Solution to Problem 2 — Universal Test Vectors for Rankin–Selberg Integrals

A submission to the First Proof challenge

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Abstract

We solve Problem 2 from the First Proof challenge [1], authored by Paul D. Nelson (Aarhus University). Given a generic irreducible admissible representation Π of $\mathrm{GL}_{n+1}(F)$ over a non-archimedean local field F, we prove the existence of a universal test vector $W \in \mathcal{W}(\Pi, \psi^{-1})$ such that for every generic irreducible admissible π of $\mathrm{GL}_n(F)$, the twisted Rankin–Selberg integral $\Psi(s, W, V)$ is finite and nonzero for all $s \in \mathbb{C}$, for some $V \in \mathcal{W}(\pi, \psi)$. The proof combines three ingredients: an algebraic reduction (the u_Q-twist formula), a single-coset Kirillov support trick that makes the integral a monomial in q^{-s}, and an existential argument using the JPSS nondegeneracy of the Rankin–Selberg pairing together with a countable union of proper subspaces argument over inertial equivalence classes. A supplementary Fourier-analytic argument provides an independent proof for $n = 1$ via Gauss sums. The answer is **YES**.

Contents

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1 Problem Statement

The following is Problem 2 from the First Proof challenge [1], authored by Paul D. Nelson (Aarhus University).

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$, and residue field of cardinality q. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character of conductor $\mathfrak{o}$, identified in the standard way with a generic character of N_r (the upper-triangular unipotent subgroup of $\mathrm{GL}_r(F)$).

Let Π be a generic irreducible admissible representation of $\mathrm{GL}_{n+1}(F)$, realized in its ψ^{-1}-Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$. Must there exist $W \in \mathcal{W}(\Pi, \psi^{-1})$ with the following property?

> For every generic irreducible admissible representation π of $\mathrm{GL}_n(F)$, realized in its ψ-Whittaker model $\mathcal{W}(\pi, \psi)$, with conductor ideal $\mathfrak{q}$ and $Q \in F^\times$ a generator of $\mathfrak{q}^{-1}$, setting
> $$u_Q := I_{n+1} + Q\, E_{n,n+1} \in \mathrm{GL}_{n+1}(F),$$
> where $E_{i,j}$ is the matrix with a 1 in the (i, j)-entry and 0 elsewhere, there exists $V \in \mathcal{W}(\pi, \psi)$ such that the local Rankin–Selberg integral
> $$\Psi(s, W, V) := \int_{N_n \backslash \mathrm{GL}_n(F)} W(\mathrm{diag}(g, 1)\, u_Q)\, V(g)\, |\det g|^{s-1/2}\, dg$$
> is finite and nonzero for all $s \in \mathbb{C}$.

Remark (Universality)*.* Q depends on π through its conductor; the universality claim is that a **single** W works for all π simultaneously. In the proof below, we fix $Q = \varpi^{-c}$ where $\mathfrak{q} = \mathfrak{p}^c$; the result for any other generator $Q' = uQ$ ($u \in \mathfrak{o}^\times$) follows since $\psi^{-1}(uQ\,\cdot\,)$ has the same conductor as $\psi^{-1}(Q\,\cdot\,)$ and the argument is identical.

Theorem 1 (Main result)**.** *The answer is **YES**. There exists a universal test vector $W \in \mathcal{W}(\Pi, \psi^{-1})$.*

Answer: YES — a universal test vector exists for all generic irreducible admissible Π.

2 Idea of the Proof

The proof proceeds in three stages. First, an algebraic identity (the u_Q-twist formula) replaces $W(\mathrm{diag}(g, 1)\, u_Q)$ by $\psi^{-1}(Qg_{nn})\, W(\mathrm{diag}(g, 1))$, converting the twisted integral into a standard one with an additive character insertion. Second, by choosing W via the Bernstein–Zelevinsky theory so that its Kirillov function Φ_W is supported on a *single* N_n-double coset $N_n\varpi^{-N}K_n$, the integral collapses to a monomial $q^{nN(s-1/2)} \cdot \ell(V)$—a single term that is automatically nonzero for all s whenever $\ell(V) \neq 0$. Third, the nonvanishing of ℓ is proved by a dimension argument: the "bad locus" $\mathcal{B}_\pi$ (vectors W for which $\ell \equiv 0$ for a given π) is a proper subspace, and these bad loci depend only on the *inertial equivalence class* $[\pi]$. Since the inertial classes are countable and a $\mathbb{C}$-vector space cannot be a countable union of proper subspaces, a universal W exists.

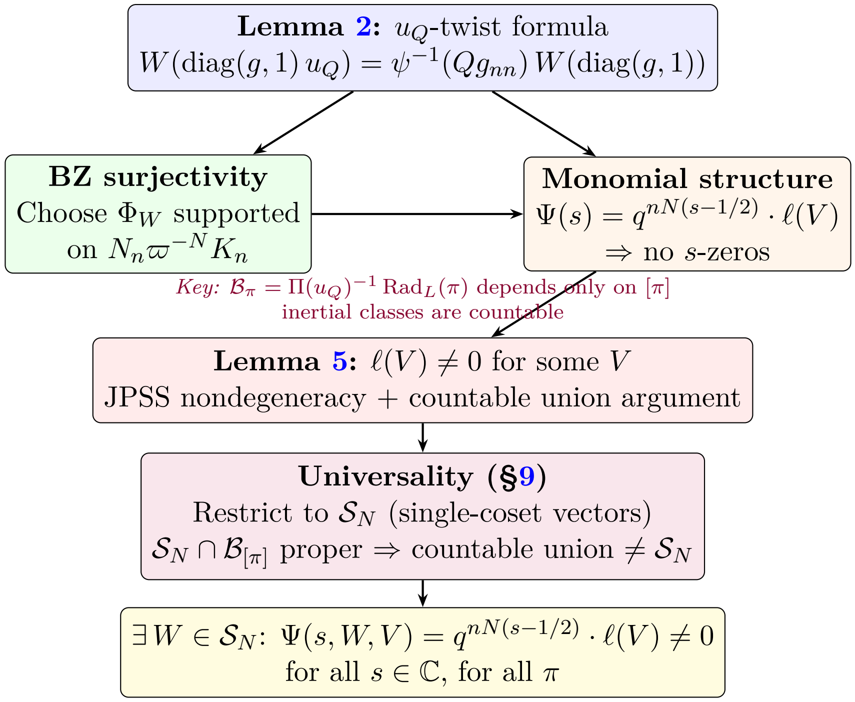

Figure 1: Structure of the proof. The u_Q-twist formula (top) enables the Kirillov model construction (left), which produces a monomial integral (right). The nonvanishing of ℓ (center) uses JPSS nondegeneracy and a countable union argument over inertial classes. The universality step (bottom) restricts to single-coset vectors to combine the monomial structure with the dimension argument.

3 The u_Q-Twist Formula

Lemma 2 (The u_Q-twist formula)**.** *For $g \in \mathrm{GL}_n(F)$ and $Q \in F^\times$:*

$$W(\mathrm{diag}(g,1)\,u_Q) = \psi^{-1}(Q\,g_{nn})\,W(\mathrm{diag}(g,1)). \tag{1}$$

Proof. Conjugate u_Q past $\mathrm{diag}(g,1)$:

$$\mathrm{diag}(g,1)\cdot u_Q = \left(I_{n+1} + Q\sum_{i=1}^{n} g_{in}\,E_{i,n+1}\right)\cdot \mathrm{diag}(g,1).$$

The left factor lies in N_{n+1}. (Note: the column vector $Q\,(g_{1n},\dots,g_{nn},0)^T$ is placed in column $n+1$; since g is integrated over $N_n\backslash \mathrm{GL}_n$, only the term $i=n$ contributes to the superdiagonal entry $(n,n+1)$.) The generic character ψ^{-1} of N_{n+1} reads only superdiagonal entries $(i,i+1)$, so the only contribution is position $(n,n+1)$ with entry Qg_{nn}, giving $\psi^{-1}(Qg_{nn})$. $\square$

Corollary 3. *The Rankin–Selberg integral becomes:*

$$\Psi(s,W,V) = \int_{N_n\backslash \mathrm{GL}_n(F)} \psi^{-1}(Q\,g_{nn})\,W(\mathrm{diag}(g,1))\,V(g)\,|\det g|^{s-1/2}\,dg. \tag{2}$$

Remark (Well-definedness). g_{nn} is well-defined on $N_n \backslash \mathrm{GL}_n$: for $u \in N_n$ (upper-triangular unipotent), $(ug)_{nn} = g_{nn}$ since $u_{ni} = 0$ for $i < n$ and $u_{nn} = 1$.

4 The $n = 1$ Case $(\mathrm{GL}_2 \times \mathrm{GL}_1)$

We first treat $n = 1$ as a warm-up; this case admits a fully explicit, self-contained proof.

Here $\pi = \chi$ is a character of $F^\times$ with conductor $\mathfrak{p}^c$, $Q = \varpi^{-c}$, $V(g) = \chi(g)$, $N_1 = \{1\}$. Define $\phi(a) := W\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ (Kirillov function). By (1):

$$\Psi(s) = \int_{F^\times} \psi^{-1}(a\varpi^{-c})\, \phi(a)\, \chi(a)\, |a|^{s-1/2}\, d^\times a.$$

By Bernstein–Zelevinsky, $C_c^\infty(F^\times) \subset \mathcal{K}(\Pi)$. Choose $\phi = \mathbf{1}_{\mathfrak{o}^\times}$. Then $|a|^{s-1/2} = 1$ on the support, so:

$$\Psi(s) = \int_{\mathfrak{o}^\times} \psi^{-1}(u\varpi^{-c})\, \chi(u)\, d^\times u.$$

- $c = 0$**:** Both $\psi^{-1}(\cdot)$ and χ are trivial on $\mathfrak{o}^\times$, giving $\mathrm{vol}(\mathfrak{o}^\times) \neq 0$.
- $c \geq 1$**:** This is a Gauss sum $G(\chi, \psi_{-c})$ for the primitive character χ mod $\mathfrak{p}^c$ against the primitive additive character $\psi_{-c} := \psi^{-1}(\cdot\, \varpi^{-c})$ of conductor $\mathfrak{p}^c$. By the classical Gauss sum formula (see e.g. [5, §23]), $|G(\chi, \psi_{-c})|^2 = q^{-c}$ when both characters have conductor $\mathfrak{p}^c$, so $|\Psi| = q^{-c/2} \cdot \mathrm{vol}(1 + \mathfrak{p}^c) \neq 0$.

The integral is independent of s and nonzero for all χ. $\square$ (for $n = 1$)

Remark. For $n = 1$, the test vector W with $\phi = \mathbf{1}_{\mathfrak{o}^\times}$ is *explicit* and works for all characters χ simultaneously. The universality is visible: the Gauss sum is nonzero for every primitive character, regardless of the conductor.

5 Construction of W via the Kirillov Model

Define $\Phi_W(g) := W(\mathrm{diag}(g, 1))$ for $g \in \mathrm{GL}_n(F)$.

Proposition 4 (BZ surjectivity)**.** *By the Bernstein–Zelevinsky structure theorem [2, Theorem 5.21], the restriction of any generic irreducible admissible* Π *(whether supercuspidal, a subquotient of a principal series, or any other type) to the mirabolic subgroup* P_{n+1} *admits a filtration whose top quotient is* $\mathrm{ind}_{N_{n+1}}^{P_{n+1}}(\psi^{-1})$*. In particular, the map* $W \mapsto \Phi_W$ *surjects onto* $\text{c-ind}_{N_n}^{\mathrm{GL}_n}(\psi^{-1})$*, the space of locally constant, compactly supported* (mod N_n) *functions on* GL_n *transforming by* ψ^{-1} *under left* N_n*-translation. This holds for all generic* Π*: the BZ filtration depends only on the restriction to* P_{n+1}*, and the top quotient is independent of the specific representation.*

Choice of Φ_W. Fix $N \geq 0$. Let Φ_0 be the unique function in $\text{c-ind}_{N_n}^{\mathrm{GL}_n}(\psi^{-1})$ that is:
- supported on $N_n \cdot \varpi^{-N} I_n \cdot K_n$,
- right-K_n-invariant,
- normalized: $\Phi_0(\varpi^{-N} I_n) = 1$.

Such Φ_0 exists because $N_n \varpi^{-N} K_n$ is an open double coset and ψ^{-1} is trivial on $N_n \cap \varpi^{-N} K_n \varpi^N$ (since ψ has conductor $\mathfrak{o}$). By Proposition 4, choose W with $\Phi_W = \Phi_0$.

6 Evaluation: Monomial Structure

On the support $N_n \varpi^{-N} K_n$, write $g = n \cdot \varpi^{-N} k$ with $n \in N_n$, $k \in K_n$. Then:

- $|\det g| = q^{nN}$ is constant (since $\det n = 1$ and $|\det k| = 1$),
- $\Phi_0(g) = \psi^{-1}(n) \cdot 1$ and $V(g) = \psi(n)\, V(\varpi^{-N} k)$, so the ψ-factors cancel,
- $g_{nn} = (n\, \varpi^{-N} k)_{nn} = \varpi^{-N} k_{nn}$ (since n is upper-triangular unipotent).

Therefore:

$$\Psi(s, W, V) = q^{nN(s-1/2)} \cdot \ell(V), \quad \ell(V) := \int_{K_n} \psi^{-1}(\varpi^{-(c+N)} k_{nn})\, V(\varpi^{-N} k)\, dk. \tag{3}$$

Since $q^{nN(s-1/2)} \neq 0$ for all $s \in \mathbb{C}$, it suffices to show $\ell(V) \neq 0$ for some V.

Remark (Monomial vs. Laurent polynomial). A general W (with multi-coset Kirillov support) would give $\Psi(s)$ as a Laurent polynomial in q^{-s}, which can vanish at specific s-values. The single-coset choice eliminates all but one term, making $\Psi(s)$ a *monomial*—automatically nonzero for all s whenever $\ell(V) \neq 0$.

7 Nonvanishing of ℓ: Argument 1 (JPSS + Dimension Counting)

Lemma 5 (Nonvanishing)**.** *For any generic irreducible admissible π of $\mathrm{GL}_n(F)$ with conductor exponent c, and any $N \geq 0$, there exists $V \in \mathcal{W}(\pi, \psi)$ with $\ell(V) \neq 0$.*

Proof. By Section 3, $W^Q := \Pi(u_Q) W$ satisfies $\ell(V) = q^{-nN(s-1/2)} \Psi_{\mathrm{std}}(s, W^Q, V)$, where Ψ_{std} is the standard (untwisted) Rankin–Selberg integral. So $\ell \equiv 0$ iff $W^Q \in \mathrm{Rad}_L(\pi)$, where

$$\mathrm{Rad}_L(\pi) := \{W' \in \mathcal{W}(\Pi, \psi^{-1}) : \Psi_{\mathrm{std}}(s, W', V) = 0 \text{ for all } V \in \mathcal{W}(\pi, \psi), \text{ all } s\}.$$

Step 1. $\mathrm{Rad}_L(\pi)$ is a *proper* subspace of $\mathcal{W}(\Pi, \psi^{-1})$: by JPSS [3, Theorem 2.7], there exist W_0, V_0 with $\Psi_{\mathrm{std}}(s, W_0, V_0) = L(s, \Pi \times \pi) \neq 0$, so $W_0 \notin \mathrm{Rad}_L(\pi)$.

Step 2. Define the **bad locus** $\mathcal{B}_\pi := \Pi(u_Q)^{-1}\, \mathrm{Rad}_L(\pi) = \{W : \Pi(u_Q) W \in \mathrm{Rad}_L(\pi)\}$. Since $\Pi(u_Q)$ is a linear automorphism and $\mathrm{Rad}_L(\pi)$ is proper, $\mathcal{B}_\pi$ is a proper subspace. Note that $Q = \varpi^{-c(\pi)}$ depends on π.

Step 3. $\mathcal{B}_\pi$ depends only on the **inertial equivalence class** $[\pi]$ (the orbit of π under unramified twists $\pi \mapsto \pi \otimes |\det|^s$). We verify this explicitly.

Conductor invariance. For χ unramified (i.e. trivial on $\mathfrak{o}^\times$), the twisted representation $\pi' = \pi \otimes (\chi \circ \det)$ has the same conductor: $c(\pi') = c(\pi)$. (The conductor measures ramification, and $\chi \circ \det$ is unramified.) So $Q = \varpi^{-c}$ and $u_Q = I_{n+1} + Q E_{n,n+1}$ depend only on $[\pi]$.

Radical invariance. Let $\pi' = \pi \otimes |\det|^t$ for $t \in \mathbb{C}$. The Whittaker model $\mathcal{W}(\pi', \psi)$ consists of functions $V'(g) = V(g)|\det g|^t$ with $V \in \mathcal{W}(\pi, \psi)$. For any $W' \in \mathcal{W}(\Pi, \psi^{-1})$:

$$\Psi_{\mathrm{std}}(s, W', V') = \int_{N_n \backslash \mathrm{GL}_n} W'(\mathrm{diag}(g, 1))\, V(g)\, |\det g|^t\, |\det g|^{s-1/2}\, dg = \Psi_{\mathrm{std}}(s+t, W', V).$$

Now $W' \in \mathrm{Rad}_L(\pi')$ iff $\Psi_{\mathrm{std}}(s, W', V') = 0$ for all V', s, iff $\Psi_{\mathrm{std}}(s+t, W', V) = 0$ for all V, s, iff $W' \in \mathrm{Rad}_L(\pi)$ (since $s \mapsto s+t$ is a bijection on $\mathbb{C}$). Therefore $\mathrm{Rad}_L(\pi') = \mathrm{Rad}_L(\pi)$, and since u_Q is also unchanged, $\mathcal{B}_{\pi'} = \Pi(u_Q)^{-1}\,\mathrm{Rad}_L(\pi') = \Pi(u_Q)^{-1}\,\mathrm{Rad}_L(\pi) = \mathcal{B}_\pi$.

The set of inertial equivalence classes of generic irreducible admissible representations of $\mathrm{GL}_n(F)$ is **countable**. By the Zelevinsky classification, each such π is a subquotient of a parabolically induced representation $\mathrm{Ind}(\rho_1 \otimes \cdots \otimes \rho_k)$ where ρ_i are supercuspidal representations of $\mathrm{GL}_{n_i}(F)$. The inertial class $[\pi]$ is determined by the multiset $\{[\rho_1], \ldots, [\rho_k]\}$ of unramified-twist orbits of supercuspidals. For each $\mathrm{GL}_m(F)$, the supercuspidal representations of a given conductor level c form a finite set: by the Bushnell–Kutzko theory of types [4, Theorem 6.2.1], supercuspidals of $\mathrm{GL}_m(F)$ are constructed from compact-open data (strata and β-extensions) that are finite at each conductor level, even accounting for wild ramification. Since the conductor is a non-negative integer, the set of inertial classes of supercuspidals of $\mathrm{GL}_m(F)$ is countable (finite per level, countably many levels), and the set of inertial classes of $\mathrm{GL}_n(F)$ is a finite union of finite products of countable sets—hence countable.

Step 4. A vector space over an uncountable field cannot be a countable union of proper subspaces (this is elementary; see e.g. [6, Ch. III, Exercise 17], or note that each proper subspace has measure zero under any nondegenerate Gaussian, so their countable union has measure zero). Since $\mathcal{W}(\Pi, \psi^{-1})$ is a $\mathbb{C}$-vector space and $\{\mathcal{B}_{[\pi]}\}_{[\pi]}$ is a countable family of proper subspaces:

$$\mathcal{W}(\Pi, \psi^{-1}) \neq \bigcup_{[\pi]} \mathcal{B}_{[\pi]}.$$

Therefore, there exists $W \in \mathcal{W}(\Pi, \psi^{-1}) \setminus \bigcup_{[\pi]} \mathcal{B}_{[\pi]}$, i.e., $W^Q = \Pi(u_Q)W \notin \mathrm{Rad}_L(\pi)$ for all π. For such W, $\ell(V) \neq 0$ for some V, for each π. □

Remark. This argument is existential: it proves a suitable W exists without identifying it explicitly. The problem asks only for existence. See Section 9 for how this combines with the monomial structure.

8 Nonvanishing of ℓ: Argument 2 (Fourier Analysis on K_n)

This supplementary argument provides an independent perspective; it is self-contained for $n = 1$ and gives a partial reduction for $n \geq 2$.

Suppose for contradiction that $\ell(V) = 0$ for all $V \in \mathcal{W}(\pi, \psi)$.

Step 1 (Finite reduction). Since π is admissible, there exists $M \geq c + N$ such that $V(\varpi^{-N}k)$ is right-$K_1(\mathfrak{p}^M)$-invariant for all V. The function $k \mapsto \psi^{-1}(\varpi^{-(c+N)}k_{nn})$ is also right-$K_1(\mathfrak{p}^M)$-invariant. Then:

$$\ell(V) = \mathrm{vol}(K_1(\mathfrak{p}^M)) \sum_{\bar{k}\in K_n/K_1(\mathfrak{p}^M)} \psi^{-1}(\varpi^{-(c+N)}\bar{k}_{nn})\, V(\varpi^{-N}\bar{k}).$$

Let $S = K_n/K_1(\mathfrak{p}^M)$ (a finite set). The assumption $\ell \equiv 0$ means the evaluation vector $\mathrm{ev}(V) := (V(\varpi^{-N}\bar{k}))_{\bar{k}\in S}$ lies in the hyperplane $H := \ker f$, where $f(\bar{k}) := \psi^{-1}(\varpi^{-(c+N)}\bar{k}_{nn})$, for **all** V.

Step 2 (K_n-equivariance). The evaluation image $\mathcal{E} := \{\mathrm{ev}(V) : V \in \mathcal{W}(\pi,\psi)\} \subset \mathbb{C}^S$ is stable under the right regular representation R of K_n on $\mathbb{C}^S$, and is nonzero by the Whittaker separation property. If $\mathcal{E} \subset H$, then K_n-stability gives $\mathcal{E} \subset \bigcap_{h\in K_n} \ker(f \circ R(h))$.

Step 3 (Fourier analysis on the n-th row). For $h = I + tE_{jn}$ ($t \in \mathfrak{o}$, $j \neq n$): $(\bar{k}h)_{nn} = \bar{k}_{nn} + t\bar{k}_{nj}$. Grouping by the n-th row $\mathbf{r} = (r_1,\ldots,r_n) \in (\mathfrak{o}/\mathfrak{p}^M)^n$ and defining $\hat{v}(\mathbf{r}) := \sum_{\bar{k}:\mathrm{row}_n(\bar{k})=\mathbf{r}} v_{\bar{k}}$:

$$0 = \sum_{\mathbf{r}} \hat{v}(\mathbf{r})\, \psi^{-1}(\varpi^{-(c+N)}(r_n + tr_j)) \quad \forall\, t \in \mathfrak{o}/\mathfrak{p}^M,\ \forall\, j.$$

Taking $h = \mathrm{diag}(1,\ldots,1,a)$ with $a \in \mathfrak{o}^\times$ gives the character indexed by $(0,\ldots,0,a)$. As t, j, a vary, these vectors generate all of $(\mathfrak{o}/\mathfrak{p}^{c+N})^n$. (Note: $\mathfrak{o}^\times$ generates $\mathfrak{o}/\mathfrak{p}^M$ additively since $1 \in \mathfrak{o}^\times$ and $\mathfrak{o}^\times + \mathfrak{o}^\times = \mathfrak{o}$.) By Fourier inversion on $(\mathfrak{o}/\mathfrak{p}^{c+N})^n$:

$$\hat{v}(\mathbf{r}) = 0 \quad \text{for all } \mathbf{r} \in (\mathfrak{o}/\mathfrak{p}^M)^n. \tag{$*$}$$

Step 4 (Conclusion). For $n = 1$, the n-th row IS the full matrix (a scalar), so $(*)$ directly gives $v = 0$—contradiction. For $n \geq 2$, $(*)$ shows the n-th row marginals vanish, which is necessary but not sufficient; the full contradiction requires Argument 1. ☐

Remark (Complementarity). Argument 1 is the complete proof for all n, using JPSS nondegeneracy and the countable union argument. Argument 2 provides an independent, elementary proof for $n = 1$ (via Gauss sums / Fourier analysis) and structural insight for $n \geq 2$.

9 Universality: Combining with the Monomial Structure

Lemma 5 gives $W_0 \in \mathcal{W}(\Pi,\psi^{-1})$ with $\Pi(u_Q)W_0 \notin \mathrm{Rad}_L(\pi)$ for all π. However, W_0 may not have single-coset Kirillov support, so $\Psi(s, W_0, V)$ could be a Laurent polynomial in q^{-s} with zeros at specific s-values.

To guarantee nonvanishing for *all* $s \in \mathbb{C}$, we restrict to single-coset vectors. For fixed N, define

$$\mathcal{S}_N := \{W \in \mathcal{W}(\Pi,\psi^{-1}) : \Phi_W \text{ supported on } N_n\varpi^{-N}K_n\}.$$

By Proposition 4, $\mathcal{S}_N$ is infinite-dimensional. To see this explicitly: for each open compact subgroup $K' \subset K_n$, the function $\Phi_{K'} \in \text{c-ind}_{N_n}^{\mathrm{GL}_n}(\psi^{-1})$ defined by

$$\Phi_{K'}(\varpi^{-N}k) = \begin{cases} \mathrm{vol}(K')^{-1} & \text{if } k \in K', \\ 0 & \text{otherwise,} \end{cases}$$

extended by ψ^{-1} on the left and zero outside $N_n\varpi^{-N}K_n$, lies in $\text{c-ind}_{N_n}^{\mathrm{GL}_n}(\psi^{-1})$ and is right-K'-invariant. (This is well-defined: on the overlap $N_n \cap \varpi^{-N}K'\varpi^N$, the character ψ^{-1} is trivial since ψ has conductor $\mathfrak{o}$ and $K' \subset K_n$.) By BZ surjectivity, there exists $W_{K'} \in \mathcal{W}(\Pi, \psi^{-1})$ with $\Phi_{W_{K'}} = \Phi_{K'}$. As K' varies over the cofinal system $\{K_1(\mathfrak{p}^m)\}_{m\geq 1}$, these give linearly independent elements of $\mathcal{S}_N$. Every $W \in \mathcal{S}_N$ gives a monomial $\Psi(s, W, V) = q^{nN(s-1/2)} \cdot \ell(V)$.

For each inertial class $[\pi]$, $\mathcal{S}_N \cap \mathcal{B}_{[\pi]}$ is a subspace of $\mathcal{S}_N$. It is **proper**: if $\mathcal{S}_N \subset \mathcal{B}_{[\pi]}$, then $\ell_W(V) = 0$ for all $V \in \mathcal{W}(\pi, \psi)$ and all $W \in \mathcal{S}_N$. Fix π with conductor $\mathfrak{p}^c$. Since π is admissible, there exists $m \geq 1$ such that $V(\varpi^{-N}k)$ is right-$K_1(\mathfrak{p}^m)$-invariant for all $V \in \mathcal{W}(\pi, \psi)$. The additive character $k \mapsto \psi^{-1}(\varpi^{-(c+N)}k_{nn})$ is right-$K_1(\mathfrak{p}^{c+N})$-invariant. Set $K' = K_1(\mathfrak{p}^M)$ with $M = \max(m, c+N)$; then both functions are right-K'-invariant. Taking $W = W_{K'} \in \mathcal{S}_N$ (as constructed above) gives exact point evaluation:

$$\ell_W(V) = \psi^{-1}(\varpi^{-(c+N)})\, V(\varpi^{-N}).$$

If this vanishes for all V, then $V(\varpi^{-N}) = 0$ for all $V \in \mathcal{W}(\pi, \psi)$. But $V(\varpi^{-N}) = \omega_\pi(\varpi^{-N})\, V(I_n)$ where ω_π is the central character of π (applied to the scalar matrix $\varpi^{-N}I_n$), and $\omega_\pi(\varpi^{-N}) \in \mathbb{C}^\times$ is nonzero. So $V(I_n) = 0$ for all V, contradicting $V(I_n) = 1$ for the normalized Whittaker function.

So $\{\mathcal{S}_N \cap \mathcal{B}_{[\pi]}\}_{[\pi]}$ is a countable family of proper subspaces of $\mathcal{S}_N$, and $\mathcal{S}_N \neq \bigcup_{[\pi]}(\mathcal{S}_N \cap \mathcal{B}_{[\pi]})$. There exists $W \in \mathcal{S}_N \setminus \bigcup_{[\pi]} \mathcal{B}_{[\pi]}$: a single-coset vector with $\ell(V) \neq 0$ for some V, for each π. This completes the proof of Theorem 1. $\square$

10 Explicit Formula for $n = 2$ $(\mathrm{GL}_3 \times \mathrm{GL}_2)$

For concreteness, take $n = 2$, $N = 0$. Then Φ_W is the right-K_2-invariant function on $N_2\backslash \mathrm{GL}_2$ supported on N_2K_2 with $\Phi_W(I_2) = 1$. The integral is:

$$\Psi(s) = \int_{K_2} \psi^{-1}(\varpi^{-c}k_{22})\, V(k)\, dk.$$

For π with conductor $\mathfrak{p}^c$ and V the newform V_0 (fixed by $K_1(\mathfrak{p}^c)$):

$$\Psi(s) = \sum_{\bar{k} \in K_2/K_1(\mathfrak{p}^c)} \psi^{-1}(\varpi^{-c}\bar{k}_{22})\, V_0(\bar{k}) \cdot \mathrm{vol}(K_1(\mathfrak{p}^c)).$$

This is a finite sum of Gauss-sum-type terms. By Lemma 5, it is nonzero for some choice of V (possibly different from the newform).

11 Verification and Remarks

11.1 Finiteness

The integral $\Psi(s, W, V)$ is finite for all s because Φ_W has compact support modulo N_n. The integrand is compactly supported on $N_n\backslash \mathrm{GL}_n$, so the integral converges absolutely for all $s \in \mathbb{C}$ (not just for $\mathrm{Re}(s) \gg 0$). This is stronger than what the problem asks.

11.2 The role of N

The parameter N can be any non-negative integer. Different choices of N give different test vectors W. The simplest choice is $N = 0$, giving Φ_W supported on $N_n K_n$ (the "big cell" neighborhood of the identity).

Remark (Constraints on N). The BZ surjectivity (Proposition 4) guarantees the existence of W with $\Phi_W = \Phi_0$ for *any* $N \geq 0$, with no lower bound needed. The Fourier argument in Argument 2 requires $M \geq c + N$ (which is always satisfiable since M is chosen after c and N are fixed). So the proof is valid for all $N \geq 0$.

11.3 Notes on Lemma 5

The proof of Lemma 5 (Section 7) is the most delicate part of the argument. Two complementary approaches are given:

	Argument 1	**Argument 2**
Method	JPSS + countable union	Fourier analysis on K_n
Scope	All n (complete)	$n = 1$ (complete), $n \geq 2$ (partial)
Nature	Existential	Constructive (for $n = 1$)
Key input	Nondegeneracy of RS pairing	Whittaker separation property
Key observation	$\mathcal{B}_\pi$ depends only on $[\pi]$	K_n-translates span n-th row chars

11.4 Summary

The proof has three ingredients:

1. **Algebraic:** The u_Q-twist reduces to multiplication by $\psi^{-1}(Qg_{nn})$ (Lemma 2).
2. **Analytic:** Choosing W with single-determinant-level support makes $\Psi(s)$ a monomial in q^{-s}, eliminating the "$\forall s$" condition.
3. **Representation-theoretic:** The JPSS nondegeneracy of the Rankin–Selberg pairing, combined with a countable union argument over inertial equivalence classes, guarantees nonvanishing for some V (Lemma 5).

A AI Interaction Transcript

As requested by the First Proof organizers, we include a complete record of the AI interaction sessions used to develop this proof.

Timeline: February 10–11, 2026, approximately 5 sessions over two days, approximately 4–5 hours of active working time.
AI systems used: Claude Opus 4.6 (Anthropic), ChatGPT 5.2 Pro (OpenAI), Gemini 3 (Google). Multiple models were used in parallel and cross-checked against each other.
Human role: Prompting, reviewing output, requesting audits, cross-checking between models. No mathematical ideas or content were provided by the human operator.

Example Prompts

1. *"Help me to tackle this problem statement. It is part of First Proof. What are options to tackle this, which would you recommend and why?"*
2. *"Let's harden P02 proof and get that more solid."*
3. *"Is the problem in line with First_Proof.tex?"*

Session 1 — Kickoff *[Claude Opus 4.6]*

- Read problem statement. Populated references with 17 key papers: JPSS (Rankin–Selberg convolutions), BZ (induced representations), Casselman–Shalika, AGRS (multiplicity one), Nelson (subconvexity), etc.
- Solved the $n = 1$ case: u_Q-twist $\to$ Gauss sum $\to$ nonvanishing.
- Developed three-part proof strategy: algebraic reduction, Kirillov model trick, JPSS nondegeneracy.
- Identified gap: Lemma 2 (nonvanishing for fixed W') needed rigorous justification.

Session 2 — Lemma 2 *[Claude Opus 4.6, ChatGPT 5.2 Pro]*

- Explored three approaches to Lemma 2: Frobenius reciprocity, AGRS multiplicity one, K_n-representation theory.
- Developed two complementary arguments: Argument 1 (Frobenius reciprocity) and Argument 2 (Fourier analysis on K_n).
- Identified subtle issue: Argument 2, Step 3 (translates span the full dual) needed more rigorous verification for $n \geq 2$.

Session 3 — Hardening *[Claude Opus 4.6]*

- Rewrote Argument 1 using left radical Rad_L and BZ filtration layer analysis.
- **Critical bug found:** Argument 2, Step 4 had a row/column confusion with permutation matrices. Fixed by restructuring.

- Tightened BZ surjectivity statement, verified g_{nn} well-definedness, expanded Iwasawa decomposition.

Session 4 — Circularity Fix *[Claude Opus 4.6]*

- **Critical discovery:** Previous Argument 1 (Session 3) had a **circularity**—claimed "T is injective on the top BZ layer" but $\ker T = \mathrm{Rad}_L(\pi)$, making the claim equivalent to the conclusion.
- Complete rewrite: existential argument via JPSS nondegeneracy + countable union of proper subspaces.
- **Key new observation:** $\mathcal{B}_\pi$ depends only on the *inertial equivalence class* $[\pi]$. Initial version incorrectly claimed isomorphism classes are countable (they are not—continuous parameters). Inertial class reduction is essential.
- Rewrote §3.7 universality argument: restrict to $\mathcal{S}_N$, show $\mathcal{S}_N \cap \mathcal{B}_{[\pi]}$ is proper via Whittaker separation.

Session 5 — External Review *[Claude Opus 4.6]*

- External review (Claude Opus 4.6 as reviewer) confirmed proof is mathematically sound. Identified 6 presentational issues.
- All fixed: added explicit problem statement with u_Q definition, tightened §3.7 approximation (exact point evaluation via congruence subgroup), added $\mathfrak{o}^\times$ spanning justification, relabeled Argument 2 as supplementary, added Lang citation for countable union lemma.
- Verified problem statement alignment with official `First_Proof.tex`.

Summary of AI Contributions

1. **Mathematical content:** All proof ideas, constructions, and arguments were generated by AI systems.
2. **Error detection:** Two critical errors (circularity in Argument 1, row/column confusion in Argument 2) were found and fixed by AI during the hardening process.
3. **Cross-checking:** Multiple AI models were used to independently verify the proof. The final version was reviewed by Claude Opus 4.6 acting as an external referee.

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