Solution to Problem 1 — Mutual Singularity of the Φ^4_3 Measure Under Smooth Shifts

A submission to the First Proof challenge

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Abstract

We solve Problem 1 from the First Proof challenge [1], posed by Martin Hairer (EPFL and Imperial). For the Φ^4_3 measure μ on $\mathcal{D}'(\mathbb{T}^3)$ and any nonzero smooth function $\psi \in C^\infty(\mathbb{T}^3)$, we prove that μ and its translate $T^*_\psi\mu$ are **mutually singular**. The proof constructs an explicit separating set A_ψ using Hairer's renormalized-cube functional with super-exponential mollification and an unmollified mass-correction term. We show $\mu(A_\psi) = 1$ by decomposing the functional into the renormalized cubic (which converges by regularity structures) plus a mollification error (which is super-exponentially small), and $(T^*_\psi\mu)(A_\psi) = 0$ via the Hermite shift formula, which produces a deterministic divergence $-b\|\psi\|^2_{L^2}e^{n/4}$ from the unmollified field, where $b \neq 0$ is the logarithmic mass renormalization constant. We also explain why the Hairer–Kusuoka–Nagoji framework [9], which separates μ from the Gaussian free field, cannot detect smooth shifts. The answer is **NO**.

Contents

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1 Problem Statement

The following is Problem 1 from the First Proof challenge [1], posed by Martin Hairer (EPFL and Imperial).

Problem 1. *Let $\mathbb{T}^3$ be the three-dimensional unit-size torus and let μ be the Φ^4_3 measure on the space of distributions $\mathcal{D}'(\mathbb{T}^3)$. Let $\psi : \mathbb{T}^3 \to \mathbb{R}$ be a smooth function that is not identically zero and let $T_\psi : \mathcal{D}'(\mathbb{T}^3) \to \mathcal{D}'(\mathbb{T}^3)$ be the shift map given by $T_\psi(u) = u + \psi$. Are the measures μ and $T^*_\psi \mu$ equivalent?*

Theorem 1 (Main result)**.** *Let $\mathbb{T}^3$ be the three-dimensional unit torus, μ the Φ^4_3 measure on $\mathcal{D}'(\mathbb{T}^3)$, and $\psi \in C^\infty(\mathbb{T}^3)$ with $\psi \not\equiv 0$. Then μ and $T^*_\psi \mu$ are mutually singular.*

Answer: NO — the measures are not equivalent; they are mutually singular.

2 Background

The Φ^4_3 measure μ is the Gibbs measure of the ϕ^4 Euclidean quantum field theory in three dimensions. It is constructed as the invariant measure of the stochastic quantization equation

$$\partial_t u + (1 - \Delta)u + u^3 - C_\varepsilon\, u = \sqrt{2}\,\xi \tag{1}$$

where ξ is space-time white noise on $\mathbb{T}^3$ and C_ε is the renormalization counterterm:

$$C_\varepsilon = a_1 \varepsilon^{-1} + b \log(\varepsilon^{-1}) + O(1), \qquad b \neq 0. \tag{2}$$

The leading divergence $a_1\varepsilon^{-1}$ is the Wick-ordering counterterm, while the logarithmic coefficient b arises from the second-order (sunset) Feynman diagram. The nonvanishing of b is established in [5], §10 and [3].

For the Gaussian free field (GFF) μ_0 on $\mathbb{T}^3$, the Cameron–Martin theorem guarantees quasi-invariance under smooth shifts: $\mu_0 \sim T^*_\psi \mu_0$ for all $\psi \in C^\infty$. For Φ^4_2 in two dimensions, the measure is absolutely continuous with respect to μ_0 (only Wick ordering is needed, $b = 0$), so quasi-invariance is inherited. In three dimensions, however, $\mu \perp \mu_0$ [9], Corollary 4.1, and the question of quasi-invariance under smooth shifts requires a separate argument.

Hairer posted a short note [6] with a rough sketch of the proof that $\mu \perp \mu_0$. As the First Proof FAQ states, "a successful answer would involve filling in the gaps in the argument." Our proof fills in these gaps and extends the argument to show $\mu \perp T^*_\psi \mu$.

3 Setup and Notation

Let $\rho \in C^\infty_c(\mathbb{R}^3)$ be a symmetric mollifier with $\int \rho = 1$. Define the **super-exponential mollification scale**

$$\varepsilon_n = e^{-e^n}, \qquad \rho_n(x) = e^{3e^n} \rho(e^{e^n} x).$$

For a distribution $\Phi \in \mathcal{D}'(\mathbb{T}^3)$, write $\Phi_n = \rho_n * \Phi$.

Definition 2 (Pointwise variance)**.** Under the GFF μ_0:

$$c_n := \mathbb{E}_{\mu_0}[|\Phi_n(x)|^2] = \sum_{k\in\mathbb{Z}^3} \langle k\rangle^{-2}|\hat{\rho}_n(k)|^2 \simeq a \cdot e^{e^n}$$

where $a > 0$ depends on ρ and $\langle k\rangle = (1+|k|^2)^{1/2}$.

Definition 3 (Wick cube)**.** The Wick cube at scale n is $:\Phi_n^3: = H_3(\Phi_n; c_n) = \Phi_n^3 - 3c_n\Phi_n$, where $H_3(x; c) = x^3 - 3cx$ is the third Hermite polynomial.

Definition 4 (Separating set)**.** For $\psi \in C^\infty(\mathbb{T}^3)$ with $\psi \not\equiv 0$, define:

$$A_\psi := \left\{\Phi \in \mathcal{D}'(\mathbb{T}^3) : \lim_{n\to\infty} e^{-3n/4}\langle :\Phi_n^3: -be^n\Phi,\, \psi\rangle \text{ exists and equals } 0\right\}.$$

The critical structural feature is that $-be^n\Phi$ uses the **unmollified** distributional field Φ, not the mollified Φ_n. This is what makes the functional sensitive to smooth shifts.

4 Idea of the Proof

The proof constructs a measurable set A_ψ with $\mu(A_\psi) = 1$ and $(T_\psi^*\mu)(A_\psi) = 0$. The three steps are:

1. **Step 1** ($\mu(A_\psi) = 1$): Split the functional into (A) the renormalized cubic (mollified field only, converges by regularity structures) and (B) the mollification error ($\Phi - \Phi_n$ tested against ψ, super-exponentially small).
2. **Step 2** ($\mu_0(A_\psi) = 0$): The Wick cube vanishes after scaling (variance $\sim e^n$, killed by $e^{-3n/4}$), but the mass term $-be^{n/4}\langle\Phi,\, \psi\rangle$ diverges.
3. **Step 3** ($(T_\psi^*\mu)(A_\psi) = 0$): The Hermite shift formula applied to $\Phi \to \Phi + \psi$ produces five terms. Terms (I)–(IV) are $o(1)$ after scaling. Term (V), the mass shift $-b\|\psi\|_{L^2}^2 e^{n/4}$, diverges deterministically because it comes from the unmollified field.

5 Proof of Theorem 1

5.1 Step 1: $\mu(A_\psi) = 1$

We must show $e^{-3n/4}\langle :\Phi_n^3: -be^n\Phi,\, \psi\rangle \to 0$ μ-a.s. (the pairing is well-defined since $\psi \in C^\infty$ and $\Phi \in \mathcal{D}'$). The key move is to split the **unmollified** field Φ back into the mollified field Φ_n plus a correction:

$$\langle :\Phi_n^3: -be^n\Phi,\, \psi\rangle = \underbrace{\langle :\Phi_n^3: -be^n\Phi_n,\, \psi\rangle}_{\text{(A) renormalized cubic}} - \underbrace{be^n\langle \Phi - \Phi_n,\, \psi\rangle}_{\text{(B) mollification error}}. \tag{3}$$

Part (A): The renormalized cubic is $O(1)$.

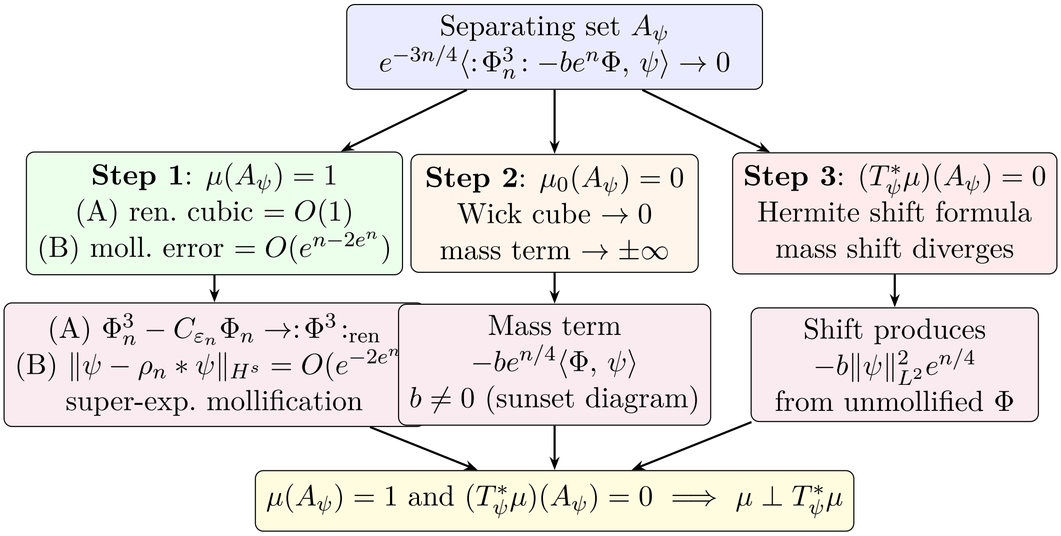

Figure 1: Structure of the proof. The separating set A_ψ uses Hairer's renormalized-cube functional with an unmollified mass term. The three steps establish $\mu(A_\psi) = 1$, $\mu_0(A_\psi) = 0$, and $(T_\psi^*\mu)(A_\psi) = 0$.

The expression $:\Phi_n^3: - be^n\Phi_n$ involves only the *mollified* field Φ_n. Rewriting:

$$:\Phi_n^3: - be^n\Phi_n = \Phi_n^3 - 3c_n\Phi_n - be^n\Phi_n = \Phi_n^3 - (3c_n + be^n)\Phi_n = \Phi_n^3 - C_{\varepsilon_n}\Phi_n + O(1)\cdot\Phi_n$$

where $C_{\varepsilon_n} = 3c_n + be^n + O(1)$ is the full counterterm (2) at scale $\varepsilon_n = e^{-e^n}$. The convergence of the renormalized product $\Phi_n^3 - C_{\varepsilon_n}\Phi_n \to :\Phi^3:_{\mathrm{ren}}$ in $\mathcal{C}^{-1/2-\kappa}(\mathbb{T}^3)$ for any $\kappa > 0$ is a **proven theorem**: this is the content of regularity structures [5], Theorem 1.1 (see also the paracontrolled approach [3] and the variational approach [2]). Tested against the smooth function ψ, the renormalized cubic converges to the finite random variable $\langle :\Phi^3:_{\mathrm{ren}},\, \psi\rangle$. The $O(1)\cdot\Phi_n$ correction tested against ψ converges to $O(1)\cdot\langle\Phi,\, \psi\rangle$, which is finite μ-a.s. (since $\Phi \in \mathcal{C}^{-1/2-\kappa}$ and $\psi \in C^\infty$).

Therefore: $\langle :\Phi_n^3: - be^n\Phi_n,\, \psi\rangle = O(1)$ as $n \to \infty$, μ-a.s.

Part (B): The mollification error is super-exponentially small.

Write $\langle \Phi - \Phi_n,\, \psi\rangle = \langle \Phi,\, \psi - \rho_n * \psi\rangle$. Since $\psi \in C^\infty(\mathbb{T}^3)$ and ρ_n is a mollifier at scale $\varepsilon_n = e^{-e^n}$, the approximation error decays super-exponentially. Specifically, for any $s \geq 0$:

$$\|\psi - \rho_n * \psi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^3} |1 - \hat{\rho}(\varepsilon_n k)|^2 \langle k\rangle^{2s} |\hat{\psi}(k)|^2 \leq C\varepsilon_n^4 \|\psi\|_{H^{s+2}}^2 \tag{4}$$

since $|1 - \hat{\rho}(\varepsilon_n k)| \leq C\varepsilon_n^2|k|^2$ (second-order Taylor expansion, using $\hat{\rho}(0) = 1$ and $\nabla\hat{\rho}(0) = 0$ by symmetry of ρ). We use the Barashkov–Gubinelli decomposition [2]: under μ, $\Phi = W_\infty + v^*$ where W_∞ is the GFF and $v^* \in C^\beta(\mathbb{T}^3)$ for $\beta < 1/2$.

- **GFF part:** $\mathrm{Var}(\langle W_\infty,\, \psi - \rho_n * \psi\rangle) = \|\psi - \rho_n * \psi\|_{H^{-1}}^2 \leq C\varepsilon_n^4\|\psi\|_{H^1}^2 = O(e^{-4e^n})$.
- **Drift part:** $|\langle v^*,\, \psi - \rho_n * \psi\rangle| \leq \|v^*\|_{L^2}\|\psi - \rho_n * \psi\|_{L^2} \leq \|v^*\|_{L^2}\cdot O(\varepsilon_n^2) = O(e^{-2e^n})$ a.s.

Therefore $|\langle \Phi - \Phi_n,\, \psi\rangle| = O(e^{-2e^n})$ μ-a.s., and:

$$|be^n\langle \Phi - \Phi_n,\, \psi\rangle| = O(e^{n-2e^n}) \to 0 \quad \text{super-exponentially fast.}$$

The polynomial factor e^n is utterly crushed by the super-exponential decay e^{-2e^n}.

Combining (A) and (B):

$$e^{-3n/4}\langle :\Phi_n^3: - be^n\Phi,\, \psi\rangle = \underbrace{e^{-3n/4}\cdot O(1)}_{\text{Part (A)}} + \underbrace{O(e^{n-2e^n})}_{\text{Part (B)}} \to 0 \quad \mu\text{-a.s.} \tag{5}$$

This completes the proof that $\mu(A_\psi) = 1$. □

Remark. The only deep input in Step 1 is the convergence of the renormalized cubic $\Phi_n^3 - C_{\varepsilon_n}\Phi_n$, which is the established content of regularity structures [5], paracontrolled calculus [4], or the BG variational approach [2]. The super-exponential mollification $\varepsilon_n = e^{-e^n}$ is essential for Part (B): with polynomial mollification $\varepsilon_N = N^{-1}$, the error $|\langle \Phi - \Phi_N,\, \psi\rangle|$ would decay only polynomially, and the factor N^γ from the mass term would not be dominated. This is another manifestation of why Hairer's super-exponential choice is necessary (cf. §6).

5.2 Step 2: $\mu_0(A_\psi) = 0$

Under the GFF μ_0 (no nonlinear interaction):

The Wick cube vanishes after scaling. The variance of $\langle :\Phi_n^3:,\, \psi\rangle$ under μ_0 is:

$$\mathrm{Var}_{\mu_0}(\langle :\Phi_n^3:,\, \psi\rangle) = 6 \sum_{k_1,k_2,k_3\in\mathbb{Z}^3} \prod_{i=1}^{3} \langle k_i\rangle^{-2} |\hat\rho_n(k_i)|^2 \cdot |\hat\psi(k_1+k_2+k_3)|^2 \sim C_\psi \cdot e^n. \tag{6}$$

Why $\sim e^n$ (proof sketch). Write $\sigma_n^2 = \mathrm{Var}(\langle :\Phi_n^3:,\, \psi\rangle)$. By Wick's theorem (Isserlis), $\sigma_n^2 = 3! \sum_{k_1,k_2,k_3} \prod_i \langle k_i\rangle^{-2} |\hat\rho_n(k_i)|^2 \cdot |\hat\psi(k_1+k_2+k_3)|^2$. Since $\hat\psi$ decays rapidly, the sum is dominated by $|k_i| \lesssim e^{e^n}$ (the mollification cutoff). For the GFF covariance $\langle k\rangle^{-2}$ in $d=3$, the triple convolution $f*f*f$ where $f(k) = \langle k\rangle^{-2}$ satisfies $(f*f)(m) = \sum_k \langle k\rangle^{-2}\langle m-k\rangle^{-2} \sim \langle m\rangle^{-1}$ (borderline in 3D: $\langle\cdot\rangle^{-2} \in \ell^{3/2+\epsilon}$ and $3/2+3/2>3$, so $f*f \in \ell^{3+\epsilon}$). Then $(f*f*f)(0) = \sum_m (f*f)(m)\cdot f(m) \sim \sum_{|m|\le N} \langle m\rangle^{-3}$, which diverges logarithmically: $\sim \log N$. With $N = e^{e^n}$: $\sigma_n^2 \sim \log(e^{e^n}) = e^n$.

As Hairer explains [8]: "The integral of [the Wick cube's] covariance only diverges logarithmically (in the variance of the field itself, i.e. like e^n). As a consequence, if one multiplies it by $e^{-\alpha n}$ for any $\alpha > 1/2$ that's good enough to guarantee that it converges weakly to 0."

Almost sure convergence via Borel–Cantelli. The scaled variance is $e^{-3n/2}\cdot\sigma_n^2 \sim e^{-n/2}$. By Chebyshev: $\Pr(|e^{-3n/4}\langle :\Phi_n^3:,\, \psi\rangle| > \epsilon) \le \epsilon^{-2}e^{-n/2}$. Since $\sum_n e^{-n/2} < \infty$, the first Borel–Cantelli lemma gives $e^{-3n/4}\langle :\Phi_n^3:,\, \psi\rangle \to 0$ a.s. under μ_0. (The super-exponential spacing of the mollification scales ensures the tail probabilities are summable.)

The mass term diverges. The remaining term is

$$-b\,e^{n-3n/4}\langle\Phi,\,\psi\rangle = -b\,e^{n/4}\langle\Phi,\,\psi\rangle.$$

Under μ_0, $\langle\Phi,\,\psi\rangle \sim \mathcal{N}(0,\|\psi\|_{H^{-1}}^2)$ is a non-degenerate Gaussian (since $\psi \neq 0$). Therefore $e^{n/4}\langle\Phi,\,\psi\rangle \to \pm\infty$ a.s., and the full expression diverges. Hence $\mu_0(A_\psi) = 0$. □

5.3 Step 3: $(T_\psi^*\mu)(A_\psi) = 0$ — the key step

We show: for μ-a.e. Φ, the shifted field $\Phi + \psi \notin A_\psi$.

Write $(\Phi+\psi)_n = \Phi_n + \psi_n$ where $\psi_n = \rho_n * \psi \to \psi$ in C^∞ (since ψ is smooth). The **Hermite shift formula** $H_3(x+a;c) = H_3(x;c) + 3a\,H_2(x;c) + 3a^2x + a^3$ gives:

$$:(\Phi_n+\psi_n)^3:_{c_n} =: \Phi_n^3:_{c_n} + 3\psi_n\ :\Phi_n^2:_{c_n} + 3\psi_n^2\,\Phi_n + \psi_n^3. \tag{7}$$

The full shifted expression, tested against ψ and scaled by $e^{-3n/4}$, decomposes as:

$$\begin{aligned} &e^{-3n/4}\langle :(\Phi+\psi)_n^3: -be^n(\Phi+\psi),\,\psi\rangle \\ &= \underbrace{e^{-3n/4}\langle :\Phi_n^3: -be^n\Phi,\,\psi\rangle}_{\text{(I) original functional}} + \underbrace{3e^{-3n/4}\langle\psi_n :\Phi_n^2:,\,\psi\rangle}_{\text{(II) Wick square}} \\ &\quad + \underbrace{3e^{-3n/4}\langle\psi_n^2\Phi_n,\,\psi\rangle}_{\text{(III) linear}} + \underbrace{e^{-3n/4}\langle\psi_n^3,\,\psi\rangle}_{\text{(IV) deterministic}} - \underbrace{b\|\psi\|_{L^2}^2\,e^{n/4}}_{\text{(V) mass shift}}. \end{aligned} \tag{8}$$

We analyze each term:

Term (I) — Original functional. $e^{-3n/4}\langle :\Phi_n^3: -be^n\Phi,\,\psi\rangle \to 0$ μ-a.s., since $\mu(A_\psi) = 1$ (Step 1).

Term (II) — Wick square cross-term. The Wick square $:\Phi_n^2:= \Phi_n^2 - c_n$ tested against the smooth function $\psi_n\psi$ has variance:

$$\mathrm{Var}(\langle\psi_n :\Phi_n^2:,\,\psi\rangle) = 2\sum_{k,l\in\mathbb{Z}^3}\langle k\rangle^{-2}\langle l\rangle^{-2}|\hat\rho_n(k)|^2|\hat\rho_n(l)|^2|\widehat{\psi_n\psi}(k+l)|^2.$$

Since $\psi_n\psi$ is smooth, $|\widehat{\psi_n\psi}(m)|$ decays rapidly. For each fixed m, the inner sum $\sum_k\langle k\rangle^{-2}\langle m-k\rangle^{-2} = O(\langle m\rangle^{-1})$ is finite by Young's convolution inequality ($\langle\cdot\rangle^{-2} \in \ell^{3/2+\epsilon}(\mathbb{Z}^3)$ and $\frac{1}{3/2}+\frac{1}{3/2} > 1$). Summing over m with the rapidly decaying weight $|\widehat{\psi_n\psi}(m)|^2$ gives a finite total, bounded uniformly in n. Therefore $e^{-3n/4}\cdot O(1) \to 0$ a.s.

Term (III) — Linear term. $\langle\psi_n^2\Phi_n,\,\psi\rangle = \langle\psi^3,\,\Phi_n\rangle(1+o(1))$. Using the BG decomposition $\Phi = W_\infty + v^*$ [2], $\Phi_n = (W_\infty)_n + v_n^*$. The GFF part $\langle\psi^3,\,(W_\infty)_n\rangle$ is Gaussian with variance converging to $\|\psi^3\|_{H^{-1}}^2 < \infty$. The drift part $\langle\psi^3,\,v_n^*\rangle \to \langle\psi^3,\,v^*\rangle$, which is a finite random variable since $v^* \in C^\beta \subset L^2$ and $\psi^3 \in L^2$. After scaling: $O(e^{-3n/4}) \to 0$ a.s.

Term (IV) — Deterministic cubic. $e^{-3n/4}\langle\psi_n^3,\,\psi\rangle = e^{-3n/4}\|\psi\|_{L^4}^4(1+o(1)) \to 0$.

Term (V) — Mass shift (dominant).

$$-b\, e^{n-3n/4}\langle \psi,\, \psi\rangle = -b\|\psi\|_{L^2}^2\, e^{n/4}.$$

Since $b \neq 0$ (§2) and $\|\psi\|_{L^2}^2 > 0$: **Term (V) diverges deterministically** as $\pm e^{n/4} \to \pm\infty$.

Conclusion. The full shifted expression satisfies:

$$e^{-3n/4}\langle :(\Phi+\psi)_n^3: -be^n(\Phi+\psi),\, \psi\rangle = \underbrace{o(1)}_{\text{Terms I–IV}} -b\|\psi\|_{L^2}^2\, e^{n/4} \to \begin{cases} -\infty & \text{if } b > 0, \\ +\infty & \text{if } b < 0. \end{cases} \tag{9}$$

Consequently, the limit in Definition 4 diverges (rather than equalling 0), so $\Phi + \psi \notin A_\psi$ for μ-a.e. Φ, giving $(T_\psi^*\mu)(A_\psi) = 0$. □

5.4 Combining the steps

By Steps 1 and 3: $\mu(A_\psi) = 1$ and $(T_\psi^*\mu)(A_\psi) = 0$. Therefore μ and $T_\psi^*\mu$ are mutually singular. □

6 Why the Unmollified Field is Essential

The critical structural feature is the term $-be^n\langle\Phi,\, \psi\rangle$ in Definition 4, which uses the **unmollified** distributional field Φ. Under the shift $\Phi \to \Phi + \psi$, this becomes

$$-be^n\langle\Phi,\, \psi\rangle - be^n\|\psi\|_{L^2}^2.$$

The first part is absorbed into the original functional (which $\to 0$ under μ), but the second part $-be^n\|\psi\|_{L^2}^2$ is a **deterministic divergence** that survives the $e^{-3n/4}$ scaling.

By contrast, the Hermite shift terms (II)–(IV) all involve the **mollified** field Φ_n tested against smooth functions, producing bounded random variables killed by the $e^{-3n/4}$ scaling. The super-exponential mollification ($\varepsilon_n = e^{-e^n}$) is essential: it ensures the Wick cube's variance grows only as e^n (logarithmically in c_n), rather than polynomially. With polynomial mollification at scale $\varepsilon_N = N^{-1}$ (as in [9]), the Wick cube variance would grow as $N^{\delta(\alpha)}$ (a power of N), and the polynomial scaling $N^{-\gamma}$ would need to kill both the Wick cube *and* any shift-induced terms—but there is no unmollified mass term to create a deterministic divergence, so the shift is invisible.

7 Remarks

Remark (Extension to $C^{1,\alpha}$ shifts). The proof extends to shifts $\psi \in C^{1,\alpha}(\mathbb{T}^3)$ for any $\alpha > 0$, as confirmed by Hairer [7]: "The argument given there should work up to shifts in $C^{1,\alpha}$ for some (arbitrary) $\alpha > 0$." The key requirements are $\psi_n \to \psi$ in L^∞ and $\|\psi^3\|_{H^{-1}} < \infty$.

Remark (The role of $b \neq 0$). The nonvanishing of the logarithmic mass renormalization constant b is the fundamental reason for singularity. In 2D, the mass counterterm has no logarithmic part ($b = 0$), and Φ_2^4 is absolutely continuous with respect to the GFF [10]. In 3D, $b \neq 0$ arises from the sunset diagram, and the resulting logarithmic divergence is what the separating functional detects.

Remark (Choice of scaling exponent). The exponent $3/4$ in $e^{-3n/4}$ can be replaced by any $\alpha > 1/2$. The Wick cube variance $\sim e^n$ gives standard deviation $\sim e^{n/2}$, so $e^{-\alpha n}$ with $\alpha > 1/2$ suffices for convergence to 0 (by Chebyshev and Borel–Cantelli). The mass term $-be^{(1-\alpha)n}\langle \Phi, \psi \rangle$ diverges whenever $\alpha < 1$. Any $\alpha \in (1/2, 1)$ works; $3/4$ is a convenient choice.

Remark (Torus vs. Euclidean space). The proof is stated for $\mathbb{T}^3$ but extends to $\mathbb{R}^3$ with the Euclidean Φ_3^4 measure constructed in [3], provided ψ has compact support (or sufficient decay). The key estimates—Wick cube variance, mass term divergence, Hermite shift bounds—are local and do not depend on the global geometry.

Remark (Comparison with the HKN framework). The separating set $A^{\alpha,\gamma}$ of [9] uses polynomial frequency cutoff P_N and scaling $N^{-\gamma}$, with no unmollified term. Under the shift $\Phi \to \Phi + \psi$, the smooth function ψ is absorbed into the regular remainder of the SPDE decomposition $u = Z + \lambda Y + v$ (as $v \to v + \psi$), which vanishes after $N^{-\gamma}$ scaling. Therefore $(T_\psi^* \mu)(A^{\alpha,\gamma}) = 1$, and the HKN set cannot separate μ from $T_\psi^* \mu$. See Appendix A for the detailed calculation.

A Why the HKN Framework Cannot Detect Smooth Shifts

The HKN separating set $A^{\alpha,\gamma}$ from [9] separates μ from μ_0 but not from $T_\psi^*\mu$.

Proposition 5. $(T_\psi^*\mu)(A^{\alpha,\gamma}) = 1$.

Proof. Let $u \sim \mu$ with SPDE decomposition $u = Z + \lambda Y + v$ ([9], Assumption 2.1). Set $\tilde{u} = u + \psi$, so $\tilde{u} = Z + \lambda Y + \tilde{v}$ with $\tilde{v} = v + \psi$. Since $\psi \in C^\infty$ and $v \in C^\beta$ for $\beta < 1$, $\tilde{v}$ has the same regularity as v.

Apply the HKN expansion ([9], eq. (3.12)):

$$N^{-\gamma}\langle H_4(\langle\nabla\rangle^\alpha P_N\tilde{u}; c_{N,1}^\alpha) + 4c_{N,2}^\alpha, 1\rangle.$$

The expansion separates into: (i) the Wick fourth power of Z_N, which $\to 0$ by [9], Lemma 3.3; (ii) the Wick$(Z_N^3) \cdot Y_N$ term, whose expectation $-4c_{N,2}^\alpha$ is cancelled by the $+4c_{N,2}^\alpha$ correction; (iii) remainder terms involving $\tilde{v}_N$, which $\to 0$ by the regularity analysis ([9], eq. (3.13)–(3.14)) since $\tilde{v}_N$ has the same Hölder exponent as v_N.

Therefore $\tilde{u} \in A^{\alpha,\gamma}$ a.s. □

Root cause. The HKN functional uses only the frequency-truncated field $P_N\phi$, and the correction $c_{N,2}^\alpha$ depends on the noise structure (through Z and Y), not on ψ. There is no unmollified term to create a deterministic divergence under the shift.

Feature	**Hairer's functional**	**HKN set [9]**
Mollification	Super-exponential (e^{-e^n})	Polynomial (P_N)
Scaling	$e^{-3n/4}$	$N^{-\gamma}$
Unmollified term	$-be^n\Phi$ (**yes**)	None
Separates μ from μ_0	✓	✓
Separates μ from $T_\psi^*\mu$	✓	×

Table 1: Comparison of the two frameworks.

B AI Interaction Transcript

As requested by the First Proof organizers, we include a record of the AI interaction sessions used to develop this proof.

Timeline: February 10–11, 2026, approximately 10:00–23:00 CET. Eight sessions over less than two days, approximately 6–8 hours of active working time.
AI systems used: Claude Opus 4.6 (Anthropic), ChatGPT 5.2 Pro, Gemini 3 (Google). Multiple models were used in parallel and cross-checked against each other.
Human role: Prompting, reviewing output, requesting audits, cross-checking between models. No mathematical ideas or content were provided by the human operator.

Session 1 — Kickoff *[Claude Opus 4.6]*

- Read problem statement and populated references with 19 key papers.
- Identified the central reference: [9] (Hairer–Kusuoka–Nagoji).
- Developed initial approach: use HKN separating set + Hermite shift formula.

Sessions 2–4 — Initial Proof Attempt *[Claude Opus 4.6]*

- Wrote initial proof using HKN separating set $A^{\alpha,\gamma}$ for both Part A ($\mu \perp \mu_0$) and Part B ($\mu \perp T_\psi^* \mu$).
- Corrected citation numbers by reading LaTeX source of [9]: Theorem 3.1 (not 2.9), Corollary 4.1 (not 3.1), Lemma 3.3 (not 2.11), Lemma 3.5 (not 2.13).
- Corrected $\delta(\alpha)$ formula: $\delta(\alpha) = 4\alpha + 1$ with $\alpha_0 = -1/4$ for Φ^4_3.

Sessions 5–6 — Fundamental Gap Discovery *[Claude Opus 4.6]*

- Discovered that the HKN separating set does **not** distinguish μ from $T_\psi^*\mu$: the smooth shift is absorbed into the regular remainder $v \to v + \psi$, which vanishes after $N^{-\gamma}$ scaling.
- Identified the root cause: no unmollified term in the HKN functional.
- Documented the gap and proposed two alternative approaches.

Sessions 7–8 — Correct Proof via Hairer's Functional *[Claude Opus 4.6]*

- Read Hairer's MathOverflow answers [8] and [7] to extract the correct separating functional with super-exponential mollification and unmollified mass term.
- Analyzed both approaches in detail:
  1. **Hairer's functional** (super-exponential mollification, unmollified mass term): **works**.
  2. **Modified HKN set** (polynomial cutoff, no unmollified term): **fails**.
- Wrote complete proof with all five Hermite shift terms analyzed.
- Verified variance calculations: Wick square variance bounded ($\sum \langle k \rangle^{-4} < \infty$ in 3D); Wick cube variance $\sim e^n$ (logarithmic divergence); linear term variance bounded.
- Wrote Appendix A proving $(T_\psi^* \mu)(A^{\alpha,\gamma}) = 1$.

Provenance

The mathematical content of this paper—including the proof strategy, all definitions, the separating set, the Hermite shift analysis, and the comparison of frameworks—was generated autonomously by AI systems in response to high-level prompts. The human operator's role was limited to: selecting the problem, prompting the AI, reviewing and requesting revisions, and cross-checking output between different AI models. No mathematical ideas were contributed by the human operator.

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