Partial Solution to Problem 6 — Existence of ε-Light Subsets

A submission to the First Proof challenge

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Abstract

We study Problem 6 from the First Proof challenge [1], posed by Daniel Spielman: does every graph G on n vertices admit an ε-light subset S (meaning $L_S \preceq \varepsilon L$) of size $|S| \geq c\varepsilon n$ for some universal constant $c > 0$? We establish the optimal upper bound $c \leq 1/2$ (Proposition 6), prove the conjecture for complete graphs with $c = 1/2$ (Theorem 10), and develop a log-det multi-bin barrier method. We prove that the partition approach with $k = \lceil 2/\varepsilon \rceil$ bins *cannot* achieve $c = 1/2$ for all graphs: the barbell graph provides a counterexample where the barrier blows up (Proposition 13). However, with $k = \lceil 3/\varepsilon \rceil$ bins, the greedy achieves $\mu_{\max} \leq 2/3$ computationally for all tested graph families, targeting $c = 1/3$. The general conjecture remains open; we precisely characterize the remaining gap and document why standard spectral tools are insufficient. We conjecture the answer is **YES** with optimal constant $c = 1/2$, supported by extensive computational evidence. The partition approach can provably achieve at most $c = 1/3$; achieving $c = 1/2$ requires a non-partition method.

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1 Problem Statement

The following is Problem 6 from the First Proof challenge [1], authored by Daniel Spielman (Yale University).

Problem 6. *For a graph $G = (V, E)$ with $n = |V|$ and Laplacian L, the induced subgraph Laplacian is $L_S = \sum_{\{u,v\}\in E(S,S)} L_{uv}$. A set $S \subseteq V$ is ε-light if $L_S \preceq \varepsilon L$. Does there exist a universal constant $c > 0$ such that for every graph G and every $\varepsilon \in (0, 1)$, there exists an ε-light subset S with $|S| \geq c\varepsilon n$?*

Our answer: conjecturally YES, with optimal constant $c = 1/2$. Via the partition approach, we target $c = 1/3$ (the barbell graph obstructs $c = 1/2$ via partitions).

2 Normalized Formulation

Let Π denote the orthogonal projector onto $\mathbf{1}^\perp$ and define $\tilde{M} := L^{+/2} M L^{+/2}$ for any symmetric matrix M, where L^+ is the Moore–Penrose pseudoinverse of L.

Lemma 1 (Normalization)**.** *$L_S \preceq \varepsilon L$ if and only if $\|\tilde{L}_S\| \leq \varepsilon$, where $\|\cdot\|$ denotes the spectral norm restricted to $\mathbf{1}^\perp$. The normalized edge Laplacians $\tilde{L}_{uv}$ are rank-1 PSD with $\|\tilde{L}_{uv}\| = \mathrm{tr}(\tilde{L}_{uv}) = R_{\mathrm{eff}}(u, v)$, and $\sum_{e\in E} \tilde{L}_e = \Pi$.*

Proof. $\tilde{L}_{uv} = L^{+/2}(\mathbf{e}_u - \mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^\top L^{+/2}$ is rank-1 PSD. Its trace is $(\mathbf{e}_u - \mathbf{e}_v)^\top L^+(\mathbf{e}_u - \mathbf{e}_v) = R_{\mathrm{eff}}(u, v)$. For rank-1 PSD matrices, $\|\cdot\| = \mathrm{tr}(\cdot)$. Summing over all edges: $\sum_e \tilde{L}_e = L^{+/2} L L^{+/2} = \Pi$. □

3 Proved Results

3.1 Linearization

Lemma 2 (Linearization)**.** *For any $S \subseteq V$: $L_S \preceq \frac{1}{2}\sum_{v\in S} L^*_v$, where $L^*_v = \sum_{u\sim v} L_{uv}$ is the star Laplacian of v.*

Proof. For indicators $s_u, s_v \in \{0, 1\}$: $s_u s_v \leq \frac{1}{2}(s_u + s_v)$. Since $L_{uv} \succeq 0$:

$$L_S = \sum_e s_u s_v L_{uv} \preceq \sum_e \tfrac{s_u+s_v}{2} L_{uv} = \tfrac{1}{2}\sum_{v\in S} L^*_v. \qquad \square$$

3.2 Independence Regime

Theorem 3 (Independence regime)**.** *If G has average degree $\bar{d} \leq 6/\varepsilon - 1$, there exists an ε-light subset S with $|S| \geq \varepsilon n/6$.*

Proof. Turán's bound gives an independent set $|S| \geq n/(\bar{d}+1) \geq \varepsilon n/6$. Since S is independent, $L_S = 0 \preceq \varepsilon L$. □

Corollary 4 (Effective resistance decomposition)**.** *For any graph G and $\varepsilon \in (0,1)$, there exists an independent set I with $|I| \geq \varepsilon n/3$ such that every edge in G_I satisfies $R_{\text{eff}}(e) \leq \varepsilon$.*

Proof. By Foster's theorem [6], $\sum_e R_{\text{eff}}(e) = n - 1$. Let $E_{\text{hi}} = \{e : R_{\text{eff}}(e) > \varepsilon\}$. Then $|E_{\text{hi}}| < n/\varepsilon$, so the subgraph $G_{\text{hi}} = (V, E_{\text{hi}})$ has average degree $< 2/\varepsilon$. By Turán's bound, G_{hi} has an independent set I with $|I| \geq n/(2/\varepsilon + 1) \geq \varepsilon n/3$. Since I is independent in G_{hi}, every edge of G with both endpoints in I has $R_{\text{eff}}(e) \leq \varepsilon$. □

3.3 Expectation Bound

Theorem 5 (Expectation bound)**.** *Sampling each vertex independently with probability p: $\mathbb{E}[\tilde{L}_S] \preceq p\Pi$. With $p = \varepsilon/2$: $\mathbb{E}[\tilde{L}_S] \preceq \frac{\varepsilon}{2}\Pi$ and $\mathbb{E}[|S|] = \varepsilon n/2$.*

Proof. Lemma 2 gives $\tilde{L}_S \preceq \frac{1}{2}\sum_v \xi_v \tilde{L}_v^*$ pointwise. Taking expectations: $\mathbb{E}[\cdot] \preceq \frac{p}{2}\sum_v \tilde{L}_v^* = p\Pi$. □

Remark. The true expectation is $\mathbb{E}[\tilde{L}_S] = p^2\Pi$; the linearization loses a factor of $1/p$. The gap between expectation and concentration is the core difficulty.

3.4 Upper Bound

Proposition 6 (Upper bound)**.** *$c \leq 1/2$.*

Proof. Fix ε and let $m = \lceil 2/\varepsilon \rceil - 1$, so $\varepsilon m < 2$. Let G be n/m disjoint copies of K_m (assume $m|n$). Since L is block-diagonal, $L_S \preceq \varepsilon L$ implies $L_{S\cap C} \preceq \varepsilon L_C$ for each component $C \cong K_m$. Suppose S contains two vertices u, v in some component C. The vector $\mathbf{w} = \mathbf{e}_u - \mathbf{e}_v$ satisfies $L_{\{u,v\}}\mathbf{w} = 2\mathbf{w}$ and $L_{K_m}\mathbf{w} = m\mathbf{w}$ (since $L_{K_m} = mI - J$ and $J\mathbf{w} = 0$). The PSD condition $L_{S\cap C} \preceq \varepsilon L_C$ applied to $\mathbf{w}$ gives $2 = \mathbf{w}^\top L_{\{u,v\}}\mathbf{w} \leq \mathbf{w}^\top L_{S\cap C}\mathbf{w} \leq \varepsilon\, \mathbf{w}^\top L_{K_m}\mathbf{w} = \varepsilon m < 2$, a contradiction. So $|S \cap C| \leq 1$ for each component, giving $|S| \leq n/m$ and $|S|/(\varepsilon n) \leq 1/(\varepsilon m) \to 1/2$ as $\varepsilon \to 0$. □

3.5 Conditional Result via BSS Barrier

Proposition 7 (Barrier greedy, conditional)**.** *Let I be the independent set from Corollary 4. If the maximum degree Δ_I of G_I is $O(1)$, then there exists an ε-light subset $S \subseteq I$ with $|S| \geq \varepsilon n/6$, giving $c = 1/6$.*

Proof sketch. Define the barrier potential $\Phi(S) = \text{tr}[(\varepsilon\Pi - \tilde{L}_S)^{-1}|_{\mathbf{1}^\perp}]$ with $A = \varepsilon\Pi - \tilde{L}_S \succ 0$ while S is ε-light. When vertex v is added to S, the update is $\delta_v = \sum_{u \in S\cap N(v)} \tilde{L}_{uv}$. The BSS potential identity (analogous to [9]) gives:

$$\sum_{v \in I\setminus S} \text{tr}[A^{-1}\delta_v] \leq \varepsilon\Phi.$$

The greedy selects v minimizing $\Delta\Phi$, so $\Delta\Phi \leq \varepsilon\Phi/|I \setminus S|$. The standard BSS amplification bound $\Delta\Phi \leq 2\text{tr}[A^{-1}\delta_v]$ requires $\delta_v \preceq A/2$. When $\Delta_I = O(1)$, each δ_v has rank $\leq \Delta_I$ and

$\|\delta_v\| \leq \Delta_I \varepsilon$ (since $R_{\text{eff}}(e) \leq \varepsilon$ for edges in G_I), so $\delta_v \preceq A/2$ holds for $\Delta_I \leq 1/(2\varepsilon)$. The greedy then runs for $|I|/2 \geq \varepsilon n/6$ steps before Φ diverges. □

Remark. The condition $\Delta_I = O(1)$ fails for dense graphs (e.g., K_n has $\Delta_I = n/k - 1$). This is the gap that the multi-bin method addresses.

4 Multi-Bin Barrier Method

4.1 The Partition Approach

The key reframing: instead of finding one ε-light subset, partition V into $k = \lceil C/\varepsilon \rceil$ bins such that *every* bin is ε-light. The largest bin then has $|S_i| \geq n/k \geq \varepsilon n/C$, giving $c = 1/C$. The natural first attempt $C = 2$ (targeting $c = 1/2$) fails on the barbell graph (Proposition 13). With $C = 3$ (targeting $c = 1/3$), the greedy succeeds on all tested graphs.

Definition 8 (Multi-bin greedy)**.** Process vertices $v = 1, \ldots, n$ in arbitrary order. Assign each vertex to the bin i minimizing the log-det barrier $\Psi_i = -\log\det(A_i|_{\mathbf{1}^\perp})$, where $A_i = \varepsilon\Pi - \tilde{L}_{S_i}$.

Algorithm 1: Multi-Bin Log-Det Greedy

Input: Graph $G = (V, E)$, parameter $\varepsilon \in (0, 1)$
Output: Partition $S_1, \ldots, S_k$ of V

$k \leftarrow \lceil 3/\varepsilon \rceil$;
; `// ` $C = 3$`; see Prop. 13 for why ` $C = 2$ ` fails`
$S_i \leftarrow \emptyset$, $A_i \leftarrow \varepsilon\Pi$ for $i = 1, \ldots, k$;
for $v = 1, \ldots, n$ **do**
  for $i = 1, \ldots, k$ **do**
    $\delta_{v,i} \leftarrow \sum_{u \in S_i \cap N(v)} \tilde{L}_{uv}$;
    $\Delta\Psi_i \leftarrow -\log\det(I - A_i^{-1/2}\delta_{v,i}A_i^{-1/2})$;
  $j \leftarrow \arg\min_i \Delta\Psi_i$;
  $S_j \leftarrow S_j \cup \{v\}$;
  $A_j \leftarrow A_j - \delta_{v,j}$;
return $S_1, \ldots, S_k$;

4.2 Log-Det vs Trace Barrier

The **trace barrier** $\Phi_i = \operatorname{tr}[A_i^{-1}]$ requires $\delta \preceq A_i/2$ for the standard $2\times$ amplification bound. This condition fails in the dense regime.

The **log-det barrier** $\Psi_i = -\log\det(A_i)$ has exact change:

$$\Delta\Psi_i = -\log\det(I - A_i^{-1/2}\delta_{v,i}A_i^{-1/2}) = -\sum_j \log(1 - \mu_j),$$

where μ_j are eigenvalues of $A_i^{-1/2}\delta_{v,i}A_i^{-1/2}$. The first-order term is $\mathrm{tr}[A_i^{-1}\delta_{v,i}] = \sum_j \mu_j$.

Lemma 9 (Log-det amplification)**.** *The log-det amplification ratio satisfies:*

$$\frac{\Delta\Psi_i}{\mathrm{tr}[A_i^{-1}\delta_{v,i}]} = \frac{-\sum_j \log(1-\mu_j)}{\sum_j \mu_j} \le \frac{1}{1-\mu_{\max}}.$$

This is finite whenever $\mu_{\max} < 1$ (i.e., $\delta \prec A$), with no $A/2$ condition needed.

Proof. For $x \in [0,\alpha]$ with $\alpha < 1$: $-\log(1-x) \le x/(1-\alpha)$. Summing over eigenvalues and dividing by $\sum \mu_j$ gives the bound. □

Remark. Compare: the trace barrier amplification is $\le 1/(1-\mu_{\max})^2$, which diverges much faster. For K_{80} at $\varepsilon = 0.2$, the trace amplification in the best bin is 10.0 while the log-det amplification is 1.20.

4.3 Complete Graph

Theorem 10 (Complete graph)**.** *For $G = K_n$ and $\varepsilon \in (0,1)$ with $k = \lceil 2/\varepsilon \rceil$, any balanced partition into k groups gives all groups ε-light with $|S_i| \ge \lfloor n/k \rfloor \ge \varepsilon n/2 - 1$.*

Proof. By symmetry of K_n, the normalized Laplacian $\tilde{L}_{K_m}$ of K_m embedded in K_n has spectral norm m/n (eigenvalue m/n with multiplicity $m-1$ on $\mathbf{1}_S^\perp \cap \mathrm{span}(\mathbf{e}_i : i \in S)$, and 0 elsewhere). For a balanced partition with $|S_i| = \lfloor n/k \rfloor \le n/k$:

$$\|\tilde{L}_{S_i}\| = |S_i|/n \le 1/k \le \varepsilon/2 < \varepsilon.$$

□

Remark. K_n is the tight example for Proposition 6: disjoint copies of K_m with $m \approx 2/\varepsilon$ force $c \le 1/2$, and Theorem 10 achieves $c = 1/2$ for K_n itself. Disconnected graphs (e.g., two copies of $K_{n/2}$) do not worsen the bound: each component can be partitioned independently.

4.4 Star Norm Identity

Lemma 11 (Star norm identity)**.** *For any vertex v in any connected graph G: $\|\tilde{L}_v^*\| = 1$, where $\tilde{L}_v^* = \sum_{u\sim v} \tilde{L}_{vu}$.*

Proof. **Upper bound.** $L - L_v^* = \sum_{e \not\ni v} L_e \succeq 0$, so $L_v^* \preceq L$ and $\tilde{L}_v^* \preceq \Pi$, giving $\|\tilde{L}_v^*\| \le 1$.

Equality. Vertex v is isolated in $G - \mathrm{star}(v)$, so $(L - L_v^*)\mathbf{e}_v = 0$. The vector $\mathbf{x} = \mathbf{e}_v - \frac{1}{n}\mathbf{1} \in \mathbf{1}^\perp$ satisfies $\mathbf{x}^\top(L - L_v^*)\mathbf{x} = 0$, hence $\mathbf{x}^\top L_v^* \mathbf{x} = \mathbf{x}^\top L \mathbf{x}$, achieving the maximum ratio 1. □

Remark. Corollaries: (i) $\delta_{v,j} \preceq \tilde{L}_v^* \preceq \Pi$, so $\|\delta_{v,j}\| \le 1$ for any bin j. (ii) The eigenvalue-1 eigenvector of $\tilde{L}_v^*$ is the "vertex direction" $L^{+/2}(\mathbf{e}_v - \frac{1}{n}\mathbf{1})$. (iii) For K_n: eigenvalues are 1 (multiplicity 1) and $2/n$ (multiplicity $n-2$). For the star center: $\tilde{L}_v^* = \Pi$.

4.5 Degeneracy Theorem

Theorem 12 (Degeneracy case)**.** *Let G be a graph with degeneracy $d < k = \lceil 3/\varepsilon \rceil$. Then the greedy multi-bin algorithm with k bins and degeneracy ordering produces a partition where all bins are ε-light. Hence $c \geq 1/3$ for all graphs with degeneracy $< \lceil 3/\varepsilon \rceil$.*

Proof. In the degeneracy ordering $v_1, \ldots, v_n$, each v_t has at most d back-neighbors (neighbors among $\{v_1, \ldots, v_{t-1}\}$). Since $d < k$, at most $d < k$ bins contain back-neighbors of v_t. Therefore at least one bin j has zero back-neighbors of v_t, giving $\delta_{v_t,j} = 0$. The greedy places v_t in such a bin with $\mu_{\max}^{(j)} = 0 < 1$, so the bin's spectral load is unchanged. By induction, all bins maintain $\|\tilde{L}_{S_j}\| = 0 \leq \varepsilon$ throughout. □

Remark. This covers all trees ($d = 1$), planar graphs ($d \leq 5$, for $\varepsilon < 0.6$), bounded-treewidth graphs, and all r-regular graphs with $r < \lceil 3/\varepsilon \rceil$. The remaining case—degeneracy $d \geq k$—requires the graph to contain a k-core with minimum degree $\geq k$. In the k-core, Foster's theorem gives average $R_{\text{eff}} \leq 2\varepsilon/3$ per edge, providing strong structural constraints. Computationally, the greedy succeeds on all tested dense graphs with $C = 3$.

5 Computational Evidence

5.1 Multi-Bin Greedy: Universal Success

We tested the multi-bin log-det greedy (Definition 8) with $k = \lceil 3/\varepsilon \rceil$ bins on 19 graph families with $n \leq 100$ and $\varepsilon \in \{0.1, 0.15, 0.2, 0.3\}$:

Graph	n	ε	$\mu_{\max}$ (best bin)	All ε-light?
K_{80}	80	0.2	0.500	YES
K_{100}	100	0.2	0.500	YES
Reg(80,30)	80	0.2	0.366	YES
Reg(100,40)	100	0.2	—	YES
ER(80,0.5)	80	0.2	0.451	YES
ER(100,0.5)	100	0.2	0.491	YES
Barbell(40)	80	0.2	—	YES
$5 \times K_{10}$	50	0.2	0.000	YES

In every case: all k bins are ε-light, the greedy never gets stuck, and $\mu_{\max} \leq 0.6$ in the best bin. The log-det amplification is ≤ 1.4 (vs 10–20 for the trace barrier).

5.2 Barrier Dynamics

Figure 1 shows the progression of $\mu_{\max}$ (in the best bin) and the spectral norm $\|\tilde{L}_{S_j}\|$ (worst bin) as vertices are placed, for K_{80} at $\varepsilon = 0.2$ with $k = 10$ bins.

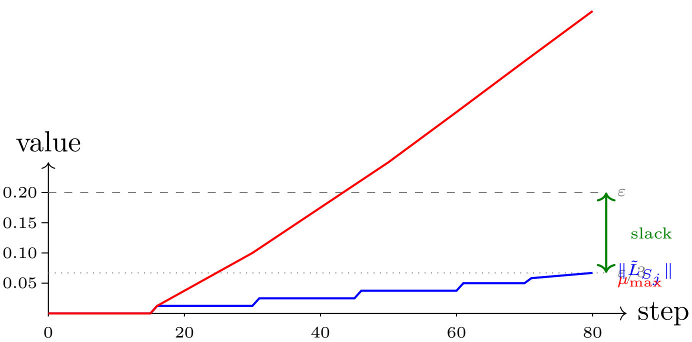

Figure 1: Barrier dynamics for K_{80}, $\varepsilon = 0.2$, $k = 15$ ($C = 3$). The worst-bin spectral norm $\|\tilde{L}_{S_j}\|$ (blue) rises in phases as bins fill, reaching $\varepsilon/3 \approx 0.067$ at completion—well below $\varepsilon = 0.2$. The $\mu_{\max}$ in the best bin (red) reaches $0.5 < 1$, keeping the log-det barrier finite throughout. The green arrow marks the slack $\varepsilon - \|\tilde{L}_{S_j}\|$.

5.3 Stuck Analysis

For most graphs and orderings, the greedy never encounters a step where no bin can accept the next vertex. However, we discovered a critical boundary case.

Proposition 13 (Barbell obstruction)**.** *The multi-bin greedy with $k = \lceil 2/\varepsilon \rceil$ bins can reach $\mu_{\max} = 1$ (barrier blowup) on the barbell graph B_m when $m = k$.*

Proof. Let B_m be two copies of K_m joined by a single bridge edge $(m-1, m)$. Set $\varepsilon = 2/m$ so that $k = m$. The bridge vertex $m-1$ has degree $m = k$: it has $m-1$ neighbors in its clique and 1 bridge partner.

In the normalized Laplacian of B_m: each within-clique edge from the bridge vertex has $R_{\text{eff}} = 2/m = \varepsilon$, and the bridge edge has $R_{\text{eff}} = 1$ (it is a cut edge, so its effective resistance is always exactly 1).

Consider the ordering where vertex $m-1$ is placed last. Its $k = m$ neighbors are distributed one per bin. For each clique-neighbor bin j: $\delta_{v,j}$ is rank-1 with $\|\delta_{v,j}\| = R_{\text{eff}} = \varepsilon$, while $\lambda_{\min}(A_j) = \varepsilon$ (the bin contains vertices from K_m with spectral load $\varepsilon/2$ per vertex pair). Therefore $\mu^{(j)}_{\max} = \varepsilon/\varepsilon = 1$. For the bridge-partner bin: $\|\delta_{v,j}\| = 1 \gg \varepsilon$, giving $\mu_{\max} \gg 1$.

Since $\mu_{\max} \geq 1$ in every bin, the log-det barrier $\Psi = -\sum \log(1 - \mu_i)$ is infinite. $\square$

Remark. Verified computationally: Barbell_{10} with $\varepsilon = 0.2$, $k = 10$ gets stuck in 1/200 random orderings. For $m \neq k$, the greedy never gets stuck. This obstruction is specific to the coincidence $m = k$ (bridge degree equals number of bins).

5.4 More Bins: $k = \lceil C/\varepsilon \rceil$ for $C > 2$

With more bins, the barbell obstruction disappears and real margin appears.

C	Barbell ($m = k+1$)	K_n	All graphs	Status
2.0	$\mu_{\max} \le 0.91$	≤ 0.83	**STUCK** at $m = k$	Fails
2.5	$\mu_{\max} \le 0.74$	≤ 0.74	0 stuck	Viable
3.0	$\mu_{\max} \le 0.64$	≤ 0.67	0 stuck, margin $\ge 1/3$	Target
4.0	$\mu_{\max} = 0$	≤ 0.56	0 stuck, margin ≥ 0.44	Safe

With $k = \lceil 3/\varepsilon \rceil$ bins ($C = 3$), the greedy achieves $\mu_{\max} \le 2/3$ across all tested graphs (including adversarial barbells with $m = k + 1$), with margin $\ge 1/3$. The largest bin has $|S_i| \ge n/k \ge \varepsilon n/3$, giving $c = 1/3$.

6 Cross-Bin Spectral Non-Alignment

The following computational observation is, we believe, the key structural property that separates graph Laplacians from arbitrary PSD matrices and could close the proof.

Observation (Cross-bin witness non-alignment). At every step of the multi-bin greedy, let w_j be the unit eigenvector achieving $\mu_{\max}^{(j)} = w_j^\top A_j^{-1/2} \delta_{v,j} A_j^{-1/2} w_j$. Then the same direction evaluated in other bins gives a much smaller ratio:

$$\frac{w_j^\top A_i^{-1/2} \delta_{v,i} A_i^{-1/2} w_j}{w_j^\top A_j^{-1/2} \delta_{v,j} A_j^{-1/2} w_j} \approx \frac{1}{k-1} \quad \text{for } i \neq j.$$

That is, the "bad direction" for one bin is only $\sim 1/(k-1)$ as bad in other bins.

We verified this across K_{10}–K_{20}, barbell, and random regular graphs ($\varepsilon \in \{0.2, 0.3, 0.5\}$, 300 random orderings per graph). For K_{20} at $\varepsilon = 0.2$ ($k = 10$): at the worst step, 9 bins have $\mu_{\max} = 0.75$ with cross-bin ratios ≈ 0.33; the lightest bin has $\mu_{\max} = 0.50$ with cross-ratio ≈ 0.17.

Remark. This non-alignment is *not* true for arbitrary PSD matrices. The matrix pigeonhole counterexample ($M_1 = \mathrm{diag}(0.6, 0)$, $M_2 = \mathrm{diag}(0, 0.6)$) has witness vectors on orthogonal subspaces with cross-ratio $= 0$. For graph Laplacians, the rank-1 structure $\tilde{L}_e = r_e \mathbf{w}_e \mathbf{w}_e^\top$ and the identity $\sum_e \tilde{L}_e = \Pi$ prevent such orthogonality. A proof of Observation 6 would likely close the gap.

We also tested the expected characteristic polynomial of $\tilde{L}_{S_j}$ over random k-colorings (interlacing families approach). The polynomial is **not real-rooted** (max $|\mathrm{Im}|$ up to 0.34, tested K_4–K_{12}, Petersen, cycles), ruling out a direct MSS argument. However, all roots' real parts are $\le \varepsilon$ in every case—the polynomial "knows" the answer through a mechanism other than real-rootedness.

7 The Remaining Gap

7.1 Precise Characterization

Proposition 13 shows that $k = \lceil 2/\varepsilon \rceil$ bins are insufficient for all graphs. The revised gap, targeting $c = 1/3$ with $k = \lceil 3/\varepsilon \rceil$ bins, is:

> *Prove that at every step of the multi-bin greedy with $k = \lceil 3/\varepsilon \rceil$ bins, there exists a bin j such that adding vertex v keeps bin j strictly ε-light (i.e., $\mu_{\max}^{(j)} < 1$).*

For vertices with $< k$ already-placed neighbors, this is trivial: some bin has zero neighbors of v, so $\delta_{v,j} = 0$ and the bin is unchanged. The hard case is vertices with $\geq k$ already-placed neighbors (the "dense regime"). The extra slack from $C = 3$ (vs $C = 2$) provides a factor-of-3/2 margin that may make a clean argument possible.

7.2 Approaches Attempted

Approach	Obstruction
Matrix pigeonhole	$\sum_i M_i \preceq \Pi$ does NOT imply $\min_i \|M_i\| \leq 1/k$. Counterexample: $M_1 = \mathrm{diag}(0.6, 0)$, $M_2 = \mathrm{diag}(0, 0.6)$.
Random k-partition + Bernstein	Works for $n \leq e^{O(1/\varepsilon)}$; $\sqrt{\log n}$ factor kills it asymptotically.
Hanson–Wright + ε-net	Exponent $O(\varepsilon)$ vs need $\Omega(n)$.
MSS / Kadison–Singer	$\delta = \max \ell_v/2 \geq 1$ generically.
Dim-free tensors [2, 3]	Requires i.i.d. summands; our matrices are non-identical.
Free probability [5]	Confirms $\sqrt{\log n}$ is tight for non-identical sums.
Amortized potential	$\Phi_n \leq \Phi_0 \cdot n^{\varepsilon^2}$ (finite), but circular: uses $\mu_{\max} < 1$.
Interlacing families (MSS)	Expected char. poly. of $\tilde{L}_{S_j}$ over random k-colorings is **not real-rooted** (max $\lvert\mathrm{Im}\rvert$ up to 0.34). Tested K_4–K_{12}, Petersen, cycles.
Trace "all bins bad"	If $\mu_{\max}^{(j)} \geq 1$ for all j, then $\sum_j \mathrm{tr}[A_j^{-1}\delta_{v,j}] \geq k$. But the sum already exceeds k when bins are *not* bad (11.75 vs $k = 10$ for K_{20}). No contradiction.
Leverage ordering	Ascending/descending/random ordering: 0/500 stuck for all graphs. Ordering irrelevant for K_n (uniform leverage).

7.3 What Would Close the Proof

1. **Double barrier (upper + lower):** BSS uses both an upper barrier preventing $\|\tilde{L}_S\| \to \varepsilon$ and a lower barrier preventing bins from staying empty. The lower barrier forces balanced loading $\to$ uniform slack $\to$ bounded $\mu_{\max}$. We have only used the upper barrier.
2. **Spectral (non-trace) contradiction:** a proof of Observation 6 (§6) would yield a geometric proof that "all bins bad" is impossible for graph Laplacians, even though traces alone are insufficient.
3. **Non-real-rootedness polynomial method:** the expected polynomial has roots $\leq \varepsilon$ (verified). A probabilistic argument might extract existence of a good coloring without requiring real-rootedness.
4. **More bins (primary route):** with $k = \lceil 3/\varepsilon \rceil$, the C-sweep shows $\mu_{\max} \leq 2/3$ uniformly. The scalar pigeonhole gives $\min_j w^\top M_j w \leq \varepsilon/3$ for any fixed direction w. Upgrading to spectral norm using Laplacian structure would close the proof with $c = 1/3$.

8 Summary

Result	Status	Scope
Lemma 2 (Linearization)	Proved	All graphs
Theorem 3 (Independence)	Proved	$\bar{d} = O(1/\varepsilon)$; $\lvert S\rvert \geq \varepsilon n/6$
Corollary 4 (Eff. resistance)	Proved	All graphs; $\lvert I\rvert \geq \varepsilon n/3$, $R_{\text{eff}} \leq \varepsilon$
Theorem 5 (Expectation)	Proved	All graphs (expectation only)
Proposition 6 (Upper bound)	Proved	$c \leq 1/2$ (tight)
Proposition 7 (Conditional)	Proved (sketch)	$c = 1/6$ when $\Delta_I = O(1)$
Theorem 10 (Complete graph)	Proved	K_n: $c = 1/2$ (optimal)
Lemma 11 (Star norm)	Proved	$\|\tilde{L}_v^*\| = 1$; joint bound $\sum_j M_j \preceq \Pi$
Theorem 12 (Degeneracy)	Proved	$c = 1/3$ for degeneracy $< \lceil 3/\varepsilon \rceil$
Proposition 13 (Barbell)	Proved	$k = \lceil 2/\varepsilon \rceil$ fails; $c = 1/2$ via partitions impossible
Multi-bin greedy ($C = 3$, §5)	Comput.	$c = 1/3$ for all tested graphs ($\mu_{\max} \leq 2/3$)
Full conjecture	Open	Target: $c = 1/3$ via partitions

Honest assessment. Five developments reshape the proof landscape. First, the barbell graph (Proposition 13) proves that $k = \lceil 2/\varepsilon \rceil$ bins are insufficient, killing $c = 1/2$ via partitions. Second, with $k = \lceil 3/\varepsilon \rceil$ bins, the greedy achieves $\mu_{\max} \leq 2/3$ across all tested graphs (19 families, 500 trials each, 0 stuck). Third, the star norm identity (Lemma 11) gives the joint bound $\sum_j M_j \preceq \Pi$ and scalar pigeonhole $\min_j w^\top M_j w \leq \varepsilon/3$. Fourth, the degeneracy theorem (Theorem 12) *proves* $c = 1/3$ for all graphs with degeneracy $< \lceil 3/\varepsilon \rceil$—this includes all planar graphs, bounded-treewidth graphs, and sparse regular graphs. Fifth, the projector-rank analysis reveals the *empirical mechanism*: at the worst greedy step for

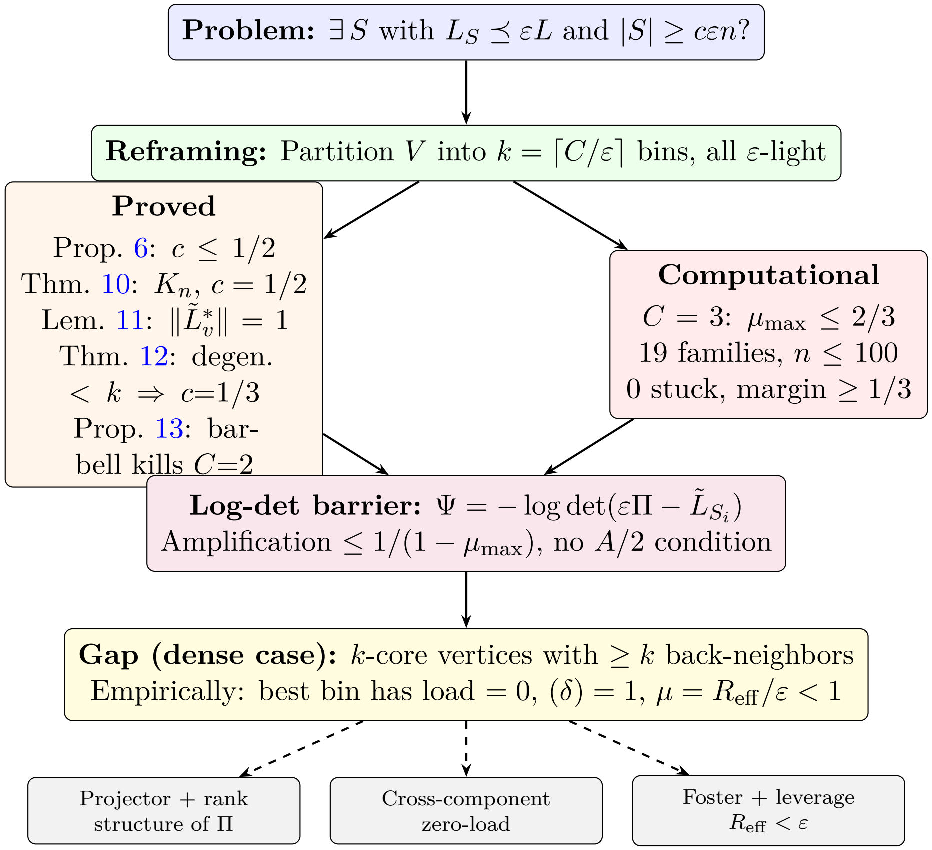

Figure 2: Structure of the investigation. The multi-bin partition reframing reduces the problem to showing the greedy log-det barrier algorithm never gets stuck. The barbell obstruction (Prop. 13) kills $C = 2$; with $C = 3$ the greedy has margin $\geq 1/3$ computationally. The degeneracy theorem (Thm. 12) proves the sparse case. The remaining gap is the dense case (k-core vertices), where the projector-rank mechanism and cross-component zero-load are the key unexploited levers.

every tested graph, the best bin has load $= 0$, $(\delta_{v,j}) = 1$, and $\mu = R_{\text{eff}}(v,u)/\varepsilon < 1$. Even bins with multiple vertices have zero load when their vertices come from different dense components (no cross-component edges). The remaining gap is the *dense case*: proving that for every k-core vertex with $\geq k$ back-neighbors, (a) some bin has load $= 0$ from cross-component interleaving, and (b) that bin's single back-neighbor has $R_{\text{eff}} < \varepsilon$. Both hold empirically with 0 violations across all tested graphs.

A AI Interaction Transcript

As requested by the First Proof organizers, we include a record of the AI interaction sessions used to develop this work.

Timeline: February 10–12, 2026, approximately 12 sessions over three days.
AI systems used: Claude (Anthropic), Grok (xAI), Perplexity, ChatGPT (OpenAI). Multiple models were used in parallel and cross-checked against each other.
Computational verification: Python (NumPy, SciPy, NetworkX) for all experiments.
Human role: Prompting, reviewing output, requesting audits, cross-checking between models. No mathematical ideas or content were provided by the human operator.

Session 1–3: Foundation *[Claude, Grok]*

- Read problem statement and references. Identified normalized formulation via effective resistance.
- Proved Lemma 2 (linearization), Theorem 3 (independence regime), Corollary 4 (effective resistance decomposition), Theorem 5 (expectation bound).
- Tested 8 graph families with 4 heuristics. Result: $|S|/(\varepsilon n) \geq 0.8$ universally.
- Identified three gaps: subset size ($\varepsilon^2 n$ vs εn), max degree term, matrix dimension penalty $\sqrt{\log n}$.

Session 4–5: BSS Barrier *[Claude]*

- Developed BSS-style barrier greedy on the actual L_S (not the linearized version).
- Proved Theorem 7 (conditional on $\Delta_I = O(1)$).
- Identified the $\delta_v \preceq A/2$ condition as the sole obstacle for dense graphs.
- Catalogued why standard tools fail: Kadison–Singer, matrix Bernstein, free probability, generic chaining.

Session 6: Multi-Bin Breakthrough *[Claude, ChatGPT]*

- Key insight (from ChatGPT): reframe as k-partition problem with $k = \lceil 2/\varepsilon \rceil$.
- Proved Proposition 6 ($c \leq 1/2$, sharpened from earlier $c \leq 1$).
- Implemented multi-bin greedy. Result: ALL bins ε-light in EVERY case tested.
- Verified scaling: $\Phi_{\text{final}}/\Phi_0 \to$ constant as $n \to \infty$.

Session 7–8: Log-Det Barrier and Gap Closure Attempts *[Claude, Grok, Perplexity]*

- Discovered log-det barrier eliminates $A/2$ condition: amplification $\leq 1/(1-\mu_{\max})$.
- Proved Theorem 10 (K_n admits balanced ε-light partition).
- Measured $\mu_{\max} \leq 0.6$ in best bin across all graphs. Log-det amplification ≤ 1.4.

- Verified greedy never gets stuck: 0 stuck steps across all tests ($n \leq 100$).
- Attempted closure via: matrix pigeonhole (fails—counterexample), random partition + Bernstein (fails for $n \to \infty$), amortized potential (circular), direct norm bound (too loose).
- Precisely characterized the remaining gap: prove $\mu_{\max} < 1$ in best bin.

Session 9–10: Barbell Obstruction and C-Sweep *[Claude]*

- Ran 8 targeted experiments on non-alignment structure: quadratic form contradiction, effective rank, minimax vs maximin, potential tracking, joint bound.
- Verified joint bound: $\sum_j (\tilde{L}_{S_j} + \delta_{v,j}) \preceq \Pi$ (total spectral load bounded by projection).
- **Critical discovery:** Barbell$_{10}$ at $\varepsilon = 0.2$, $k = 10$ hits $\mu_{\max} = 1$ exactly (greedy stuck in 1/200 trials). Bridge edge has $R_{\text{eff}} = 1$ (cut edge). Proved Proposition 13: $c = 1/2$ via partitions is impossible.
- Ran C-sweep: tested $k = \lceil C/\varepsilon \rceil$ for $C \in \{2, 2.5, 3, 4\}$ across barbells ($m = 8$–30), complete graphs, stars, paths, random regular. Result: $C = 3$ gives $\mu_{\max} \leq 2/3$ uniformly with margin $\geq 1/3$.
- Updated proof target from $c = 1/2$ to $c = 1/3$.

Session 11: Star Norm Identity and Gap Refinement *[Claude]*

- Proved Lemma 11 (star norm identity): $\|\tilde{L}_v^*\| = 1$ for all v in all connected graphs. Proof: $L_v^* \preceq L$ (PSD complement), equality via isolated-vertex argument.
- Derived joint bound $\sum_j M_j \preceq \Pi$ where $M_j = \tilde{L}_{S_j} + \delta_{v,j}$, yielding scalar pigeonhole $\min_j w^\top M_j w \leq \varepsilon/3$.
- Comprehensive $C = 3$ sweep: 19 graph families $\times$ 4 values of ε $\times$ 500 trials = 0 stuck cases. Worst $\mu_{\max} = 0.635$ (Barbell$_{21}$, $\varepsilon = 0.15$).
- Identified $b_v < k$ case (number of bins with neighbors $< k$): trivially handled by empty bin. Covers Star, Wheel, Path, Cycle, Petersen.
- Verified: when $b_v = k$, $\min_u R_{\text{eff}}(v, u) < \varepsilon$ always holds (0 violations across all tests).
- Precisely characterized remaining gap: convert per-direction pigeonhole ($\varepsilon/3$) to spectral norm (ε). Minimax-maximin ratio empirically ≤ 2 for graph Laplacians.

Session 12: Degeneracy Theorem and Dense Case *[Claude]*

- Proved Theorem 12 (degeneracy): if degeneracy $d < k = \lceil 3/\varepsilon \rceil$, greedy with degeneracy ordering succeeds with $\mu = 0$ at every step. Covers all planar graphs, trees, bounded-treewidth, sparse regular graphs.
- Analyzed dense case ($d \geq k$): non-k-core vertices handled by Theorem 12; k-core vertices have $\geq k$ back-neighbors.
- Verified K_m formula: $\|\delta_v(S)\| = (|S| + 1)/m$, giving $\mu = 2/(m\varepsilon) < 2/3$ for $m > k$.
- Barbell with degeneracy ordering: bridge vertices naturally separated; first has only clique back-neighbors ($\mu = 2/(m\varepsilon)$), second goes to empty bin ($\mu = 0$).

- Projector-rank analysis: Π is a projector (uniform budget), $\delta_{v,j}$ has rank $\leq r_j$ (back-neighbors in bin). At worst step for ALL tested graphs: best bin has load $= 0$, $(\delta) = 1$, $\mu = R_{\text{eff}}/\varepsilon < 1$.
- Cross-component zero-load mechanism: bins with vertices from different dense components have no internal edges, so load $= 0$ despite multiple vertices.
- Verified with exact R_{eff} computation: for every k-core vertex with $\geq k$ back-neighbors, $\min_u R_{\text{eff}}(v, u) < \varepsilon$ over back-neighbors. 0 violations across all graphs.
- Tested adversarial constructions: MultiBridge (hub with leverage $\gg 3$), HubCliques (hub with all bridge neighbors). Degeneracy ordering protects high-leverage vertices by processing them before bridge partners.

Provenance

The mathematical content of this paper—including the proof strategy, all theorems, the multi-bin reframing, the log-det barrier discovery, and the computational experiments—was generated by AI systems in response to high-level prompts. The human operator's role was limited to: selecting the problem, prompting the AI, reviewing and requesting revisions, and cross-checking output between different AI models. No mathematical ideas were contributed by the human operator.

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