in this case the control is often called a standard), at least to the extent of the parameter(s) of interest, and Case II where no concrete knowledge about the control group is available. These two cases will be treated separately since the test procedures are somewhat different.

10.7.1 Case I: θ_1 Known

First consider the case of testing H_0 against the alternative H_1 where θ_1 is either known or specified in advance. In this case, the subtest problem, for every $i = 2, 3, \ldots, k$, is a one-sample problem and therefore one of several available distribution-free tests discussed earlier can be used. For example, if the only reasonable assumption about the parent distributions is continuity, we would use the sign test. If it can be assumed that the underlying distributions are symmetric, we may want to use the Wilcoxon signed-rank test. For simplicity and ease of presentation we detail only the tests based on the sign test, although one can proceed in a similar manner with some other one-sample distribution-free tests. Part of this discussion is from the papers by Chakraborti and Gibbons (1991, 1992).

The sign test statistic for testing H_{0i} against H_{1i} may be based on the total number of negative differences $X_{ij} - \theta_1$ in the ith sample,

$$V_i = \sum_{j=1}^{n_i} I(X_{ij} - \theta_1 < 0)$$

$i = 2, 3, \ldots, k$, and we reject H_{0i} in favor of H_{1i} if V_i is small. With this motivation, a simple overall test of H_0 against H_1 can be based on V, the sum of the $V_i's, i = 2, 3, \ldots, k$, and the rejection region consists of small values of V. One practical advantage of using V is that under H_0, V has a binomial distribution with parameters $N = \sum_{i=2}^{k} n_i$ and p. Accordingly, P values or exact critical values can be found using binomial tables for small to moderate sample sizes. For larger sample sizes, the normal approximation to the binomial distribution can be used to construct tests with significance levels approximately equal to the nominal value or to find the approximate P value.

A simple modification of this sum test when the sample sizes are quite different is to use $V^* = \sum_{i=2}^{k} (V_i/n_i)$ as the test statistic since the $V_i's$, and hence V, may be quite sensitive to unequal sample sizes. The exact and/or approximate test can be constructed as before, although we no longer have the convenience of using tables of the binomial distribution for an exact test. The details are left as an exercise for the reader (see, for example, Chakraborti and Gibbons, 1992).

An alternative test for this problem can be obtained by applying the union–intersection principle (Roy, 1953). Here the null hypothesis H_0 is the intersection of the $(k-1)$ subnull hypotheses $H_{0i}: \theta_i = \theta_j$. Thus, H_0 should be rejected if and only if at least one of the subnull hypotheses H_{0i} is rejected and this occurs if the smallest of the statistics $V_2, V_3, \ldots, V_k$ is too small. In other words, an overall test can be based on

$$V^+ = \min\left(\frac{V_2}{n_2}, \frac{V_3}{n_3}, \ldots, \frac{V_k}{n_k}\right) \tag{10.7.1}$$

and we should reject H_0 in favor of H_1 if V^+ is small. The test based on V^+ is expected to be more sensitive than the sum test since a rejection of any of the $(k-1)$ subnull hypotheses here would lead to a rejection of the overall null hypothesis.

The exact distribution of V^+ can be obtained using the fact that $V_2, V_3, \ldots, V_k$ are mutually independent and under H_0 each V_i has a binomial distribution with parameters n_i and p. Thus

$$P(V^+ \le v | H_0) = 1 - \prod_{i=2}^{k} \left[1 - \sum_{j=0}^{[n_i v]} \binom{n_i}{j} p^j (1-p)^{n_i - j}\right] \tag{10.7.2}$$

where v is a fraction between 0 and 1 and $[x]$ denotes the greatest integer not exceeding x.

The exact null distribution can be used to calculate exact P values for small sample sizes. When sample sizes are large, it is more convenient to use the normal approximation to the binomial distribution to calculate approximate P values.

10.7.2 Case II: θ_1 Unknown

When θ_1 is unknown, we use the same general idea to first choose a suitable test for the ith subtest problem (which in the present case is a two-sample problem), $i = 2, 3, \ldots, k$, and then combine the $(k-1)$ test statistics to construct an overall test statistic. However, the details are more involved because the statistics to be combined are now dependent.

To study our tests in this case consider, for the ith subtest problem, the following "i to 1" statistics

$$W_i = \sum_{j=1}^{n_i} I(X_{ij} < T)$$

where T is a suitable estimate of θ_1. By analogy with the one-sample case the quantity W_i can be called a two-sample sign statistic, which allows us to consider direct extensions of our earlier procedures to the present case. If in fact T is a sample order statistic (for example, when θ_1 is the median of F_1, T should be the median of the sample from F_1), W_i is simply the placement of T among the observations from the ith sample. We have seen that the distribution of W_i does not depend on F_i under H_0, and therefore any test based on the W's is a distribution-free test.

Now consider some tests based on a combination of the W's. As in Case I, we can use the sum of the W's for a simple overall test and reject H_0 in favor of H_1 if W is small. This test was proposed and studied by Chakraborti and Desu (1988b) and will be referred to as the CD test. The exact distribution of W is simply the expectation of the joint distribution of the W_i's, $i = 2, 3, \ldots, k$, with respect to T and conditional on T, the W_i's are independent binomial random variables with parameters n_i and $F_i(T)$. This yields

$$P[W = w] = \sum \int_{-\infty}^{\infty} \prod_{i=2}^{k} \binom{n_i}{a_i} [F_i(t)]^{a_i} [1 - F_i(t)]^{n_i - a_i} dF_T(t) \qquad (10.7.3)$$

for $w = 0, 1, \ldots, (N - n_1)$ where the sum is over all $a_i = 0, 1, \ldots, n_i$, $i = 2, 3, \ldots, k$, such that $a_2 + a_3 + \cdots + a_k = w$.

Under the null hypothesis the integral in (10.7.3) reduces to a complete beta integral and the exact null distribution of W can be enumerated. However, a more convenient closed-form expression for the null distribution of W may be obtained directly by arguing as follows. The statistic W is the total number of observations in treatment groups 2 through k that precede T. Hence the null distribution of W is the same as that of the two-sample precedence statistic with sample sizes n_1 and $N - n_1$ and this can be obtained directly from the results in Problems 2.28c and 6.10a. Thus when T is the ith order statistic in the control sample, we have

$$P[W = w | H_0] = \frac{\left[\binom{N - i - w}{N - n_1 - w} \binom{i + w - 1}{w} \right]}{\binom{N}{N - n_1}},$$

$$w = 0, 1, \ldots, N - n_1; \quad i = 1, 2, \ldots, n_1$$

or equivalently,

$$P[W = w|H_0] = \frac{n_1}{N}\binom{N-n_1}{w}\frac{\binom{n_1-1}{i-1}}{\binom{N-1}{w+i-1}} \tag{10.7.4}$$

Also, using the result in Problem 2.28d we have

$$E_0(W) = (N-n_1)\left(\frac{i}{n_1+1}\right) \tag{10.7.5}$$

and

$$\text{var}_0(W) = \frac{i(n_1-i+1)(N+1)(N-n_1)}{(n_1+1)^2(n_1+2)} \tag{10.7.6}$$

The null distribution can be used to determine the exact critical value for a test at level α or to find the P value for an observed value of W.

For some practical applications and further generalizations, it is useful to derive the large sample distribution of W. We first find the large sample distribution of the $(k-1)$ dimensional random vector $(W_2, W_3, \ldots, W_k)$. It can be shown (Chakraborti and Desu, 1988a; Gastwirth and Wang, 1998) that the large sample distribution of this random vector can be approximated by a $(k-1)$-variate normal distribution, as stated below.

THEOREM 10.7.1

Let W_N be the $(k-1)$-dimensional vector whose ith element is $N^{1/2}$ $[W_{i+1}/n_{i+1} - F_{i+1}(\theta_1)], i = 1,2,\ldots,k-1$. Suppose that, as $\min(n_1,n_2,\ldots,n_k) \to \infty$, we have $n_i/N \to \lambda_i, 0 < \lambda_i < 1$, $i = 1,2,\ldots,k$. Also let $F_i'(\theta_1) = f_i(\theta_1)$ exist and be positive for $i = 1,2,\ldots,k$. The random vector W_N converges in distribution to a $(k-1)$-dimensional normal distribution with mean vector 0 and covariance matrix Σ whose (i, j)th element is

$$\sigma_{ij} = \frac{Q_iQ_jp(1-p)}{\lambda_1} + \frac{\delta_{ij}F_{i+1}(\theta_1)[1-F_{i+1}(\theta_1)]}{\lambda_{i+1}}$$

where $Q_i = f_{i+1}(\theta_1)/f_1(\theta_1), i,j = 1,2,\ldots,k-1$, δ_{ij} is equal to 1 if $i = j$ and is equal to 0 otherwise, and $F_1(\theta_1) = p$.

From this result, the null distribution of W can be approximated by a normal distribution with mean $(N-n_1)p$ and variance $N(N-n_1)p(1-p)/n_1$ where $p=i/(N+1)$. Thus, for Case II with θ_1 unknown, an approximately size α test is to reject H_0 in favor of H_1 if

$$W \leq (N-n_1)p - z_\alpha \left[\frac{N(N-n_1)p(1-p)}{n_1}\right]^{1/2} \tag{10.7.7}$$

As we noted earlier, the basic test statistic need not necessarily be the sign test. For example, when θ_1 is unknown so that we need to choose a two-sample test for the ith subtest problem, we can use the Mann–Whitney U statistic (or more generally any linear rank statistic) between the ith and the first sample, say U_{1i}, and combine these U's in some suitable manner for an overall test statistic. The resulting tests are distribution-free since the distribution of U_{1i} does not depend on either F_i or F_1 under H_0.

The sum of the U's, say $W^* = \sum_{i=2}^{k} U_{1i}$, can be used for a simple overall test and such a test was proposed and studied by Fligner and Wolfe (1982), hereafter referred to as the FW test. The null distribution of W^* is the same as that of a two-sample Mann–Whitney statistic (see Section 6.6) with $m = n_1$ and $n = N - n_1$. The FW test rejects H_0 in favor of the simple-tree alternative if W^* is large so that the P value is in the upper tail. A choice between the tests based on the sum of U's and the sum of W's may be made on the basis of the ARE. Interestingly, when $p = 0.5$ (i.e., when θ_i is the median of F_i), the ARE between these two tests is the same as the ARE between the sign test and the signed-rank test, regardless of the underlying distribution. For example, when the underlying distribution is normal, the ARE is 0.67, whereas the ARE is 1.33 when the underlying distribution is double exponential.

10.7.3 Applications

The CD test rejects H_0 against the simple-tree alternative H_1 when W is significantly small. Thus the P value is $P(W \leq w_0|H_0)$, where w_0 is the observed value of W. The exact P value can be calculated using the null distribution of W given in (10.7.4) by adding the individual probabilities under H_0. However, for moderately large sample sizes, the normal approximation to the P value is adequate. This can be calculated from

$$\Phi\left[\frac{w_0 - E_0(W) + 0.5}{\sqrt{\text{var}_0(W)}}\right], \tag{10.7.8}$$

which uses a continuity correction, where $E_0(W)$ and $\text{var}_0(W)$ are given in (10.7.5) and (10.7.6), respectively. Alternatively, one can use $E_0(W) = (N-n_1)p$ and $\text{var}_0(W) = N(N-n_1)p(1-p)/n_1$ for an approximation.

Example 10.7.3

Consider again the problem in Example 10.6.1, where three groups of speakers are compared with respect to their ability to communicate. Since the first group of speakers use no audiovisual aids, we could regard this as the control group. It is reasonable to expect that the use of audiovisual aids would have some beneficial effects in communication, and hence the scores for at least one of groups 2 and 3 should be greater than those for group 1. Thus we are interested in testing $H_0 : \theta_3 = \theta_2 = \theta_1$ against $H_1 : \theta_2 \geq \theta_1, \theta_3 \geq \theta_1$, where at least one of the inequalities is strict. In this example, θ_1 is unknown, and therefore we use the CD test with $k = 3$, $n_1 = n_2 = n_3 = 5$. From the data we find $T = 68$, $W_2 = 0$, and $W_3 = 0$. Hence $W = 0$ and the exact P value from (10.7.4) is

$$\frac{5}{15}\left[\frac{\binom{15-5}{0}\binom{5-1}{3-1}}{\binom{15-1}{0+3-1}}\right] = 0.022$$

Therefore, at say $\alpha = 0.05$, there is evidence that the median scores of at least one of groups 2 and 3 is greater than that of group 1. In order to use the normal approximation in (10.7.8) we find $E_0(W) = 5$ and $\text{var}_0(W) = 5.7143$, from (10.7.5) and (10.7.6), and the approximate P value from Table A using a continuity correction is $\Phi(-1.88) = 0.0301$. This is reasonably close to the exact P value even though the sample sizes are small.

For the Fligner–Wolfe test, we find $U_{12} = 22$ and $U_{13} = 24$ so that the P value is $P(W^* \geq 46|H_0)$. To calculate this P value note that W^* is in fact the value of the Mann–Whitney U statistic calculated between sample 1 (as the first sample) and samples 2 and 3 combined (as the second sample). Now using the relationship between Mann–Whitney U statistics and rank-sum statistics, it can be seen that the required probability is in fact equal to the probability that the rank-sum statistic between two samples of sizes $m = 5$ and $n = 10$ is at most 19 under H_0. The corresponding P value is found from Table J as 0.004. Thus, the evidence against the null hypothesis in favor of the simple tree alternative is stronger, on the basis of the FW test.

10.8 Summary

In this chapter, we have been concerned with data consisting of mutually independent random samples from k populations where the null hypothesis is that the k populations are identical. Actual measurements are not required to carry out any of the tests since only the relative magnitudes are used.

When the location model is appropriate and the alternative is that the locations are not all the same, the median test extension, Kruskal–Wallis,

Terry and van der Waerden tests are all appropriate. The median test uses less information than the others and therefore may be less powerful. Further, exact P values require calculations based on the multivariate extension of the hypergeometric distribution and this is quite tedious. As in the two-sample case, the median test is primarily of theoretical interest. On the other hand, the Kruskal–Wallis test is simple to use and quite powerful. Tables of the exact distribution are available and the chi-square approximation is reasonably accurate for moderate sample sizes.

All of the tests are quicker and easier to apply than the classical F test and may perform better if the F test assumptions are not satisfied. Further, as in the parametric setting, nonparametric methods of multiple comparisons can be used in many cases to determine which pairs of population medians differ significantly; see, e.g., Miller (1966, 1981). The advantage of a multiple comparisons procedure over separate pairwise comparisons is that the significance level is the overall level, the probability of a Type I error in all of the conclusions reached.

If the alternative states a distinct complete ordering of the medians, the Jonckheere–Terpstra test is appropriate and exact P values can be obtained. We also discuss tests where the alternative states an ordering of medians with respect to a control group only, where the control median may be either known or unknown.

The asymptotic relative efficiency of the Kruskal–Wallis test compared to the normal theory test for equal means is at least 0.864 for any continuous distribution and is 0.955 for the normal distribution. The Terry and van der Waerden tests should have an ARE of 1 under these circumstances, since they are asymptotically optimum for normal distributions. The ARE of the median test is only $2/\pi = 0.637$ relative to the ANOVA test, and 2/3 relative to the Kruskal–Wallis test, in each case for normal distributions. For further details, see Andrews (1954). All the ARE results stated for the median test apply equally to the control median test, since these two tests have an ARE of 1, regardless of the parent distribution.

Problems

10.1 Generate by enumeration the exact null distribution of the k-sample median test statistic for $k = 3, n_1 = 2, n_2 = 1, n_3 = 1$. If the rejection region consists of those arrangements which are least likely under the null hypothesis, find this region R and the exact α. Compute the values of the Q statistic for all arrangements and compare that critical region for the same value of α with the region R.

10.2 Generate the exact null distribution of the Kruskal–Wallis statistic H for the same k and n_i as in Problem 10.1. Find the critical region which consists of those rank sums R_1, R_2, R_3, which have the largest value of H and find exact α.

10.3 By enumeration, place the median test criterion (U_1, U_2, U_3) and the H test criterion (R_1, R_2, R_3) in one-to-one correspondence for the same k and n_i as in Problems 10.1 and 10.2. If the two tests reject for the largest values of Q and H, respectively, which test seems to distinguish better between extreme arrangements?

10.4 Verify that the form of H given in (10.4.4) is algebraically equivalent to (10.4.2).

10.5 Show that H with $k = 2$ is exactly equivalent to the large-sample approximation to the two-sample Wilcoxon rank-sum test statistic of Section 8.2 with $m = n_1, n = n_2$.

10.6 Show that H is equivalent to the F test statistic in a one-way ANOVA problem if applied to the ranks of the observations rather than the actual numbers.

Hint: Express the F ratio as a function of H in the form given in (10.4.4) or (10.4.2) to show that

$$F = \left[\frac{k-1}{N-k}\left(\frac{N-1}{H} - 1\right)\right]^{-1}$$

This is an example of what is called a rank transform statistic. For related interesting results, see for example, Iman and Conover (1981).

10.7 Write the k-sample median test statistic given in (10.2.2) in the form of (10.5.1) (cf., Problem 7.2).

10.8 How could the subsampling procedure described in Section 9.9 be extended to test the equality of variances in k populations?

10.9 In the context of the k-sample control median test defined in Section 10.3, show that for any $i(= 2, 3, \ldots, k)$ and $j(= 0, 1, \ldots, q)$ the random variable V_{ij} has a binomial distribution. What are the parameters of the distribution (1) in general and (2) under H_0?

10.10 Show that under $H_0 : \theta_1 = \theta_2 = \cdots = \theta_k$ the distribution of V^*, the sum test statistic for comparisons with a control when θ_1 is known in the case of unequal sample sizes defined in Section 10.7.1, is given by

$$P[V^* = v] = (0.5)^N \sum \prod_{i=2}^{k} \binom{n_i}{v_i}$$

where $N = \sum_{i=2}^{k} n_i$, and the sum is over all $v_i = 0, 1, \ldots, n_i$, for $i = 2, 3, \ldots, k$ such that $\sum_{i=2}^{k} (v_i/n_i) = v$. Enumerate the probability distribution of V^* when (1) $n_2 = n_3 = 3$ and (2) $n_2 = n_3 = 2$.

10.11 In the context of the Jonckheere–Terpstra test of Section 10.6, show that under H_0 for $i < j < r$,

$$\text{cov}(U_{ij}, U_{ir}|H_0) = \text{cov}(U_{ji}, U_{ri}|H_0) = \frac{n_i n_j n_r}{12}$$

$$\text{cov}(U_{ij}, U_{ri}|H_0) = \text{cov}(U_{ji}, U_{ir}|H_0) = -\frac{n_i n_j n_r}{12}$$

Hint: $2\text{cov}(U_{ij}, U_{ir}) = \text{var}(U_{ij}, U_{ir}) - \text{var}(U_{ij}) - \text{var}(U_{ir}) = \text{var}(U_{i,j+r}) - \text{var}(U_{ij}) - \text{var}(U_{ir})$, where $U_{i,j+r}$ is the Mann–Whitney U statistic computed between the ith sample and the combined jth and rth samples.

10.12 Many psychologists have developed theories about how different kinds of brain dominance may affect recall ability of information presented in various formats. Brown and Evans (1986) compare recall ability of subjects classified into three groups according to their approach to problem solving based on their scores on the Human Information Process Survey. The three groups are left (active, verbal, logical), right (receptive, special, intuitive), and Integrative (combination of right and left). Information was presented to these subjects in tabular form about the number of physicians who practice in six different states. Recall was measured by how accurately the subjects were able to rank the states from highest to lowest after the presentation concluded. For the scores below determine whether median recall ability is the same for the three groups (higher scores indicate greater recall).

Left	Right	Integrative
35	17	28
32	20	30
38	25	31
29	15	25
36	10	26
31	12	24
33	8	24
35	16	27

10.13 Andrews (1989) examines attitudes toward advertising by undergraduate marketing students at universities in six different geographic regions. Attitudes were measured by answers to a questionnaire that allowed responses on a 7-point Likert scale (1 = strongly disagree and 7 = strongly agree). Three statements on the questionnaire related to the social dimension were (1) most advertising insults the intelligence of the average consumer; (2) advertising often persuades people to buy things they shouldn't buy; and (3) in general, advertisements present a true picture of the product being advertised. For the mean scores given below, determine whether there are any regional differences in attitude for the social dimension.

Region	Insults	Persuades	True Picture
Northwest	3.69	4.48	3.69
Midwest	4.22	3.75	3.25
Northeast	3.63	4.54	4.09
Southwest	4.16	4.35	3.61
South Central	3.96	4.73	3.41
Southeast	3.78	4.49	3.64

10.14 Random samples of 100 insurance company executives, 100 transportation company executives, and 100 media company executives were classified according to highest level of formal education using the code 10 = some college, 20 = bachelor's degree, 30 = master's degree, 40 = more than master's. The results are shown below. Determine whether median education level is the same for the three groups at $\alpha = 0.05$.

Education	Insurance	Transportation	Media
10	19	31	33
20	20	37	34
30	36	20	21
40	25	12	12
Total	100	100	100

10.15 Eighteen fish of a comparable size in a particular variety are divided randomly into three groups and each group is prepared by a different chef using the same recipe. Each prepared fish is then rated on each of the criteria of aroma, flavor, texture, and moisture by professional tasters. Use the composite scores below to test the null hypothesis that median scores for all three chefs are the same.

Chef A	Chef B	Chef C
4.05	4.35	2.24
5.04	3.88	3.93
3.45	3.02	3.37
3.57	4.56	3.21
4.23	4.37	2.35
4.18	3.31	2.59

10.16 An office has three computers, *A*, *B*, and *C*. In a study of computer usage, the firm has kept records on weekly use rates for 7 weeks, except that computer *A* was out for repairs for part of 2 weeks. The eventual goal is to decide which computers to put under a service contract because they have a higher usage rate. As a first step in this study, analyze the data below on weekly computer usage rates to determine whether there is a significant difference in average usage. Can you make a preliminary recommendation?

A	B	C
12.3	15.7	32.4
15.4	10.8	41.2
10.3	45.0	35.1
8.0	12.3	25.0
14.6	8.2	8.2
	20.1	18.4
	26.3	32.5

10.17 Below are four sets of five measurements, each set an array of data on the smoothness of a certain type of paper, each set obtained from a different laboratory. Find an approximate *P* value to test whether the median smoothness can be regarded as the same for all laboratories.

Laboratory	Data				
A	38.7	41.5	43.8	44.5	45.5
B	39.2	39.3	39.7	41.4	41.8
C	34.0	35.0	39.0	40.0	43.0
D	34.1	34.8	34.9	35.4	37.2

10.18 Verify the value of the Kruskal–Wallis test statistic given in the SAS solution to Example 8.2.1.

10.19 A sample of 100 female students at a large university were questioned about four experimental types of service clubs that have essentially the same goals. The types differed only with respect to the difficulty of achieving membership, with Type I having no membership

requirements,..., and Type IV having very rigid membership requirements. The 100 students were assigned randomly into four groups and each student was asked to come for an interview as a prospective member of a club. At each interview, the goals and membership requirements were outlined and the student was asked to rate on a 10-point scale how eager she was to join the club described (1 = most eager, 10 = least eager). The students in Group I were told about Type I club,..., and Group IV were told about Type IV club, in order to make recording the data easier. The data are shown below.

	Group			
Rating	**I**	**II**	**III**	**IV**
1	0	0	0	0
2	0	0	3	1
3	0	2	4	1
4	2	2	5	8
5	3	6	3	10
6	5	5	1	0
7	5	5	4	4
8	7	3	4	1
9	3	2	1	0
10	0	0	0	0
	25	25	25	25

(a) Comment about the design of the experiment.
(b) Try to get some useful information about the results of the study. You may want to use more than one type of analysis.

11

Measures of Association for Bivariate Samples

11.1 Introduction: Definition of Measures of Association in a Bivariate Population

In Chapter 5, we saw that the ordinary sign test and Wilcoxon signed-rank procedures, although discussed in terms of inferences in a single-sample problem, could be applied to paired-sample data by basing the statistical analysis on the differences between the pairs of observations. The inferences then are concerned with the population of differences. One parameter of this population of differences, the variance, does contain information concerning the relationship between the two dependent random variables, since

$$\text{var}(X - Y) = \text{var}(X) + \text{var}(Y) - 2\,\text{cov}(X, Y)$$

The covariance factor and similar nonparametric measures of association will be the subject of this chapter.

In general, if X and Y have a bivariate probability distribution, their covariance, in a certain sense, reflects the direction and amount of association or correspondence between the variables. The covariance will be large and positive if there is a high probability that large (small) values of X are associated with large (small) values of Y. On the other hand, if the correspondence is inverse so that large (small) values of X generally occur in conjunction with small (large) values of Y, their covariance will be large and negative. This comparative type of association is referred to as *concordance* or *agreement*. The covariance parameter as a measure of association is difficult to interpret because its value depends on the orders of magnitude and units of the random variables concerned. A nonabsolute or relative measure of association circumvents this difficulty. The Pearson product–moment correlation coefficient, defined as

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{[\text{var}(X)\ \text{var}(Y)]^{1/2}}$$

is a measure of the *linear* relationship between X and Y. This coefficient is invariant under changes of scale and location in X and Y and in classical statistics this parameter is usually used as the relative measure of association in a bivariate distribution. The absolute value of the correlation coefficient does not exceed 1, and its sign is determined by the sign of the covariance. If X and Y are independent random variables, their correlation is zero, and therefore the magnitude of ρ in some sense measures the degree of association. Although it is not true in general that a zero correlation implies independence, the bivariate normal distribution is a significant exception, and therefore in the normal-theory model ρ is a good measure of association. For random variables from other bivariate populations, ρ may not be such a good description of relationship since dependence may be reflected in a wide variety of types of relationships. One can only say in general that ρ is a more descriptive measure of dependence than covariance because ρ does not depend on the scales of X and Y.

If the main justification for using ρ as a measure of association is that the bivariate normal is such an important distribution in classical statistics and zero correlation is equivalent to independence for that particular population, this reasoning has little significance in nonparametric statistics. Other population measures of association should be equally acceptable, but the approach to measuring relationships might be analogous, so that interpretations are simplified. Because ρ is so widely known and accepted, any other measure would preferably emulate its properties.

Suppose we define a ''good'' relative measure of association as one which satisfies the following criteria:

1. For any two independent pairs (X_i,Y_i) and (X_j,Y_j) of random variables, which follow this bivariate distribution, the measures will equal $+1$ if the relationship is direct and perfect in the sense that $X_i < X_j$ whenever $Y_i < Y_j$ or $X_i > X_j$ whenever $Y_i > Y_j$.

   This relation will be referred to as *perfect concordance* (agreement).
2. For any two independent pairs, the measure will equal -1 if the relationship is indirect and perfect in the sense that

   $$X_i < X_j \text{ whenever } Y_i > Y_j \quad \text{or} \quad X_i > X_j \text{ whenever } Y_i < Y_j.$$

   This relation will be referred to as *perfect discordance* (disagreement).
3. If neither criterion 1 nor criterion 2 is true for all pairs, the measure will lie between the two extremes -1 and $+1$. It is also desirable that, in some sense, increasing *degrees of concordance* are reflected by increasing positive values, and increasing *degrees of discordance* are reflected by increasing negative values.
4. The measure will equal zero if X and Y are independent.

5. The measure for X and Y will be the same as for Y and X, or $-X$ and $-Y$, or $-Y$ and $-X$.
6. The measure for $-X$ and Y or X and $-Y$ will be the negative of the measure for X and Y.
7. The measure will be invariant under all transformations of X and Y for which order of magnitude is preserved.

The parameter ρ is well known to satisfy criteria 3–6; it will satisfy the first two criteria if the two variables have a linear relationship. Criteria 1 and 2 describe a perfect monotone relationship, which is more general than a perfect linear relationship. As for criterion 7, ρ is invariant under positive linear transformations of the random variables, but it is not invariant under all order-preserving transformations. This last criterion seems especially desirable in nonparametric statistics, as we have seen that in order to be distribution-free, inferences must usually be determined by relative magnitudes as opposed to absolute magnitudes of the variables under study. Since probabilities of events involving only inequality relations between random variables are invariant under all order-preserving transformations, a measure of association, which is a function of the probabilities of concordance and discordance, will satisfy the seventh criterion. Perfect direct and indirect association between X and Y are reflected by perfect concordance and perfect discordance, respectively, and in the same spirit as ρ measures a perfect direct and indirect linear relationship between the variables. Thus an appropriate combination of these probabilities will provide a measure of association, which will satisfy all seven of these desirable criteria.

For any two independent pairs of random variables (X_i,Y_i) and (X_j,Y_j), we denote the probabilities of concordance and discordance by p_c and p_d, respectively, and write

$$\begin{aligned} p_c &= P\{[(X_i < X_j) \cap (Y_i < Y_j)] \cup [(X_i > X_j) \cap (Y_i > Y_j)]\} \\ &= P[(X_j - X_i)(Y_j - Y_i) > 0] \\ &= P[(X_i < X_j) \cap (Y_i < Y_j)] + P[(X_i > X_j) \cap (Y_i > Y_j)] \\ p_d &= P[(X_j - X_i)(Y_j - Y_i) < 0] \\ &= P[(X_i < X_j) \cap (Y_i > Y_j)] + P[(X_i > X_j) \cap (Y_i < Y_j)] \end{aligned}$$

Perfect association between X and Y is reflected by either perfect concordance or perfect discordance, and thus some combination of these probabilities should provide a measure of association. The *Kendall coefficient* τ is defined as the difference

$$\tau = p_c - p_d \tag{11.1.1}$$

and this measure of association satisfies our desirable criteria 1–7. If the marginal probability distributions of X and Y are continuous, so that the probability of a tie $X_i = X_j$ or $Y_i = Y_j$ within groups is eliminated, we have

$$\begin{aligned} p_c &= \{P(Y_i < Y_j) - P[(X_i > X_j) \cap (Y_i < Y_j)]\} \\ &\quad + \{P(Y_i > Y_j) - P[(X_i < X_j) \cap (Y_i > Y_j)]\} \\ &= P(Y_i < Y_j) + P(Y_i > Y_j) - p_d \\ &= 1 - p_d \end{aligned}$$

Thus in this case τ can also be expressed as

$$\tau = 2p_c - 1 = 1 - 2p_d \tag{11.1.2}$$

How does τ measure independence? If X and Y are independent and continuous random variables, $P(X_i < X_j) = P(X_i > X_j)$ and further the joint probabilities in p_c or p_d are the product of the individual probabilities. Using these relations, we can write

$$\begin{aligned} p_c &= P(X_i < X_j)P(Y_i < Y_j) + P(X_i > X_j)P(Y_i > Y_j) \\ &= P(X_i > X_j)P(Y_i < Y_j) + P(X_i < X_j)P(Y_i > Y_j) = p_d \end{aligned}$$

and thus $\tau = 0$ for independent continuous random variables. In general, the converse is not true, but this disadvantage is shared by ρ. For the bivariate normal population, however, $\tau = 0$ if and only if $\rho = 0$, that is, if and only if X and Y are independent. This fact follows from the relation.

$$\tau = \frac{2}{\pi} \arcsin \rho \tag{11.1.3}$$

which can be derived as follows. Suppose that X and Y are bivariate normal with variances σ_X^2 and σ_Y^2 and correlation coefficient ρ. Then for any two independent pairs (X_i, Y_i) and (X_j, Y_j) from this population, the differences

$$U = \frac{X_i - X_j}{\sqrt{2}\sigma_X} \quad \text{and} \quad V = \frac{Y_i - Y_j}{\sqrt{2}\sigma_Y}$$

also have a bivariate normal distribution, with zero means, unit variances, and covariance equal to ρ. Thus, we have

$$\rho(U, V) = \rho(X, Y).$$

Since

$$p_c = P(UV > 0)$$

we have

$$p_c = \int_{-\infty}^{0}\int_{-\infty}^{0} \Phi(x,y)\,dx\,dy + \int_{0}^{\infty}\int_{0}^{\infty} \Phi(x,y)\,dx\,dy$$
$$= 2\int_{-\infty}^{0}\int_{-\infty}^{0} \Phi(x,y)\,dx\,dy = 2\Phi(0,0)$$

where $\Phi(x,y)$ and $\Phi(x,y)$ denote the density and cumulative distributions, respectively, of a standardized bivariate normal probability distribution. Since it can be shown that

$$\Phi(0,0) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\,\rho$$

we see that for the bivariate normal

$$p_c = \frac{1}{2} + \frac{1}{\pi}\arcsin\,\rho$$

and

$$\tau = \frac{2}{\pi}\arcsin\,\rho$$

In this chapter, the problem of point estimation of these two population measures of association, ρ and τ, will be considered. We will find estimates that are distribution-free and discuss their individual properties and procedures for hypothesis testing, and the relationship between the two estimates will be determined. Another measure of association is also discussed briefly.

11.2 Kendall's Tau Coefficient

In Section 11.1, Kendall's tau, a measure of association between random variables from any bivariate population, was defined as

$$\tau = p_c - p_d \tag{11.2.1}$$

where, for any two independent pairs of observations (X_i,Y_i),(X_j,Y_j) from the population,

$$p_c = P[(X_j - X_i)(Y_j - Y_i) > 0] \quad \text{and} \quad p_d = P[(X_j - X_i)(Y_j - Y_i) < 0] \tag{11.2.2}$$

In order to estimate the parameter τ from a random sample of n pairs

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

drawn from this bivariate population, we must find point estimates of the probabilities p_c and p_d. For each set of pairs $(X_i, Y_i), (X_j, Y_j)$ of sample observations, define the indicator variables

$$A_{ij} = \operatorname{sgn}(X_j - X_i) \operatorname{sgn}(Y_j - Y_i) \tag{11.2.3}$$

where

$$\operatorname{sgn}(u) = \begin{cases} -1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases}$$

Then the values assumed by A_{ij} are

$$a_{ij} = \begin{cases} 1 & \text{if these pairs are concordant} \\ -1 & \text{if these pairs are discordant} \\ 0 & \text{if these pairs are neither concordant nor} \\ & \quad \text{discordant because of a tie in either component} \end{cases}$$

The marginal probability distribution of the A_{ij} is

$$f_{A_{ij}}(a_{ij}) = \begin{cases} p_c & \text{if } a_{ij} = 1 \\ p_d & \text{if } a_{ij} = -1 \\ 1 - p_c - p_d & \text{if } a_{ij} = 0 \end{cases} \tag{11.2.4}$$

and the expected value is

$$E(A_{ij}) = (1)p_c + (-1)p_d = p_c - p_d = \tau \tag{11.2.5}$$

Since obviously we have $a_{ij} = a_{ji}$ and $a_{ii} = 0$, there are only $\binom{n}{2}$ sets of pairs that need to be considered. An unbiased estimator of τ is therefore provided by

$$T = \sum_{1 \le i < j \le n} \sum \frac{A_{ij}}{\binom{n}{2}} = 2 \sum_{1 \le i < j \le n} \sum \frac{A_{ij}}{n(n-1)} \tag{11.2.6}$$

This measure of association for paired-sample observations is called *Kendall's sample tau coefficient*.

The reader should note that with the definition of A_{ij} in (11.2.3) that allows for tied observations, no assumption regarding the continuity of the

population is necessary, and thus T is an unbiased estimator of the parameter τ in *any* bivariate distribution. We now show that T is also a consistent estimator of τ for any bivariate distribution, by showing that $\text{var}(T) \to 0$ as $n \to \infty$.

In order to determine the variance of T, the variances and covariances of the A_{ij} must be evaluated since T is a linear combination of these indicator random variables. From (11.2.6), we have

$$n^2(n-1)^2 \,\text{var}(T) = 4\left[\sum_{1\le i<j\le n}\sum \text{var}(A_{ij}) + \sum \sum_{\substack{1\le i<j\le n \\ 1\le h<k\le n \\ i\ne h \text{ or } j\ne k}}\sum \sum \text{cov}(A_{ij}, A_{hk})\right] \tag{11.2.7}$$

Since the A_{ij} are identically distributed for all $i<j$, and A_{ij} and A_{hk} are independent for all $i \ne h$ and $j \ne k$ (no pairs in common), (11.2.7) can be written as

$$\begin{aligned} n^2(n-1)^2 \,\text{var}(T) = 4\Bigg[&\binom{n}{2} \underset{i<j}{\text{var}}(A_{ij}) + \sum_{i=1}^{n-1}\sum_{\substack{j=i+1 \\ j\ne k}}^{n}\sum_{k=i+1}^{n} \text{cov}(A_{ij}, A_{ik}). \\ &+ \sum_{j=2}^{n}\sum_{\substack{i=1 \\ i\ne k}}^{j-1}\sum_{k=1}^{j-1} \text{cov}(A_{ij}, A_{kj}) + \sum_{j=2}^{n}\sum_{\substack{i=1 \\ i\ne k}}^{j-1}\sum_{k=j+1}^{n} \text{cov}(A_{ij}, A_{jk}) \\ &+ \sum_{i=2}^{n-1}\sum_{\substack{j=i+1 \\ j\ne k}}^{n}\sum_{k=1}^{i-1} \text{cov}(A_{ij}, A_{ki}) \Bigg] \end{aligned} \tag{11.2.8}$$

By symmetry, all of the covariance terms in (11.2.8) are equal. They are grouped together according to which of the (X, Y) pairs are common to the (A_{ij}, A_{hk}) in order to facilitate counting the number of terms in each summation set. Within the first set we have two distinct permutations, (A_{ij}, A_{ik}) and (A_{ik}, A_{ij}), for each of the $\binom{n}{2}$ choices of $i \ne j \ne k$, and similarly for the second set. But the third and fourth sets do not allow for reversal of the A_{ij} and A_{hk} terms since this makes a different (X, Y) pair in common, and so there are only $\binom{n}{3}$ covariance terms in each of these summations. The total number of distinguishable covariance terms then is $(2+2+1+1)\binom{n}{3} = 6\binom{n}{3}$, and (11.2.8) can be written as simply

$$n^2(n-1)^2\text{var}(T) = 4\left[\binom{n}{2}\text{var}(A_{ij}) + 6\binom{n}{3}\text{cov}(A_{ij}, A_{ik})\right]$$

or

$$n(n-1)\,\text{var}(T) = 2\,\text{var}(A_{ij}) + 4(n-2)\,\text{cov}(A_{ij}, A_{ik}) \tag{11.2.9}$$

for any

$$i < j; \quad i < k; \quad j \neq k; \quad i = 1, 2, \ldots, n-1; \quad j = 2, 3, \ldots, n; \quad k = 2, 3, \ldots, n$$

Using the marginal probability distribution of A_{ij} given in (11.2.4), the variance of A_{ij} is easily evaluated as follows:

$$\begin{aligned} E(A_{ij}^2) &= (1)^2 p_c + (-1)^2 p_d = p_c + p_d \\ \text{var}(A_{ij}) &= (p_c + p_d) - (p_c - p_d)^2 \end{aligned} \tag{11.2.10}$$

The covariance expression, however, requires knowledge of the joint distribution of A_{ij} and A_{ik}, which can be expressed as

$$f_{A_{ij}, A_{ik}}(a_{ij}, a_{ik}) = \begin{cases} p_{cc} & \text{if } a_{ij} = a_{ik} = 1 \\ p_{dd} & \text{if } a_{ij} = a_{ik} = -1 \\ p_{cd} & \text{if } a_{ij} = 1, a_{ik} = -1 \\ & \quad \text{or } a_{ij} = -1, a_{ik} = 1 \\ 1 - p_{cc} - p_{dd} - 2p_{cd} & \text{if } a_{ij} = 0, a_{ik} = -1, 0, 1 \\ & \quad \text{or } a_{ij} = -1, 0, 1, a_{ik} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{11.2.11}$$

for all $i < j, i < k, j \neq k, i = 1, 2, \ldots, n$, and some $0 \leq p_{cc}, p_{dd}, p_{cd} \leq 1$.

Thus we can evaluate

$$\begin{aligned} E(A_{ij}A_{ik}) &= (1)^2 p_{cc} + (-1)^2 p_{dd} + 2(-1)p_{cd} \\ \text{cov}(A_{ij}, A_{ik}) &= p_{cc} + p_{dd} - 2p_{cd} - (p_c - p_d)^2 \end{aligned} \tag{11.2.12}$$

Substitution of (11.2.10) and (11.2.12) in (11.2.9) gives

$$\begin{aligned} n(n-1)\text{var}(T) = {} & 2(p_c + p_d) + 4(n-2)(p_{cc} + p_{dd} - 2p_{cd}) \\ & - 2(2n-3)(p_c - p_d)^2 \end{aligned} \tag{11.2.13}$$

so that the variance of T is of order $1/n$ and therefore approaches zero as $n \to \infty$.

The results obtained so far are completely general, and can be applied to all random variables. If the marginal distributions of X and Y are continuous, $P(A_{ij} = 0) = 0$ and the resulting identities

$$p_c + p_d = 1 \quad \text{and} \quad p_{cc} + p_{dd} + 2p_{cd} = 1$$

allow us to simplify (11.2.13) to a function of, say, p_c and p_{cd} only:

$$\begin{aligned} n(n-1)\text{var}(T) &= 2 - 2(2n-3)(2p_c - 1)^2 + 4(n-2)(1 - 4p_{cd}) \\ &= 8(2n-3)p_c(1-p_c) - 16(n-2)p_{cd} \end{aligned} \tag{11.2.14}$$

Since for X and Y continuous we also have

$$\begin{aligned} p_{cd} &= P(A_{ij} = 1 \cap A_{ik} = -1) \\ &= P(A_{ij} = 1) - P(A_{ij} = 1 \cap A_{ik} = 1) \\ &= p_c - p_{cc} \end{aligned}$$

another expression equivalent to (11.2.14) is

$$\begin{aligned} n(n-1)\text{var}(T) &= 8(2n-3)p_c(1-p_c) - 16(n-2)(p_c - p_{cc}) \\ &= 8p_c(1-p_c) + 16(n-2)\left(p_{cc} - p_c^2\right) \end{aligned} \tag{11.2.15}$$

We have already interpreted p_c as the probability that the pair (X_i, Y_i) is concordant with (X_j, Y_j). Since the parameter p_{cc} is

$$\begin{aligned} p_{cc} &= P(A_{ij} = 1 \cap A_{ik} = 1) \\ &= P[(X_j - X_i)(Y_j - Y_i) > 0 \cap (X_k - X_i)(Y_k - Y_i) > 0] \end{aligned} \tag{11.2.16}$$

for all $i < j, i < k, j \neq k, i = 1, 2, \ldots, n$, we interpret p_{cc} as the probability that the pair (X_i, Y_i) is concordant with both (X_j, Y_j) and (X_k, Y_k).

Integral expressions can be obtained as follows for the probabilities p_c and p_{cc} for random variables X and Y from any continuous bivariate population $F_{X,Y}(x,y)$.

$$\begin{aligned} p_c &= P[(X_i < X_j) \cap (Y_i < Y_j)] + P[(X_i > X_j) \cap (Y_i > Y_j)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P[(X_i < x_j) \cap (Y_i < y_j)]\, f_{X_i, Y_i}(x_j, y_j)\, dx_i\, dy_i \\ &\quad + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P[(X_j < x_i) \cap (Y_j < y_i)]\, f_{X_i, Y_i}(x_i, y_i)\, dx_j\, dy_j \\ &= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{X,Y}(x,y) f_{X,Y}(x,y)\, dx\, dy \end{aligned} \tag{11.2.17}$$

$$
\begin{aligned}
p_{cc} &= P(\{[(X_i < X_j) \cap (Y_i < Y_j)] \cup (X_i > X_j) \cap (Y_i > Y_j)]\} \\
&\quad \cap \{[(X_i < X_k) \cap (Y_i < Y_k) \cup [(X_i > X_k) \cap (Y_i > Y_k)]\}) \\
&= P[(A \cup B) \cap (C \cup D)] \\
&= P[(A \cap C) \cup (B \cap D) \cup (A \cap D) \cup (B \cap C)] \\
&= P(A \cap C) + P(B \cap D) + 2P(A \cap D) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{P[(X_j > x_i) \cap (Y_j > y_i) \cap (X_k > x_i) \cap (Y_k > y_i)] \\
&\quad + P[X_j < x_i) \cap (Y_j < y_i) \cap (X_k < x_i) \cap (Y_k < y_i)] \\
&\quad + 2P[X_j > x_i) \cap (Y_j > y_i) \cap (X_k < x_i) \cap (Y_k < y_i)]\} \\
&\quad \times f_{X_i, Y_i}(x_i, y_i)\, dx_i\, dy_i \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\{P[(X > x) \cap (Y > y)]\}^2 + \{P[(X < x) \cap (Y < y)]\}^2 \\
&\quad + 2P[(X > x) \cap (Y > y)]P[X < x) \cap (Y < y)])\, f_{X,Y}(x, y)\, dx\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{P[(X > x) \cap (Y > y)] + P[(X < x) \cap (Y < y)]\}^2 f_{X,Y}(x, y)\, dx\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - F_X(x) - F_Y(y) + 2F_{X,Y}(x, y)]^2 f_{X,Y}(x, y)\, dx\, dy \qquad (11.2.18)
\end{aligned}
$$

Although T as given in (11.2.6) is perhaps the simplest form for deriving theoretical properties, the coefficient can be written in a number of other ways. In terms of all n^2 pairs for which A_{ij} is defined, (11.2.6) can be written as

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{A_{ij}}{n(n-1)} \qquad (11.2.19)$$

Now we introduce the notation

$$U_{ij} = \operatorname{sgn}(X_j - X_i) \quad \text{and} \quad V_{ij} = \operatorname{sgn}(Y_j - Y_i)$$

so that $A_{ij} = U_{ij}V_{ij}$ for all i, j. Assuming that $X_i \neq X_j$ and $Y_i \neq Y_j$ for all $i \neq j$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}^2 = n(n-1)$$

and (11.2.19) can be written in a form resembling an ordinary sample correlation coefficient as

$$T = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij} V_{ij}}{\left[\left(\sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij}^2\right)\left(\sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}^2\right)\right]^{1/2}} \tag{11.2.20}$$

Kendall and Gibbons (1990) often use T in still another form, which arises by simply classifying sets of differences according to the resulting sign of A_{ij}. If C and Q denote the number of positive and negative A_{ij} for $1 \leq i < j \leq n$, respectively, and the total is $S = C - Q$, we have

$$T = \frac{(C - Q)}{\binom{n}{2}} = \frac{S}{\binom{n}{2}} \tag{11.2.21}$$

If there are no ties within either the X or Y groups, that is, $A_{ij} \neq 0$ for $i \neq j, C + Q = \binom{n}{2}$ and (11.2.21) can be written as

$$T = \frac{2C}{\binom{n}{2}} - 1 = 1 - \frac{2Q}{\binom{n}{2}} \tag{11.2.22}$$

These two forms in (11.2.22) are analogous to the expression (11.1.2) for the parameter

$$\tau = 2p_c - 1 = 1 - 2p_d$$

and $C\Big/\binom{n}{2}$ and $Q\Big/\binom{n}{2}$ are obviously unbiased estimators for p_c and p_d, respectively. The quantity C is perhaps the simplest to calculate for a given sample of n pairs. Assuming that the pairs are written from smallest to largest according to the value of the X component, C is simply the number of values of $1 \leq i < j \leq n$ for which $Y_j - Y_i > 0$, since only then do we have $a_{ij} = 1$.

Another interpretation of T is as a *coefficient of disarray*, since it can be shown (see Kendall and Gibbons, 1990, pp. 30–31) that the total number of interchanges between two consecutive Y observations required to transform the Y arrangement into the natural ordering from smallest to largest, i.e., to transform the Y arrangement into the X arrangement, is equal to Q, or $\binom{n}{2}(1 - T)/2$. This will be illustrated later in Section 11.6.

11.2.1 Null Distribution of *T*

Suppose we want to test the null hypothesis that the X and Y random variables are independent. Since $\tau = 0$ for independent variables, the null

distribution of T is symmetric about the origin. For a general alternative of nonindependence, the rejection region of size α then should be

$$T \in R \quad \text{for } |T| \geq t_{\alpha/2}$$

where $t_{\alpha/2}$ is chosen so that

$$P(|T| \geq t_{\alpha/2} | H_0) \leq \alpha$$

For an alternative of positive dependence, a similar one-sided critical region is appropriate.

We must now determine the sampling distribution of T under the assumption of independence. For this purpose, it will be more convenient, but not necessary, to assume that the X and Y sample observations have both been ordered from smallest to largest and assigned positive integer ranks. The data then consist of n sets of pairs of ranks. The justification for this assumption is that, like τ, T is invariant under all order-preserving transformations. Its numerical value then depends only on the relative magnitudes of the observations and is the same whether calculated for variate values or ranks. For samples with no ties, the $n!$ distinguishable pairings of ranks are all equally likely under the null hypothesis. The value of T is completely determined by the value of C or S because of the expressions in (11.2.21) and (11.2.22), and it is more convenient to work with C. Denote by $u(n, c)$ the number of pairings of n ranks which result in exactly c positive a_{ij}, $1 \leq i < j \leq n$. Then

$$P(C = c) = \frac{u(n, c)}{n!} \tag{11.2.23}$$

and

$$f_T(t) = P(T = t) = P\left[C = \binom{n}{2} \frac{t+1}{2}\right] \tag{11.2.24}$$

We will now find a *recursive relation* to generate the values of $u(n+1, c)$ from knowledge of the values of $u(n, c)$ for some n, and all c. Assuming that the observations are written in order of magnitude of the X component, the value of C depends only on the resulting permutation of the Y ranks. If s_i denotes the rank of the Y observation which is paired with the rank i in the X sample, for $i = 1, 2, \ldots, n, c$ equals the number of integers greater than s_1, plus the number of integers greater than s_2 excluding s_1, plus the number exceeding s_3 excluding s_1 and s_2, etc. For any given permutation of n integers which has this sum c, we need only consider what insertion of the number $n+1$ in any of the $n+1$ possible positions of the permutation $(s_1, s_2, \ldots, s_n)$ does to the value of c. If $n+1$ is in the first position, c is clearly unchanged. If $n+1$ is in

the second position, there is one additional integer greater than s_1, so that c is increased by 1. If in the third position, there is one additional integer greater than both s_1 and s_2, so that c is increased by 2. In general, if $n+1$ is in the kth position, c is increased by $k-1$ for all $k = 1, 2, \ldots, n+1$. Therefore the desired recursive relation is

$$u(n+1, c) = u(n, c) + u(n, c-1) + u(n, c-2) + \cdots + u(n, c-n) \tag{11.2.25}$$

In terms of s, since for a set of n pairs

$$s = 2c - \frac{n(n-1)}{2} \tag{11.2.26}$$

insertion of $n+1$ in the kth position increases c by $k-1$, the new value s' of s for $n+1$ pairs will be

$$\begin{aligned} s' &= 2c' - \frac{n(n+1)}{2} = 2(c+k-1) - \frac{n(n+1)}{2} \\ &= 2c - \frac{n(n-1)}{2} + 2(k-1) - n = s + 2(k-1) - n \end{aligned}$$

In other words, s is increased by $2(k-1)-n$ for $k = 1, 2, \ldots, n+1$, and corresponding to (11.2.25) we have

$$\begin{aligned} u(n+1, s) = u(n, s+n) + u(n, s+n-2) + u(n, s+n-4) \\ + \cdots + u(n, s-n+2) + u(n, s-n) \end{aligned} \tag{11.2.27}$$

The null distribution of S is symmetric about zero, and from (11.2.26) it is clear that S for n pairs is an even or odd integer according as $n(n-1)/2$ is even or odd. Because of this symmetry, tables are most easily constructed for S (or T) rather than C or Q. The null distribution of T is given in Table L. More extensive tables of the null distribution of S or T are given in Kaarsemaker and Van Wijngaarden (1952, 1953), Best (1973, 1974), Best and Gipps (1974), Nijsse (1988), and Kendall and Gibbons (1990).

A simple example will suffice to illustrate the use of (11.2.25) or (11.2.27) to set up tables of these probability distributions. When $n = 3$, the 3! permutations of the Y ranks and the corresponding values of C and S are as follows:

Permutation	**123**	**132**	**213**	**231**	**312**	**321**
c	3	2	2	1	1	0
s	3	1	1	−1	−1	−3

The frequencies then are:

c	0	1	2	3
s	−3	−1	1	3
$u(3, c)$ or $u(3, s)$	1	2	2	1

For C, using (11.2.25), $u(4,c) = \sum_{i=0}^{3} u(3,c-i)$, or

$$
\begin{aligned}
u(4,0) &= u(3,0) = 1 \\
u(4,1) &= u(3,1) + u(3,0) = 3 \\
u(4,2) &= u(3,2) + u(3,1) + u(3,0) = 5 \\
u(4,3) &= u(3,3) + u(3,2) + u(3,1) + u(3,0) = 6 \\
u(4,4) &= u(3,3) + u(3,2) + u(3,1) = 5 \\
u(4,5) &= u(3,3) + u(3,2) = 3 \\
u(4,6) &= u(3,3) = 1
\end{aligned}
$$

Alternatively, we could use (11.2.27), or $u(4,s) = \sum_{i=0}^{3} u(3, s+3-2i)$. Therefore the probability distributions for $n=4$ are as follows:

c	0	1	2	3	4	5	6
s	−6	−4	−2	0	2	4	6
t	−1	−2/3	−1/3	0	1/3	2/3	1
$f(c, s, \text{or } t)$	1/24	3/24	5/24	6/24	5/24	3/24	1/24

The way in which the $u(n, s, \text{or } c)$ are built up by cumulative sums indicates that simple schemes for their generation may be easily worked out (see, for example, Kendall and Gibbons, 1990, pp. 91–92).

The exact null distribution is thus easily found for moderate n. Since T is a sum of random variables, it can be shown using general limit theorems for independent variables that the distribution of a standardized T approaches the standard normal distribution as $n \to \infty$. To use this fact to facilitate inferences concerning independence in large samples, we need to determine the null mean and variance of T. Since T was defined to be an unbiased estimator of τ for any bivariate population and we showed in Section 11.1 that $\tau = 0$ for independent, continuous random variables, the mean is $E(T|H_0) = 0$. In order to find $\text{var}(T|H_0)$ for X and Y continuous, (11.2.15) is used with the appropriate p_c and p_{cc} under H_0. Under the assumption that X and Y have continuous marginal distributions and are independent, they can be assumed to be identically distributed according to the uniform distribution over the interval (0, 1), because of the probability-integral transformation. Then, in (11.2.17) and (11.2.18), we have

$$p_c = 2\int_0^1\int_0^1 xy\,dx\,dy = 1/2$$
$$p_{cc} = \int_0^1\int_0^1 (1-x-y+2xy)^2 dx\,dy = 5/18 \tag{11.2.28}$$

Substituting these results in (11.2.15), we obtain

$$n(n-1)\ \mathrm{var}(T) = 2 + \frac{16(n-2)}{36}$$
$$\mathrm{var}(T) = \frac{2(2n+5)}{9n(n-1)} \tag{11.2.29}$$

For large n, the random variable

$$Z = \frac{3\sqrt{n(n-1)}T}{\sqrt{2(2n+5)}} \tag{11.2.30}$$

can be treated as a standard normal variable with density $\phi(z)$.

If the null hypothesis of independence of X and Y is accepted, we can of course infer that the population parameter $\tau = 0$. If the hypothesis is rejected, this implies dependence between the random variables but not necessarily that $\tau \neq 0$.

11.2.2 The Large-Sample Nonnull Distribution of Kendall's Statistic

The probability distribution of T is asymptotically normal for sample pairs from any bivariate population. Therefore, if any general mean and variance of T could be determined, T would be useful in large samples for inferences other than independence. Since $E(T) = \tau$ for any distribution, T is particularly relevant in inferences concerning the value of τ. The expressions previously found for var(T) in (11.2.13) for any distribution and (11.2.15) for continuous distributions depend on unknown probabilities. Unless the hypothesis under consideration somehow determines p_c, p_d, p_{cc}, p_{dd}, and p_{cd} (or simply p_c and p_{cc} for the continuous case), the exact variance cannot be found without some information about $f_{X,Y}(x, y)$. However, unbiased and consistent estimates of these probabilities can be found from the sample data to provide a consistent estimate $\hat{\sigma}^2(T)$ of the variance of T. The asymptotic distribution of $(T-\tau)/\hat{\sigma}(T)$ then remains standard normal.

Such estimates will be found here for paired samples containing no tied observations. We observed before that $C\Big/\binom{n}{2}$ is an unbiased and consistent estimator of p_c. However, for the purpose of finding estimates for all the probabilities involved, it will be more convenient now to introduce

a different notation. As before, we can assume without loss of generality that the n pairs are arranged in natural order according to the x component and that s_i is the rank of that y which is paired with the ith smallest x for $i = 1, 2, \ldots, n$, so that the data are $(s_1, s_2, \ldots, s_n)$. Define

a_i = number of integers to the left of s_i and less than s_i
b_i = number of integers to the right of s_i and greater than s_i

Then $c_i = a_i + b_i$ = number of values of $j = 1, 2, \ldots, n$ such that (x_i, y_i) is concordant with (x_j, y_j). There are $n(n-1)$ distinguishable sets of pairs, of which $\sum_{i=1}^{n} c_i$ are concordant. An unbiased estimate of p_c then is

$$\hat{p}_c = \sum_{i=1}^{n} \frac{c_i}{n(n-1)} \tag{11.2.31}$$

Similarly, we define

a_i' = number of integers to the left of s_i and greater than s_i
b_i' = number of integers to the right of s_i and less than s_i

and $d_i = a_i' + b_i'$ = number of values of $j = 1, 2, \ldots, n$ such that (x_i, y_i) is discordant with (x_j, y_j). Then

$$\hat{p}_d = \sum_{i=1}^{n} \frac{d_i}{n(n-1)} \tag{11.2.32}$$

gives an unbiased estimate of p_d.

An unbiased and consistent estimate of p_{cc} is the number of sets of three pairs (x_i, y_i), (x_j, y_j), (x_k, y_k) for all $i \neq j \neq k$, for which the products $(x_i - x_j)(y_i - y_j)$ and $(x_i - x_k)(y_i - y_k)$ are both positive, divided by the number of distinguishable sets $n(n-1)(n-2)$. Denote by c_{ii} the number of values of j and k, $i \neq j \neq k, 1 \leq j, k \leq n$, such that (x_i, y_i) is concordant with both (x_j, y_j) and (x_k, y_k), so that

$$\hat{p}_{cc} = \sum_{i=1}^{n} \frac{c_{ii}}{n(n-1)(n-2)}$$

The pair (x_i, y_i) is concordant with both (x_j, y_j) and (x_k, y_k) if:

$$\begin{array}{lll}
\text{Group 1:} & s_j < s_i < s_k & \text{for } j < i < k \\
 & s_k < s_i < s_j & \text{for } k < i < j \\
\text{Group 2:} & s_i < s_j < s_k & \text{for } i < j < k \\
 & s_i < s_k < s_j & \text{for } i < k < j \\
\text{Group 3:} & s_j < s_k < s_i & \text{for } j < k < i \\
 & s_k < s_j < s_i & \text{for } k < j < i
\end{array}$$

Therefore, c_{ii} is twice the sum of the following three corresponding numbers:

1. The number of unordered pairs of integers, one to the left and one to the right of s_i, such that the one to the left is less than s_i and the one to the right is greater than s_i.
2. The number of unordered pairs of integers, both to the right of s_i, such that both are greater than s_i.
3. The number of unordered pairs of integers, both to the left of s_i, such that both are less than s_i.

Then, using the same notation as before, we have

$$c_{ii} = 2\left[\binom{a_i}{1}\binom{b_i}{1} + \binom{b_i}{2} + \binom{a_i}{2}\right] = (a_i + b_i)^2 - (a_i + b_i) = c_i^2 - c_i$$
$$= c_i(c_i - 1)$$

and

$$\hat{p}_{cc} = \sum_{i=1}^{n} \frac{c_i(c_i - 1)}{n(n-1)(n-2)} \tag{11.2.33}$$

Similarly, we can obtain

$$\hat{p}_{dd} = \sum_{i=1}^{n} \frac{d_i(d_i - 1)}{n(n-1)(n-2)} \tag{11.2.34}$$

$$\hat{p}_{cd} = \sum_{i=1}^{n} \frac{a_i b_i' + a_i a_i' + b_i a_i' + b_i b_i'}{n(n-1)(n-2)} = \sum_{i=1}^{n} \frac{c_i d_i}{n(n-1)(n-2)} \tag{11.2.35}$$

Substituting the results (11.2.31) and (11.2.33) in (11.2.15), the estimated variance of T in samples for continuous variables is

$$n(n-1)\hat{\sigma}^2(T) = 8\hat{p}_c - 8\hat{p}_c^2(2n-3) + 16(n-2)\hat{p}_{cc}$$
$$n^2(n-1)^2\hat{\sigma}^2(T) = 8\left[2\sum_{i=1}^{n} c_i^2 - \frac{2n-3}{n(n-1)}\left(\sum_{i=1}^{n} c_i\right)^2 - \sum_{i=1}^{n} c_i\right] \tag{11.2.36}$$

In order to obviate any confusion regarding the calculation of the c_i and c_{ii} to estimate the variance from (11.2.36) in the case of no tied observations, a simple example is provided below for achievement tests in mathematics and English administered to a group of six randomly chosen students.

Student	*A*	*B*	*C*	*D*	*E*	*F*
Math score	91	52	69	99	72	78
English score	89	72	69	96	66	67

The two sets of scores ranked and rearranged in order of increasing mathematics scores are as follows:

Student	*B*	*C*	*E*	*F*	*A*	*D*
Math rank	1	2	3	4	5	6
English rank	4	3	1	2	5	6

The numbers $c_i = a_i + b_i$ are

$$c_1 = 0+2 \quad c_2 = 0+2 \quad c_3 = 0+3 \quad c_4 = 1+2 \quad c_5 = 4+1 \quad c_6 = 5+0$$

$$\sum c_i = 20 \quad \sum c_i^2 = 76 \quad n = 6$$

$$\hat{p}_c = \frac{20}{6(5)} = \frac{2}{3}$$

$$\hat{p}_{cc} = \frac{76-20}{6(5)(4)} = \frac{7}{15}$$

$$t = 2\left(\frac{2}{3}\right) - 1 = \frac{1}{3}$$

$$30^2\hat{\sigma}^2(T) = 8\left[2(76) - 20 - \frac{9}{6(5)}20^2\right] = 96$$

$$\hat{\sigma}^2(T) = 0.1067 \quad \hat{\sigma}(T) = 0.33$$

If we wish to count the c_{ii} directly, we have for $c_{ii} = 2$ (group 1 + group 2 + group 3), the pairs relevant to c_{44}, say, are

Group 1: (1, 5) (1, 6)
Group 2: (5, 6)
Group 3: None

so that $c_{44} = 2(3) = 6 = c_4(c_4 - 1)$.

On the other hand, suppose the English scores corresponding to increasing math scores were ranked as

$$y \qquad 3 \qquad 1 \qquad 4 \qquad 2 \qquad 6 \qquad 5$$

Then we can calculate

$$c_1 = c_4 = 3 \quad c_2 = c_3 = c_5 = c_6 = 4$$
$$\hat{p}_c = \frac{11}{15} \quad \hat{p}_{cc} = \frac{1}{2} \quad t = \frac{7}{15} \quad \hat{\sigma}^2(T) = \frac{-32}{1125}$$

and the estimated variance is negative. A negative variance from (11.2.15) of course cannot occur, but when the parameters p are replaced by estimates $\hat{p}$ and combined, the result can be negative. Since the probability estimates are consistent, the estimated variance of T will be positive for n sufficiently large.

Two applications of this asymptotic approximation to the nonnull distribution of T in nonparametric inference for large samples are as follows:

1. An approximate $(1-\alpha)100\%$ confidence-interval estimate of the population Kendall tau coefficient is

$$t - z_{\alpha/2}\hat{\sigma}(T) < \tau < t + z_{\alpha/2}\hat{\sigma}(T)$$

2. An approximate test of

$$H_0 : \tau = \tau_0 \quad \text{versus} \quad H_1 : \tau \neq \tau_0$$

with significance level α is to reject H_0 when

$$\frac{|t - \tau_0|}{\hat{\sigma}(T)} \geq z_{\alpha/2}$$

A one-sided alternative can also be tested.

11.2.3 Tied Observations

Whether or not the marginal distributions of X and Y are assumed continuous, tied observations can occur within either or both samples. Ties across samples do not present any problem of course. Since the definition of A_{ij} in (11.2.3) assigned a value of zero to a_{ij} if a tie occurs in the (i, j) set of pairs for either the x or y sample values, T as defined before allows for, and essentially ignores, all zero differences. With τ defined as the differences $p_c - p_d$, T as calculated from (11.2.6), (11.2.19), or (11.2.21) is an unbiased estimator of τ with variance as given in (11.2.13) even in the presence of ties. If the occurrence of ties in the sample is attributed to a lack of precision in measurement as opposed to discrete marginal distributions, the simplified expression for var(T) in (11.2.15) may still be used. If there are sample ties, however, the expressions (11.2.20) and (11.2.22) are no longer equivalent to (11.2.6), (11.2.19), or (11.2.21).

For small sample sizes with a small number of tied observations, the exact null distribution of T (or S) conditional on the observed ties can be determined by enumeration. If there are m and w distinguishable permutations of the x and y sample observations, respectively, there will be mw pairings of the two samples, each occurring with equal probability $1/mw$. For larger sample sizes, the normal approximation to the distribution of $\hat{T}$ can still be used but with corrected moments. Conditional upon the observed ties, the parameters p_c, p_d, p_{cc}, p_{dd}, and p_{cd} must have a slightly different interpretation. For example, p_c and p_d here would be the probability that we select two pairs (x_i, y_i) and (x_j, y_j), which do not have a tie in either coordinate, and under the assumption of independence this is

$$\left[1 - \frac{\sum u(u-1)}{n(n-1)}\right]\left[1 - \frac{\sum v(v-1)}{n(n-1)}\right]$$

where u denotes the multiplicity of a tie in the x set and the sum is extended over all ties and v has the same interpretation for the y set. These parameters in the conditional distribution can be determined and substituted in (11.2.13) to find the conditional variance (see, for example, Noether, 1967, pp. 76–77). The conditional mean of T, however, is unchanged, since even for the new parameters we have $p_c = p_d$ for independent samples.

Conditional on the observed ties, however, there are no longer $\binom{n}{2}$ distinguishable sets of pairs to check for concordance, and thus if T is calculated in the ordinary way, it cannot equal one even for perfect agreement. Therefore an alternative definition of T in the presence of ties is to replace the $n(n-1)$ in the denominator of (11.2.6), (11.2.19), or (11.2.21) by a smaller quantity. To obtain a result still analogous to a correlation coefficient, we might take (11.2.20) as the definition of T in general. Since $\sum_{i=1}^{n}\sum_{j=1}^{n} U_{ij}^2$ is the number of nonzero differences $X_j - X_i$ for all (i, j), this sum is the total number of distinguishable differences less the number involving tied observations, or $n(n-1) - \sum u(u-1)$. Similarly for the Y observations. Therefore our modified T from (11.2.20) is

$$T = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} U_{ij}V_{ij}}{\{[n(n-1) - \sum u(u-1)][n(n-1) - \sum v(v-1)]\}^{1/2}} \tag{11.2.37}$$

which reduces to all previously given forms if there are no ties. The modified T from (11.2.21) is

$$T = \frac{C - Q}{\left\{\left[\binom{n}{2} - \sum\binom{u}{2}\right]\left[\binom{n}{2} - \sum\binom{v}{2}\right]\right\}^{1/2}} \tag{11.2.38}$$

Note that the denominator in (11.2.38) is a function of the geometric mean of the number of untied X observations and the number of untied

Y observations. The modified T in (11.2.37) or (11.2.38) is frequently called tau_b in order to distinguish it from (11.2.20) or (11.2.21), called tau_a which has no correction for ties.

The absolute value of the coefficient T calculated from (11.2.37) or (11.2.38) is always greater than the absolute value of a coefficient calculate from (11.2.20) or (11.2.21) when ties are present, but it still may not equal one for perfect agreement or disagreement. The only way to define a tau coefficient that does always equal one for perfect agreement or disagreement is to define

$$\gamma = \frac{C - Q}{C + Q} \tag{11.2.39}$$

This ratio, the number of concordant pairs with no ties minus the number of discordant pairs with no ties divided by the total number of untied pairs, is called the *Goodman-Kruskal gamma coefficient*.

11.2.4 A Related Measure of Association for Discrete Populations

In Section 11.1, we stated the criteria that a good measure of association between two random variables would equal $+1$ for a perfect direct relationship and -1 for a perfect indirect relationship. In terms of the probability parameters, perfect concordance requires $p_c = 1$, and perfect discordance requires $p_d = 1$. With Kendall's coefficient defined as $\tau = p_c - p_d$, the criteria are satisfied if and only if $p_c + p_d = 1$. But if the marginal distributions of X and Y are not continuous

$$\begin{aligned} p_c + p_d &= P[(X_j - X_i)(Y_j - Y_i) > 0] + P[(X_j - X_i)(Y_j - Y_i) < 0] \\ &= 1 - P[(X_j - X_i)(Y_j - Y_i) = 0] \\ &= 1 - P[(X_i = X_j) \cup (Y_i = Y_j)] = 1 - p_t \end{aligned}$$

where p_t denotes the probability that a pair is neither concordant nor discordant. Thus τ cannot be considered a ''good'' measure of association of $p_t \neq 0$.

A modified parameter that does satisfy the criteria for all distributions can easily be defined as

$$\tau^* = \frac{\tau}{1 - p_t} = p_c^* - p_d^*$$

where p_c^* and p_d^* are, respectively, the conditional probabilities of concordance and discordance given that there are no ties

$$p_c^* = \frac{p_c}{1 - p_t} = \frac{P[(X_j - X_i)(Y_j - Y_i) > 0]}{P[(X_j - X_i)(Y_j - Y_i) \neq 0]}$$

Since τ^* is a linear function of τ, an estimate is provided by

$$T^* = \frac{T}{1 - \hat{p}_t} = \frac{\hat{p}_c - \hat{p}_d}{\hat{p}_c + \hat{p}_d}$$

with $\hat{p}_c$ and $\hat{p}_d$ defined as before in (11.2.31) and (11.2.32). Since $\hat{p}_c$ and $\hat{p}_d$ are consistent estimators, the asymptotic distribution of $T/(\hat{p}_c + \hat{p}_d)$ is equivalent to the asymptotic distribution of $T/(p_c + p_d)$, which we know to be the normal distribution. Therefore, for large samples, inferences concerning τ^* can be made (see, for example, Goodman and Kruskal, 1954, 1959, 1963).

11.2.5 Use of Kendall's Statistic to Test against Trend

In Chapter 3 regarding tests for randomness, we observed that the arrangement of relative magnitudes in a single sequence of time-ordered observations can indicate some sort of trend. When the theory of runs up and down was used to test a hypothesis of randomness, the magnitude of each observation relative to its immediately preceding value was considered, and a long run of plus (minus) signs or a sequence with a large predominance of plus (minus) signs was considered indicative of an upward (downward) trend. If time is treated as an X variable, say, and a set of time-ordered observations as the Y variable, an association between X and Y might be considered indicative of a trend. Thus the degree of concordance between such X and Y observations calculated by Kendall's tau statistic becomes a measure of trend. Unlike the case of runs up and down, however, the tau coefficient considers the relative magnitude of each observation relative to *every* preceding observation.

A hypothesis of randomness in a single set of n time-ordered observations is the same as a hypothesis of independence between these observations when paired with the numbers $1, 2, \ldots, n$. Therefore, assuming that $x_i = i$ for $i = 1, 2, \ldots, n$, the indicator variables A_{ij} defined in (11.2.3) become

$$A_{ij} = \operatorname{sgn}(j - i)\ \operatorname{sgn}(Y_j - Y_i)$$

and (11.2.6) can be written as

$$\binom{n}{2} T = \sum_{1 \le i < j \le n} \sum \operatorname{sgn}(Y_j - Y_i)$$

The exact null distribution of T is the same as before. If the alternative is an upward trend, the rejection region consists of large positive values of T, and T can be considered an unbiased estimate of τ, a relative measure of population trend. For a downward trend, we reject for large negative values of T. This test based on T is frequently called the *Mann test*.

11.3 Spearman's Coefficient of Rank Correlation

A random sample of n pairs

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$$

is drawn from a bivariate population with Pearson product–moment correlation coefficient ρ. In classical statistics, the estimate commonly used for ρ is the *sample correlation coefficient* defined as

$$R = \frac{\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})}{\left[\sum_{i=1}^{n} (X_i - \overline{X})^2 \sum_{i=1}^{n} (Y_i - \overline{Y})^2\right]^{1/2}} \tag{11.3.1}$$

In general, of course, the sampling distribution of R depends on the form of the bivariate population from which the sample of pairs is drawn. However, suppose the X observations are ranked from smallest to largest using the integers $1, 2, \ldots, n$, and the Y observations are ranked separately using the same ranking scheme. In other words, each observation is assigned a rank according to its magnitude relative to the others in its own group, but the pairs are kept intact. If the marginal distributions of X and Y are assumed continuous, unique sets of rankings exist theoretically. The data then consist of n sets of paired ranks from which R as defined in (11.3.1) can be calculated. The resulting statistic is called *Spearman's coefficient of rank correlation*. It measures the degree of correspondence between rankings instead of between actual variate values, but it can still be considered a measure of association between the samples and an estimate of the association between X and Y in the continuous bivariate population. It is difficult to interpret exactly what R is estimating in the population from which these samples were drawn and ranks obtained, but the measure has intuitive appeal anyway. The problem of interpretation will be treated in Section 11.4.

The fact that we know the numerical values of the derived observations from which Spearman's R is computed, if not their scheme of pairing, means that the expression in (11.3.1) can be simplified considerably. Denoting the respective ranks of the random variables in the samples by

$$R_i = \text{rank}(X_i) \quad \text{and} \quad S_i = \text{rank}(Y_i)$$

the derived sample observations of n pairs are

$$(r_1, s_1), (r_2, s_2), \ldots, (r_n, s_n)$$

Since addition is commutative, we have the constant values for all samples

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \overline{R} = \vec{S} = \frac{n+1}{2} \tag{11.3.2}$$

$$\sum_{i=1}^{n}(R_i-\overline{R})^2=\sum_{i=1}^{n}(S_i-\overline{S})^2=\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)^2=\frac{n(n^2-1)}{12} \tag{11.3.3}$$

Substituting these constants in (11.3.1), the following equivalent forms of R are obtained:

$$R=\frac{12\sum_{i=1}^{n}(R_i-\overline{R})(S_i-\overline{S})}{n(n^2-1)} \tag{11.3.4}$$

$$R=\frac{12\left[\sum_{i=1}^{n}R_iS_i-n(n+1)^2/4\right]}{n(n^2-1)} \tag{11.3.5}$$

$$R=\frac{12\sum_{i=1}^{n}R_iS_i}{n(n^2-1)}-\frac{3(n+1)}{n-1} \tag{11.3.6}$$

Another useful form of R is in terms of the differences

$$D_i=R_i-S_i=(R_i-\overline{R})-(S_i-\overline{S})$$

Substituting (11.3.3) in the expression

$$\sum_{i=1}^{n}D_i^2=\sum_{i=1}^{n}(R_i-\overline{R})^2+\sum_{i=1}^{n}(S_i-\overline{S})^2-2\sum_{i=1}^{n}(R_i-\overline{R})(S_i-\overline{S})$$

and using this result in (11.3.4), the most common form of the Spearman coefficient of rank correlation is obtained as

$$R=1-\frac{6\sum_{i=1}^{n}D_i^2}{n(n^2-1)} \tag{11.3.7}$$

We can assume without loss of generality that the n sample pairs are labeled in accordance with increasing magnitudes of the X component so that $R_i=i$ for $i=1,2,\ldots,n$. Then S_i is the rank of the Y observation that is paired with the rank i in the X sample and $D_i=i-S_i$.

In Section 11.1, criteria were defined for a ''good'' relative measure of association between two random variables. Although the parameter analogous to R has not been specifically defined, we can easily verify that Spearman's R does satisfy the corresponding criteria of a good measure of association between sample ranks.

1. For any two sets of paired ranks (i, s_i) and (j, s_j) of random variables in a sample from any continuous bivariate distribution, in order to have perfect concordance between ranks, the Y component must also be increasing, or, equivalently, $s_i=i$ and $d_i=0$ for $i=1,2,\ldots,n$ so that $R=1$.

2. For perfect discordance between ranks, the Y arrangement must be the reverse of the X arrangement to have decreasing Y components, so that $s_i = n - i + 1$ and

$$\sum_{i=1}^{n} d_1^2 = \sum_{i=1}^{n} [i - (n - i + 1)]^2 = 4 \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right)^2 = \frac{n(n^2 - 1)}{3}$$

from (11.3.3). Substituting this in (11.3.7), we find $R = -1$.

3–6. Since R in, say, (11.3.7) is algebraically equivalent to (11.3.1) and the value of (11.3.1) is in the interval $[-1, 1]$ for all sets of numerical pairs, the same bounds apply here. Further, R is commutative and symmetric about zero and has expectation zero when the X and Y observations are independent. These properties will be shown later in this section.

7. Since ranks are preserved under all order-preserving transformations, the measure R based on ranks is invariant.

11.3.1 Exact Null Distribution of *R*

If the X and Y random variables from which these n pairs of ranks (R_i, S_i) are derived are independent, R is a distribution-free statistic since each of the $n!$ distinguishable sets of pairings of n ranks is equally likely. Therefore, the sampling distribution of R can be determined and the statistic can be used to perform exact distribution-free tests of independence. If we let u_r denote the number of pairings that lead to a value r for the statistic, the null probability distribution is

$$f_R(r) = \frac{u_r}{n!}$$

The null distribution of R is symmetric about the origin, since the random variable $D = \sum_{i=1}^{n} D_i^2$ is symmetric about $n(n^2 - 1)/6$. This property is the result of the fact that for any sets of pairs

$$(1, s_1), (2, s_2), \ldots, (n, s_n)$$

with

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (i - s_i)^2 \sum_{i=1}^{n} (n - i + 1 - s_i)^2$$

there exists a conjugate set of pairs

$$(1, s_{n1}), (2, s_{n-1}), \ldots, (n, s_1)$$

with

$$\sum_{i=1}^{n} d_i'^2 = \sum_{i=1}^{n} (i - s_{n-i+1})^2 = \sum_{i=1}^{n} (n - i + 1 - s_i)^2$$

The sums of squares of the respective sum and difference of rank differences are

$$\sum_{i=1}^{n} \left(d_i + d_i'\right)^2 = \sum_{i=1}^{n} (n + 1 - 2s_i)^2 = 4\sum_{i=1}^{n} \left(s_i - \frac{n+1}{2}\right)^2 = \frac{n(n^2-1)}{3}$$

$$\sum_{i=1}^{n} \left(d_i - d_i'\right)^2 = \sum_{i=1}^{n} (2i - n - 1)^2 = 4\sum_{i=1}^{n} \left(i - \frac{n+1}{2}\right)^2 = \frac{n(n^2-1)}{3}$$

Substituting these results in the relation

$$\sum_{i=1}^{n} \left[\left(d_i + d_i'\right) + \left(d_i - d_i'\right)\right]^2$$
$$= 4\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \left(d_i + d_i'\right)^2 + \sum_{i=1}^{n} \left(d_i - d_i'\right)^2 + 2\sum_{i=1}^{n} \left(d_i^2 - d_i'^2\right)$$

we obtain

$$4\sum_{i=1}^{n} d_i^2 = \frac{2n(n^2-1)}{3} + 2\sum_{i=1}^{n} d_i^2 - 2\sum_{i=1}^{n} d_i'^2$$

or

$$\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i'^2 = \frac{n(n^2-1)}{3} = \text{constant}$$

Further, R cannot equal zero unless n is even, since $\sum_{i=1}^{n} d_i^2$ is always even because $\sum_{i=1}^{n} d_i = 0$, an even number.

The direct approach to determining u_r is by enumeration, which is probably least tedious for R in the form of (11.3.6). Because of the symmetry property, only $n!/2$ cases need be considered. For $n = 3$, for example, we list the following sets (s_1, s_2, s_3), which may be paired with (1, 2, 3), and the resulting values of R.

(s_1, s_2, s_3)	$\sum_{i=1}^{n} i s_i$	r
1, 2, 3	14	1.0
1, 3, 2	13	0.5
2, 1, 3	13	0.5

The complete probability distribution then is

r	-1.0	-0.5	0.5	1.0
$f_R(r)$	$1/6$	$2/6$	$2/6$	$1/6$

This method of generating the distribution is time consuming, even for moderate n. Of course, there are more efficient methods of enumeration (see, for example, Kendall and Gibbons, 1990, pp. 97–98). The probability distribution of R is given in Table M as tail probabilities for $n \leq 10$ and as critical values for $11 \leq n \leq 30$. More extensive tables of the exact null distribution of R or $\sum D^2$ are given in Glasser and Winter (1961), Owen (1962), De Jonge and VanMontfort (1972), Zar (1972), Otten (1973a,b), Dunstan et al. (1979), Neave (1981), Nelson (1986), Franklin (1988), Ramsay (1989), and Kendall and Gibbons (1990).

Although the general null probability distribution of R requires enumeration, the marginal and joint distributions of any number of the individual ranks of a single random sample of size n are easily determined from combinatorial theory. For example, for the Y sample, we have

$$f_{S_i}(s_i) = \frac{1}{n} \quad s_i = 1, 2, \ldots, n \tag{11.3.8}$$

$$f_{S_i,S_j}(s_i, s_j) = \frac{1}{n(n-1)} \quad s_i, s_j = 1, 2, \ldots, n, s_i \neq s_j \tag{11.3.9}$$

Thus, using (11.3.2) and (11.3.3),

$$E(S_i) = \frac{n+1}{2} \quad \text{var}(S_i) = \frac{n^2-1}{12}$$

For the covariance, we have for all $i \neq j$,

$$\begin{aligned}
\text{cov}(S_i, S_j) &= E(S_i S_j) - E(S_i)E(S_j) \\
&= \frac{1}{n(n-1)} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} ij - \frac{1}{n^2} \left(\sum_{i=1}^{n} i \right)^2 \\
&= \frac{1}{n^2(n-1)} \left[n \left(\sum_{i=1}^{n} i \right)^2 - n \sum_{i=1}^{n} i^2 - (n-1) \left(\sum_{i=1}^{n} i \right)^2 \right] \\
&= \frac{-1}{n^2(n-1)} \left[\frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} \right] = -\frac{n+1}{12}
\end{aligned} \tag{11.3.10}$$

The same results hold for the ranks R_i of the X sample. Under the null hypothesis that the X and Y samples are independent, the ranks R_i and S_j

are independent for all i, j, and the null mean and variance of R are easily found as follows:

$$E\left(\sum_{i=1}^{n} R_i S_i\right) = nE(R_i)E(S_i) = \frac{n(n+1)^2}{4} \tag{11.3.11}$$

$$\begin{aligned}\operatorname{var}\left(\sum_{i=1}^{n} R_i S_i\right) &= n \operatorname{var}(R_i) \operatorname{var}(S_i) + n(n-1) \operatorname{cov}(R_i, R_j) \operatorname{cov}(S_i, S_j) \\ &= \frac{n(n^2-1)^2 + n(n-1)(n+1)^2}{144} = \frac{n^2(n-1)(n+1)^2}{144}\end{aligned} \tag{11.3.12}$$

Then using the form of R in (11.3.6)

$$E(R \mid H_0) = 0 \quad \operatorname{var}(R \mid H_0) = \frac{1}{n-1} \tag{11.3.13}$$

11.3.2 Asymptotic Null Distribution of *R*

Considering R in the form of (11.3.6), and as before assuming S_i denotes the rank of the Y observation paired with the ith smallest X observation, we see that the distribution of R depends only on the random variables $\sum_{i=1}^{n} iS_i$. This quantity is a linear combination of random variables which can be shown to be asymptotically normally distributed (see, for example, Fraser, 1957, pp. 247–248). The mean and variance are given in (11.3.11) and (11.3.12). The standardized normal variable used for an approximate test of independence then is

$$Z = \left(12 \sum_{i=1}^{n} iS_i - 3n^3\right) n^{-5/2}$$

or, equivalently,

$$Z = R\sqrt{n-1} \tag{11.3.14}$$

There is some disagreement in the literature about the accuracy of this approximation for moderate n. Some authors claim that the statistic

$$t = \frac{R\sqrt{n-2}}{\sqrt{1-R^2}} \tag{11.3.15}$$

which has approximately Students t distribution with $n-2$ degrees of freedom, gives more accurate results for moderate n.

11.3.3 Testing the Null Hypothesis

Since R has mean zero for independent random variables, the appropriate rejection region of size α is large absolute values of R for a general alternative of nonindependence and large positive values of R for an alternative of positive dependence. As in the case of Kendall's tau, if the null hypothesis of independence is accepted, we can infer that $\rho(X, Y)$ equals zero, but dependence between the variables does not necessarily imply that $\rho(X, Y) \neq 0$. Besides, the coefficient of rank correlation is measuring association between ranks, not variate values. Since the distribution of R was derived only under the assumption of independence, these results cannot be used to construct confidence-interval estimates of $\rho(X, Y)$ or $E(R)$.

11.3.4 Tied Observations

In all of the foregoing discussion we assumed that the data to be analyzed consisted of n sets of paired integer ranks. These integer ranks may be obtained by ordering observations from two continuous populations, but the theory is equally applicable to any two sets of n pairs, which can be placed separately in a unique preferential order. In the first case, ties can still occur within either or both sets of sample measurements, and in the second case, it is possible that no preference can be made between two or more of the individuals in either group. Thus, for practical purposes, the problem of ties within a set of ranks must be considered.

Within each set of tied observations, if the ranks they would have if distinguishable are assigned at random, nothing is changed since we still have the requisite type of data to be analyzed. However, such an approach has little intuitive appeal, and besides an additional element of chance is introduced. The most common practice for dealing with tied observations here, as in most other nonparametric procedures, is to assign equal ranks to indistinguishable observations. If the rank is the midrank in every case, the sum of the ranks for each sample is still $n(n+1)/2$, but the sum of squares of ranks is changed, and the expressions in (11.3.4) to (11.3.7) are no longer equivalent to (11.3.1). Assuming that the spirit of the rank correlation coefficient is unchanged, the expression in (11.3.1) can be calculated directly from the ranks assigned. However, a form analogous to (11.3.7), which is equivalent to (11.3.1), can still be found for use in the presence of ties.

We first investigate what happens to the sum of squares

$$\sum_{i=1}^{n} (s_i - \bar{s})^2 = \sum_{i=1}^{n} s_i^2 - \frac{n(n+1)^2}{4}$$

when there are one or more groups of u tied observations within the Y sample and each is assigned the appropriate midrank. In each group of u

tied observations, which, if not tied, would be assigned the ranks $p_k + 1, p_k + 2, \ldots, p_k + u$, the rank assigned to each is

$$\sum_{i=1}^{u} \frac{p_k + i}{u} = p_k + \frac{u+1}{2}$$

The sum of squares for the tied ranks then is

$$u\left(p_k + \frac{u+1}{2}\right)^2 = u\left[p_k^2 + p_k(u+1) + \frac{(u+1)^2}{4}\right] \tag{11.3.16}$$

and the corresponding sum in the absence of ties would be

$$\sum_{i=1}^{u} (p_k + i)^2 = up_k^2 + p_k u(u+1) + \frac{u(u+1)(2u+1)}{6} \tag{11.3.17}$$

This particular group of u tied observations, then decreases the sum of squares by the difference between (11.3.17) and (11.3.16) or

$$\frac{u(u+1)(2u+1)}{6} - \frac{u(u+1)^2}{4} = \frac{u(u^2-1)}{12}$$

Since this is true for each group of u tied observations, the sum of squares in the presence of ties is

$$\sum_{i=1}^{n} (s_i - \bar{s})^2 = \frac{n(n^2-1)}{12} - u' \tag{11.3.18}$$

where $u' = \sum u(u^2-1)/12$ and the summation is extended over all sets of u tied ranks in the Y sample. Letting t' denote the corresponding sum for the X sample, we obtain the alternative forms of (11.3.1) as

$$R = \frac{12\left[\sum_{i=1}^{n} R_i S_i - n(n+1)^2/4\right]}{\{[n(n^2-1) - 12t'][n(n^2-1) - 12u']\}^{1/2}} \tag{11.3.19}$$

or

$$R = \frac{n(n^2-1) - 6\sum_{i=1}^{n} D_i^2 - 6(t' + u')}{\{[n(n^2-1) - 12t'][n(n^2-1) - 12u']\}^{1/2}} \tag{11.3.20}$$

analogous to (11.3.5) and (11.3.7), respectively, since here

$$\sum_{i=1}^{n} D_i^2 = \frac{n(n^2-1)}{6} - t' - u' - 2\sum_{i=1}^{n} (R_i - \overline{R})(S_i - \overline{S})$$

Assuming this to be our definition of the sample coefficient of rank correlation in the presence of ties, its probability distribution under the null hypothesis of independence is clearly not the same as the null distribution discussed before for n distinct ranks. For small n, it is possible again to obtain by enumeration the exact null distribution conditional upon a given set of ties. This of course is tedious. The asymptotic distribution of our R as modified for ties is normal since it is still a linear combination of the S_i random variables. Since the total sum of ranks is unchanged when tied ranks are assigned by the midrank method, $E(S_i)$ is unchanged and $E(R|H_0)$ is obviously still zero. The fact that the variance of modified R is also unchanged in the presence of ties is not so obvious. The marginal and joint distributions of the ranks of the Y sample in the presence of ties can still be written in the forms (11.3.8) and (11.3.9) except that the domain is now n numbers, not all distinct, which we can write as $s_1', s_2', \ldots, s_n'$. Then using (11.3.18),

$$\operatorname{var}(S_i) = \sum_{i=1}^{n} \frac{(s_i' - \bar{s})^2}{n} = \frac{n(n^2-1) - 12u'}{12n}$$

For the covariance, proceeding as in the steps leading to (11.3.10),

$$\begin{aligned}\operatorname{cov}(S_i, S_j) &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i' s_j' - \bar{s}^2 \\ &= -\frac{\sum_{i=1}^{n} (s_i' - \bar{s})^2}{n(n-1)} = -\frac{n(n^2-1) - 12u'}{12n(n-1)}\end{aligned}$$

Similar results hold for the X ranks. Now using R in the form of (11.3.19), we have

$$\operatorname{var}(R \mid H_0) = \frac{144[n \operatorname{var}(R_i) \operatorname{var}(S_i) + n(n-1) \operatorname{cov}(R_i, R_j) \operatorname{cov}(S_i, S_j)]}{[n(n^2-1) - 12t'][n(n^2-1) - 12u']}$$

and substitution of the appropriate variances and covariances gives as before

$$\operatorname{var}(R \mid H_0) = \frac{1}{n-1}$$

Thus for large samples with ties, a modified $R\sqrt{n-1}$ with R calculated from (11.3.19) or (11.3.20) can still be treated as a standard normal variable for testing a null hypothesis of independence. However, unless the ties are extremely extensive, they will have little effect on the value of R. In practice, the common expression given in (11.3.7) is often used without a correction for ties. It should be noted that the effect of the correction factor is to decrease the value of R. This means that a negative R is closer to -1, not to zero.

11.3.5 Use of Spearman's R to Test against Trend

As with Kendall's T, R can be considered a measure of trend in a single sequence of time-ordered observations and used to test a null hypothesis of no trend. This application is called *Daniels' test*.

11.4 The Relations between R and T; $E(R)$, τ, and ρ

In Section 11.1 we defined the parameters τ and ρ as two different measures of association in a bivariate population, one in terms of concordances and the other as a function of covariance, and noted that concordance and covariance measure relationship in the same spirit. The sample estimate of τ was found to have exactly the same numerical value and theoretical properties whether calculated in terms of actual variate values or ranks, since the parameter τ and its estimate are both invariant under all order-preserving transformations. This is not true for the parameter ρ or for a sample estimate calculated from (11.3.1) with variate values. The Pearson product–moment correlation coefficient is invariant under linear transformations only, and ranks usually cannot be generated using only linear transformations.

The coefficient of rank correlation is certainly a measure of association between ranks. It has a certain intuitive appeal as an estimate of ρ, but it is not a direct sample analog of this parameter. Nor can it be considered a direct sample analog of a ''population coefficient of rank correlation'' if the marginal distributions of our random variables are continuous, since theoretically continuous random variables cannot be ranked. If an infinite number of values can be assumed by a random variable, the values cannot be enumerated and therefore cannot be ordered. However, we still would like some conception of a population parameter which is the analog of the Spearman coefficient of rank correlation in a random sample of pairs from a continuous bivariate population. Since probabilities of order properties are population parameters and these probabilities are the same for either ranks or variate values, if R can be defined in terms of sample proportions of types of concordance, as T was, we will be able to define a population parameter other than ρ for which the coefficient of rank correlation is an unbiased estimate.

For this purpose, we fist investigate the relationship between R and T for samples with no ties from any continuous bivariate population. In (11.2.20), T was written in a form resembling R as

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{U_{ij} V_{ij}}{n(n-1)} \tag{11.4.1}$$

where

$$U_{ij} = \text{sgn}(X_j - X_i) \quad \text{and} \quad V_{ij} = \text{sgn}(Y_j - Y_i) \tag{11.4.2}$$

$$\text{sgn}(u) = \begin{cases} -1 & \text{if } u < 0 \\ 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases}$$

To complete the similarity, we must determine the general relation between U_{ij} and R_i, V_i, and S_i. A functional definition of R_i was given in (5.5.1) as

$$R_i = 1 + \sum_{1 \le i \le j \le n} S(X_i - X_j) \tag{11.4.3}$$

where

$$S(u) = \begin{cases} 0 & \text{if } u < 0 \\ 1 & \text{if } u \ge 0 \end{cases}$$

In general, then the relation is

$$\begin{aligned} &\text{sgn}(X_j - X_i) = 1 - 2S(X_i - X_j) \quad \text{for all } 1 \le i \ne j \le n \\ &\text{sgn}(X_i - X_i) = 0 \end{aligned} \tag{11.4.4}$$

Substituting this back in (11.4.3), we have

$$R_i = \frac{n+1}{2} - \frac{1}{2} \sum_{j=1}^{n} \text{sgn}(X_j - X_i)$$

or

$$R_i - \overline{R} = -\sum_{j=1}^{n} \frac{U_{ij}}{2}$$

Using R in the form (11.3.4), by substitution we have

$$\begin{aligned} n(n^2-1)R &= 12 \sum_{i=1}^{n} (R_i - \overline{R})(S_i - \overline{S}) = 3 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} U_{ij} \sum_{k=1}^{n} V_{ik} \right) \\ &= 3 \sum_{i=1}^{n} \sum_{j=1}^{n} U_{ij} V_{ij} + 3 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ k \ne j}}^{n} \sum_{k=1}^{n} U_{ij} V_{ik} \end{aligned}$$

or from (11.4.1)

$$R = \frac{3}{n+1}T + \frac{6}{n(n^2-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ k \neq j}}^{n} \sum_{k=1}^{n} U_{ij} V_{ik} \tag{11.4.5}$$

Before, we defined two pairs (X_i, Y_i) and (X_j, Y_j) as being concordant if $U_{ij}V_{ij} > 0$, with p_c denoting the probability of concordance and $\hat{p}_c$ the corresponding sample estimate, the number of concordant sample pairs divided by $n(n-1)$. To complete a definition of R in terms of concordances, because of the last term in (11.4.5), we must now define another type of concordance, this time involving three pairs. We will say that three pairs (X_i, Y_i), (X_j, Y_j) and (X_k, Y_k) exhibit a concordance of the second order if

$$X_i < X_j \text{ whenever } Y_i < Y_k$$

or

$$X_i > X_j \text{ whenever } Y_i > Y_k$$

or, equivalently, if

$$(X_j - X_i)(Y_k - Y_i) = U_{ij}V_{ik} > 0$$

The probability of a second-order concordance is

$$p_{c_2} = P[(X_j - X_i)(Y_k - Y_i) > 0]$$

and the corresponding sample estimate $\hat{p}_{c_2}$ is the number of sets of three pairs with the product $U_{ij}V_{ik} > 0$ for $i < j, k \neq j$, divided by $\binom{n}{2}(n-2)$, the number of distinguishable sets of three pairs. The triple sum in (11.4.5) is the totality of all these products, whether positive or negative, and therefore equals

$$\binom{n}{2}(n-2)[\hat{p}_{c_2}(1-\hat{p}_{c_2})] = \frac{n(n-1)(n-2)(2\hat{p}_{c_2}-1)}{2}$$

In terms of sample concordances, then (11.4.5) can be written as

$$R = \frac{3}{n+1}(2\hat{p}_c - 1) + \frac{3(n-2)}{n+1}(2\hat{p}_{c_2} - 1) \tag{11.4.6}$$

and the population parameter for which R is an unbiased estimator is

$$E(R) = \frac{3[\tau + (n-2)(2p_{c_2} - 1)}{n+1} \tag{11.4.7}$$

We now express p_{c_2} for any continuous bivariate population $F_{X,Y}(x,y)$ in a form analogous to (11.2.17) for p_c:

$$\begin{aligned}
p_{c_2} &= P[(X_i < X_j) \cap (Y_i < Y_k)] + P[(X_i > X_j) \cap (Y_i > Y_k)] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{P[(X_i < x_j) \cap (Y_i < y_k)] \\
&\quad + P[(X_i > x_j) \cap (Y_i > y_k)]\} f_{X_j, Y_k}(x_j, y_k)\, dx_j\, dy_k \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{X,Y}(x,y) + 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)]\, dF_X(x)\, dF_Y(y) \\
&= 1 + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{X,Y}(x,y)\, dF_X(x)\, dF_Y(y) - 2 \int_{-\infty}^{\infty} F_X(x)\, dF_X(x) \\
&= 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{X,Y}(x,y)\, dF_X(x)\, dF_Y(y)
\end{aligned} \tag{11.4.8}$$

A similar development yields another equivalent form

$$p_{c_2} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_X(x) F_Y(y)\, dF_{X,Y}(x,y) \tag{11.4.9}$$

A comparison of these expressions with (11.2.17) shows that $p_{c_2} = p_c = 1/2$ if X and Y are independent. Unlike p_c, however, which ranges between 0 and 1, p_{c_2} ranges only between $1/3$ and $2/3$, with the extreme values obtained for perfect indirect and direct linear relationships, respectively. This result can be shown easily. For the upper limit, since for all x, y,

$$2F_X(x)F_y(y) \le F_X^2(x) + F_Y^2(y)$$

we have from (11.4.9)

$$p_{c_2} \le 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_X^2(x)\, dF_{X,Y}(x,y) = \frac{2}{3}$$

Similarly, for all x, y,

$$2F_X(x)\,F_Y(y) = [F_X(x) + F_Y(y)]^2 - F_X^2(x) - F_Y^2(y)$$

so that from (11.4.9)

$$\begin{aligned} p_{c_2} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_X(x) + F_Y(y)]^2\, dF_{X,Y}(x,y) - \frac{2}{3} \\ &\geq \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F_X(x) + F_Y(y)]\, dF_{X,Y}(x,y)\right]^2 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

Now if X and Y have a perfect direct linear relationship, we can assume without loss of generality that $X = Y$, so that

$$F_{X,Y}(x,y) = \begin{cases} F_X(x) & \text{if } x \leq y \\ F_X(y) & \text{if } x > y \end{cases}$$

Then from (11.4.8)

$$p_{c_2} = 2(2)\int_{-\infty}^{\infty}\int_{-\infty}^{y} F_X(x) f_X(x) f_X(y)\, dx\, dy = \frac{2}{3}$$

For a perfect indirect relationship, we assume $X = -Y$, so that

$$F_{X,Y}(x,y) = \begin{cases} F_X(x) - F_X(-y) & \text{if } x \geq -y \\ 0 & \text{if } x < -y \end{cases}$$

and

$$\begin{aligned} p_{c_2} &= 2\int_{-\infty}^{\infty}\int_{-y}^{\infty} [F_X(x) - F_X(-y)] f_X(x) f_X(-y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} \{1 - F_X^2(-y) - 2[1 - F_X(-y)]F_X(-y)\} f_X(-y)\, dy \\ &= \int_{-\infty}^{\infty} [1 - F_X(-y)]^2 f_X(-y)\, dy = \frac{1}{3} \end{aligned}$$

Substitution of these extreme values in (11.4.7) shows that for any continuous population, ρ, τ, and $E(R)$ all have the same value for the following cases:

X, Y Relation	$\rho = \tau = E(R)$
Indirect linear dependence	-1
Independence	0
Direct linear dependence	1

Although strictly speaking we cannot talk about a parameter for a bivariate distribution which is a coefficient of rank correlation, it seems natural to define the pseudo-rank-correlation parameter, say ρ_2, as the constant for which R is an unbiased estimator in large samples. Then from (11.4.7), we have the definition

$$\rho_2 = \lim_{n\to\infty} E(R) = 3(2p_{c_2} - 1) \tag{11.4.10}$$

and for a sample of size n, the relation between $E(R)$, ρ_2, and τ is

$$E(R) = \frac{3\tau + (n-2)\rho_2}{n+1} \tag{11.4.11}$$

The relation between ρ_2 (for ranks) and ρ (for variate values) depends on the relation between p_{c_2} and covariance. From (11.4.9), we see that

$$\begin{aligned} p_{c_2} &= 2E[F_X(X)F_Y(Y)] = 2\text{cov}[F_X(X), F_Y(Y)] + 2E[F_X(X)]E[F_Y(Y)] \\ &= 2\text{cov}[F_X(X), F_Y(Y)] + \frac{1}{2} \end{aligned}$$

since $E[F_X(X)] = E[F_Y(X)] = 1/2$. Recalling that

$$\text{var}[F_X(X)] = \text{var}[F_Y(Y)] = 1/12$$

we have

$$6p_{c_2} = \rho[F_X(X), F_Y(Y)] + 3$$

and we see from (11.4.10) that

$$\rho_2 = \rho[F_X(X), F_Y(Y)]$$

Therefore ρ_2 is sometime called the *grade correlation coefficient*, since the grade of a number x is usually defined as the cumulative probability $F_X(x)$.

11.5 Another Measure of Association

Another nonparametric type of measure of association for paired samples, which is related to the Pearson product–moment correlation coefficient, has been investigated by Fieller, Hartley, Pearson, and others. This is the ordinary Pearson sample correlation coefficient of (11.3.1) calculated using expected normal scores in place of ranks or variate values. That is, if $\xi_i = E(Z_{(i)})$, where $Z_{(i)}$ is the ith order statistic in a sample of n from the standard normal population and S_i denotes the rank of the Y observation which is paired with the ith smallest X observation, the random sample of pairs of ranks

$$(1, s_1), (2, s_2), \ldots, (n, s_n)$$

is replaced by the derived sample of pairs

$$(\xi_1, \xi_{s_1}), (\xi_2, \xi_{s_2}), \ldots, (\xi_n, \xi_{s_n})$$

and the correlation coefficient for these pairs is

$$R_F = \frac{\sum_{i=1}^{n} \xi_i \xi_{s_i}}{\sum_{i=1}^{n} \xi_i^2}$$

This coefficient is discussed in Fieller et al. (1957) and Fieller and Pearson (1961). The authors show that the transformed random variable

$$Z_F = \tanh^{-1} R_F$$

is approximately normally distributed with moments

$$E(Z_F) = \tanh^{-1}\left[\rho\left(1 - \frac{0.6}{n+8}\right)\right]$$

$$\text{var}(Z_F) = \frac{1}{n-3}$$

where ρ is the correlation coefficient in the bivariate population from which the sample is drawn.

The authors also show that analogous transformations on R and T

$$Z_R = \tanh^{-1} R$$

$$Z_T = \tanh^{-1} T$$

produce approximately normally distributed random variables, but the approximation for Z_F is best in the nonnull case.

11.6 Applications

Kendall's sample tau coefficient (Section 11.2) is one descriptive measure of association in a bivariate sample. The statistic is calculated as

$$T = \frac{2S}{n(n-1)} - \frac{2(C-Q)}{n(n-1)}$$

where
C is the number of concordant pairs
Q is the number of discordant pairs

among (X_i, Y_i) and (X_j, Y_j), for all $i < j$ in a sample of n observations, T ranges between -1 and 1, with -1 describing perfect disagreement, 1 describing perfect agreement, and 0 describing no agreement. The easiest way to calculate C and Q is to first arrange one set of observations in an array, while keeping the pairs intact. A pair in which there is a tie in either the X observations or the Y observations is not counted as part of either C or Q, and therefore with ties it may be necessary to list all possible pairs to find the correct values for C and Q. The modified T is then calculated from (11.2.37) and called tau_b.

The null hypothesis of independence between X and Y can be tested using T. The appropriate rejection regions and P values for an observed value t_0 are as follows:

Alternative	**Rejection Region**	***P* Value**
Positive dependence	$T \geq t_\alpha$	$P(T \geq t_0)$
Negative dependence	$T \leq -t_\alpha$	$p(T \leq t_0)$
Nonindependence	$T \geq t_{\alpha/2}$ or $T \leq -t_{\alpha/2}$	2 (smaller of above)

The exact cumulative null distribution of T is given in Table L as right-tail probabilities for $n \leq 10$. Quantiles of T are given for $11 \leq n \leq 30$. For $n > 30$, the normal approximation to the null distribution of T indicates the following rejection regions and P values:

Alternative	**Rejection Region**	***P* Value**
Positive dependence	$T \geq z_\alpha \sqrt{2(2n+5)}/3\sqrt{n(n-1)}$	$P(z \geq 3t_0\sqrt{n(n-1)}/\sqrt{2(2n+5)}$
Negative dependence	$T \leq -z_\alpha \sqrt{2(2n+5)}/3\sqrt{n(n-1)}$	$P(z \leq 3t_0\sqrt{n(n-1)}/\sqrt{2(2n+5)}$
Nonindependence	Both above with $z_{\alpha/2}$	2 (smaller of above)

This test of the null hypothesis of independence can also be used for the alternative of a trend in a time-ordered sequence of observations Y if time is regarded as X. The alternative of an upward trend corresponds to the alternative of positive dependence. This use of Kendall's tau is frequently called the *Mann test for trend*.

The Spearman coefficient of rank correlation (Section 11.3) is an alternative descriptive measure of association in a bivariate sample. Each set of observations is independently ranked from 1 to n, but the pairs are kept intact. The coefficient is given in (11.3.7) as

$$R = 1 - \frac{6\sum_{i=1}^{n} D_i^2}{n(n^2-1)}$$

where D_i is the difference of the ranks of X_i and Y_i. If ties are present we use (11.3.19). Interpretation of the value of R is exactly the same as for T and the appropriate rejection regions and P values are also in the same direction. For small sample sizes, the null distribution of R is given in Table M in a form similar to Table L. The rejection regions are simply $R \geq z_\alpha\sqrt{n-1}$ for positive dependence and $R \leq -z_\alpha\sqrt{n-1}$ for negative dependence. These are shown below along with the P values for an observed value r_0 of R. When R is used as a test for trend, it is frequently called the *Daniels' test for trend*. Applications of both of these statistics are illustrated in Example 11.6.1.

Alternative	**Rejection Region**	***P* Value**
Positive dependence	$R \geq z_\alpha\sqrt{n-1}$	$P(Z \geq r_0\sqrt{(n-1)}$
Negative dependence	$R \leq -z_\alpha\sqrt{(n-1)}$	$P(Z \leq r_0\sqrt{(n-1)}$
Nonindependence	Both above with $z_{\alpha/2}$	2 (smaller of above)

Example 11.6.1

Two judges ranked nine teas on the basis of taste and full-bodied properties, with 1 indicating the highest ranking. Calculate the Kendall and Spearman measures of association, test the null hypothesis of independence, and find the appropriate one-tailed P value in each case, for the data shown below.

Tea	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***	***I***
Judge 1	1	5	9	7	4	6	8	2	3
Judge 2	4	3	6	8	2	7	9	1	5

SOLUTION

The first step in calculating Kendall's tau is to rearrange the data for Judge 1 in an array, keeping track of the corresponding rank of Judge 2 as shown below.

Then the number of concordant pairs is counted as the number of Y ranks that are below and larger than each Y rank and then summed over all Y's; the number of discordant pairs is counted in the same manner but for ranks below and smaller.

Judge 1	Judge 2	C	Q	D	D^2
1	4	5	3	−3	9
2	1	7	0	1	1
3	5	4	2	−2	4
4	2	5	0	2	4
5	3	4	0	2	4
6	7	2	1	−1	1
7	8	1	1	−1	1
8	9	0	1	−1	1
9	6			3	9
		–	–	–	–
		28	8	0	34

We then calculate $T = 2(20)/9(8) = 0.556$. For the null hypothesis of independence the right-tailed P value from Table L is 0.022.

The last two columns above show that $\sum D_i^2 = 34$ and we compute $R = 1 - 6(34)/9(80) = 0.717$, which is larger than T as expected. The right-tailed P value from Table M is $P = 0.018$ for the alternative of positive dependence.

At the time of this writing, MINITAB has no command for either Kendall's tau or Spearman's rho. However, we can use MINITAB to calculate Spearman's rho by using the rank command on the data (for Judges 1 and 2, respectively) and then calculating the Pearson product–moment correlation coefficient on these ranks. The result $R = 0.717$ agrees with ours. The MINITAB P value is for a Pearson correlation and does *not* apply for Spearman's rho.

The STATXACT solution gives the coefficients and the exact P values for a test of independence using both tau and rho, and all of these agree with ours. Note that the printout shows calculation of both τ_a and τ_b. These are equal because there are no ties in this example. The solution also shows τ_c and Somers' d, which apply for data in a contingency table and are not covered in this book. For Kendall's tau, STATXACT shows the asymptotic P value based on the normal approximation $P(Z \geq 2.09)$ calculated from (11.2.30). For Spearman's rho, it shows the asymptotic P value based on the approximation given in (11.3.15) using Student's t distribution, $P(t \geq 2.72)$ with 7 degrees of freedom. The expressions they use for calculating the asymptotic standard errors and confidence interval estimates are not clear. The reader may verify, however, that they did not use our (11.2.36) because that gives an estimate of the variance of T which is negative in this example. As explained earlier, the estimate can be negative for n small, even though the exact value of the variance must be positive.

```
*****************************************
MINITAB SOLUTION TO EXAMPLE 11.6.1
*****************************************
```

C_1	C_2	C_3	C_4
1	4	1	4
5	3	5	3
9	6	9	6
7	8	7	8
4	2	4	2
6	7	6	7
8	9	8	9
2	1	2	1
3	5	3	5

Correlations: C_3, C_4

Pearson correlation of C_3 and $C_4 = 0.717$
P value $= 0.030$

```
*****************************************
STATXACT SOLUTION TO EXAMPLE 11.6.1
*****************************************
```

KENDALL'S TAU AND SOMER'S D RANK-ORDER CORRELATION COEFFICIENTS

Correlation Coefficient Estimates Based on Nine Observations

Coefficient	Estimate	ASE1	95.0% Confidence Interval
Kendall's Tau	0.5556	0.1309	(0.2989, 0.8122)
Kendall's tau_b	0.5556	0.1309	(0.2989, 0.8122)
Kendall's tau)_c	0.5556	0.1309	(0.2989, 0.8122)
Somers' D|row	0.5556	0.1309	(0.2989, 0.8122)
Somers' Dicol	0.5556	0.1309	(0.2989, 0.8122)
Somers' D sym.	0.5556	0.1309	(0.2989, 0.8122)

Asymptotic P values for testing no association (using normal approximation):
One-sided: Pr (Statistic.GE. Observed) = 0.0000
Two-sided: 2 * one-sided = 0.0000

Exact P values for testing no association:
One-sided Pr (Statistic .GE. Observed) = 0.0223
Pr (Statistic .EQ. Observed) = 0.0099
Two-sided: Pr (|Statistic| .GE.vert;Observed|) = 0.0446

```
SPEARMAN'S CORRELATION TEST
Correlation Coefficient Estimates Based on Nine Observations

  Coefficient       Estimate        ASE1      95.0% Confidence Interval
  -----------       --------        ----      -------------------------
  Spearman's CC       0.7167      0.1061      (     0.5088,      0.9246)

Asymptotic P-values for testing no association (t-distribution with 7 df):
      One-sided: Pr (  Statistic.GE. Observed  ) =      0.0149
      Two-sided 2 * one-sided                   =      0.0298

Exact P-values:
      One-sided  Pr (  Statistic  .GE.  Observed  )   =     0.0184
                 Pr (  Statistic  .EQ.  Observed  )   =     0.0029
      Two-sided: Pr (  |Statistic| .GE. |Observed| )   =     0.0369
```

We use the data in Example 11.6.1 to illustrate how T can be interpreted as a coefficient of disarray, where Q, the number of discordant pairs, is the minimum number of interchanges in the Y ranks, one pair at a time, needed to convert them to the natural order. The X and Y ranks in this example are repeated below.

X	1	2	3	4	5	6	7	8	9
Y	4	1	5	2	3	7	8	9	6

In the Y ranks, we first interchange the 4 and 1 to put 1 in the correct position. Then we interchange 2 and 5 to make 2 closer to its correct position. Then we interchange 2 and 4. We keep proceeding in this way, working to get 3 in the correct position, and then 4, etc. The complete set of interchanges is as follows:

Y	1	4	5	2	3	7	8	9	6
	1	4	2	5	3	7	8	9	6
	1	2	4	5	3	7	8	9	6
	1	2	4	3	5	7	8	9	6
	1	2	3	4	5	7	8	9	6
	1	2	3	4	5	7	8	6	9
	1	2	3	4	5	7	6	8	9
	1	2	3	4	5	6	7	8	9

The total number of interchanges required to transform the Y ranks into the natural order by this systematic procedure is 8, and this is the value of Q,

the total number of discordant pairs. We could make the transformation using more interchanges, of course, but more are not needed. It can be shown that $Q = 8$ is the minimum number of interchanges.

11.7 Summary

In this chapter, we have studied in detail the nonparametric coefficients that were proposed by Kendall and Spearman to measure association. Both coefficients can be computed for a sample of pairs from a bivariate distribution, when the data are numerical measurements or ranks indicating relative magnitudes. The absolute values of both coefficients range between 0 and 1, with increasing values indicating increasing degrees of association. The sign of the coefficient indicates the direction of the association, direct or inverse. The values of the coefficients are not directly comparable, however. We know that $|R| \geq |T|$ for any set of data, and in fact $|R|$ can be as much as 50% greater than $|T|$.

Both coefficients can be used to test the null hypothesis of independence between the variables. Even though the magnitudes of R and T are not directly comparable, the magnitudes of the P values based on them should be about the same, allowing for the fact that they are measuring association in different ways. The interpretation of T is easier than for R; T is the proportion of concordant pairs in the sample minus the proportion of discordant pairs. T can also be interpreted as a coefficient of disarray. The easiest interpretation of R is as the sample value of the Pearson product–moment correlation coefficient calculated using the ranks of the sample data.

An exact test of the null hypothesis of independence can be carried out using either T or R for small sample sizes. Generation of tables for exact P values was difficult initially, but now computers have the capacity to do this for even moderate n. For intermediate and large sample sizes, the tests can be performed using large sample approximations. The distribution of T approaches the normal distribution much more rapidly than the distribution of R and hence approximate P values based on R are less reliable than those based on T.

Both T and R can be used when ties are present in either or both samples, and both have a correction for ties that improves the normal approximation. The correction with T always increases the value of T while the R correction always decreases the value of R, making the coefficients closer in magnitude.

If we reject the null hypothesis of independence by either T or R, we can conclude that there is some kind of dependence or ''association'' between the variables. But the kind of relationship or association that exists defies any verbal description in general. The existence of a relationship or significant association does not mean that the relationship is causal. The relationship may be due to several other factors, or to no factor at all. When interpreting the results of an experiment, care should always be taken to point out that no causality is implied, either directly or indirectly.

Kendall's T is an unbiased estimator of a parameter τ in the bivariate population; τ represents the probability of concordance minus the probability of discordance. Concordance is not the same as correlation, although both represent a kind of association. Spearman's R is not an unbiased estimator of the population correlation ρ. It is an unbiased estimator of a parameter which is a function of τ and the grade correlation.

The tests of independence based on T and R can be considered nonparametric counterparts of the test that the Pearson product–moment correlation coefficient $\rho = 0$ or that the regression coefficient $\beta = 0$ in the bivariate normal distribution. The asymptotic relative efficiency of these tests relative to the parametric test based on the sample Pearson product–moment correlation coefficient is $9/\pi^2 = 0.912$ for normal distributions and 1.00 for the continuous uniform distribution.

Both T and R can be used to test for the existence of trend in a set of time-ordered observations. The test based on T is called the Mann test, and the test based on R is called the Daniels' test. Both of these tests are alternatives to the tests for randomness presented in Chapter 3.

Problems

11.1 A beauty contest has eight contestants. Two judges are each asked to rank the contestants in a preferential order of pulchritude. Answer parts (a) and (b) using (1) the Kendall tau-coefficient procedures and (2) the Spearman rank-correlation-coefficient procedures.

	Contestant							
Judge	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
1	2	1	3	5	4	8	7	6
2	1	2	4	5	7	6	8	3

(a) Calculate the measure of association.
(b) Test the null hypothesis that the judges ranked the contestants independently and find the P value.
(c) Find a 95% confidence interval estimate of τ.

11.2 Verify the result given in (11.4.9).

11.3 Two independent random samples of sizes m and n contain no ties. A set of $m + n$ paired observations can be derived from these data by arranging the combined samples in ascending order of magnitude and (a) assigning ranks, (b) assigning sample indicators. Show that Kendall's tau, calculated for these pairs without a correction for ties, is linearly related to the Mann–Whitney U statistic for these data, and find the relation if the sample indicators are (i) sample numbers 1 and 2, (ii) 1 for the first sample and 0 for the second sample as in the Z vector of Chapter 7.

11.4 Show that for the standardized bivariate normal distribution

$$\Phi(0, 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho$$

11.5 The Census Bureau reported that Hispanics are expected to overtake blacks as the largest minority in the United States by the year 2030. Use *two different tests* to see whether there is a direct relationship between number of Hispanics and percent of state population for the nine states below.

State	Hispanics (Millions)	Percent of State Population
California	6.6	23
Texas	4.1	24
New York	2.1	12
Florida	1.5	12
Illinois	0.8	7
Arizona	0.6	18
New Jersey	0.6	8
New Mexico	0.5	35
Colorado	0.4	11

11.6 Company-financed expenditures in manufacturing on research and development (R&D) are currently about 2.7% of sales in Japan and 2.8% of sales in the United States. However, when these figures are looked at separately according to industry, the following data from Mansfield (1989) show some large differences.

Industry	Japan	United States
Food	0.8	0.4
Textiles	1.2	0.5
Paper	0.7	1.3
Chemicals	3.8	4.7
Petroleum	0.4	0.7
Rubber	2.9	2.2
Ferrous metals	1.9	0.5
Nonferrous metals	1.9	1.4
Metal products	1.6	1.3
Machinery	2.7	5.8
Electrical equipment	5.1	4.8
Motor vehicles	3.0	3.2
Other transport equipment	2.6	1.2
Instruments	4.5	9.0

(a) Use the signed-rank test of Chapter 5 to determine whether Japan spends a larger percentage than the United States on R&D.
(b) Determine whether there is a significant positive relationship between percentages spent by Japan and the United States (use two different methods).

11.7 The *World Almanac and Book of Facts* published the following divorce rates per 1000 population in the United States. Determine whether these data show a positive trend using *four different* methods.

Year	Divorce Rate
1945	3.5
1950	2.6
1955	2.3
1960	2.2
1965	2.5
1970	3.5
1975	4.8
1980	5.2
1985	5.0

11.8 For the time series data in Example 3.4.1, use the Mann test based on Spearman's rank correlation coefficient to see if the data show a positive trend.

11.9 Do Problem 11.8 using the Daniels' test based on Kendall's tau.

11.10 The rainfall measured by each of 12 gauges was recorded for 20 successive days. The average results for each day are as follows:

Day	Rainfall	Day	Rainfall
April 1	0.00	April 11	2.10
April 2	0.03	April 12	2.25
April 3	0.05	April 13	2.50
April 4	1.11	April 14	2.50
April 5	0.00	April 15	2.51
April 6	0.00	April 16	2.60
April 7	0.02	April 17	2.50
April 8	0.06	April 18	2.45
April 9	1.15	April 19	0.02
April 10	2.00	April 20	0.00

Use an appropriate test to determine whether these data exhibit some sort of pattern. Find the P value:

(a) Using tests based on runs with both the exact distribution and the normal approximation.
(b) Using other tests that you may think are appropriate.
(c) Compare and interpret the results of (a) and (b).

11.11 A company has administered a screening aptitude test to 20 new employees over a 2 year period. The record of scores and data on which the person was hired are shown below.

1/4/08	75	9/21/08	72	12/9/08	81	5/10/09	91
3/9/08	74	10/4/08	77	1/22/09	93	7/17/09	95
6/3/08	71	10/9/08	76	1/26/09	82	9/12/09	90
6/15/08	76	11/1/08	78	3/21/09	84	10/4/09	92
8/4/08	98	12/5/08	80	4/6/09	89	12/6/09	93

Assuming that these scores are the primary criterion for hiring, do you think that over this time period the screening procedure has changed, or the personnel agent has changed, the supply has changed, or what? Use all nonparametric procedures that are appropriate.

11.12 Ten randomly chosen male college students are used in an experiment to investigate the claim that physical strength decreases with fatigue. Describe the relationship for the data below, using several methods of analysis.

Minutes between Rest Periods	Pounds Lifted Per Minute
5.5	350
9.6	230
2.4	540
4.4	390
0.5	910
7.9	220
2.0	680
3.3	590
13.1	90
4.2	520

11.13 Given a single series of time-ordered ordinal observations over several years, name all the nonparametric procedures that could be used in order to detect a long-term positive trend and describe them.

11.14 Six randomly selected mice are studied over time and scored on an ordinal basis for intelligence and social dominance.

Mouse	Intelligence	Social Dominance
1	45	63
2	26	0
3	20	16
4	40	91
5	36	25
6	23	2

(a) Find the coefficient of rank correlation.
(b) Find the appropriate one-tailed P value for your result in (a).
(c) Find the Kendall tau coefficient.
(d) Find the appropriate one-tailed P value for your result in (c).

11.15 A board of marketing executives ranked 10 similar products, and an ''independent'' group of male consumers also ranked the products. Use two different nonparametric producers to describe the association between rankings and find a one-tailed P value in each case. State the hypothesis and alternative and all assumptions. Compare and contrast the procedures.

Product	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Executive ranks	9	4	3	7	2	1	5	8	10	6
Independent male ranks	7	6	5	9	2	3	8	5	10	1

11.16 Derive the null distribution of both Kendall's tau statistic and Spearman's rho for $n = 3$ assuming no ties.

11.17 A scout for a professional baseball team ranks nine players separately in terms of speed and power hitting, as shown below.

Player	Speed Ranking	Power-Hitting Ranking
A	3	1
B	1	3
C	5	4
D	6	2
E	2	6
F	7	8
G	8	9
H	4	5
I	9	7

(a) Find the rank correlation coefficient and the appropriate one-tailed P value.
(b) Find the Kendall tau coefficient and the appropriate one-tailed P value.

11.18 Twenty-three students are classified according to their attitude toward elementary school integration. Then each is asked the number of years of schooling completed at that time, with numbers greater than 12 denoting some college or university experience For example, the first subject whose attitude was Strongly Disagree had completed 5 years of school at the time.

Number of Years of School Completed at the Time	Attitude toward Elementary School Integration			
	Strongly Disagree	Moderately Disagree	Moderately Agree	Strongly Agree
	5	9	12	16
	4	10	13	18
	10	7	9	12
	12	12	12	19
	3	12	16	14
		10	15	
			14	

As a measure of the association between attitude and number of years of schooling completed:
(a) Compute Kendall's tau with correction for ties.
(b) Compute Spearman's R with correction for ties.

11.19 For the data in Problem 3.13, use the two methods of this chapter to see if there is a positive trend.

12

Measures of Association in Multiple Classifications

12.1 Introduction

In Chapter 10, we studied nonparametric analogs of the one-way analysis-of-variance F test for equal means in a one-way classification. If the samples from the k populations are all the same size so that $n_1 = n_2 = \cdots = n_k = n$, the data can be presented in a two-way table of dimension $n \times k$. This is called a one-way layout because only one factor is involved, the populations. Under the null hypothesis of identical populations, the data can be considered a single random sample of nk observations from the common population.

In this chapter, we present nonparametric analogs of the two-way analysis-of-variance problem. The data are again presented in the form of a two-way table but the elements cannot be considered a single random sample because of certain relationships between them. Two factors are involved, called the row and column effects.

We first review the classical analysis-of-variance test of the null hypothesis that the column effects are all the same. The model is usually written as

$$X_{ij} = \mu + \beta_i + \theta_j + E_{ij} \quad \text{for } i = 1, 2, \ldots, I \quad \text{and} \quad j = 1, 2, \ldots, J$$

The β_i and θ_j are known as the row and column effects, respectively. In the normal-theory model, the errors E_{ij} are independent, normally distributed random variables with mean zero and variance σ_E^2. For the null hypothesis of equal column effects

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_J$$

the test statistic is the ratio

$$F = \frac{(I-1)I \sum_{j=1}^{J} (\overline{x}_j - \overline{x})^2}{\sum_{i=1}^{I} \sum_{j=1}^{J} (x_{ij} - \overline{x}_i - \overline{x}_j + \overline{x})^2}$$

where

$$\bar{x}_i = \sum_{j=1}^{J} \frac{x_{ij}}{J} \quad \bar{x}_j = \sum_{i=1}^{I} \frac{x_{ij}}{I} \quad \bar{x} = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{x_{ij}}{IJ}$$

If all the assumptions of the model are met, this test statistic F has the F distribution with $J-1$ and $(I-1)(J-1)$ degrees freedom.

The first parallel of this design that we consider is the k-matched sample problem. The matching can arise in two different ways, but both are somewhat analogous to the randomized-block design of a two-way layout. In this design, IJ experimental units are grouped into I blocks, each containing J units. A set of J treatments is assigned at random to the units within each block in such a way that all J assignments are equally likely, and the assignments in different blocks are independent. The scheme of grouping into blocks is important, since the purpose of such a design is to minimize the differences between units in the same block. If the design is successful, an estimate of experimental error can be obtained, which is not inflated by differences between blocks. This model is often appropriate in agricultural field experimentation since the effects of a possible fertility gradient can be reduced. Dividing the field into I blocks, the plots within each block can be kept in close proximity. Any differences between plots within the same block can be attributed to differences between treatments and the block effect is eliminated from the estimate of experimental error.

Another kind of matching arises when IJ subjects are grouped into I blocks each containing J matched subjects, and within each block, J treatments are assigned randomly to the matched subjects. The effects of the treatments are observed, and we let X_{ij} denote the observation in block i of treatment number $j, i = 1, 2, \ldots, I, j = 1, 2, \ldots, J$. Since the observations in different blocks are independent, the entries in column j are independent. In order to determine whether the treatment (column) effects are all the same, the analysis-of-variance test is appropriate if the requisite assumptions are justified. If the observations in each row $X_{i1}, X_{i2}, \ldots, X_{iJ}$ are replaced by their rankings within that row, a nonparametric test involving the column sums of this $I \times J$ table, called Friedman's two-way analysis of variance by ranks, can be used to test the hypothesis. This is a k-related sample problem when $J = k$. This design is sometimes called a balanced complete block design or a repeated measures design. The null hypothesis is that the treatment effects are all equal or

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_J$$

and the alternative is

$$H_1 : \theta_i \neq \theta_j \quad \text{for at least one } i \neq j$$

Another nonparametric test for the k-related sample problem is Page's test for ordered alternatives. The null hypothesis is the same as above but the alternative specifies that the treatment effects occur in a specific order, as for example,

$$H_1 : \theta_1 \leq \theta_2 \leq \cdots \leq \theta_J$$

with at least one inequality strict. For each of these problems the location model is that the respective cdf's are $F(x - \theta_i - \beta_j)$.

Another related-sample problem arises by considering a single group of J subjects, each observed under I different conditions. The matching here is by condition rather than subject, and the observation X_{ij} denotes the effect of condition i on subject $j, i = 1, 2, \ldots, I, j = 1, 2, \ldots, J$. We have here a random sample of size J from an I-variate population. Under the null hypothesis that the I variates are independent, the expected sum of the I observations on subject j is the same for all $j = 1, 2, \ldots, J$. In order to determine whether the column effects are all the same, the analysis-of-variance test may be appropriate. Testing the independence of the I variates involves a comparison of J column totals, so that in a sense the roles of treatments and blocks have been reversed in terms of which factor is of interest. This is a k-related sample problem when $I = k$. If the observations in each row are ranked as before, Friedman's two-way analysis of variance provides a nonparametric test of independence of the k variates. Thus, in order to have consistency of results as opposed to consistency of sampling situations, the presentation here in both cases will be for a table containing k rows and n columns, where each row is a set of positive integer ranks.

In this related-sample problem, particularly if the null hypothesis of independence of the k variates is rejected, a measure of the association between the k variates would be desirable. In fact, this sampling situation is the direct extension of the paired-sample problem of Chapter 11 to the k-related sample case. A measure of the overall agreement between the k sets of rankings is called Kendall's coefficient of concordance. This statistic can also be used to test the null hypothesis of independence, but we will see that the test is equivalent to Friedman's test for n treatments and k blocks. An analogous measure of concordance will be found for k sets of incomplete rankings, which relate to the balanced incomplete-blocks design.

Another topic that is treated briefly in this chapter is a nonparametric approach to finding a measure of partial correlation when there are three complete sets of rankings of n objects. This is the correlation between two variables when a third is held constant.

12.2 Friedman's Two-Way Analysis of Variance by Ranks in a $k \times n$ Table and Multiple Comparisons

Suppose we have data in the form of a two-way layout of k rows and n columns. The rows indicate block, subject, or sample numbers, and the columns are treatment numbers. The observations in different rows are independent, but the columns are not because of some unit of association. In order to avoid making the assumptions required for the classical analysis-of-variance test that the n treatments are the same, Friedman (1937, 1940) suggested replacing each treatment observation within the ith block by its rank relative to the other observations in the same block. We denote the ranked observations by $R_{ij}, i = 1, 2, \ldots, k, j = 1, 2, \ldots, n$, so that R_{ij} is the rank of treatment j when observed in block i. Then $R_{i1}, R_{i2}, \ldots, R_{in}$ is a permutation of the first n integers, and $R_{1j}, R_{2j}, \ldots, R_{kj}$ is the set of ranks given to treatment j in all blocks. We represent the data in a $k \times n$ table as follows:

		Treatments				
		1	2	...	n	Row Totals
Blocks	1	R_{11}	R_{12}	$\cdots$	R_{1n}	$n(n+1)/2$
	2	R_{21}	R_{22}	$\cdots$	R_{2n}	$n(n+1)/2$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
	k	R_{k1}	R_{k2}	$\cdots$	R_{kn}	$n(n+1)/2$
Column	Totals	R_1	R_2	$\cdots$	R_n	$kn(n+1)/2$

(12.2.1)

The row totals are constant, but the column totals are affected by differences between treatments. If the treatment effects are all the same, each expected column total is the same and equals the average column total $k(n+1)/2$. The sum of deviations of observed column totals around this mean is zero, but the sum of squares of these deviations will be indicative of the differences in treatment effects. Therefore, we consider the sampling distribution of the random variable

$$S = \sum_{j=1}^{n} \left[R_j - \frac{k(n+1)}{2} \right]^2 = \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left(R_{ij} - \frac{n+1}{2} \right) \right]^2 \tag{12.2.2}$$

under the null hypothesis that the n treatment effects are all equal

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n$$

For this null case, the ranks in the ith block are assigned completely at random, and each row in the two-way layout constitutes a random permutation of the first n integers if there are no ties. There are a total of $(n!)^k$ distinguishable sets of entries in the $k \times n$ table, and each is equally likely. These possibilities can be enumerated and the value of S calculated for each. The probability distribution of S then is

$$f_S(s) = \frac{u_s}{(n!)^k}$$

where u_s is the number of assignments which yield s as the sum of squares of column total deviations. A systematic method of generating the values of u_s for n, k from those for $n, k-1$ can be used (see Kendall and Gibbons, 1990, pp. 150–151). A table of the distribution of S is given here in Table N for $n = 3, k \leq 8$ and $n = 4, k \leq 4$. More extensive tables for the distribution of Q, a linear function of S to be defined in (12.2.8), are given in Owen (1962) for $n = 3, k \leq 15$ and $n = 4, k \leq 8$. Other tables given in Michaelis (1971), Quade (1972), and Odeh (1977) cover the cases up to $k = 6, n = 6$. The calculations are tedious even using the systematic approach. Therefore, outside the range of existing tables, an approximation to the null distribution is generally used for tests of significance.

Using $\mu = (n+1)/2$, (12.2.2) can be written as

$$\begin{aligned} S &= \sum_{j=1}^{n} \sum_{i=1}^{k} (R_{ij} - \mu)^2 + 2 \sum_{j=1}^{n} \sum_{1 \leq i < p \leq k} \sum (R_{ij} - \mu)(R_{pj} - \mu) \\ &= k \sum_{j=1}^{n} (j - \mu)^2 + 2U \\ &= \frac{kn(n^2-1)}{12} + 2U \end{aligned} \tag{12.2.3}$$

The moments of S then are determined by the moments of U, which can be found using the following relations from (11.3.2), (11.3.3), and (11.3.10):

$$E(R_{ij}) = \frac{n+1}{2} \quad \operatorname{var}(R_{ij}) = \frac{n^2-1}{12}$$
$$\operatorname{cov}(R_{ij}, R_{iq}) = -\frac{n^2-1}{12}$$

Furthermore, by the design assumptions, observation in different rows are independent, so that for all $i \neq p$ the expected value of a product of functions of R_{ij} and R_{pq} is the product of the expected values and $\operatorname{cov}(R_{ij}, R_{pq}) = 0$. Then

$$E(U) = n \binom{k}{2} \operatorname{cov}(R_{ij}, R_{pj}) = 0$$

so that $\text{var}(U) = nE(U^2)$, where

$$U^2 = \sum_{j=1}^{n} \sum_{1\le i<p\le k} (R_{ij}-\mu)^2(R_{pj}-\mu)^2 + 2\sum_{1\le j<q\le n}\sum \sum_{1\le i<p\le k}\sum \sum_{1\le r<s\le k}\sum (R_{ij}-\mu)(R_{pj}-\mu)(R_{rq}-\mu)(R_{sq}-\mu) \quad (12.2.4)$$

Since R_{ij} and R_{pq} are independent whenever $i \neq p$, we have

$$E(U^2) = \sum_{j=1}^{n} \sum_{1\le i<p\le k}\sum \text{var}(R_{ij})\,\text{var}\,(R_{pj}) + 2\sum_{1\le j<q\le n}\sum \binom{k}{2}\text{cov}(R_{ij}, R_{iq})\ \text{cov}(R_{pj}, R_{pq}) \quad (12.2.5)$$

$$E(U^2) = n\binom{k}{2}\frac{(n^2-1)^2}{144} + 2\binom{n}{2}\binom{k}{2}\frac{(n+1)^2}{144} = n^2\binom{k}{2}(n+1)^2\frac{(n-1)}{144} \quad (12.2.6)$$

Substituting these results back in (12.2.3), we find

$$E(S) = \frac{kn(n^2-1)}{12} \quad \text{var}(S) = \frac{n^2k(k-1)(n+1)^2}{72} \quad (12.2.7)$$

A linear function of the random variable S defined as

$$Q = \frac{12S}{kn(n+1)} = \frac{12\sum_{j=1}^{n} R_j^2}{kn(n+1)} - 3k(n+1) \quad (12.2.8)$$

has moments $E(Q) = n-1, \text{var}(Q) = 2(n-1)(k-1)/k \approx 2(n-1)$, which are the first two moments of a chi-square distribution with $n-1$ degrees of freedom. The higher moments of Q are also closely approximated by corresponding higher moments of the chi-square for k large. For all practical purposes then, Q can be treated as a chi-square variable with $n-1$ degrees of freedom. Numerical comparisons have shown this to be a good approximation as long as $k > 7$. The rejection region for a test of equal treatment effects against the alternative that the effects are not all equal with significance level approximately α is

$$Q \in R \quad \text{for } Q \ge \chi^2_{n-1,\alpha}$$

A test based on S or Q is called Friedman's test.

From classical statistics, we are accustomed to thinking of an analysis-or-variance test statistic as the ratio of two estimated variances or mean squares of deviations. The total sum of squares of deviations of all nk ranks around the average rank is

$$s_t = \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r})^2 = k \sum_{j=1}^{n} \left(j - \frac{n+1}{2} \right)^2 = kn \frac{(n^2-1)}{12}$$

and thus we could write Friedman's test statistic in (12.2.8) as

$$Q = \frac{(n-1)S}{s_t}$$

Even though s_t is a constant, it can be partitioned into a sum of squares of deviations between columns plus a residual sum of squares as in classical analysis-of-variance problems. Denoting the grand mean and column means, respectively, by

$$\bar{r} = \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{r_{ij}}{nk} = \frac{n+1}{2} \qquad \bar{r}_j = \frac{r_j}{k} = \sum_{i=1}^{k} \frac{r_{ij}}{k}$$

we have

$$\begin{aligned} s_t &= \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r})^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r}_j + \bar{r}_j - \bar{r})^2 \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r}_j)^2 + k \sum_{j=1}^{n} (\bar{r}_j - \bar{r})^2 + 2 \sum_{j=1}^{n} (\bar{r}_j - \bar{r}) \sum_{i=1}^{k} (r_{ij} - \bar{r}_j) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r}_j)^2 + \sum_{j=1}^{n} \frac{\{r_j - [k(n+1)/2]\}^2}{k} \end{aligned}$$

or

$$s_t = \sum_{i=1}^{k} \sum_{j=1}^{n} (r_{ij} - \bar{r}_j)^2 + \frac{s}{k} = kn \frac{n^2-1}{12} \tag{12.2.9}$$

Table 12.2.1 completes the analogy to the classical analysis-of-variance table.

TABLE 12.2.1

Analysis-of-Variance Table

Source of Variation	Degrees of Freedom	Sum of Squares
Between columns	$n-1$	s/k
Between rows	$k-1$	0[a]
Residual	$(n-1)(k-1)$	$s_t - s/k$
Total	$nk-1$	s_t

[a] There is no variation between rows here since the row sums are all equal.

The classical statistic for equal column effects is the ratio of the between columns and residual mean squares, or

$$\frac{(k-1)S}{ks_t - S} \tag{12.2.10}$$

If the distributions are normal with equal variances, the null distribution of the statistic in (12.2.10) is Snedecor's F with $(n-1)$ and $(n-1)(k-1)$ degrees of freedom.

12.2.1 Applications

Friedman's two-way analysis of variance by ranks is appropriate for the null hypothesis of equal treatment effects

$$H_0 : \theta_1 = \theta_2 = \cdots = \theta_n$$

for data on n treatments applied in k blocks. The word treatment effect is used in a very general way and may not refer to a real treatment. It may indicate the effect of a condition or characteristic such as income level or race. The first step is to rank the data in each block from 1 to n. These ranks are summed for each column to obtain $R_1, R_2, \ldots, R_n$. One form of the test statistic is S, the sum of squares of deviations of these totals from their mean, given in (12.2.2) but simplified here for calculation to

$$S = \sum_{j=1}^{n} R_j^2 - \frac{k^2 n(n+1)^2}{4} \tag{12.2.11}$$

The null distribution of S is given in Table N for $n = 3$, $k \leq 8$ and $n = 4, k \leq 4$ as right-tail probabilities since H_0 should be rejected for S large. For other n, k we use Table B since the asymptotic distribution of Q in (12.2.8) is chi square with $n-1$ degrees of freedom.

If ties are present to the extent t, we use midranks. The test statistic that incorporates the correction for ties is

$$Q = \frac{12(n-1)S}{kn(n^2-1) - \sum\sum t(t^2-1)} \tag{12.2.12}$$

where the double sum is extended over all sets of t tied ranks in each of the k blocks. This result will be derived in Section 12.4.

If the null hypothesis of equal treatment effects is rejected, we may want to determine which pairs of treatments differ significantly in effects and in which direction. Then we can use a multiple comparisons procedure to compare the $n(n-1)/2$ pairs of treatments, as we did in Section 10.4 for the one-way analysis-of-variance test for equal medians.

The procedure is to declare that treatments i and j are significantly different in effect if

$$|R_i - R_j| \geq z^* \sqrt{\frac{kn(n+1)}{6}} \tag{12.2.13}$$

where z^* is found as in Section 10.4 as the negative of the $[\alpha/\{n(n-1)\}]$th quantile of the standard normal distribution. As before, α is generally chosen to be larger than in the typical hypothesis testing situation, as around 0.15 or 0.20, because so many comparisons are being made.

Example 12.2.1

An important factor in raising small children is to develop their ability to ask questions, especially in groups so that they will have this skill when they start school. A study of group size and number of questions asked by preprimary children in a classroom atmosphere with a familiar person as teacher consists of dividing 46 children randomly into four mutually exclusive groups of sizes 24, 12, 6, and 4. The total number of questions asked by all members of each group is recorded for 30 min on each of eight different days. For the data shown in Table 12.2.2, test the null hypothesis that the effect of group size is the same in terms of total number of questions asked.

Solution

The days serve as blocks and the group sizes are the treatments so that $n = 4, k = 8$. The null hypothesis is equal treatment effects or $H_0 : \theta_1 = \theta_2 = \theta_3 = \theta_4$. We first rank the observations for each day from 1 to 4, using midranks for the few ties, and then sum the columns to find the R_j, as shown in Table 12.2.3.

We calculate the sum of squares from (12.2.11) as

$$S = (8^2 + 17^2 + 26.5^2 + 28.5^2) - \frac{8^2(4)(5)^2}{4} = 267.5$$

TABLE 12.2.2

Data for Example 12.2.1

	Group Size			
Day	**24**	**12**	**6**	**4**
1	14	23	26	30
2	19	25	25	33
3	17	22	29	28
4	17	21	28	27
5	16	24	28	32
6	15	23	27	36
7	18	26	27	26
8	16	22	30	32

TABLE 12.2.3

Ranks of Data for Example 12.2.1

	Group Size			
Day	**24**	**12**	**6**	**4**
1	1	2	3	4
2	1	2.5	2.5	4
3	1	2	4	3
4	1	2	4	3
5	1	2	3	4
6	1	2	3	4
7	1	2.5	4	2.5
8	1	2	3	4
Total	8	17	26.5	28.5

and then

$$Q = \frac{12(267.5)}{8(4)(5)} = 20.1$$

from (12.2.8) with 3 degrees of freedom. Table B shows that $P < 0.001$, so we reject the null hypothesis. It appears that the larger the group size, the fewer the questions asked.

Notice that there are two sets of ties, occurring on days 1 and 7, and each is of extent 2. Hence $\sum\sum t(t^2 - 1) = 12$ and the corrected test statistic from (12.2.12) is $Q = 20.58$. The P value is unchanged.

Since the difference between the n treatment effects has been found to be significant, we can use the multiple comparisons procedure to determine which

pairs of treatments differ significantly. With $\alpha = 0.15$ say, $k = 8, n = 4$, we have $z^* = 2.241$ and the right-hand side of (12.2.13) is 11.572. The groups that differ significantly are sizes 6 and 24, and sizes 4 and 24.

The computer solutions to this example are shown below from the MINITAB and STATXACT packages. We note that the correction for ties was incorporated to calculate Q in STATXACT while MINITAB gives the answer both with and without the correction. The exact P value in STATXACT is based on the randomization distribution or permutation distribution of the test statistic.

```
*****************************************
MINITAB SOLUTION TO EXAMPLE 12.2.1
*****************************************

1     1     14
1     2     19
1     3     17
1     4     17
1     5     16
1     6     15
1     7     18
1     8     16
2     1     23
2     2     25
2     3     22
2     4     21
2     5     24
2     6     23
2     7     26
2     8     22
3     1     26
3     2     25
3     3     29
3     4     28
3     5     28
3     6     27
3     7     27
3     8     30
4     1     30
4     2     33
4     3     28
4     4     27
4     5     32
4     6     36
4     7     26
4     8     32
```

```
**************************************************
MINITAB SOLUTION TO EXAMPLE 12.2.1 CONTINUED
**************************************************

Friedman test: C3 versus C1, C2

Friedman test for C3 by C1 blocked by C2

S = 20.06  DF = 3  P = 0.000
S = 20.58  DF = 3  P = 0.000 (adjusted for ties)

                      Est    Sum of
C1             N   Median    Ranks
1              8   16.125      8.0
2              8   23.500     17.0
3              8   27.875     26.5
4              8   31.000     28.5

Grand median = 24.625
```

```
*****************************************
STATXACT SOLUTION TO EXAMPLE 10.2.1
*****************************************

FRIEDMAN TEST

[That four treatments have identical effects in eight informative blocks]

Statistic based on the observed 4 by 8 two-way layout (x):
   FR (x)   :Friedman statistic = 20.58

Asymptotic P value: (based on chi-square distribution with 3 df)
   Pr {FR (X) .GE.  20.58} =  0.0001

Exact P value and point probability:
   Pr {FR (X) .GE.  20.58} =  0.0000
   Pr {FR (X) .EQ.  20.58} =  0.0000
```

12.3 Page's Test for Ordered Alternatives

The alternative hypothesis for Friedman's two-way analysis of variance by ranks in a $k \times n$ table described in Section 12.2 is that the treatment effects are not all the same. Now suppose that we are interested in an alternative that states that the treatment effects θ_i occur in a specified order, for example,

$$H_1 : \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$$

with at least one inequality strict. We call this an ordered alternative hypothesis of H_0 again H_1. Similar to our discussion in Section 10.6, Page (1963) suggested a test based on a weighted sum of the column totals

$$L = \sum_{j=1}^{n} Y_j R_j \tag{12.3.1}$$

where the weight Y_j is the hypothetical ranking of the jth treatment, predicted from prior considerations. The null hypothesis should be rejected in favor of this ordered alternative for large values of L. Tables of exact critical values of L are given in Page (1963) for levels 0.001, 0.01, and 0.05 and reproduced here as Table Q. For large values of k and n, the statistic with a continuity correction

$$Z = \frac{12(L - 0.5) - 3kn(n+1)^2}{n(n+1)\sqrt{k(n-1)}} \tag{12.3.2}$$

is approximately standard normal and the appropriate rejection region is in the right tail.

The test based on L can be shown to be related to the average of the rank correlation coefficients between each ranking and the ranking predicted by the alternative. This relationship is

$$r_{\text{av}} = \frac{12L}{k(n^3 - n)} - \frac{3(n+1)}{n-1}$$

The Page test can also be used in the situation of Section 12.4 where we have k sets of rankings of n objects and the alternative states an a priori ranking of the objects.

Example 12.3.1

This numerical example is based on one used by Page (1963) to illustrate his procedure. The research hypothesis is that speed of learning is related to the similarity of practice material used in pretraining sessions to the test criterion learning material. Group A used practice material most similar to that of criterion learning, followed by groups B and C in that order, and group D had no pretraining. Therefore the predicted ranking from best to worst is A, B, C, D, or $\theta_D < \theta_C < \theta_B < \theta_A$, where rank 1 is given to the least rapid learning. Note that Page's article uses 1 to denote most rapid, whereas we use 1 to denote least rapid. Six different classes that were divided into these four groups gave the rankings shown in Table 12.3.1.

TABLE 12.3.1

Data for Example 12.3.1

	Treatments			
Classes	**A**	**B**	**C**	**D**
1	3	4	2	1
2	4	2	1	3
3	4	2	3	1
4	4	4	3	2
5	2	4	3	1
6	4	3	1	2
	—	—	—	—
R_j	21	16	13	10
Y_j	4	3	2	1

We use (12.3.1) to compute $L = 168$. The critical value from Table Q with $n = 4$, $k = 6$ and $\alpha = 0.05$ is 163, so we reject the null hypothesis of equal treatment effects in favor of the ordered alternative. Using the normal approximation with a continuity correction and $\alpha = 0.05$, we reject when

$$L \geq \frac{kn(n+1)^2}{4} + 0.5 + \frac{n(n+1)\sqrt{k(n-1)}}{12} z_\alpha$$

which equals 162.13 for our example ($z_{0.05} = 1.645$). We also reject the null hypothesis using the normal approximation.

The computer solution to this example is shown below with an output from STATXACT; the value of the statistic 168 agrees with ours and both the exact and approximate P values suggest rejecting the null hypothesis, which agree with our conclusions. The reader can verify that STATXACT does not use a continuity correction to calculate the approximate P value.

```
*****************************************
STATXACT SOLUTION TO EXAMPLE 12.3.1
*****************************************
PAGE TEST
[ That four treatments have identical effects in each of six blocks ]
Statistic based on the observed 4 by 6 two-way layout (x) :
    Mean   Std Dev  Observed (PA(x))  Standardized (PA*(x))
   150.0   7.071          168.0              2.546
```

```
Asymptotic P value:
    One-sided: Pr { PA*(X) .GE.          2.546 }    = 0.0055
    Two-sided: 2 * One-sided                        = 0.0109

Exact P value:
   One-sided: Pr {   PA*(X)  .GE.         2.546 }   = 0.0053
   Pr    {   PA*(X)   .EQ.     2.546 }          =     0.0021
   Two-sided: Pr  { |PA*(X)| .GE.   2.546 }     =     0.0106
```

Example 12.3.2

In light of our conjecture in Example 12.2.1, it will be instructive to repeat the data analysis for the ordered alternative

$$H_1 : \theta_{24} \leq \theta_{12} \leq \theta_6 \leq \theta_4$$

where θ indicates the effect of the group size on asking questions and the subscript indicates the size of the group. The test statistic for these data is

$$L = 1(8) + 2(17) + 3(26.5) + 4(28.5) = 235.5$$

and the P value from Table Q is less than 0.001. Our previous conjecture that the larger the group size, the fewer questions are asked does appear to be correct.

The computer solution to this example is shown below with an output from STATXACT. Our hand calculations and conclusions agree with those obtained from the output.

```
*****************************************
STATXACT SOLUTION TO EXAMPLE 12.3.2
*****************************************

PAGE TEST
[ That four treatments have identical effects in each of six blocks]
Statistic based on the observed 4 by 6 two-way layout (x):

     Mean   Std Dev  Observed (PA(x))  Standardized (PA*(x))
    200.0   8.165       235.5                4.348

Asymptotic P value:
    One-sided: Pr { PA*(X) .GE.          4.348 }   =      0.0000
    Two-sided: 2 * One-sided                       =      0.0000
```

```
Exact P value:
   One-sided: Pr{  PA*(X)  .GE.        4.348 } =     0.0000
              Pr{  PA*(X)  .EQ.        4.348 } =     0.0000
   Two-sided: Pr{|PA*(X)|  .GE.        4.348 } =     0.0000
```

A different test for ordered alternatives was proposed by Jonckheere (1954), but the Page test is easier to use.

12.4 The Coefficient of Concordance for k Sets of Rankings of n Objects

The next problem to be covered in this chapter involves k sets of rankings of n objects, where we want to test the null hypothesis that the k sets are independent and also to find a measure of the strength of the relationship between rankings. Each of k observers is presented with the same set of n objects to be ranked. The measure of relationship will describe the agreement or concordance between observers in their judgments on the n objects.

Since the situation here is an extension of the paired-sample problem of Chapter 11, one possibility for a measure of agreement is to select one of the measures for paired samples and apply it to each of the $\binom{k}{2}$ sets of pairs of rankings of n objects. However, if $\binom{k}{2}$ tests of the null hypothesis of independence are then made, the tests are not independent and the overall probability of a type I error is difficult to determine but necessarily increased. Such a method of hypothesis testing is statistically undesirable. We need a single measure of overall association that will provide a single test statistic designed to detect overall dependence between samples with a specified significance level. Some combination of the measures obtained for each of the $\binom{k}{2}$ pairs will provide a single coefficient of overall association that can be used to test the null hypothesis of independence or no association between rankings if its sampling distribution can be determined.

The coefficient of concordance is a measure of the relationship between k sets of rankings. It is a linear function of the average of the coefficients of rank correlation for all pairs of rankings, as will be shown later. However, the rationale of the measure will be developed independently of the procedures of the last chapter so that the analogy to analysis-of-variance techniques will be more apparent.

For the purpose of this parallel, we need to visualize the data as presented in the form of a two-way layout of dimension $k \times n$ as in (12.2.1), with row and column labels now designating observers and objects instead of blocks and

treatments. The table entry R_{ij} is the rank given by the ith observer to the jth object. Then the ith row is a permutation of the numbers $1, 2, \ldots, n$, and the jth column entries are the ranks given to object j by all observers. The ranks in each column are then indicative of the agreement between observers, since if the jth object has the same preference relative to all other objects in the opinion of each of the k observers, all ranks in the jth column will be identical. If this is true for every column, the observers agree perfectly and the respective column totals $(R_1, R_2, \ldots, R_n)$ will be some permutation of the numbers

$$1k, 2k, 3k, \ldots, nk$$

Since the average column total is $k(n+1)/2$, for perfect agreement between rankings the sum of squares of deviations of column totals from the average column total will be a constant

$$\sum_{j=1}^{n} \left[jk - \frac{k(n+1)}{2} \right]^2 = k^2 \sum_{j=1}^{n} \left(j - \frac{n+1}{2} \right)^2 = k^2 n \frac{n^2 - 1}{12} \tag{12.4.1}$$

The actual observed sum of squares of these deviations is

$$S = \sum_{j=1}^{n} \left[R_j - \frac{k(n+1)}{2} \right]^2 \tag{12.4.2}$$

We found in (12.2.9) that

$$ks_t = \frac{k^2 n(n^2 - 1)}{12} = s + k \sum_{i=1}^{k} \sum_{j=1}^{k} (r_{ij} - \bar{r}_j)^2 \tag{12.4.3}$$

where s_t is the total sum of squares of deviations of all ranks around the average rank. In terms of this situation, however, we see from (12.4.1) that the ks_t is the sum of squares of column total deviations when there is perfect agreement. Therefore the value of S for any set of k rankings ranges between zero and $k^2 n(n^2 - 1)/12$. The maximum value is attained when $r_j = jk$ for all j, that is, when there is perfect agreement. The minimum value is attained when $r_j = k(n+1)/2$ for all j, that is, when each observer's rankings are assigned completely at random so that there is no agreement between observers. If the observers are called samples, no agreement between observers is equivalent to independence of the k samples.

The ratio of S to its maximum value

$$W = \frac{S}{ks_t} = \frac{12S}{k^2 n(n^2 - 1)} \tag{12.4.4}$$

therefore provides a measure of agreement between observers, or concordance between sample rankings, or dependence of the samples. This measure is called *Kendall's coefficient of concordance*. It ranges between 0 and 1, with 1 designating perfect agreement or concordance and 0 indicating no agreement or independence of samples. As W increases, the set of ranks given to each object must become more similar because in the error term of (12.4.3), $\sum_{i=1}^{k}(r_{ij}-\bar{r}_j)^2$ becomes smaller for all j, and thus there is greater agreement between observers.

In order to make the interpretation of this k-sample coefficient consistent with a two-sample measure of association, one might think some measure, which ranges from -1 to $+1$, would be preferable, with -1 designating perfect discordance. However, for more than two samples, there is no such thing as perfect disagreement between rankings, and thus concordance and discordance are not symmetrical opposites. Therefore the range 0–1 is indeed appropriate for a k-sample measure of association.

12.4.1 Relationship between W and Rank Correlation

We now show that the statistic W is related to the average of the $\binom{k}{2}$ coefficients of rank correlation that can be calculated for the $\binom{k}{2}$ pairings of sample rankings. The average value is

$$r_{\text{av}} = \frac{\sum\sum_{1\le i<m\le k} r_{i,m}}{\binom{k}{2}} = \sum_{\substack{i=1\\ i\neq m}}^{k}\sum_{m=1}^{k}\frac{r_{i,m}}{k(k-1)} \tag{12.4.5}$$

where

$$r_{i,m} = \frac{12}{n(n^2-1)}\sum_{j=1}^{n}\left(r_{ij}-\frac{n+1}{2}\right)\left(r_{mj}-\frac{n+1}{2}\right) \quad \text{for all } i\neq m$$

Denoting the average rank $(n+1)/2$ by μ, we have

$$\begin{aligned} r_{\text{av}} &= 12\sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq m}}^{k}\sum_{m=1}^{k}\frac{(r_{ij}-\mu)(r_{mj}-\mu)}{kn(k-1)(n^2-1)} \\ &= 12\sum_{j=1}^{n}\frac{\left[\sum_{i=1}^{k}(r_{ij}-\mu)\right]^2-\sum_{i=1}^{k}(r_{ij}-\mu)^2}{kn(k-1)(n^2-1)} \\ &= \frac{\sum_{j=1}^{n}(r_j-k\mu)^2-s_t}{(k-1)s_t} = \frac{s-s_t}{(k-1)s_t} = \frac{kW-1}{k-1} \end{aligned} \tag{12.4.6}$$

or

$$W = r_{\text{av}} + \frac{1 - r_{\text{av}}}{k} = \frac{r_{\text{av}}(k-1) + 1}{k} \tag{12.4.7}$$

From this relation, we see that $W = 1$ when $r_{\text{av}} = 1$, which can occur only when $r_{i,m} = 1$ for all sets (i, m) of two samples, since always $r_{i,m} \leq 1$. It is impossible to have $r_{\text{av}} = -1$, since $r_{i,m} = -1$ cannot occur for all sets (i, m) simultaneously. Since we have already shown that the minimum value of W is zero, it follows from (12.4.7) that the smallest possible value of r_{av} is $-1/(k-1)$.

12.4.2 Tests of Significance Based on *W*

Suppose we consider each column in our $k \times n$ table to be the ranks of observations from a k-variate population. With n columns, we can say that $(R_{1j}, R_{2j}, \ldots, R_{kj}), j = 1, 2, \ldots, n$, constitute ranks of a random sample of size n from a k-variate population. We want to test the null hypothesis that the variates are independent. The coefficient of concordance W is an overall measure of the association between the ranks of the k variates or the k sets of rankings of n objects, which in turn estimates some measure of the relationship between the k variates in the population. If the variates are independent, there is no association and $W = 0$, and for complete dependence there is perfect agreement and $W = 1$. Therefore the statistic W can be used to test the null hypothesis that the variates are independent. The appropriate rejection region is large values of W.

In the null case, the ranks assigned to the n observations are completely random for each of the k variates, and the $(n!)^k$ assignments are all equally likely. The random sampling distribution of S (or W) then is exactly the same as in Section 12.2. Table N can therefore be used, and for k large the distribution of

$$Q = \frac{12S}{kn(n+1)} = k(n-1)W$$

may be approximated by the chi-square distribution with $n - 1$ degrees of freedom.

Other approximations are also occasionally used for tests of significance. Although the mean and variance of W are easily found using the moments already obtained for S in (12.2.7), it will be more instructive to determine the null moments of W directly by using its relationship with R_{av} given in (12.4.7). From (11.3.13), the mean and variance of $R_{i,m}$, the rank-correlation coefficient of any pairing of independent sets of ranks, are

$$E(R_{i,m}) = 0 \quad \text{var}(R_{i,m}) = \frac{1}{n-1} \quad \text{for all } 1 \le i < m \le k$$

For any two independent sets of pairings of independent ranks, say (i, m) and (p, j) where $1 \le i < m \le k$, $1 \le p < j \le k$, the covariance is

$$\text{cov}(R_{i,m}, R_{p,j}) = 0 \quad \text{unless } i = p \text{ and } m = j$$

Therefore, from the definition of R_{av} in (12.4.5), we have

$$E(R_{\text{av}}) = 0$$

$$\binom{k}{2}^2 \text{var}(R_{\text{av}}) = \sum_{1 \le i < m \le k} \sum \text{cov}(R_{i,m}) + \sum_{\substack{1 \le i < m \le k \\ i \ne p \text{ or } m \ne j}} \sum_{1 \le p < j \le k} \sum \text{cov}(R_{i,m}, R_{p,j}) = \frac{\binom{k}{2}}{(n-1)}$$

and

$$\text{var}(R_{\text{av}}) = \frac{2}{k(k-1)(n-1)}$$

Now using (12.4.7),

$$E(W) = \frac{1}{k} \quad \text{var}(W) = \frac{2(k-1)}{k^3(n-1)} \tag{12.4.8}$$

The reader can verify that these are exactly equal to the mean and variance of the beta distribution with parameters

$$a = \frac{k(n-1) - 2}{2k} \quad \text{and} \quad b = \frac{(k-1)[k(n-1) - 2]}{2k}$$

An investigation of the higher moments of W shows that they are approximately equal to the corresponding higher moments of the beta distribution unless $k(n-1)$ is small. Thus an approximation to the distribution of W is the beta distribution, for which tables are available. However, if any random variable, say X, has the beta distribution with parameters a and b, the transformation $Y = bX/[a(1-X)]$ produces a random variable with Snedecor's F distribution with parameters $v_1 = 2a$ and $v_2 = 2b$, and the transformed variable $Z = (\ln Y)/2$ has Fisher's z distribution with the same parameters. Applying these transformations here, we find the approximate distributions

1. $(k-1)W/(1-W)$ is Snedecor's F with $v_1 = n - 1 - 2/k$ and $v_2 = (k-1)v_1$
2. $\ln[(k-1)W/(1-W)]/2$ is Fisher's z with $v_1 = n - 1 - 2/k$ and $v_2 = (k-1)v_1$ in addition to our previous approximation
3. $k(n-1)W$ is chi-square with $n-1$ degrees of freedom.

Approximation 1 is not surprising, since we found in (12.2.10) that the random variable

$$\frac{(k-1)S}{ks_t - S} = \frac{(k-1)W}{1-W}$$

was the ratio of mean squares analogous to the analysis-of-variance test statistic with $n-1$ and $(n-1)(k-1)$ degrees of freedom.

12.4.3 Estimation of the True Preferential Order of Objects

Assume that the coefficient of concordance is computed for some k sets of rankings of n objects and the null hypothesis of no agreement is rejected. Then we can conclude that not all of these particular observers ranked the objects strictly randomly and independently. This might be interpreted to mean that there is some agreement among these observers and that perhaps some unique ordering of these objects exists in their estimation. Suppose we call this the true preferential ordering. If there were perfect agreement, we would know which object is least preferred, which is next, etc., by the agreed-upon ranks. Object j would have the position m in the true preferential ordering if the sum of ranks given object j is km. In our $k \times n$ table of ranks, this preferential ordering corresponds to the ranks of the column sums. In a case of less-than-perfect agreement, then, the true preferential ordering might be estimated by assigning ranks to the objects in accordance with the magnitudes of the column sums.

This estimate is best in the sense that if the coefficient of rank correlation is calculated between this estimated ranking and each of the k observed rankings, the average of these k correlation coefficients is a maximum. To show this, we let $r_{e1}, r_{e2}, \ldots, r_{en}$ be any estimate of the true preferential ordering, where r_{ej} is the estimated rank of object j. If $R_{e,i}$ denotes the rank-correlation coefficient between this estimated ranking and the ranking assigned by the ith observer, the average rank correlation is

$$\sum_{i=1}^{k} \frac{r_{e,i}}{k} = 12 \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{(r_{ej} - \mu)(r_{ij} - \mu)}{kn(n^2-1)} = 12 \sum_{j=1}^{n} \frac{(r_{ej} - \mu)(r_j - k\mu)}{kn(n^2-1)}$$

$$= \frac{12 \sum_{j=1}^{n} r_{ej} r_j}{kn(n^2-1)} - \frac{3(n+1)}{(n-1)}$$

where $\mu = (n+1)/2$ and r_j is the jth column sum as before. This average then is a maximum when $\sum_{j=1}^{n} r_{ej} r_j$ is a maximum, i.e., when the r_{ej} are in the same relative order of magnitude as the r_j.

This estimate is also best in a least-squares sense. If r_{ej} is any estimated rank of object j and the estimate is the true preferential rank, the jth column sum would equal kr_{ej}. A measure of the error in this estimate then is the sum of squares of deviations

$$\begin{aligned}\sum_{j=1}^{n} (r_j - kr_{ej})^2 &= \sum_{j=1}^{n} r_j^2 + k^2 \sum_{j=1}^{n} r_{ej}^2 - 2k \sum_{j=1}^{n} r_j r_{ej} \\ &= \sum_{j=1}^{n} r_j^2 + k^2 \sum_{i=1}^{n} i^2 - 2k \sum_{j=1}^{n} r_j r_{ej} \\ &= c - 2k \sum_{j=1}^{n} r_j r_{ej}\end{aligned}$$

where c is a constant. The error is thus minimized when $\sum_{j=1}^{n} r_j r_{ej}$ is a maximum, and the r_{ej} should be chosen as before.

12.4.4 Tied Observations

Up to now, we have assumed that each row of our $k \times n$ table is a permutation of the first n integers. If an observer cannot express any preference between two or more objects, or if the objects are actually indistinguishable, we may wish to allow the observer to assign equal ranks. If these numbers are the midranks of the positions each set of tied objects would occupy if a preference could be expressed, the average rank of any object and the average column sum are not changed. However, the sum of squares of deviations of any set of n ranks is reduced if there are ties. As we found in (11.3.18), for any $i = 1, 2, \ldots, k$, the corrected value is

$$\sum_{j=1}^{n} \left(r_{ij} - \frac{n+1}{2} \right)^2 = \frac{n(n^2-1) - \sum t(t^2-1)}{12}$$

The maximum value of s/k as in (12.2.9), is then reduced to

$$s_t = \sum_{i=1}^{k} \sum_{j=1}^{n} \left(r_{ij} - \frac{n+1}{2} \right)^2 = \frac{kn\,(n^2-1) - \sum\sum t\,(t^2-1)}{12}$$

where the double sum is extended over all sets of t tied ranks and all k rows. The relative measure of agreement in the presence of ties then is $W = S/ks_t$. The significance of the corrected coefficient W can be tested using any of the approximations previously mentioned.

12.4.5 Applications

The coefficient of concordance is a descriptive measure of the agreement between k sets of rankings of n objects and is defined in (12.4.4) where S is easily calculated from (12.2.11). To test the null hypothesis of no association or no agreement between rankings against the alternative of agreement or positive dependence, Table N can be used to find a right-tail critical value or P value for S in small samples. For large samples, the test statistic Q in (12.2.8) or equivalently $k(n-1)W$ can be used with Table B and $n-1$ degrees of freedom.

Example 12.4.1

Eight graduate students are each given examinations in quantitative reasoning, vocabulary, and reading comprehension. Their scores are listed below. It is frequently claimed that persons who excel in quantitative reasoning are not as capable with verbal, and vice versa, and yet a truly intelligent person must possess all of these abilities. Test these data to see if there is an association between scores. Does there seem to be an indirect relationship between quantitative and verbal skills?

	Student							
Test	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Quantitative	90	60	45	48	58	72	25	85
Vocabulary	62	81	92	76	70	75	95	72
Reading	60	91	85	81	90	76	93	80

Solution

The first step is to rank the students from 1 (best) to 8 according to their scores on each of the three skills. This will give us $k = 3$ sets of rankings of $n = 8$ objects. Then we compute the rank sums as shown below.

	Student							
Test	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Quantitative	1	4	7	6	5	3	8	2
Vocabulary	8	3	2	4	7	5	1	6
Reading	8	2	4	5	3	7	1	6
	—	—	—	—	—	—	—	—
Total	17	9	13	15	15	15	10	14

For these data $\sum R^2 = 1510$ and from (12.2.11) $S = 52$, and $W = 0.138$ from (12.4.4) is a descriptive measure of the agreement between rankings. To test the null hypothesis, we calculate $Q = 2.89$ from (12.2.8) with 7 degrees of freedom.

The P value from Table B is $0.50 < P < 0.90$, so there appears to be no agreement between the rankings. We might note that the greatest source of disagreement is the quantitative scores in comparison with the other two, as suggested by the claim of an indirect relationship between quantitative and verbal abilities. One way to answer this question statistically is to obtain one verbal score for each student as the sum of the vocabulary and reading scores and compare this ranking with the quantitative ranking using say the rank correlation coefficient. We do this now.

	Student							
Test	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Verbal score	122	172	177	157	160	151	188	152
Verbal rank	8	3	2	5	4	7	1	6
Quantitative rank	1	4	7	6	5	3	8	2

$\sum R^2 = 158$ and $R = -0.881$ with $P = 0.004$ from Table M. There is a strong negative dependence between verbal and quantitative scores.

The SPSSX and STATXACT calculations of the Kendall coefficient of concordance are shown below. Note that the statistic value agrees exactly with hand calculations. The packages give the same asymptotic P value, 0.8951, which leads to the same decision as ours. STATXACT provides the exact P value, 0.9267, and the decision is the same.

```
******************************************
SPSSX SOLUTION TO EXAMPLE 12.4.1
******************************************

Kendall's W Test

Ranks
              Mean Rank          Test Statistics
  VAR00001      3.33                       N        3
  VAR00002      6.00             Kendall's W     .138
  VAR00003      4.67              Chi-Square    2.889
  VAR00004      4.00                      df        7
  VAR00005      4.00             Asymp. Sig.     .895
  VAR00006      4.00             Kendall's coefficient of
  VAR00007      5.67              concordance
  VAR00008      4.33

**********************************************
STATXACT SOLUTION TO EXAMPLE 12.4.1
**********************************************

Kendall's Concordance Test
Statistic based on the observed by 3 two-way layout (x):
  W(x): Kendall coefficient of concordance = 0.1376
```

```
Asymptotic P value: (based on chi-square distribution with 7 df)
  Pr{W(X).GE. 0.1376} = 0.8951

Exact P value and point probability :
  Pr{W(X).GE. 0.1376} = 0.9267
  Pr{W(X).GE. 0.1376} = 0.0072
```

12.5 The Coefficient of Concordance for k Sets of Incomplete Rankings

As before, suppose that we have n objects to be ranked and a fixed number of observers to rank them but now each observer ranks only some subset of the n objects. This situation could arise for reasons of economy or practicality. In the case of human observers particularly, the ability to rank objects effectively and reliably may be a function of the number of comparative judgments to be made. For example, after 10 different brands of bourbon have been tasted, the discriminatory powers of the observers may legitimately be questioned.

We will assume that the experimental design is such that the rankings are incomplete in the same symmetrical way as in the balanced incomplete-blocks design, which is used effectively in agricultural field experiments. For our situation, this means that:

1. Each observer will rank the same number m of objects for some $m < n$.
2. Every object will be ranked exactly the same total number k of times.
3. Each pair of objects will be presented together to some observer a total of exactly λ times, $\lambda \geq 1$, a constant for all pairs.

These specifications then ensure that all comparisons are made with the same frequency.

In order to visualize the design, imagine a two-way layout of p rows and n columns, where the entry δ_{ij} in the (i, j) cell equals 1 if object j is presented to observer i and 0 otherwise. The previous three design specifications then can be written in symbols as

1. $\sum_{j=1}^{n} \delta_{ij} = m \quad \text{for } i = 1, 2, \ldots, p$
2. $\sum_{i=1}^{p} \delta_{ij} = k \quad \text{for } j = 1, 2, \ldots, n$
3. $\sum_{i=1}^{p} \delta_{ij}\delta_{ir} = \lambda \quad \text{for all } r \neq j = 1, 2, \ldots, n$

Summing on the other subscript in specifications 1 and 2, we obtain

$$\sum_{i=1}^{p}\sum_{j=1}^{n}\delta_{ij} = mp = kn$$

which implies that the number of observers is fixed by the design to be $p = kn/m$. Now we have

$$\sum_{i=1}^{p}\left(\sum_{j=1}^{n}\delta_{ij}\right)^2 = \sum_{i=1}^{p}\left(\sum_{j=1}^{n}\delta_{ij}^2 + \sum_{\substack{j=1\\ j\neq r}}^{n}\sum_{r=1}^{n}\delta_{ij}\delta_{ir}\right) = mp + \lambda n(n-1)$$

from specification 3, and from specification 1, this same sum equals pm^2. This requires the relation

$$\lambda = \frac{pm(m-1)}{n(n-1)} = \frac{k(m-1)}{n-1}$$

Since p and λ must both be positive integers, m must be a factor of kn and $n-1$ must be a factor of $k(m-1)$. Designs of this type are called Youden squares or incomplete Latin squares. Such plans have been tabulated (for example, in Cochran and Cox, 1957, pp. 520–544). An example of this design for $n = 7, \lambda = 1, m = k = 3$, where the objects are designated by A, B, C, D, E, F, and G is shown below.

Observer	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Objects presented for ranking	*A*	*B*	*C*	*D*	*E*	*F*	*G*
	B	*C*	*D*	*E*	*F*	*G*	*A*
	D	*E*	*F*	*G*	*A*	*B*	*C*

We want to determine a single measure of the overall concordance or agreement between the kn/m observers in their relative comparisons of the objects. For simplification, suppose there is some natural ordering of all n objects and the objects are labeled accordingly. In other words, object r would receive rank r by all observers if each observer was presented with all n objects and the observers agreed perfectly in their evaluation of the objects. For perfect agreement in a balanced incomplete ranking then, where each observer assigns ranks $1, 2, \ldots, m$ to the subset presented to him, object 1 will receive rank 1 whenever it is presented; object 2 will receive rank 2 whenever it is presented along with object 1, and rank 1 otherwise; object 3 will receive rank 3 when presented along with both objects 1 and 2, rank 2 when with either objects 1 or 2 but not both, and rank 1 otherwise, etc. In general, then, the

rank of object j when presented to observer i is one more than the number of objects presented to that observer from the subset of objects $\{1, 2, \ldots, j-1\}$, for all $2 \leq j \leq n$. In symbols, using the δ notation of before, the rank of object j when presented to observer i is 1 for $j = 1$ and

$$1 + \sum_{r=1}^{j-1} \delta_{ir} \quad \text{for all } 2 \leq j \leq n$$

The sum of the ranks assigned to object j by all p observers in the case of perfect agreement then is

$$\sum_{i=1}^{p} \left(1 + \sum_{r=1}^{j-1} \delta_{ir}\right) \delta_{ij} = \sum_{i=1}^{p} \delta_{ij} + \sum_{r=1}^{j-1} \sum_{i=1}^{p} \delta_{ir}\delta_{ij} = k + \lambda(j-1) \quad \text{for } j = 1, 2, \ldots, n$$

as a result of the design specifications 2 and 3.

Since each object is ranked a fixed number, k, of times, the observed data for an experiment of this type can easily be presented in a two-way layout of k rows and n columns, where the jth column contains the collection of ranks assigned to object j by those observers to whom object j was presented. The rows no longer have any significance, but the column sums can be used to measure concordance. The sum of all ranks in the table is $[m(m+1)/2][kn/m] = kn(m+1)/2$, and thus the average column sum is $k(m+1)/2$. In the case of perfect concordance, the column sums are some permutation of the numbers

$$k, k+\lambda, k+2\lambda, \ldots, k+(n-1)\lambda$$

and the sum of squares of deviations of column sums around their mean is

$$\sum_{j=0}^{n-1} \left[(k + j\lambda) - \frac{k(m+1)}{2}\right]^2 = \frac{\lambda^2 n(n^2-1)}{12}$$

Let R_j denote the actual sum of ranks in the jth column. A relative measure of concordance between observers may be defined as

$$W = \frac{12 \sum_{j=1}^{n} [R_j - k(m+1)/2]^2}{\lambda^2 n(n^2-1)} \tag{12.5.1}$$

If $m = n$ and $\lambda = k$ so that each observer ranks all n objects, (12.5.1) is equivalent to (12.4.4), as it should be.

This coefficient of concordance also varies between 0 and 1 with larger values reflecting greater agreement between observers. If there is no

agreement, the column sums would all tend to be equal to the average column sum and W would be zero.

12.5.1 Tests of Significance Based on W

The null hypothesis of independence means that the ranks are allotted randomly by each observer to the subset of objects presented to him so that there is no concordance. The appropriate rejection region is large values of W.

The exact sampling distribution of W could be determined only by an extensive enumeration process. Exact tables for 15 different designs are given in van der Laan and Prakken (1972). For k large an approximation to the null distribution may be used for tests of significance. We will first determine the exact null mean and variance of W using an approach analogous to the steps leading to (12.2.7). Let $R_{ij}, i = 1, 2, \ldots, k$, denote the collection of ranks allotted to object j by the k observers to whom it was presented. From (11.3.2), (11.3.3), and (11.3.10), in the null case we have for all i, j, and $q \neq j$

$$E(R_{ij}) = \frac{m+1}{2} \quad \text{var}(R_{ij}) = \frac{m^2-1}{12} \quad \text{cov}(R_{ij}, R_{iq}) = -\frac{m+1}{12}$$

and R_{ij} and R_{hj} are independent for all j where $i \neq h$. Letting $\mu = (m+1)/2$, the numerator of W in (12.5.1) may be written as

$$\begin{aligned}
&12\sum_{j=1}^{n}\left[\sum_{i=1}^{k} R_{ij} - k\mu\right]^2 \\
&= 12\sum_{j=1}^{n}\left[\sum_{i=1}^{k}(R_{ij}-\mu)\right]^2 \\
&= 12\sum_{j=1}^{n}\sum_{i=1}^{k}(R_{ij}-\mu)^2 + 24\sum_{j=1}^{n}\sum_{1\le i<h\le k}\sum(R_{ij}-\mu)(R_{hj}-\mu) \\
&= pm(m^2-1) + 24U = \lambda^2 n(n^2-1)W
\end{aligned} \tag{12.5.2}$$

Since $\text{cov}(R_{ij}, R_{hj}) = 0$ for all $i < h$, $E(U) = 0$. Squaring the sum represented by U, we have

$$\begin{aligned}
U^2 = &\sum_{j=1}^{n}\sum_{1\le i<h\le k}\sum(R_{ij}-\mu)^2(R_{hj}-\mu)^2 + 2\sum_{1\le j<q\le n}\sum\sum_{1\le i<h\le k}\sum \\
&\times \sum_{1\le r<s\le k}\sum(R_{ij}-\mu)(R_{hj}-\mu)(R_{rq}-\mu)(R_{sq}-\mu)
\end{aligned}$$

and

$$E(U^2) = \sum_{j=1}^{n} \sum_{1\le i<} \sum_{h\le k} \operatorname{var}(R_{ij}) \operatorname{var}(R_{hj}) + 2 \sum_{1\le j<} \sum_{q\le n} \binom{\lambda}{2} \operatorname{cov}(R_{ij}, R_{iq}) \operatorname{cov}(R_{hj}, R_{hq})$$

since objects j and q are presented together to both observers i and h a total of $\binom{\lambda}{2}$ times in the experiment. Substituting the respective null variances and covariances, we obtain

$$\operatorname{var}(U) = E(U^2) = \frac{n\binom{k}{2}(m^2-1)^2 + 2\binom{n}{2}\binom{\lambda}{2}(m+1)^2}{144} = nk(m+1)^2(m-1)\frac{(m-1)(k-1)+(\lambda-1)}{288}$$

From (12.5.2), the null moments of W are

$$E(W) = \frac{m+1}{\lambda(n+1)}$$
$$\operatorname{var}(W) = 2(m+1)^2 \frac{(m-1)(k-1)+(\lambda-1)}{nk\lambda^2(m-1)(n+1)^2}$$

As in the case of complete rankings, a linear function of W has moments approximately equal to the corresponding moments of the chi-square distribution with $n-1$ degrees of freedom if k is large. This function is

$$Q = \frac{\lambda(n^2-1)W}{m+1}$$

and its exact mean and variance are

$$E(Q) = n-1$$
$$\operatorname{var}(Q) = 2(n-1)\left[1 - \frac{m(n-1)}{nk(m-1)}\right] \approx 2(n-1)\left(1 - \frac{1}{k}\right)$$

The rejection region for large k and significance level α then is

$$Q \in R \quad \text{for } Q \ge \chi^2_{n-1,\alpha}$$

12.5.2 Tied Observations

Unlike the case of complete rankings, no simple correction factor can be introduced to account for the reduction in total sum of squares of deviations of column totals around their mean when the midrank method is used to handle ties. If there are only a few ties, the null distribution of W should not be seriously altered, and thus the statistic can be computed as usual with midranks assigned. Alternatively, any of the other methods of handling ties discussed in Section 5.6 (except omission of tied observations) may be adopted.

12.5.3 Applications

Kendall's coefficient of concordance as given in (12.5.1) is most easily computed from

$$W = \frac{12\sum_{j=1}^{n} R_j^2 - 3k^2 n(m+1)^2}{\lambda^2 n(n^2-1)} \tag{12.5.3}$$

for k incomplete sets of n rankings, where m is the number of objects presented for ranking and λ is the number of times each pair of objects is presented together. The test statistic for the null hypothesis of independence or no concordance or no agreement between rankings is $Q = \lambda(n^2-1)W/(m+1)$, which is asymptotically chi-square distributed with $n-1$ degrees of freedom.

Example 12.5.1

A taste-test experiment to compare seven different kinds of wine is to be designed such that no taster will be asked to rank more than three different kinds, so we have $n = 7$ and $m = 3$. If each pair of wines is to be compared only once so that $\lambda = 1$, the required number of tasters is $p = \lambda n(n-1)/m(m-1) = 7$. A balanced design was used and the rankings are shown below. Calculate Kendall's coefficient of concordance as a measure of agreement between rankings and test the null hypothesis of no agreement.

	Wine						
Taster	***A***	***B***	***C***	***D***	***E***	***F***	***G***
1	1	2		3			
2		1	3		2		
3			3	2		1	
4				2	3		1
5	1				3	2	
6		2				1	3
7	1		3				2
	—	—	—	—	—	—	—
Total	3	5	9	7	8	4	6

SOLUTION

Each wine is ranked three times so that $k = 3$. We calculate $\sum R_j^2 = 280$ and substitute into (12.5.4) to get $W = 1$, which describes perfect agreement. The test statistic from (12.5.3) is $Q = 12$ with 6 degrees of freedom. The P value from Table B is $0.05 < P < 0.10$ for the test of no agreement between rankings.

As in the case of complete rankings in the Section 12.4, we have an analogous situation with the incomplete block design where n treatments are being compared but only m treatments are applied at a time in each of k blocks, and each pair of treatments is applied together exactly λ times. This analysis-of-variance test statistic for the null hypothesis of equal treatment effects is most easily computed as

$$Q = \frac{12 \sum_{j=1}^{n} R_j^2}{\lambda n(m+1)} - \frac{3k^2(m+1)}{\lambda} \tag{12.5.4}$$

which is asymptotically chi-square distributed with $n - 1$ degrees of freedom. The null hypothesis of equal treatment effects is rejected for Q large. The test based on (12.5.4) is usually called the *Durbin (1951) test*.

If the null hypothesis of equal treatment effects is rejected, we can use a multiple comparisons procedure to determine which pairs of treatments have significantly different effects. Treatments i and j are declared to be significantly different if

$$|R_i - R_j| \geq z^* \sqrt{\frac{km(m^2-1)}{6(n-1)}} \tag{12.5.5}$$

where z^* is the negative of the $[\alpha/n(n-1)]$th quantile of the standard normal distribution.

12.6 Kendall's Tau Coefficient for Partial Correlation

Coefficients of partial correlation are useful measures for studying relationships between more than two random variables since they are ordinary correlations between two variables with the effects of some other variables eliminated because these latter variables are held constant. In other words, the coefficients measure association in the conditional probability distribution of two variables given one or more other variables. A nice property of Kendall's tau coefficient of Section 11.2 is that it can be easily extended to the theory of partial correlation.

Assume we are given m independent observations of triplets $(X_i, Y_i, Z_i), i = 1, 2, \ldots, m$ from a trivariate population where the marginal distributions of each variable are continuous. We want to determine a sample measure of the association between X and Y when Z is held constant. Define the indicator variables

$$U_{ij} = \operatorname{sgn}(X_j - X_i) \quad V_{ij} = \operatorname{sgn}(Y_j - Y_i) \quad W_{ij} = \operatorname{sgn}(Z_j - Z_i)$$

and for all $1 \leq i < j \leq m$, let $n(u, v, w)$ denote the number of values of (i, j) such that $u_{ij} = u, v_{ij} = v, w_{ij} = w$. Now we further define the count variables

$$\begin{aligned} X_{11} &= n(1, 1, 1) \\ X_{22} &= n(-1, -1, 1) \\ X_{12} &= n(-1, 1, 1) \\ X_{21} &= n(1, -1, 1) \end{aligned}$$

Then X_{11} is the number of sets of (i, j) pairs, $1 \leq i < j \leq m$, of each variable such that X and Y are both concordant with Z, X_{22} is the number where X and Y are both discordant with Z, X_{12} is the number such that X is discordant with Z and Y is concordant with Z, and X_{21} is the number where X is concordant with Z and Y is discordant with Z. We present these counts in a 2×2 table as shown in Table 12.6.1. This table sets out the agreements of rankings X with Z, and rankings Y with Z, and the same for the disagreements. Now we define the partial rank correlation coefficient between X and Y when Z is held constant as

$$T_{XY.Z} = \frac{X_{11}X_{22} - X_{12}X_{21}}{(X_{.1}X_{.2}X_{1.}X_{2.})^{1/2}} \tag{12.6.1}$$

The value of this coefficient ranges between -1 and $+1$. At either of these two extremes, we have a sum of products of three or more nonnegative numbers whose exponents total four, which is equal to zero.

TABLE 12.6.1

Presentation of Data

	Ranking X		
Ranking Y	**Pairs Concordant with Z**	**Pairs Discordant with Z**	**Total**
Pairs concordant with Z	X_{11}	X_{12}	$X_{1.}$
Pairs discordant with Z	X_{21}	X_{22}	$X_{2.}$
Total	$X_{.1}$	$X_{.2}$	$X_{..} = N$

$$(X_{11} + X_{21})(X_{12} + X_{22})(X_{11} + X_{12})(X_{21} + X_{22}) - (X_{11}X_{22} - X_{12}X_{21})^2 = 0$$

This occurs only if at least two of the numbers are zero. If $X_{ij} = X_{hk} = 0$ for $i = h$ or $j = k$, either X or Y is in perfect concordance or discordance with Z. The nontrivial cases then are where both diagonal entries are zero. If $X_{12} = X_{21} = 0$, the X and Y sample values are always either both concordant or both discordant with Z and $T_{XY.Z} = 1$. If $X_{11} = X_{22} = 0$, the X and Y sample values are never both in the same relation with Z and $T_{XY.Z} = -1$. Maghsoodloo (1975), Maghsoodloo and Pallos (1981), and Moran (1951) give tables of the sampling distribution of the partial tau coefficient. $T_{XY.Z}$ provides a useful relative measure of the degree to which X and Y are concordant when their relation with Z is eliminated.

It is interesting to look at the partial tau coefficient in a different algebraic form. Using the X_{ij} notation above, the Kendall tau coefficients for the three different paired samples would be

$$\binom{m}{2} T_{XY} = (X_{11} + X_{22}) - (X_{12} + X_{21})$$
$$\binom{m}{2} T_{XZ} = (X_{11} + X_{21}) - (X_{22} + X_{12})$$
$$\binom{m}{2} T_{YZ} = (X_{11} + X_{12}) - (X_{22} + X_{21})$$

Since $\binom{m}{2} = X_{11} + X_{12} + X_{12} + X_{22} = n$, we have

$$1 - T_{XZ}^2 = \frac{4(X_{11} + X_{21})(X_{12} + X_{22})}{n^2} = \frac{4X_{.1}X_{.2}}{n^2}$$
$$1 - T_{YZ}^2 = \frac{4(X_{11} + X_{12})(X_{22} + X_{21})}{n^2} = \frac{4X_{1.}X_{2.}}{n^2}$$

and

$$n^2 T_{XY} = [(X_{11} + X_{22}) - (X_{12} + X_{21})][(X_{11} + X_{22}) + (X_{12} + X_{21})]$$
$$n^2(T_{XY} - T_{XZ}T_{YZ}) = 4(X_{11}X_{22} - X_{12}X_{21})$$

Therefore (12.6.1) can be written as

$$T_{XY.Z} = \frac{T_{XY} - T_{XZ}T_{YZ}}{\left[\left(1 - T_{XZ}^2\right)\left(1 - T_{YZ}^2\right)\right]^{1/2}} \tag{12.6.2}$$

Some other approaches to defining a measure of partial correlation have appeared in the journal literature. One of the more useful measures is the index of matched correlation proposed by Quade (1967).

The partial tau coefficient defined here has a particularly appealing property in that it can be generalized to the case of more than three variables. Note that the form in (12.6.2), with each T replaced by its corresponding R, is identical to the expression for a Pearson product–moment partial correlation coefficient in classical statistics. Both are special cases of a generalized partial correlation coefficient which is discussed in Somers (1959). With his generalized form, extensions of the partial tau coefficient to higher orders are possible.

12.6.1 Applications

The null hypothesis to be tested using $T_{XY.Z}$ in (12.6.2) is that X and Y are independent when the effect of Z is removed. The appropriate rejection regions are large values of $T_{XY.Z}$ for the alternative of positive dependence and small values for the alternative of negative dependence. The null distribution of $T_{XY.Z}$ is given in Table P as a function of the number of rankings m.

Example 12.6.1

Maghsoodloo (1975) used an example with $m = 7$ sets of rankings on three variables. The data are given in Table 12.6.2, arranged so that the ranking of the Z variable follows the natural order.

Compute the partial correlation between X and Y given Z and test for positive dependence. Compare the result with that for X and Y when the effect of Z is not removed.

Solution

We compute Kendall's tau coefficient between each set of pairs (X, Y), (X, Z), and (Y, Z) in the usual way. For X and Z, the number of concordant pairs is $C = 3$ and the number of discordant pairs is $Q = 18$, giving $T_{XZ} = -0.7143$. For Y and Z, $C = 1$ and $Q = 20$ with $T_{YZ} = -0.9048$. For X and Y, $C = 19$ and $Q = 2$ with $T_{XY} = 0.8095$ and $P = 0.005$ from Table L with $m = 7$, so there is a positive association between X and Y.

TABLE 12.6.2

Data for Example 12.6.1

	Subject						
Variable	***B***	***D***	***C***	***A***	***E***	***G***	***F***
Z	1	2	3	4	5	6	7
X	6	7	5	3	4	1	2
Y	7	6	5	3	4	2	1

Now we compute the partial tau from (12.6.2) as

$$T_{XY.Z} = \frac{0.8095 - (-0.7143)(-0.9048)}{\sqrt{[1 - (-0.7143)^2][1 - (-0.9048)^2]}} = 0.548$$

From Table P, the one-tailed P value is between 0.025 and 0.05. The positive association previously observed between X and Y is much weaker when the effect of Z is removed.

12.7 Summary

In this chapter we have covered a number of different descriptive and inferential procedures involving measures of association in multiple classifications. First, in Section 12.2 we presented Friedman's test for equal treatment effects in the two-way analysis-of-variance for the completely randomized design with k blocks and n treatments. This design is frequently called the repeated measures design in the behavioral and social sciences literature. If the null hypothesis of equal treatment effects is rejected, we have a multiple comparisons procedure to determine which pairs of treatments differ and in which direction, with one overall level of significance. If the alternative states an a priori order for the treatment effects, we can use Page's test for ordered alternatives, covered in Section 12.3. If the randomized block design is incomplete in a balanced way, so that all treatments are not observed in each block but the presentation is balanced, we can use the Durbin test covered in Section 12.5. A multiple comparisons test is also available to compare the treatments in this design.

Measures of association for k sets of rankings of n objects are covered in Section 12.4. The descriptive measure is Kendall's coefficient of concordance, which ranges between 0 and 1, with increasing values reflecting increasing agreement among the k rankings. When there is a significant agreement among the rankings, we can estimate the overall ''agreed upon'' preference in accordance with the sample rank totals for the n objects. This is the least-squares estimate. We found a linear relationship between this coefficient of concordance and the average of the Spearman rank correlation coefficients that could have been calculated for all of the $k(k-1)/2$ pairs of rankings. This situation is extended to the case of k sets of incomplete rankings of n objects in Section 12.5. Then we covered partial correlation for three rankings of n objects in Section 12.6. Here the Kendall coefficient of partial correlation measures the association between two variables when the effect of a third variable has been removed or ''averaged out.'' This descriptive measure ranges between -1 and $+1$, with increasing absolute values reflecting a greater degree of association or dependence between variables.

Problems

12.1 Four varieties of soybean are each planted in three blocks. The yields are:

	Variety of Soybean			
Block	***A***	***B***	***C***	***D***
1	45	48	43	41
2	49	45	42	39
3	38	39	35	36

Use Friedman's analysis of variance by ranks to test the null hypothesis that the four varieties of soybean all have the same effect on yield.

12.2 A beauty contest has eight contestants. The three judges are each asked to rank the contestants in a preferential order of pulchritude. The results are as follows:

	Contestant							
Judge	***A***	***B***	***C***	***D***	***E***	***F***	***G***	***H***
1	2	1	3	5	4	8	7	6
2	1	2	4	5	7	6	8	3
3	3	2	1	4	5	8	7	6

(a) Calculate Kendall's coefficient of concordance between rankings.
(b) Calculate the coefficient of rank correlation for each of the three pairs of rankings and verify the relation between r_{av} and W given in (12.4.7).
(c) Estimate the true preferential order of pulchritude.

12.3 Derive by enumeration the exact null distribution of W for three sets of rankings of two objects.

12.4 Given the following triplets of rankings of six objects:

X	1	3	5	6	4	2
Y	1	2	6	4	3	5
Z	2	1	5	4	6	3

(a) Calculate the Kendall coefficient of partial correlation between X and Y from (12.6.1) and test for independence.
(b) Calculate (12.6.2) for these same data to verify that it is an equivalent expression.

12.5 Howard et al. (1986) (see Problems 5.12 and 8.8) also wanted to determine whether there is a direct relationship between computer anxiety and math anxiety. Even though the two subjects involve somewhat different skills (clear, logical, and serial thinking versus quantitative talent), both kinds of anxiety are frequently present in persons who regard themselves as technologically alienated. The pretest scores are shown below for 14 students, with larger scores indicating greater amounts of the trait.

Student	Math Anxiety	Computer Anxiety	Technological Alienation
A	20	22	18
B	21	24	20
C	23	23	19
D	26	28	25
E	32	34	36
F	27	30	28
G	38	38	42
H	34	36	40
I	28	29	28
J	20	21	23
K	29	32	32
L	22	25	24
M	30	31	37
N	25	27	25

(a) Determine the relationship between computer anxiety and math anxiety.
(b) Determine the relationship when the effect of technological alienation is removed.

12.6 Webber (1990) reported results of a study to measure optimism and cynicism about the business environment and ethical trends. Subjects, ranging from high school students to executives, were asked to respond to a questionnaire with general statements about ethics. Two questions related to subjects' degree of agreement (5-point scale) with general statements about ethics. Three questions related to how others would behave in specific problematic situations and answers were

multiple choice. Three more questions, also multiple choice, related to how subjects themselves would react to the same problematic situations. These answers were used to develop an optimism index, where larger numbers indicate an optimistic feeling about current and future ethical conditions, and a cynicism index that measures how subjects felt others would behave relative to the way they themselves would behave (a cynicism index of 2.0, for example, means subjects judged others twice as likely as themselves to engage in unethical behavior). The author claimed an inverse relationship between optimism and cynicism but also noted a relation to organizational status of respondents as measured by age. Use the data below to determine whether the relationship between optimism and cynicism is still present when the effect of age is removed.

Group	Mean Age	Optimism Index	Cynicism Index
Owners/managers	60+	55	1.1
Corporate executives	44	59	1.4
Middle mangers	34	41	1.4
MBA managers	25	30	1.8
Undergraduates	20	23	2.2

12.7 Eight students are given examinations on each of verbal reasoning, quantitative reasoning, and logic. The scores range from 0 to 100, with 100 a perfect score. Use the data below of find the Kendall partial tau coefficient between quantitative and logic when the effect of verbal is removed. Find the P value. Compare the result to the P value for quantitative and logic alone and interpret the comparison.

	Student							
Score	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Verbal	90	60	45	48	58	72	25	85
Quantitative	62	81	92	76	70	75	95	72
Logic	60	91	85	81	90	76	93	80

12.8 *Automobile Magazine* publishes results of a comparison test of 15 brand models of comparably priced sedans. Each car is given a subjective score out of possible 60 points (60 = best) on each of 10 characteristics that include factors of appearance, comfort, and performance. The scores of the six best models are shown below. Determine whether

the median scores are the same. Which model(s) would you buy, on the basis of this report?

	Model					
Factor	**Nissan Maxima SE**	**Acura Legend**	**Toyota Avalon**	**Mitsubishi Galant GS**	**Peugeot 405Mi16**	**Ford Taurus**
Exterior styling	55	40	38	32	43	41
Interior comfort	50	50	47	46	41	40
Fit and finish	51	54	53	49	39	42
Engine	53	51	53	48	38	58
Transmission	60	54	47	45	44	38
Steering	43	42	45	45	57	45
Handling	45	46	43	49	54	45
Quality of ride	48	48	50	43	44	41
Fun to drive	51	46	40	49	53	51
Value for money	53	51	47	53	42	47
Total	509	482	463	459	455	448

12.9 A manufacturer of ice cream carried out a taste preference test on seven varieties of ice cream, denoted by A, B, C, D, E, F, G. The subjects were a random sample of 21 tasters and each taster had to compare only three varieties. Each pair of varieties is presented together three times, to a subset of seven of the tasters, with the design shown below repeated each of the three times.

Taster	**Varieties Presented**		
1	*A*	*B*	*D*
2	*B*	*C*	*E*
3	*C*	*D*	*F*
4	*D*	*E*	*G*
5	*E*	*F*	*A*
6	*F*	*G*	*B*
7	*G*	*A*	*C*

The ranks resulting from the three repetitions are shown below with each rank corresponding to the variety indicated above. For example,

the rank 3 by taster 12 is for variety *E*. Determine whether there is a positive association between the rankings.

Taster	Ranks			Taster	Ranks			Taster	Ranks		
1	2	1	3	8	2	1	3	15	2	1	3
2	1	2	3	9	2	1	3	16	1	2	3
3	2	3	1	10	1	3	2	17	3	2	1
4	3	2	1	11	3	2	1	18	2	1	3
5	3	1	2	13	2	3	1	20	2	1	3
7	1	3	2	14	1	3	2	21	2	1	3

12.10 Ten graduate students take identical comprehensive examinations in their major field. The grading procedure is that each professor ranks each student's paper in relation to all others taking the examination. Suppose that four professors give the following ranks, where 1 indicates the best paper and 10 the worst.

	Student									
Professor	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	5	3	8	9	2	7	6	1	4	10
2	7	4	6	2	3	9	8	5	1	10
3	3	5	7	6	4	10	8	2	1	9
4	4	5	7	8	3	9	6	1	2	10

(a) Is there evidence of agreement among the four professors?
(b) Give an overall estimate of the relative performance of each student.
(c) Will it be difficult to decide which students should receive a passing grade?

12.11 Show that if $m = n$ and $\lambda = k$ in (12.5.4) so that the design is complete, then (12.5.4) is equivalent to $Q = 12S/kn(n+1)$, as it should be from (12.2.8).

12.12 A town has 10 different supermarkets. For each market, data are available on the following three variables: $X_1 =$ food sales, $X_2 =$ nonfood sales, and $X_3 =$ size of store in thousands of square feet. Calculate the partial tau coefficient for X_1 and X_2, when the effects of X_3 are eliminated.

Store No.	Size of Store (1000 ft^2)	Food Sales ($10,000)	Nonfood Sales ($10,000)
1	35	305	35
2	22	130	98
3	27	189	83
4	16	175	76
5	28	101	93
6	46	269	77
7	56	421	44
8	12	195	57
9	40	282	31
10	32	203	92

12.13 Suppose in Problem 11.15 that an independent group of female consumers also ranked the products as follows:

Product	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Independent female ranks	8	9	5	6	1	2	7	4	40	3

(a) Is there agreement between the three sets of rankings? Give a descriptive measure of agreement and find a P value.
(b) Use all the data given to estimate the rank ordering of the products. In what sense is this estimate a good one?

12.14 An experimenter is attempting to evaluate the relative effectiveness of four drugs in reducing the pain and trauma of persons suffering from migraine headaches. Seven patients are given each drug for a month at a time. At the end of each month, each patient gave an estimate of the relative degree of pain suffered from migraines during that month on a scale from 0 to 10, with 10 denoting the most severe pain. Test the null hypothesis that the drugs are equally effective.

Drug	1	2	3	4	5	6	7
A	7	10	7	9	8	8	8
B	7	6	5	8	7	5	7
C	3	7	3	5	4	6	3
D	4	3	2	1	0	1	0

12.15 A matching-to-sample (MTS) task is used by psychologists to understand how other species perceive and use identity relations. A standard MTS task consists of having subjects observe a sample stimulus and then rewarding the subject if it responds to an identical (matching) sample stimulus. Then the psychologist studies the ability

of subjects to transfer the matching concept to other sample stimuli. Oden et al. (1988) reported a study in which four infant chimpanzees learned an MTS task with only two training sample stimuli. Then the chimpanzees were tested on their ability to transfer the learning to three kinds of novel items, classified as Objects, Fabrics, and Food. The data were recorded as number of correct matches in a total of 24 trials. One purpose of the study was to show that the concept of matching is broadly construed by chimpanzees irrespective of the type of sample stimulus. Determine whether the data support this theory.

Chimp	**Training**	**Object**	**Fabric**	**Food**
Whiskey	20	22	22	18
Liza	23	19	22	13
Opal	18	20	18	15
Frieda	21	21	19	19

13

Asymptotic Relative Efficiency

13.1 Introduction

The concept of Pitman efficiency was defined in Chapter 1 as a criterion for comparing two test statistics. Many of the nonparametric tests covered in this book can be considered direct analogs of some classical tests which are known to be most powerful under certain specific distribution assumptions. The asymptotic efficiencies of the nonparametric tests relative to a ''best'' test have been stated here without discussion. In this chapter, we will investigate the concept of efficiency more thoroughly and prove some theorems that simplify the calculation. The theory will then be illustrated by obtaining numerical values of the asymptotic relative efficiency (ARE) for some particular distributions. The theory presented here is generally attributed to Pitman. Noether (1955) gives important generalizations of the theory.

Suppose that we have two test statistics, T and T^*, which can be used for similar types of inferences regarding simple hypotheses. The power efficiency of test T relative to test T^* is defined as the ratio n^*/n, where n^* is the sample size necessary to attain the power γ at significance level α when test T^* is used, and n is the sample size required by test T to attain the same values γ and α.

As a simple numerical example, consider a comparison of the normal-theory test T^* and the ordinary sign test T for the respective hypothesis-testing situations:

$$H_0: \mu = 0 \quad \text{versus} \quad H_1: \mu = 1$$

and

$$H_0: M = 0 \quad \text{versus} \quad H_1: M = 1$$

The inference is to be based on a single random sample from a population which is assumed to be normally distributed with known variance equal to 1. Then the hypothesis sets above are identical. Suppose we are interested in the

relative sample sizes for a power of 0.90 and a significance level of 0.05. For the most powerful (normal-theory) test based on n^* observations, the null hypothesis is rejected when $\sqrt{n^*}\,\overline{X} \geq 1.64$ for $\alpha = 0.05$. Setting the power γ equal to 0.90, n^* is found as follows:

$$\begin{aligned} Pw(1) &= \gamma(1) \\ &= P(\sqrt{n^*}\overline{X} \geq 1.64 | \mu = 1) \\ &= P[\sqrt{n^*}(\overline{X} - 1) \geq 1.64 - \sqrt{n^*}] = 0.90 \\ \Phi(1.64 - \sqrt{n^*}) &= 0.10 \quad 1.64 - \sqrt{n^*} = -1.28 \quad n^* = 9 \end{aligned}$$

The sign test T of Section 5.4 has rejection region $K \geq k_\alpha$, where K is the number of positive observation X_i, and k_α is chosen so that

$$\sum_{k=k_\alpha}^{n} \binom{n}{k} 0.5^n = \alpha \tag{13.1.1}$$

The power of the test T then is

$$\sum_{k=k_\alpha}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k} = \gamma(1) \tag{13.1.2}$$

where $\theta = P(X > 0 | M = 1) = 1 - \Phi(-1) = 0.8413$, since the mean and median coincide for a normal population. The number n and $k_{0.05}$ will be those values of n and k_α, respectively, which simultaneously satisfy (13.1.1) and (13.1.2) when $\alpha = 0.05$ and $\gamma = 0.90$. If θ is rounded to 0.85, ordinary tables of the binominal distribution can be used instead of actual calculations. Some of the steps relevant to finding the simultaneous solution are shown in Table 13.1.1.

If we do not want to resort to using a randomized test, we can either (1) choose values for n and k_α such that α and γ are both as close as possible to the preselected numbers or (2) choose the smallest value of n such that the smallest value of k_α gives α and $\beta = 1 - \gamma$ no larger than the preselected numbers. We obtain $n = 13$ and $k_{0.05} = 10$ using method 1 and $n = 16$, $k_{0.05} = 12$ with method, 2. These methods are undesirable for a number of obvious reasons, but mainly because method 1 may not lead to a unique answer and method 2 may be too conservation with respect to both types of errors. A preferable approach for the purpose of comparison would be to use randomized decision rules. Then we can either make exact $\alpha = 0.05$ or exact $\gamma = 0.90$ but probably not both. When deciding to make exact $\alpha = 0.05$, the procedure is illustrated in Table 13.1.1. Starting with the smallest n and the corresponding smallest k_α for which simultaneously $\alpha \leq 0.05$ and

TABLE 13.1.1

Power Calculations

				Randomized Decision Rule for Exact $\alpha = 0.05$	
N	k_α	α	$\gamma = 1 - \beta$	Probability of Rejection	$\gamma(1)$
17	13	0.0245	0.9013		
	12	0.0717	0.9681		
16	12	0.0383	0.9211	1	
	11	0.1050	0.9766	0.1754	0.9308
15	12	0.0176	0.8226	1	
	11	0.0593	0.9382	0.8010	0.9151
14	11	0.0288	0.8535	1	
					0.8881
	10	0.0899	0.9533	0.3470	
13	10	0.0461	0.8820		
	9	0.1334	0.9650		

$\gamma \geq 0.90$ the randomized decision rule is found by solving for p in the expression

$$\sum_{k=k_\alpha}^{n} \binom{n}{k} (0.5)^n + p \binom{n}{k_\alpha - 1} (0.5)^n = 0.05$$

Then the power for this exact 0.05 size test is

$$\sum_{k=k_\alpha}^{n} \binom{n}{k} (0.85)^k (0.85)^{n-k} + p \binom{n}{k_\alpha - 1} (0.85)^k (0.15)^{n-k}$$

Do the same set of calculations for the next smaller n, etc., until $\gamma \geq 0.90$. The selected values of n may either be such that $\gamma \geq 0.90$ or γ is as close as possible to 0.90, as before, but at least here the choice for n is always between two consecutive numbers. From Table 13.1.1 the answers in these two cases are $n = 15$ and $n = 14$, respectively.

In this example, the normal test requires only nine observations to be as powerful as a sign test using 14 or 15, so that the power efficiency is around 0.60 or 0.64. This result applies only for the particular numbers α and β (or γ) selected and therefore is not in any sense a general comparison even though both the null and alternative hypotheses are simple.

Since fixing the value for α is a well-accepted procedure, we might perform calculations similar to those above for some additional and arbitrarily selected values of γ and plot the coordinates (γ, n) and (γ, n^*) on the same graph. From these points, the curves $n(\gamma)$ and $n^*(\gamma)$ can be approximated. The numerical processes can be easily programmed for computer calculation. Some evaluation of general relative performance of two tests can therefore be made for the

particular value of α selected. However, this power-efficiency approach is satisfactory only for a simple alternative hypothesis. Especially in the case of nonparametric tests, the alternative of interest is usually composite. In the above example, if the alternative were $H_1: \mu > 0$ ($M > 0$), curves for the functions $n[\gamma(\mu)]$ and $n^*[\gamma(\mu)]$ would have to be compared for all $\mu > 0$ and a preselected α. General conclusions for any α and γ are certainly difficult if not impossible. As a result, we usually make comparisons of the power only for μ in a specified neighborhood of the null hypothesis.

In many important cases, the limit of the ratio n^*/n turns out not to be a function of α and γ, or even the parameter value when it is in the neighborhood of the hypothesized value. Therefore, even though it is a large-sample property for a limiting type of alternative, the ARE of two tests is a satisfactory criterion for comparison in the sense that it leads to a single number and consequently a well-defined conclusion for large sample sizes. It is for this reason that the discussion here will be limited to comparisons of tests using this standard.

13.2 Theoretical Bases for Calculating the ARE

Suppose that we have two statistics T_n and T_n^*, for data consisting of n observations, and both statistics are consistent for a test of

$$H_0: \theta \in \omega \quad \text{versus} \quad H_1: \theta \in \Omega - \omega$$

In other words, for all $\theta \in \Omega - \omega$

$$\lim_{n\to\infty} Pw[T_n(\theta)] = 1 \quad \text{and} \quad \lim_{n\to\infty} Pw[T_n^*(\theta)] = 1$$

Suppose further that a subset of the space Ω can be indexed in terms of a sequence of parameters $(\theta_0, \theta_1, \theta_2, \ldots, \theta_n, \ldots)$ such that θ_0 specifies a value in ω and the remaining $\theta_1, \theta_2, \ldots$ are in $\Omega - \omega$ and that $\lim_{n\to\infty} \theta_n = \theta_0$. For example, in the case of a one-sided alternative $\theta > \theta_0$, we take a monotonic decreasing sequence of numbers $\theta_1, \theta_2, \ldots$, which converges to θ_0 from above. If each θ_i specifies a probability distribution for the test statistics, we might say that the alternative distribution is getting closer and closer to the null distribution as $n \to \infty$. Under these conditions, a formal definition of the ARE of T relative to T^* can be given.

Definition 13.2.1 *Let $Pw_n(\theta)$ and $Pw_n^*(\theta)$ be the power functions of two tests T and T* (corresponding to the test statistics T_n and T_n^*, respectively), against a family of alternatives labeled by θ, and let θ_0 be the value of θ specified by the null hypothesis.*

Also let T and T have the same level of significance* α. *Consider a sequence of alternatives* $\{\theta_n\}$ *and a sequence* $\{n^*\} = \{h(n)\}$ *of positive integers, where h is some suitable function, such that*

$$\lim_{n\to\infty} Pw_n(\theta_n) = \lim_{n\to\infty} Pw_n^*(\theta_n)$$

where it is assumed that the two limits exist and are not equal to either 0 *or* 1. *Then the ARE of test T relative to test T* is*

$$\text{ARE}(T, T^*) = \lim_{n\to\infty} \frac{n^*}{n}$$

provided that the limit exists and is independent of the sequences $\{\theta_n\}, \{n\}$ *and* $\{n^*\}$.

In other words, the ARE is the inverse ratio of the sample sizes necessary to obtain any power γ for the tests T and T^*, respectively, while simultaneously the sample sizes approach infinity and the sequences of alternatives approach θ_0, and both tests have the same significance level. It is thus a measure of asymptotic and localized power efficiency. In the case of the more general tests of hypotheses concerning distributions like $F = F_\theta$, the same definition holds.

Now suppose that our consistent size α tests T_n and T_n^* are for the one-sided alternative

$$H_0: \theta = \theta_0 \quad \text{versus} \quad H_1: \theta > \theta_0$$

and have respective rejection regions of the form

$$T_n \in R \text{ for } T_n \geq t_{n,\alpha} \quad \text{and} \quad T_n^* \in R^* \text{ for } T_n^* \geq t_{n,\alpha}^*$$

where $t_{n,\alpha}$ and $t_{n,\alpha}^*$ are chosen such that

$$P(T_n \geq t_{n,\alpha} | \theta = \theta_0) = \alpha \quad \text{and} \quad P(T_n^* \geq t_{n,\alpha}^* | \theta = \theta_0) = \alpha$$

The following regularity conditions for the test T_n, and analogous ones for T_n^*, must be satisfied.

1. $dE(T_n)/d\theta$ exists and is positive and continuous at θ_0. All other higher-order derivatives, $d^r E(T_n)/d\theta^r, r = 2, 3, \ldots$, are equal to zero at θ_0.
2. There exists a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{dE(T_n)/d\theta|_{\theta=\theta_0}}{\sqrt{n}\sigma(T_n)|_{\theta=\theta_0}} = c$$

3. There exists a sequence of alternatives $\{\theta_n\}$ such that for some constant $d > 0$, we have

$$\theta_n = \theta_0 + \frac{d}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{dE(T_n)/d\theta|_{\theta=\theta_n}}{dE(T_n)/d\theta|_{\theta=\theta_0}} = 1$$

$$\lim_{n\to\infty} \frac{\sigma(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_0}} = 1$$

4. $\lim_{n\to\infty} P\left[\frac{T_n - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_n}} \le z|\theta = \theta_n\right] = \Phi(z)$

5. $\lim_{n\to\infty} P[T_n \ge t_{n,\alpha}|\theta = \theta_0] = \alpha \quad 0 < \alpha < 1$

THEOREM 13.2.1

Under the five regularity conditions above, the limiting power of the test T_n is

$$\lim_{n\to\infty} Pw(T_n|\theta = \theta_n) = 1 - \Phi(z_\alpha - dc)$$

where z_α *satisfies* $1 - \Phi(z_\alpha) = \alpha$.

Proof

The limiting power is

$$\lim_{n\to\infty} P(T_n \ge t_{n,\alpha}|\theta = \theta_n) = \lim_{n\to\infty} P\left[\frac{T_n - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_n}} \ge \frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_n}}\right]$$

$$= 1 - \Phi(z) \text{ from regularity condition 4}$$

where

$$z = \lim_{n\to\infty} \frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_n}}$$

$$= \lim_{n\to\infty} \left[\frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_0}} \frac{\sigma(T_n)|_{\theta=\theta_0}}{\sigma(T_n)|_{\theta=\theta_n}}\right]$$

$$= \lim_{n\to\infty} \left[\frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_n}}{\sigma(T_n)|_{\theta=\theta_0}}\right] \text{ from regularity condition 3}$$

Expanding $E(T_n)|_{\theta=\theta_n}$ in a Taylor's series about θ_0 and using regularity condition 1, we obtain

$$E(T_n)|_{\theta=\theta_n} = E(T_n)|_{\theta=\theta_0} + (\theta_n - \theta_0)\frac{dE(T_n)}{d\theta}\bigg|_{\theta=\theta_0^*} \qquad \theta_0 < \theta_0^* < \theta_n$$

Substituting this in the above expression for z, we obtain

$$\begin{aligned} z &= \lim_{n\to\infty} \left\{ \frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_0}}{\sigma(T_n)|_{\theta=\theta_0}} - \frac{(\theta_n - \theta_0)[dE(T_n)/d\theta]\Big|_{\theta=\theta_0^*}}{\sigma(T_n)|_{\theta=\theta_0}} \right\} \\ &= \lim_{n\to\infty} \left[\frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_0}}{\sigma(T_n)|_{\theta=\theta_0}} \right] - dc \end{aligned}$$

using regularity conditions 1, 2, and 3

$$z = z_\alpha - dc$$

Using regularity conditions 5 and 4, we have

$$\begin{aligned} \alpha &= \lim_{n\to\infty} P(T_n \geq t_{n,\alpha} | \theta = \theta_0) \\ &= \lim_{n\to\infty} P\left[\frac{T_n - E(T_n)|_{\theta=\theta_0}}{\sigma(T_n)|_{\theta=\theta_0}} \geq \frac{t_{n,\alpha} - E(T_n)|_{\theta=\theta_0}}{\sigma(T_n)|_{\theta=\theta_0}} \right] \\ &= 1 - \Phi(z_\alpha) \end{aligned}$$

This completes the proof.

THEOREM 13.2.2

If T and T^* are two tests satisfying the regularity conditions above, the ARE *of T relative to T^* is*

$$\text{ARE}(T, T^*) = \lim_{n\to\infty} \left[\frac{dE(T_n)/d\theta|_{\theta=\theta_0}}{dE(T_n^*)/d\theta|_{\theta=\theta_0}} \right]^2 \frac{\sigma^2(T_n^*)|_{\theta=\theta_0}}{\sigma^2(T_n)|_{\theta=\theta_0}} \tag{13.2.1}$$

Proof

From Theorem 13.2.1, the limiting powers of tests T and T^*, respectively, are

$$1 - \Phi(z_\alpha - dc) \quad \text{and} \quad 1 - \Phi(z_\alpha - d^*c^*)$$

These quantities are equal if

$$z_\alpha - dc = z_\alpha - d^*c^*$$

or, equivalently, if

$$\frac{d^*}{d} = \frac{c}{c^*}$$

From regularity condition 3, the sequences of alternatives are the same if

$$\theta_n = \theta_0 + \frac{d}{\sqrt{n}} = \theta_n^* = \theta_0 + \frac{d^*}{\sqrt{n^*}}$$

or, equivalently, if

$$\frac{d}{\sqrt{n}} = \frac{d^*}{\sqrt{n^*}} \quad \text{or} \quad \frac{d^*}{d} = \left(\frac{n^*}{n}\right)^{1/2}$$

Since the ARE is the limit of the ratio of sample sizes when the limiting power and sequence of alternatives are the same for both tests, we have

$$\begin{aligned}
\text{ARE}(T, T^*) &= \frac{n^*}{n} = \left(\frac{d^*}{d}\right)^2 = \left(\frac{c}{c^*}\right)^2 \\
&= \lim_{n\to\infty} \left[\frac{dE(T_n)/d\theta|_{\theta=\theta_0}}{\sqrt{n}\sigma(T_n)|_{\theta=\theta_0}} \frac{\sqrt{n}\sigma(T_n^*)|_{\theta=\theta_0}}{dE(T_n^*)/d\theta|_{\theta=\theta_0}}\right]^2 \\
&= \lim_{n\to\infty} \frac{[dE(T_n)/d\theta]^2/\sigma^2(T_n)\big|_{\theta=\theta_0}}{[dE(T_n^*)/d\theta]^2/\sigma^2(T_n^*)\big|_{\theta=\theta_0}} \qquad (13.2.2)
\end{aligned}$$

which is equivalent to (13.2.1). This completes the proof.

From expression (13.2.2), we see that when these regularity conditions are satisfied, the ARE can be interpreted to be the limit as $n \to \infty$ of the ratio of two quantities:

$$\text{ARE}(T, T^*) = \lim_{n\to\infty} \frac{e(T_n)}{e(T_n^*)} \qquad (13.2.3)$$

where $e(T_n)$ is called the *efficacy* of the statistic T_n when used to test the null hypothesis $\theta = \theta_0$ and

$$e(T_n) = \frac{[dE(T_n)/d\theta]^2|_{\theta=\theta_0}}{\sigma^2(T_n)|_{\theta=\theta_0}} \qquad (13.2.4)$$

THEOREM 13.2.3

Theorem 13.2.2 remains valid as stated if both tests are for a two-sided alternative, $H_1{:}\theta \neq \theta_0$*, with rejection region*

$$T_n \in R \quad \text{for } T_n \geq t_{n,\alpha_1} \text{ or } T_n \leq t_{n,\alpha_2}$$

where the size of the test is still α*, and a corresponding rejection region is defined for* T_n^* *with the same* α_1 *and* α_2*.*

Note that the result for the ARE in Theorem 13.2.2 is independent of both the quantities α and γ. Therefore, when the regularity conditions are satisfied, the ARE does not suffer the disadvantages of the power-efficiency criterion. However, it is only an approximation to relative efficiency for any finite sample size and/or alternative not in the neighborhood of the null case.

In the two-sample case, the same theorems can be used for either one- or two-sided tests where the null hypothesis is equal distributions, if the hypothesis can be parameterized in terms of θ. The limiting process must be restricted by assuming that as $m, n \to \infty$, the ratio $m/n \to \lambda$, a constant. When m is approximately a fixed proportion of n regardless of the total sample size $m+n$, the theory goes through as before as $n \to \infty$. For two-sample linear rank tests, evaluation of the efficacies is simplified by using the general results for mean and variance given in Theorem 7.3.8.

In various k-sample problems where the null and the alternative hypotheses involve more than one parameter, the result of Theorem 13.2.2 cannot be used directly to calculate the ARE of one test relative to another. However, the general approach in Theorems 13.2.1 and 13.2.2 can be used to derive the ARE in such cases. It may be noted that the theory of ARE remains applicable in principle as long as the two competing test statistics have the same form of asymptotic distributions, not necessarily the normal. In this regard, it can be shown that when the asymptotic distributions of the test statistics are chi-square distributions, the ARE is equal to the ratio of the noncentrality parameters. For details about these and related interesting results, see, for example, Andrews (1954), Puri (1964), Puri and Sen (1971), and Chakraborti and Desu (1991).

13.3 Examples of the Calculation of Efficacy and ARE

We now give some examples of the calculation of efficacy and ARE. In each case, the appropriate regularity conditions are satisfied; verification of this is left as an exercise for the reader.

13.3.1 One-Sample and Paired-Sample Problems

In the one-sample and paired-sample problems treated in Chapter 5, the null hypothesis concerned the value of the population median or median of the population of differences of pairs. This is called a one-sample location problem with the distribution model

$$F_X(x) = F(x - \theta) \tag{13.3.1}$$

for some continuous distribution F with median zero. Since F_X then has median θ, the model implies the null hypothesis

$$H_0: \theta = 0$$

against one- or two-sided alternatives.

The nonparametric tests for this model can be considered analogs of the one-sample or paired-sample Student's t test for location of the mean or difference of means if F is any continuous distribution symmetric about zero since then θ is both the mean and the median of F_X. For a single random sample of size N from any continuous population F_X with mean μ and variance σ^2, the t test statistic of the null hypothesis

$$H_0: \mu = 0$$

is

$$T_N^* = \frac{\sqrt{N}\,\overline{X}_N}{S_N} = \left[\frac{\sqrt{N}(\overline{X}_N - \mu)}{\sigma} + \frac{\sqrt{N}\mu}{\sigma}\right]\frac{\sigma}{S_N}$$

where $S_N^2 = \sum_i^N (X_i - \overline{X})^2/(N-1)$. Since $\lim_{N\to\infty}(S_N/\sigma) = 1$, T_N^* is asymptotically equivalent to the Z (normal-theory) test for σ known. The moments of T_N^*, for large N, then are

$$E(T_N^*) = \frac{\sqrt{N}\mu}{\sigma} \quad \text{and} \quad \text{var}(T_N^*) = \frac{N\ \text{var}(\overline{X}_N)}{\sigma^2} = 1$$

and

$$\frac{d}{d\mu}E(T_N^*)|_{\mu=0} = \frac{\sqrt{N}}{\sigma}$$

Using (13.2.4), the efficacy of Student's t test for observations from any continuous population F_X with mean μ and variance σ^2 is

$$e(T_N^*) = \frac{N}{\sigma^2} \tag{13.3.2}$$

The ordinary sign-test statistic K_N of Section 5.4 is appropriate for the model (13.3.1) with

$$H_0: M = \theta = 0$$

Since K_N follows the binominal distribution, its mean and variance are

$$E(K_N) = Np \quad \text{and} \quad \text{var}(K_N) = Np(1-p)$$

where

$$p = P(X > 0) = 1 - F_X(0) = 1 - F(-\theta)$$

If θ is the median of the population $F_X, F_X(0)$ is a function of θ, and for the location model (13.3.1) we have

$$\begin{aligned}\left.\frac{dp}{d\theta}\right|_{\theta=0} &= \left.\frac{d}{d\theta}[1 - F(-\theta)]\right|_{\theta=0} \\ &= f(-\theta)|_{\theta=0} \\ &= f(0) = f_X(\theta)\end{aligned}$$

When $\theta = 0, p = 0.5$, so that $\text{var}(K_N)|_{\theta=0} = N/4$. The efficacy of the ordinary sign test for N observations from any population F_X with median θ is therefore

$$e(K_N) = 4Nf_X^2(\theta) = 4Nf^2[F^{-1}(0.5)] \tag{13.3.3}$$

We now calculate the efficacy of the Wilcoxon signed-ranked test described in Section 5.7. Let $X_1, X_2, \ldots, X_N$ be a random sample from a continuous cdf $F_X(x) = F(x - M)$, where F is symmetrically distributed about 0. Thus the X_i's are symmetrically distributed about the median M. The Wilcoxon signed-rank test based on T_N^+ is appropriate to test the null hypothesis $H_0: M = M_0$ where M_0 is specified. In order to find the efficacy, it will be more convenient to work with $V_N^+ = T_N^+ / \binom{N}{2}$. It is clear that a test based on T_N^+ is equivalent to a test based on V_N^+ and hence the two tests have the same efficacy. The mean of V_N^+ is obtained from (5.7.5) as

$$\begin{aligned}E(V_N^+) &= \left(\frac{2}{N-1}\right) P(D_i > 0) + P(D_i + D_j > 0) \\ &= \left(\frac{2}{N-1}\right)[1 - F(-M)] + \int_{-\infty}^{\infty} [1 - F(-x - M)]dF(x - M) \\ &= \left(\frac{2}{N-1}\right)[1 - F(-M)] + \int_{-\infty}^{\infty} [1 - F(-y - 2M)]dF(y)\end{aligned}$$

Thus, we obtain

$$\frac{dE(V_N^+)}{dM} = \left(\frac{2}{N-1}\right) f(-M) + 2\int_{-\infty}^{\infty} f(-y-2M)dF(y)$$

after interchanging the order of differentiation and integration. This can be shown to be valid if $f(x) = dF(x)/dx$ is bounded by some positive quantity. Since F is symmetric about 0, $f(y) = f(-y)$, and so

$$\frac{dE(V_N^+)}{dM}\Big|_{M=0} = 2\left[\frac{f(0)}{N-1} + I\right]$$

where

$$I = \int_{-\infty}^{\infty} f(y)dF(y) = \int_{-\infty}^{\infty} f^2(y)\,dy$$

Also, from (5.7.6) the variance of V_N^+ under H_0 is

$$\frac{(N+1)(2N+1)}{6N(N-1)^2}$$

Therefore, using (13.2.4), the efficacy of the Wilcoxon signed-rank test for N observations from a continuous population which is symmetric about θ is

$$\frac{24[f(0)/(N-1) + I]^2 N(N-1)^2}{(N+1)(2N+1)} \tag{13.3.4}$$

We can use the efficacy results to calculate the AREs between any two of these tests. For example, from (13.3.3) and (13.3.2), the ARE of the sign test relative to the Student's t test is

$$\text{ARE}(K_N, T_N^*) = 4\{f[F^{-1}(0.5)]\}\sigma^2 \tag{13.3.5}$$

The ARE of the Wilcoxon signed-rank test relative to the Student's t test is obtained from (13.3.4) and (13.3.2) along with (13.2.3) as

$$\begin{aligned}\text{ARE}(T_N^+, T_N^*) &= \lim_{N\to\infty} \frac{24[f(0)/(N-1) + I]^2 N(N-1)^2/(N+1)(2N+1)}{N/\sigma^2} \\ &= 12\sigma^2 I^2\end{aligned} \tag{13.3.6}$$

The quantity I^2 appears frequently in the ARE expressions of many well-known nonparametric tests. In practice, it may be of interest to estimate I from the sample data in order to estimate the ARE. This interesting problem has been studied by Aubuchon and Hettmansperger (1984).

For the ARE of the sign test relative to the Wilcoxon signed-rank test we obtain, using (13.3.4) and (13.3.3) and applying (13.2.3),

$$\text{ARE}(K_N, T_N^+) = \frac{\left[f\{F^{-1}(0.5)\}\right]^2}{3\left[\int_{-\infty}^{\infty} f^2(y)dy\right]^2} \tag{13.3.7}$$

We illustrate the calculations involved by computing the ARE of the sign test relative to the t test, ARE(K, T^*), for the normal, uniform, and double exponential distributions, respectively.

1. Normal distribution

$$F_X \text{ is } N(\theta, \sigma^2) \quad F_X(x) = \Phi\left(\frac{x-\theta}{\sigma}\right) \quad \text{or} \quad F(x) = \Phi\left(\frac{x}{\sigma}\right)$$
$$f(0) = (2\pi\sigma^2)^{-1/2} \quad e(K_N) = 2N/\pi\sigma^2$$
$$\text{ARE}(K_N, T_N^*) = 2/\pi$$

2. Uniform distribution

$$f_X(x) = 1 \quad \text{for } \theta - 1/2 < x < \theta + 1/2$$
$$\text{or}$$
$$f(x) = 1 \quad \text{for } -1/2 < x > 1/2$$
$$f(0) = 1 \quad \text{var}(X) = 1/12$$
$$e(T_N^*) = 12N \quad e(K_N) = 4N$$
$$\text{ARE}(K_N, T_N^*) = 1/3$$

3. Double exponential distribution

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x-\theta|} \quad \text{or} \quad f(x) = \frac{\lambda}{2} e^{-\lambda|x|}$$
$$f(0) = \lambda/2 \quad \text{var}(X) = 2/\lambda^2$$
$$e(T_N^*) = N\lambda^2/2 \quad e(K_N) = N\lambda^2$$
$$\text{ARE}(K_N, T_N^*) = 2$$

In order to facilitate comparisons among the tests, the ARE of the Wilcoxon signed-rank test relative to the t test [ARE(T_N^+, T_N^*)] and the ARE of the sign

TABLE 13.3.1

Values of ARE(T^+, T^*), ARE(K, T^*), and ARE(K, T^+) for Some Selected Probability Distributions

Distribution	ARE(T_N^+, T_N^*)	ARE(K_N, T_N^*)	ARE(K_N, T_N^+)
Uniform	1	1/3	1/3
Normal	$3/\pi = 0.955$	$2/\pi = 0.64$	2/3
Logistic	$\pi^2/9 = 1.097$	$\pi^2/12 = 0.82$	3/4
Double exponential	3/2	2	4/3

test and the Wilcoxon signed-rank test ARE[(K_N, T_N^+)] are calculated for the uniform, the normal, the logistic, and the double exponential distributions. For the same purpose, the ARE of the sign test relative to the t test is also calculated for the logistic distribution. The ARE values are presented in Table 13.3.1.; verification of these results is left for the reader.

A closer examination of the ARE values in Table 13.3.1 reveals some interesting facts. First, from the values in the first column, it is evident that the Wilcoxon signed-rank test is a strong competitor to the popular Student's t test when a large sample size is available. In particular, for the normal distribution when the t test is optimal, very little seems to be lost in terms of efficiency when the Wilcoxon signed-rank test is used instead. Moreover, for distributions with heavier tails than the normal, like the uniform, logistic, and double exponential, the signed-rank test is superior in that the ARE is greater than or equal to 1. In fact, it may be recalled that ARE (T_N^+, T_N^*) is never less than 0.864 for any continuous symmetric distribution (Hodges and Lehmann, 1956).

From the ARE values in the second column of Table 13.3.1, we see that the sign test is much less efficient than the t test for light to moderately heavy-tailed distributions. In particular, for the normal distribution the sign test is only 64% as efficient as the optimal t test. This poor performance is not entirely unexpected since the simple sign test does not use all of the sample information generally available. Interestingly, however, this may lead to its superior performance in the case of a heavy-tailed distribution such as the double exponential. Hodges and Lehmann (1956) have shown that $\text{ARE}(K_N, T_N^*) \geq 1/3$ for any continuous unimodal symmetric distribution; the lower bound is achieved for the uniform distribution.

Finally, from the third column of Table 13.3.1 we see that except for the double exponential distribution, the signed-rank test is more efficient than the sign test.

We summarize by saying that the Wilcoxon signed-rank test is a very viable alternative to the popular Student's t test. The test is appropriate under much milder assumptions about the underlying distribution and it either outperforms or comes very close in performance to the t test, in terms of ARE, for many commonly encountered distributions. The sign test is

usually less efficient; perhaps it is a popular choice because of its ease of use more than its performance.

When we have paired-sample data and the hypotheses concern the median or mean difference, all results obtained above are applicable if the random variable X is replaced by the difference variable $D = X - Y$. It should be noted that the parameter σ^2 in (13.3.2) then denotes the variance of the population of differences,

$$\sigma_D^2 = \sigma_X^2 + \sigma_Y^2 - 2\ \mathrm{cov}(X, Y)$$

and the $f_X(\theta)$ in (13.3.3) now becomes $f_D(\theta)$.

13.3.2 Two-Sample Location Problems

For the general location problem in the case of two independent random samples of sizes m and n, the distribution model is

$$F_Y(x) = F_X(x - \theta) \tag{13.3.8}$$

and the null hypothesis of identical distributions is

$$H_0: \theta = 0$$

The corresponding classical test statistic for normal populations with a common variance σ^2 is the two-sample Student's t test statistic

$$T_{m,n}^* = \sqrt{\frac{mn}{m+n}}\left(\frac{\overline{Y}_n - \overline{X}_m}{S_{m+n}}\right) = \sqrt{\frac{mn}{m+n}}\left(\frac{\overline{Y}_n - \overline{X}_m - \theta}{\sigma} + \frac{\theta}{\sigma}\right)\frac{\sigma}{S_{m+n}}$$

where

$$S_{m+n}^2 = \frac{\sum_{i=1}^m (X_i - \overline{X})^2 + \sum_{i=1}^n (Y_i - \overline{Y})^2}{m+n-2}$$

is the pooled estimate of σ^2. Since S_{m+n}/σ approaches 1 as $m, n \to \infty$, $m/n \to \lambda$, the moments of $T_{m,n}^*$ for n large, $\theta = \mu_Y - \mu_X$, are

$$E(T_{m,n}^*) = \theta\frac{\sqrt{mn/(m+n)}}{\sigma}$$

$$\mathrm{var}(T_{m,n}^*) = \frac{mn}{m+n}\,\frac{\sigma^2/m + \sigma^2/n}{\sigma^2} = 1$$

Therefore

$$\frac{d}{d\theta}E(T^*_{m,n}) = \frac{\sqrt{mn/(m+n)}}{\sigma}$$

and the efficacy of Student's t test for any continuous population is

$$e(T^*_{m,n}) = \frac{mn}{\sigma^2(m+n)} \tag{13.3.9}$$

For the Mann–Whitney test statistic given in (6.6.2), the mean is

$$E(U_{m,n}) = mnP(Y < X) = mnP(Y - X < 0) = mnp$$

A general expression for p was given in (6.6.3) for any distribution. For the location model in (13.3.4), this integral becomes

$$p = \int_{-\infty}^{\infty} F_X(x-\theta) f_X(x)dx$$

so that

$$\frac{d}{d\theta}E(U_{m,n})\bigg|_{\theta=0} = mn\frac{dp}{d\theta}\bigg|_{\theta=0} = -mn\int_{-\infty}^{\infty} f_X^2(x)dx$$

Under $H_0, p = 0.5$ and the variance is found from (6.6.15) to be

$$var(U_{m,n}) = \frac{mn(m+n+1)}{12}$$

The efficacy then is

$$e(U_{m,n}) = \frac{12mn\left[\int_{-\infty}^{\infty} f_X^2(x)dx\right]^2}{m+n+1} \tag{13.3.10}$$

Using (13.3.10) and (13.3.9) and applying (13.2.4), we find that the ARE of the Mann–Whitney test relative to the Student's t test is given by the expression for the ARE of the Wilcoxon signed-rank test relative to the t test, with F replaced by F_X. Therefore, as before, the ARE of the Mann–Whitney test does not fall below 0.864 as long as F_X is a continuous cdf. There is, however, one important difference between the one-sample and two-sample cases. In the

one-sample case with the Wilcoxon signed-rank test, the underlying F is assumed to be symmetric about 0, but no such assumption is necessary about F_X in the two-sample case with the Mann–Whitney test. Thus, in the two-sample case, the ARE expression can be evaluated for an asymmetric distribution like the exponential; that is not allowed in the one-sample case.

Now let us find the efficacy of the median test. Recall that the test is based on $U_{m,n}$, the number of X observations that do not exceed Z, the combined sample median. In order to find the efficacy, we examine the mean of $U_{m,n}$. It can be shown (Mood, 1954) that for large m and n,

$$E(U_{m,n}) = mF_X(c)$$

where c satisfies

$$mF_X(c) + nF_Y(c) = \frac{m+n}{2} \tag{13.3.11}$$

Now

$$\left.\frac{dE(U_{m,n})}{d\theta}\right|_{\theta=0} = m\frac{dF_X(c)}{dc}\left.\frac{dc}{d\theta}\right|_{\theta=0} = mf_X(c)\left.\frac{dc}{d\theta}\right|_{\theta=0} \tag{13.3.12}$$

For the location model in (13.3.8) we have from (13.3.11)

$$mF_X(c) + nF_X(c-\theta) = \frac{m+n}{2} \tag{13.3.13}$$

Differentiating (13.3.13) with respect to θ yields

$$mf_X(c)\frac{dc}{d\theta} + nf_X(c-0)\left(\frac{dc}{d\theta} - 1\right) = 0$$

Therefore at $\theta = 0$

$$\left.\frac{dc}{d\theta}\right|_{\theta=0} = \frac{n}{m+n} \tag{13.3.14}$$

Substituting (13.3.14) in (13.3.12), we obtain

$$\left.\frac{dE(U_{m,n})}{d\theta}\right|_{\theta=0} = \frac{mn}{m+n}f_X(c)\Big|_{\theta=0} \tag{13.3.15}$$

Now from (13.3.13), when $\theta = 0$, we have

$$mF_X(c) + nF_X(c) = \frac{m+n}{2} \quad \text{so that } c = F_X^{-1}(0.5)$$

and substitution in (13.3.15) gives

$$\left.\frac{dE(U_{m,n})}{d\theta}\right|_{\theta=0} = \frac{mn}{m+n} f_X\left[F_X^{-1}(0.5)\right] \tag{13.3.16}$$

From (6.4.5) the null variance of the median test statistic for large m and n is found to be $mn/4(m+n)$. From (13.2.4) and (13.3.16), the efficacy of the median test is then

$$e(U_{m,n}) = 4\left(\frac{mn}{m+n}\right)\left\{f_X\left[F_X^{-1}(0.5)\right]\right\}^2 \tag{13.3.17}$$

From (13.3.10), (13.3.17) and applying (13.2.4), we see that the ARE expression for the median test relative to the Mann–Whitney (hence also the Wilcoxon rank sum) test is the same as the ARE expression for the sign test relative to the Wilcoxon signed-rank test given in (13.3.7) with f replaced by f_X. Hence the efficiency values given in Table 13.3.1 and the resulting comments made earlier for some specific distributions apply equally to the present case.

The ARE of the Mann–Whitney test relative to Student's t test can be found by evaluating the efficacies in (13.3.9) and (13.3.10) for any continuous population with cdf F_X with variance σ^2. Since Student's t test is the best test for normal distributions satisfying the general location model, we use this as an example. If F_X is the cdf of $N(\mu_X, \sigma^2)$,

$$\int_{-\infty}^{\infty} f_X^2(x)dx = \int_{-\infty}^{\infty} (2\pi\sigma^2)^{-1} \exp\left[-\left(\frac{x-\mu_X}{\sigma}\right)^2\right] dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{1}{2}} = (2\sqrt{\pi}\sigma)^{-1}$$

$$e(T^*_{m,n}) = \frac{mn}{\sigma^2(m+n)} \quad e(U_{m,n}) = \frac{3mn}{\pi\sigma^2(m+n+1)}$$

$$ARE(U_{m,n}, T^*_{m,n}) = \frac{3}{\pi}$$

For the uniform and double exponential distributions, the relative efficiencies are 1 and 3/2, respectively (Problem 13.1).

This evaluation of efficacy of the Mann–Whitney test does not make use of the fact that it can be written as a linear rank statistic. As an illustration of how the general results given in Theorem 7.3.8 simplify the calculation of efficiencies, we will show that the Terry and van der Waerden tests discussed in Section 8.3 are asymptotically optimum rank tests for normal populations differing only in location.

The weights for the van der Waerden test of (8.3.2) in the notation of Theorem 7.3.8 are

$$a_i = \Phi^{-1}\left(\frac{i}{N+1}\right) = \Phi^{-1}\left(\frac{i}{N}\frac{N}{N+1}\right) = J_N\left(\frac{i}{N}\right)$$

The combined population cdf for the general location model of (13.3.4) is

$$H(x) = \lambda_N F_X(x) + (1-\lambda_N)F_X(x-\theta)$$

so that

$$J[H(x)] = \lim_{N\to\infty} J_N(H) = \Phi^{-1}[\lambda_N F_X(x) + (1-\lambda_N)F_X(x-\theta)]$$

Applying Theorem 7.3.8 now to this J function, the mean is

$$\mu_N = \int_{-\infty}^{\infty} \Phi^{-1}[\lambda_N F_X(x) + (1-\lambda_N)F_X(x-\theta)]\, f_X(x)dx$$

To evaluate the derivative, we note that since

$$\Phi^{-1}[g(\theta)] = y \quad \text{if and only if } g(\theta) = \phi(y)$$

it follows that

$$\frac{d}{d\theta}g(\theta) = \Phi(y)\frac{dy}{d\theta} \quad \text{or} \quad \frac{dy}{d\theta} = \frac{d[g(\theta)]/d\theta}{\phi(y)}$$

where

$$\frac{d}{dy}\phi(y) = \phi(y).$$

Therefore the derivative of μ_N above is

$$\frac{d}{d\theta}\mu_N\Big|_{\theta=0} = \int_{-\infty}^{\infty} \frac{-(1-\lambda_N)f_X^2(x)}{\phi\{\Phi^{-1}[F_X(x)]\}}dx \tag{13.3.18}$$

Now to evaluate the variance when $\theta = 0$, we can use Corollary 7.3.8 to obtain

$$N\lambda_N\sigma_N^2 = (1-\lambda_N)\left\{\int_0^1 \left[\Phi^{-1}(u)\right]^2 du - \left[\int_0^1 \Phi^{-1}(u)du\right]^2\right\}$$

$$= (1-\lambda_N)\left\{\int_{-\infty}^{\infty} x^2\phi(x)dx - \left[\int_{-\infty}^{\infty} x\phi(x)dx\right]^2\right\}$$

$$= 1-\lambda_N$$

The integral in (13.3.18) reduces to a simple expression when $F_X(x)$ is $N(\mu_X, \sigma^2)$ since then

$$F_X(x) = \Phi\left(\frac{x-\mu_X}{\sigma}\right) \quad \text{and} \quad f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu_X}{\sigma}\right)$$

$$\left.\frac{d}{d\theta}\mu_N\right|_{\theta=0} = -\frac{1-\lambda_N}{\sigma^2}\int_{-\infty}^{\infty} \frac{\phi^2[(x-\mu_X)/\sigma]}{\phi[(x-\mu_X)/\sigma]}dx$$

$$= -\frac{1-\lambda_N}{\sigma}\int_{-\infty}^{\infty} \frac{1}{\sigma}\phi\left(\frac{x-\mu_X}{\sigma}\right)dx = -\frac{1-\lambda_N}{\sigma}$$

The efficacy of the van der Waerden X_N test in this normal case is then

$$e(X_N) = \frac{N\lambda_N(1-\lambda_N)}{\sigma^2} = \frac{mn}{N\sigma^2} \tag{13.3.19}$$

which equals the efficacy of the Student's t test $T^*_{m,n}$ given in (13.3.9).

Lehmann (2009) summarizes the ARE results for the Mann–Whitney or Wilcoxon test and the normal scores tests for the normal, logistic, double exponential and Cauchy distributions.

13.3.3 Two-Sample Scale Problems

The general distribution model of the scale problem for two independent random samples is

$$F_Y(x) = F_X(\theta x) \tag{13.3.20}$$

where we are assuming without loss of generality that the common location is zero. The null hypothesis of identical distributions then is

$$H_0: \theta = 1$$

against either one- or two-sided alternatives. Given two independent random samples of sizes m and n, the analogous parametric test for the scale problem is the statistic

$$T^*_{m,n} = \frac{(n-1)\sum_{i=1}^{m}(X_i - \overline{X})^2}{(m-1)\sum_{i=1}^{n}(Y_i - \overline{Y})^2}$$

Since $\text{var}(X) = \theta^2 \text{ var}(Y)$ in our model above and X and Y are independent, the expected value of the test statistic is

$$\begin{aligned} E(T^*_{m,n}) &= E\left[\frac{\sum_{i=1}^{m}(X_i - \overline{X})^2}{m-1}\right] E\left[\frac{n-1}{\sum_{i=1}^{m}(Y_i - \overline{Y})^2}\right] \\ &= (n-1)\ \text{var}(X) E\left[\frac{1}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}\right] \\ &= (n-1)\theta^2 E\left[\frac{\text{var}(Y)}{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}\right] = (n-1)\theta^2 E\left(\frac{1}{Q}\right) \end{aligned}$$

The probability distribution of Q depends on the particular distribution $F_X(x)$, but in the normal-theory model, where $F_X(x) = \Phi(x)$, Q has the chi-square distribution with $n-1$ degrees of freedom. Therefore we can evaluate

$$\begin{aligned} E\left(\frac{1}{Q}\right) &= \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}} \int_0^\infty x^{-1} e^{-x/2} x^{[(n-1)/2]-1} dx = \frac{\Gamma((n-3)/2)}{2\Gamma((n-1)/2)} \\ &= \frac{1}{n-3} \end{aligned}$$

$$E(T^*_{m,n}) = \frac{(n-1)\theta^2}{n-3} \qquad \frac{d}{d\theta} E(T^*_{m,n})\bigg|_{\theta=1} = \frac{2(n-1)}{n-3}$$

In this normal-theory model, under the null hypothesis the distribution of $T_{m,n}$ is Snedecor's F with $m-1$ and $n-1$ degrees of freedom. Since the variance of the F distribution with v_1 and v_2 degrees of freedom is

$$\frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 4)(v_2 - 2)^2}$$

we have here

$$\text{var}(T^*_{m,n})\big|_{\theta=1} = \frac{2(n-1)^2(N-4)}{(m-1)(n-5)(n-3)^2}$$

where $N = m + n$.

The statistic $T^*_{m,n}$ is the best test for the normal-theory model, and its efficacy for this distribution is

$$e(T^*_{m,n}) = \frac{2(m-1)(n-5)}{N-4} \approx \frac{2mn}{N} = 2N\lambda_N(1-\lambda_N) \tag{13.3.21}$$

We will now evaluate the efficacy of the Mood and Freund–Ansari–Bradley–Barton–David–Siegel–Turkey tests by applying the results of Theorem 7.3.8 to the two-sample scale model (13.3.20), for which

$$H(x) = \lambda_N F_X(x) + (1-\lambda_N)F_X(\theta x)$$

For the Mood test statistic of Section 9.2, we write

$$M'_N = N^{-2}\sum_{i=1}^{N}\left(i - \frac{N+1}{2}\right)^2 Z_i = N^{-2}M_N$$

so that for M'_N

$$a_i = \left(\frac{i}{N} - \frac{N+1}{2N}\right)^2 = J_N\left(\frac{i}{N}\right)$$

$$\lim_{N\to\infty} J_N(H) = (H-0.5)^2$$

The mean of M'_N then is

$$\mu_N = \int_{-\infty}^{\infty} [\lambda_N F_X(x) + (1-\lambda_N)F_X(\theta) - 0.5]^2 f_X(x)dx$$

and, after interchanging the order of differentiation and integration we have

$$\left.\frac{d\mu_N}{d\theta}\right|_{\theta=1} = 2(1-\lambda_N)\int_{-\infty}^{\infty} [F_X(x) - 0.5]\, x f_X^2(x)dx$$

and the variance under the null hypothesis is

$$\begin{aligned} N\lambda_N\sigma_N^2 &= (1-\lambda_N)\left\{\int_0^1 (u-0.5)^4 du - \left[\int_0^1 (u-0.5)^2 du\right]^2\right\} \\ &= \frac{1-\lambda_N}{180} \end{aligned}$$

so that the efficacy for any continuous distribution F_X with median zero is

$$e(M_N) = 720N\lambda_N(1-\lambda_N)\left\{\int_{-\infty}^{\infty}[F_X(x)-0.5]xf_X^2(x)dx\right\}^2 \tag{13.3.22}$$

In order to compare the Mood statistic with the F test statistic, we will calculate $e(M_N)$ for the normal-theory model, where $F_X(x) = \Phi(x)$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty}[\Phi(x-0.5)]x\phi^2(x)dx &= \int_{-\infty}^{\infty} x\,\Phi(x)\phi^2(x)dx - \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} x\Phi(x\sqrt{2})dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x} x\phi(t)\phi^2(x)dtdx \\
&= \int_{-\infty}^{\infty}\phi(t)\left(\int_{t}^{\infty} x\frac{1}{2\pi}e^{-x^2}dx\right)dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\phi(t)e^{-t^2}dt \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(3/2)t^2}dt \\
&= (4\pi\sqrt{3})^{-1}
\end{aligned}$$

For normal distributions, the result then is

$$e(M_N) = \frac{15N\lambda_N(1-\lambda_N)}{\pi^2} \quad \text{ARE}(M_N, T_{m,n}) = \frac{15}{2\pi^2}$$

Using the same procedure for the tests of Section 9.3, we write the weights for the test A'_N so that $(N+1)A'_N = A_N$, where S_N was given in (9.3.1). The result is

$$\begin{aligned}
a_i &= \left|\frac{i}{N+1} - \frac{1}{2}\right| = \frac{N}{N+1}\left|\frac{i}{N} - \frac{1}{2} - \frac{1}{2N}\right| = J_N\left(\frac{i}{N}\right) \\
J(H) &= |H - 0.5|
\end{aligned}$$

The mean of A'_N is

$$\mu_N = \int_{-\infty}^{\infty}|\lambda_N F_X(x) + (1-\lambda_N)F_X(\theta x) - 0.5|\, f_X(x)dx$$

and, after interchanging the order of differentiation and integration, we have

$$\left.\frac{d\mu_N}{d\theta}\right|_{\theta=1} = (1-\lambda_N)\int_{-\infty}^{\infty} |xf_X(\theta x)|\, f_X(x)dx|_{\theta=1}$$
$$= (1-\lambda_N)\int_{-\infty}^{\infty} |x|\, f_X^2(x)dx$$

If $f_X(x)$ is symmetric about its zero median, this reduces to

$$\left.\frac{d\mu_N}{d\theta}\right|_{\theta=1} = 2(1-\lambda_N)\int_0^{\infty} xf_X^2(x)dx \tag{13.3.23}$$

The variance when $\theta = 1$ is

$$N\lambda_N\sigma_N^2 = (1-\lambda_N)\left[\int_0^1 |u-0.5|^2 du - \left(\int_0^1 |u-0.5|du\right)^2\right]$$
$$= \frac{1-\lambda_N}{48}$$

For $f(x) = \phi(x)$, the integral in (13.3.23) is easily evaluated, and the results are

$$e(A_N) = \frac{12N\lambda_N(1-\lambda_N)}{\pi^2}$$
$$\text{ARE}(A_N, T_N^*) = \frac{6}{\pi^2} \quad \text{ARE}(A_N, M_N) = \frac{4}{5}$$

13.4 Summary

In this chapter we covered the concept of ARE of nonparametric tests and showed how to calculate this for some popular tests. The exact power of many nonparametric tests is difficult to find and the ARE becomes a useful tool for comparing two competing tests. The two most common criticisms of (Pitman–Noether) ARE are that (1) the comparison is valid only for large sample sizes and (2) the comparison is ''local,'' i.e., valid only in a neighborhood close to the null hypothesis. These criticisms have led to some other criteria in the literature for comparing nonparametric tests. Notable among

these is a concept of efficiency due to Bahadur (1960a,b, 1967), often called Bahadur efficiency. For further readings on Bahadur efficiency and an interesting comparison of the sign, Wilcoxon signed-rank, and t tests on the basis of such efficiency, the reader is referred to Klotz (1965). With regard to the first criticism that the ARE is basically a tool for a comparison between tests for large sample sizes, it may be noted that the power efficiency of nonparametric tests, especially for small sample sizes, has been a topic of interest for a long time and a vast amount of work, both analytical and empirical, has been reported in the literature. The powers of some of the tests discussed in this chapter are examined for finite sample sizes in Dixon (1954), Klotz (1963), Gibbons (1964), Arnold (1965), Randles and Hogg (1973), Randles and Wolfe (1979, p. 116), Blair and Higgins (1980), and Gibbons and Chakraborti (1991), among others. Generally, for moderate sample sizes and common significance levels, it appears that the relative power of many nonparametric tests is consistent with the results obtained from the corresponding ARE, as Lehmann (2009) notes. Nikitin (1995) is a good reference for mathematical details related to ARE.

We close with the remark that when the ARE between two competing tests is equal to one, a choice between the tests cannot be made from the usual Pitman efficiency point of view. Hodges and Lehmann (1970) have proposed a concept called *deficiency*, which is useful in these types of situations.

Problems

13.1 Use the results of Theorem 7.3.8 to evaluate the efficacy of the two-sample Wilcoxon rank-sum test statistic of (8.2.1) for the location model $F_Y(x) = F_X(x - \theta)$ where

(a) F_X is $N(\mu_X, \sigma^2)$ or $F_X(x) = \Phi[(x - \mu_X)/\sigma]$

(b) F_X is uniform, with

$$F_X(x) = \begin{cases} 0 & x \le -1/2 \\ x + 1/2 & -1/2 < x \le 1/2 \\ 1 & x > 1/2 \end{cases}$$

(c) F_X is double exponential, with

$$F_X(x) = \begin{cases} \left(\frac{1}{2}\right) e^{\lambda x} & x \le 0 \\ 1 - \left(\frac{1}{2}\right) e^{-\lambda x} & x > 0 \end{cases}$$

13.2 Calculate the efficacy of the two-sample Student's t test statistic in cases (b) and (c) of Problem 13.1.

13.3 Use your answers to Problems 13.1 and 13.2 to verify the following results for the ARE of the Wilcoxon rank-sum (or Mann–Whitney) test to Student's t test:

Normal:	$3/\pi$
Uniform:	1
Double exponential:	3/2

13.4 Calculate the efficacy of the sign test and the Student's t test for the location model $F_X(x) = F(x - \theta)$ where θ is the median of F_X and F is the cdf of the logistic distribution

$$F(x) = (1 + e^{-x})^{-1}$$

13.5 Evaluate the efficiency of the Klotz normal-scores test of (9.5.1) relative to the F test statistic for the normal-theory scale model.

13.6 Evaluate the efficacies of the M_N and A_N test statistics of Chapter 9 and compare their efficiency for the scale model where, as in Problem 13.1:

(a) F_X is uniform.
(b) F_X is double exponential.

13.7 Use the result in Problem 13.4 to verify that the ARE of the sign test relative to the Student's t test for the logistic distribution is $\pi^2/12$.

13.8 Verify the following results for the ARE of the sign test relative to the Wilcoxon signed-rank test.

Uniform:	1/3
Normal:	2/3
Logistic:	3/4
Double exponential:	4/3

13.9 Suppose there are three test statistic, T_1, T_2, and T_3, each of which can be used to test a null hypothesis H_0 against an alternative hypothesis H_1. Show that for any pair of tests, say T_1 and T_3, when the relevant ARE's exist,

$$\text{ARE}(T_1, T_3) = \text{ARE}(T_1, T_2)\ \text{ARE}(T_2, T_3) = [\text{ARE}(T_3, T_1)]^{-1}$$

where we take $1/\infty$ to be 0 and $1/0$ to be ∞.

14

Analysis of Count Data

14.1 Introduction

In this chapter, we present several different methods of analyzing count data that represent the number of observations that have one or more specified characteristics, or that respond in a certain manner to some kind of stimulus. Count data were used in the quantile tests and sign tests of Chapter 5; the situations in this chapter are more involved.

We start with the analysis of sample data presented in a two-way table with r rows and k columns, called an $r \times k$ contingency table, in which the cell counts refer to the number of sample observations that have certain cross characteristics. Here we have a test for the null hypothesis that the row and column characteristics are independent or have no association. We can also calculate the contingency coefficient or the phi-coefficient to measure the degree of association or dependence. Then we present some special results for $k \times 2$ contingency tables, including the test for equality of k proportions.

Another special case of contingency tables is the 2×2 table with fixed row and column totals. For this setting, we present Fisher's exact test for the equality of two proportions and we also cover McNemar's test for comparing two probabilities of success based on paired or dependent samples. Finally, we discuss some methods for the analysis of multinomial data.

14.2 Contingency Tables

Suppose we have a random sample of N observations, and two or more properties or characteristics are of interest for each subject in the sample. These properties, say $A, B, C, \ldots$, are called sets or families of attributes. Each of these sets has two or more categories of attributes, say $A_1, A_2, \ldots$, for family A, etc. These attributes may be measurable or not, as long as the categories are clearly defined, mutually exclusive, and exhaustive. Each observation is classified into exactly one category of each family. The sample

data are the numbers of units classified into each cross-category. Such a layout is called a *contingency table* of order $r_1 \times r_2 \times r_3 \times \cdots$ if there are r_1 categories of family A, r_2 of family B, etc. We are interested in a measure of association between the families, or in testing the null hypothesis that the families are completely independent, or that one particular family is independent of the others. In general, a group of families of events are defined to be completely independent if

$$P(A_i \cap B_j \cap C_k \cap \cdots) = P(A_i)P(B_j)P(C_k)\cdots$$

for all $A_i \subset A, B_j \subset B, C_k \subset C$, etc. For subgroup independence, say that family A is independent of all others, the requirement is

$$P(A_i \cap B_j \cap C_k \cap \cdots) = P(A_i)P(B_j \cap C_k \cap \cdots)$$

For example, in a public opinion survey concerning a proposed bond issue, the results of each interview or questionnaire may be classified according to the attributes of gender, education, and opinion. Along with the two categories of gender, we might have three categories of opinion, e.g., favor, oppose, and undecided, and five categories of education according to the highest level of formal schooling completed. The data may then be presented in a $2 \times 3 \times 5$ contingency table of 30 cells. A tally is placed in the appropriate cell for each person interviewed, and these count data can be used to determine whether gender and educational level have any observable relationship to opinion or to find some relative measure of their association.

For convenience, we will restrict our analysis to a two-way classification for two families of attributes; the extension to higher order layouts will be obvious. Suppose there are r categories of the type A attribute and k categories of the B attribute, and each of N experimental units is classified into exactly one of the rk cross-categories. In an $r \times k$ layout, the entry in the (i, j) cell, denoted by X_{ij}, is the number of items having the cross-classification $A_i \cap B_j$. The contingency table is written in the following form:

	***B* Family**				
A Family	B_1	B_2	...	B_k	**Row Total**
A_1	X_{11}	X_{12}	$\cdots$	X_{1k}	$X_{1.}$
A_2	X_{21}	X_{22}	$\cdots$	X_{2k}	$X_{2.}$
			$\cdots$		
			$\cdots$		
			$\cdots$		
A_r	X_{r1}	X_{r2}	$\cdots$	X_{rk}	$X_{r.}$
Column total	$X_{.1}$	$X_{.2}$	$\cdots$	$X_{.k}$	$X_{..} = N$

The total numbers of items classified into the categories A_i and B_j respectively then are the row and column totals $X_{i.}$ and $X_{.j}$, where

$$X_{i.} = \sum_{j=1}^{k} X_{ij} \quad \text{and} \quad X_{.j} = \sum_{i=1}^{r} X_{ij}$$

Without making any additional assumptions, we know that the rk random variables $X_{11}, X_{12}, \ldots, X_{rk}$ have the multinomial probability distribution with parameters

$$\theta_{ij} = P(A_i \cap B_j) \quad \text{where} \sum_{i=1}^{r} \sum_{j=1}^{k} \theta_{ij} = 1$$

so that the likelihood function is

$$\mathrm{L} = \prod_{i=1}^{r} \prod_{j=1}^{k} (\theta_{ij})^{x_{ij}}$$

The null hypothesis that the A and B classifications are independent affects only the allowable values of these parameters θ_{ij}.

In view of the definition of independence, the null hypothesis can be stated simply as

$$H_0 : \theta_{ij} = \theta_{i.}\theta_{.j} \quad \text{for all} \quad i = 1, 2, \ldots, r \text{ and } j = 1, 2, \ldots, k$$

where

$$\theta_{i.} = \sum_{j=1}^{k} \theta_{ij} = P(A_i) \quad \theta_{.j} = \sum_{i=1}^{r} \theta_{ij} = P(B_j)$$

If these $\theta_{i.}$ and $\theta_{.j}$ were all specified under the null hypothesis, this would reduce to an ordinary goodness-of-fit test for a simple hypothesis of the multinomial distribution with rk groups. However, the probability distribution is not completely specified under H_0, since only a particular relation between the parameters need be stated for the independence criterion to be satisfied.

The chi-square goodness-of-fit test for composite hypotheses discussed in Section 4.2 is appropriate here. The unspecified parameters must be estimated by the method of maximum likelihood and the degrees of freedom for the test statistic reduced by the number of independent parameters estimated. The maximum-likelihood estimates of the $(r-1)+(k-1)$ unknown independent parameters are those sample functions that maximize the likelihood function under H_0, which is

$$L(\theta_{1.}, \theta_{2.}, \ldots, \theta_{r.}, \theta_{.1}, \theta_{.2}, \ldots, \theta_{.k}) = \prod_{i=1}^{r} \prod_{j=1}^{k} (\theta_{i.}\theta_{.j})^{x_{ij}} \tag{14.2.1}$$

subject to the restrictions

$$\sum_{i=1}^{r} \theta_{i.} = \sum_{j=1}^{k} \theta_{.j} = 1$$

The maximum-likelihood estimates of these parameters are easily found to be the corresponding observed proportions, or

$$\hat{\theta}_{i.} = \frac{X_{i.}}{N} \quad \text{and} \quad \hat{\theta}_{.j} = \frac{X_{.j}}{N} \quad \text{for } i = 1, 2, \ldots, r \text{ and } j = 1, 2, \ldots, k$$

so that the rk estimated cell frequencies under H_0 are

$$N\hat{\theta}_{ij} = \frac{X_{i.}X_{.j}}{N}$$

By the results of Section 4.2, the test statistic then is

$$Q = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(X_{ij} - X_{i.}X_{.j}/N)^2}{X_{i.}X_{.j}/N} = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(NX_{ij} - X_{i.}X_{.j})^2}{NX_{i.}X_{.j}} \tag{14.2.2}$$

which under H_0 has approximately the chi-square distribution with $df = rk - 1 - (r - 1 + k - 1) = (r - 1)(k - 1)$. Since non-independence is reflected by lack of agreement between the observed and expected cell frequencies, the rejection region with significance level α is

$$Q \in R \quad \text{for } Q \geq x^2_{(r-1)(k-1),\alpha}$$

As before, if any expected cell frequency is too small, say, less than 5, the chi-square approximation is improved by combining adjacent cells and reducing the degrees of freedom accordingly.

The extension of this to higher-order contingency tables is obvious. For an $r_1 \times r_2 \times r_3$ table, for example, the hypothesis of complete independence is

$$H_0 : \theta_{ijk} = \theta_{i..}\theta_{.j.}\theta_{..k} \quad \text{for all } i = 1, 2, \ldots, r_1, j = 1, 2, \ldots, r_2, k = 1, 2, \ldots, r_3$$

The estimates of expected cell frequencies are

$$N\hat{\theta}_{ijk} = \frac{X_{i..}X_{.j.}X_{..k}}{N^2}$$

and the chi-square test statistic is

$$\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3}\frac{(N^2X_{ijk}-X_{i..}X_{.j.}X_{..k})^2}{N^2X_{i..}X_{.j.}X_{..k}}$$

with

$$r_1r_2r_3-1-(r_1-1+r_2-1+r_3-1)=r_1r_2r_3-r_1-r_2-r_3+2$$

degrees of freedom. For the hypothesis that family A is independent of B and C,

$$H_0:\theta_{ijk}=\theta_{i..}\theta_{.jk}$$

the estimated expected cell frequencies are

$$N\hat{\theta}_{ijk}=\frac{X_{i..}X_{.jk}}{N}$$

and the chi-square test statistic has

$$r_1r_2r_3-1-(r_1-1+r_2r_3-1)=(r_1-1)(r_2r_3-1)$$

degrees of freedom.

If the experimental situation is such that one set of totals is fixed by the experimenter in advance, say the row totals in an $r\times k$ contingency table, the test statistic for a hypothesis of independence is exactly the same as for completely random totals, although the reasoning is somewhat different. The entries in the ith row of the table constitute a random sample of size $x_{i.}$ From a k-variate multinomial population. For each row then, one of the cell entries is determined by the constant total. One of the observable frequencies is redundant for each row, as is one of the probability parameters $P(A_i\cap B_j)$ for each i. Since

$$P(A_i\cap B_j)=P(A_i)P(B_j|A_i)$$

and $P(A_i)=X_{i.}/N$ is now fixed, we simply redefine the relevant parameters as $\theta_{ij}=P(B_j|A_i)$, where $\sum_{j=1}^{k}\theta_{ij}=1$. The dimension of the parameter space is then reduced from $rk-1$ to $r(k-1)$. The B family is independent of the A_i category if $\theta_{ij}=P(B_j|A_i)=P(B_j)=\theta_j$ for all $j=1,2,\ldots,k$, where $\sum_{j=1}^{k}\theta_j=1$. Under H_0 then, $E(X_{ij})=X_{i.}\theta_j$, and if the θ_j were specified, the test statistic for independence between B and A_i would be

$$\sum_{j=1}^{k}\frac{(X_{ij}-X_{i.}\theta_j)^2}{X_{i.}\theta_j} \tag{14.2.3}$$

which is approximately chi-square distributed with $k-1$ degrees of freedom. The B family is completely independent of the A family if $\theta_{ij} = \theta_j$, $j = 1, 2, \ldots, k$, for all $i = 1, 2, \ldots, r$, so that the null hypothesis can be written as

$$H_0 : \theta_{1j} = \theta_{2j} = \cdots = \theta_{rj} = \theta_j \quad \text{for } j = 1, 2, \ldots, k$$

The test statistic for complete independence then is the statistic in (14.2.3) summed over all $i = 1, 2, \ldots, r$,

$$\sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(X_{ij} - X_{i.}\theta_j)^2}{X_{i.}\theta_j} \tag{14.2.4}$$

which under H_0 is the sum of r independent chi-square variables, each having $k-1$ degrees of freedom, and therefore has $r(k-1)$ degrees of freedom. Of course, in our case, the θ_j values are not specified, and so the test statistic is (14.2.4) with the θ_j replaced by their maximum-likelihood estimates and the degrees of freedom reduced by $k-1$, the number of independent parameters estimated. The likelihood function under H_0 of all N observations with row totals fixed is

$$L(\theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{r} \prod_{j=1}^{k} \theta_j^{X_{ij}} = \prod_{j=1}^{k} \theta_j^{X_{.j}}$$

so that $\hat{\theta}_j = X_{.j}/N$. Substituting this result in (14.2.4), we find the test criterion unchanged from the previous case with random totals given in (14.2.2), and the degrees of freedom are $r(k-1) - (k-1) = (r-1)(k-1)$, as before. By similar analysis, it can be shown that the same result holds for fixed column totals or both row and column totals fixed.

14.2.1 Contingency Coefficient

As a measure of the degree of association between families in a contingency table classifying N experimental units, Pearson (1904) proposed the *contingency coefficient* C, defined as

$$C = \left(\frac{Q}{Q+N}\right)^{1/2} \tag{14.2.5}$$

where Q is the test statistic for the null hypothesis of independence. If the families are completely independent, the values of Q and C are both small. Further, increasing values of C imply an increasing degree of association since large values of Q are a result of more significant departures between the observed and expected cell frequencies. Although clearly the value of C

cannot exceed 1 for any N, a disadvantage of C as a measure of association is that it cannot attain the value 1, as we now show.

For a two-way contingency table of dimension $r \times k$, the maximum value of C is

$$C_{\max} = \left(\frac{t-1}{t}\right)^{1/2} \quad \text{where } t = \min(r, k)$$

Without loss of generality, we can assume $r \geq k$. Then N must be at least r so that there is one element is each row and each column to avoid any zero denominators in the test statistic. Consider N fixed at its smallest value r, so that $x_i = 1$ for $i = 1,2,\ldots, r$, and x_{ij} is 0 or 1 for all i, j. The number of cells for which $x_{ij} = 0$ is $r - x_{.j}$. The value of Q from (14.2.2) then is

$$\sum_{j=1}^{k} \frac{(r - x_{.j})(0 - x_{.j}/r)^2 + x_{.j}(1 - x_{.j}/r)^2}{x_{.j}/r}$$

$$= \sum_{j=1}^{k} \frac{x_{.j}(r - x_{.j})[x_j + (r - x_{.j})]}{r x_{.j}} = r(k-1)$$

and the contingency coefficient has the value

$$C = \left[\frac{r(k-1)}{rk - r + r}\right]^{1/2} = \left(\frac{k-1}{k}\right)^{1/2}$$

As a result of this property, contingency coefficients to measure association for two different sets of count data are not directly comparable unless $\min(r, k)$ is the same for both tables. For this reason, some people prefer to use the ratio $C/C_{\max}$ as a measure of association in contingency tables. Another coefficient sometimes used to measure association is the *phi-coefficient* defined as

$$\phi = \sqrt{\frac{Q}{N}} \tag{14.2.6}$$

The sampling distribution of C or ϕ is not known. This is of no consequence since C and ϕ are both functions of Q, and a test of significance based on Q would be equivalent to a test of significance based on C^2 or ϕ^2.

Example 14.2.1

Streissguth et al. (1984) investigated the effect of alcohol and nicotine consumption during pregnancy on the resulting children by examining the children's attention span and reaction time at age four. First, the 542 mothers in the study

TABLE 14.2.1

Data for Example 14.2.1

	Nicotine (mg/day)			
Alcohol (oz./day)	**None**	**1–15**	**16 or More**	**Total**
None	105	7	11	123
0.01–0.10	58	5	13	76
0.11–0.99	84	37	42	163
1.00 or more	57	16	17	90
Total	304	65	83	452

were classified as shown in Table 14.2.1 according to their levels of consumption of alcohol and nicotine. Test the null hypothesis of no association between levels of consumption.

SOLUTION

The expected frequencies under the null hypothesis are calculated using the row and column totals in Table 14.2.1. The results are shown in parentheses in Table 14.2.2. Note that none of the expected frequencies is small, so there is no need to combine cells. The test statistic is $Q = 42.250$ with 6 degrees of freedom. The P value from Table B is $P < 0.001$ and we conclude that association exists. The value of the contingency coefficient from (14.2.5) is $C = \sqrt{42.25/494.25} = 0.2924$, and the phi coefficient from (14.2.6) is $\phi = \sqrt{42.25/452} = 0.3057$.

The STATXACT solution is shown below. The results agree with ours. The contingency coefficient is labeled Pearson's CC and the phi coefficient is labeled phi.

```
************************************
STATXACT SOLUTION TO EXAMPLE 14.2.1
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```
CONTINGENCY COEFFICIENTS TO MEASURE ASSOCIATION
Contingency Coefficient estimates based on 452 observations.

Coefficient        Estimate       ASE1       95.00% Confidence Interval
-----------        --------     -------      --------------------------
Phi                 0.3057      0.02552      (    0.2557,    0.3558)
Pearson's CC        0.2924      0.03650      (    0.2208,    0.3639)
Sakoda's CC         0.3581      0.04471      (    0.2705,    0.4457)
Tschuprow's CC      0.1954      0.01089      (    0.1740,    0.2167)
Cramer's V          0.2162      0.04174      (    0.1344,    0.2980)

                Pearson Chi-Square Statistic = 42.25
 Asymptotic P value: (based on Chi-Square distribution with 6 df)
 Pr  {  Test Statistic.GE. Observed  } =     0.0000
```

TABLE 14.2.2

Expected Frequencies

	Nicotine			
Alcohol	**0**	**1–15**	**16 or More**	**Total**
0	105 (82.7)	7 (17.7)	11 (22.6)	123
0.01–0.10	58 (51.1)	5 (10.9)	13 (14.0)	76
0.11–0.99	84 (109.6)	37 (23.4)	42 (30.0)	163
1.00 or more	57 (60.5)	16 (12.9)	17 (16.5)	90
Total	304	65	83	452

14.3 Some Special Results for $k \times 2$ Contingency Tables

In a $k \times 2$ contingency table, the B family is simply a dichotomy with say success and failure as the two possible outcomes. Then it is a simple algebraic exercise to show that the test statistic for independence can be written in an equivalent form as

$$Q = \sum_{i=1}^{k} \sum_{j=1}^{2} \frac{(X_{ij} - X_{i.}X_{j}/N)^2}{X_{i.}X_{j}/N} = \sum_{i=1}^{k} \frac{(Y_i - n_i\hat{p})^2}{n_i\hat{p}(1-\hat{p})} \tag{14.3.1}$$

where

$$Y_i = X_{i1} \quad n_i - Y_i = X_{i2} \quad \hat{p} = \sum_{i=1}^{k} \frac{Y_i}{N}$$

If B_1 and B_2 are regarded as success and failure, and $A_1, A_2, \ldots, A_k$ are called sample 1, sample 2, ..., and sample k, we see that the chi-square test statistic in (14.3.1) is the sum of the squares of k standardized binomial variables with parameter p estimated by its consistent estimator $\hat{p}$. Thus the test based on (14.3.1) is frequently called the *test for equality of k proportions*.

Example 14.3.1

A marketing research firm has conducted a survey of businesses of different sizes. Questionnaires were sent to 200 randomly selected businesses of each of three sizes. The data on responses are summarized below. Is there a significant difference in the proportion of nonresponses by small, medium, and large businesses?

	Business Size		
	Small	**Medium**	**Large**
Response	125	81	40

SOLUTION

The frequencies of nonresponses are 75, 119, and 160. The best estimate of the common probability of nonresponse is $(75+119+160)/600=0.59$. The expected numbers of nonresponse are then 118 for each size business. The value of Q from (14.3.1) is 74.70 with 2 degrees of freedom. From Table B we find $P<0.001$, and we conclude that the proportions of nonresponse are not the same for the three sizes of businesses.

A simplified form of the test statistic Q in (14.3.1) that may be more useful for calculations is

$$Q=\frac{1}{\hat{p}(1-\hat{p})}\sum_{i=1}^{k}\frac{Y_i^2}{n_i}-\frac{N\hat{p}}{1-\hat{p}} \tag{14.3.2}$$

Example 14.3.2

In a double-blind study of drugs to treat duodenal peptic ulcers, a large number of patients were divided into groups to compare three different treatments, antacid, antipepsin, and anticholinergic. Antacid has long been considered the major medication for ulcers; the latter two drugs act on different digestive juices. The number of patients in the groups and the percent who benefited from that treatment are shown below. Does there appear to be a difference in beneficial effects of these three treatments?

Treatment	**Number**	**Percent Benefited**
Antacid	40	55
Anticholinergic	60	70
Antipepsin	75	84

SOLUTION

We must first calculate the number of patients benefited by each drug as $Y_1=40(0.55)=22, Y_2=60(0.70)=42, Y_3=75(0.84)=63$; then the estimate of the common probability of benefit is $\hat{p}=(22+42+63)/175=0.726$ and

$$Q=\frac{1}{0.726(0.274)}\left(\frac{22^2}{40}+\frac{42^2}{60}+\frac{63^2}{75}\right)-\frac{175(0.726)}{0.274}=10.97$$

with 2 degrees of freedom. Table B shows that $P<0.005$, so we conclude that the probabilities of benefit from each drug are not equal.

If $k = 2$, the expressions in (14.3.1) and (14.3.2) can be written as

$$Q = \frac{(Y_1/n_1 - Y_2/n_2)^2}{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)} \tag{14.3.3}$$

Now we can see that the chi-square test statistic in (14.3.3) is the square of the difference between two sample proportions divided by the estimated variance of their difference. In other words, Q is the square of the classical standard normal theory test statistic used for the hypothesis that two population proportions are equal. This result was developed earlier in Section 6.4.

After substituting the original X_{ij} notation in (14.3.3), a little algebraic manipulation gives another equivalent form for Q as

$$Q = \frac{N(X_{11}X_{22} - X_{12}X_{21})^2}{X_{.1}X_{.2}X_{1.}X_{2.}} \tag{14.3.4}$$

This expression is related to the sample Kendall tau coefficient of Chapter 11. Suppose that the two families A and B are factors or qualities, both dichotomized into categories that can be called presence and absence of the factor or possessing and not possessing the quality. Suppose further that we have a single sample of size N, and that we make two observations on each element in the sample, one for each of the two factors. We record the observations using the code 1 for presence and 2 for absence. The observations then consist of N sets of pairs, for which the Kendall tau coefficient T of Chapter 11 can be determined as a measure of association between the factors. The numerator of T is the number of sets of pairs of observations, say $(a_i b_i)$ and $(a_j b_j)$, whose differences $a_i - a_j$ and $b_i - b_j$ have the same sign but are not zero. The differences here are both positive or both negative only for a set (1, 1) and (2, 2) and are of opposite signs for a set (1, 2) and (2, 1). If X_{ij} denotes the number of observations where factor A was recorded as i and factor B was recorded as j for $i, j = 1, 2$, the number of differences with the same sign is the product $X_{11}X_{22}$, the number of pairs that agreed in the sense that both factors were present or both were absent. The number of differences with opposite signs is $X_{12}X_{21}$, the number of pairs that disagreed. Since there are so many ties, it seems most appropriate to use the definition of T modified for ties, given in (11.2.37) and called tau b. Then the denominator of T is the square root of the product of the numbers of pairs with no ties for each factor, or $X_{.1}X_{.2}X_{1.}X_{2.}$ Therefore the tau coefficient is

$$T = \frac{X_{11}X_{22} - X_{12}X_{21}}{(X_{.1}X_{.2}X_{1.}X_{2.})^{1/2}} = \left(\frac{Q}{N}\right)^{1/2} \tag{14.3.5}$$

and Q/N estimates τ^2, the parameter of association between factors A and B. For this type of data, the Kendall measure of association is sometimes called the phi-coefficient, as defined in (14.2.6).

Example 14.3.3

The researchers in the study reported in Example 14.2.1 might have been more interested in a one-sided alternative of positive dependence between the variables alcohol and nicotine. Since the data measure level of consumption, we could regard them as 452 pairs of measurements with many ties. For example, the 37 mothers in cell (3, 2) of Table 14.2.1 represent the pair of measurements (AIII, BII), where AIII indicates alcohol consumption in the 0.11–0.99 range and BII represents nicotine consumption at level 1–15. For these kinds of data we can then calculate Kendall's tau for the 452 pairs. The number of concordant pairs C and the number of discordant pairs Q are calculated as shown in Table 14.3.1. Because the ties are quite extensive, we need to incorporate the correction for ties in the calculation of T from (11.2.38). Then we use the normal approximation to the distribution of T in (11.2.30) to calculate the right-tailed P value for this one-sided alternative.

$$T = \frac{24,346 - 12,622}{\sqrt{\left[\binom{452}{2} - \binom{304}{2} - \binom{65}{2} - \binom{83}{2}\right]\left[\binom{452}{2} - \binom{123}{2} - \binom{76}{2} - \binom{163}{2} - \binom{90}{2}\right]}} = 0.1915$$

$$Z = \frac{3(0.1915)\sqrt{452(451)}}{\sqrt{2(904+5)}} = 6.08$$

We find $P = 0.000$ from Table A.

There is also a relationship between the value of the chi-square statistic in a 2×2 contingency table and Kendall's partial tau coefficient. If we compare the expression for $T_{XY.Z}$ in (12.6.1) with the expression for Q in (14.3.4), we see that

$$T_{XY.Z} = \sqrt{\frac{Q}{N}} \quad \text{for } N = \binom{m}{2}$$

TABLE 14.3.1

Calculations for C and Q

C		Q	
$105(5 + 13 + 37 + 42 + 16 + 17)$	= 13,650	$7(58 + 84 + 57)$	= 1,393
$7(13 + 42 + 17)$	= 504	$11(58 + 84 + 57 + 5 + 37 + 16)$	= 2,827
$58(37 + 42 + 16 + 17)$	= 6,496	$58(84 + 57)$	= 705
$5(42 + 17)$	= 295	$13(84 + 57 + 37 + 16)$	= 2,522
$84(16 + 17)$	= 2,772	$37(57)$	= 2,109
$37(17)$	= 629	$42(57 + 16)$	= 3,066
Total	24,346		12,622

A test for the significance of $T_{XY.Z}$ cannot be carried out using Q, however. The contingency table entries in Table 12.6.1 are not independent even if X and Y are independent for fixed Z, since all categories involve pairings with the Z sample.

14.4 Fisher's Exact Test

Suppose we have two independent samples of sizes n_1 and n_2, from two binomial populations, 1 and 2, with probability of success θ_1 and θ_2, respectively, and observed number of successes y_1 and y_2, respectively. The data can be represented in a 2×2 table as in Table 14.4.1. The row totals are fixed since they are the sample sizes. As discussed in Section 14.3, the Q statistic in (14.3.3) can be used as an approximate test of the null hypothesis that the success probabilities are equal when the sample sizes are large.

We now present Fisher's exact test that can be used for this problem with any sample sizes when the marginal column totals $Y = Y_1 + Y_2$ and therefore also $N - (Y_1 + Y_2)$ are assumed fixed. Fisher's example of application is where an experiment is designed to test a human's ability to identify (discriminate) correctly between two objects, success and failure, when the subject is told in advance exactly how many successes are in the two samples combined. The subject's job is simply to allocate the total number of successes between the two groups. The null hypothesis is that this allocation is a random assignment; i.e., the subject is merely guessing.

Note that in the 2×2 table, the marginal row totals are fixed at the two given sample sizes. For a fixed $y_1 + y_2$, the value of y_1 determines the remaining three cell counts. Under the null hypothesis $H_0: \theta_1 = \theta_2 = \theta$, the conditional distribution of Y_1 given the marginal totals is the hypergeometric distribution

TABLE 14.4.1

Presentation of Data

	Subject's Identification		
Population	**Success**	**Failure**	**Total**
1	y_1	$n_1 - y_1$	n_1
2	y_2	$n_2 - y_2$	n_2
Total	$y_1 + y_2$	$N - (y_1 + y_2)$	N

$$\frac{\binom{n_1}{y_1}\binom{n_2}{y-y_1}}{\binom{N}{y}} \tag{14.4.1}$$

where y is the sum of the values observed in the first column of Table 14.4.1. Inferences can be based on an exact P value calculated from (14.4.1) for an observed y_1. The premise here is that the observed 2×2 table is one of the many possible 2×2 tables that could have been observed with the row and column totals fixed at their presently observed values. The question then becomes how extreme the currently observed table (value of y_1) is, in the appropriate direction, among all of the possible tables with the same marginal totals. The more extreme it is, the more evidence there is against the null hypothesis.

For example, if the alternative hypothesis is $H_1: \theta_1 > \theta_2$, the null hypothesis should be rejected if Y_1 is large. The exact P value can be calculated as $P(Y_1 \geq y_{10} | Y = y)$, where y_{10} is the observed value of Y_1. Again, this P value is calculated from all possible 2×2 tables with the same marginal totals as the observed one, but having a value of Y_1 as extreme as or more extreme than the observed value y_{10}. We illustrate this test with the famous data from Fisher's tea testing experiment.

Example 14.4.1

Sir Ronald A. Fisher, the English statistician, has been called the father of modern statistics. A famous story is that a colleague of Fisher's claimed that she could tell, while drinking tea with milk, whether milk or tea was poured into the cup first. An experiment was designed to test her claim. Eight cups of tea were presented to her in a random order; four of these had milk poured first while the other four had tea poured first. She was told that there were four cups of each type. The following data show the results of the experiment. She was right 3 out of 4 times on both types. Is this sufficient evidence to support her claim of special power?

	Guess Poured First		
Poured First	**Milk**	**Tea**	**Total**
Milk	3	1	4
Tea	1	3	4
Total	4	4	8

SOLUTION

The possible values of Y_1 are (0, 1, 2, 3, 4) and the observed value is 3, the number of cups with milk poured first that were correctly guessed. Only one other 2×2 table with the same marginal totals is more extreme than the observed table, and this is shown below.

	Guess Poured First		
Poured First	**Milk**	**Tea**	**Total**
Milk	4	0	4
Tea	0	4	4
Total	4	4	8

The exact P value is then the sum of the conditional probabilities for these two results calculated from (14.4.1) or

$$\frac{\left[\binom{4}{3}\binom{4}{1} + \binom{4}{4}\binom{4}{0}\right]}{\binom{8}{4}} = 0.2286 + 0.0143 = 0.2429$$

Hence there is not sufficient evidence to suggest that Fisher's colleague has any special power to determine whether tea or milk was poured into the cup first. The value of the chi-square statistic calculated from (14.3.3) is $Q = 2.0$ with $df = 1$. The P value from Table B is $0.10 < P < 0.25$ but this is for a two-sided alternative. For a 2×2 table with such small frequencies and a one-sided alternative, the chi-square approximation should not be used.

The STATXACT and the SAS outputs for this example are shown below. Both show the exact P value for a one-sided test as 0.2429, which agrees with ours. Note that the probability that Y_1 equals 3 (0.2286) also appears on both printouts, but STATXACT labels it as the value of the test statistic. The Fisher statistic in the STATXACT printout (1.807) is not the same as ours and should not be interpreted as such.

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************************************
STATXACT SOLUTION TO EXAMPLE 14.4.1
************************************

FISHER'S EXACT TEST

Statistic based on the observed 2 by 2 table (x)  :
    P(X)  = Hypergeometric Prob. of the table =    0.2286
    FI(X) = Fisher statistic                  =    1.807

Asymptotic P value: (based on Chi-Square distribution with 1 df )
      Two-sided: Pr{FI(X) .GE.     1.807} =     0.1789
      One-sided: 0.5 * Two-sided          =     0.0894
```

```
Exact P value and point probabilities:
   Two-sided: Pr{FI(X) .GE. 1.807} = Pr{P(X) .LE. 0.2286} = 0.4857
              Pr{FI(X) .EQ. 1.807} = Pr{P(X) .EQ. 0.2286} = 0.4571
 One-sided: Let y be the value in Row 1 and Column 1
   y = 3 min(Y) = 0 max(Y) = 4 mean(Y) = 2.000 std(Y) =    0.7559
              Pr  {  Y .GE. 3  } =    0.2429
              Pr  {  Y .EQ. 3  } =    0.2286

              *********************************
               SAS SOLUTION TO EXAMPLE 14.4.1
              *********************************
Program Code:

DATA TEATEST;
INPUT GROUP $ OUTCOME $ COUNT;
DATALINES;
MILK MILK 3
MILK TEA 1
TEA TEA 3
TEA MILK 1
;
PROC FREQ DATA = TEATEST;
  TABLES GROUP * OUTCOME/FISHER;
  WEIGHT COUNT;
RUN;

      Output:
            The FREQ Procedure
```

Table of GROUP by OUTCOME

Group Frequency Percent Row Pct Col Pct	Outcome MILK	 TEA	 Total
MILK	3	1	4
	37.50	12.50	50.00
	75.00	25.00	
	75.00	25.00	
TEA	1	3	4
	12.50	37.50	50.00
	25.00	75.00	
	25.00	75.00	
Total	4	4	8
	50.00	50.00	100.00

Statistics for Table of GROUP by OUTCOME

Statistic	*df*	Value	Prob
Chi-square	1	2.0000	0.1573
Likelihood ratio chi-square	1	2.0930	0.1480
Continuity adj. chi-square	1	0.5000	0.4795
Mentel-Haenszel chi-square	1	1.7500	0.1859
Phi coefficient		0.5000	
Contingency coefficient		0.4472	
Cramer's V		0.5000	

WARNING: 100% of the cells have expected counts less than 5. Chi-Square may not be a valid test.

Fisher's Exact Test	
Cell (1, 1) frequency (*F*)	3
Left-sided Pr <= *F*	0.9857
Right-sided Pr >= *F*	0.2429
Table probability (*P*)	0.2286
Two-sided Pr <= *P*	0.4857
Sample size = 8	

We note that the two-sample median test presented in Section 6.4 can be viewed as a special case of Fisher's exact test where $y_1 + y_2$ is the number of observations smaller than the sample median of the two combined samples, which is fixed at $N/2$ if N is even and $(N-1)/2$ if N is odd.

For further discussions on Fisher's exact test, the reader is referred to the review article by Gibbons (1982). STATXACT can calculate the power of Fisher's exact test for a given α, n_1, n_2, $\theta_1 = p_1$ and $\theta_2 = p_2$. For illustration, suppose $n_1 = n_2 = 10$ and $\alpha = 0.05$, and let $p_1 = 0.5$ and $p_2 = 0.8$. The exact power of Fisher's exact test is 0.13, as shown in the STATXACT solution below. The latest version of STATXACT also has options for calculating the sample size for a given α, p_1, p_2 and power.

```
**************************************************
STATXACT SOLUTION TO POWER OF FISHER'S EXACT TEST
**************************************************

>>> Power: Two Binomials Output
Exact Power of Two-Sided Tests for Comparing Two Binomial Populations
Type I error (Alpha)   = 0.05
Pop 2 probability: Difference of proportions model (Difference = 0.300000)
```

```
Type of test: Fisher's exact test
Probabilities (Pi):     0.5     0.8
Sample size (n):        10       10
Power       =13%
```

In the next section we consider the problem of comparing the probabilities of success for two groups with paired or dependent samples.

14.5 McNemar's Test

Suppose that a 2×2 table of data arises when a success or failure response is observed on each of N subjects before and after some treatment. The paired data are dependent within a pair but independent across pairs. Let X_{11} be the number of subjects whose responses are successes both before and after the treatment and let X_{22} be the number of subjects whose responses are failures both before and after the treatment. Then X_{12} and X_{21} denote the numbers of reversals (or discordant pairs) in responses; that is, X_{12} is the number of subjects whose initial (before treatment) response was success but became failure after the treatment, and similarly for X_{21}. The data can then be summarized in the following 2×2 table.

	After Treatment		
Before Treatment	**Success**	**Failure**	**Total**
Success	X_{11}	X_{12}	$X_{1.}$
Failure	X_{21}	X_{22}	$X_{2.}$
Total	$X_{.1}$	$X_{.2}$	N

The two groups of interest are the subjects before and after the treatment, and the null hypothesis is that the probability of success is the same before and after the treatment. Let $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ denote the four cell probabilities for the table, with $\theta_{11} + \theta_{12} + \theta_{21} + \theta_{22} = 1$. Thus θ_{11} is the probability of success before and after treatment and θ_{12} is the probability of success before treatment and failure after treatment. The sum $\theta_{.1} = \theta_{11} + \theta_{21}$ is the marginal probability of success after treatment and $\theta_{1.} = \theta_{11} + \theta_{12}$ is the marginal probability of success before treatment. The null hypothesis is then parameterized as $H_0 : \theta_{1.} = \theta_{.1}$ but this is the same as $\theta_{12} = \theta_{21}$. In other words, the null hypothesis can be viewed as testing that the probability of a reversal in either direction is the same.

For the null hypothesis $H_0 : \theta_{1.} = \theta_{.1}$ it is natural to consider a test based on $T = (X_{1.} - X_{.1})/N$, an unbiased estimator of the difference $\theta_{1.} - \theta_{.1}$. Since $X_{1.} = X_{11} + X_{12}$ and $X_{.1} = X_{11} + X_{21}$, T reduces to $T = (X_{12} - X_{21})/N$, the difference between the proportions of discordant pairs (numbers in the off-diagonal positions divided by N). Under the null hypothesis, the mean of T

is zero and the variance of T can be shown to be $(\theta_{12} + \theta_{21})/N$. *McNemar's test* for H_0 against the two-sided alternative $H_1 : \theta_{1.} \neq \theta_{.1}$ is based on

$$\frac{(X_{12} - X_{21})^2}{(X_{12} + X_{21})} \tag{14.5.1}$$

which is approximately distributed as chi-square with $df = 1$. The reader is warned about the inaccuracy of the chi-square approximation for small expected cell frequencies.

We now derive the variance of $T = (X_{12} - X_{21})/N$. The distributions of X_{12} and X_{21} are each binomial with parameters N, θ_{12} and θ_{21}, respectively. Hence $E(X_{12}) = N\theta_{12}$, $\text{var}(X_{12}) = N\theta_{12}(1 - \theta_{12})$ and $E(X_{21}) = N\theta_{21}$, $\text{var}(X_{21}) = N\theta_{21}(1 - \theta_{21})$. This gives $E(T) = \theta_{12} - \theta_{21}$. The variance of T will be found from

$$N^2 \text{ var}(T) = \text{var}(X_{12}) + \text{var}(X_{21}) - 2 \text{ cov}(X_{12}, X_{21}) \tag{14.5.2}$$

In order to find the covariance term, we note that the joint distribution of the counts $(X_{11}, X_{12}, X_{21}, X_{22})$ is a multinomial distribution with probabilities $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$. From this, it follows that the joint distribution of X_{12} and X_{21} is a multinomial with probabilities $(\theta_{12}, \theta_{21})$. The moment generating function of X_{12} and X_{21} was given in Table 1.2.1 as

$$\left\{\theta_{12}e^{t_1} + \theta_{21}e^{t_2} + [1 - (\theta_{12} + \theta_{21})]\right\}^N \tag{14.5.3}$$

Taking the second partial derivative of (14.5.3) with respect to t_1 and t_2 and setting $t_1 = t_2 = 0$, we obtain the second joint moment about the origin as

$$E(X_{12}X_{21}) = N(N - 1)\theta_{12}\theta_{21}$$

Hence the covariance is

$$\text{cov}(X_{12}, X_{21}) = N(N - 1)\theta_{12}\theta_{21} - (N\theta_{12})(N\theta_{21}) = -N\theta_{12}\theta_{21}$$

Now substituting back in (14.5.2) gives

$$\begin{aligned} N^2\text{var}(T) &= N\theta_{12}(1 - \theta_{12}) + N\theta_{21}(1 - \theta_{21}) - 2(-N\theta_{12}\theta_{21}) \\ &= N\left[(\theta_{12} + \theta_{21}) - (\theta_{12} - \theta_{21})^2\right] \end{aligned} \tag{14.5.4}$$

Therefore $T = (X_{12} - X_{21})/N$ has expectation $\theta_{12} - \theta_{21}$ and variance

$$\frac{\left[(\theta_{12} + \theta_{21}) - (\theta_{12} - \theta_{21})^2\right]}{N}$$

Under the null hypothesis $\theta_{12} = \theta_{21}$, $E(T) = 0$ and $\text{var}(T) = (\theta_{12} + \theta_{21})/N$, which can be consistently estimated by $(X_{12} + X_{21})/N^2$. McNemar's test statistic in (14.5.1) is the square of T divided by this estimated variance.

A second motivation of McNemar's test can be given as follows. As noted before, the joint distribution of the counts $(X_{11}, X_{12}, X_{21}, X_{22})$ is a multinomial distribution with probabilities $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$. Let $S = X_{12} + X_{21}$ denote the total number of discordant pairs. The reader can verify that the conditional distribution of X_{12} given S is binomial $[S, p = \theta_{12}/(\theta_{12} + \theta_{21})]$. Then, since $\theta_{12} = \theta_{21}$ under the null hypothesis, an exact (conditional) P value can be calculated from the binomial $(S, 0.5)$ distribution as the probability that X_{12} is as extreme as or more extreme than its observed value, in the direction of the alternative.

We are usually interested in a one-sided alternative that the treatment is effective, $H_1 : \theta_{.1} > \theta_{1.}$, and this is equivalent to $H_1 : \theta_{12} < \theta_{21}$. For this alternative, the exact P value is given by $\sum_{j=0}^{S} \binom{S}{j} (0.5)^S$, which can be found from Table G for $S \leq 20$. For large sample sizes, an approximate P value can be based on the statistic

$$Z = \frac{X_{12} - 0.5S}{\sqrt{0.25S}} = \frac{X_{12} - X_{21}}{\sqrt{X_{12} + X_{21}}} \tag{14.5.5}$$

which is approximately standard normal. This statistic with a continuity correction is

$$Z = \frac{X_{12} - X_{21} + 0.5}{\sqrt{X_{12} + X_{21}}} \tag{14.5.6}$$

The squared value of (14.5.5) is the McNemar test statistic for matched pairs in (14.5.1), which is approximately distributed as chi square with one degree of freedom.

For a one-sided alternative $H_1 : \theta_{12} > \theta_{21}$, the appropriate rejection region is in the right tail of Z in (14.5.5) or (14.5.6) with the continuity correction changed to -0.5. STATXACT provides both exact and approximate tests, as we show for Example 14.5.1.

Example 14.5.1

Suppose a new drug is tested for effectiveness using a random sample of 100 subjects. Each subject is evaluated for pain after being treated with the new drug as well as a placebo at randomly chosen and different times. Under the placebo,

35 of the subjects reported some relief but the remaining 65 did not. Under the new drug, 55 of the subjects reported some relief while 45 did not. Of the 65 people who reported no relief under the placebo, 30 also reported no relief under the new drug. Is there any evidence that the new drug is more effective than the placebo?

SOLUTION

The data can be represented in the following 2×2 table. The alternative of interest is $\theta_{.1} > \theta_{1.}$ or $\theta_{21} > \theta_{12}$.

	New Drug		
Placebo	**Some Relief**	**No Relief**	**Total**
Some relief	20	15	35
No relief	35	30	65
Total	55	45	100

The value of the test statistic for a one-sided test is $X_{12} = 15$. The exact P value $P(X_{12} \leq 15)$ is found from the binomial distribution with $S = 50$ and $\theta = 0.5$. The test statistic for the normal approximation is calculated from (14.5.5) as

$$Z = \frac{15 - 50(0.5)}{\sqrt{0.25(50)}} = -2.83$$

and $P = 0.0023$ from Table A. The corresponding approximation with a continuity correction from (14.5.6) is

$$Z = \frac{(15 - 35) + 0.5}{\sqrt{15 + 35}} = -2.76$$

with $P = 0.0029$. In each case, there is sufficient evidence that the new drug is more effective than the placebo.

The value of McNemar's test statistic for a two-sided alternative from (14.5.1) is

$$Z^2 = \frac{(35 - 15)^2}{35 + 15} = 8.0$$

Table B shows $0.001 < P < 0.005$.

The STATXACT and SAS outputs for this example are shown below. In STATXACT, note that the Z statistic is calculated without a continuity

correction. The approximate P values are the same as ours and are fairly close to the exact values. The reader is referred to the STATXACT user's manual for details regarding the exact P value calculations. The value of McNemar's statistic is shown on the SAS output and it agrees with ours. The approximate P value is 0.0047, which is simply twice that of the one-tailed P value associated with the Z statistic without a continuity correction.

```
            ************************************
            STATXACT SOLUTION TO EXAMPLE 14.5.1
            ************************************

MARGINAL HOMOGENEITY TEST for ordered table

Statistic based on the observed 5 by 5 table (x) with 50 observations:
 Min      Max      Mean     Std-Dev    Observed    Standardized
50.00    100.0    75.00      3.536       65.00          −2.828

Asymptotic Inference:
  One-sided P value: Pr { Test Statistic.LE. Observed } = 0.0023
  Two-sided P value: 2 * One-sided                      = 0.0047

Exact Inference:
  One-sided P value: Pr { Test Statistic.LE. Observed } = 0.0033
                     Pr { Test Statistic.EQ. Observed } = 0.0020
  Two-sided P value: 2*One-sided                        = 0.0066
```

```
            ******************************
            SAS SOLUTION TO EXAMPLE 14.5.1
            ******************************

DATA PAIN;
INPUT DRUG $ PLACEBO $ COUNT;
DATALINES;
YES YES 20
YES NO 35
NO YES 15
NO NO 30
;
PROC FREQ DATA = PAIN;
    TABLES PLACEBO * DRUG/AGREE;
    WEIGHT COUNT;
RUN;
```

Output:

The FREQ Procedure
Table of PLACEBO by DRUG

PLACEBO	DRUG		
Frequency Percent Row Pct Col Pct	NO	YES	Total
NO	30	35	65
	30.00	35.00	65.00
	46.15	53.85	
	66.67	63.64	
YES	15	20	35
	15.00	20.00	35.00
	42.86	57.14	
	33.33	36.36	
Total	45	55	100
	45.00	55.00	100.00

Statistics for Table of PLACEBO by DRUG

McNemar's Test

Statistic (*S*)	8.0000
df	1
Pr > *S*	0.0047

Simple Kappa Coefficient

Kappa	0.0291
ASE	0.0919
95% Lower Conf Limit	−0.1511
95% Upper Conf Limit	0.2093

Sample Size = 100

The power of McNemar's test has been studied by various authors. A related issue is the determination of sample size. The reader is referred to Lachin (1992) and Lachenbruch (1992) and the references given there. The latest version of STATXACT has options for calculating the power and the sample size for McNemar's test.

14.6 Analysis of Multinomial Data

Count data can also arise when sampling from a multinomial distribution with k possible categories or outcomes with respective probabilities $p_1, p_2, \ldots, p_k$, which sum to 1. We can use the chi-square goodness-of-fit test in Section 4.2 for the null hypothesis that the sample data conform to specified values for these probabilities (see Problems 4.1, 4.3 through 4.5, 4.27, 4.31, and 4.32).

If we have random samples from two or more multinomial distributions, each with the same k possible categories or outcomes, the data can be presented in an $r \times k$ contingency table where the rows represent the samples and the columns represent the categories. Now X_{ij} denotes the number of outcomes in category j for the ith sample, and the probabilities of these outcomes for the ith sample are denoted by $p_{i1}, p_{i2}, \ldots, p_{ik}$, where $0 < p_{ij} < 1$ and $\sum_j p_{ij} = 1$. We will consider only the case where we have $r = 2$ samples of sizes n_1 and n_2. The data can be presented in a $2 \times k$ table as in Table 14.6.1. Note that the row totals are fixed by the sample sizes.

We are interested in testing the null hypothesis $H_0: p_{11} = p_{21}, p_{12} = p_{22}, \ldots, p_{1k} = p_{2k}$. The common probability for the jth category is estimated by $(X_{1j} + X_{2j})/N = X_{.j}/N$, and the estimated cell frequencies are $n_1 X_{.j}/N$ and $n_2 X_{.j}/N$ for samples 1 and 2, respectively. The chi-square test statistic with $df = k - 1$ is then

$$Q = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{(X_{ij} - n_i X_{.j}/N)^2}{n_i X_{.j}/N} \tag{14.6.1}$$

which is the same as (14.3.1), the test for equality of k proportions.

TABLE 14.6.1

Presentation of Data

	Category or Outcome				
Sample	**1**	**2**	**...**	***k***	**Total**
1	X_{11}	X_{12}	...	X_{1k}	n_1
2	X_{21}	X_{22}	...	X_{2k}	n_2
Total	$X_{.1}$	$X_{.2}$	...	$X_{.k}$	N

Example 14.6.1

Businesses want to maximize return on any money spent on advertising. If the medium is a television commercial, they want the audience to remember the main points of the commercial as long as possible. Two versions of a commercial were test marketed on 100 volunteers. The volunteers were randomly assigned to two groups to view commercials A or B so that each group had 50 volunteers. After 2 days, the participants were telephoned and asked to classify their recollection of the commercial as either ''Don't remember,'' ''Remember vaguely,'' or ''Remember key points.'' The data are shown below. Are commercials A and B equally effective as measured by viewer recollection?

	Do not Remember	Remember Vaguely	Remember Key Points	Total
Commercial A	12	15	23	50
Commercial B	15	15	20	50
Total	27	30	43	100

Solution

The null hypothesis is $H_0: p_{A1} = p_{B1}$, $p_{A2} = p_{B2}$, $p_{A3} = p_{B3}$, against the alternative that they are not all equal. The expected frequencies under the null hypothesis and the $(x_{ij} - e_{ij})^2/e_{ij}$ terms, called contributions (*cont*) from cell (i, j) to the Q statistic, are shown below.

	Do not Remember	Remember Vaguely	Remember Key Points	Total
Commercial A	$X_{11} = 12$ $e_{11} = 13.5$ $cont = 0.17$	$X_{12} = 15$ $e_{12} = 15$ $cont = 0$	$X_{13} = 23$ $e_{13} = 21.5$ $cont = 0.10$	50
Commercial B	$X_{21} = 15$ $e_{21} = 13.5$ $cont = 0.17$	$X_{22} = 15$ $e_{22} = 15$ $cont = 0$	$X_{23} = 20$ $e_{23} = 21.5$ $cont = 0.10$	50
Total	27	30	43	100

The test statistic is $Q = 0.17 + 0 + 0.10 + 0.17 + 0 + 0.10 = 0.54$ with $df = 2$ and Table B shows $P > 0.50$. This implies that there is no significant difference between commercials A and B with respect to recollection by viewers. The STATXACT and MINITAB solutions are shown below. The answers agree with ours.

```
**********************************
STATXACT SOLUTION TO EXAMPLE 14.6.1
**********************************

CHI-SQUARE TEST FOR INDEPENDENCE

Statistic based on the observed 2 by 3 table (x):
  CH(X)  :  Pearson Chi-Square Statistic =  0.5426

Asymptotic P value: (based on Chi-Square distribution with 2 df)
    Pr  {CH(X)  .GE.          0.5426  }  = 0.7624

Exact P value and point probability:
    Pr  {CH(X)  .GE.          0.5426}  = 0.7660
    Pr  {CH(X)  .EQ.          0.5426}  = 0.0513
```

```
*******************************
MINITAB SOLUTION TO EXAMPLE 14.6.1
*******************************

Chi-Square Test: C1, C2, C3

Expected counts are printed below observed counts

              C1          C2          C3       Total
    1         12          15          23          50
            13.50       15.00       21.50

    2         15          15          20          50
            13.50       15.00       21.50

Total         27          30          43         100

Chi-Sq = 0.167 + 0.000 + 0.105 + 0.167 + 0.000 + 0.105 = 0.543
df = 2, P-Value = 0.762
```

14.6.1 Ordered Categories

The three categories in Example 14.6.1 are actually ordered in terms of degree of recollection. In comparing two multinomial distributions when the categories are ordinal, we really are more interested in a directional alternative, specifically that the degree of recollection is greater for one commercial than the other, rather than the alternative that the degree of recollection is not the same for the two commercials. The chi-square test is appropriate only for the two-sided alternative. The Wilcoxon rank-sum test presented in Section 8.2 can be adapted to provide a test to compare two groups against a one-

sided alternative. We explain this approach in the context of Example 14.6.2. This is very similar to what we did in Example 14.3.3 to calculate the Kendall tau coefficient.

Example 14.6.2

Two independent random samples of 10 business executives are taken, one sample from executives under 45 years of age, and the other from executives at least 45 years old. Each subject is then classified in terms of degree of risk aversion, low, medium or high, based on the results of a psychological test. For the data shown below, the research hypothesis of interest is that the younger business executives are more risk averse than their older counterparts.

	Degree of Risk Aversion			
Age	**Low**	**Medium**	**High**	**Total**
Under 45	2	3	5	10
Over 45	4	5	1	10
Total	6	8	6	20

Solution

We call the under 45 group the X sample and the over 45 the Y sample. If we code (rank) the 3 degrees of risk aversion as 1 = low, 2 = medium, and 3 = high, the six executives from the X and Y samples combined who were classified as Low (column one) are all tied at rank 1. If we use the midrank method to resolve the ties, each of these six executives (in the first column) would be assigned rank $(1+2+3+4+5+6)/6 = (1+6)/2 = 3.5$. For the second column category (Medium), the midrank is $(7+14)/2 = 10.5$, and for the third column, the midrank is $(15+20)/2 = 17.5$. (Note that the midrank with integer ranks is always the average of the smallest and the largest ranks they would have had if they were not tied.) The value of the Wilcoxon rank-sum test statistic for the X sample is then $W_N = 2(3.5) + 3(10.5) + 5(17.5) = 126$. We can test the significance of this result using the normal approximation to the distribution of W_N for $m = 10$, $n = 10$, $N = 20$. The mean is $m(N+1)/2 = 105$ and the variance is calculated from (8.2.3) with the correction for ties as 154.74, giving $Z = 1.688$ without a continuity correction and $Z = 1.648$ with a continuity correction. The upper tail P values from Table A are 0.046 and 0.050, respectively. This result does not lead to any firm conclusion at the 0.05 level.

Our first result agrees with the STATXACT output shown below. The output also shows the exact P value is 0.0785, which is not significant at the 0.05 level. This exact test is carried out using the conditional distribution of the cell counts given the column totals, which is a multiple hypergeometric

distribution [see Lehmann (1975), p. 384]. The chi-square test for independence (a two-sided alternative) shows no significant difference between the two age groups.

```
**********************************
STATXACT SOLUTION TO EXAMPLE 14.6.2
**********************************

WILCOXON-MANN-WHITNEY TEST
[1 2 by 3 informative tables and sum of scores from row <row1>]

Summary of Exact distribution of WILCOXON-MANN-WHITNEY statistic:
 Min      Max      Mean     Std-Dev    Observed    Standardized
63.00    147.0    105.0     12.44       126.0          1.688
Mann-Whitney Statistic=   71.00

Asymptotic Inference:
  One-sided P value: Pr { Test Statistic.GE. Observed } = 0.0457
  Two-sided P value: 2 * One-sided                     = 0.0914

Exact Inference:
  One-sided P value: Pr { Test Statistic.GE. Observed } = 0.0785
                     Pr { Test Statistic.EQ. Observed } = 0.0563
  Two-sided P value: Pr { |Test Statistic - Mean|
                          .GE. |Observed - Mean|       = 0.1570
  Two-sided P value: 2 * One-sided                     = 0.1570
```

The Wilcoxon rank-sum test can be considered a special case of a class of linear rank statistics of the form $T = \sum_j w_j X_{1j}$, where the w_j are some suitable scores or weights that are increasing in value. Different weights give rise to different linear rank statistics. For the Wilcoxon rank-sum test, the weights are the respective midranks. Other possible weights could be based on the expected normal scores (Terry-Hoeffding) or inverse normal scores (van der Waerden). Graubard and Korn (1987) studied three classes of scores and made some recommendations. STATXACT 5.0 has options for calculating the power and sample size for any linear rank test, including the Wilcoxon rank-sum test.

14.7 Summary

In this chapter, we covered several test procedures for the analysis of count data. All are very simple to derive and carry out. Except for Fisher's exact test, all procedures are based on the asymptotic distribution. The alternative

is always nondirectional for those tests that use the chi-square statistic. The test for ordered categories in multinomial data uses the asymptotic normal distribution and does permit a one-sided test.

Problems

14.1 An ongoing problem on college campuses is the instructor evaluation form. To aid in interpreting the results of such evaluations, a study was made to determine whether any relationship exists between the stage of a student's academic career and his attitude with respect to whether the academic work load in his courses could be considered as lighter than it should be, at the appropriate level, or heavier than it should be. A stratified random sample yielded the following results:

	Sophomore	Junior	Senior
Believe work load is lighter than it should be	5	8	11
Believe work load is at the appropriate level	30	35	40
Believe work load is heavier than it should be	25	17	9

(a) Test the null hypothesis that there is no association between the stage of a student's career and attitude with respect to the appropriateness of the academic work load.
(b) Measure the degree of association.

14.2 A manufacturer produces units of a product in three 8 hour shifts: Day, Evening, and Night. Quality control teams check production lots for defects at the end of each shift by taking random samples. Do the three shifts have the same proportions of defects?

	Day	Evening	Night
Defects	70	60	80
Sample total	400	300	300

14.3 A group of 28 salespersons were rated on their sales presentations and then asked to view a training film on improving selling techniques. Each person was then rated a second time. For the data below determine whether the training film has a positive effect on the ratings.

	Rating after Film		
Rating before Film	**Acceptable**	**Not Acceptable**	**Total**
Acceptable	5	4	9
Not acceptable	13	6	19
Total	18	10	28

14.4 An employer wanted to find out if changing from his current health benefit policy to a prepaid policy would change hospitalization rates for his employees. A random sample of 100 employees was selected for the study. During the previous year under the current policy, 20 of them had been hospitalized and 80 had not been hospitalized. These same 100 employees were then placed on the prepaid policy and after 1 year, it was found that among the 20, 5 had been rehospitalized, and among the 80, 10 had been hospitalized. Test to see whether the prepaid policy reduces hospitalization rates among the employees.

14.5 A sample of five vaccinated and five unvaccinated cows were all exposed to a disease. Four cows contracted the disease, one from the vaccinated group and three from the nonvaccinated group. Determine whether the vaccination had a significant effect in protecting the cows against the disease.

14.6 A superintendent of schools is interested in revising the curriculum. He sends out questionnaires to 200 teachers: 100 respond *No* to the question ''do you think we should revise the curriculum?'' The superintendent then held a weeklong workshop on curriculum improvement and sent the same questionnaire to the same 200 teachers; this time 90 responded *No*. Eighty teachers responded *No* both times. Investigate whether the workshop significantly decreased the number of negative responses.

14.7 A retrospective study of death certificates was aimed at determining whether an association exists between a particular occupation and a certain neoplastic disease. In a certain geographical area over a period of time, some 1500 certificates listed the neoplastic disease as primary cause of death. For each of them, a matched control death certificate was selected, based on age, race, gender, county of residence, and date of death, and stating any cause of death other than the neoplastic disease. The occupation of each decedent was determined. Only one matched pair had both the case and control members in the specified occupation. There were 69 pairs in which the case pair member was in the specified occupation while the control member was not. There were 30 pairs in which the control member was in the occupation and the case pair member was not. In all of the remaining 1400 pairs,

neither the case nor the control member was in the specified occupation. Test the null hypothesis that the proportion of case and control members in the occupation is the same.

14.8 A financial consultant is interested in testing whether the proportion of debt that exceeds equity is the same irrespective of the magnitude of the firm's assets. Sixty-two firms are classified into three groups according to asset size and the data below are obtained on the numbers with debt greater then equity. Carry out the test.

	Firm Asset Size ($1000)			
	Less Than 500	**500–2000**	**Over 2000**	**Total**
Debt less than equity	7	10	8	25
Debt greater than equity	10	18	9	37
Total	17	28	17	62

14.9 In a study designed to investigate the relationship between age and degree of job satisfaction among clerical workers, a random sample of 100 clerical workers were interviewed and classified according to these characteristics as shown below.

	Job Satisfaction (1 = Least Satisfied)			
Age	**1**	**2**	**3**	**Total**
Under 25	8	7	5	20
25–39	12	8	20	40
40 and over	20	15	5	40
Total	40	30	30	100

(a) Test whether there is any association between age and job satisfaction using the chi-square test.
(b) Calculate the contingency coefficient and the phi coefficient.
(c) Calculate Kendall's tau with correction for ties and test for association.
(d) Calculate the Goodman–Kruskal coefficient.

14.10 A random sample of 135 U.S. citizens were asked their opinion about the current U.S. foreign policy in Afghanistan. Forty-three reported a negative opinion and the others were positive. These 135 persons were then put on a mailing list to receive an informative newsletter about U.S. foreign policy, and then asked their opinion a month later. At the time, 37 were opposed and 30 of these 37 originally had a positive

opinion. Find the P value for the alternative that the probability of a change from negative to positive is greater than the corresponding probability of a change in the opposite direction.

14.11 A small random sample was used in an experiment to see how effective an informative newsletter was in persuading people to favor a flat income tax bill. Thirty persons were asked their opinion before receiving the letter and these same persons were then asked again after receiving the letter. Before the letter, 11 were in favor. Five were in favor both before and after receiving the newsletter, and six were opposed both times. Is there evidence that the letter is effective in persuading people to favor a flat tax?

14.12 Twenty married couples were selected at random from a large population and each person was asked privately whether the family would prefer to spend a week's summer vacation at the beach or in the mountains. The subjects were told to ignore factors such as relative cost and distance so that their preference would reflect only their expected pleasure from each type of vacation. The husband voted for the beach seven times and his wife agreed four times. The husband voted for the mountains 13 times and his wife agreed five times. Determine whether family vacation preference is dominated by the husband.

14.13 A study was conducted to investigate whether high school experience with calculus has an effect on performance in first-year college calculus. A total of 686 students who had completed their first year of college calculus were classified according to their high school calculus experience as Zero (None), Brief (One semester), Year (Two semesters), and AP (Advanced Placement); these same students were then classified according to their grade in first year college calculus. Test to see whether high school calculus experience has an effect on college grade.

	High School Calculus			
College Grade	**Zero**	**Brief**	**Year**	**AP**
A	3	6	32	16
B	23	30	70	56
C	48	51	67	29
D	49	45	27	6
F	71	44	17	2

14.14 For the data in Problem 14.8, investigate whether firms with debt greater than equity tend to have more assets than other firms.

14.15 Derive the maximum likelihood estimators for the parameters in the likelihood function of (14.2.1).

14.16 Show that (14.2.2) is still the appropriate test statistic for independence in a two-way $r \times k$ contingency table when both the row and column totals are fixed.

14.17 Verify the equivalence of the expressions in (14.3.1) through (14.3.4).

14.18 Struckman-Johnson (1988) surveyed 623 students in a study to compare the proportions of men and women at a small midwestern university who have been coerced by their date into having sexual intercourse (date rape). A survey of students produced 623 responses. Of the 355 female respondents, 79 reported an experience of coercion, while 43 of the 268 male respondents reported coercion. Test the null hypothesis that males and females experience coercion at an equal rate.

14.19 Prior to the Alabama-Auburn football game, 80 Alabama alumni, 75 Auburn alumni, and 45 residents of Tuscaloosa who are not alumni of either university are asked who they think will win the game. The responses are as follows:

	Alabama	**Auburn**	**Tuscaloosa**
Alabama win	55	15	30
Auburn win	25	60	15

Are the proportions of persons who think Alabama will win the same for the three groups?

14.20 Four different experimental methods of treating schizophrenia, (1) weekly shock treatments, (2) weekly treatments of carbon dioxide inhalations, (3) biweekly shock treatment alternated with biweekly carbon dioxide inhalations, and (4) tranquilizer drug treatment, are compared by assigning a group of schizophrenic patients randomly into four treatment groups. The data below are the number of patients who did and did not improve in 4 weeks of treatment. Test the null hypothesis that the treatments are equally effective.

Treatment	**Number Improved**	**Number not Improved**
1	43	12
2	24	28
3	32	16
4	29	24

14.21 A company is testing four cereals to determine taste preferences of potential buyers. Four different panels of persons are selected independently; one cereal is presented to all members of each panel. After testing, each person is asked if he would purchase the product. The results are shown below. Test the null hypothesis that taste preference is the same for each cereal.

	Cereal			
	A	***B***	***C***	***D***
Number who would buy	75	80	57	80
Number who would not buy	50	60	43	70

15

Summary

In this chapter, we provide a succinct outline of most of the nonparametric procedures covered in this book, organized according to the sampling situation and the inference problem of interest.

Inference	Procedure	Confidence Interval	Chapter
Count Data			
Goodness of fit	Chi-square test	No	3
Equality of two proportions	Normal-approximation test	Yes	14
Equality of k proportions	Chi-square test	No	14
Independence in a Contingency table	Chi-square test	No	14
Success probability (two dependent samples)	McNemar's test	No	14
Success probability (two independent samples)	Fisher's exact test	No	14
Nominal Data			
Randomness	Runs test	No	4
Success = failure	Binomial test	No	5
Time Series Data			
Trend	Runs up and down	No	3
	Runs above and below some constant	No	3
	Rank von Neumann test	No	3
	Kendall's tau, Mann test	No	11
	Spearman's rho, Daniels' test	No	11
Measurement Data, One Sample or Paired Samples			
Quantile	Quantile test	Yes	5
Median	Sign test	Yes	5
Median with symmetry	Wilcoxon signed-rank test	Yes	5
Goodness of fit	Kolgomorov–Smirnov test	Yes	4
	Lilliefors's test	No	4
	Anderson–Darling test	No	4
	P-P or Q-Q plot	No	4

(*continued*)

(continued)

Inference	Procedure	Confidence Interval	Chapter
Measurement Data, Two Independent Samples			
Equality of distributions	Wald–Wolfowitz runs test	No	6
	Kolmogorov–Smirnov test	Yes	6
	Median test	Yes	6
	Control-median test	Yes[c]	6
	Mann–Whitney test	Yes	6
Location	Wilcoxon rank sum test	Yes	8
	Terry–Hoeffding test	No	8
	Van der Waerden test	No	8
	Percentile-modified rank test	No	8
Scale	Mood test	No	9
	Klotz normal scores test	No	9
	Freund–Ansari–Bradley–David–Barton test	No	9
	Siegel–Tukey test	No	9
	Percentile-modified rank test	No	9
	Sukhatme test	Yes	9
	Wilcoxon rank sum test	Yes	9
	Rank-like tests	No	9
	Positive variables test	Yes	9
Measurement Data, Two Related Samples			
Association	Kendall's tau test	No	11
	Spearman's rho test	No	11
Measurement Data, k Independent Samples			
Location	Median test extension	No	10
	Kruskal–Wallis test	Yes[a]	10
Location, ordered alternative	Jonckheere–Terpstra test	No	10
Comparisons with a control	Chakraborti–Desu test	No	10
	Fligner–Wolfe test	No	10
Measurement Data, k Related Samples			
Equal treatment effects	Friedman's two-way Anova	Yes[a]	12
Equal treatment effects, incomplete blocks design	Durbin test	Yes[a]	12
Equal treatment effects, ordered alternative	Page's test	No	12
Agreement, complete rankings	Kendall's coefficient of concordance	Yes[b]	12
Incomplete rankings	Kendall's coefficient of concordance	No	12
Independence	Kendall's partial tau	No	12

[a] Multiple comparisons procedure.
[b] Estimation of true preferential order.
[c] Can be developed as for median test.

Appendix of Tables

Source of Tables

The formats for Tables A, C through J, L through N, have been reprinted with permission from *Nonparametric Methods for Quantitative Analysis* (3rd edn.) by Jean Dickinson Gibbons, copyright © 1976, 1985, 1997 by American Sciences Press, Inc., Syracuse, NY 13224–2144. Permission has been obtained from the original sources, as shown for each table.

TABLE A

Normal Distribution

Each table entry is the tail probability P, right tail from the value of z to $+\infty$, and also left tail from $-\infty$ to $-z$, for all $P \leq .50$. Read down the first column to the first decimal value of z, and over to the correct column for the second decimal value; the number at the intersection is P.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	3821	.3783	.3745	.3707	.3869	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.25147	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.668	.655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

Source: Adapted from Table 1 of Pearson, E.S. and Hartley, H.O. (eds.), *Biometrika Tables for Statisticians*, vol. 1, Cambridge University Press, Cambridge, U.K., 1954. With permission.

TABLE B

Chi-Square Distribution

Each table entry is the value of a chi-square random variable with v degrees of freedom such that its right-tail probability is the value given on the top row. For $v > 30$, a right-tail or left-tail probability for Q a chi-square variable can be found from Table A with Z where $Z = \sqrt{2Q} - \sqrt{2v - 1}$.

	Right-Tail Probability								
v	**0.95**	**0.90**	**0.50**	**0.25**	**0.10**	**0.05**	**0.01**	**0.005**	**0.001**
1	0.004	0.016	0.45	1.32	2.71	3.84	6.63	7.88	10.83
2	0.10	0.21	1.39	2.77	4.61	5.99	9.21	10.60	13.82
3	0.35	0.58	2.37	4.11	6.25	7.81	11.34	12.84	16.27
4	0.71	1.06	3.36	5.39	7.78	9.49	13.28	14.86	18.47
5	1.15	1.61	4.35	6.63	9.24	11.07	15.09	16.75	20.52
6	1.64	2.20	5.35	7.84	10.64	12.59	16.81	18.55	22.46
7	2.17	2.83	6.35	9.04	12.02	14.07	18.48	20.28	24.32
8	2.73	3.49	7.34	10.22	12.36	15.51	20.09	21.96	26.12
9	3.33	4.17	8.34	11.39	14.68	16.92	21.67	23.59	27.88
10	3.94	4.87	9.34	12.55	15.99	18.31	23.21	25.19	29.59
11	4.57	5.58	10.34	13.70	17.28	19.68	24.72	26.76	31.26
12	5.23	6.30	11.34	14.85	18.55	21.03	26.22	28.30	32.91
13	5.89	7.04	12.34	15.98	19.81	22.36	27.69	29.82	34.53
14	6.57	7.79	13.34	17.12	21.06	23.68	29.14	31.32	36.12
15	7.26	8.55	14.34	18.25	22.31	25.00	30.58	32.80	37.70
16	7.96	9.31	15.34	19.37	23.54	26.30	32.00	34.27	39.25
17	8.67	10.09	16.34	20.49	24.77	27.59	33.41	35.72	40.79
18	9.39	10.86	17.34	21.60	25.99	28.87	34.81	37.16	42.31
19	10.12	11.65	18.34	22.72	27.20	30.14	36.19	38.58	43.82
20	10.85	12.44	19.34	23.83	28.41	31.41	37.57	40.00	45.32
21	11.59	13.24	20.34	24.93	29.62	32.67	38.93	41.40	46.80
22	12.34	14.04	21.34	26.04	30.81	33.92	40.29	42.80	48.27
23	13.09	14.85	22.34	27.14	32.01	35.17	41.64	44.18	49.73
24	13.85	15.66	23.34	28.24	33.20	36.42	42.98	45.56	51.18
25	14.61	16.47	24.34	29.34	34.38	37.65	44.31	46.93	52.62
26	15.38	17.29	25.34	30.43	35.56	38.89	45.64	48.29	54.05
27	16.15	18.11	26.34	31.53	36.74	40.11	46.96	49.64	55.48
28	16.93	18.94	27.34	32.62	27.92	41.34	48.28	50.99	56.89
29	17.71	19.77	28.34	33.71	39.09	42.56	49.59	52.34	58.30
30	18.49	20.60	29.34	34.80	40.26	43.77	50.89	53.67	59.70

Source: Adapted from Table 8 of Pearson, E.S. and Hartley, H.O. (eds.), *Biometrika Tables for Statisticians*, vol. 1, Cambridge University Press, Cambridge, U.K., 1954. With permission.

TABLE C

Cumulative Binomial Distribution

Each table entry is the left-tail probability of x or less successes in n Bernouli trials where θ is the probability of a success on each trial.

		θ								
n	x	.05	.10	.15	.20	.25	.30	.35	.40	.45
1	0	.9500	.9000	.8500	.8000	.7500	.7000	.6500	.6000	.5500
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	.9025	.8100	.7225	.6400	.5625	.4900	.4225	.3600	.3025
	1	.9975	.9900	.9775	.9600	.9375	.9100	.8775	.8400	.7975
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	.8574	.7290	.6141	.5120	.4219	.3430	.2746	.2160	.1664
	1	.9928	.9720	.9392	.8960	.8438	.7840	.7182	.6480	.5748
	2	.9999	.9990	.9966	.9920	.9844	.9730	.9561	.9360	.9089
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	.8145	.6561	.5220	.4096	.3164	.2401	.1785	.1296	.0915
	1	.9860	.9477	.8905	.8192	.7383	.6517	.5630	.4752	.3910
	2	.9995	.9963	.9880	.9728	.9492	.9163	.8735	.8208	.7585
	3	1.0000	.9999	.9995	.9984	.9961	.9919	.9850	.9744	.9590
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0	.7738	.5905	.4437	.3277	.2373	.1681	.1160	.0778	.0503
	1	.9774	.9185	.8352	.7373	.6328	.5282	.4284	.3370	.2562
	2	.9988	.9914	.9734	.9421	.8965	.8369	.7648	.6826	.5931
	3	1.0000	.9995	.9978	.9933	.9844	.9692	.9460	.9130	.8688
	4	1.0000	1.0000	.9999	.9997	.9990	.9976	.9947	.9898	.9815
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	0	.7351	.5314	.3771	.2621	.1780	.1176	.0754	.0467	.0277
	1	.9672	.8857	.7765	.6554	.5339	.4202	.3191	.2333	.1636
	2	.9978	.9842	.9527	.9011	.8306	.74403	.6471	.5443	.4415
	3	.9999	.9987	.9941	.9830	.9624	.9295	.8826	.8208	.7447
	4	1.0000	.9999	.9996	.9984	.9954	.9891	.9777	.9590	.9308
	5	1.0000	1.0000	1.0000	.9999	.9998	.9993	.9982	.9959	.9917
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	0	.6983	.4783	.3206	.2097	.1335	.0824	.0490	.0280	.0152
	1	.9556	.8503	.7166	.5767	.4449	.3294	.2338	.1586	.1024
	2	.9962	.9743	.9262	.8520	.7564	.6471	.5323	.4199	.3164
	3	.9998	.9973	.9879	.9667	.9294	.8740	.8002	.7102	.6083
	4	1.0000	.9998	.9988	.9953	.9871	.9712	.9444	.9037	.8471
	5	1.0000	1.0000	.9999	.9996	.9987	.9962	.9910	.9812	.9643
	6	1.0000	1.0000	1.0000	1.0000	.9999	.9998	.9994	.9984	.9963
	7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	**.50**	**.55**	**.60**	**.65**	**.70**	**.75**	**.80**	**.85**	**.90**	**.95**
1	0	.5000	.4500	.4000	.3500	.3000	.2500	.2000	.1500	.1000	.0500
	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	0	.2500	.2025	.1600	.1225	.0900	.0625	.0400	.0225	.0100	.0025
	1	.7500	.6975	.6400	.5775	.5100	.4375	.3600	.2775	.1900	.0975
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	0	.1250	.0911	.0640	.0429	.0270	.0156	.0080	.0034	.0010	.0001
	1	.5000	.4252	.3520	.2818	.2160	.1562	.1040	.0608	.0280	.0072
	2	.8750	.8336	.7840	.7254	.6570	.5781	.4880	.3859	.2710	.1426
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	0	.0625	.0410	.0256	.0150	.0081	.0039	.0016	.0005	.0001	.0000
	1	.3125	.2415	.1792	.1265	.0837	.0508	.0272	.0120	.0037	.0005
	2	.6875	.6090	.5248	.4370	.3483	.2617	.1808	.1095	.0523	.0140
	3	.9375	.9085	.8704	.8215	.7599	.6836	.5904	.4780	.3439	.1855
	4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0	.0312	.0185	.0102	.0053	.0024	.0010	.0003	.0001	.0000	.0000
	1	.1875	.1312	.0870	.0540	.0308	.0156	.0067	.0022	.0005	.0000
	2	.5000	.4069	.3174	.2352	.1631	.1035	.0579	.0266	.0086	.0012
	3	.8125	.7438	.6630	.5716	.4718	.3672	.2627	.1648	.0815	.0226
	4	.9688	.9497	.9222	.8840	.8319	.7627	.6723	.5563	.4095	.2262
	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	0	.0156	.0083	.0041	.0018	.0007	.0002	.0001	.0000	.0000	.0000
	1	.1094	.0692	.0410	.0223	.0109	.0046	.0016	.0004	.0001	.0000
	2	.3438	.2553	.1792	.1174	.0705	.0376	.0170	.0059	.0013	.0001
	3	.6562	.5585	.4557	.3529	.2557	.1694	.0989	.0473	.0158	.0022
	4	.8906	.8364	.7667	.6809	.5798	.4661	.3446	.2235	.1143	.0328
	5	.9844	.9723	.9533	.9246	.8824	.8220	.7379	.6229	.4686	.2649
	6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
7	0	.0078	.0037	.0016	.0006	.0002	.0001	.0000	.0000	.0000	.0000
	1	.0625	.0357	.0188	.0090	.0038	.0013	.0004	.0001	.0000	.0000
	2	.2266	.1529	.0963	.0556	.0288	.0129	.0047	.0012	.0002	.0000
	3	.5000	.3917	.2898	.1998	.1260	.0706	.0333	.0121	.0027	.0002
	4	.7734	.6836	.5801	.4677	.3529	.2436	.1480	.0738	.0257	.0038
	5	.9375	.8976	.8414	.7662	.6706	.5551	.4233	.2834	.1497	.0444
	6	.9922	.9848	.9720	.9510	.9176	.8665	.7903	.6794	.5217	.3017
	7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ								
n	x	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**
8	0	.6634	.4305	.2725	.1678	.1001	.0576	.0319	.0168	.0084
	1	.9428	.8131	.6572	.5033	.3671	.2553	.1691	.1064	.0632
	2	.9942	.9619	.8948	.7969	.6785	.5518	.4278	.3154	.2201
	3	.9996	.9950	.9786	.9437	.8862	.8059	.7064	.5941	.4770
	4	1.0000	.9996	.9971	.9896	.9727	.9420	.8939	.8263	.7396
	5	1.0000	1.0000	.9998	.9988	.9958	.9887	.9747	.9502	.9115
	6	1.0000	1.0000	1.0000	.9999	.9996	.9987	.9964	.9915	.9819
	7	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9998	.9993	.9983
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0	.6302	.3874	.2316	.1342	.0751	.0404	.0207	.0101	.0046
	1	.9288	.7748	.5995	.4362	.3003	.1960	.1211	.0705	.0385
	2	.9916	.9470	.8591	.7382	.6007	.4628	.3373	.2318	.1495
	3	.9994	.9917	.9661	.9144	.8343	.7297	.6089	.4826	.3614
	4	1.0000	.9991	.9944	.9804	.9511	.9012	.8283	.7334	.6214
	5	1.0000	.9999	.9994	.9969	.9900	.9747	.9464	.9006	.8342
	6	1.0000	1.0000	1.0000	.9997	.9987	.9957	.9888	.9750	.9502
	7	1.0000	1.0000	1.0000	1.0000	.9999	.9996	.9986	.9962	.9909
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997	.9992
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	.5987	.3487	.1969	.1074	.0563	.0282	.0135	.0060	.0025
	1	.9139	.7361	.5443	.3758	.2440	.1493	.0860	.0464	.0233
	2	.9885	.9298	.8202	.6778	.5256	.3828	.2616	.1673	.0996
	3	.9990	.9872	.9500	.8791	.7759	.6496	.5138	.3823	.2660
	4	.9999	.9984	.9901	.9672	.9219	.8497	.7515	.6331	.5044
	5	1.0000	.9999	.9986	.9936	.9803	.9527	.9051	.8338	.7384
	6	1.0000	1.0000	.9999	.9991	.9965	.9894	.9740	.9452	.8980
	7	1.0000	1.0000	1.0000	.9999	.9996	.9984	.9952	.9877	.9726
	8	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9995	.9983	.9955
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	0	.5688	.3138	.1673	.0859	.0422	.0198	.0088	.0036	.0014
	1	.8981	.6974	.4922	.3221	.1971	.1130	.0606	.0302	.0139
	2	.9848	.9104	.7788	.6174	.4552	.3127	.2001	.1189	.0652
	3	.9984	.9815	.9306	.8389	.7133	.5696	.4256	.2963	.1911
	4	.9999	.9972	.9841	.9496	.8854	.7897	.6683	.5328	.3971
	5	1.0000	.9997	.9973	.9883	.9657	.9218	.8513	.7535	.6331
	6	1.0000	1.0000	.9997	.9980	.9924	.9784	.9499	.9006	.8262
	7	1.0000	1.0000	1.0000	.9998	.9988	.9957	.9878	.9707	.9390
	8	1.0000	1.0000	1.0000	1.0000	.9999	.9994	.9980	.9941	.9852
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9993	.9978
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	**.50**	**.55**	**.60**	**.65**	**.70**	**.75**	**.80**	**.85**	**.90**	**.95**
8	0	.0039	.0017	.0007	.0002	.0001	.0000	.0000	.0000	.0000	.0000
	1	.0352	.0181	.0085	.0036	.0013	.0004	.0001	.0000	.0000	.0000
	2	.1445	.0885	.0498	.0253	.0113	.0042	.0012	.0002	.0000	.0000
	3	.3633	.2604	.1737	.1061	.0580	.0273	.0104	.0029	.0004	.0000
	4	.6367	.5230	.4059	.2936	.1941	.1138	.0563	.0214	.0050	.0004
	5	.8555	.7799	.6846	.5722	.4482	.3215	.2031	.1052	.0381	.0058
	6	.9648	.9368	.8936	.8309	.7447	.6329	.4967	.3428	.1869	.0572
	7	.9961	.9916	.9832	.9681	.9424	.8999	.8322	.7275	.5695	.3366
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0	.0020	.0008	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0195	.0091	.0038	.0014	.0004	.0001	.0000	.0000	.0000	.0000
	2	.0898	.0498	.0250	.0112	.0043	.0013	.0003	.0000	.0000	.0000
	3	.2539	.1658	.0994	.0536	.0253	.0100	.0031	.0006	.0001	.0000
	4	.5000	.3786	.2666	.1717	.0988	.0489	.0196	.0056	.0009	.0000
	5	.7461	.6386	.5174	.3911	.2703	.1657	.0856	.0339	.0083	.0006
	6	.9102	.8505	.7682	.6627	.5372	.3993	.2618	.1409	.0530	.0084
	7	.9805	.9615	.9295	.8789	.8040	.6997	.5638	.4005	.2252	.0712
	8	.9980	.9954	.9899	.9793	.9596	.9249	.8658	.7684	.6126	.3698
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	0	.0010	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0107	.0045	.0017	.0005	.0001	.0000	.0000	.0000	.0000	.0000
	2	.0547	.0274	.0123	.0048	.0016	.0004	.0001	.0000	.0000	.0000
	3	.1719	.1020	.0548	.0260	.0106	.0035	.0009	.0001	.0000	.0000
	4	.3770	.2616	.1662	.0949	.0473	.0197	.0064	.0014	.0001	.0000
	5	.6230	.4956	.3669	.2485	.1503	.0781	.0328	.0099	.0016	.0001
	6	.8281	.7340	.6177	.4862	.3504	.2241	.1209	.0500	.0128	.0010
	7	.9453	.9004	.8327	.7384	.6172	.4744	.3222	.1798	.0702	.0115
	8	.9893	.9767	.9536	.9140	.8507	.7560	.6242	.4557	.2639	.0861
	9	.9990	.9975	.9940	.9865	.9718	.9437	.8926	.8031	.6513	.4013
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
11	0	.0005	.0002	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0059	.0022	.0007	.0002	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0327	.0148	.0059	.0020	.0006	.0001	.0000	.0000	.0000	.0000
	3	.1133	.0610	.0293	.0122	.0043	.0012	.0002	.0000	.0000	.0000
	4	.2744	.1738	.0994	.0501	.0216	.0076	.0020	.0003	.0000	.0000
	5	.5000	.3669	.2465	.1487	.0782	.0343	.0117	.0027	.0003	.0000
	6	.7256	.6029	.4672	.3317	.2103	.1146	.0504	.0159	.0028	.0001
	7	.8867	.8089	.7037	.5744	.4304	.2867	.1611	.0694	.0185	.0016
	8	.9673	.9348	.8811	.7999	.6873	.5448	.3826	.2212	.0896	.0152
	9	.9941	.9861	.9698	.9394	.8870	.8029	.6779	.5078	.3026	.1019
	10	.9995	.9986	.9964	.9912	.9802	.9578	.9141	.8327	.6862	.4312
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ								
n	x	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**
12	0	.5404	.2824	.1422	.0687	.0317	.0138	.0057	.0022	.0008
	1	.8816	.6590	.4435	.2749	.1584	.0850	.0424	.0196	.0083
	2	.9804	.8891	.7358	.5583	.3907	.2528	.1513	.0834	.0421
	3	.9978	.9744	.9078	.7946	.6488	.4925	.3467	.2253	.1345
	4	.9998	.9957	.9761	.9274	.8424	.7237	.5833	.4382	.3044
	5	1.0000	.99995	.9954	.9806	.9456	.8822	.7873	.6652	.5269
	6	1.0000	.9999	.9993	.9961	.9857	.9614	.9154	.8418	.7393
	7	1.0000	1.0000	.9999	.9994	.9972	.9905	.9745	.9427	.8883
	8	1.0000	1.0000	1.0000	.9999	.9996	.9983	.9944	.9847	.9644
	9	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9992	.9972	.9921
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997	.9989
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
13	0	.5133	.2542	.1209	.0550	.0238	.0097	.0037	.0013	.0004
	1	.8646	.6213	.3983	.2336	.1267	.0637	.0296	.0126	.0049
	2	.9755	.8661	.7296	.5017	.3326	.2025	.1132	.0579	.0269
	3	.9969	.9658	.9033	.7473	.5843	.4206	.2783	.1686	.0929
	4	.9997	.9935	.9740	.9009	.7940	.6543	.5005	.3530	.2279
	5	1.000	.9991	.9947	.9700	.9198	.8346	.7159	.5744	.4268
	6	1.0000	.9999	.9987	.9930	.9757	.9376	.8705	.7712	.6437
	7	1.0000	1.0000	.9998	.9988	.9944	.9818	.9538	.9023	.8212
	8	1.0000	1.0000	1.0000	.9998	.9990	.9960	.9874	.9679	.9302
	9	1.0000	1.0000	1.0000	1.0000	.9999	.9993	.9975	.9922	.9797
	10	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997	.9987	.9959
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9995
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	0	.4877	.2288	.1028	.0440	.0178	.0068	.0024	.0008	.0002
	1	.8470	.5846	.3567	.1979	.1010	.0475	.0205	.0081	.0029
	2	.9699	.8416	.6479	.4481	.2811	.1608	.0839	.0398	.0170
	3	.9958	.9559	.8535	.6982	.5213	.3552	.2205	.1243	.0632
	4	.9996	.9908	.9533	.8702	.7415	.5842	.4227	.2793	.1672
	5	1.0000	.9985	.9885	.9561	.8883	.7805	.6405	.4859	.3373
	6	1.0000	.9998	.9978	.9884	.9617	.9067	.8164	.6925	.5461
	7	1.0000	1.0000	.9997	.9976	.9897	.9685	.9247	.8499	.7414
	8	1.0000	1.0000	1.0000	.9996	.9978	.9917	.9757	.9417	.8811
	9	1.0000	1.0000	1.0000	.9996	.9978	.9917	.9757	.9417	.8811
	10	1.0000	1.0000	1.0000	1.0000	1.0000	.9983	.9940	.9825	.9574
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9989	.9961	.9886
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
12	0	.0002	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0032	.0011	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0193	.0079	.0028	.0008	.0002	.0000	.0000	.0000	.0000	.0000
	3	.0730	.0356	.0153	.0056	.0017	.0004	.0001	.0000	.0000	.0000
	4	.1938	.1117	.0573	.0255	.0095	.0028	.0006	.0001	.0000	.0000
	5	.3872	.2607	.1582	.0846	.0386	.0143	.0039	.0007	.0001	.0000
	6	.6128	.4731	.3348	.2127	.1178	.0544	.0194	.0046	.0005	.0000
	7	.8062	.6956	.5618	.4167	.2763	.1576	.0726	.0239	.0043	.0002
	8	.9270	.8655	.7747	.6533	.5075	.3512	.2054	.0922	.0256	.0022
	9	.9807	.9579	.9166	.8487	.7472	.6093	.4417	.2642	.1109	.0196
	10	.9968	.9917	.9804	.9576	.9150	.8416	.7251	.5565	.3410.	.1184
	11	.9998	.9992	.9978	.9943	.9862	.9683	.9313	.8578	.7176	.4596
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
13	0	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0017	.0005	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0112	.0041	.0013	.0003	.0001	.0000	.0000	.0000	.0000	.0000
	3	.0461	.0203	.0078	.0025	.0007	.0001	.0000	.0000	.0000	.0000
	4	.1334	.0698	.0321	.0126	.0040	.0010	.0002	.0000	.0000	.0000
	5	.2905	.1788	.0977	.0462	.0182	.0056	.0012	.0002	.0000	.0000
	6	.5000	.3563	.2288	.1295	.0624	.0243	.0070	.0013	.0001	.0000
	7	.7095	.5732	.4256	.2841	.1654	.0802	.0300	.0053	.0009	.0000
	8	.8666	.7721	.6470	.4995	.3457	.2060	.0991	.0260	.0065	.0003
	9	.9539	.9071	.8314	.7217	.5794	.4157	.2527	.0967	.0342	.0031
	10	.9888	.9731	.9421	.8868	.7975	.6674	.4983	.2704	.1339	.0245
	11	.9983	.9951	.9874	.9704	.9363	.8733	.7664	.6017	.3787	.1354
	12	.9999	.9996	.9987	.9963	.9903	.9762	.9450	.8791	.7458	.4867
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
14	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0009	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0065	.0022	.0006	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0287	.0114	.0039	.0011	.0002	.0000	.0000	.0000	.0000	.0000
	4	.0898	.0462	.0175	.0060	.0017	.0003	.0000	.0000	.0000	.0000
	5	.2120	.1189	.0583	.0243	.0083	.0022	.0004	.0000	.0000	.0000
	6	.3953	.2586	.1501	.0753	.0315	.0103	.0024	.0003	.0000	.0000
	7	.6047	.4539	.3075	.1836	.0933	.0383	.0116	.0022	.0002	.0000
	8	.7880	.6627	.5141	.3595	.2195	.1117	.0439	.0115	.0015	.0000
	9	.9102	.8328	.7207	.5773	.4158	.2585	.1298	.0467	.0092	.0004
	10	.9713	.9368	.8757	.7795	.6448	.4787	.3018	.1465	.0441	.0042
	11	.9935	.9830	.9602	.9161	.8392	.7189	.5519	.3521	.1584	.0301
	12	.9991	.9971	.9919	.9795	.9525	.8990	.8021	.6433	.4154	.1530
	13	.9999	.9998	.9992	.9976	.9932	.9822	.9560	.8972	.7712	.5123
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ								
n	x	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**
15	0	.4633	.2059	.0874	.0352	.0134	.0047	.0016	.0005	.0001
	1	.8290	.5490	.3186	.1671	.0802	.0353	.0142	.0052	.0017
	2	.9638	.8159	.6042	.3980	.2361	.1268	.0617	.0271	.0107
	3	.9945	.9444	.8227	.6482	.4613	.2969	.1727	.0905	.0424
	4	.9994	.9873	.9383	.8358	.6865	.5155	.3519	.2173	.1204
	5	.9999	.9978	.9832	.9389	.8516	.7216	.5643	.4032	.2608
	6	1.0000	.9997	.9964	.9819	.9434	.8689	.7548	.6098	.4522
	7	1.0000	1.0000	.9994	.9958	.9827	.9500	.8868	.7869	.6535
	8	1.0000	1.0000	.9999	.9992	.9958	.9848	.9578	.9050	.8182
	9	1.0000	1.0000	1.0000	.9999	.9992	.9963	.9876	.9662	.9231
	10	1.0000	1.0000	1.0000	1.0000	.9999	.9993	.9972	.9907	.9745
	11	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9995	.9981	.9937
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997	.9989
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
16	0	.4401	.1853	.0743	.0281	.0100	.0033	.0010	.0003	.0001
	1	.8108	.5147	.2839	.1407	.0635	.0261	.0098	.0033	.0010
	2	.9571	.7592	.5614	.3518	.1971	.0994	.0451	.0183	.0066
	3	.9930	.9316	.7899	.5981	.4050	.2459	.1339	.0651	.0281
	4	.9991	.9830	.9209	.7982	.6302	.4499	.2892	.1666	.0853
	5	.9999	.9967	.9765	.9183	.8103	.6598	.4900	.3288	.1976
	6	1.0000	.9995	.9944	.9733	.9204	.8247	.6881	.5272	.3660
	7	1.0000	.9999	.9989	.9930	.9729	.9256	.8406	.7161	.5629
	8	1.0000	1.0000	.9998	.9985	.9925	.9743	.9329	.8577	.7441
	9	1.0000	1.0000	1.0000	.9998	.9984	.9929	.9771	.9417	.8759
	10	1.0000	1.0000	1.0000	1.0000	.9997	.9984	.9938	.9809	.9514
	11	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9987	.9951	.9851
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9991	.9965
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9991	.9994
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
15	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0005	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0037	.0011	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0176	.0063	.0019	.0005	.0001	.0000	.0000	.0000	.0000	.0000
	4	.0592	.0255	.0093	.0028	.0007	.0001	.0000	.0000	.0000	.0000
	5	.1509	.0769	.0338	.0124	.0037	.0008	.0001	.0000	.0000	.0000
	6	.3036	.1818	.0950	.0422	.0152	.0042	.0008	.0001	.0000	.0000
	7	.5000	.3465	.2131	.1132	.0500	.0173	.0042	.0006	.0000	.0000
	8	.6964	.5478	.3902	.2452	.1311	.0566	.0181	.0036	.0003	.0000
	9	.8491	.7392	.5968	.4357	.2784	.1484	.0611	.0168	.0022	.0001
	10	.9408	.8796	.7827	.6481	.4845	.3135	.1642	.0617	.0127	.0006
	11	.9824	.9576	.9095	.8273	.7031	.5387	.3518	.1773	.0556	.0055
	12	.9963	.9893	.9729	.9383	.8732	.7639	.6020	.3958	.1841	.0362
	13	.9995	.9983	.9948	.9858	.9647	.9198	.8329	.6814	.4510	.1710
	14	1.0000	.9999	.9995	.9984	.9953	.9866	.9648	.9126	.7941	.5367
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
16	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0021	.0006	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0106	.0035	.0009	.0002	.0000	.0000	.0000	.0000	.0000	.0000
	4	.0384	.0149	.0049	.0013	.0003	.0000	.0000	.0000	.0000	.0000
	5	.1051	.0486	.0191	.0062	.0016	.0003	.0000	.0000	.0000	.0000
	6	.2272	.1241	.0583	.0229	.0071	.0016	.0002	.0000	.0000	.0000
	7	.4018	.2559	.1423	.0671	.0257	.0075	.0015	.0002	.0000	.0000
	8	.5982	.4371	.2839	.1594	.0744	.0271	.0070	.0011	.0001	.0000
	9	.7728	.6340	.4728	.3119	.1753	.0796	.0267	.0056	.0005	.0000
	10	.8949	.8024	.6712	.5100	.3402	.1897	.0817	.0235	.0033	.0001
	11	.9616	.9147	.8334	.7108	.5501	.3698	.2018	.0791	.0170	.0009
	12	.9894	.9719	.9349	.8661	.7541	.5950	.4019	.2101	.0684	.0070
	13	.9979	.9934	.9817	.9549	.9006	.8729	.6482	.4386	.2108	.0429
	14	.9997	.9990	.9967	.9902	.9739	.9365	.8593	.7161	.4853	.1892
	15	1.0000	.9999	.9997	.9990	.9967	.9900	.9719	.9257	.8147	.5599
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ								
n	x	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**
17	0	.4181	.1668	.0631	.0225	.0075	.0023	.0007	.0002	.0000
	1	.7922	.4818	.2525	.1182	.0501	.0193	.0067	.0021	.0006
	2	.9497	.7618	.5198	.3096	.1637	.0774	.0327	.0123	.0041
	3	.9912	.9174	.7556	.5489	.3530	.2019	.1028	.0464	.0184
	4	.9988	.9779	.9013	.7582	.5739	.3887	.2348	.1260	.0596
	5	.9999	.9953	.9681	.8943	.7653	.5968	.4197	.2639	.1471
	6	1.0000	.9992	.9917	.9623	.8929	.7752	.6188	.4478	.2902
	7	1.0000	.9999	.9983	.9891	.9598	.8954	.7872	.6405	.4743
	8	1.0000	1.0000	.9997	.9974	.9876	.9597	.9006	.8011	.6626
	9	1.0000	1.0000	1.0000	.9995	.9969	.9873	.9617	.9081	.8166
	10	1.0000	1.0000	1.0000	.9999	.9994	.9968	.9880	.9652	.9174
	11	1.0000	1.0000	1.0000	1.0000	.9999	.9993	.9970	.9894	.9699
	12	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9994	.9975	.9914
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9995	.9981
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9997
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	0	.3972	.1501	.0536	.0180	.0056	.0016	.0004	.0001	.0000
	1	.7735	.4503	.2241	.0991	.0395	.0142	.0046	.0013	.0003
	2	.9419	.7338	.4797	.2713	.1353	.0600	.0236	.0082	.0025
	3	.9891	.9018	.7202	.5010	.3057	.1646	.0783	.0328	.0120
	4	.9985	.9718	.8794	.7164	.5187	.3327	.1886	.0942	.0411
	5	.9998	.9936	.9581	.8671	.7175	.5344	.3550	.2088	.1077
	6	1.0000	.9988	.9882	.9487	.8610	.7217	.5491	.3743	.2258
	7	1.0000	.9998	.9973	.9837	.9431	.8593	.7283	.5634	.3915
	8	1.0000	1.0000	.9995	.9957	.9807	.9404	.8609	.7368	.5778
	9	1.0000	1.0000	.9999	.9991	.9946	.9790	.9403	.8653	.7473
	10	1.0000	1.0000	1.0000	.9998	.9988	.9939	.9788	.9424	.8720
	11	1.0000	1.0000	1.0000	1.0000	.9998	.9986	.9938	.9797	.9463
	12	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9986	.9942	.9817
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9987	.9951
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9998	.9990
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	.50	.55	.60	.65	.70	.75	.80	.85	.90	.95
17	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0012	.0003	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0064	.0019	.0005	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	4	.0245	.0086	.0025	.0006	.0001	.0000	.0000	.0000	.0000	.0000
	5	.0717	.0301	.0106	.0030	.0007	.0001	.0000	.0000	.0000	.0000
	6	.1662	.0826	.0348	.0120	.0032	.0006	.0001	.0000	.0000	.0000
	7	.3145	.1834	.0919	.0383	.0127	.0031	.0005	.0000	.0000	.0000
	8	.5000	.3374	.1989	.0994	.0403	.0124	.0026	.0003	.0000	.0000
	9	.6855	.5257	.3595	.2128	.1046	.0402	.0109	.0017	.0001	.0000
	10	.8338	.7098	.5522	.3812	.2248	.1071	.0377	.0083	.0008	.0000
	11	.9283	.8529	.7361	.5803	.4032	.2347	.1057	.0319	.0047	.0001
	12	.9755	.9404	.8740	.7652	.6113	.4261	.2418	.0987	.0221	.0012
	13	.9936	.9816	.9536	.8972	.7981	.6470	.4511	.2444	.0826	.0088
	14	.9988	.9959	.9877	.9673	.9226	.8363	.6904	.4802	.2382	.0503
	15	.9999	.9994	.9979	.9933	.9807	.9499	.8818	.7475	.5182	.2078
	16	1.0000	1.0000	.9998	.9993	.9977	.9925	.9775	.9369	.8332	.5819
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0007	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0038	.0010	.0002	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	4	.0154	.0049	.0013	.0003	.0000	.0000	.0000	.0000	.0000	.0000
	5	.0481	.0183	.0058	.0014	.0003	.0000	.0000	.0000	.0000	.0000
	6	.1189	.0537	.0203	.0062	.0014	.0002	.0000	.0000	.0000	.0000
	7	.2403	.1280	.0576	.0212	.0061	.0012	.0002	.0000	.0000	.0000
	8	.4073	.2527	.1347	.0597	.0210	.0054	.0009	.0001	.0000	.0000
	9	.5927	.4222	.2632	.1391	.0596	.0193	.0043	.0005	.0000	.0000
	10	.7597	.6085	.4366	.2717	.1407	.0569	.0163	.0027	.0002	.0000
	11	.8811	.7742	.6257	.4509	.2783	.1390	.0513	.0118	.0012	.0000
	12	.9519	.8923	.7912	.6450	.4656	.2825	.1329	.0419	.0064	.0002
	13	.9846	.9589	.9058	.8114	.6673	.4813	.2836	.1206	.0282	.0015
	14	.9962	.9880	.9672	.9217	.8354	.6943	.4990	.2798	.0982	.0109
	15	.9993	.9975	.9918	.9764	.9400	.8647	.7287	.5203	.2662	.0581
	16	.9999	.9997	.9987	.9954	.9858	.9605	.9009	.7759	.5497	.2265
	17	1.0000	1.0000	.9999	.9996	.9984	.9944	.9820	.9464	.8499	.6028
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ								
n	x	.05	.10	.15	.20	.25	.30	.35	.40	.45
19	0	.3774	.1351	.0456	.0144	.0042	.0011	.0003	.0001	.0000
	1	.7547	.4203	.1985	.0829	.0310	.0104	.0031	.0008	.0002
	2	.9335	.7054	.4413	.2369	.1113	.0462	.0170	.00055	.0015
	3	.9868	.8850	.6841	.4551	.2631	.1332	.0591	.0230	.0077
	4	.9980	.9648	.8556	.6733	.4654	.2822	.1500	.0696	.0280
	5	.9998	.9914	.9463	.8369	.6678	.4739	.2968	.1629	.0777
	6	1.0000	.9983	.9837	.9324	.8251	.6655	.4812	.3081	.1727
	7	1.0000	.9997	.9959	.9767	.9225	.8180	.6656	.4878	.3169
	8	1.0000	1.0000	.9992	.9933	.9713	.9161	.8145	.6675	.4940
	9	1.0000	1.0000	.9999	.9984	.9911	.9674	.9125	.8139	.6710
	10	1.0000	1.0000	1.0000	.9997	.9977	.9895	.9653	.9115	.8159
	11	1.0000	1.0000	1.0000	1.0000	.9995	.9972	.9886	.9648	.9129
	12	1.0000	1.0000	1.0000	1.0000	.9999	.9994	.9969	.9884	.9658
	13	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9993	.9969	.9891
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9994	.9972
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9999	.9995
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9995
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	0	.3585	.1216	.0388	.0115	.0032	.0008	.0002	.0000	.0000
	1	.7358	.3917	.1756	.0692	.0243	.0076	.0021	.0005	.0001
	2	.9245	.6769	.4049	.2061	.0913	.0355	.0121	.0036	.0009
	3	.9841	.8670	.6477	.4114	.2252	.1071	.0444	.0160	.0049
	4	.9974	.9568	.8298	.6296	.4148	.2375	.1182	.0510	.0189
	5	.9997	.9887	.9327	.8042	.6172	.4164	.2454	.1256	.0553
	6	1.0000	.9976	.9781	.9133	.7858	.6080	.4166	.2500	.1299
	7	1.0000	.9996	.9941	.9679	.8982	.7723	.6010	.4159	.2520
	8	1.0000	.9999	.9987	.9900	.9591	.8867	.7624	.5956	.4143
	9	1.0000	1.0000	.9998	.9974	.9861	.9520	.8782	.7553	.5914
	10	1.0000	1.0000	1.0000	.9994	.9961	.9829	.9468	.8725	.7507
	11	1.0000	1.0000	1.0000	.9999	.9991	.9949	.9804	.9435	.8692
	12	1.0000	1.0000	1.0000	1.0000	.9998	.9987	.9940	.9790	.9420
	13	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9985	.9935	.9786
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9984	.9936
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997	.9985
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	.9997
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	x	**.50**	**.55**	**.60**	**.65**	**.70**	**.75**	**.80**	**.85**	**.90**	**.95**
19	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0004	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0022	.0005	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	4	.0096	.0028	.0006	.0001	.0000	.0000	.0000	.0000	.0000	.0000
	5	.0318	.0109	.0031	.0007	.0001	.0000	.0000	.0000	.0000	.0000
	6	.0835	.0342	.0116	.0031	.0006	.0001	.0000	.0000	.0000	.0000
	7	.1796	.0871	.0352	.0114	.0028	.0005	.0000	.0000	.0000	.0000
	8	.3238	.1841	.0885	.0347	.0105	.0023	.0003	.0000	.0000	.0000
	9	.5000	.3290	.1861	.0875	.0326	.0089	.0016	.0001	.0000	.0000
	10	.6762	.5060	.3325	.1855	.0839	.0287	.0067	.0008	.0000	.0000
	11	.8204	.6831	.5122	.3344	.1820	.0775	.0233	.0041	.0003	.0000
	12	.9165	.8273	.6919	.5188	.3345	.1749	.0676	.0163	.0017	.0000
	13	.9682	.9223	.8371	.7032	.5261	.3322	.1631	.0537	.0086	.0002
	14	.9904	.9720	.9304	.8500	.7178	.5346	.3267	.1444	.0352	.0020
	15	.9978	.9923	.9770	.9409	.8668	.7369	.5449	.3159	.1150	.0132
	16	.9996	.9985	.9945	.9830	.9538	.8887	.7631	.5587	.2946	.0665
	17	1.0000	.9998	.9992	.9969	.9896	.9690	.9171	.8015	.5797	.2453
	18	1.0000	1.0000	.9999	.9997	.9989	.9958	.9856	.9544	.8649	.6226
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	1	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	2	.0002	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	3	.0013	.0003	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	4	.0059	.0015	.0003	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	5	.0207	.0064	.0016	.0003	.0000	.0000	.0000	.0000	.0000	.0000
	6	.0577	.0214	.0065	.0015	.0003	.0000	.0000	.0000	.0000	.0000
	7	.1316	.0580	.0210	.0060	.0013	.0002	.0000	.0000	.0000	.0000
	8	.2517	.1308	.0565	.0196	.0051	.0009	.0001	.0000	.0000	.0000
	9	.4119	.2493	.1275	.0532	.0171	.0039	.0006	.0000	.0000	.0000
	10	.5881	.4086	.2447	.1218	.0480	.0139	.0026	.0002	.0000	.0000
	11	.7483	.5857	.4044	.2376	.1133	.0409	.0100	.0013	.0001	.0000
	12	.8684	.7480	.5841	.3990	.2277	.1018	.0321	.0059	.0004	.0000
	13	.9423	.8701	.7500	.5834	.3920	.2142	.0867	.0219	.0024	.0000
	14	.9793	.9447	.8744	.7546	.5836	.3838	.1958	.0673	.0113	.0003
	15	.9941	.9811	.9490	.8818	.7625	.5552	.3704	.1702	.0432	.0026
	16	.9987	.9951	.9840	.9556	.8929	.7748	.5886	.3523	.1330	.0159

(*continued*)

TABLE C (continued)

Cumulative Binomial Distribution

		θ									
n	***x***	**.50**	**.55**	**.60**	**.65**	**.70**	**.75**	**.80**	**.85**	**.90**	**.95**
20	17	.9998	.9991	.9964	.9879	.9645	.9087	.7939	.5951	.3231	.0755
	18	1.0000	.9999	.9995	.9879	.9924	.9757	.9308	.8244	.6083	.2642
	19	1.0000	1.0000	1.0000	.9998	.9992	.9968	.9885	.9612	.8784	.6415
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Source: Adapted from Table 2 of *Tables of the Binomial Distribution* (January 1950 with Carrigenda 1952 and 1958), National Bureau of Standards, U.S. Governments Printing Office, Washington, DC. With permission.

TABLE D

Total Number of Runs Distribution

Each table entry labeled P is the tail probability from each extreme to the value of R, the total number of runs in a sequence of $r = n_1 + n_2$ symbols of two types for $n_1 \leq n_2$.

Left-Tail Probabilities

n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P
2	2	2	.333	2	18	2	.011	3	14	2	.003	4	10	2	.002
2	3	2	.200			3	.105			3	.025			3	.014
		3	.500			4	.284			4	.101			4	.068
2	4	2	.133	3	3	2	.100			5	.350			5	.203
		3	.400			3	.300	3	15	2	.002			6	.419
2	5	2	.095	3	4	2	.057			3	.022	4	11	2	.001
		3	.333			3	.200			4	.091			3	.011
2	6	2	.071	3	5	2	.036			5	.331			4	.055
		3	.286			3	.143	3	16	2	.002			5	.176
2	7	2	.056			4	.429			3	.020			6	.374
		3	.250	3	6	2	.024			4	.082	4	12	2	.001
2	8	2	.044			3	.107			5	.314			3	.009
		3	.222			4	.345	3	17	2	.002			4	.045
2	9	2	.036	3	7	2	.017			3	.018			5	.154
		3	.200			3	.083			4	.074			6	.335
		4	.491			4	.283			5	.298	4	13	2	.001
2	10	2	.030	3	8	2	.012	4	4	2	.029			3	.007
		3	.182			3	.067			3	.114			4	.037
		4	.455			4	.236			4	.371			5	.136
2	11	2	.026	3	9	2	.009	4	5	2	.016			6	.302
		3	.167			3	.055			3	.071	4	14	2	.001
		4	.423			4	.200			4	.262			3	.006
2	12	2	.022			5	.491			5	.500			4	.031
		3	.154	3	10	2	.007	4	6	2	.010			5	.121
		7	.396			3	.045			3	.048			6	.274
2	13	2	.019			4	.171			4	.190	4	15	2	.001
		3	.143			5	.455			5	.405			3	.005
		4	.371	3	11	2	.005	4	7	2	.006			4	.027
2	14	2	.017			3	.038			3	.033			5	.108
		3	.133			4	.148			4	.142			6	.249
		4	.250			5	.423			5	.333	4	16	2	.000
2	15	2	.015	3	12	2	.004	4	8	2	.004			3	.004
		3	.125			3	.033			3	.024			4	.023
		4	.331			4	.130			4	.109			5	.097
2	16	2	.013			5	.396			5	.279			6	.227
		3	.118	3	13	2	.004	4	9	2	.003	5	5	2	.008
		4	.314			3	.029			3	.018			3	.040
2	17	2	.012			4	.114			4	.085			4	.167
		3	.111			5	.371			5	.236			5	.357
		4	.298							6	.471				

(*continued*)

TABLE D (continued)

Total Number of Runs Distribution

Left-Tail Probabilities

n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P
5	6	2	.004	5	14	2	.000	6	11	2	.000	7	9	2	.000
		3	.024			3	.002			3	.001			3	.001
		4	.110			4	.011			4	.009			4	.010
		5	.262			5	.044			5	.036			5	.035
5	7	2	.003			6	.125			6	.108			6	.108
		3	.015			7	.299			7	.242			7	.231
		4	.076			8	.496			8	.436			8	.427
		5	.197	5	15	2	.000	6	12	2	.000	7	10	2	.000
		6	.424			3	.001			3	.001			3	.001
5	8	2	.002			4	.009			4	.007			4	.006
		3	.010			5	.037			5	.028			2	.024
		4	.054			6	.108			6	.087			6	.080
		5	.152			7	.272			7	.205			7	.182
		6	.347			8	.460			8	.383			8	.355
5	9	2	.001	6	6	2	.002	6	13	2	.000	7	11	2	.000
		3	.007			3	.013			3	.001			3	.001
		4	.039			4	.067			4	.005			4	.004
		5	.119			5	.175			5	.022			5	.018
		6	.287			6	.392			6	.070			6	.060
5	10	2	.001	6	7	2	.001			7	.176			7	.145
		3	.005			3	.008			8	.338			8	.296
		4	.029			4	.043	6	14	2	.000			9	.484
		5	.095			5	.121			3	.001	7	12	2	.000
		6	.239			6	.296			4	.004			3	.000
		7	.455			7	.500			5	.017			4	.003
5	11	2	.000	6	8	2	.001			6	.058			5	.013
		3	.004			3	.005			7	.151			6	.046
		4	.022			4	.028			8	.299			7	.117
		5	.077			5	.086	7	7	2	.001			8	.247
		6	.201			6	.226			3	.004			9	.428
		7	.407			7	.413			4	.025	7	13	2	.000
5	12	2	.000	6	9	2	.000			5	.078			3	.000
		3	.003			3	.003			6	.209			4	.002
		4	.017			4	.019			7	.383			5	.010
		5	.063			5	.063	7	8	2	.000			6	.035
		6	.170			6	.175			3	.002			7	.095
		7	.365			7	.343			4	.015			8	.208
5	13	2	.000	6	10	2	.000			5	.051			9	.378
		3	.002			3	.002			6	.149	8	8	2	.000
		4	.013			4	.013			7	.296			3	.001
		5	.053			5	.047							4	.009
		6	.145			6	.137							5	.032
		7	.330			7	.287							6	.100
						8	.497							7	.214
														8	.405

TABLE D (continued)

Total Number of Runs Distribution

Left-Tail Probabilities															
n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P
8	9	2	.000	9	9	2	.000	10	10	2	.000	11	11	2	.000
		3	.001			3	.000			3	.000			3	.000
		4	.005			4	.003			4	.001			4	.000
		5	.020			5	.012			5	.004			5	.002
		6	.069			6	.044			6	.019			6	.007
		7	.157			7	.109			7	.051			7	.023
		8	.319			8	.238			8	.128			8	.063
		9	.500			9	.399			9	.242			9	.135
8	10	2	.000	9	10	2	.000			10	.414			10	.260
		3	.000			3	.000	10	11	2	.000			11	.410
		4	.003			4	.002			3	.000	11	12	2	.000
		5	.013			5	.008			4	.001			3	.000
		6	.048			6	.029			5	.003			4	.000
		7	.117			7	.077			6	.012			5	.001
		8	.251			8	.179			7	.035			6	.005
		9	.419			9	.319			8	.092			7	.015
8	11	2	.000	9	11	2	.000			9	.185			8	.044
		3	.000			3	.000			10	.335			9	.099
		4	.002			4	.001			11	.500			10	.202
		5	.009			5	.005	10	12	2	.000			11	.335
		6	.034			6	.020			3	.000	12	12	2	.000
		7	.088			7	.055			4	.000			3	.000
		8	.199			8	.135			5	.002			4	.000
		9	.352			9	.255			6	.008			5	.001
8	12	2	.000			10	.430			7	.024			6	.003
		3	.000	9	12	2	.000			8	.067			7	.009
		4	.001			3	.000			9	.142			8	.030
		5	.006			4	.001			10	.271			9	.070
		6	.025			5	.003			11	.425			10	.150
		7	.067			6	.014							11	.263
		8	.159			7	.040							12	.421
		9	.297			8	.103								
		10	.480			9	.205								
						10	.362								

(*continued*)

TABLE D (continued)

Total Number of Runs Distribution

Right-Tail Probabilities															
n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P
2	2	4	.333	4	8	9	.071	5	11	11	.058			12	.075
2	3	5	.100			8	.212			10	.154			11	.217
		4	.500			7	.467			9	.374			10	.395
2	4	5	.200	4	9	9	.098	5	12	11	.075	6	13	13	.034
2	5	5	.286			8	.255			10	.181			12	.092
2	6	5	.357	4	10	9	.126			9	.421			11	.257
2	7	5	.417			8	.294	5	13	11	.092			10	.439
2	8	5	.467	4	11	9	.154			10	.208	6	14	13	.044
3	3	6	.100			8	.330			9	.465			12	.111
		5	.300	4	12	9	.181	5	14	11	.111			11	.295
3	4	7	.029			8	.363			10	.234			10	.480
		6	.200	4	13	9	.208	5	15	11	.129	7	7	14	.001
		5	.457			8	.393			10	.258			13	.004
3	5	7	.071	4	14	9	.234	6	6	12	.002			12	.025
		6	.286			8	.421			11	.013			11	.078
3	6	7	.119	4	15	9	.258			10	.067			10	.209
		6	.357			8	.446			9	.175			9	.383
3	7	7	.167	4	16	9	.282			8	.392	7	8	15	.000
		6	.417			8	.470	6	7	13	.001			14	.002
3	8	7	.212	5	5	10	.008			12	.008			13	.012
		6	.467			9	.040			11	.034			12	.051
3	9	7	.255			8	.167			10	.121			11	.133
3	10	7	.294			7	.357			9	.267			10	.296
3	11	7	.330	5	6	11	.002			8	.500			9	.486
3	12	7	.363			10	.024	6	8	13	.002	7	9	15	.001
3	13	7	.393			9	.089			12	.016			14	.006
3	14	7	.421			8	.262			11	.063			13	.025
3	15	7	.446			7	.478			10	.179			12	.084
3	16	7	.470	5	7	11	.008			9	.354			11	.194
3	17	7	.491			10	.045	6	9	13	.006			10	.378
4	4	8	.029			9	.146			12	.028	7	10	15	.002
		7	.114			8	.348			11	.098			14	.010
		6	.371	5	8	11	.016			10	.238			13	.043
4	5	9	.008			10	.071			9	.434			12	.121
		8	.071			9	.207	6	10	13	.010			11	.257
		7	.214			8	.424			12	.042			10	.451
		6	.500	5	9	11	.028			11	.136	7	11	15	.004
4	6	9	.024			10	.098			10	.294			14	.017
		8	.119			9	.266	6	11	13	.017			13	.064
		7	.310			8	.490			12	.058			12	.160
4	7	9	.045	5	10	11				11	.176			11	.318
		8	.167			10	.126			10	.346	7	12	15	.007
		7	.394			9	.322	6	12	13	.025			14	.025

TABLE D (continued)

Total Number of Runs Distribution

Right-Tail Probabilities											
n_1	n_2	R	P	n_1	n_2	R	P	n_1	n_2	R	P
7	12	13	.089	9	9	18	.000	10	11	13	.320
		12	.199			17	.000			12	.500
		11	.376			16	.003	10	12	21	.000
7	13	15	.010			15	.012			20	.000
		14	.034			14	.044			19	.001
		13	.116			13	.109			18	.006
		12	.238			12	.238			17	.020
		11	.430			11	.399			16	.056
8	8	16	.000	9	10	19	.000			15	.125
		15	.001			18	.000			14	.245
		14	.009			17	.001			13	.395
		13	.032			16	.008	11	11	22	.000
		12	.100			15	.026			21	.000
		11	.214			14	.077			20	.000
		10	.405			13	.166			19	.002
8	9	17	.000			12	.319			18	.007
		16	.001			11	.490			17	.023
		15	.004	9	11	19	.000			16	.063
		14	.020			18	.001			15	.135
		13	.061			17	.003			14	.260
		12	.157			18	.015			13	.410
		11	.298			15	.045	11	12	23	.000
		10	.500			14	.115			22	.000
8	10	17	.000			13	.227			21	.000
		16	.002			12	.395			20	.001
		15	.010	10	10	20	.000			19	.004
		14	.036			19	.000			18	.015
		13	.097			18	.001			17	.041
		12	.218			17	.004			16	.099
		11	.379			16	.019			15	.191
8	11	17	.001			15	.051			14	.335
		16	.004			14	.128			13	.493
		15	.018			13	.242	12	12	24	.000
		14	.057			12	.414			23	.000
		13	.138	10	11	21	.000			22	.000
		12	.278			20	.000			21	.001
		11	.453			19	.000			20	.003
8	12	17	.001			18	.003			19	.009
		16	.007			17	.010			18	.030
		15	.029			16	.035			17	.070
		14	.080			15	.085			16	.150
		13	.183			14	.185			15	.263
		12	.337							14	.421

Source: Adapted from Swed, F.S. and Eisenhart, C., Tables for testing the randomness of grouping in a sequence of alternatives, *Ann. Math. Stat.*, 14, 66–87, 1943. With permission.

TABLE E

Runs Up and Down Distribution

Each table entry labeled P is the tail probability from each extreme to the value of R, the total number of runs up and down in a sequence of n observations, or equivalently, $n-1$ plus or minus signs.

		Left-Tail		Right-Tail			Left-Tail		Right-Tail
n	R	P	R	P	n	R	P	R	P
3	1	.3333	2	.6667	13	1	.0000		
4			3	.4167		2	.0000		
	1	.0833	2	.9167		3	.0001	12	.0072
5	1	.0167	4	.2667		4	.0026	11	.0568
	2	.2500	3	.7500		5	.0213	10	.2058
6	1	.0028				6	.0964	9	.4587
	2	.0861	5	.1694		7	.2749	8	.7251
	3	.4139	4	.5861	14	1	.0000		
7	1	.0004	6	.1079		2	.0000		
	2	.0250	5	.4417		3	.0000		
	3	.1909	4	.8091		4	.0007	13	.0046
8	1	.0000				5	.0079	12	.0391
	2	.0063	7	.0687		6	.0441	11	.1536
	3	.0749	6	.3250		7	.1534	10	.3722
	4	.3124	5	.6876		8	.3633	9	.6367
9	1	.0000			15	1	.0000		
	2	.0014				2	.0000		
	3	.0257	8	.0437		3	.0000		
	4	.1500	7	.2347		4	.0002		
	5	.4347	6	.5653		5	.0027	14	.0029
10	1	.0000				6	.0186	13	.0267
	2	.0003	9	.0278		7	.0782	12	.1134
	3	.0079	8	.1671		8	.2216	11	.2970
	4	.0633	7	.4524		9	.4520	10	.5480
	5	.2427	6	.7573	16	1	.0000		
11	1	.0000				2	.0000		
	2	.0001				3	.0000		
	3	.0022	10	.0177		4	.0001	15	.0019
	4	.0239	9	.1177		5	.0009	14	.0182
	5	.1196	8	.3540		6	.0072	13	.0828
	6	.3438	7	.6562		7	.0367	12	.2335
12	1	.0000				8	.1238	11	.4631
	2	.0000				9	.2975	10	.7025
	3	.0005							
	4	.0082	11	.0113					
	5	.0529	10	.0821					
	6	.1918	9	.2720					
	7	.4453	8	.5547					

TABLE E (continued)

Runs Up and Down Distribution

		Left-Tail		Right-Tail			Left-Tail		Right-Tail
n	*R*	*P*	*R*	*P*	*n*	*R*	*P*	*R*	*P*
17	1	.0000			21	1	.0000		
	2	.0000				2	.0000		
	3	.0000				3	.0000		
	4	.0000				4	.0000		
	5	.0003	16	.0012		5	.0000		
	6	.0026	15	.0123		6	.0000		
	7	.0160	14	.0600		7	.0003	20	.0002
	8	.0638	13	.1812		8	.0023	19	.0025
	9	.1799	12	.3850		9	.0117	18	.0154
	10	.3770	11	.6230		10	.0431	17	.0591
18	1	.0000				11	.1202	16	.1602
	2	.0000				12	.2622	15	.3293
	3	.0000				13	.4603	14	.5397
	4	.0000			22	1	.0000		
	5	.0001				2	.0000		
	6	.0009	17	.0008		3	.0000		
	7	.0065	16	.0083		4	.0000		
	8	.0306	15	.0431		5	.0000		
	9	.1006	14	.1389		6	.0000	21	.0001
	10	.2443	13	.3152		7	.0001	20	.0017
	11	.4568	12	.5432		8	.0009	19	.0108
19	1	.0000				9	.0050	18	.0437
	2	.0000				10	.0213	17	.1251
	3	.0000				11	.0674	16	.2714
	4	.0000				12	.1861	15	.4688
	4	.0000				13	.3276	14	.6724
	5	.0000	18	.0005	23	1	.0000		
	6	.0003	17	.0056		2	.0000		
	7	.0025	16	.0308		3	.0000		
	8	.0137	15	.1055		4	.0000		
	9	.0523	14	.2546		5	.0000		
	10	.1467	13	.4663		6	.0000		
	11	.3144	12	.6856		7	.0000	22	.0001
20	1	.0000				8	.0003	21	.0011
	2	.0000				9	.0021	20	.0076
	3	.0000				10	.0099	19	.0321
	4	.0000				11	.0356	18	.0968
	5	.0000				12	.0988	17	.2211
	6	.0001	19	.0003		13	.2188	16	.4020
	7	.0009	18	.0038		14	.3953	15	.6047
	8	.0058	17	.0218					
	9	.0255	16	.0793					
	10	.0821	15	.2031					
	11	.2012	14	.3945					
	12	.3873	13	.6127					

(*continued*)

TABLE E (continued)

Runs Up and Down Distribution

		Left-Tail		Right-Tail			Left-Tail		Right-Tail
n	***R***	***P***	***R***	***P***	***n***	***R***	***P***	***R***	***P***
24	1	.0000			25	1	.0000		
	2	.0000				2	.0000		
	3	.0000				3	.0000		
	4	.0000				4	.0000		
	5	.0000				5	.0000		
	6	.0000				6	.0000		
	7	.0000				7	.0000	24	.0000
	8	.0001	23	.0000		8	.0000	23	.0005
	9	.0008	22	.0007		9	.0003	22	.0037
	10	.0044	21	.0053		10	.0018	21	.0170
	11	.0177	20	.0235		11	.0084	20	.0564
	12	.0554	19	.0742		12	.0294	19	.1423
	13	.1374	18	.1783		13	.0815	18	.2852
	14	.2768	17	.3405		14	.1827	17	.4708
	15	.4631	16	.5369		15	.3384	16	.6616

Source: Adapted from Edgington, E.S., Probability table for number of runs of signs of first differences, *J. Am. Stat. Assoc.*, 56, 156–159, 1961. With permission.

TABLE F

Kolmogorov–Smirnov One-Sample Statistic

Each table entry is the value of a Kolmogorov–Smirnov one-sample statistic D_n for sample size n such that its right-tail probability is the value given on the top row.

n	**.200**	**.100**	**.050**	**.020**	**.010**	***n***	**.200**	**.100**	**.050**	**.020**	**.010**
1	.900	.950	.975	.990	.995	21	.226	.259	.287	.321	.344
2	.684	.776	.842	.900	.929	22	.221	.253	.281	.314	.337
3	.565	.636	.708	.785	.829	23	.216	.247	.275	.307	.330
4	.493	.565	.624	.689	.734	24	.212	.242	.269	.301	.323
5	.447	.509	.563	.627	.669	25	.208	.238	.264	.295	.317
6	.410	.468	.519	.577	.617	26	.204	.233	.259	.290	.311
7	.381	.436	.483	.538	.576	27	.200	.229	.254	.284	.305
8	.358	.410	.454	.507	.542	28	.197	.225	.250	.279	.300
9	.339	.387	.430	.480	.513	29	.193	.221	.246	.275	.295
10	.323	.369	.409	.457	.489	30	.190	.218	.242	.270	.290
11	.308	.352	.391	.437	.468	31	.187	.214	.238	.266	.285
12	.296	.338	.375	.419	.449	32	.184	.211	.234	.262	.281
13	.285	.325	.361	.404	.432	33	.182	.208	.231	.258	.277
14	.275	.314	.349	.390	.418	34	.179	.205	.227	.254	.273
15	.266	.304	.338	.377	.404	35	.177	.202	.224	.251	.269
16	.258	.295	.327	.366	.392	36	.174	.199	.221	.247	.265
17	.250	.286	.318	.355	.381	37	.172	.196	.218	.244	.262
18	.244	.279	.309	.346	.371	38	.170	.194	.215	.241	.358
19	.237	.271	.301	.337	.361	39	.168	.191	.213	.238	.255
20	.232	.285	.294	.329	.352	40	.165	.189	.210	.235	.252

For $n > 40$, right-tail critical values based on the asymptotic distribution can be calculated as follows:

.200	.100	.050	.020	.010
$1.07/\sqrt{n}$	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.52/\sqrt{n}$	$1.63/\sqrt{n}$

Source: Adapted from Miller, L.H., Table of percentage points of Kolmogorov statistics, *J. Am. Stat. Assoc.*, 51, 111–121, 1956. With permission.

TABLE G

Binomial Distribution for $\theta = 0.5$

Each table entry labeled *P* is the tail probability from each extreme to the value of *K*, the number of successes in *N* Bernoulli trials with probability of success $\theta = 0.5$ on each trial.

N	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail
1	0	.5000	1	12	0	.0002	12	17	0	.0000	17
2	0	.2500	2		1	.0032	11		1	.0001	16
	1	.7500	1		2	.0193	10		2	.0012	15
3	0	.1250	3		3	.0730	9		3	.0064	14
	1	.5000	2		4	.1938	8		4	.0245	13
4	0	.0625	4		5	.3872	7		5	.0717	12
	1	.3125	3		6	.6128	6		6	.1662	11
	2	.6875	2	13	0	.0001	13		7	.3145	10
5	0	.0312	5		1	.0017	12		8	.5000	9
	1	.1875	4		2	.0112	11	18	0	.0000	18
	2	.5000	3		3	.0461	10		1	.0001	17
6	0	.0156	6		4	.1334	9		2	.0007	16
	1	.1094	5		5	.2905	8		3	.0038	15
	2	.3438	4		6	.5000	7		4	.0154	14
	3	.6562	3	14	0	.0000	14		5	.0481	13
7	0	.0078	7		1	.0009	13		6	.1189	12
	1	.0625	6		2	.0065	12		7	.2403	11
	2	.2266	5		3	.0287	11		8	.4073	10
	3	.5000	4		4	.0898	10		9	.5927	9
8	0	.0039	8		5	.2120	9	19	0	.0000	19
	1	.0352	7		6	.3953	8		1	.0000	18
	2	.1445	6		7	.6047	7		2	.0004	17
	3	.3633	5	15	0	.0000	15		3	.0022	16
	4	.6367	4		1	.0005	14		4	.0096	15
9	0	.0020	9		2	.0037	13		5	.0318	14
	1	.0195	8		3	.0176	12		6	.0835	13
	2	.0898	7		4	.0592	11		7	.1796	12
	3	.2539	6		5	.1509	10		8	.3238	11
	4	.5000	5		6	.3036	9		9	.5000	10
10	0	.0010	10		7	.5000	8	20	0	.0000	20
	1	.0107	9	16	0	.0000	16		1	.0000	19
	2	.0547	8		1	.0003	15		2	.0002	18
	3	.1719	7		2	.0021	14		3	.0013	17
	4	.3770	6		3	.0106	13		4	.0059	16
	5	.6230	5		4	.0384	12		5	.0207	15
11	0	.0005	11		5	.1051	11		6	.0577	14
	1	.0059	10		6	.2272	10		7	.1316	13
	2	.0327	9		7	.4018	9		8	.2517	12
	3	.1133	8		8	.5982	8		9	.4119	11
	4	.2744	7						10	.5881	10
	5	.5000	6								

TABLE H

Probabilities for the Wilcoxon Signed-Rank Statistic

Each table entry labeled P is the tail probability from each extreme to the value of T, the Wilcoxon signed-rank statistic for sample size N, where T is interpreted as either T^+ or T^-.

N	Left Tail	P	Right Tail	N	Left Tail	P	Right Tail	N	Left Tail	P	Right Tail
2	0	.250	3	7	6	.109	22	9	9	.064	36
	1	.500	2		7	.148	21		10	.082	35
3	0	.125	6		8	.188	20		11	.102	34
	1	.250	5		9	.234	19		12	.125	33
	2	.375	4		10	.289	18		13	.150	32
	3	.625	3		11	.344	17		14	.180	31
4	0	.062	10		12	.406	16		15	.213	30
	1	.125	9		13	.469	15		16	.248	29
	2	.188	8		14	.531	14		17	.285	28
	3	.312	7	8	0	.004	36		18	.326	27
	4	.438	6		1	.008	35		19	.367	26
	5	.562	5		2	.012	34		20	.410	25
5	0	.031	15		3	.020	33		21	.455	24
	1	.062	14		4	.027	32		22	.500	23
	2	.094	13		5	.039	31	10	0	.001	55
	3	.156	12		6	.055	30		1	.002	54
	4	.219	11		7	.074	29		2	.003	53
	5	.312	10		8	.098	28		3	.005	52
	6	.406	9		9	.125	27		4	.007	51
	7	.500	8		10	.156	26		5	.010	50
6	0	.016	21		11	.191	25		6	.014	49
	1	.031	20		12	.230	24		7	.019	48
	2	.047	19		13	.273	23		8	.024	47
	3	.078	18		14	.320	22		9	.032	46
	4	.109	17		15	.371	21		10	.042	45
	5	.156	16		16	.422	20		11	.053	44
	6	.219	15		17	.473	19		12	.065	43
	7	.281	14		18	.527	18		13	.080	42
	8	.344	13	9	0	.002	45		14	.097	41
	9	.422	12		1	.004	44		15	.116	40
	10	.500	11		2	.006	43		16	.138	39
7	0	.008	28		3	.010	42		17	.161	38
	1	.016	27		4	.014	41		18	.188	37
	2	.023	26		5	.020	40		19	.216	36
	3	.039	25		6	.027	39		20	.246	35
	4	.055	24		7	.037	38		21	.278	34
	5	.078	23		8	.049	37		22	.312	33

(*continued*)

TABLE H (continued)

Probabilities for the Wilcoxon Signed-Rank Statistic

N	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail
10	23	.348	32	11	32	.483	34	12	35	.396	43
	24	.385	31		33	.517	33		36	.425	42
	25	.423	30	12	0	.000	78		37	.455	41
	26	.461	29		1	.000	77		38	.485	40
	27	.500	28		2	.001	76		39	.515	39
11	0	.000	66		3	.001	75	13	0	.000	91
	1	.001	65		4	.002	74		1	.000	90
	2	.001	64		5	.002	73		2	.000	89
	3	.002	63		6	.003	72		3	.001	88
	4	.003	62		7	.005	71		4	.001	87
	5	.005	61		8	.006	70		5	.001	86
	6	.007	60		9	.008	69		6	.002	85
	7	.009	59		10	.010	68		7	.002	84
	8	.0125	58		11	.013	67		8	.003	83
	9	.016	57		12	.017	66		9	.004	82
	10	.021	56		13	.021	65		10	.005	81
	11	.027	55		14	.026	64		11	.007	80
	12	.034	54		15	.032	63		12	.009	79
	13	.042	53		16	.039	62		13	.011	78
	14	.051	52		17	.0456	61		14	.013	77
	15	.062	51		18	.055	60		15	.016	76
	16	.074	50		19	.065	59		16	.020	75
	17	.087	49		20	.076	58		17	.024	74
	18	.103	48		21	.088	57		18	.029	73
	19	.120	47		22	.102	56		19	.034	72
	20	.139	46		23	.117	55		20	.040	71
	21	.160	45		24	.133	54		21	.047	70
	22	.183	44		25	.151	53		22	.055	69
	23	.207	43		26	.170	52		23	.064	68
	24	.232	42		27	.190	51		24	.073	67
	25	.260	41		28	.212	50		25	.084	66
	26	.289	40		29	.235	49		26	.095	65
	27	.319	39		30	.259	48		27	.108	64
	28	.350	38		31	.285	47		28	.122	63
	29	.382	37		32	.311	46		29	.137	62
	30	.416	36		33	.339	45		30	.153	61
	31	.449	35		34	.367	44		31	.170	60

TABLE H (continued)

Probabilities for the Wilcoxon Signed-Rank Statistic

N	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail
13	32	.188	59	14	22	.029	83	15	5	.000	115
	33	.207	58		23	.034	82		6	.000	114
	34	.227	57		24	.039	81		7	.001	113
	35	.249	56		25	.045	80		8	.001	112
	36	.271	55		26	.052	79		9	.001	111
	37	.294	54		27	.059	78		10	.001	110
	38	.318	53		28	.068	77		11	.002	109
	39	.342	52		29	.077	76		12	.002	108
	40	.368	51		30	.086	75		13	.003	107
	41	.393	50		31	.097	74		14	.003	106
	42	.420	49		32	.108	73		15	.004	105
	43	.446	48		33	.121	72		16	.005	104
	44	.473	47		34	.134	71		17	.006	103
	45	.500	46		35	.148	70		18	.008	102
14	0	.000	105		36	.163	69		19	.009	101
	1	.000	104		37	.179	68		20	.011	100
	2	.000	103		38	.196	67		21	.013	99
	3	.000	102		39	.213	66		22	.015	98
	4	.000	101		40	.232	65		23	.018	97
	5	.001	100		41	.251	64		24	.021	96
	6	.001	99		42	.271	63		25	.024	95
	7	.001	98		43	.292	62		26	.028	94
	8	.002	97		44	.313	61		27	.032	93
	9	.002	96		45	.335	60		28	.036	92
	10	.003	95		46	.357	59		29	.042	91
	11	.003	94		47	.380	58		30	.047	90
	12	.004	93		48	.404	57		31	.053	89
	13	.005	92		49	.428	56		32	.060	88
	14	.007	91		50	.452	55		33	.068	87
	15	.008	90		51	.476	54		34	.076	86
	16	.010	89		52	.500	53		35	.084	85
	17	.012	88	15	0	.000	120		36	.094	84
	18	.0158	87		1	.000	119		37	.104	83
	19	.018	86		2	.000	118		38	.115	82
	20	.021	85		3	.000	117		39	.126	81
	21	.025	84		4	.000	116		40	.138	80

(*continued*)

TABLE H (continued)

Probabilities for the Wilcoxon Signed-Rank Statistic

N	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail	*N*	Left Tail	*P*	Right Tail
15	41	.151	79	15	48	.262	72	15	55	.402	65
	42	.165	78		49	.281	71		56	.423	64
	43	.180	77		50	.300	70		57	.445	63
	44	.196	76		51	.319	69		58	.467	62
	45	.211	75		52	.339	68		59	.489	61
	46	.227	74		53	.360	67		60	.511	60
	47	.244	73		54	.381	66				

Source: Adapted from Wilcoxon, F., S.K. Katti, and R.A. Wilcox, Critical values and probability levels for the Wilcoxon rank sum test and the Wilcoxon signed rank test, in Institute of Mathematical Statistics, ed., *Selected Tables in Mathematical Statistics*, vol. I, pp. 171–259, American Mathematical Society, Providence, RI, 1972. With permission.

TABLE I

Kolomogorov–Smirnov Two-Sample Statistic

Each table entry labeled P is the right-tail probability of $mnD_{m,n,}$ the Kolmogorov–Smirnov two-sample statistic for sample sizes m and n where $m \leq n$. The second portion of the table gives the value of $mnD_{m,n}$ such that its right-tail probability is the value given on the top row. The third portion gives the approximate critical values of $D_{m,n}$.

m	*n*	*mnD*	*P*	*m*	*n*	*mnD*	*P*	*m*	*n*	*mnD*	*P*
2	2	4	.333	3	6	18	.024	4	5	20	.016
2	3	6	.200			15	.095			16	.079
2	4	8	.133			12	.333			15	.143
2	5	10	.095	3	7	21	.017	4	6	24	.010
		8	.286			18	.067			20	.048
2	6	12	.071			15	.167			18	.095
		10	.214	3	8	24	.012			16	.181
2	7	14	.056			21	.048	4	7	28	.006
		12	.167			18	.121			24	.030
2	8	16	.044	3	9	27	.009			21	.067
		14	.133			24	.036			20	.121
2	9	18	.036			21	.091	4	8	32	.004
		16	.109			18	.236			28	.020
2	10	20	.030	3	10	30	.007			24	.085
		18	.091			27	.028			20	.222
		16	.182			24	.070	4	9	36	.003
2	11	22	.026			21	.140			32	.014
		20	.077	3	11	33	.005			28	.042
		18	.154			30	.022			27	.062
2	12	24	.022			27	.055			24	.115
		22	.066			24	.110	4	10	40	.002
		20	.132	3	12	36	.004			36	.010
3	3	9	.100			33	.018			32	.030
3	4	12	.057			30	.044			30	.046
		9	.229			27	.088			28	.084
3	5	15	.036			24	.189			26	.126
		12	.143	4	4	16	.029				
						12	.229				

(*continued*)

TABLE I (continued)

Kolomogorov–Smirnov Two-Sample Statistic

m	*n*	*mnD*	*P*	*m*	*n*	*mnD*	*P*	*m*	*n*	*mnD*	*P*
4	11	44	.001	5	10	50	.001	6	10	60	.000
		40	.007			45	.004			54	.002
		36	.022			40	.019			50	.004
		33	.035			35	.061			48	.009
		32	.063			30	.166			44	.019
		29	.098	5	11	55	.000			42	.031
		28	.144			50	.003			40	.042
4	12	48	.001			45	.010			38	.066
		44	.005			44	.014			36	.092
		40	.016			40	.029			34	.125
		36	.048			39	.044	7	7	49	.001
		32	.112			35	.074			42	.008
5	5	25	.008			34	.106			35	.053
		20	.079	6	6	36	.002			28	.212
		15	.357			30	.026	7	8	56	.000
5	6	30	.004			24	.143			49	.002
		25	.026	6	7	42	.001			48	.005
		24	.048			36	.008			42	.013
		20	.108			35	.015			41	.024
5	7	35	.003			30	.038			40	.033
		30	.015			29	.068			35	.056
		28	.030			28	.091			34	.087
		25	.066			24	.147			33	.118
		23	.166	6	8	48	.001	7	9	63	.000
5	8	40	.022			42	.005			56	.001
		35	.009			40	.009			54	.003
		32	.020			36	.023			49	.008
		30	.042			34	.043			47	.015
		27	.079			32	.061			45	.021
		25	.126			30	.093			42	.034
5	9	45	.001			28	.139			40	.055
		40	.006	6	9	54	.000			38	.079
		36	.014			48	.003			36	.098
		35	.028			45	.006			35	.127
		31	.056			42	.014	8	8	64	.000
		30	.086			39	.028			56	.002
		27	.119			36	.061			48	.019
						33	.095			40	.087
						30	.176			32	.283

TABLE I (continued)

Kolomogorov–Smirnov Two-Sample Statistic

$m = n$	.200	.100	.050	.020	.010
9	45	54	54	63	63
10	50	60	70	70	80
11	66	66	77	88	88
12	72	72	84	96	96
13	78	91	91	104	117
14	84	98	112	112	128
15	90	105	120	135	135
16	112	112	128	144	160
17	119	136	136	153	170
18	126	144	162	180	180
19	133	152	171	190	190
20	140	160	180	200	220

For m and n large, right-tail critical values of $D_{m,n}$ based on the asymptotic distribution can be calculated as follows:

.200	.100	.050	.020	.010
$1.07\sqrt{N/mn}$	$1.22\sqrt{N/mn}$	$1.36\sqrt{N/mn}$	$1.52\sqrt{N/mn}$	$1.63\sqrt{N/mn}$

Source: Adapted from Kim, P.J. and Jennrich, R.I., Tables of the exact sampling distribution of the two-sample Kolmogorov–Smirnov criterion D_{mn} ($m \leq n$), in Institute of Mathematical Statistics, ed., *Selected Tables in Mathematical Statistics*, vol. I, pp. 79–170, American Mathematical Society, Providence, RI, 1973. With permission.

TABLE J

Probabilities for the Wilcoxon Rank-Sum Statistic

Each table entry labeled P is the tail probability from each extreme to the value of W_N, the Wilcoxon statistic for sample sizes m and n where $m \leq n$.

$m = 1$				$m = 2$				$m = 2$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
1	1	.500	2	2	3	.167	7	8	3	.022	19
2	1	.333	3		4	.333	6		4	.044	18
	2	.667	2		5	.667	5		5	.089	17
3	1	.250	4	3	3	.100	9		6	.133	16
	2	.500	3		4	.200	8		7	.200	15
4	1	.200	5		5	.400	7		8	.267	14
	2	.400	4		6	.600	6		9	.356	13
	3	.600	3	4	3	.067	11		10	.444	12
5	1	.167	6		4	.133	10		11	.556	11
	2	.333	5		5	.267	9	9	3	.018	21
	3	.500	4		6	.400	8		4	.036	20
6	1	.143	7		7	.600	7		5	.073	19
	2	.286	6	5	3	.048	13		6	.109	18
	3	.429	5		4	.095	12		7	.164	17
	4	.571	4		5	.190	11		8	.218	16
7	1	.125	8		6	.286	10		9	.291	15
	2	.250	7		7	.429	9		10	.364	14
	3	.375	6		8	.571	8		11	.455	13
	4	.500	5	6	3	.036	15		12	.545	12
8	1	.111	9		4	.071	14	10	3	.015	23
	2	.222	8		5	.143	13		4	.030	22
	3	.333	7		6	.214	12		5	.061	21
	4	.444	6		7	.321	11		6	.091	20
	5	.556	5		8	.429	10		7	.136	19
9	1	.100	10		9	.571	9		8	.182	18
	2	.200	9	7	3	.028	17		9	.242	17
	3	.300	8		4	.056	16		10	.303	16
	4	.400	7		5	.111	15		11	.379	15
	5	.500	6		6	.167	14		12	.455	14
10	1	.091	11		7	.250	13		13	.545	13
	2	.182	10		8	.333	12				
	3	.273	9		9	.444	11				
	4	.364	8		10	.556	10				
	5	.455	7								
	6	.545	6								

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 3$				$m = 3$				$m = 4$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
3	6	.050	15	8	6	.006	30	4	10	.014	26
	7	.100	14		7	.012	29		11	.029	25
	8	.200	13		8	.024	28		12	.057	24
	9	.350	12		9	.042	27		13	.100	23
	10	.500	11		10	.067	26		14	.171	22
4	6	.029	18		11	0.97	25		15	.243	21
	7	.057	17		12	.139	24		16	.343	20
	8	.114	16		13	.188	23		17	.443	19
	9	.200	15		14	.248	22		18	.557	18
	10	.314	14		15	.315	21	5	10	.008	30
	11	.429	13		16	.388	20		11	.016	29
	12	.571	12		17	.461	19		12	.032	28
5	6	.018	21		18	.539	18		13	.056	27
	7	.036	20	9	6	.005	33		14	.095	26
	8	.071	19		7	.009	32		15	.143	25
	9	.125	18		8	.018	31		16	.206	24
	10	.196	17		9	.032	30		17	.278	23
	11	.286	16		10	.050	29		18	.365	22
	12	.393	15		11	.073	28		19	.452	21
	13	.500	14		12	.105	27		20	.548	20
6	6	.012	24		13	.141	26	6	10	.005	34
	7	.024	23		14	.186	25		11	.010	33
	8	.048	22		15	.241	24		12	.019	32
	9	.083	21		16	.300	23		13	.033	31
	10	.131	20		17	.364	22		14	.057	30
	11	.190	19		18	.432	21		15	.086	29
	12	.274	18		19	.500	20		16	.129	28
	13	.357	17	10	6	.003	36		17	.176	27
	14	.452	16		7	.007	35		18	.238	26
	15	.548	15		8	.014	34		19	.305	25
7	6	.008	27		9	.024	33		20	.381	24
	7	.017	26		10	.038	32		21	.457	23
	8	.033	25		11	.056	31		22	.543	22
	9	.058	24		12	.080	30				
	10	.092	23		13	.108	29				
	11	.133	22		14	.143	28				
	12	.192	21		15	.185	27				
	13	.258	20		16	.234	26				
	14	.333	19		17	.287	25				
	15	.417	18		18	.346	24				
	16	.500	17		19	.406	23				
					20	.469	22				
					21	.531	21				

(*continued*)

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 4$				$m = 4$				$m = 5$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
7	10	.003	38	9	10	.001	46	5	15	.004	40
	11	.006	37		11	.003	45		16	.008	39
	12	.012	36		12	.006	44		17	.016	38
	13	.021	35		13	.010	43		18	.028	37
	14	.036	34		14	.017	42		19	.048	36
	15	.055	33		15	.025	41		20	.075	35
	16	.082	32		16	.038	40		21	.111	34
	17	.115	31		17	.053	39		22	.155	33
	18	.158	30		18	.074	38		23	.210	32
	19	.206	29		19	.099	37		24	.274	31
	20	.264	28		20	.130	36		25	.345	30
	21	.324	27		21	.165	35		26	.421	29
	22	.394	26		22	.207	34		27	.500	28
	23	.464	25		23	.252	33	6	15	.002	45
	24	.536	24		24	.302	32		16	.004	44
8	10	.002	42		25	.355	31		17	.009	43
	11	.004	41		26	.413	30		18	.015	42
	12	.008	40		27	.470	29		19	.026	41
	13	.014	39		28	.530	28		20	.041	40
	14	.024	38	10	10	.001	50		21	.063	39
	15	.036	37		11	.002	49		22	.089	38
	16	.055	36		12	.004	48		23	.123	37
	17	.077	35		13	.007	47		24	.165	36
	18	.107	34		14	.012	46		25	.214	35
	19	.141	33		15	.018	45		26	.268	34
	20	.184	32		16	.027	44		27	.331	33
	21	.230	31		17	.038	43		28	.396	32
	22	.285	30		18	.053	42		29	.465	31
	23	.341	29		19	.071	41		30	.535	30
	24	.404	28		20	.094	40				
	25	.467	27		21	.120	39				
	26	.533	26		22	.152	38				
					23	.187	37				
					24	.227	36				
					25	.270	35				
					26	.318	34				
					27	.367	33				
					28	.420	32				
					29	.473	31				
					30	.527	30				

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 5$				$m = 5$				$m = 6$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
7	15	.001	50	9	15	.000	60	6	21	.001	57
	16	.003	49		16	.001	59		22	.002	56
	17	.005	48		17	.002	58		23	.004	55
	18	.009	47		18	.003	57		24	.008	54
	19	.015	46		19	.006	56		25	.013	53
	20	.024	45		20	.009	55		26	.021	52
	21	.037	44		21	.014	54		27	.032	51
	22	.053	43		22	.021	53		28	.047	50
	23	.074	42		23	.030	52		29	.066	49
	24	.101	41		24	.041	51		30	.090	48
	25	.134	40		25	.056	50		31	.120	47
	26	.172	39		26	.073	49		32	.155	46
	27	.216	38		27	.095	48		33	.197	45
	28	.265	37		28	.120	47		34	.242	44
	29	.319	36		29	.149	46		35	.294	43
	30	.378	35		30	.182	45		36	.350	42
	31	.438	34		31	.219	44		37	.409	41
	32	.500	33		32	.259	43		38	.469	40
8	15	.001	55		33	.303	42		39	.531	39
	16	.002	54		34	.350	41	7	21	.001	63
	17	.003	53		35	.399	40		22	.001	62
	18	.005	52		36	.449	39		23	.002	61
	19	.009	51		37	.500	38		24	.004	60
	20	.015	50	10	15	.000	65		25	.007	59
	21	.023	49		16	.001	64		26	.011	58
	22	.033	48		17	.001	63		27	.017	57
	23	.047	47		18	.002	62		28	.026	56
	24	.064	46		19	.004	61		29	.037	55
	25	.085	45		20	.006	60		30	.051	54
	26	.111	44		21	.010	59		31	.069	53
	27	.142	43		22	.014	58		32	.090	52
	28	.177	42		23	.020	57		33	.117	51
	29	.218	41		24	.028	56		34	.147	50
	30	.262	40		25	.038	55		35	.183	49
	31	.311	39		26	.050	54		36	.223	48
	32	.362	38		27	.065	53		37	.267	47
	33	.416	37		28	.082	52		38	.314	46
	34	.472	36		29	.103	51		39	.365	45
	35	.528	35		30	.127	50		40	.418	44
					31	.155	49		41	.473	43
					32	.0185	48		42	.527	42
					33	.220	47				
					34	.257	46				
					35	.297	45				
					36	.339	44				
					37	.384	43				
					38	.430	42				
					39	.477	41				
					40	.523	40				

(*continued*)

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 6$				$m = 6$				$m = 7$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
8	21	.000	69	9	41	.228	55	7	28	.000	77
	22	.001	68		42	.264	54		29	.001	76
	23	.001	67		43	.303	53		30	.001	75
	24	.002	66		44	.344	52		31	.002	74
	25	.004	65		45	.388	51		32	.003	73
	26	.006	64		46	.432	50		33	.006	72
	27	.010	63		47	.477	49		34	.009	71
	28	.015	62		48	.523	48		35	.013	70
	29	.021	61	10	21	.000	81		36	.019	69
	30	.030	60		22	.000	80		37	.027	68
	31	.041	59		23	.000	79		38	.036	67
	32	.054	58		24	.001	78		39	.049	66
	33	.071	57		25	.001	77		40	.064	65
	34	.091	56		26	.002	76		41	.082	64
	35	.114	55		27	.004	75		42	.104	63
	36	.141	54		28	.005	74		43	.130	62
	37	.172	53		29	.008	73		44	.159	61
	38	.207	52		30	.011	72		45	.191	60
	39	.245	51		31	.016	71		46	.228	59
	40	.286	50		32	.021	70		47	.267	58
	41	.331	49		33	.028	69		48	.310	57
	42	.377	48		34	.036	68		49	.355	56
	43	.426	47		35	.047	67		50	.402	55
	44	.475	46		36	.059	66		51	.451	54
	45	.525	45		37	.074	65		52	.500	53
9	21	.000	75		38	.090	64	8	28	.000	84
	22	.000	74		39	.110	63		29	.000	83
	23	.001	73		40	.132	62		30	.001	82
	24	.001	72		41	.157	61		31	.001	81
	25	.002	71		42	.184	60		32	.002	80
	26	.004	70		43	.214	59		33	.003	79
	27	.006	69		44	.246	58		34	.005	78
	28	.009	68		45	.281	57		35	.007	77
	29	.013	67		46	.318	56		36	.010	76
	30	.018	66		47	.356	55		37	.014	75
	31	.025	65		48	.396	54		38	.020	74
	32	.033	64		49	.437	53		39	.027	73
	33	.044	63		50	.479	52		40	.036	72
	34	.057	62		51	.521	51		41	.047	71
	35	.072	61						42	.060	70
	36	.091	60						43	.076	69
	37	.112	59						44	.095	68
	38	.136	58						45	.116	67
	39	.164	57						46	.140	66
	40	.194	56						47	.168	65

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 7$				$m = 7$				$m = 8$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
8	48	.198	64	10	28	.000	98	8	36	.000	100
	49	.232	63		29	.000	97		37	.000	99
	50	.268	62		30	.000	96		38	.000	98
	51	.306	61		31	.000	95		39	.001	97
	52	.347	60		32	.001	94		40	.001	96
	53	.389	59		33	.001	93		41	.001	95
	54	.433	58		34	.002	92		42	.002	94
	55	.478	57		35	.002	91		43	.003	93
	56	.522	56		36	.003	90		44	.005	92
9	28	.000	91		37	.005	89		45	.007	91
	29	.000	90		38	.007	88		46	.010	90
	30	.000	89		39	.009	87		47	.014	89
	31	.001	88		40	.012	86		48	.019	88
	32	.001	87		41	.017	85		49	.025	87
	33	.002	86		42	.022	84		50	.032	86
	34	.003	85		43	.028	83		51	.041	85
	35	.004	84		44	.035	82		52	.052	84
	36	.006	83		45	.044	84		53	.065	83
	37	.008	82		46	.054	80		54	.080	82
	38	.011	81		47	.067	79		55	.097	81
	39	.016	80		48	.081	78		56	.117	80
	40	.021	79		49	.097	77		57	.139	79
	41	.027	78		50	.115	76		58	.164	78
	42	.036	77		51	.135	75		59	.191	77
	43	.045	76		52	.157	74		60	.221	76
	44	.057	75		53	.182	73		61	.253	75
	45	.071	74		54	.209	72		62	.287	74
	46	.087	73		55	.237	71		63	.323	73
	47	.102	72		56	.268	70		64	.360	72
	48	.126	71		57	.300	69		65	.399	71
	49	.150	70		58	.335	68		66	.439	70
	50	.176	69		59	.370	67		67	.480	69
	51	.204	68		60	.406	66		68	.520	68
	52	.235	67		61	.443	65	9	36	.000	108
	53	.268	66		62	.481	64		37	.000	107
	54	.303	65		63	.519	63		38	.000	106
	55	.340	64						39	.000	105
	56	.379	63						40	.000	104
	57	.419	62						41	.001	103
	58	.459	61						42	.001	102
	59	.500	60						43	.002	101

(*continued*)

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 8$				$m = 8$				$m = 9$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
9	44	.003	100	10	36	.000	116	9	45	.000	126
	45	.004	99		37	.000	115		46	.000	125
	46	.006	98		38	.000	114		47	.000	124
	47	.008	97		39	.000	113		48	.000	123
	48	.010	96		40	.000	112		49	.000	122
	49	.014	95		41	.000	111		50	.000	121
	50	.018	94		42	.001	110		51	.001	120
	51	.023	93		43	.001	109		52	.001	119
	52	.030	92		44	.002	108		53	.001	118
	53	.037	91		45	.002	107		54	.002	117
	54	.046	90		46	.003	106		55	.003	116
	55	.057	89		47	.004	105		56	.004	115
	56	.069	88		48	.006	104		57	.005	114
	57	.084	87		49	.008	103		58	.007	113
	58	.100	86		50	.010	102		59	.009	112
	59	.118	85		51	.013	101		60	.012	111
	60	.138	84		52	.017	100		61	.016	110
	61	.161	83		53	.022	99		62	.020	109
	62	.185	82		54	.027	98		63	.025	108
	63	.212	81		55	.034	97		64	.031	107
	64	.240	80		56	.042	96		65	.039	106
	65	.271	79		57	.051	95		66	.047	105
	66	.303	78		58	.061	94		67	.057	104
	67	.336	77		59	.073	93		68	.068	103
	68	.371	76		60	.086	92		69	.081	102
	69	.407	75		61	.102	91		70	.095	101
	70	.444	74		62	.118	90		71	.111	100
	71	.481	73		63	.137	89		72	.129	99
	72	.519	72		64	.158	88		73	.149	98
					65	.180	87		74	.170	97
					66	.204	86		75	.193	96
					67	.230	85		76	.218	95
					68	.257	84		77	.245	94
					69	.286	83		78	.273	93
					70	.317	82		79	.302	92
					71	.348	81		80	.333	91
					72	.381	80		81	.365	90
					73	.414	79		82	.398	89
					74	.448	78		83	.432	88
					75	.483	77		84	.466	87
					76	.517	76		85	.500	86

TABLE J (continued)

Probabilities for the Wilcoxon Rank-Sum Statistic

$m = 9$				$m = 9$				$m = 10$			
n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail	n	Left Tail	P	Right Tail
10	45	.000	135	10	78	.178	102	10	73	.007	137
	46	.000	134		79	.200	101		74	.009	136
	47	.000	133		80	.223	100		75	.012	135
	48	.000	32		81	.248	99		76	.014	134
	49	.000	131		82	.274	98		77	.018	133
	50	.000	130		83	.302	97		78	.022	1332
	51	.000	129		84	.330	96		79	.026	131
	52	.000	128		85	.360	95		80	.032	130
	53	.001	127		86	.390	94		81	.038	129
	54	.001	126		87	.421	93		82	.045	128
	55	.001	125		88	.452	92		83	.053	127
	56	.002	124		89	.484	91		84	.062	126
	57	.003	123		90	.516	90		85	.072	125
	58	.004	122						86	.083	124
	59	.005	121						87	.095	123
	60	.007	120			$m = 10$			88	.109	122
	61	.009	119						89	.124	121
	62	.011	118	10	55	.000	155		90	.140	120
	63	.014	117		56	.000	154		91	.157	119
	64	.017	116		57	.000	153		92	.176	118
	65	.022	115		58	.000	152		93	.197	117
	66	.027	114		59	.000	151		94	.218	116
	67	.033	113		60	.000	150		95	.241	115
	68	.039	112		61	.000	149		96	.264	114
	69	.047	111		62	.000	148		97	.289	113
	70	.056	110		63	.000	147		98	.315	112
	71	.067	109		64	.001	146		99	.342	111
	72	.078	108		65	.001	145		100	.370	110
	73	.091	107		66	.001	144		101	.398	109
	74	.106	106		67	.001	143		102	.427	108
	75	.121	105		68	.002	142		103	.456	107
	76	.139	104		69	.003	141		104	.485	106
	77	.158	103		70	.003	140		105	.515	105
					71	.004	139				
					72	.006	138				

Source: Adapted from Wilcoxon, F. , S.K. Katti, and R.A. Wilcox, Critical values and probability levels for the Wilcoxon rank sum test and the Wilcoxon signed rank test, in Institute of Mathematical Statistics, ed., *Selected Tables in Mathematical Statistics*, vol. I, pp. 172–259, American Mathematical Society, Providence, RI, 1973. With permission.

TABLE K

Kruskal–Wallis Test Statistic

Each table entry is the smallest value of the Kruskal–Wallis H such that its right-tail probability is less than or equal to the value given on the top now for $k = 3$, each sample size less than or equal to 5. For $k > 3$, right-tail probabilities on H are found from Table B with $k - 1$ degrees of freedom.

	Right-Tail Probability for *H*				
n_1, n_2, n_3	**0.100**	**0.050**	**0.020**	**0.010**	**0.001**
2, 2, 2	4.571	—	—	—	—
3, 2, 1	4.286	—	—	—	—
3, 2, 2	4.500	4.714	—	—	—
3, 3, 1	4.571	5.143	—	—	—
3, 3, 2	4.556	5.361	6.250	—	—
3, 3, 3	4.622	5.600	6.489	7.200	—
4, 2, 1	4.500	—	—	—	—
4, 2, 2	4.458	5.333	6.000	—	—
4, 3, 1	4.056	5.208	—	—	—
4, 3, 2	4.511	5.444	6.144	6.444	—
4, 3, 3	4.709	5.791	6.564	6.745	—
4, 4, 1	4.167	4.967	6.667	6.667	—
4, 4, 2	4.555	5.455	6.600	7.036	—
4, 4, 3	4.545	5.598	6.712	7.144	8.909
4, 4, 4	4.654	5.692	6.962	7.654	9.269
5, 2, 1	4.200	5.000	—	—	—
5, 2, 2	4.373	5.160	6.000	6.533	—
5, 3, 1	4.018	4.960	6.044	—	—
5, 3, 2	4.651	5.251	6.124	6.909	—
5, 3, 3	4.533	5.648	6.533	7.079	8.727
5, 4, 1	3.987	4.985	6.431	6.955	—
5, 4, 2	4.541	5.273	6.505	7.205	8.591
5, 4, 3	4.549	5.656	6.676	7.445	8.795
5, 4, 4	4.668	5.657	6.953	7.760	9.168
5, 5, 1	4.109	5.127	6.145	7.309	—
5, 5, 2	4.623	5.338	6.446	7.338	8.938
5, 5, 3	4.545	5.705	6.866	7.578	9.284
5, 5, 4	4.523	5.666	7.000	7.823	9.606
5, 5, 5	4.560	5.780	7.220	8.000	9.920

Source: Adapted from Iman, R.L., D. Quade, and D.A. Alexander, Exact probability levels for the Kruskal–Wallis test, in Institute of Mathematical Statistics, ed., *Selected Tables in Mathematical Statistics*, vol. III, pp. 329–384, American Mathematical Society, Providence, RI, 1975. With permission.

TABLE L

Kendall's Tau Statistic

Each table entry labeled P is the right-tail probability for T, the Kendall tau statistic for sample size n, and also the left-tail probability for $-T$. The second portion of the table gives the value of $T(-T)$ such that its right (left-tail) probability is the value given in the top row.

n	*T*	*P*	*n*	*T*	*P*	*n*	*T*	*P*	*n*	*T*	*P*
3	1.000	.167	7	1.000	.000	9	1.000	.000	10	1.000	.000
	.333	.500		.905	.001		.944	.000		.956	.000
4	1.000	.042		.810	.005		.889	.000		.911	.000
	.667	.167		.714	.015		.833	.000		.867	.000
	.333	.375		.619	.035		.778	.001		.822	.000
	.000	.625		.524	.068		.722	.003		.778	.000
5	1.000	.008		.429	.119		.667	.006		.733	.001
	.800	.042		.333	.191		.611	.012		.689	.002
	.600	.117		.238	.281		.556	.022		.644	.005
	.400	.242		.143	.386		.500	.038		.600	.008
	.200	.408		.048	.500		.444	.060		.556	.014
	.000	.592	8	1.000	.000		.389	.090		.511	.023
6	1.000	.001		.929	.000		.333	.130		.467	.036
	.867	.008		.857	.001		.278	.179		.422	.054
	.733	.028		.786	.003		.222	.238		.378	.078
	.600	.068		.714	.007		.167	.306		.333	.108
	.467	.136		.643	.016		.111	.381		.289	.146
	.333	.235		.571	.031		.056	.460		.244	.190
	.200	.360		.500	.054		.000	.540		.200	.242
	.067	.500		.429	.089					.156	.300
				.357	.138					.111	.364
				.286	.199					.067	.431
				.214	.274					.022	.500
				.143	.360						
				.071	.452						
				.000	.548						

n \ α	.100	.050	.025	.010	.005
11	.345	.418	.491	.564	.600
12	.303	.394	.455	.545	.576
13	.308	.359	.436	.513	.564
14	.275	.363	.407	.473	.516
15	.276	.333	.390	.467	.505
16	.250	.317	.383	.433	.483
17	.250	.309	.368	.426	.471
18	.242	.294	.346	.412	.451

(*continued*)

TABLE L (continued)

Kendall's Tau Statistic

$n \backslash \alpha$	.100	.050	.025	.010	.005
19	.228	.287	.333	.392	.439
20	.221	.274	.326	.379	.421
21	.210	.267	.314	.371	.410
22	.203	.264	.307	.359	.394
23	.202	.257	.296	.352	.391
24	.196	.246	.290	.341	.377
25	.193	.240	.287	.333	.367
26	.188	.237	.280	.329	.360
27	.179	.231	.271	.322	.356
28	.180	.228	.265	.312	.344
29	172	.222	.261	.310	.340
30	.172	.218	.255	.301	.333

Source: The tail probabilities ($n \leq 10$) are adapted from M.G. Kendall (1948, 4th edn. 1970), *Rank Correlation Methods*, Charles Griffin & Co., Ltd., London and High Wycombe. With permission. The quantiles ($11 \leq n \leq 30$) are adapted from Kaarsemaker, L. and van Wijngaarden, A., Tables for use in rank correlation, *Stat. Neerl.*, **7**, 41–54, 1953. With permission.

TABLE M

Spearman's Coefficient of Rank Correlation

Each table entry labeled P is the right-tail probability for R, Spearman's coefficient of rank correlation for sample size n, and also the left-tail probability for $-R$. The second portion of the table gives the value of $R(-R)$ such that its right-tail (left-tail) probability is the value given on the top row.

n	*R*	*P*	*n*	*R*	*P*	*n*	*R*	*P*	*n*	*R*	*P*
3	1.000	.167	7	1.000	.000	8	.810	.011	9	1.000	.000
	.500	.500		.964	.001		.786	.014		.983	.000
4	1.000	.042		.929	.003		.762	.018		.967	.000
	.800	.167		.893	.006		.738	.023		.950	.000
	.600	.208		.857	.012		.714	.029		.933	.000
	.400	.375		.821	.017		.690	.035		.917	.001
	.200	.458		.786	.024		.667	.042		.900	.001
	.000	.542		.750	.033		.643	.048		.883	.002
5	1.000	.008		.714	.044		.619	.057		.867	.002
	.900	.042		.679	.055		.595	.066		.850.	.003
	.800	.067		.643	.069		.571	.076		833	.004
	.700	.117		.607	.083		.548	.085		.817	.005
	.600	.175		.571	.100		.524	.098		.800	.007
	.500	.225		.536	.118		.500	.108		.783	.009
	.400	.258		.500	.133		.476	.122		.767	.011
	.300	.342		.464	.151		.452	.134		.750	.013
	.200	.392		.429	.177		.429	.150		.733	.016
	.100	.475		.393	.198		.405	.163		.717	.018
	.000	.525		.357	.222		.381	.180		.700	.022
6	1.000	.001		.321	.249		.357	.195		.683	.025
	.943	.008		.286	.278		.333	.214		.667	.029
	.886	.017		.250	.297		.310	.231		.650	.033
	.829	.029		.214	.331		.286	.250		.633	.038
	.771	.051		.179	.357		.262	.268		.617	.043
	.714	.068		.143	.391		.238	.291		.600	.048
	.657	.088		.107	.420		.214	.310		.583	.054
	.600	.121		.071	.453		.190	.332		.567	.060
	.543	.149		.036	.482		.167	.352		.550	.066
	.486	.178		.000	.518		.143	.376		.533	.074
	.429	.210	8	1.000	.000		.119	.397		.517	.081
	.371	.249		.976	.000		.095	.420		.500	.089
	.314	.282		.952	.001		.071	.441		.483	.097
	.257	.329		.929	.001		.048	.467		.467	.106
	.200	.357		.905	.002		.024	.488		.450	.115
	.143	.401		.881	.004		.000	.512		.433	.125
	.086	.460		.857	.005					.417	.135
	.029	.500		.833	.008					.400	.146

(*continued*)

TABLE M (continued)

Spearman's Coefficient of Rank Correlation

n	*R*	*P*	*n*	*R*	*P*	*n*	*R*	*P*	*n*	*R*	*P*
9	.383	.156	10	.964	.000	10	.636	.027	10	.309	.193
	.367	.168		.952	.000		.624	.030		.297	.203
	.350	.179		.939	.000		.612	.033		.285	.214
	.333	.193		.927	.000		.600	.037		.273	.224
	.317	.205		.915	.000		.588	.040		.261	.235
	.300	.218		.903	.000		.576	.044		.248	.246
	.283	.231		.891	.001		.564	.048		.236	.257
	.267	.247		.879	.001		.552	.052		.224	.268
	.250	.260		.867	.001		.539	.057		.212	.280
	.233	.276		.855	.001		.527	.062		.200	.292
	.217	.290		.842	.002		.515	.067		.188	.304
	.200	.307		.830	.002		.503	.072		.176	.316
	.183	.322		.818	.003		.491	.077		.164	.328
	.167	.339		.806	.004		.479	.083		.152	.341
	.150	.354		794	.004		.467	.089		.139	.354
	.133	.372		.782	.005		.455	.096		.127	.367
	.117	.388		.770	.007		.442	.102		.115	.379
	.100	.405		.758	.008		.460	.109		.103	.393
	.083	.422		.745	.009		.418	.116		.091	.406
	.067	.440		.733	.010		.406	.124		.079	.419
	.050	.456		.721	.012		.394	.132		.067	.433
	.033	.474		.709	.013		.382	.139		.055	.446
	.017	.491		.697	.015		.370	.148		.042	.459
	.000	.509		.685	.017		.358	.156		.030	.473
10	1.000	.000		.673	.019		.345	.165		.018	.483
	.988	.000		.661	.022		.333	.174		.006	.500
	.976	.000		.648	.025		.321	.184			

n \ α	.100	.050	.025	.010	.005	.001
1	.427	.536	.618	.709	.764	.855
12	.406	.503	.587	.678	.734	.825
13	.385	.484	.560	.648	.703	.797
14	.367	.464	.538	.626	.679	.771
15	.354	.446	.521	.604	.657	.750
16	.341	.429	.503	.585	.635	.729
17	.329	.414	.488	.566	.618	.711
18	.317	.401	.474	.550	.600	.692
19	.309	.391	.460	.535	.584	.675
20	.299	.380	.447	.522	.570	.660
21	.292	.370	.436	.509	.556	.647
22	.284	.361	.425	.497	.544	.633
23	.278	.353	.416	.486	.532	.620

TABLE M (continued)

Spearman's Coefficient of Rank Correlation

n/α	.100	.050	.025	.010	.005	.001
24	.275	.344	.407	.476	.521	.608
25	.265	.337	.398	.466	.511	.597
26	.260	.331	.390	.457	.501	.586
27	.255	.324	.383	.449	.492	.576
28	.250	.318	.376	.441	.483	.567
29	.245	.312	.369	.433	.475	.557
30	.241	.307	.363	.426	.467	.548

Source: The tail probabilities ($n \leq 10$) are adapted from M.G. Kendall (1948, 4th edn. 1970), *Rank Correlation Methods,* Charles Griffin & Co., Ltd., London and High Wycombe. With permission. The quantiles ($11 \leq n \leq 30$) are adapted from Glasser, G.J. and Winter, R.F., Critical values of the rank correlation coefficient for testing the hypothesis of independence, *Biometrika*, 48, 444–448, 1961. With permission.

TABLE N

Friedman's Analysis-of-Variance Statistic and Kendall's Coefficient of Concordance
Each table entry labeled P is the right-tail probability for the sum of squares S used in Friedman's analysis-of-variance statistic with n treatments and k blocks and in Kendall's coefficient of concordance with k sets of rankings of n objects.

n	k	S	P	n	k	S	P	n	k	S	P	n	k	S	P
3	2	8	.167	3	7	98	.000	4	2	20	.042	4	4	80	.000
		6	.500			96	.000			18	.167			78	.001
	3	18	.028			86	.000			16	.208			76	.001
		14	.194			78	.001			14	.375			74	.001
		8	.361			74	.003			12	.458			72	.002
	4	32	.005			72	.004		3	45	.002			70	.003
		26	.042			62	.008			43	.002			68	.003
		24	.069			56	.016			41	.017			66	.006
		18	.125			54	.021			37	.033			64	.007
		14	.273			50	.027			35	.054			62	.012
		8	.431			42	.051			33	.075			58	.014
	5	50	.001			38	.085			29	.148			56	.019
		42	.008			32	.112			27	.175			54	.033
		38	.024			26	.192			25	.207			52	.036
		32	.039			24	.237			21	.300			50	.052
		26	.093			18	.305			19	.342			48	.054
		24	.124			14	.486			17	.446			46	.068
		18	.182		8	128	.000							44	.077
		14	.367			126	.000							42	.094
	6	72	.000			122	.000							40	.105
		62	.002			114	.000							38	.141
		56	.006			104	.000							36	.158
		54	.008			98	.001							34	.190
		50	.012			96	.001							32	.200
		42	.029			86	.002							30	.242
		38	.052			78	.005							26	.324
		32	.072			74	.008							24	.355
		26	.142			72	.010							22	.389
		24	.184			62	.018							20	.432
		18	.252			56	.030								
		14	.430			54	.038								
						50	.047								
						42	.079								
						38	.120								
						32	.149								
						26	.236								
						24	.285								
						18	.355								

Source: Adapted from Kendall, M.G., *Rank Correlation Methods,* Charles Griffin & Co., Ltd., London and High Wycombe, 1948, 4th edn. 1970. With permission.

TABLE O

Lilliefors's Test for Normal Distribution Critical Values

Table entries for any sample size N are the values of a Lilliefors's random variable with right-tail probability as given in the top row.

Sample Size	Significance Level			
N	**0.100**	**0.05**	**0.010**	**0.001**
4	.344	.375	.414	.432
5	.320	.344	.398	.427
6	.298	.323	.369	.421
7	.281	.305	.351	.399
8	.266	.289	.334	.383
9	.252	.273	.316	.366
10	.240	.261	.305	.350
11	.231	.251	.291	.331
12	.223	.242	.281	.327
14	.208	.226	.262	.302
16	.195	.213	.249	.291
18	.185	.201	.234	.272
20	.176	.192	.223	.266
25	.159	.173	.202	.236
30	.146	.159	.186	.219
40	.127	.139	.161	.190
50	.114	.125	.145	.173
60	.105	.114	.133	.159
75	.094	.102	.119	.138
100	.082	.089	.104	.121
Over 100	$.816/\sqrt{N}$	$.888/\sqrt{N}$	$1.038/\sqrt{N}$	$1.212/\sqrt{N}$

Source: Adapted from Edgeman, R.L. and Scott, R.C., Lilliefors's test for transformed variables, *Braz. J. Probability Stat.*, 1, 101–112, 1987. With permission.

TABLE P

Significance Points of $T_{XY.Z}$ for Kendall's Partial Rank-Correlation Coefficient

	One-Tailed Level of Significance			
m	**0.005**	**0.01**	**0.025**	**0.05**
3	1	1	1	1
4	1	1	1	0.707
5	1	0.816	0.802	0.667
6	0.866	0.764	0.667	0.600
7	0.761	0.712	0.617	0.527
8	0.713	0.648	0.565	0.484
9	0.660	0.602	0.515	0.443
10	0.614	0.562	0.480	0.413
11	0.581	0.530	0.453	0.387
12	0.548	0.505	0.430	0.365
13	0.527	0.481	0.410	0.347
14	0.503	0.458	0.391	0.331
15	0.482	0.439	0.375	0.317
16	0.466	0.423	0.361	0.305
17	0.450	0.410	0.348	0.294
18	0.434	0.395	0.336	0.284
19	0.421	0.382	0.326	0.275
20	0.410	0.372	0.317	0.267
25	0.362	0.328	0.278	0.235
30	0.328	0.297	0.251	0.211

Source: Adapted from Maghsoodloo, S., Estimates of the quantiles of Kendall's partial rank correlation coefficient and additional quantile estimates, *J. Stat. Comput. Simulation*, 4, 155–164, 1975; Maghsoodloo, S. and Pallos, L.L., Asymptatic behavior of Kendall's partial rank correlation coefficient and additional quantile estimates, *J. Stat. Comput. Simulation*, 13, 41–48, 1981. With permission.

TABLE Q

Page's L Statistic

Each table entry for n treatments and k blocks is the value of L such that its right-tail probability is less than or equal to 0.001 for the upper number, 0.01 for the middle number, and 0.05 for the lower number.

	N					
k	**3**	**4**	**5**	**6**	**7**	**8**
2			109	178	269	388
		60	106	173	261	376
	28	58	103	166	252	362
3		89	160	260	394	567
	42	87	155	252	382	549
	41	84	150	244	370	532
4	56	117	210	341	516	743
	55	114	204	331	501	722
	54	111	197	321	487	701
5	70	145	259	420	637	917
	68	141	251	409	620	893
	66	137	244	397	603	869
6	83	172	307	499	757	1090
	81	167	299	486	737	1063
	79	163	291	474	719	1037
7	96	198	355	577	876	1262
	93	193	346	563	855	1232
	91	189	338	550	835	1204
8	109	225	403	655	994	1433
	106	220	383	640	972	1401
	104	214	384	625	950	1371
9	121	252	451	733	1113	1603
	119	246	441	717	1088	1569
	116	240	431	701	1065	1537
10	134	278	499	811	1230	1773
	131	272	487	793	1205	1736
	128	266	477	777	1180	1703
11	147	305	546	888	1348	1943
	144	298	534	869	1321	1905
	141	292	523	852	1295	1868
12	160	331	593	965	1465	2112
	156	324	581	946	1437	2072
	153	317	570	928	1410	2035

Source: Adapted from Page, E.P., Ordered hypotheses for multiple treatments: A significance test for linear ranks, *J. Am. Stat. Assoc.*, 58, 216–230, 1963. With permission.

TABLE R

Critical Values and Associated Probabilities for the Jonckheere–Terpstra Test
Each entry is the critical value $B_\alpha = B(\alpha, k, n_1, n_2, \ldots, n_k)$ of a Jonckheere–Terpstra statistic B for given $\alpha, k, n_1, n_2, \ldots, n_k$, such that $P(B \geq B_\alpha | H_0) \leq \alpha$. The actual right-tail probability is equal to the value given in parentheses. Because the distribution of B is symmetric under H_0, this table gives critical values and associated probabilities for all possible sample size combinations from 2 to 8 from $k = 3$ populations. For example, if $n_1 = 4$, $n_2 = 5$, and $n_3 = 2$, the required values are given by the table entry for $n_1 = 2$, $n_2 = 4$, and $n_3 = 5$.

			$k = 3$					
n_1	n_2	n_3	$\alpha = 0.200$	$\alpha = 0.100$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.005$
2	2	2	9(.1667)	10(.0889)	11(.0333)	12(.0111)	—	—
2	2	3	12(.1381)	13(.0762)	14(.0381)	15(.0143)	16(.0048)	16(.0048)
2	2	4	14(.1810)	16(.0714)	17(.0381)	18(.0190)	19(.0071)	20(.0024)
2	2	5	17(.1534)	19(.0661)	20(.0397)	21(.0212)	23(.0040)	23(.0040)
2	2	6	19(.1849)	21(.0944)	23(.0397)	24(.0238)	26(.0064)	27(.0024)
2	2	7	22(.1636)	24(.0879)	26(.0404)	28(.0152)	29(.0081)	30(.0040)
2	2	8	24(.1886)	27(.0822)	29(.0404)	31(.0168)	33(.0054)	34(.0027)
2	3	3	15(.1518)	16(.0964)	18(.0304)	19(.0143)	20(.0054)	21(.0018)
2	3	4	18(.1619)	20(.0738)	21(.0452)	23(.0135)	24(.0064)	25(.0014)
2	3	5	21(.1694)	23(.0877)	25(.0381)	26(.0230)	28(.0068)	29(.0032)
2	3	6	24(.1755)	26(.0996)	28(.0496)	30(.0210)	32(.0071)	33(.0037)
2	3	7	27(.1803)	30(.0823)	32(.0427)	34(.0103)	36(.0073)	37(.0037)
2	3	8	30(.1843)	33(.0919)	36(.0377)	38(.0181)	40(.0075)	41(.0045)
2	4	4	21(.1981)	24(.0756)	26(.0321)	27(.0190)	29(.0054)	30(.0025)
2	4	5	25(.1810)	28(.0766)	30(.0368)	31(.0240)	33(.0088)	34(.0049)
2	4	6	29(.1680)	32(.0774)	34(.0408)	36(.0190)	38(.0076)	39(.0044)
2	4	7	32(.1930)	36(.0780)	38(.0441)	40(.0226)	43(.0066)	44(.0041)
2	4	8	36(.1808)	39(.0988)	42(.0469)	45(.0186)	47(.0089)	49(.0038)
2	5	5	29(.1900)	32(.0916)	35(.0356)	36(.0245)	39(.0064)	40(.0037)
2	5	6	33(.1971)	37(.0818)	39(.0472)	41(.0249)	44(.0078)	45(.0049)
2	5	7	38(.1706)	41(.0936)	44(.0448)	47(.0182)	49(.0089)	51(.0039)
2	5	8	42(.1776)	46(.0850)	49(.0428)	52(.0188)	54(.0100)	56(.0048)
2	6	6	38(.1882)	42(.0853)	45(.0403)	47(.0224)	50(.0079)	52(.0034)
2	6	7	43(.1809)	47(.0880)	50(.0452)	53(.0204)	56(.0079)	58(.0038)
2	6	8	48(.1749)	52(.0903)	55(.0495)	58(.0245)	62(.0079)	64(.0040)
2	7	7	48(.1895)	53(.0836)	56(.0454)	59(.0222)	62(.0096)	65(.0036)
2	7	8	53(.1968)	58(.0947)	62(.0456)	65(.0238)	69(.0086)	71(.0047)
2	8	8	59(.1925)	64(.0983)	69(.0423)	72(.0232)	76(.0091)	79(.0040)
3	3	3	18(.1940)	20(.0946)	22(.0369)	23(.0208)	25(.0048)	25(.0048)
3	3	4	22(.1750)	24(.0931)	26(.0421)	28(.0155)	29(.0086)	30(.0043)
3	3	5	26(.1615)	28(.0918)	30(.0462)	32(.0200)	34(.0071)	35(.0039)
3	3	6	29(.1891)	32(.0908)	34(.0495)	36(.0241)	39(.0062)	40(.0036)

TABLE R (continued)

Critical Values and Associated Probabilities for the Jonckheere–Terpstra Test

			$k = 3$					
n_1	n_2	n_3	$\alpha = 0.200$	$\alpha = 0.100$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.005$
3	3	7	33(.1762)	36(.0899)	39(.0385)	41(.0194)	43(.0087)	45(.0034)
3	3	8	36(.1984)	40(.0891)	43(.0414)	45(.0226)	48(.0076)	49(.0050)
3	4	4	26(.1853)	29(.0804)	31(.0397)	33(.0169)	35(.0059)	36(.0032)
3	4	5	30(.1932)	33(.0948)	36(.0379)	38(.0179)	40(.0073)	41(.0044)
3	4	6	34(.1997)	38(.0843)	40(.0492)	43(.0186)	45(.0086)	47(.0034)
3	4	7	39(.1728)	42(.0957)	45(.0464)	48(.0193)	50(.0096)	52(.0044)
3	4	8	43(.1795)	47(.0867)	50(.0442)	53(.0197)	56(.0075)	58(.0035)
3	5	5	35(.1836)	38(.0971)	41(.0438)	43(.0232)	46(.0074)	47(.0048)
3	5	6	40(.1761)	43(.0988)	46(.0489)	49(.0208)	52(.0074)	54(.0033)
3	5	7	44(.1985)	49(.0822)	52(.0421)	55(.0190)	58(.0074)	60(.0036)
3	5	8	49(.1906)	54(.0846)	57(.0463)	60(.0228)	64(.0074)	66(.0038)
3	6	6	45(.1853)	49(.0923)	52(.0486)	55(.0227)	58(.0092)	60(.0046)
3	6	7	50(.1932)	55(.0870)	58(.0482)	61(.0242)	65(.0081)	67(.0042)
3	6	8	55(.1998)	60(.0977)	64(.0479)	68(.0203)	71(.0095)	74(.0040)
3	7	7	56(.1887)	61(.0911)	65(.0442)	68(.0234)	72(.0086)	74(.0049)
3	7	8	62(.1849)	67(.0946)	71(.0492)	75(.0226)	79(.0091)	82(.0041)
3	8	8	68(.1929)	74(.0920)	79(.0425)	82(.0248)	87(.0087)	90(.0042)
4	4	4	31(.1756)	34(.0844)	36(.0463)	38(.0229)	40(.0099)	42(.0037)
4	4	5	36(.1682)	39(.0874)	42(.0387)	44(.0203)	46(.0096)	48(.0040)
4	4	6	40(.1929)	44(.0898)	47(.0438)	49(.0250)	52(.0093)	54(.0043)
4	4	7	45(.1849)	49(.0918)	52(.0482)	55(.0224)	58(.0091)	60(.0045)
4	4	8	50(.1783)	54(.0935)	58(.0420)	61(.0205)	64(.0088)	66(.0047)
4	5	5	41(.1787)	45(.0818)	48(.0393)	50(.0222)	53(.0082)	55(.0037)
4	5	6	46(.1875)	50(.0944)	54(.0397)	56(.0238)	59(.0099)	62(.0035)
4	5	7	51(.1949)	56(.0886)	59(.0496)	63(.0196)	66(.0086)	68(.0046)
4	5	8	57(.1772)	61(.0992)	65(.0495)	69(.0210)	72(.0100)	75(.0042)
4	6	6	52(.1831)	56(.0981)	60(.0455)	63(.0229)	67(.0077)	69(.0041)
4	6	7	58(.1794)	63(.0862)	67(.0417)	70(.0221)	74(.0082)	76(.0046)
4	6	8	63(.1982)	69(.0897)	73(.0456)	77(.0214)	81(.0086)	84(.0039)
4	7	7	64(.1880)	69(.0976)	74(.0432)	77(.0243)	82(.0078)	84(.0046)
4	7	8	70(.1955)	76(.0945)	81(.0444)	85(.0217)	89(.0095)	92(.0047)
4	8	8	74(.1932)	83(.0987)	88(.0497)	93(.0219)	98(.0083)	101(.0043)
5	5	5	77(.1748)	51(.0867)	54(.0456)	57(.0214)	60(.0087)	62(.0044)
5	5	6	52(.1971)	57(.0908)	61(.0415)	64(.0207)	67(.0092)	70(.0036)
5	5	7	58(.1920)	63(.0943)	67(.0466)	71(.0201)	74(.0096)	77(.0041)
5	5	8	64(.1877)	69(.0973)	74(.0430)	77(.0241)	81(.0099)	84(.0046)
5	6	6	59(.1812)	64(.0879)	68(.0430)	71(.0230)	75(.0087)	77(.0050)
5	6	7	65(.1895)	70(.0991)	75(.0443)	79(.0204)	83(.0082)	85(.0049)

(*continued*)

TABLE R (continued)

Critical Values and Associated Probabilities for the Jonckheere–Terpstra Test

			$k = 3$					
n_1	n_2	n_3	$\alpha = 0.200$	$\alpha = 0.100$	$\alpha = 0.050$	$\alpha = 0.025$	$\alpha = 0.010$	$\alpha = 0.005$
5	6	8	71(.1968)	77(.0958)	82(.0454)	86(.0224)	90(.0098)	93(.0049)
5	7	7	72(.1874)	78(.0902)	82(.0498)	86(.0250)	91(.0090)	94(.0045)
5	7	8	79(.1856)	85(.0944)	90(.0474)	94(.0249)	99(.0008)	103(.0041)
5	8	8	86(.1934)	93(.0932)	98(.0492)	103(.0232)	108(.0097)	112(.0043)
6	6	6	66(.1796)	71(.0928)	75(.0490)	79(.0231)	83(.0095)	86(.0045)
6	6	7	72(.1983)	78(.0972)	83(.0464)	87(.0231)	92(.0083)	95(.0041)
6	6	8	79(.1956)	86(.0891)	91(.0444)	95(.0231)	100(.0090)	103(.0047)
6	7	7	80(.1869)	86(.0956)	91(.0484)	96(.0215)	101(.0083)	104(.0043)
6	7	8	87(.1946)	94(.0943)	100(.0435)	104(.0238)	110(.0083)	113(.0045)
6	8	8	95(.1936)	102(.0988)	108(.0487)	113(.0244)	119(.0092)	123(.0044)
7	7	7	88(.1864)	95(.0894)	100(.0471)	105(.0223)	110(.0093)	114(.0042)
7	7	8	96(.1860)	103(.0941)	109(.0460)	114(.0229)	120(.0086)	123(.0049)
7	8	8	104(.1938)	112(.0940)	118(.0483)	124(.0221)	130(.0088)	134(.0044)
8	8	8	113(.1939)	121(.0989)	128(.0480)	134(.0231)	140(.0099)	145(.0044)
n_1	n_2	n_3	n_4	$k = 4$				
2	2	2	2	16(.1929)	18(.0829)	19(.0484)	21(.0123)	22(.0052)	23(.0016)
3	3	3	3	34(.1823)	37(.0907)	40(.0374)	42(.0183)	44(.0080)	45(.0050)
4	4	4	4	58(.1910)	63(.0895)	67(.0420)	70(.0215)	73(.0100)	76(.0041)
5	5	5	5	89(.1846)	95(.0962)	100(.0498)	105(.0230)	110(.0093)	114(.0040)
6	6	6	6	126(.1863)	134(.0961)	141(.0474)	147(.0234)	154(.0089)	158(.0048)
n_1	n_2	n_3	n_4	n_5	$k = 5$				
2	2	2	2	2	26(.1625)	28(.0878)	30(.0412)	32(.0162)	33(.0094)	35(.0026)
3	3	3	3	3	54(.1982)	59(.08734)	62(.0475)	65(.0234)	69(.0076)	71(.0039)
4	4	4	4	4	94(.1876)	100(.0991)	106(.0452)	110(.0245)	116(.0084)	119(.0046)
5	5	5	5	5	144(.1903)	153(.0954)	160(.0497)	167(.0232)	174(.0096)	179(.0047)
6	6	6	6	6	204(.1972)	216(.0984)	226(.0488)	235(.0229)	244(.0097)	251(.0046)
n_1	n_2	n_3	n_4	n_5	n_6	$k = 6$				
2	2	2	2	2	2	37(.1871)	40(.0953)	43(.0404)	45(.0207)	47(.0094)	49(.0038)
3	3	3	3	3	3	80(.1806)	85(.0969)	90(.0452)	94(.0220)	98(.0096)	101(.0047)
4	4	4	4	4	4	138(.1909)	147(.0918)	154(.0457)	160(.0227)	167(.0089)	171(.0049)
5	5	5	5	5	5	212(.1938)	224(.0968)	234(.0478)	242(.0248)	252(.0096)	259(.0046)
6	6	6	6	6	6	302(.1930)	317(.0998)	330(.0449)	342(.0236)	354(.0100)	363(.0048)

Source: Adapted from Tables 1 and 2 of Odeh, R.E., On Jonckheere's *k*-sample test against ordered alternatives, *Technometrics*, 13, 912–918, 1971. With permission.

TABLE S

Rank von Neumann Statistic

Each table entry for $n \leq 10$ is the exact left-tail or right-tail P value of the corresponding listed value of NM. Only those values of NM that are close to the typical values of $\alpha = 0.005$, 0.01, 0.025, 0.05, and 0.10 are included. The table entries for $n > 10$ are the left-tail critical values of RVN for the same typical α values. Since these entries are based on a beta approximation which is symmetric about 2, corresponding right-tail critical values are easily found. For example if $n = 40$, $\alpha = 0.005$, the left-tail critical value of RVN is 1.22 and hence the right-tail critical value is 2.78.

	***P* Values for Selected Values of NM**			
n	**NM**	**Left-Tail *P***	**NM**	**Right-Tail *P***
4	3	0.0833	17	0.0833
	6	0.2500	14	0.2500
5	4	0.0167	35	0.0333
	7	0.0500	33	0.0667
	10	0.1333	30	0.1333
6	5	0.0028	65	0.0028
	8	0.0083	63	0.0083
	11	0.0250	62	0.0139
	14	0.0472	60	0.0194
	16	0.0750	59	0.0306
	17	0.0806	56	0.0361
	19	0.1306	55	0.0694
			52	0.0972
			51	0.1139
7	14	0.0048	101	0.0040
	15	0.0079	100	0.0056
	17	0.0119	98	0.0087
	18	0.0151	97	0.0103
	20	0.0262	93	0.0206
	24	0.0444	92	0.0254
	25	0.0563	88	0.0464
	31	0.0988	87	0.0536
	32	0.1155	81	0.0988
			80	0.1115
8	23	0.0049	149	0.0043
	24	0.0073	148	0.0052
	26	0.0095	144	0.0084
	27	0.0111	143	0.0105
	32	0.0221	136	0.0249

(*continued*)

TABLE S (continued)

Rank von Neumann Statistic

	***P* Values for Selected Values of NM**			
n	**NM**	**Left-Tail *P***	**NM**	**Right-Tail *P***
8	33	0.0264	135	0.0286
	39	0.0481	129	0.0481
	40	0.0526	128	0.0530
	48	0.0978	120	0.0997
	49	0.1049	119	0.1074
9	34	0.0045	208	0.0046
	35	0.0055	207	0.0053
	40	0.0096	202	0.0091
	41	0.0109	201	0.0104
	49	0.0236	191	0.0245
	50	0.0255	190	0.0262
	59	0.0486	181	0.0499
	60	0.0516	180	0.0528
	71	0.0961	169	0.0978
	72	0.1010	168	0.1030
10	51	0.0050	282	0.0046
	59	0.0100	281	0.0051
	72	0.0242	273	0.0097
	73	0.0260	272	0.0103
	85	0.0493	259	0.0240
	86	0.0517	258	0.0252
	101	0.0985	246	0.0475
	102	0.1017	245	0.0504
			229	0.0990
			228	0.1023

	Left-Tail Critical Values of RVN				
n	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**
10	0.62	0.72	0.89	1.04	1.23
11	0.67	0.77	0.93	1.08	1.26
12	0.71	0.81	0.96	1.11	1.29
13	0.74	0.84	1.00	1.14	1.32
14	0.78	0.87	1.03	1.17	1.34
15	0.81	0.90	1.05	1.19	1.36
16	0.84	0.93	1.08	1.21	1.38
17	0.87	0.96	1.10	1.24	1.40
18	0.89	0.98	1.13	1.26	1.41
19	0.92	1.01	1.15	1.27	1.43
20	0.94	1.03	1.17	1.29	1.44

TABLE S (continued)

Rank von Neumann Statistic

	Left-Tail Critical Values of RVN				
n	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**
21	0.96	1.05	1.18	1.31	1.45
22	0.98	1.07	1.20	1.32	1.46
23	1.00	1.09	1.22	1.33	1.48
24	1.02	1.10	1.23	1.35	1.49
25	1.04	1.12	1.25	1.36	1.50
26	1.05	1.13	1.26	1.37	1.51
27	1.07	1.15	1.27	1.38	1.51
28	1.08	1.16	1.28	1.39	1.52
29	1.10	1.18	1.30	1.40	1.53
30	1.11	1.19	1.31	1.41	1.54
32	1.13	1.21	1.33	1.43	1.55
34	1.16	1.23	1.35	1.45	1.57
36	1.18	1.25	1.36	1.46	1.58
38	1.20	1.27	1.38	1.48	1.59
40	1.22	1.29	1.39	1.49	1.60
42	1.24	1.30	1.41	1.50	1.61
44	1.25	1.32	1.42	1.51	1.62
46	1.27	1.33	1.43	1.52	1.63
48	1.28	1.35	1.45	1.53	1.63
50	1.29	1.36	1.46	1.54	1.64
55	1.33	1.39	1.48	1.56	1.66
60	1.35	1.41	1.50	1.58	1.67
65	1.38	1.43	1.52	1.60	1.68
70	1.40	1.45	1.54	1.61	1.70
75	1.42	1.47	1.55	1.62	1.71
80	1.44	1.49	1.57	1.64	1.71
85	1.45	1.50	1.58	1.65	1.72
90	1.47	1.52	1.59	1.66	1.73
95	1.48	1.53	1.60	1.66	1.74
100	1.49	1.54	1.61	1.67	1.74
100[a]	1.48	1.53	1.61	1.67	1.74
100[b]	1.49	1.54	1.61	1.67	1.74

Source: Adapted from Bartels, R., The rank version of von Neumann's ratio test for randomness, *J. Am. Stat. Assoc.*, 77, 40–46, 1982. With permission.

[a] Using the $N(2, 4/n)$ approximation.

[b] Using the $N(2, 20/(5n+7)]$ approximation.

TABLE T

Lilliefors's Test for Exponential Distribution Critical Values
Table entries for any sample size N are the values of a Lilliefors's random variable with right-tail probability as given in the top row.

	Significance Level			
Sample Size N	**0.100**	**0.050**	**0.010**	**0.001**
4	.444	.483	.556	.626
5	.405	.443	.514	.585
6	.374	.410	.477	.551
7	.347	.381	.444	.509
8	.327	.359	.421	.502
9	.310	.339	.399	.460
10	.296	.325	.379	.444
11	.284	.312	.366	.433
12	.271	.299	.350	.412
14	.252	.277	.325	.388
16	.237	.261	.311	.366
18	.224	.247	.293	.328
20	.213	.234	.279	.329
25	.192	.211	.251	.296
30	.176	.193	.229	.270
40	.153	.168	.201	.241
50	.137	.150	.179	.214
60	.125	.138	.164	.193
75	.113	.124	.146	.173
100	.098	.108	.127	.150
Over 100	$.980/\sqrt{N}$	$1.077/\sqrt{N}$	$1.274/\sqrt{N}$	$1.501/\sqrt{N}$

Source: Adapted from Edgeman, R.L. and Scott, R.C., Lilliefors's tests for transformed variables, *Braz. J. Probability Stat.*, 1, 101–112, 1987. With permission.

Answers to Selected Problems

Chapter 2

2.6 $Y = 4X - 2X^2$

2.7 $\left(\dfrac{7-x}{6}\right)^5 - \left(\dfrac{6-x}{6}\right)^5, \quad x = 1, 2, \ldots, 6$

2.8 $X_{(1)} - \ln(20)/3$

2.10 (a) $1 - (0.9)^{10}$ (b) $1 - (0.5)^{1/10}$

2.11 (a) $11/6$ (b) $3/(2\sqrt{\pi})$

2.12 $\begin{cases} 8u^2(3-4u), & 0 < u < 1/2 \\ 32u^3 - 72u^2 + 48u - 8, & 1/2 < u < 1 \end{cases}$

2.13 (a) $1/2$, $1/4(n+2)$ (b) $1/2$, $n/4[(n+1)(n+2)]$

2.14 $(n-1)(e^8 - 1)^2/2$

2.15 $4(n-1)e^{4u}(e^{4u} - 1)$

2.16 $\begin{cases} n(2u)^{n-1}, & 0 < u < 1/2 \\ n[2(1-u)]^{n-1}, & 1/2 < u < 1 \end{cases}$

2.18 (a) $\mu, \pi\sigma^2/2(2m+3)$ (b) 0.2877, 0.016

2.23 0.896

2.24 0.50

2.25 0.05

2.26 $n(n-1)$; $2(n-1)/(n+1)^2(n+2)$

Chapter 3

3.13 (c) $RVN = 0.7413$, $0.005 < P < 0.01$

3.15 (a) $Z = -2.79$, $P = 0.0028$ (b) No, too many zeros

3.16 (a) $R = 2$, $P = 0.095$ (b) $R = 2$, $P = 0.025$

3.17 (a) $R = 6$, $P = 0.069$; $R = 11$, $P = 0.3770$

(b) $R = 4$, $P = 0.054$; $R = 10$, $P = 0.452$

(c) $R = 5$, $P = 0.024$; $R = 12$, $P = 0.3850$

3.18 $R = 6$, $P > 0.5$; $R = 5$, $P > 0.7573$

Chapter 4

4.1 $Q = 3.1526$, $0.25 < P < 0.50$

4.2 $Q = 7.242$, $0.10 < P < 0.25$

4.12 $n = 1063$

4.18 $D = 0.2934$, $0.10 < P < 0.20$

4.20 (a) $D = 0.3120$, $0.10 < P < 0.15$

(b) $D = 0.1994$, $P > 0.20$

(c) (i) 28 (ii) 47

4.21 (a) $Q = 76.89$, df $= 9$, $P < 0.001$ or $Q = 61.13$, df $= 7$, $P < 0.001$

4.25 $Q = 0.27$, df $= 1$, $P > 0.50$

4.27 $Q = 35.54$, $P < 0.001$

4.28 K–S

4.30 (a) $D = 0.1813$, $P > 0.20$ (b) $D = 0.1948$, $P > 0.20$

4.34 $D = 0.400$, $0.05 < P < 0.10$

4.35 $Q = 18.69$, df $= 2$, $P < 0.001$

4.36 $Q = 18.0$, df $= 2$, $P < 0.001$

Chapter 5

5.2 (a) $[(N-1)/(N+1)]^{1/2}$ (b) $1/(2\sqrt{\pi})$

(c) $[3(N-1)/4(N+1)]^{1/2}$

5.4 (1) (a) Reject, if $K \geq 6$ (b) 0.0625

(c) $K = 4$, do not reject H_0 (d) 0.0704

(e) $-4 \leq M_D \leq 16$

(2) (a) Do not reject $H_0, T^- = 9.5$ ($T^+ = 18.5$) (b) 0.078

(c) Do not reject $H_0, T^+ = 16.5$ (d) 0.078

(e) $-3.5 \leq M_D \leq 11$

5.10 $T^+ = 53$, $P = 0.003$; $K = 9$, $P = 0.0107$

5.11 $T^+ = 103$

5.12 $K = 12$, $P = 0.0017$

5.13 $K = 13$, $P = 0.0835$

5.14 $249 \leq M_D \leq 1157, \gamma = 0.9376$

$273 \leq M_D \leq 779, \gamma = 0.876$

5.15 $K = 4$, $P = 0.1875$

5.16 (a) $-8 \leq M \leq 10, \gamma = 0.961$

(b) $-7 \leq M \leq 7.5, \gamma = 0.96$

5.20 (a) $T^+ = 48$, $P = 0.019$

(b) $0.5 \leq M_D \leq 6$, $\gamma = 0.916$

(c) $K = 7$, $P = 0.1719$

(d) $-2 \leq M_D \leq 6$, $\gamma = 0.9786$

5.21 $P = 0.0202$

5.22 (a) $K = 4$, $P = 0.3438$
(b) $-4 \leq M_D \leq 13$, $\gamma = 0.9688$
(c) $T^+ = 18$, $P = 0.078$
(d) $-3 \leq M_D \leq 9.5$, $\gamma = 0.906$

5.23 $P = 0.0176$

5.29 (a) $15/16$ (b) $5^2 3^3/4^5$ (c) $(0.8)^4$

5.31 (a) $n = 7$ (b) $n = 8$

Chapter 6

6.1 0.75

6.6 (i) (a) $U = 6$, do not reject H_0 (b) 130/12, 870
(c) $-27 \leq \theta \leq 80$
(ii) (a) $U = 12.5$, reject H_0 (b) $\alpha = 0.082$
(c) $12 \leq M_Y - M_X \leq 65$

6.9 $mnD = 54$, $P = 0.0030$
$R = 4$, $P = 0.010$

6.14 $P = 0.07$, $3 \leq \theta \leq 9$, $\gamma = 0.857$

Chapter 8

8.9 $-7 \leq M_X - M_Y \leq 19$, $\gamma = 0.97$
$-6 \leq M_X - M_Y \leq 15$, $\gamma = 0.948$

8.10 $W_N = 60$, $P = 0.086$

8.13 $W_N = 54$, $P = 0.866$

8.14 (a) $u = 3$, $\gamma = 0.9346$
(b) $u = 14$, $\gamma = 0.916$
(c) $u = 6$, $\gamma = 0.918$

8.15 (a) $W_N = 14$, $P = 0.452$
(b) $-17 \leq M_D \leq 24$, $\gamma = 0.904$

Chapter 9

9.11 $P = 0.042$

9.13 (a) $0.228 < P < 0.267$
(b) $0.159 < P < 0.191$
(c) K–S or chi square

Chapter 10

10.12 $H = 18.91$

10.14 $H = 16.7$, $H_c = 17.96$, df $= 2$, $P < 0.001$

10.17 $H = 10.5$, $0.001 < P < 0.01$

Chapter 11

11.1 (1) (a) $T = 0.5$ (b) Do not reject H_0, $P = 0.108$
(c) $0.43 \leq \tau \leq 0.57$
(2) (a) $R = 0.69$ (b) Do not reject H_0, $P = 0.070$

11.3 (a) $T = 2(mn - 2u)/N(N-1)$ (b) $T = (2u - mn)/(N/2)$

11.5 $R = 0.25$, $P = 0.26$
$R_c = 0.244$, $P \approx 0.268$
$T = 0.17$, $P = 0.306$
$T_c = 0.17$, $P \approx 0.3$

11.6 (a) $T^+ = 57$, $z = 0.25$, $P = 0.4013$
(b) $T = 0.648$, $P < 0.005$
$R = 0.8038$, $P < 0.001$

11.7 $R = 0.661$, $0.029 < P < 0.033$
$T = 0.479$, $0.038 < P < 0.06$
NM $= 24.75$, $P < 0.0045$
R (runs up and down) $= 3$, $P = 0.0257$

11.14 (a) $R = 0.7143$
(b) $P = 0.068$
(c) $T = 0.4667$
(d) $P = 0.136$

11.15 $R = 0.6363$, $P = 0.027$
$T = 0.51$, $P = 0.023$

11.18 (a) $T = 0.6687$
(b) $R = 0.7793$

Chapter 12

12.1 $S = 37$, $P = 0.033$

12.2 (a) $S = 312$, $W = 0.825$

12.4 $T_{XY} = 7/15$, $T_{XZ} = 5/15$
$T_{YZ} = 5/15$, $T_{XY\ Z} = 2/5$

12.5 (a) $R = 0.977$

12.6 $T_{XZ} = 0.80$, $T_{YZ} = -0.9487$
$T_{XY} = -0.7379$, $T_{XY.Z} = 0.1110$
12.7 $T_{12} = -0.7143$, $T_{13} = -0.5714$
$T_{23} = 0.5714$, $T_{23.1} = 0.2842$
$P > 0.05$
12.8 $Q = 4.814$, df $= 5$, $P > 0.30$
$Q_C = 4.97$, $P > 0.30$
12.9 $Q = 19.14$, df $= 6$, $0.001 < P < 0.005$
12.13 (a) $W = 0.80$, $P < 0.02$
(b) IADGBHCJFE
12.14 $S = 43.5$, $0.072 < P < 0.094$

Chapter 13

13.1 (a) $1/2\sigma\sqrt{\pi}$ (b) $3N\lambda_N(1-\lambda_N)$ (c) $3N\lambda_N(1-\lambda_N)\lambda^2/4$
13.2 (b) $1/12$ (c) σ^2
13.5 1

Chapter 14

14.2 $Q = 8.94$, df $= 2$, $0.01 < P < 0.05$
14.3 $Z_c = 1.94$, $P = 0.0262$
14.4 $Z_c = 0.80$, $P = 0.2119$
14.5 Exact $P = 0.2619$
14.6 $Z_c = 1.643$, $P = 0.0505$
14.7 $Z_c = 3.82$, $P < 0.001$
14.8 $Q = 0.5759$, df $= 2$, $0.50 < P < 0.70$
14.9 (a) $Q = 13.83$, df $= 4$, $0.005 < P < 0.01$
(b) $C = 0.35$, $\phi = 0.37$
(c) $T = -0.16$, $Z = -2.39$, $P = 0.0091$
(d) $\gamma = -0.2366$
14.10 $Z_c = 0.62$, $P = 0.2676$
14.11 Exact $P = 0.0835$
14.12 Exact $P = 0.1133$
14.13 $Q = 261.27$, df $= 12$, $P < 0.001$
14.14 $Z = 2.01$, $P = 0.0222$
14.18 $Q = 3.24$
14.19 $Q = 43.25$, df $= 2$, $P < 0.001$

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