Problem 9 — Quadrilinear Determinantal Tensors

Theorem

Let $n \geq 5$. Let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, let $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ with entries

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl} = \det\bigl[A^{(\alpha)}(i,:);\ A^{(\beta)}(j,:);\ A^{(\gamma)}(k,:);\ A^{(\delta)}(l,:)\bigr].$$

Then there exists a polynomial map $\mathbf{F}\colon \mathbb{R}^{81n^4} \to \mathbb{R}^N$ such that:

1. $\mathbf{F}$ does not depend on $A^{(1)}, \ldots, A^{(n)}$.
2. The degrees of the coordinate functions of $\mathbf{F}$ are at most 4, independent of n.
3. Let $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for all non-identical $(\alpha, \beta, \gamma, \delta)$ (i.e., not all four equal). Then $\mathbf{F}(\lambda \cdot Q) = 0$ if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ with $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all non-identical $(\alpha, \beta, \gamma, \delta)$.

Remark (Identical tuples). The entries $\lambda_{\alpha\alpha\alpha\alpha}$ are unconstrained: since each $A^{(\alpha)} \in \mathbb{R}^{3\times 4}$ has only 3 rows, $Q^{(\alpha\alpha\alpha\alpha)}_{ijkl} = 0$ (repeated rows in a 4×4 matrix), so $T^{(\alpha\alpha\alpha\alpha)} \equiv 0$ regardless of $\lambda_{\alpha\alpha\alpha\alpha}$. No camera-independent polynomial in T can constrain these entries. The rank-1 characterization is therefore stated only on the non-identical support, which is the natural observable domain.

Answer: YES.

Notation

- $a^{(\alpha)}_i \in \mathbb{R}^4$: i-th row of camera $A^{(\alpha)}$, for $i \in [3] := \{1, 2, 3\}$.
- $Q^{(\alpha\beta\gamma\delta)}_{ijkl} = \det[a^{(\alpha)}_i;\ a^{(\beta)}_j;\ a^{(\gamma)}_k;\ a^{(\delta)}_l]$: quadrifocal tensor.
- $T^{(\alpha\beta\gamma\delta)}_{ijkl} = \lambda_{\alpha\beta\gamma\delta} \cdot Q^{(\alpha\beta\gamma\delta)}_{ijkl}$: observed tensor.
- Non-identical: not all four camera indices equal.
- For $\{p, q\} \subset \{1, 2, 3, 4\}$, write $\{r, s\} = \{1, 2, 3, 4\} \setminus \{p, q\}$ for the complementary positions.

1 Construction of F

The map $\mathbf{F}$ consists of six families of Plücker equations, one for each pair of positions $\{p, q\} \subset \{1, 2, 3, 4\}$.

Fix a pair $\{p, q\}$ with complement $\{r, s\}$ ($r < s$). For cameras c_p, c_q (shared) and v_1, v_2, v_3, v_4 (varying in positions r, s), define six 4-tuples by placing c_p in position p, c_q in position q, and the indicated cameras in positions r, s:

$$\begin{aligned} \tau_1 &= (c_p, c_q;\ v_1, v_2), & \tau_2 &= (c_p, c_q;\ v_3, v_4),\\ \tau_3 &= (c_p, c_q;\ v_1, v_3), & \tau_4 &= (c_p, c_q;\ v_2, v_4),\\ \tau_5 &= (c_p, c_q;\ v_1, v_4), & \tau_6 &= (c_p, c_q;\ v_2, v_3), \end{aligned}$$

where $(c_p, c_q; v_r, v_s)$ denotes the 4-tuple with camera c_p in position p, c_q in position q, v_r in position r, v_s in position s. Require all six 4-tuples to be non-identical.

For row-index 6-tuples $I_m = (i_p^m, i_q^m, i_1^m, i_2^m, i_3^m, i_4^m) \in [3]^6$ $(m = 1, 2)$, the six components correspond to the row indices for cameras $c_p, c_q, v_1, v_2, v_3, v_4$ respectively. Define:

$$P_{\{p,q\}}(I_m) := T^{\tau_1}_{(i_p^m, i_q^m, i_1^m, i_2^m)} \cdot T^{\tau_2}_{(i_p^m, i_q^m, i_3^m, i_4^m)},$$
$$S_{\{p,q\}}(I_m) := T^{\tau_3}_{(i_p^m, i_q^m, i_1^m, i_3^m)} \cdot T^{\tau_4}_{(i_p^m, i_q^m, i_2^m, i_4^m)} - T^{\tau_5}_{(i_p^m, i_q^m, i_1^m, i_4^m)} \cdot T^{\tau_6}_{(i_p^m, i_q^m, i_2^m, i_3^m)},$$

where each subscript 4-tuple is ordered by position $(1, 2, 3, 4)$, with the shared indices in positions p, q and the varying indices in positions r, s. The equation is:

$$F^{\{p,q\}}_{(c_p, c_q, v_1, v_2, v_3, v_4),\, I_1, I_2} := P_{\{p,q\}}(I_1) \cdot S_{\{p,q\}}(I_2) - P_{\{p,q\}}(I_2) \cdot S_{\{p,q\}}(I_1). \qquad (\mathcal{P}_{pq})$$

The map $\mathbf{F}$ is the collection of all $F^{\{p,q\}}$ equations over all six pairs $\{p, q\}$, all camera choices, and all row-index pairs.

Explicit instantiation for $\{p, q\} = \{1, 2\}$. Complement $\{r, s\} = \{3, 4\}$. Shared cameras $(c_1, c_2) = (\alpha, \beta)$. Varying cameras $(v_1, v_2, v_3, v_4) = (\gamma, \delta, \gamma', \delta')$. Row indices $(i, j, k, l, k', l') \in [3]^6$. Then:

$$\begin{aligned}
\tau_1 &= (\alpha, \beta, \gamma, \delta), & J_1 &= (i, j, k, l),\\
\tau_2 &= (\alpha, \beta, \gamma', \delta'), & J_2 &= (i, j, k', l'),\\
\tau_3 &= (\alpha, \beta, \gamma, \gamma'), & J_3 &= (i, j, k, k'),\\
\tau_4 &= (\alpha, \beta, \delta, \delta'), & J_4 &= (i, j, l, l'),\\
\tau_5 &= (\alpha, \beta, \gamma, \delta'), & J_5 &= (i, j, k, l'),\\
\tau_6 &= (\alpha, \beta, \delta, \gamma'), & J_6 &= (i, j, l, k'),
\end{aligned}$$

and $P(I) = T^{\tau_1}_{J_1} T^{\tau_2}_{J_2}$, $\quad S(I) = T^{\tau_3}_{J_3} T^{\tau_4}_{J_4} - T^{\tau_5}_{J_5} T^{\tau_6}_{J_6}$.

Properties 1 and 2 are immediate: each $F^{\{p,q\}}$ is a degree-4 polynomial in the entries of T only; no camera matrix appears. The unknown ratio μ between $P(I)$ and $S(I)$ (see Section 2) is eliminated by cross-multiplying two instances I_1, I_2, ensuring $\mathbf{F}$ is polynomial (no denominators) and quartic in T.

Idea of the Proof

Each Plücker relation gives a quadratic identity among the Q-tensors. After scaling by λ, it becomes $P(I) = \mu \cdot S(I)$ where μ depends only on λ-ratios, not on row indices. Cross-multiplying two instances I_1, I_2 removes μ and yields quartic, camera-free polynomials in T. For generic cameras, these force vanishing of all 2×2 minors of certain λ-slices (pairwise rank-1 conditions). The six families—one per pair of positions—yield six such conditions, and a six-step algebraic peeling argument shows these jointly force $\lambda = u \otimes v \otimes w \otimes x$ on the non-identical support.

2 Necessity (λ rank-1 $\Rightarrow \mathbf{F} = 0$)

Lemma 1 (Plücker identity). *For any vectors $r_1, r_2, v_1, v_2, v_3, v_4 \in \mathbb{R}^4$:*

$$\begin{aligned}
&\det[r_1, r_2, v_1, v_2] \cdot \det[r_1, r_2, v_3, v_4]\\
&\qquad = \det[r_1, r_2, v_1, v_3] \cdot \det[r_1, r_2, v_2, v_4] - \det[r_1, r_2, v_1, v_4] \cdot \det[r_1, r_2, v_2, v_3].
\end{aligned}$$

Proof. This is the Plücker relation for Gr(2, 4); see [1], Ch. 1, §5, eq. (5.1), p. 211, or [2], Vol. I, Ch. VII, §4, Theorem I. ☐

Lemma 2. *If $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$, then $F^{\{p,q\}} = 0$ for all six pairs $\{p, q\}$.*

Proof. We prove the case $\{p, q\} = \{1, 2\}$; the other five follow by the same argument with the shared rows in positions p, q. (For $\{p, q\} \neq \{1, 2\}$, reordering rows of the 4×4 determinant to place the shared rows first introduces a sign $(-1)^\sigma$ in each Q-factor, but these signs cancel in the degree-4 expression $F^{\{p,q\}}$ since each term is a product of two Q-factors sharing the same row permutation.)

With shared cameras (α, β) in positions $(1, 2)$ and varying cameras $(\gamma, \delta, \gamma', \delta')$ in positions $(3, 4)$, Lemma 1 with $r_1 = a_i^{(\alpha)}$, $r_2 = a_j^{(\beta)}$ gives at the Q-level:

$$Q_{ijkl}^{(\alpha\beta\gamma\delta)} \cdot Q_{ijk'l'}^{(\alpha\beta\gamma'\delta')} = Q_{ijkk'}^{(\alpha\beta\gamma\gamma')} \cdot Q_{ijll'}^{(\alpha\beta\delta\delta')} - Q_{ijkl'}^{(\alpha\beta\gamma\delta')} \cdot Q_{ijlk'}^{(\alpha\beta\delta\gamma')}.$$

Substituting $T = \lambda \cdot Q$:

$$P(I) = \lambda_{\alpha\beta\gamma\delta} \cdot \lambda_{\alpha\beta\gamma'\delta'} \cdot \left[Q^{(\alpha\beta\gamma\gamma')}Q^{(\alpha\beta\delta\delta')} - Q^{(\alpha\beta\gamma\delta')}Q^{(\alpha\beta\delta\gamma')}\right].$$

For $S(I) = \lambda_{\alpha\beta\gamma\gamma'}\lambda_{\alpha\beta\delta\delta'} \cdot Q^{(\alpha\beta\gamma\gamma')}Q^{(\alpha\beta\delta\delta')} - \lambda_{\alpha\beta\gamma\delta'}\lambda_{\alpha\beta\delta\gamma'} \cdot Q^{(\alpha\beta\gamma\delta')}Q^{(\alpha\beta\delta\gamma')}$, the λ-ratios for rank-1 λ are:

$$\mu_1 := \frac{\lambda_{\alpha\beta\gamma\delta}\lambda_{\alpha\beta\gamma'\delta'}}{\lambda_{\alpha\beta\gamma\gamma'}\lambda_{\alpha\beta\delta\delta'}} = \frac{x_\delta w_{\gamma'}}{x_{\gamma'} w_\delta}, \qquad \mu_2 := \frac{\lambda_{\alpha\beta\gamma\delta}\lambda_{\alpha\beta\gamma'\delta'}}{\lambda_{\alpha\beta\gamma\delta'}\lambda_{\alpha\beta\delta\gamma'}} = \frac{x_\delta w_{\gamma'}}{w_\delta x_{\gamma'}}.$$

Since $\mu_1 = \mu_2 =: \mu$, we have $P(I) = \mu \cdot S(I)$ for all I. Therefore:

$$F^{\{1,2\}} = P(I_1)\, S(I_2) - P(I_2)\, S(I_1) = \mu\, S(I_1)\, S(I_2) - \mu\, S(I_2)\, S(I_1) = 0.$$ ☐

3 Sufficiency — Plücker Equations Imply Pairwise Rank-1

Proposition 3. *For Zariski-generic cameras, if $F^{\{p,q\}} = 0$ for all camera and row-index choices, then for every pair $\{p, q\}$ and every choice of cameras c_p, c_q, the matrix $M_{v_r,v_s}^{(c_p,c_q)} := \lambda_{(c_p,c_q;\, v_r,v_s)}$ has rank 1 in (v_r, v_s).*

Proof. We prove the case $\{p, q\} = \{1, 2\}$; the others follow by the same argument with positions relabeled.

Factoring $F^{\{1,2\}}$. Write $\Lambda_P := \lambda_{\alpha\beta\gamma\delta}\lambda_{\alpha\beta\gamma'\delta'}$, $\Lambda_{S_1} := \lambda_{\alpha\beta\gamma\gamma'}\lambda_{\alpha\beta\delta\delta'}$, $\Lambda_{S_2} := \lambda_{\alpha\beta\gamma\delta'}\lambda_{\alpha\beta\delta\gamma'}$. Define:

$$S_{Q,1}(I) := Q_{I_3}^{(\alpha\beta\gamma\gamma')} \cdot Q_{I_4}^{(\alpha\beta\delta\delta')}, \qquad S_{Q,2}(I) := Q_{I_5}^{(\alpha\beta\gamma\delta')} \cdot Q_{I_6}^{(\alpha\beta\delta\gamma')}.$$

By Lemma 1, $P_Q(I) := Q_{I_1}^{(\alpha\beta\gamma\delta)}Q_{I_2}^{(\alpha\beta\gamma'\delta')} = S_{Q,1}(I) - S_{Q,2}(I)$. Then:

$$P(I) = \Lambda_P \cdot [S_{Q,1}(I) - S_{Q,2}(I)], \qquad S(I) = \Lambda_{S_1} \cdot S_{Q,1}(I) - \Lambda_{S_2} \cdot S_{Q,2}(I).$$

Expanding $F^{\{1,2\}} = P(I_1)\, S(I_2) - P(I_2)\, S(I_1)$:

$$F^{\{1,2\}} = \Lambda_P \cdot (\Lambda_{S_1} - \Lambda_{S_2}) \cdot \left[S_{Q,1}(I_1) \cdot S_{Q,2}(I_2) - S_{Q,1}(I_2) \cdot S_{Q,2}(I_1)\right]. \tag{†}$$

☐

Lemma 4 (Genericity)**.** *For any $n \geq 5$ and any choice of cameras $\gamma, \delta, \gamma', \delta'$ (with possible repetitions among α, β), the bracket in (†) is a polynomial in the camera entries that is not identically zero.*

Proof. It suffices to exhibit one camera configuration where the bracket is nonzero (since a polynomial that is nonzero at a point is nonzero on a Zariski-open dense set). Consider cameras $A^{(m)}$ with rows

$$a_1^{(m)} = (1, 0, 0, m), \quad a_2^{(m)} = (0, 1, 0, m^2), \quad a_3^{(m)} = (0, 0, 1, m^3)$$

for $m = 1, \ldots, n$. Take $\alpha = 1, \beta = 2$ (shared), $\gamma = 3, \delta = 4, \gamma' = 5, \delta' = 6$ (or $\delta' = 1$ if $n = 5$). Choose $I_1 = (1, 1, 2, 2, 3, 3)$ and $I_2 = (1, 1, 2, 3, 2, 3)$.

For I_1: the shared rows are $a_1^{(1)} = (1, 0, 0, 1)$ and $a_1^{(2)} = (1, 0, 0, 2)$, with varying rows of type $a_2^{(\cdot)} = (0, 1, 0, \cdot)$ and $a_3^{(\cdot)} = (0, 0, 1, \cdot)$. By cofactor expansion along the first two rows (which differ only in the last entry), every determinant of the form $\det[a_1^{(1)}, a_1^{(2)}, a_2^{(\cdot)}, a_3^{(\cdot)}]$ equals $t_\beta - t_\alpha = 1$, independent of the varying cameras. Therefore $S_{Q,1}(I_1) = 1$ and $S_{Q,2}(I_1) = 1$.

For I_2: the varying rows are $(2, 3, 2, 3)$ instead of $(2, 2, 3, 3)$. The factors of $S_{Q,1}(I_2)$ involve determinants $\det[a_1^{(1)}, a_1^{(2)}, a_2^{(\gamma)}, a_2^{(\gamma')}]$ and $\det[a_1^{(1)}, a_1^{(2)}, a_3^{(\delta)}, a_3^{(\delta')}]$. Each has two rows of the form $(0, 1, 0, \cdot)$ or $(0, 0, 1, \cdot)$, giving a 4×4 matrix with rank ≤ 3. Hence both determinants are 0, so $S_{Q,1}(I_2) = 0$.

The factor $S_{Q,2}(I_2)$ involves $\det[a_1^{(1)}, a_1^{(2)}, a_2^{(\gamma)}, a_3^{(\delta')}] \cdot \det[a_1^{(1)}, a_1^{(2)}, a_3^{(\delta)}, a_2^{(\gamma')}]$. The first determinant equals 1 (same structure as I_1). The second has rows $a_3^{(\cdot)}$ before $a_2^{(\cdot)}$ in positions 3,4, giving -1 (one row swap). So $S_{Q,2}(I_2) = -1$.

Therefore the bracket equals $1 \cdot (-1) - 0 \cdot 1 = -1 \neq 0$. The same computation applies for $n = 5$ with $\delta' = \alpha$ (repeated index), since the determinant structure depends only on the row types. □

Proof of Proposition 3 (continued). Since the bracket is a nonzero polynomial, it is nonzero on a Zariski-open dense subset of the camera parameter space. Combined with $\Lambda_P \neq 0$ (all λ-values on non-identical tuples are nonzero), we conclude from (†):

$$\Lambda_{S_1} = \Lambda_{S_2}, \quad \text{i.e.,} \quad \lambda_{\alpha\beta\gamma\gamma'}\lambda_{\alpha\beta\delta\delta'} = \lambda_{\alpha\beta\gamma\delta'}\lambda_{\alpha\beta\delta\gamma'}. \tag{$\star_{12}$}$$

This is the vanishing of all 2×2 minors of $M^{(\alpha,\beta)}_{\gamma,\delta} = \lambda_{\alpha\beta\gamma\delta}$, i.e., $\operatorname{rank} M^{(\alpha,\beta)} = 1$. □

Applying Proposition 3 to all six pairs yields six pairwise rank-1 conditions:

- $(\star_{12})$: For fixed (α, β), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (γ, δ).
- $(\star_{13})$: For fixed (α, γ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (β, δ).
- $(\star_{14})$: For fixed (α, δ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (β, γ).
- $(\star_{23})$: For fixed (β, γ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (α, δ).
- $(\star_{24})$: For fixed (β, δ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (α, γ).
- $(\star_{34})$: For fixed (γ, δ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (α, β).

4 Sufficiency — Pairwise Rank-1 Implies Rank-1

Proposition 5. *If $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for all non-identical $(\alpha, \beta, \gamma, \delta)$ and the six conditions $(\star_{12})$–$(\star_{34})$ hold, then there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all non-identical $(\alpha, \beta, \gamma, \delta)$.*

Proof. We extract the rank-1 factorization in six steps, each using one pairwise condition. Fix five distinct reference indices $a_0, b_0, c_0, d_0, e_0 \in [n]$ (possible since $n \geq 5$).

Step 1. By $(\star_{12})$, for each fixed (α, β), the matrix $\lambda_{\alpha\beta\gamma\delta}$ has rank 1 in (γ, δ) on non-identical tuples. So:

$$\lambda_{\alpha\beta\gamma\delta} = \varphi(\alpha, \beta, \gamma) \cdot \psi(\alpha, \beta, \delta)$$

for all non-identical $(\alpha,\beta,\gamma,\delta)$. Explicitly: for each (α,β), choose a pivot $c_*(\alpha,\beta)\in\{c_0,e_0\}$ such that $c_*\neq\alpha$ (possible since $c_0\neq e_0$). Then $(\alpha,\beta,c_*,\delta)$ is non-identical for all δ (since $c_*\neq\alpha$), so we may define:

$$\psi(\alpha,\beta,\delta):=\lambda_{\alpha\beta\, c_*\,\delta},\qquad \varphi(\alpha,\beta,\gamma):=\frac{\lambda_{\alpha\beta\gamma d_0}}{\psi(\alpha,\beta,d_0)}.$$

All evaluations are at non-identical tuples (since $c_*\neq\alpha$ and $d_0\notin\{c_0,e_0\}$ can be chosen distinct from c_*).

Step 2. By $(\star_{23})$, for each fixed (β,γ), $\lambda_{\alpha\beta\gamma\delta}$ is rank-1 in (α,δ). Substituting Step 1: $\varphi(\alpha,\beta,\gamma)\cdot\psi(\alpha,\beta,\delta)$ must be rank-1 in (α,δ) for fixed (β,γ). The φ-factors cancel (nonzero), leaving:

$$\psi(\alpha_1,\beta,\delta_1)\,\psi(\alpha_2,\beta,\delta_2)=\psi(\alpha_1,\beta,\delta_2)\,\psi(\alpha_2,\beta,\delta_1).$$

So for fixed β, $\psi(\alpha,\beta,\delta)$ is rank-1 in (α,δ): $\psi(\alpha,\beta,\delta)=\psi_0(\alpha,\beta)\cdot X(\beta,\delta)$, where $X(\beta,\delta):=\psi(a_0,\beta,\delta)/\psi(a_0,\beta,d_0)$ and $\psi_0(\alpha,\beta):=\psi(\alpha,\beta,d_0)$.

Substituting: $\lambda_{\alpha\beta\gamma\delta}=F(\alpha,\beta,\gamma)\cdot X(\beta,\delta)$, where $F:=\varphi\cdot\psi_0$.

Step 3. By $(\star_{14})$, for fixed (α,δ), λ is rank-1 in (β,γ). The X-factors cancel in the 2×2 minor, so F is rank-1 in (β,γ) for fixed α:

$$F(\alpha,\beta,\gamma)=G(\alpha,\beta)\cdot H(\alpha,\gamma).$$

Therefore $\lambda_{\alpha\beta\gamma\delta}=G(\alpha,\beta)\cdot H(\alpha,\gamma)\cdot X(\beta,\delta)$.

Step 4. By $(\star_{34})$, for fixed (γ,δ), λ is rank-1 in (α,β). The H and X factors cancel: $G(\alpha_1,\beta_1)\,G(\alpha_2,\beta_2)=G(\alpha_1,\beta_2)\,G(\alpha_2,\beta_1)$. So G is rank-1:

$$G(\alpha,\beta)=g(\alpha)\cdot g'(\beta).$$

Therefore $\lambda_{\alpha\beta\gamma\delta}=g(\alpha)\cdot g'(\beta)\cdot H(\alpha,\gamma)\cdot X(\beta,\delta)$.

Step 5. By $(\star_{24})$, for fixed (β,δ), λ is rank-1 in (α,γ). The bracket $g'(\beta)\,X(\beta,\delta)$ is a scalar, so $g(\alpha)\,H(\alpha,\gamma)$ must be rank-1 in (α,γ). The g-factors cancel, so H is rank-1:

$$H(\alpha,\gamma)=h(\alpha)\cdot w(\gamma).$$

Therefore $\lambda_{\alpha\beta\gamma\delta}=g(\alpha)\,h(\alpha)\cdot g'(\beta)\cdot w(\gamma)\cdot X(\beta,\delta)$.

Step 6. By $(\star_{13})$, for fixed (α,γ), λ is rank-1 in (β,δ). The bracket $g(\alpha)\,h(\alpha)\,w(\gamma)$ is a scalar, so $g'(\beta)\,X(\beta,\delta)$ must be rank-1 in (β,δ). The g'-factors cancel, so X is rank-1:

$$X(\beta,\delta)=\tilde v(\beta)\cdot x(\delta).$$

Conclusion. Setting $u(\alpha):=g(\alpha)\,h(\alpha)$, $v(\beta):=g'(\beta)\,\tilde v(\beta)$:

$$\lambda_{\alpha\beta\gamma\delta}=u(\alpha)\cdot v(\beta)\cdot w(\gamma)\cdot x(\delta)$$

for all non-identical $(\alpha,\beta,\gamma,\delta)$. All factors are nonzero since $\lambda\neq 0$ on non-identical tuples. $\square$

Remark. The rank-1 tensors form the Segre variety $\mathrm{Seg}(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}\times\mathbb{P}^{n-1})$, cut out by 2×2 minors of all mode flattenings [5]. Proposition 5 shows that the six pairwise rank-1 conditions $(\star_{12})$–$(\star_{34})$—which are precisely the 2×2 minors of the six $\binom{4}{2}$ flattenings—suffice to place λ on the Segre variety (restricted to the non-identical support).

Summary

The polynomial map $\mathbf{F}$ consists of six families of Plücker equations $\mathcal{P}_{12}, \mathcal{P}_{13}, \mathcal{P}_{14}, \mathcal{P}_{23}, \mathcal{P}_{24}, \mathcal{P}_{34}$ (degree 4 each), totaling $O(n^6 \cdot 3^{12})$ equations. The proof establishes:

1. **Camera-independence**: $\mathbf{F}$ is expressed purely in terms of T-entries.
2. **Bounded degree**: all coordinate functions have degree 4, independent of n.
3. **Exact characterization** (for Zariski-generic cameras): $\mathbf{F}(\lambda \cdot Q) = 0 \iff \lambda$ is rank-1.

The assumption $n \geq 5$ is used in two places: (i) the Plücker equations require six camera slots (two shared, four varying), and for $n = 5$ some varying cameras must repeat a shared camera—this is valid since the problem defines $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for non-identical (not all-same) tuples, so e.g. $(\alpha, \beta, \gamma, \alpha)$ is permitted; (ii) the factor extraction in Proposition 5 requires five distinct reference indices a_0, b_0, c_0, d_0, e_0. The genericity witness in Lemma 4 applies equally to $n = 5$ (with $v_4 = c_p$), as verified by explicit computation. For $n \leq 4$, the problem is open under this proof technique. □

References

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