Exercises - Numerical Analysis - 2016-2017
Section AMMA

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Session 1 - Non-Linear Equations

Exercise 1
We want to compute the zero α of the function $f(x) = x^3 - 2$ with the fixed-point method $x^{(k+1)} = \phi(x^{(k)})$ given as :

$$x^{(k+1)} = x^{(k)} \left(1 - \frac{\omega}{3}\right) + (x^{(k)})^3(1-\omega) + \frac{2\omega}{3(x^{(k)})^2} + 2(\omega - 1), \quad k \geq 0,$$

$\omega \in \mathbb{R}$ is a Real parameter.

a) For which values of the parameter ω the root of the function f is a fixed point ?

b) For which ω the method is at least of order 2 ?

c) Is there a value for ω such that the order of the fixed point is greater than 2 ?

Exercise 2
Let α be a double root of f, i.e. $f(\alpha) = f'(\alpha) = 0$.

a) If we can write the function f as

$$f(x) = (x-\alpha)^2 h(x) \qquad \text{where} \quad h(\alpha) \neq 0\ ,$$

verify that the Newton method for the approximation of the root α is only of order 1. [Hint : write the method as a fix-point method and compute $\Phi'(\alpha)$]

b) Consider the modified Newton method as it follows :

$$x^{(k+1)} = x^{(k)} - 2\frac{f(x^{(k)})}{f'(x^{(k)})}.$$

Verify the this method is at least of order 2 if we want to compute α.

Exercise 3

We want to compute the solution of equation $x/2 - \sin(x) + \pi/6 - \sqrt{3}/2 = 0$ in $[-\pi/2, \pi]$. Let $f(x) = x/2 - \sin(x) + \pi/6 - \sqrt{3}/2$; from Fig. 1, we can identify two roots $\alpha_1 \in I_1 = [-\pi/2, 0]$ and $\alpha_2 \in I_2 = [\pi/2, \pi]$:

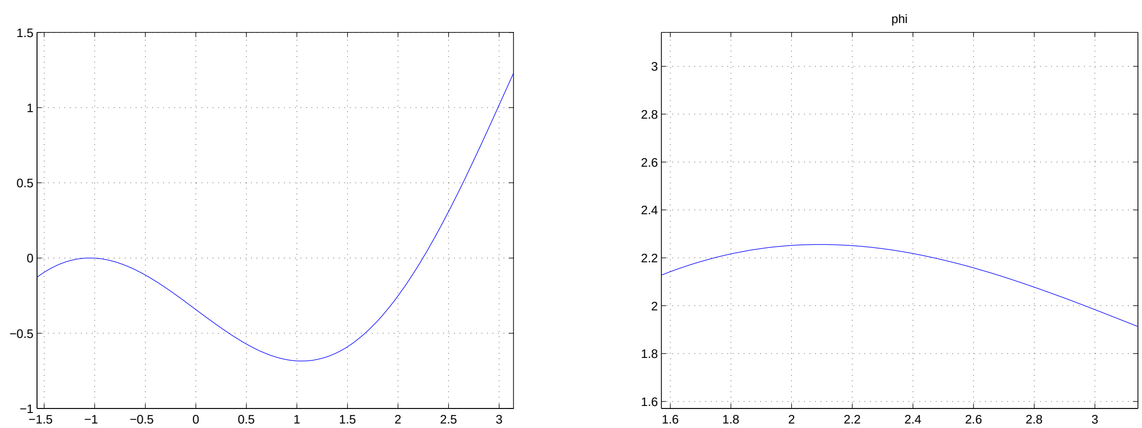
a) is it possible to use the bisection method for computing the two roots ? Why ? If it is possible, estimate the minimal number of iterations required to approximate the root(s) in I_1, I_2 with a tolerance of 10^{-10} ;

b) write the Newton method for the problem $f(x) = 0$. With the help of its graphs, find if there is any root for which the method converges with order 2 ;

c) let us use now the fixed-point method $x^{(k+1)} = \phi(x^{(k)})$ to compute $\alpha_2 \in I_2$ with

$$\phi(x^{(k)}) = \sin(x^{(k)}) + \frac{x^{(k)}}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) .$$

Check that α_2 is a fixed-point of $\phi(x)$ and establish if the method is
- *locally convergent*, i.e. the method converges to α_2 if $x^{(0)}$ is close to α_2 ;
- *globally convergent in* I_2, i.e. the method converges for each $x^{(0)} \in I_2$. (*hint : take a look to the graph of* $\phi(x)$ *in* I_2*, Fig. 1*) ;



(a) (b)

FIGURE 1 – Graphs of $f(x)$ (on the left) and $\phi(x)$ (on the right).

d) let us consider again α_2 and the fixed-point method of point c). Show that there exist a positive contant $0 < C < 1$ such that

$$|x^{(k+1)} - \alpha_2| \leq C|x^{(k)} - \alpha_2| , \tag{1}$$

and compute its value ;

e) let us consider the iterations $x^{(k)}$ of the fixed-point method of point c), initialized with $x^{(0)} = \pi/2$. Demonstrate that starting from (1) we can write

$$|x^{(k)} - \alpha_2| \leq C^k |x^{(0)} - \alpha_2|.$$

Then, use this result to find the minimal number of iterations such that the error $|x^{(k)} - \alpha_2|$ is smaller than 2^{-20}.