10.03.2022 **LSEF: Prior for local spectra** (1)

f_ℓ, $\ell = 0, 1, \dots, \ell_{max}$

Switch to log-spec as a fn of log-wn:

$$\ln f_\ell = (\ln f)(s = \ln(\ell+1)) \quad (1)$$

Subtract the <u>backgr</u> $\ln f_\ell$ (time mean or fcst):

$$\varphi(s) = \ln f_\ell - \ln f_\ell^{fc} \quad (2)$$

$s = 0 \div A \equiv s_{max} = \ln \ell_{max}$

$\varphi(s)$ can be modeled as a zero-mean <u>statio</u> rand proc. Its len sc (R) appears to be compara-ble w A. Its variance $\sigma_\varphi^2 = const(s)$.

As a minus log-prior, we take

$$\mathcal{L}^{prior} = \mathcal{L}^{prior}_{backg} + \mathcal{L}^{prior}_{smoo1} + \mathcal{L}^{prior}_{smoo2} \quad (3)$$

whr each of the 3 terms is the L^2 norm squared of $\varphi(s)$, $\varphi'(s)$, and $\varphi''(s)$, respectively:

$$\mathcal{L}_{prior}(\varphi) = \frac{1}{2}(\varphi, \varphi) + \frac{1}{2}(\varphi', \varphi') + \frac{1}{2}(\varphi'', \varphi'') \quad (4)$$

Since $\varphi(s)$ is statio, so are φ' & $\varphi''(s)$.

Let $R_{\varphi'} = R_\varphi / \sqrt{2}$, $R_{\varphi''} = R_\varphi / 2$ (5)

$R \equiv R_\varphi \approx A$, $R_{\varphi'} \ll A$, ~~$R_{\varphi''} \ll A$~~, $R_{\varphi''} \ll A$

Derive ($\cancel{1}$-4) as a special case of a general Gau minus log prior:

Dscr: $\boxed{\mathcal{L}(\varphi) = \frac{1}{2} \vec{\varphi}^T \hat{K}^{-1} \vec{\varphi}}$ (1)

drop $\frac{1}{2}$:

$\mathcal{L}(\varphi) = (\hat{K}^{-1}\vec{\varphi}, \vec{\varphi})$ (2)

Denote $\hat{Q} = \hat{K}^{-1}$ - precis mx

$\mathcal{L}(\varphi) = (\hat{Q}\vec{\varphi}, \vec{\varphi})$ (3)

Contin analogue: $\boxed{\mathcal{L}(\varphi) = (Q\varphi, \varphi)}$ (4)

Now, specify a particular form of Q:

Let $\boxed{Q = I - w_1 \mathcal{D}^2 + w_2 \mathcal{D}^4}$ (5)

diff oper $\mathcal{D} = \frac{\partial}{\partial s}$

Asst This form results in

$\mathcal{L}(\varphi) = (\varphi, \varphi) - w_1 (\mathcal{D}^2\varphi, \varphi) + w_2 (\mathcal{D}^4\varphi, \varphi)$ (6)

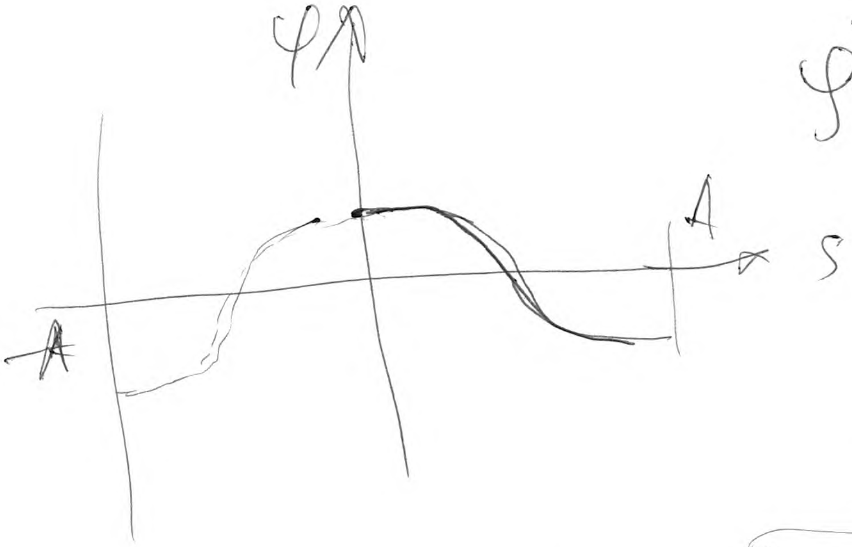
Since $\mathcal{D}^* = -\mathcal{D}$, $(\mathcal{D}^2)^* = (\mathcal{D}^2)$, we have

$\mathcal{L}(\varphi) = (\varphi, \varphi) + w_1 (\mathcal{D}\varphi, \mathcal{D}\varphi) + w_2 (\mathcal{D}^2\varphi, \mathcal{D}^2\varphi)$ (*)

This is (1-4).

Make $\varphi(s)$ periodic. To this end, extend it to $s \in [-A, 0]$ assuming that $\varphi'(0) = \varphi'(A) = 0$

$$\varphi^+(-s) := \varphi(s)$$



Consider Fourier transf of φ^+: $\tilde{\varphi}^+$ on the circle whose length $= 2A \Rightarrow R_s = \frac{2A}{2\pi} = \frac{A}{\pi}$ — radius (0)

(1)

$$\varphi(s) = \sum_{k=-\infty}^{\infty}{}' \, \tilde{\varphi}(k) e^{ik\frac{s}{R_s}} \qquad (2)$$

$$\| e^{ik\frac{s}{R}} \|^2 = \int \underbrace{| e^{ik\frac{s}{R_s}} |^2}_{=1} ds = 2A$$

$$\tilde{\varphi}(k) = \frac{1}{2A} \int \varphi(s) e^{-ik\frac{s}{R_s}} ds \qquad (3)$$

$$D\varphi = \sum \tilde{\varphi}(k) \cdot \frac{ik}{R_s} e^{ik\frac{s}{2}}$$

$$\Downarrow$$

$$\boxed{\hat{D}(k) = \frac{ik}{R_s}} \text{ — symbol } \cancel{\text{of}} \text{ of } D \qquad (4)$$

Now, using (3-4) & (2-5), we obtain (4

$$\boxed{\tilde{Q} = 1 + W_1 \frac{k^2}{R_s^2} + W_2 \frac{k^4}{R_s^4}} \qquad (*)$$

Here, it's meaningful to denote

$$\begin{cases} \dfrac{R_s^2}{W_1} = K_1^2 = \dfrac{1}{S_{[1]}^2} \iff W_1 = R_s^2 S_{[1]}^2 \equiv \dfrac{A^2}{\pi^2} S_{[1]}^2 & (1) \\ \dfrac{R_s^4}{W_2} = K_2^4 = \dfrac{1}{S_{[2]}^4} \iff W_2 = R_s^4 S_{[2]}^4 = \dfrac{A^4}{\pi^4} S_{[2]}^4 & (2) \end{cases}$$

whr $S_{[1]}$ & $S_{[2]}$ are some len scale[s]
$\in (0, A)$ $\in (0, A)$
in the s-coord scale.

To summarize,

$$\boxed{W_1 = \left(\frac{S_{max}}{\pi} \cdot S_{[1]}\right)^2} \qquad (3)$$

$$\boxed{W_2 = \left(\frac{S_{max}}{\pi} S_{[2]}\right)^4} \qquad (4)$$