13.03.2022 LSEF : Anls-err based spectral norm (1

Since an LSM r. fld is loc statio, fix x_0 & consider the statio fld on the global domain:

$$\boxed{\xi(x;x_0) = \sum_{l}^{l_{max}} \alpha_l \sigma_l(x_0) e^{ilx} \quad \text{on } S^1, \&} \quad (1)$$

$$\boxed{\xi(x;x_0) = \sum_{lm}^{l_{max}} \alpha_{lm} \sigma_l(x_0) Y_{lm}(x) \quad \text{on } S^2} \quad (2)$$

Consider, further, a regular obs network capable of observing the fld components w wns up to $\boxed{l_{obs}}$. Obs err are indep ⇒ white, with the constant modal spectrum (2)

$$S^1: \ Var\, \delta x^{obs} \equiv Var\, \eta = \sum_{-l_{obs}}^{l_{obs}} r = 2 l_{obs}\, r \quad (3)$$

$$S^2: \ Var\, \eta = \sum_{0}^{l_{obs}} \frac{2l+1}{4\pi} r \quad (4)$$

σ_{obs}^2

Turn to the spectral repres in the anls

Anls

$$x^a = x^f + K(x^o - Hx^f) = x^f + K(x^o - Hx + Hx - Hx^f) =$$

$$= x^f + K(\eta - H\xi)$$

$$\delta x^a = (I - KH)\xi + K\eta$$

$$K = BH^T(HBH^T + R)^{-1}$$

In spec space: all $\tilde{\xi}_l$ in (2

$$\xi(x;x_0) = \sum \tilde{\xi}_l e^{ilx} \qquad (1)$$

and all $\tilde{\xi}_{lm}$ in

$$\xi(x,x_0) = \sum_l \sum_m \tilde{\xi}_{lm} \cancel{Y}_{lm}(x) \qquad (2)$$

are indep. And so are $\tilde{\eta}_l$ or $\tilde{\eta}_{lm}$ - obs errs.

Obs eqn:

$$\tilde{x}_l^o = \tilde{x}_l + \eta_l \qquad (3)$$

Since $\tilde{x}_l^o$ is related only to $\tilde{x}_l$ (w. the same l) K is diagonal:

$$\tilde{x}_l^a = x_l^f + k_l(x_l^o - Hx_l^f) = x_l^f + k_l(\eta_l - H\xi_l) \quad (4$$

$$\delta x_l^a = (1 - k_l H_l)\xi_l + k_l\eta_l \qquad (5)$$

$$H_l = \begin{cases} 1, & |l| \le l_{obs} \\ 0, & > \end{cases} \qquad (6)$$

So, $\forall |l| > l_{obs}: \ \delta x^a = \delta x^f \qquad (7)$

$|l| \le l_{obs}: \ \delta x^a = (1 - k_l)\xi_l + k_l\eta_l \qquad (8)$

$$k_l = \frac{\hat{b}_l}{\hat{b}_l + 2} \qquad (9)$$

$$\delta X^a_e = \left(1 - \frac{\hat{b}_e}{\hat{b}_e + 2}\right)\xi_e + \frac{\hat{b}_e}{\hat{b}_e + 2}\eta_e =$$

$$= \frac{2}{\hat{b}_e + 2}\xi_e + \frac{\hat{b}_e}{\hat{b}_e + 2}\eta_e \qquad (*)$$

Now 2 cases: opt & sub-optim anls

1°. Optim anls $\mathbb{E}[\xi_e^2] = b_e^{tru} \Rightarrow$ from $(*)$:

$$A_e^{opt} = \mathrm{Var}\,\delta X^a_e = \frac{1}{(b_e^{true}+2)^2}\left(2^2 b_e^{tru} + b_e^{tru\,2}\, 2\right) = \frac{b_e^{tru}\, 2}{b_e^{tru} + 2} \qquad (1)$$

2°. Sub-optim anls.: $\hat{b}_e \neq b_e^{true}$:

From $(*)$:

$$\hat{A}_e = \mathrm{Var}\,\delta X^a_e = \frac{2^2 b_e^{true} + \hat{b}_e^2\, 2}{(\hat{b}_e + 2)^2} \qquad (2)$$

Now, denote $\begin{cases} t_e = b_e^{true} \\ \hat{b}_e = b_e \end{cases}$

Then, $\forall e$

$$A_e^{opt} = \frac{t2}{t+2} \qquad (3)$$

$$\hat{A}_e = \frac{2^2 t + 2 b^2}{(2+b)^2} \qquad (4)$$

Prove that $\hat{A}_\ell \geq A_\ell^{opt}$ & $\hat{A}_\ell = A_\ell^{opt} \iff t = b$ (4

Proof:

$$\frac{tr}{t+r} \overset{?}{\underset{\leq}{=}} \frac{r^2 t + r b^2}{(r+b)^2} \iff$$

$$tr\left(r^2 + 2br + b^2\right) \overset{?}{\underset{\leq}{=}} (t+r)\left(r^2 t + r b^2\right) \iff$$

$$\cancel{tr^3} + 2btr^2 + \cancel{trb^2} \overset{?}{\underset{\leq}{=}} r^2 t^2 + \cancel{trb^2} + \cancel{tr^3} + r^2 b^2$$

$$2bt \overset{?}{\underset{\leq}{=}} t^2 + b^2 \iff$$

$$t^2 - 2bt + b^2 \overset{?}{\underset{\geq}{=}} 0 \iff b = t \quad (*)$$

So, we've defined a kind of divergence : $\forall \ell$:

$$\tilde{D}(b,t) = \frac{r^2 t + r b^2}{(r+b)^2} - \frac{tr}{t+r}$$ or, rather deviance (*)

whose zero minimum implies the best ans

The phys-space divergence

$$D(\{b_\ell\},\{t_\ell\}) = \begin{cases} \displaystyle\sum_{\substack{\ell=-\ell_{obs}\\ |\ell|\neq 0}}^{\ell_{obs}} \left[\frac{r^2 t_\ell + r b_\ell^2}{(r+b)^2} - \frac{t_\ell r}{t_\ell + r}\right], & S^1 \quad (1) \\ \displaystyle\sum_{0}^{\ell_{obs}} \frac{2\ell+1}{4\pi}\left[\frac{r^2 t_\ell + r b_\ell^2}{(r+b_\ell)^2} - \frac{t_\ell r}{t_\ell + r}\right], & S^2 \quad (2) \end{cases}$$

$$D \to \min$$

Simplify (4-*) :

$$\mathcal{D}(b,t) = \frac{(t+r)(r^2t+rb^2) - (r^2+2br+b^2)tr}{(r+b)^2(t+r)} =$$

$$= \frac{t^2r^2 + \cancel{trb^2} + \cancel{tr^3} + r^2b^2 - \cancel{tr^3} - 2tbr^2 - \cancel{trb^2}}{(r+b)^2(t+r)} =$$

$$= r^2 \frac{t^2 - 2tb + b^2}{(r+b)^2(t+r)} = \frac{r^2(t-b)^2}{(r+b)^2(t+r)}$$

$$\mathcal{D}(b,t) = \frac{r^2(t-b)^2}{(r+b)^2(t+r)} \qquad (*)$$

Here t is the true spectrum (model)
b is the spectrum in question
r is found fr (1-4).

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$$\hat{A} - A^{opt} = \frac{1}{4\pi} \sum_{l=0}^{l_{obs}} (2l+1)\, \mathcal{D}(b_l, t_l) \qquad (**)$$