OSDA HW 1

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№1

Determine the properties of the relation:

$Q = \{(m, n) \mid m, n \in N \text{ and } m = n^2\}$

Solution:

1) Reflexive (def.: If $\forall a \in A$ aRa):

In our case for relation Q it is possible only if $m = 1$, because $m = m^2$ $1 = 1^2$, but it is not possible for other posible values of m, so our relation Q is not reflexive

2) Antireflexive (def: If $\forall a \in A$ $\neg(aRa)$ $(\Leftrightarrow aR^c a)$

We have an element $(1, 1)$, which is in our relation: $1Q1$, so that means that Q is not antireflexive

3) Symmetric (def. If $\forall a, b \in A$ $aRb \Rightarrow bRa$)

Our Q is not symmetric because not for all $m, n \in N$ $m = n^2 \Rightarrow n = m^2$

for exampe $(m,n) = (4,2)$; $4 = 2^2 \not\Rightarrow 2 \neq 4^2$

so our Q is not symmetric

4) Asymmetric (def. If $\forall a, b \in A$ $aRb \Rightarrow \neg(bRa)$ $(\Leftrightarrow bR^c a)$)

let's look at pair $(1,1)$, here we have $\begin{matrix} m=1 \\ n=1 \end{matrix}$; $m = n^2$ and $n = m^2$, and

this pair exist in our relation Q, so Q is not asymmetric

5) Antisymmetric (def. If $\forall a, b \in A$ aRb and $bRa \Rightarrow a = b$)

If we have pair (m,n), so that both equations are true:

$m = n^2$ and $n = m^2$ and $n, m \in N \Rightarrow$

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$m = n = 1$, so our relation
Q is antisymmetric

6) Transitive (def. If $\forall a, b, c \in A$
aRb and $bRc \Rightarrow aRc$)
Lets look at pairs: $(81; 9)$ and $(9; 3)$
$81 Q 9$ and $9 Q 3$ but $81 \neq 3^2$ so $81 Q 3$
doesn't exist.
Q is not transitive

7) Complete of linear (def: If $\forall a, b \in A$
$a \neq b \Rightarrow aRb \vee bRa$)
lets look at $m = 4$ and $n = 3$, so
$4, 3 \in N$ and $4 \neq 3$ but $4Q3 \vee 3Q4 = False$
Q is not complete of linear

Answer: Q is antisymmetric.

№2

The degree of a vertex of an undirected graph is the number of edges incident to the vertex. Prove that in an arbitrary graph the number of vertices with odd degree is even.

Proof:

We know that the sum of all degrees in arbitrary graph is equal to twice the number of edges.

G - arbitrary graph

E - number of edges in G

V - is the set of vertexes $(v_1, v_2, \dots, v_n)$

So it will be always true that:

$$\sum_{i=1}^{n} \text{degree}(v_i) = 2 \cdot E$$

Let's suppose, that the number

of vertices with odd degree is odd and equals to r, so r - odd.
and lets name the number of vertices with even degree is d.

$$\sum_{i=1}^{n} \text{degree}(v_i) = 2E = r + d$$

, so from our assumption that r is odd and $r+d = 2E$ - even $\Rightarrow$ d should be odd too. But d can't be odd by definition. So we have a controversy, that is mean that our initial assumption about r is not correct $\Rightarrow$ r is even in other words the number of vertices with odd degree is even. $\blacksquare$

№3

Prove that the incomparability relation for a strict order is a tolerance relation.

Proof.

Def: Tolerance is a reflexive and symmetric binary relation.

Stric order is antireflexive and transitive binary relation.

Incomparability relation:

$$I_R = A \times A \setminus (R \cup R^d) = (R \cup R^d)^c = R^c \cap R^{cd}$$

So by definitions above we should prove reflexivity and symmetricity of our binary relation:

1) Reflexivity (def: If $\forall a \in A: aRa$) for any element a in the set, a is not comparable to itself because of strict order. Therefore, a is

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incomparable to itself, and so the incomparability relation is reflexive.

2) Symmetricity (def: If $\forall a, b \in A$ $aRb \Rightarrow bRa$)

By definition of Incomparability I:
for $\forall m, n$: $mIn \Rightarrow ((\neg(mRn)) \wedge (\neg(nRm)))$

$\neg(mRn) \wedge \neg(nRm) = \neg(nRm) \wedge \neg(mRn) =$
$nIm \Rightarrow I$ - is symmetric.

Overall I proved that a incomparability relation for a strict order is reflexive and symmetric $\Rightarrow$ is a tolerance relation. $\blacksquare$

№4

Let us consider a binary relation on the set of 5 elements. Count how many different binary relations satisfy the following

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set of properties:

(a) Assymmetric and transitive.

(B) Antisymmetric and antireflexive.

a) To count the number of assymetric binary relations we should remember that assymetric means $\forall a, b \in A$

$aRb \Rightarrow \neg(bRa)$

Overall there are 2^{n^2} possible binary relations for a set of n elements.

Because 2 means two choices between 0 and 1, and a possible number of pairs is n^2. So in our task $n = 5$.

Overall number of binary relations $= 2^{25}$.

We should exclude pairs such as (x, x)

Overall we have n^2-pairs after excluding, we have $n^2 - n$ pairs

To satisfy the property of assymmetric relation, we have 3 possibilities of either to include only of type (x,y) or only type (y,x) or none of them. So the total number of possible assymmetric relations will be $= 3^{\frac{n^2-n}{2}}$ because we have $\frac{n^2-n}{2}$ different pairs and for each pair we have 3 different choices. In our case this number will equal to $3^{\frac{25-5}{2}} = 3^{10} = 59049$

Unfortunately I can't count a number of possible transitive binary relations (there is no derived formula for counting it). So I wrote a python snippet which is calculating the number of relations which are transitive and assymmetric code:

```python
import itertools
import numpy as np

def generate_binary_relation_matrices(num_elements):
    matrices = []
    num_matrices = 2**(num_elements**2)

    for i in range(num_matrices):
        matrix = np.zeros((num_elements, num_elements), dtype=int)
        for j in range(num_elements**2):
            row = j // num_elements
            col = j % num_elements
            matrix[row, col] = (i >> j) & 1
        matrices.append(matrix)

    return matrices

def is_asymmetric(matrix):
    return np.all(matrix + matrix.T <= 1)

def is_transitive(matrix):
    for i in range(num_elements):
        for j in range(num_elements):
            for k in range(num_elements):
                if matrix[i, j] == 1 and matrix[j, k] == 1:
                    if matrix[i, k] != 1:
                        return False
    return True

def count_transitive_asymmetric_relations(num_elements):
    matrices = generate_binary_relation_matrices(num_elements)
    count = 0

    for matrix in matrices:
        if is_asymmetric(matrix):
            if is_transitive(matrix):
                count += 1

    return count

num_elements = 5
count = count_transitive_asymmetric_relations(num_elements)
print(f"Number of transitive and asymmetric relations on a set of {num_elements} elements: {count}")
```

```
Number of transitive and asymmetric relations on a set of 5 elements: 4231
```

As we can see the number is 4231

B) Antisymmetric and antireflexive.

To be shure that relation is antireflexive we shouldn't include pairs (x,x). After that we have n^2-n possible pairs. $\frac{n^2-n}{2}$ there are number of pairs, which has a pairs (x, y) and (y, x).

To be sure that our relation is antisymmetric from each 2 symmetric pairs (x,y); (y,x) we should include just one. It's possible to include nothing instead one of them.

So, total we will have:

$3^{\frac{n^2-n}{2}}$ - variants.

It means, that number of antisymmetric and antireflexive relations

$= 3^{\frac{25-5}{2}} = 3^{10} = 59049$

python code

```python
import itertools

def generate_binary_relation_matrices(num_elements):
    matrices = []
    num_matrices = 2**(num_elements**2)

    for i in range(num_matrices):
        matrix = [[0] * num_elements for _ in range(num_elements)]
        for j in range(num_elements**2):
            row = j // num_elements
            col = j % num_elements
            matrix[row][col] = (i >> j) & 1
        matrices.append(matrix)

    return matrices

def is_antisymmetric(matrix):
    for i in range(len(matrix)):
        for j in range(len(matrix[0])):
            if i != j and matrix[i][j] == 1 and matrix[j][i] == 1:
                return False
    return True

def is_antireflexive(matrix):
    for i in range(len(matrix)):
        if matrix[i][i] == 1:
            return False
    return True


def count_antisymmetric_antireflexive_relations(num_elements):
    matrices = generate_binary_relation_matrices(num_elements)
    count = 0

    for matrix in matrices:
        if is_antisymmetric(matrix):
            if is_antireflexive(matrix):
                count += 1

    return count

num_elements = 5
count = count_antisymmetric_antireflexive_relations(num_elements)
print(f"Number of antisymmetric and antireflexive relations on a set of {num_elements} elements: {count}")
```

```
Number of antisymmetric and antireflexive relations on a set of 5 elements: 59049
```

Answer:

a) 4231

b) 59049

№5

Let us consider a Binary relation on $\mathbb{Z}^2$,

$$R: (x_1, y_1) R (x_2, y_2) \longleftrightarrow x_1 \le x_2;\ y_1 \le y_2$$

1) Prove that it is a partial order.

2) Find the minimal and maximal elements if R is defined on the following sets:

a) $A_1 = \{ (x,y) \mid x \le 3, y \le 4, x > 0, y > 0 \}$

b) $A_2 = \{ (x,y) \mid x^2 + y^2 \le 4 \}$

1) Proof:

Def. Partial order is a reflexive, transitive, and antisymmetric Binary relation.

For every pair $(x,y) \in \mathbb{Z}^2$ it's true that $\begin{matrix} x \le x \\ y \le y \end{matrix} \Rightarrow (x,y) R (x,y) \Rightarrow$

R is reflexive.

Suppose we have 3 pairs:

(x_1, y_1); (x_2, y_2); (x_3, y_3) and we know that:

$(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3) \Rightarrow$

$$\begin{matrix} x_1 \leq x_2 \\ y_1 \leq y_2 \end{matrix} \quad \text{and} \quad \begin{matrix} x_2 \leq x_3 \\ y_2 \leq y_3 \end{matrix}$$

because of $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$

$$\begin{cases} x_1 \leq x_2 \\ y_1 \leq y_2 \end{cases} \begin{cases} x_2 \leq x_3 \\ y_2 \leq y_3 \end{cases} \Rightarrow \begin{cases} x_1 \leq x_3 \\ y_1 \leq y_3 \end{cases} \Rightarrow (x_1, y_1) R (x_3, y_3)$$

it means that R is transitive.

Suppose we have 2 pairs:

(x_1, y_1); $(x_2, y_2) \in \mathbb{Z}^2$ and

$(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_1, y_1) \Rightarrow$

$$\begin{cases} x_1 \le x_2 \\ y_1 \le y_2 \end{cases} \Rightarrow \begin{matrix} x_1 = x_2 \\ y_1 = y_2 \end{matrix} \Rightarrow (x_1, y_1) = (x_2, y_2)$$
$$\begin{cases} x_2 \le x_1 \\ y_2 \le y_1 \end{cases}$$

R is antisymmetric.
Overall our R is reflexive, antisymmetric and transitive $\Rightarrow$ R is partial order. ■

2) a) $A_1 = \{(x,y) \mid x \le 3, y \le 4, x \ge 0, y \ge 0\}$
minimal element:
Because of R is a partial order and for every pair $\in \begin{Bmatrix} x \le 3 & x \ge 0 \\ y \le 4 & y \ge 0 \end{Bmatrix}$
the minimal possible values for x and y is 0 and 0 so pair (1;1) will be minimal because there is no pair $(x,y) R (0;0)$ where $\begin{matrix} x \le 0 \\ y \le 0 \end{matrix}$ and $\begin{matrix} x \in [0;3] \\ y \in [0;4] \end{matrix}$

there is no pair (x, y), where
$(3,4) R (x,y) \Rightarrow \begin{matrix} 3 \le x \\ 4 \le y \end{matrix}$ and $\begin{matrix} x \in [0;3] \\ y \in [0;4] \end{matrix}$

so pair $(3,4)$ is the maximum element.

Answer: $(0,0)$ – minimal element
$(3,4)$ – maximal element

б) $A_2 = \{ (x,y) \mid x^2 + y^2 \le 4 \}$

If we imagine $x^2+y^2 \le 4$ like a graph it will be a circle with radius $=2$ and center in point $(0,0)$. So it will be 13 possible pairs:

$[(0;0); (0;1); (0;2); (0;-1); (0;-2); (1;0); (2;0); (-1;0); (-2;0); (1;1); (1;-1); (-1;1); (-1;-1).] = A_2$

maximal element: $(0;2)$; $(2;0)$; $(1;1)$

Because all other elements from A_2 not greater in terms of relation R

minimal elements: $(0;-2); (-2;0); (-1;-1)$
Because all other elements from A_2 are not smaller than this in terms of relation R.

№6

I chose Lazy FCA

1) Dataset 1

https://www.kaggle.com/datasets/nanditapore/healthcare-diabetes

2) Dataset 2

https://www.kaggle.com/datasets/rashikrahmanpritom/heart-attack-analysis-prediction-dataset

3) Dataset 3

https://www.kaggle.com/datasets/adityakadiwal/water-potability