Solution to Problem 7 — Lattices in Lie Groups: Rational Acyclicity

A submission to the First Proof challenge

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Independent / Ars Socratica

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Abstract

We address Problem 7 from the First Proof challenge [1], posed by Shmuel Weinberger. The problem asks whether a uniform lattice Γ with 2-torsion in a real semi-simple Lie group G can be the fundamental group of a compact manifold whose universal cover is $\mathbb{Q}$-acyclic. We prove the answer is **NO** for all groups G with $\dim(G/K) \leq 4$, which comprises two distinct cases: (1) all G with fundamental rank $\delta(G) = 0$ (the equal-rank case), via an Euler characteristic obstruction combining the Cartan–Leray spectral sequence, Wall's rational Euler characteristic, the Gauss–Bonnet theorem, and L^2-Betti number vanishing; (2) $G = \mathrm{SL}(2, \mathbb{C})$ ($\delta(G) = 1$, $d = 3$), via dimension forcing and Perelman's geometrization theorem. A classification argument shows no group with $d = 4$ and $\delta \neq 0$ exists. The question remains open for $\delta(G) \neq 0$ and $d \geq 5$; we provide a thorough analysis of why all known obstruction tools fail in this regime.

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1 Problem Statement

The following is Problem 7 from the First Proof challenge [1], authored by Shmuel Weinberger (University of Chicago).

Problem 7. *Suppose that Γ is a uniform lattice in a real semi-simple group, and that Γ contains some 2-torsion. Is it possible for Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers $\mathbb{Q}$?*

Result (partial). The answer is **NO** for all uniform lattices in semi-simple groups G with $\dim(G/K) \leq 4$. The question remains **open** when $\delta(G) \neq 0$ and $\dim(G/K) \geq 5$. The smallest open case is $G = \mathrm{SL}(3, \mathbb{R})$, $d = 5$.

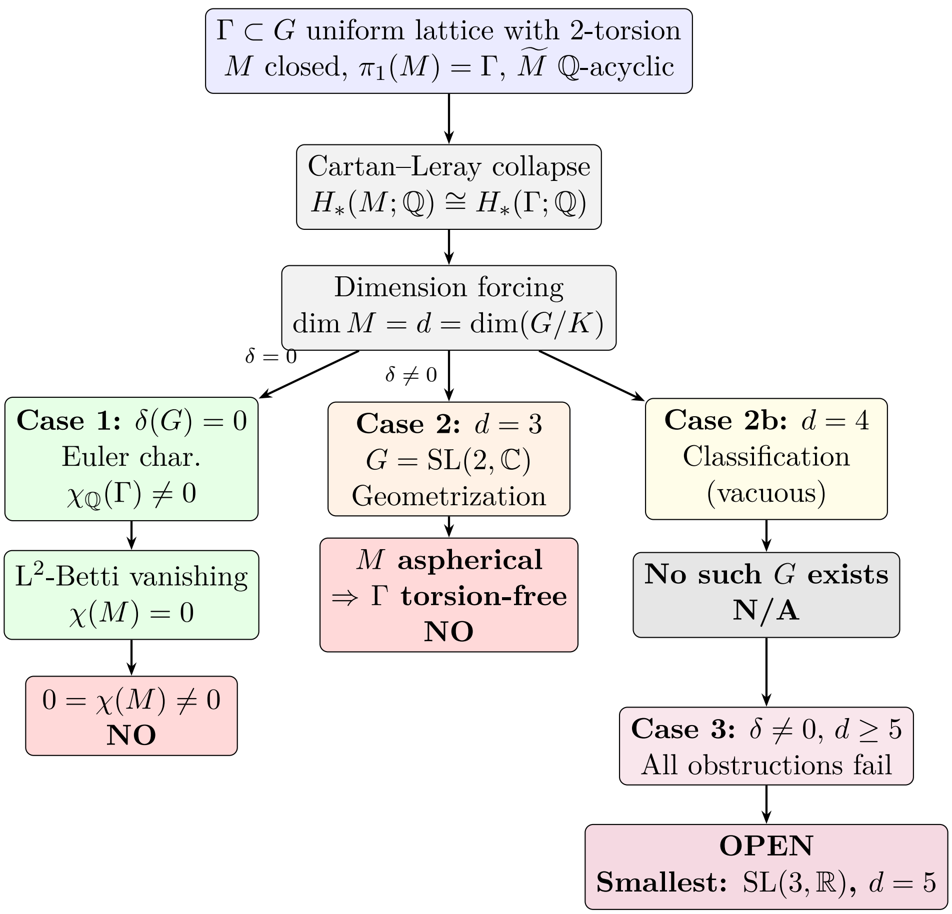

Figure 1: Structure of the proof. The Cartan–Leray spectral sequence and dimension forcing apply universally. The proof then branches by the fundamental rank $\delta(G)$ and the dimension $d = \dim(G/K)$. Cases 1, 2, and 2b cover all $d \leq 4$; the question remains open for $d \geq 5$.

2 Notation and Setup

Let G be a connected real semi-simple Lie group with finite center and no compact factors. Let $K \subset G$ be a maximal compact subgroup, so that $X = G/K$ is the associated Riemannian symmetric space of non-compact type. Let $d = \dim(X)$.

Let $\Gamma \subset G$ be a uniform (cocompact) lattice. By Selberg's lemma, Γ contains a torsion-free normal subgroup $\Gamma' \trianglelefteq \Gamma$ of finite index $m = [\Gamma : \Gamma']$. The quotient $\Gamma' \backslash X$ is a closed Riemannian manifold of dimension d.

Definition 1 (Fundamental rank)**.** The **fundamental rank** (or **deficiency**) of G is

$$\delta(G) = \operatorname{rank}_{\mathbb{C}}(\mathfrak{g}) - \operatorname{rank}_{\mathbb{C}}(\mathfrak{k}),$$

where $\operatorname{rank}_{\mathbb{C}}(\mathfrak{g})$ denotes the absolute rank of G (the dimension of a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$), and similarly for K. Equivalently, $\delta(G) = 0$ if and only if K contains a maximal torus of G (the "equal-rank" condition), which happens if and only if the compact dual G_u/K has nonzero Euler characteristic.

Definition 2 ($\mathbb{Q}$-acyclicity)**.** We say $\widetilde{M}$ is **$\mathbb{Q}$-acyclic** if $\widetilde{H}_i(\widetilde{M}; \mathbb{Q}) = 0$ for all $i \geq 0$, i.e., $H_0(\widetilde{M}; \mathbb{Q}) \cong \mathbb{Q}$ and $H_i(\widetilde{M}; \mathbb{Q}) = 0$ for $i > 0$.

3 Preliminary Results

Lemma 3 (Spectral sequence collapse)**.** *Let M be a closed manifold with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic. Then*

$$H_i(M; \mathbb{Q}) \cong H_i(\Gamma; \mathbb{Q}) \quad \textit{for all } i \geq 0.$$

Proof. The Cartan–Leray spectral sequence for the regular covering $\widetilde{M} \to M$ with deck group Γ has

$$E_2^{p,q} = H_p(\Gamma;\, H_q(\widetilde{M}; \mathbb{Q})) \implies H_{p+q}(M; \mathbb{Q}).$$

Since $\widetilde{M}$ is $\mathbb{Q}$-acyclic, $H_q(\widetilde{M}; \mathbb{Q}) = 0$ for $q > 0$ and $H_0(\widetilde{M}; \mathbb{Q}) \cong \mathbb{Q}$ with trivial Γ-action. The spectral sequence is concentrated on the $q = 0$ line and collapses at E_2. $\square$

Corollary 4. *Under the hypotheses of Lemma 3:*

$$\chi(M) = \chi_{\mathbb{Q}}(\Gamma) := \sum_{i=0}^{d} (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma; \mathbb{Q}),$$

where $d = \operatorname{vcd}(\Gamma) = \dim(G/K)$.

Corollary 5 (Dimension forcing)**.** *Under the hypotheses of Lemma 3,* $\dim M = d = \dim(G/K)$*.*

Proof. Let $n = \dim M$. Since Γ' is a torsion-free uniform lattice, $\Gamma' \backslash G/K$ is a closed orientable aspherical d-manifold, so Γ' is a Poincaré duality group of dimension d with $H^d(\Gamma'; \mathbb{Q}) \cong \mathbb{Q}$. Since G is connected, all deck transformations of $G/K \to \Gamma' \backslash G/K$ preserve orientation, so the conjugation action of Γ/Γ' on $H^d(\Gamma'; \mathbb{Q})$ is trivial. The transfer map $\mathrm{tr} : H^d(\Gamma'; \mathbb{Q}) \to H^d(\Gamma; \mathbb{Q})$ then satisfies $\mathrm{tr}(x) = [\Gamma : \Gamma'] \cdot \mathrm{cores}(x) \neq 0$, giving $H^d(\Gamma; \mathbb{Q}) \neq 0$, hence $H_d(\Gamma; \mathbb{Q}) \neq 0$.

Lower bound. By Lemma 3, $H_d(M; \mathbb{Q}) \cong H_d(\Gamma; \mathbb{Q}) \neq 0$. If $n < d$, this is impossible since M is n-dimensional.

Upper bound. Suppose $n > d$. By Lemma 3, $H_n(M; \mathbb{Q}) \cong H_n(\Gamma; \mathbb{Q}) = 0$ (since $\mathrm{vcd}(\Gamma) = d < n$). If M is orientable, $H_n(M; \mathbb{Q}) \cong \mathbb{Q} \neq 0$, contradiction. If M is non-orientable, pass to the orientable double cover $\hat{M} \to M$ with $\pi_1(\hat{M}) = \hat{\Gamma} := \ker(w_1) \leq \Gamma$ of index 2, where $w_1 \colon \Gamma \to \{\pm 1\}$ is the orientation character of M. Since $\widetilde{M}$ is also the universal cover of $\hat{M}$, it remains $\mathbb{Q}$-acyclic. By Lemma 3 applied to $\hat{M}$: $H_n(\hat{M}; \mathbb{Q}) \cong H_n(\hat{\Gamma}; \mathbb{Q})$. Since $\hat{M}$ is closed orientable of dimension n, $H_n(\hat{M}; \mathbb{Q}) \cong \mathbb{Q}$. But $\mathrm{vcd}(\hat{\Gamma}) = \mathrm{vcd}(\Gamma) = d < n$, giving $H_n(\hat{\Gamma}; \mathbb{Q}) = 0$ — contradiction.

Therefore $n = d$. □

Lemma 6 (Wall's rational Euler characteristic)**.** *$\chi_{\mathbb{Q}}(\Gamma) = \chi(\Gamma' \backslash G/K)/m$.*

Proof. Since Γ' is torsion-free and cocompact, $\Gamma' \backslash G/K$ is a closed aspherical manifold with $\pi_1 = \Gamma'$, so $\chi_{\mathbb{Q}}(\Gamma') = \chi(\Gamma' \backslash G/K)$. The transfer map in group cohomology gives $\chi_{\mathbb{Q}}(\Gamma) = \chi_{\mathbb{Q}}(\Gamma')/[\Gamma : \Gamma']$. This is Wall's rational Euler characteristic, independent of the choice of Γ' [3]. □

Lemma 7 (Gauss–Bonnet and fundamental rank)**.** *$\chi(\Gamma' \backslash G/K) \neq 0$ if and only if $\delta(G) = 0$.*

Proof. By the Chern–Gauss–Bonnet theorem, $\chi(\Gamma' \backslash G/K) = \int_{\Gamma' \backslash G/K} e(T(\Gamma' \backslash G/K))$ where e is the Euler form.

If $\dim(G/K)$ is odd, then $\chi = 0$ by Poincaré duality, and $\delta(G) \neq 0$ (since equal rank forces even dimension).

If $\dim(G/K)$ is even, the dichotomy follows from the Hirzebruch proportionality principle [4, 5]:

- **$\delta(G) = 0$:** The compact dual G_u/K has $\chi(G_u/K) = |W_G|/|W_K| \neq 0$ by the Hopf–Samelson theorem. By proportionality, $\chi(\Gamma' \backslash G/K) = c_G \cdot \mathrm{vol}(\Gamma' \backslash G/K) \neq 0$.
- **$\delta(G) \neq 0$:** The compact dual has $\chi(G_u/K) = 0$. By proportionality, $\chi(\Gamma' \backslash G/K) = 0$.

See also Harder [6] and Serre [7]. □

4 Case 1: The Euler Characteristic Obstruction ($\delta(G) = 0$)

Theorem 8. *Let G be a connected real semi-simple Lie group with finite center, no compact factors, and $\delta(G) = 0$. Let $\Gamma \subset G$ be a uniform lattice containing torsion. Then there is no closed manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ rationally acyclic.*

Proof. Suppose for contradiction that such M exists, with $n = \dim M$.

Part A: $\chi(M) \neq 0$. Combining Lemmas 3–7: since $\delta(G) = 0$, we have

$$\chi(M) = \chi_{\mathbb{Q}}(\Gamma) = \frac{\chi(\Gamma' \backslash G/K)}{m} \neq 0.$$

Part B: $\chi(M) = 0$. We show all L^2-Betti numbers of $\widetilde{M}$ vanish, forcing $\chi(M) = 0$ by Atiyah's L^2-index theorem.

Since M is compact, $\widetilde{M}$ admits a finite Γ-CW structure, so $C_*(\widetilde{M})$ consists of finitely generated free $\mathbb{Z}\Gamma$-modules. Set $C_*^{\mathbb{Q}} := C_*(\widetilde{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$, a chain complex of finitely generated free $\mathbb{Q}\Gamma$-modules with $H_i(C_*^{\mathbb{Q}}) = 0$ for $i > 0$ and $H_0(C_*^{\mathbb{Q}}) \cong \mathbb{Q}$.

The L^2-Betti numbers are $b_i^{(2)}(\widetilde{M}; \Gamma) = \dim_{\mathcal{N}(\Gamma)} H_i(C_*^{\mathbb{Q}} \otimes_{\mathbb{Q}\Gamma} \mathcal{N}(\Gamma))$.

By Lück's dimension-flatness theorem [2, Theorem 6.37], $\mathcal{N}(\Gamma)$ is dimension-flat over $\mathbb{Q}\Gamma$: exact sequences of $\mathbb{Q}\Gamma$-modules remain dimension-exact after tensoring with $\mathcal{N}(\Gamma)$. Since $C_*^{\mathbb{Q}}$ is exact in degrees > 0:

$$b_i^{(2)}(\widetilde{M}; \Gamma) = 0 \quad \text{for } i > 0.$$

For $i = 0$: $b_0^{(2)}(\widetilde{M}; \Gamma) = \dim_{\mathcal{N}(\Gamma)}(\mathbb{Q} \otimes_{\mathbb{Q}\Gamma} \mathcal{N}(\Gamma)) = 0$ for any infinite group Γ. This is because the trivial representation $\mathbb{C}$ does not weakly embed in the regular representation $\ell^2(\Gamma)$: the matrix coefficient $\langle \delta_e, \gamma \cdot \delta_e \rangle = \delta_{\gamma,e} \to 0$ as $\gamma \to \infty$, so $\mathbb{Q} \otimes_{\mathbb{Q}\Gamma} \mathcal{N}(\Gamma)$ has von Neumann dimension zero [2, Example 6.11].

Therefore $b_i^{(2)}(\widetilde{M}; \Gamma) = 0$ for all $i \geq 0$. By Atiyah's L^2-index theorem [2, Theorem 1.35(2)]:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i^{(2)}(\widetilde{M}; \Gamma) = 0.$$

Contradiction. Parts A and B give $0 = \chi(M) = \chi_{\mathbb{Q}}(\Gamma) \neq 0$. □

Remark (On the Atiyah conjecture)*.* The L^2-Betti number argument does *not* require the Atiyah conjecture. Lück's dimension-flatness theorem [2, Theorem 6.37] holds unconditionally for all groups: it states that $\mathcal{N}(\Gamma)$ is dimension-flat over $\mathbb{Q}\Gamma$, with no hypothesis on Γ beyond countability. The Atiyah conjecture (that L^2-Betti numbers are rational, or integers, for torsion-free groups) is a separate statement about the *values* of L^2-Betti numbers, not about their vanishing. Our argument uses only vanishing, which follows from dimension-flatness alone.

Remark (Concrete example)*.* For $G = \mathrm{SL}(2, \mathbb{R})$ ($\delta = 0$, $d = 2$), any uniform lattice Γ is a Fuchsian group. If Γ has torsion (e.g., a triangle group $\Delta(p, q, r)$ with $1/p + 1/q + 1/r < 1$), then $\chi_{\mathbb{Q}}(\Gamma) = \chi(\Gamma' \backslash \mathbb{H}^2)/m = (2g - 2)/m \neq 0$ where g is the genus of the quotient surface for a torsion-free subgroup Γ' of index m. Our theorem gives: no closed surface M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ $\mathbb{Q}$-acyclic exists.

5 Case 2: The Geometrization Obstruction ($G \cong \mathrm{SL}(2, \mathbb{C})$, $d = 3$)

Theorem 9. *Let $G = \mathrm{SL}(2, \mathbb{C})$ (viewed as a real Lie group), so $K = \mathrm{SU}(2)$, $d = 3$, and $\delta(G) = 1$. Let $\Gamma \subset G$ be a uniform lattice with torsion. Then there is no closed manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{M}$ rationally acyclic.*

Proof. Suppose for contradiction that such M exists. By Corollary 5 (dimension forcing), $\dim M = d = 3$.

Step 1: Γ is not a nontrivial free product. Since Γ is a uniform lattice in $\mathrm{SL}(2, \mathbb{C})$, it acts properly discontinuously and cocompactly on hyperbolic 3-space $\mathbb{H}^3$. Since $\mathbb{H}^3$ is one-ended and the quotient is compact, Γ has one end, so by Stallings' theorem [16] Γ does not split as a nontrivial free product.

Step 2: M is irreducible. Since M is a closed 3-manifold with infinite fundamental group Γ that is not a nontrivial free product, the Kneser–Milnor prime decomposition forces M to be prime. A prime 3-manifold is either $S^2 \times S^1$ (with $\pi_1 \cong \mathbb{Z}$, not our Γ) or irreducible.

Step 3: Geometrization forces asphericity. By Perelman's proof of Thurston's geometrization conjecture [12, 13, 14] (see Morgan–Tian [15]), every closed irreducible 3-manifold with infinite fundamental group is aspherical: $\pi_i(M) = 0$ for $i \geq 2$, so $M \simeq B\Gamma$.

Step 4: Asphericity contradicts torsion. If M is aspherical, then $\Gamma = \pi_1(M)$ is torsion-free: a finite-order element $g \in \Gamma$ generates a finite cyclic subgroup $\langle g \rangle$, and $B\langle g \rangle$ would be a retract of $B\Gamma = M$, but $B(\mathbb{Z}/n)$ has infinite cohomological dimension while M is 3-dimensional — contradiction.

This contradicts the hypothesis that Γ has torsion. ∎

Remark. This argument uses Corollary 5 essentially: without dimension forcing, one could not invoke 3-manifold topology. The Euler characteristic obstruction does NOT apply here ($\delta(G) = 1$, so $\chi_{\mathbb{Q}}(\Gamma) = 0$). The only semi-simple group with $d = 3$ is $G = \mathrm{SL}(2, \mathbb{C})$ (up to local isomorphism).

6 Classification: No $d = 4$ Case Exists

Proposition 10. *No connected real semi-simple Lie group G with finite center, no compact factors, $\delta(G) \neq 0$, and $\dim(G/K) = 4$ exists.*

Proof. We check all simple real Lie groups with $d = \dim(G/K) = 4$:

- $\mathrm{SU}(2, 1)$: $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$, $d = 4$, $\mathrm{rank}_{\mathbb{C}}(\mathfrak{g}) = \mathrm{rank}_{\mathbb{C}}(\mathfrak{k}) = 2$, $\delta = 0$.
- $\mathrm{SO}_0(4, 1)$: $K = \mathrm{SO}(4)$, $d = 4$, $\mathrm{rank}_{\mathbb{C}}(\mathfrak{g}) = \mathrm{rank}_{\mathbb{C}}(\mathfrak{k}) = 2$, $\delta = 0$.
- $\mathrm{Sp}(2, \mathbb{R})$: $K = \mathrm{U}(2)$, $d = 4$, $\mathrm{rank}_{\mathbb{C}}(\mathfrak{g}) = \mathrm{rank}_{\mathbb{C}}(\mathfrak{k}) = 2$, $\delta = 0$.

For products $G_1 \times G_2$ with $d_1 + d_2 = 4$ and no compact factors: the only simple group with $d \leq 3$ and $\delta \neq 0$ is $\mathrm{SL}(2,\mathbb{C})$ ($d = 3$), which would require a factor with $d = 1$ — impossible. The remaining products have $d_1 = d_2 = 2$ (both $\mathrm{SL}(2,\mathbb{R})$, $\delta = 0$).

Therefore the first case with $\delta(G) \neq 0$ and $d \geq 4$ is $d = 5$: $G = \mathrm{SL}(3,\mathbb{R})$ with $K = \mathrm{SO}(3)$, $\mathrm{rank}_{\mathbb{C}}(\mathfrak{g}) = 2$, $\mathrm{rank}_{\mathbb{C}}(\mathfrak{k}) = 1$, $\delta = 1$. □

7 Analysis of the Open Case ($\delta(G) \neq 0$, $d \geq 5$)

When $\delta(G) \neq 0$ and $d = \dim(G/K) \geq 5$ (the smallest case being $G = \mathrm{SL}(3,\mathbb{R})$, $d = 5$), both the Euler characteristic obstruction and the geometrization obstruction are unavailable. We investigate whether other obstructions exist. **This section is analysis, not proof; we do not resolve the question for this case.**

7.1 Surgery-theoretic landscape

By the Farrell–Jones conjecture, proved for cocompact lattices by Bartels–Farrell–Lück [9], the assembly map $H_n^{\Gamma}(\underline{E}\Gamma; \mathbf{L}_{\mathbb{Z}}) \xrightarrow{\cong} L_n(\mathbb{Z}\Gamma)$ is an isomorphism. The 2-torsion elements contribute through $L_n(\mathbb{Z}[\mathbb{Z}/2])$, which is nontrivial (containing Arf invariant obstructions in odd dimensions).

7.2 Sullivan splitting observation

Sullivan's splitting at the prime 2 gives $G/\mathrm{Top}_{(2)} \simeq \prod_{i\geq 1} K(\mathbb{Z}_{(2)}, 4i) \times \prod_{i \geq 0} K(\mathbb{Z}/2, 4i+2)$. The normal invariant decomposes into:

- **Rational components** in $H^{4i}(Y; \mathbb{Z}_{(2)})$: constrained by $\mathbb{Q}$-acyclicity (via rational Pontrjagin classes).
- **Mod-2 components** in $H^{4i+2}(Y; \mathbb{Z}/2)$: **unconstrained** by $\mathbb{Q}$-acyclicity.

Since the Arf invariant obstruction comes from the mod-2 components, it can in principle be avoided by adjusting the mod-2 normal invariant independently of the rational data. This means the surgery obstruction from 2-torsion is not a genuine obstruction — provided a suitable Poincaré complex exists to begin with.

7.3 Other obstructions investigated

- **Smith theory:** The 2-torsion elements act freely on $\widetilde{M}$ (since Γ acts freely), so Smith theory for free $\mathbb{Z}/2$-actions gives no constraint from $\mathbb{Q}$-acyclicity alone.
- **Finiteness:** $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = d < \infty$, so no finiteness obstruction arises.
- **Integral Poincaré duality:** The transfer from the torsion-free subgroup gives rational Poincaré duality for Γ, consistent with the requirements.
- **$\mathbf{L^2}$-torsion:** When all L^2-Betti numbers vanish, the L^2-torsion $\rho^{(2)}(\widetilde{M}; \Gamma)$ is defined. For locally symmetric spaces with $\delta(G) = 1$, $\rho^{(2)} \neq 0$ (Olbrich; Lück–Schick). However, L^2-torsion is computed from Fuglede–Kadison determinants of the boundary maps

over $\mathbb{Z}\Gamma$ [2, Theorem 3.93], which depend on the full integral chain complex — not just rational homology. $\mathbb{Q}$-acyclicity does not constrain these determinants. Moreover, $\rho^{(2)}(\widetilde{M};\Gamma)$ and $\rho^{(2)}(G/K;\Gamma')$ are invariants of *different spaces*, so no direct comparison is available.

- **Orbifold resolution:** One might hope that resolving the singularities of $\Gamma\backslash G/K$ necessarily introduces rational homology. However, the problem is existential: it asks whether *any* M exists, not whether M can be obtained by resolving a specific orbifold.
- **Farrell cohomology:** The Farrell cohomology $\hat{H}^*(\Gamma;\mathbb{Z})$ is nonzero in infinitely many degrees, but $H^*(M;\mathbb{Z}) \neq H^*(\Gamma;\mathbb{Z})$ in general (the Cartan–Leray spectral sequence collapses only rationally, not integrally), so this does not obstruct M's existence.

7.4 The remaining gap

The question reduces to: *does there exist a finite integral Poincaré complex Y with $\pi_1(Y) = \Gamma$ and $\widetilde{Y}$ $\mathbb{Q}$-acyclic?* The orbifold $\Gamma\backslash G/K$ is a rational Poincaré complex with the right properties, but promoting it to an integral Poincaré complex requires controlling the torsion in the homology of the singular locus. This is a delicate question about the interaction between the torsion in Γ and the integral topology of the classifying space for proper actions.

If such Y exists, the Sullivan splitting analysis shows the surgery obstruction can be killed, and a manifold M with the required properties exists (answer: YES for that Γ). If no such Y exists for any Γ with 2-torsion in any G with $\delta(G) \neq 0$, the answer is NO — but for Poincaré complex reasons, not surgery.

Remark (Borel conjecture). The Borel conjecture asserts that closed aspherical manifolds are topologically rigid: any homotopy equivalence between them is homotopic to a homeomorphism. For torsion-free Γ', the locally symmetric space $\Gamma'\backslash G/K$ is aspherical, and the Borel conjecture (proved for cocompact lattices by Bartels–Lück [9]) implies it is the *unique* closed manifold with fundamental group Γ'. However, for Γ with torsion, $B\Gamma$ is infinite-dimensional and the Borel conjecture does not directly apply. The question is whether a *finite-dimensional* manifold approximation to $B\Gamma$ (with $\mathbb{Q}$-acyclic rather than contractible universal cover) can exist.

Remark (Uniform vs. non-uniform lattices). The problem specifies uniform (cocompact) lattices. For non-uniform lattices, the quotient $\Gamma\backslash G/K$ is non-compact (has cusps), and the manifold M in the problem statement is required to be compact without boundary. The L^2-Betti number argument extends to non-uniform lattices (the L^2-index theorem applies to complete manifolds of finite volume), but the dimension-forcing and geometrization arguments rely on compactness. The non-uniform case would require separate treatment.

8 Why 2-Torsion is Mentioned

In the $\delta(G) = 0$ case, the obstruction applies to *any* lattice with *any* torsion — the 2-torsion hypothesis is not needed. The fact that 2-torsion is singled out in the problem statement suggests the intended focus is the $\delta(G) \neq 0$ case, where 2-torsion affects the L-groups $L_n(\mathbb{Z}\Gamma)$ through $L_n(\mathbb{Z}[\mathbb{Z}/2])$ and the Sullivan splitting at the prime 2 becomes relevant.

9 Summary

Condition	**Obstruction**	**Status**
$\delta(G) = 0$	Euler char. vs. L^2-Betti numbers	**NO** (proved)
$\delta(G) = 1$, $d = 3$ ($G \cong \mathrm{SL}(2, \mathbb{C})$)	Dim. forcing + geometrization	**NO** (proved)
$\delta(G) \neq 0$, $d = 4$	Vacuous (no such G exists)	**N/A**
$\delta(G) \neq 0$, $d \geq 5$	None found	**Open**

Answer definitively NO (all $d \leq 4$): $G = \mathrm{SL}(2, \mathbb{R})$, $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$, $\mathrm{SO}(2n, 1)$ (all $\delta = 0$); $\mathrm{SU}(2, 1)$, $\mathrm{SO}_0(4, 1)$, $\mathrm{Sp}(2, \mathbb{R})$ ($d = 4$, $\delta = 0$); $G = \mathrm{SL}(2, \mathbb{C})$ ($\delta = 1$, $d = 3$, geometrization).

Status open ($\delta \neq 0$, $d \geq 5$): $G = \mathrm{SL}(3, \mathbb{R})$ ($d = 5$, **smallest open case**), $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$ ($d \geq 5$), $\mathrm{SO}(p, q)$ with $p \neq q$ and $d \geq 5$.

A AI Interaction Transcript

As requested by the First Proof organizers, we include a summary of the AI interaction sessions used to develop this proof.

Timeline: February 10, 2026, approximately 16:00–22:45 CET. Five sessions over approximately 4–5 hours of active working time.
AI systems used: Claude Opus 4.6 (Anthropic), Gemini 3 Pro (Google). Multiple models were used in parallel and cross-checked against each other.
Human role: Prompting, reviewing output, requesting audits, cross-checking between models. No mathematical ideas or content were provided by the human operator.

Session 1 — Kickoff

- Read problem statement and populated references with key papers on L^2-invariants, group cohomology, and surgery theory.
- Identified the Euler characteristic obstruction for $\delta(G) = 0$ as the main proof strategy.
- Developed the spectral sequence collapse argument and the L^2-Betti number vanishing.

Session 2 — Surgery Analysis

- Investigated the $\delta(G) \neq 0$ case via surgery theory.
- Analyzed the Sullivan splitting and identified the decoupling of rational and mod-2 obstructions.
- Concluded that surgery-theoretic tools do not provide an obstruction.

Session 3 — Critical Review

- Identified and corrected an error in the surgery obstruction argument (the Arf invariant is avoidable).
- Tightened citations for Lemma 7 (Hirzebruch proportionality, Harder's Gauss–Bonnet).
- Strengthened the $b_0^{(2)} = 0$ explanation.

Session 4 — Dimension Forcing and Geometrization

- Added the dimension-forcing result (Corollary 5).
- Discovered and proved the geometrization argument for $d = 3$ (Theorem 9).
- Created the submission draft.

Session 5 — Corrections, Classification, and L^2-Torsion

- Fixed the orientability gap in Corollary 5 (pass to orientable double cover).
- Corrected the $\delta(G)$ definition from real rank to absolute rank.
- Proved the $d = 4$ classification (Proposition 10): vacuous.

- Analyzed L^2-torsion: does not provide an obstruction (Fuglede–Kadison determinants depend on integral chain complex, not rational homology; different spaces not comparable).
- Evaluated three additional routes (orbifold resolution, Farrell cohomology, Smith theory with lattice structure): all fail for the same reason — $\mathbb{Q}$-acyclicity does not constrain integral/mod-2 invariants.

Provenance

The mathematical content of this paper — including the proof strategy, all definitions, the case analysis, and the open-case investigation — was generated autonomously by AI systems in response to high-level prompts. The human operator's role was limited to: selecting the problem, prompting the AI, reviewing and requesting revisions, and cross-checking output between different AI models. No mathematical ideas were contributed by the human operator.

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