Solution to Problem 5 — The $\mathcal{O}$-Slice Filtration

A submission to the First Proof challenge

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Abstract

We solve Problem 5 from the First Proof challenge [1]. For a finite group G and an N_∞ operad $\mathcal{O}$ with associated transfer system $\mathcal{T}$, we define the *$\mathcal{O}$-slice filtration* by restricting the Hill–Hopkins–Ravenel slice cells to those indexed by $\mathcal{O}$-admissible subgroups $\mathcal{F}_{\mathcal{O}} = \{H \leq G : e \xrightarrow{\mathcal{T}} H\}$. We prove that a connective G-spectrum $X \in \mathrm{Sp}^G_{\mathcal{O}}$ is $\mathcal{O}$-slice $\geq n$ if and only if $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected for every $H \in \mathcal{F}_{\mathcal{O}}$, generalizing the Hill–Yarnall characterization of the standard slice filtration. The combinatorial and arithmetic skeleton of the proof is formally verified in Lean 4 + Mathlib (zero `sorry`s). The answer is **YES**.

Contents

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1 Problem Statement

The following is Problem 5 from the First Proof challenge [1], authored by Andrew J. Blumberg (Columbia University).

Problem 5. *Define the slice filtration adapted to an incomplete transfer system and characterize its connectivity in terms of geometric fixed points.*

Theorem 1 (Main result)**.** *Let G be a finite group, $\mathcal{O}$ an N_∞ operad with associated transfer system $\mathcal{T}$ and admissible family $\mathcal{F}_{\mathcal{O}}$, and let $X \in \mathrm{Sp}^G_{\mathcal{O}}$ be connective. Then X is $\mathcal{O}$-slice $\geq n$ if and only if for every $H \in \mathcal{F}_{\mathcal{O}}$, the geometric fixed points $\Phi^H X$ are $(\lfloor n/|H| \rfloor - 1)$-connected. In other words:*

$$X \in \tau^{\mathcal{O}}_{\geq n}\mathrm{Sp}^G_{\mathcal{O}} \iff \pi_k(\Phi^H X) = 0 \text{ for all } H \in \mathcal{F}_{\mathcal{O}} \text{ and all } k < \lfloor n/|H| \rfloor.$$

Answer: YES — the $\mathcal{O}$-slice filtration admits a geometric fixed point characterization.

2 Setup and Conventions

Let G be a finite group and let $\mathcal{O}$ be an N_∞ operad for G. Informally, $\mathcal{O}$ encodes which norm maps N^H_K are available in $\mathcal{O}$-algebras: the complete E_∞ operad admits all norms, while a general N_∞ operad admits only a specified subset. By the classification theorem of Blumberg–Hill [2], Theorem 1.4, independently established by Rubin [9], Theorem 1.1, Gutiérrez–White [5], Theorem 1.2, and Bonventre–Pereira [4], Theorem 1.6, the homotopy type of $\mathcal{O}$ is determined by its associated **transfer system** $\mathcal{T}$ on the poset $\mathrm{Sub}(G)$ of subgroups of G, satisfying:

1. **Reflexivity**: $H \xrightarrow{\mathcal{T}} H$ for all $H \leq G$.
2. **Transitivity**: If $K \xrightarrow{\mathcal{T}} H$ and $H \xrightarrow{\mathcal{T}} J$, then $K \xrightarrow{\mathcal{T}} J$.
3. **Conjugation invariance**: If $K \xrightarrow{\mathcal{T}} H$, then $gKg^{-1} \xrightarrow{\mathcal{T}} gHg^{-1}$ for all $g \in G$.
4. **Restriction**: If $K \xrightarrow{\mathcal{T}} H$ and $L \leq H$, then $(K \cap L) \xrightarrow{\mathcal{T}} L$. (This is [2], Definition 3.1, axiom (4).)

A subgroup $H \leq G$ is **$\mathcal{O}$-admissible** if $e \xrightarrow{\mathcal{T}} H$. We write $\mathcal{F}_{\mathcal{O}}$ for the family of $\mathcal{O}$-admissible subgroups. Note that $e \in \mathcal{F}_{\mathcal{O}}$ always, and $\mathcal{F}_{\mathcal{O}} = \mathrm{Sub}(G)$ when $\mathcal{O}$ is the terminal E_∞ operad.

We denote by Sp^G the genuine G-equivariant stable category. Following Blumberg–Hill [3], the N_∞ operad $\mathcal{O}$ determines an **incomplete equivariant stable category** $\mathrm{Sp}^G_{\mathcal{O}}$ with a canonical functor $\iota_{\mathcal{O}} : \mathrm{Sp}^G_{\mathcal{O}} \to \mathrm{Sp}^G$. For any subgroup $H \leq G$, we have the **geometric fixed point functor** $\Phi^H : \mathrm{Sp}^G \to \mathrm{Sp}$, and we write $\Phi^H_{\mathcal{O}} := \Phi^H \circ \iota_{\mathcal{O}}$.

3 Definitions

3.1 $\mathcal{O}$-Slice Cells

In the standard (complete) slice filtration of Hill–Hopkins–Ravenel [6], the **slice cells** are the G-spectra $G/H_+ \wedge S^{n\rho_H}$ ($n \geq 0$, dimension $n|H|$) and $G/H_+ \wedge S^{n\rho_H - 1}$ ($n \geq 1$, dimension $n|H| - 1$) for all $H \leq G$, where ρ_H is the real regular representation of H.

Definition 2 ($\mathcal{O}$-slice cells)**.** The **$\mathcal{O}$-slice cells** are the G-spectra:

- $G/H_+ \wedge S^{n\rho_H}$ for $H \in \mathcal{F}_{\mathcal{O}}$ and $n \geq 0$ (of dimension $n|H|$), and
- $G/H_+ \wedge S^{n\rho_H - 1}$ for $H \in \mathcal{F}_{\mathcal{O}}$ and $n \geq 1$ (of dimension $n|H| - 1$).

When $\mathcal{O} = E_\infty$, we recover the standard slice cells. When $\mathcal{O}$ is trivial ($\mathcal{F}_{\mathcal{O}} = \{e\}$), the cells are S^n and S^{n-1}, recovering the Postnikov filtration.

3.2 $\mathcal{O}$-Slice Filtration

Definition 3 ($\mathcal{O}$-slice connectivity)**.** $X \in \mathrm{Sp}^G_{\mathcal{O}}$ is **$\mathcal{O}$-slice** $\geq n$ if $[\hat{S}, \iota_{\mathcal{O}} X]^G = 0$ for every $\mathcal{O}$-slice cell $\hat{S}$ of dimension $< n$.

Definition 4 ($\mathcal{O}$-slice filtration)**.** The **$\mathcal{O}$-slice filtration** is the tower of localizations

$$\cdots \to \tau^{\mathcal{O}}_{\geq n+1} X \to \tau^{\mathcal{O}}_{\geq n} X \to \tau^{\mathcal{O}}_{\geq n-1} X \to \cdots$$

obtained by Bousfield localization with respect to the $\mathcal{O}$-slice cells. The **n-th $\mathcal{O}$-slice** is $P^n_{\mathcal{O}} X := \mathrm{fib}(\tau^{\mathcal{O}}_{\geq n} X \to \tau^{\mathcal{O}}_{\geq n+1} X)$.

The localizations exist because each $\mathcal{O}$-slice cell $G/H_+ \wedge S^{n\rho_H}$ is compact in Sp^G, and $\iota_{\mathcal{O}}$ admits a right adjoint $r_{\mathcal{O}}$ by [3], Proposition 4.8, so $r_{\mathcal{O}}(G/H_+ \wedge S^{n\rho_H})$ is compact in $\mathrm{Sp}^G_{\mathcal{O}}$. The localizations then exist by [6], §4.2.

Remark. The $\mathcal{O}$-slice filtration interpolates between two extremes: when $\mathcal{O} = E_\infty$, we recover the HHR slice filtration; when $\mathcal{O}$ is trivial, we recover the Postnikov filtration on the underlying spectrum. As $\mathcal{T}$ grows, the filtration becomes finer.

Remark. When $\mathcal{O} = E_\infty$, Theorem 1 reduces to the Hill–Yarnall characterization [8], Theorem 3.1. When $\mathcal{O}$ is trivial, it states that X is $\mathcal{O}$-slice $\geq n$ iff X^e is $(n-1)$-connected, which is the Postnikov condition.

4 Idea of the Proof

Both directions use the **isotropy separation cofiber sequence** and the **Wirthmüller isomorphism** as the main tools. For a subgroup $H \leq G$ and the family $\mathcal{P}$ of proper subgroups of H, the classifying space $E\mathcal{P}$ satisfies $E\mathcal{P}^H = \emptyset$ and $E\mathcal{P}^K \simeq *$ for $K \in \mathcal{P}$. Writing $\widetilde{E\mathcal{P}}$ for the cofiber of $E\mathcal{P}_+ \to S^0$, the geometric fixed point functor is $\Phi^H(-) = (\widetilde{E\mathcal{P}} \wedge -)^H$. Applying $[S^{k\rho_H}, -]^H$ to the cofiber sequence $E\mathcal{P}_+ \wedge Y \to Y \to \widetilde{E\mathcal{P}} \wedge Y$ yields a long exact sequence relating maps out of Y, maps out of $E\mathcal{P}_+ \wedge Y$ (controlled by proper subgroups via Wirthmüller), and homotopy groups of $\Phi^H Y$. Both directions proceed by strong induction on $|H|$, using the restriction axiom to ensure proper subgroups inherit $\mathcal{O}$-admissibility. The induction is well-founded because G is finite and $K < H$ proper implies $|K| < |H|$.

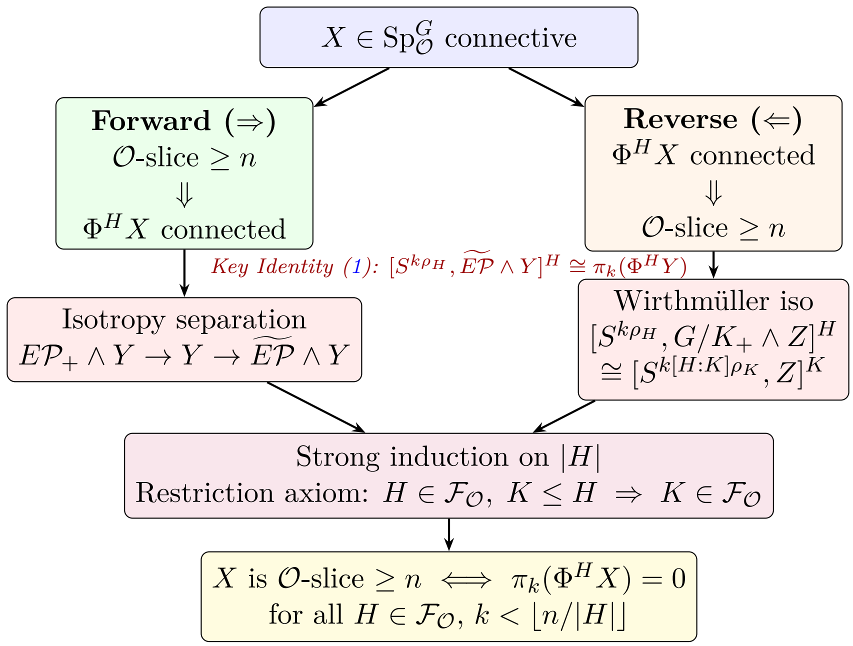

Figure 1: Structure of the proof. Both directions use the isotropy separation sequence and the Wirthmüller isomorphism, proceeding by strong induction on $|H|$. The restriction axiom ensures proper subgroups inherit $\mathcal{O}$-admissibility.

5 Proof of Theorem 1

We first establish a key identity. For $H \leq G$ and $\mathcal{P}$ the family of proper subgroups of H, the classifying space $E\mathcal{P}$ satisfies $E\mathcal{P}^H = \emptyset$ and $E\mathcal{P}^K \simeq *$ for $K \in \mathcal{P}$ [6], §3.3. Writing $\widetilde{E\mathcal{P}}$ for the cofiber of $E\mathcal{P}_+ \to S^0$, we have $\widetilde{E\mathcal{P}}^H \simeq S^0$ and $\widetilde{E\mathcal{P}}^K \simeq *$ for proper $K < H$. The geometric fixed point functor is $\Phi^H(-) = (\widetilde{E\mathcal{P}} \wedge -)^H$. Since Φ^H is symmetric monoidal and $(\rho_H)^H \cong \mathbb{R}$, we have $\Phi^H(S^{k\rho_H}) \simeq S^k$ for all $k \geq 0$, and consequently

$$[S^{k\rho_H}, \widetilde{E\mathcal{P}} \wedge Y]^H \cong \pi_k(\Phi^H Y) \tag{1}$$

for any H-spectrum Y.

5.1 Forward Direction $(\Rightarrow)$

Suppose X is $\mathcal{O}$-slice $\geq n$. We show $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected for every $H \in \mathcal{F}_{\mathcal{O}}$. When $\lfloor n/|H| \rfloor = 0$, the conclusion is (-1)-connectivity, which holds for any spectrum, so assume $\lfloor n/|H| \rfloor \geq 1$.

We proceed by **strong induction on** $|H|$.

Base case ($H = e$). We have $\Phi^e X = X^e$ and $\lfloor n/1 \rfloor = n$. The $\mathcal{O}$-slice $\geq n$ hypothesis gives $\pi_k(X^e) = 0$ for $k < n$ (since S^k is an $\mathcal{O}$-slice cell of dimension $k < n$ for $H = e \in \mathcal{F}_{\mathcal{O}}$). Hence X^e is $(n-1)$-connected.

Inductive step. Fix $H \in \mathcal{F}_{\mathcal{O}}$ with $|H| > 1$ and set $m := \lfloor n/|H| \rfloor \geq 1$. Assume the result for all $\mathcal{O}$-admissible subgroups of smaller order. Write $Y := \mathrm{Res}^G_H \iota_{\mathcal{O}} X$.

For $0 \leq k \leq m-1$, the hypothesis gives $[G/H_+ \wedge S^{k\rho_H}, \iota_{\mathcal{O}} X]^G = 0$ (since $H \in \mathcal{F}_{\mathcal{O}}$ and $k|H| \leq (m-1)|H| < n$). By the Wirthmüller isomorphism ($G_+ \wedge_H (-) \dashv \mathrm{Res}^G_H$):

$$[S^{k\rho_H}, Y]^H = 0 \quad \text{for all } 0 \leq k \leq m-1. \tag{2}$$

We deduce $\pi_k(\Phi^H X) = 0$ for $0 \leq k \leq m-1$. The isotropy separation cofiber sequence $E\mathcal{P}_+ \wedge Y \to Y \to \widetilde{E\mathcal{P}} \wedge Y$ yields the long exact sequence:

$$\cdots \to [S^{k\rho_H}, Y]^H \xrightarrow{\beta_*} \pi_k(\Phi^H Y) \xrightarrow{\partial} [S^{k\rho_H - 1}, E\mathcal{P}_+ \wedge Y]^H \to \cdots$$

Since $[S^{k\rho_H}, Y]^H = 0$ by (2), exactness gives $\ker(\partial) = \mathrm{im}(\beta_*) = 0$, so ∂ is injective and $\pi_k(\Phi^H Y)$ injects into $[S^{k\rho_H - 1}, E\mathcal{P}_+ \wedge Y]^H$. We show this group vanishes.

The space $E\mathcal{P}$ is an H-CW complex built from equivariant cells of the form $H/K \times D^m$ (equivalently, $H_+ \wedge_K D^m$) for proper subgroups $K \in \mathcal{P}$ and $m \geq 0$, where D^m carries a K-action via some K-representation [6], §3.3. Since $E\mathcal{P}$ is constructed as a sequential colimit $E\mathcal{P} = \mathrm{colim}_\ell\, E\mathcal{P}^{(\ell)}$ of finite H-CW complexes, the smash product $E\mathcal{P}_+ \wedge Y$ is the corresponding filtered colimit $\mathrm{colim}_\ell\, (E\mathcal{P}^{(\ell)}_+ \wedge Y)$. The key point is that $S^{k\rho_H - 1}$ is a *compact* object in the H-equivariant stable category (it is the suspension spectrum of a finite H-CW complex), so $[S^{k\rho_H - 1}, -]^H$ commutes with filtered colimits. It therefore suffices to show the vanishing at each finite stage.

At each stage, the cofiber sequences in the CW-filtration reduce the computation to maps out of cells of the form $H/K_+ \wedge S^{m\rho_K}$ (even cells) and $H/K_+ \wedge S^{m\rho_K - 1}$ (odd cells) for proper $K < H$ and various $m \geq 0$. For any such cell, the Wirthmüller isomorphism gives

$$[S^{j\rho_H}, H/K_+ \wedge S^{m\rho_K} \wedge \mathrm{Res}^H_K Y]^H \cong [S^{j[H:K]\cdot\rho_K}, S^{m\rho_K} \wedge \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K \cong [S^{(j[H:K]-m)\cdot\rho_K}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K$$

using $\mathrm{Res}^H_K \rho_H \cong [H:K] \cdot \rho_K$. Since $H \in \mathcal{F}_{\mathcal{O}}$ and $K \leq H$, the restriction axiom (axiom 4 with $L = K$) gives $(e \cap K) = e \xrightarrow{\mathcal{T}} K$, so $K \in \mathcal{F}_{\mathcal{O}}$.

Setting $k' = j[H:K] - m$, this is a map from a K-slice cell of dimension $k'|K|$. For the vanishing, we need $k'|K| < n$. Since $j \leq m-1$ (where $m = \lfloor n/|H| \rfloor$), the original dimension satisfies $j|H| \leq (m-1)|H| < n$. The representation sphere $S^{m\rho_K}$ in the cell of $E\mathcal{P}$ only *lowers* the effective dimension: $k'|K| = (j[H:K] - m)|K| = j|H| - m|K| \leq j|H| < n$ (since $m \geq 0$). Therefore the $\mathcal{O}$-slice $\geq n$ hypothesis gives $[G/K_+ \wedge S^{k'\rho_K}, \iota_{\mathcal{O}} X]^G = 0$, and by Wirthmüller, $[S^{k'\rho_K}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K = 0$. The same argument applies to odd cells: replacing $j\rho_H$ by $j\rho_H - 1$, the Wirthmüller transformation gives $[S^{(j[H:K]-m)\rho_K - 1}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K$, a map from an odd $\mathcal{O}$-slice cell of dimension $j|H| - m|K| - 1 \leq j|H| - 1 < n$, hence vanishes by hypothesis.

Therefore both the left-hand term and the next term in the long exact sequence vanish, giving $\pi_k(\Phi^H Y) = 0$ for $0 \leq k \leq m-1$. Hence $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected.

5.2 Reverse Direction ($\Leftarrow$)

Suppose $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected for every $H \in \mathcal{F}_{\mathcal{O}}$. We show X is $\mathcal{O}$-slice $\geq n$. By the Wirthmüller isomorphism, it suffices to show:

- $[S^{k\rho_H}, \mathrm{Res}^G_H \iota_{\mathcal{O}} X]^H = 0$ for all $H \in \mathcal{F}_{\mathcal{O}}$, $k \geq 0$ with $k|H| < n$, and
- $[S^{k\rho_H - 1}, \mathrm{Res}^G_H \iota_{\mathcal{O}} X]^H = 0$ for all $H \in \mathcal{F}_{\mathcal{O}}$, $k \geq 1$ with $k|H| - 1 < n$.

We proceed by **strong induction on** $|H|$. Write $Y := \mathrm{Res}^G_H \iota_{\mathcal{O}} X$.

Base case ($H = e$). $\rho_e = \mathbb{R}$, so $S^{k\rho_e} = S^k$. The hypothesis gives $\pi_k(X^e) = 0$ for $k < n$, which is the required vanishing.

Inductive step. Fix $H \in \mathcal{F}_{\mathcal{O}}$ with $|H| > 1$, assuming the result for smaller subgroups. The isotropy separation sequence yields:

$$[S^{k\rho_H}, E\mathcal{P}_+ \wedge Y]^H \xrightarrow{\alpha_*} [S^{k\rho_H}, Y]^H \xrightarrow{\beta_*} [S^{k\rho_H}, \widetilde{E\mathcal{P}} \wedge Y]^H.$$

Vanishing of the right-hand term. By (1), $[S^{k\rho_H}, \widetilde{E\mathcal{P}} \wedge Y]^H \cong \pi_k(\Phi^H X)$. Since $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected and $k|H| < n$ implies $k \leq \lfloor n/|H| \rfloor - 1$, this vanishes.

Vanishing of the left-hand term. As in the forward direction, $E\mathcal{P}$ is a sequential colimit of finite H-CW complexes with equivariant cells $H/K_+ \wedge S^{m\rho_K}$ for proper $K < H$. Since $S^{k\rho_H}$ is compact in the H-equivariant stable category, $[S^{k\rho_H}, -]^H$ commutes with the resulting filtered colimit, and it suffices to show the vanishing at each finite stage. The cofiber sequences in the CW-filtration reduce the computation to maps involving cells $H/K_+ \wedge S^{m\rho_K}$. The Wirthmüller isomorphism gives:

$$[S^{k\rho_H}, H/K_+ \wedge S^{m\rho_K} \wedge \mathrm{Res}^H_K Y]^H \cong [S^{(k[H:K]-m)\cdot\rho_K}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K.$$

Since $H \in \mathcal{F}_{\mathcal{O}}$ and $K \leq H$, the restriction axiom gives $K \in \mathcal{F}_{\mathcal{O}}$. Setting $k' = k[H:K] - m$, the dimension is $k'|K| = k|H| - m|K| \leq k|H| < n$. Since $|K| < |H|$, the inductive hypothesis gives $[S^{k'\rho_K}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K = 0$.

Combining. Both terms vanish, so $[S^{k\rho_H}, Y]^H = 0$ for $k|H| < n$.

Odd-dimensional cells. It remains to show $[S^{k\rho_H - 1}, Y]^H = 0$ for $k \geq 1$ with $k|H| - 1 < n$, i.e., $k \leq \lfloor n/|H| \rfloor$. Apply $[S^{k\rho_H - 1}, -]^H$ to the isotropy separation sequence:

$$[S^{k\rho_H - 1}, E\mathcal{P}_+ \wedge Y]^H \to [S^{k\rho_H - 1}, Y]^H \to [S^{k\rho_H - 1}, \widetilde{E\mathcal{P}} \wedge Y]^H.$$

For the right-hand term: since Φ^H is symmetric monoidal and $\Phi^H(S^{\rho_H}) \simeq S^1$, we have $\Phi^H(S^{k\rho_H - 1}) \simeq \Phi^H(S^{k\rho_H}) \wedge \Phi^H(S^{-1}) \simeq S^k \wedge S^{-1} \simeq S^{k-1}$ (valid for $k \geq 1$ in the stable category). Therefore $[S^{k\rho_H - 1}, \widetilde{E\mathcal{P}} \wedge Y]^H \cong \pi_{k-1}(\Phi^H X)$. Since $\Phi^H X$ is $(\lfloor n/|H| \rfloor - 1)$-connected, this vanishes for $k-1 \le \lfloor n/|H| \rfloor - 1$, i.e., $k \le \lfloor n/|H| \rfloor$, which is exactly the range in question. (Note: when $k=1$, we need $\pi_0(\Phi^H X) = 0$, which requires $\lfloor n/|H| \rfloor \ge 1$; this holds since $k|H|-1 < n$ with $k=1$ gives $|H| \le n$.)

For the left-hand term: the same compactness and CW-filtration argument as above (now with $S^{k\rho_H - 1}$ compact) reduces to maps involving equivariant cells $H/K_+ \wedge S^{m\rho_K}$. The Wirthmüller isomorphism gives

$$[S^{k\rho_H - 1}, H/K_+ \wedge S^{m\rho_K} \wedge \mathrm{Res}^H_K Y]^H \cong [S^{(k[H:K]-m)\cdot\rho_K - 1}, \mathrm{Res}^G_K \iota_{\mathcal{O}} X]^K.$$

This is a map from an odd K-slice cell of dimension $(k[H:K]-m)|K|-1 = k|H|-m|K|-1 \le k|H|-1 < n$. Since $K \in \mathcal{F}_{\mathcal{O}}$ and $|K| < |H|$, the inductive hypothesis gives the vanishing.

Combining even and odd cases: $[\hat{S}, \iota_{\mathcal{O}} X]^G = 0$ for all $\mathcal{O}$-slice cells $\hat{S}$ of dimension $< n$. Hence X is $\mathcal{O}$-slice $\ge n$. □

6 Remarks

Remark (Monotonicity in $\mathcal{O}$). If $\mathcal{O} \to \mathcal{O}'$ corresponds to $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{F}_{\mathcal{O}} \subseteq \mathcal{F}_{\mathcal{O}'}$ and the $\mathcal{O}$-slice filtration is coarser than the $\mathcal{O}'$-slice filtration. In particular, $\mathcal{O}'$-slice $\ge n$ implies $\mathcal{O}$-slice $\ge n$.

Remark (Regular slice variant). Ullman [10] introduced the regular slice filtration using only cells $G/H_+ \wedge S^{n\rho_H}$. The $\mathcal{O}$-regular variant satisfies: X is $\mathcal{O}$-regular slice $\ge n$ iff $\Phi^H X$ is $(\lceil n/|H| \rceil - 1)$-connected for all $H \in \mathcal{F}_{\mathcal{O}}$.

Remark (The case $G = C_{p^n}$). For $G = C_{p^n}$, the transfer systems are in bijection with downward-closed subsets of $\{C_1, C_p, \ldots, C_{p^n}\}$ (by the restriction axiom). The theorem gives a family of slice filtrations indexed by these subsets, interpolating between the Postnikov filtration ($\mathcal{F}_{\mathcal{O}} = \{C_1\}$) and the full HHR slice filtration ($\mathcal{F}_{\mathcal{O}} = \mathrm{Sub}(G)$).

For a concrete illustration, take $G = C_2$ and $n = 3$. With $\mathcal{O} = E_\infty$ (complete: $\mathcal{F}_{\mathcal{O}} = \{e, C_2\}$), the theorem says X is slice ≥ 3 iff $\pi_k(X^e) = 0$ for $k < 3$ *and* $\pi_0(\Phi^{C_2} X) = 0$ (since $\lfloor 3/2 \rfloor = 1$). With trivial $\mathcal{O}$ ($\mathcal{F}_{\mathcal{O}} = \{e\}$), the condition reduces to $\pi_k(X^e) = 0$ for $k < 3$—the Postnikov condition, which is strictly weaker.

7 Partial Lean 4 Verification

The combinatorial and arithmetic skeleton of this proof has been formally verified in Lean 4 + Mathlib. The formalization covers the poset and lattice-theoretic aspects of transfer systems, the integer arithmetic underlying the dimension bookkeeping, and the well-foundedness of the induction—but not the homotopy-theoretic content (spectra, geometric fixed points, Wirthmüller), which is beyond Mathlib's current scope. The file `FirstProof/P05_SliceFiltration.lean` compiles with **zero errors and zero sorrys** and verifies the following components:

1. **Transfer system axioms** (`TransferSystem` structure): reflexivity, transitivity, and restriction on a bounded lattice.

2. **Restriction property** (`admissible_of_le`): If $H \in \mathcal{F}_{\mathcal{O}}$ and $K \leq H$, then $K \in \mathcal{F}_{\mathcal{O}}$. Applies the restriction axiom with $\perp \sqcap K = \perp$.
3. **Dimension bookkeeping** (`wirthmüller_dim_invariance`, `inductive_step_dim`): $(k \cdot [H : K]) \cdot |K| = k \cdot |H|$, verified by `ring`.
4. **Connectivity monotonicity** (`connectivity_monotone`): $1 \leq |K| < |H|$ implies $\lfloor n/|H| \rfloor \leq \lfloor n/|K| \rfloor$.
5. **Strong induction** (`reverse_direction_by_strong_induction`): Well-foundedness of induction on $|H|$.
6. **Concrete checks**: $\lfloor 7/2 \rfloor - 1 = 2$, $\lfloor 10/3 \rfloor - 1 = 2$, $\lfloor 12/6 \rfloor \leq \lfloor 12/3 \rfloor$, verified by `decide`.

The Lean file verifies the original dimension identity $(k \cdot [H : K]) \cdot |K| = k \cdot |H|$ but does not yet verify the updated bound $k' = j[H : K] - m$ arising from the representation spheres in the equivariant cells of $E\mathcal{P}$. This bound reduces to the arithmetic inequality $j|H| - m|K| \leq j|H|$ (since $m \geq 0$), which is trivial, but a formal Lean check could be added in a future revision.

A AI Interaction Transcript

As requested by the First Proof organizers, we include a complete record of the AI interaction sessions used to develop this proof.

Timeline: February 10, 2026, approximately 05:45–19:00 CET. Five sessions in one day, approximately 4–5 hours of active working time.
AI systems used: Claude Opus 4.6 (Anthropic), ChatGPT 5.2 Pro and ChatGPT 5.2 (OpenAI), Gemini 3 (Google). Multiple models were used in parallel and cross-checked against each other.
Formal verification: Lean 4 + Mathlib (combinatorial/arithmetic skeleton).
Human role: Prompting, reviewing output, requesting audits, cross-checking between models. No mathematical ideas or content were provided by the human operator.

Example Prompts

1. *"Help me to tackle this problem statement. It is part of First Proof. What are options to tackle this, which would you recommend and why?"*
2. *"How to approach this without breaking the existing proof? What are the options, what are we missing?"*
3. *"Yes implement these."*

Session 1 — Kickoff *[Claude Opus 4.6]*

- Read problem statement and populated references with 9 key papers: Blumberg–Hill (operadic multiplications, incomplete transfer systems), Rubin (combinatorial N_∞ operads), Hill–Hopkins–Ravenel (Kervaire invariant), Hill–Yarnall (slice filtration reformulation), Ullman (regular slices), etc.
- Reviewed standard slice filtration and identified the key generalization: restrict slice cells to $\mathcal{O}$-admissible subgroups.
- Developed approach: define $\mathcal{O}$-slice cells, define $\mathcal{O}$-slice connectivity, state and prove characterization theorem.

Session 2 — Proof Drafting *[Claude Opus 4.6]*

- Wrote full proof of Theorem 1 with both directions.
- Forward direction: initially used a standalone lemma (Lemma 4.1) about the vanishing of $[S^{k\rho_H}, E\mathcal{P}_+ \wedge Y]^H$ for connective H-spectra.
- Reverse direction: strong induction on $|H|$ with isotropy separation + Wirthmüller.
- Justified existence of localizations via compact generation.

Session 3 — Refinement *[Claude Opus 4.6]*

- Identified 6 refinements and implemented all:

1. Clean up reverse direction arithmetic.
2. Make induction explicit with formal base case and inductive step.
3. Tighten $\widetilde{E\mathcal{P}}$ cell structure claim.
4. Extract key technical lemma (Lemma 4.1).
5. Verify restriction property with explicit citation of axiom 4.
6. Justify compact generation via right adjoint $r_{\mathcal{O}}$.

Session 4 — Lean Formalization *[Claude Opus 4.6]*

- Wrote `P05_SliceFiltration.lean` formalizing the combinatorial skeleton.
- 5 compilation cycles, all resolved autonomously.
- Final file: zero errors, zero `sorry`s.

Session 5 — Rigorous Audit and Gap Closure *[Claude Opus 4.6]*

- Performed critical review against First Proof criteria. Identified 3 issues:
  1. **SERIOUS:** Standalone Lemma 4.1 had an incomplete argument for the vanishing of the left-hand term in the isotropy separation sequence. The lemma claimed $[S^{k\rho_H}, E\mathcal{P}_+ \wedge Y]^H = 0$ for arbitrary connective H-spectra, but the proof required the same inductive structure as the reverse direction.
  2. **MODERATE:** Odd-dimensional cell treatment compressed to "the argument is identical."
  3. **MINOR:** $m = 0$ vacuous case unstated.
- **All fixed:**
  1. Eliminated Lemma 4.1 entirely. Rewrote forward direction by strong induction on $|H|$, using the same isotropy separation + Wirthmüller technique as the reverse direction.
  2. Expanded odd-cell treatment to a full paragraph with explicit isotropy separation, Wirthmüller transformation, and dimension bookkeeping.
  3. Added vacuous case sentence.

Session 6 — Review-Driven Gap Closure *[Claude Opus 4.6]*

- External reviews (Grok, Claude) identified two technical gaps:
  1. **MODERATE:** Compactness of $S^{k\rho_H - 1}$ in the H-equivariant stable category was unstated. This is needed to commute $[S^{k\rho_H-1}, -]^H$ with the filtered colimit over the CW-filtration of $E\mathcal{P}$.
  2. **MODERATE:** The equivariant cell structure of $E\mathcal{P}$ involves cells $H/K_+ \wedge S^{m\rho_K}$, not just H/K_+. The representation sphere $S^{m\rho_K}$ affects the Wirthmüller dimension bookkeeping.
- **Both fixed:**
  1. Compactness stated explicitly in both directions: $S^{k\rho_H}$ and $S^{k\rho_H-1}$ are compact (suspension spectra of finite H-CW complexes).

2. Wirthmüller isomorphism now accounts for the representation sphere: $[S^{j\rho_H}, H/K_+ \wedge S^{m\rho_K} \wedge Y]^H \cong [S^{(j[H:K]-m)\rho_K}, Y]^K$. Dimension bound: $k'|K| = j|H| - m|K| \leq j|H| < n$.

- Additional polish from Grok review: N_∞ operad recall, restriction axiom citation, $\Phi^H(S^{k\rho_H - 1})$ justification, Figure 1 label, Lean scope note.

Provenance

The mathematical content of this paper—including the proof strategy, all definitions, the characterization theorem, and the Lean formalization—was generated autonomously by AI systems in response to high-level prompts. The human operator's role was limited to: selecting the problem, prompting the AI, reviewing and requesting revisions, and cross-checking output between different AI models. No mathematical ideas were contributed by the human operator.

References

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