1. An integer $1 \leq p \leq |w|$ is called a period of a word w when $w[i] = w[i+p]$ for every $1 \leq i \leq |w| - p$. Prove that p is a period of w if and only if $|w| - p$ is a border of w.

2. Show that p being a period of a word w is equivalent to the following conditions:

   1. w is a subword of some x^k with $|x| = p$ and $k > 0$,
   2. w may be written as $(uv)^k u$ with $|uv| = p$, nonempty v and $k > 0$,
   3. for some x, y and z, $w = xy = yz$ with $|x| = |z| = p$.

3. Prove that if p and q are both periods of w such that $p + q \leq |w|$ then $\gcd(p, q)$ is also a period of w.

(2 points) 4. Prove that if p and q are both periods of w such that $p + q \leq |w| + \gcd(p, q)$ then $\gcd(p, q)$ is also a period of w.

5. Consider a modification of the failure function π known as the strong failure function π'. It is defined as follows: for each $i = 1, 2, \ldots, |w| - 1$ we choose $\pi'[i]$ to be the longest proper border of $w[1..i]$ such that $w[\pi'[i] + 1] \neq w[i + 1]$. If there is no such border, $\pi'[i] = -1$. Show how to (quickly) compute the values of π' given the values of π.

(2 points) 6. Show that, if $\pi'[i] \neq -1$ and $\pi'[\pi'[i]] \neq -1$, then $i \geq \pi'[i] + \pi'[\pi'[i]] + 2$. Now consider the following procedure: start with $j = i$ and then, as long as $\pi'[j] \neq -1$, set $j = \pi'[j]$. What is the maximum number of iterations of this process?

7. Explain how to evaluate $\sum_{k=1}^{n} a_i r^{n-k} \bmod q$ in $O(n)$ time.

8. Recall the definition of the fingerprint of a string $S[1..n]$:

$$\phi_r(S) = \sum_{k=1}^{|S|} S[k] r^{|S|-k} \bmod q.$$

   Explain how to compute $\phi_r(xy)$ from $\phi_r(x)$ and $\phi_r(y)$. Similarly, explain how to compute $\phi_r(x)$ from $\phi_r(xy)$ and $\phi_r(y)$. Extend the definition of a fingerprint so that such operations can be implemented in $O(1)$ time.