Contents

1 Introduction

How I mentioned in KT-1 report - understanding the fluctuations in human population growth and decline in natural environments, as well as the competition between species for dominance, has always been a topic of great interest. The utilization of mathematical calculations and computer models is an essential component in studying the patterns of population dynamics.

This particular research project employs the Ulf Dieckmann and Richard Law model as a valuable tool to investigate stable biological communities. The model is based on an examination of integral-differential equations that describe the dynamics of spatial moments. The simulation outcomes are subsequently compared with results obtained through numerical methods.

Ulf Dieckmann and Richard Law proposed an approach that aims to bridge the various levels of organization, starting from the known characteristics of individual organisms. These levels encompass the microscopic interactions among individuals, the mesoscopic dynamics of local environments, and the macroscopic changes that affect the entire community. This method shows promise in providing a simplified yet analytical depiction of the macroscopic spatio-temporal processes, which can lead to new insights and enable comprehensive analyses beyond the investigation of individual simulation runs.

2 Research steps

Initial step was familiarization with the materials on the topic of the study, namely with the article [1] about spatial point processes and moment dynamics in the life sciences: a parsimonious derivation and some extensions and article [2] about moment approximations of individual-based models.

2.1 Spatial moments

Here by **spatial moment** an average spatial statistical characteristics of the community is understood.
For simplicity of notation, let's consider a single-species population. The population lives in a continuous spatial area of size A; for convenience, the space is assumed to be two-dimensional and it is sufficient large so that edge effects can be neglected. An individual i at $x_i = (x_{1i}, x_{2i})$ is denoted by the Dirac delta function $\delta_{xi}(x)$ with a peak at x_i and zero at other points. Spatial pattern of individuals $p(x)$ is the sum of all delta functions. The first spatial moment $N(p)$ is simply the mean density, known as the state variable in mean field population dynamics.

$$N(p) = \frac{1}{A} \int p(x)\, dx \tag{1}$$

First moment, however, does not contain any information about how individuals are distributed in a space. To measure the spatial structure of the second order, we use the second moment.

$$C(\xi, p) = \frac{1}{A} \int p(x)(p(x+\xi) + \delta_x(x+\xi))\, dx \tag{2}$$

This measures the density of pairs with a spatial displacement $\xi = (\xi_1, \xi_2)$ from the first individual to the second in the pair. In the context of moment dynamics, the pair density has the advantage of being particularly simple and immediately intuitive measure of the spatial structure of the second order. Dirac delta function $\delta_x(x+\xi)$ removes the self-adjoint terms that occur when the spatial offset ξ is zero. The first and second spatial moments given above are only the first two terms in moment hierarchy. next member:

$$T(\xi, \xi', p) = \frac{1}{A} \int p(x)(p(x+\xi) - \delta_x(x+\xi))(p(x+\xi') - \delta_x(x+\xi') - \delta_{x+\xi}(x+\xi'))\, dx \tag{3}$$

The third moment describes the density $T(\xi, \xi', p)$ of such triplets, and third-moment closures predict this density in terms of the N and C densities. Extension by moments higher order continues in a similar way.

3 Equilibrium equation

3.1 Dynamics of spatial moments

The spatial arrangement and the aforementioned points pertain to the condition of a specific population at a given time, denoted as t. In practical terms, the spatial pattern undergoes changes over time due to the birth, death, and movement of individuals. Additionally, since these events involve stochastic processes, restarting the process with the same initial conditions will not yield an identical spatial configuration at time t. To highlight the distinction between the spatial moments of the particular spatial pattern described above and the dynamics of the moment obtained by averaging across multiple realizations of the stochastic process, the parameter p is utilized solely in the former case, but not in the latter. For simplicity, we explore these concepts within the framework of spatial dynamic systems involving a single species, specifically the logistic model of population growth that incorporates birth, death, and movement at birth. The dynamical system encompasses the first and second moments as state variables and has been derived analytically from a stochastic individual process involving births, deaths, and displacements in continuous space. The dynamics of the first moment is given as:

$$\frac{d}{dt} N = (b-d)N - d' \int_{R^n} C(\xi) w(\xi) d\xi \tag{4}$$

$$\frac{dC(\xi)}{dt} = bm(\xi)N + \int_{R^n} bm(\xi')C(\xi+\xi')d\xi' - (d + d'w(\xi))C(\xi) - \int_{R^n} d'w(\xi')T(\xi,\xi')d\xi' \tag{5}$$

Here, as above, the constants b and d' are the fecundity and aggressiveness of the species, the constant d is the mortality of the species from unfavorable environmental conditions, and the functions m and w are the nuclei birth and competition, respectively.

From an ecological point of view, the case when the community is located in state of equilibrium, that is, when its average spatial characteristics. Let us pose the problem of finding the first two spatial moments corresponding to a given state. From a mathematical point of view, this the problem is formulated as the problem of finding the stationary point of system 4, 5.

Relation

$$\lim_{||x||_{R^n}\to+\infty} C(x) = N^2 \tag{6}$$

gives us a condition on the desired solution, allowing us to select it among sets of all other solutions

Note that the problem posed always has a zero solution. Really, if the community is extinct, then the first, second, and third spatial moments are equal to zero, as well as their time derivatives, and in addition, the equation 6. Therefore, we impose one more condition on the solution - it should not be trivial. Everywhere below, we will use this condition to justify the correctness of some calculations.

3.2 Closure of spatial moments

It can be seen that the dynamics of the first spatial moment depends on the second spatial moment, and the dynamics of the second - on the third. This trend will continue, that is, no matter how much we take the equations of dynamics and no matter how we expand the system 4, 5, it will always contain one equation less than unknown quantities. So lets use the Closure Method to solve this problem.

The essence of this method is that the unknown spatial moment of the highest order among all those considered is expressed in terms of the rest, thus, the number of unknowns decreases. Such an expression of the spatial moment is called a closure. The closing method is found empirically, based on some physical or biological properties of the problem under consideration. Naturally, this approach introduces a certain amount of error, but a well-chosen closure can reduce this error to acceptable values.

Closing the second moment reduces our model to the Verhulst model (see [?]), since the only way one can express the second moment in terms of the first this is $C(x) \equiv const$. This model is well known and studied by many mathematicians. Moreover, it does not take into account the spatial structure of the community, thus we will lose all the advantages of the current model with such a closure. So, let's try to close the third spatial moment. We will only give the main conditions that the closure must satisfy:

1. $\lim_{|x|\to+\infty} T(x,y) = NC(y)$
2. $\lim_{|y|\to+\infty} T(x,y) = NC(x)$
3. If $C(x) \equiv N^2$, then $T(x,y) \equiv N^3$

Here we consider three parametrized family of closures of type:

$$T_{\alpha\beta\gamma}(\xi,\xi') = \frac{1}{\alpha+\gamma}(\alpha\frac{C(\xi)C(\xi')}{N} + \beta\frac{C(\xi)C(\xi'-\xi)}{N} + \gamma\frac{C(\xi)C(\xi'-\xi)}{N} - \beta N^3) \tag{7}$$

Where $\alpha,\beta,\gamma \in [0;+\infty)$, $\alpha+\gamma > 0$.

3.3 Derivation of the equilibrium equation

Let's denote scalar multiplication $\int_{R^n} f(x)g(x)dx$ as $\langle f, g\rangle$. Convolution $\int_{R^n} f(x-y)g(y)dy$ as $[f * g]$. And also $d'w(x) = \bar{w}(x)$ and $bm(x) = \bar{m}(x)$.

In equilibrium both derivatives became zero, $\frac{dN}{dt} = 0$ and $\frac{dC}{dt} = 0$

So getting from 4

$$N = \frac{\langle \bar{w}, C\rangle}{b-d}$$

We can pass it into 5 and join the similar parts, finally obtaining

$$C = \frac{\bar{m}N + [\bar{m} * C] - \bar{w}C - \frac{\beta}{\alpha+\gamma} * \frac{C[\bar{w}*C]}{N} - \frac{\gamma}{\alpha+\gamma} * \frac{[C\bar{w}*C]}{N} + \frac{\beta d'}{\alpha+\gamma}N^3}{d + \frac{\alpha(b-d)}{\alpha+\gamma}} \tag{8}$$

4 Implementation of the numerical method

The Neiman's Method is a quick approach for obtaining results, but the equation at hand can also be solved using Euler's Method, although it may not be the most convenient method. However, it is still interesting to compare the results obtained from these two methods. One notable difference between the two methods is the calculation of convolution. In Neiman's Method ($O(n)$), we optimize the convolution calculation by transforming it into Fourier series. On the other hand, in Euler's Method ($O(n^2)$), we employ a relatively straightforward technique of numerically integrating using rectangle fitting.

4.1 Experiments

4.1.1 Euler's Method

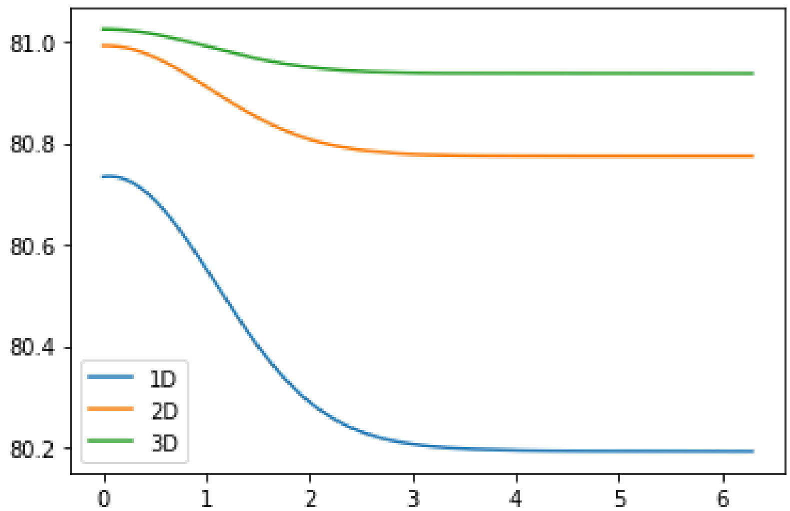

Figure 1: Experiment 1

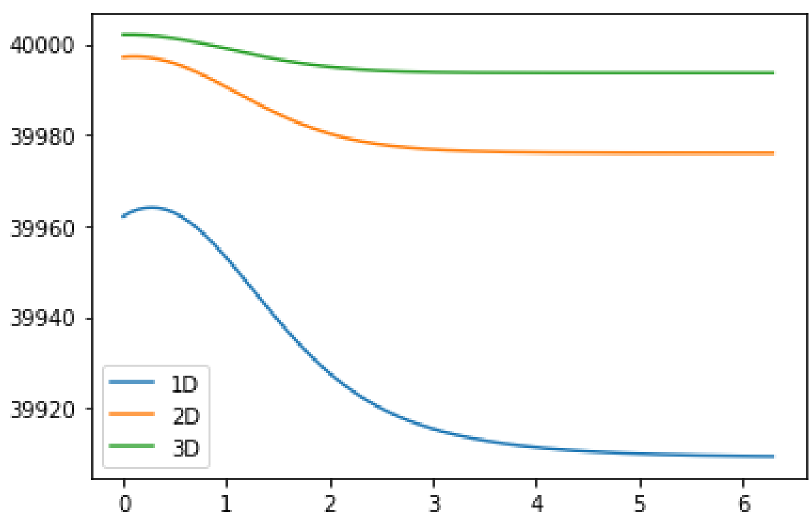

Figure 2: Experiment 2

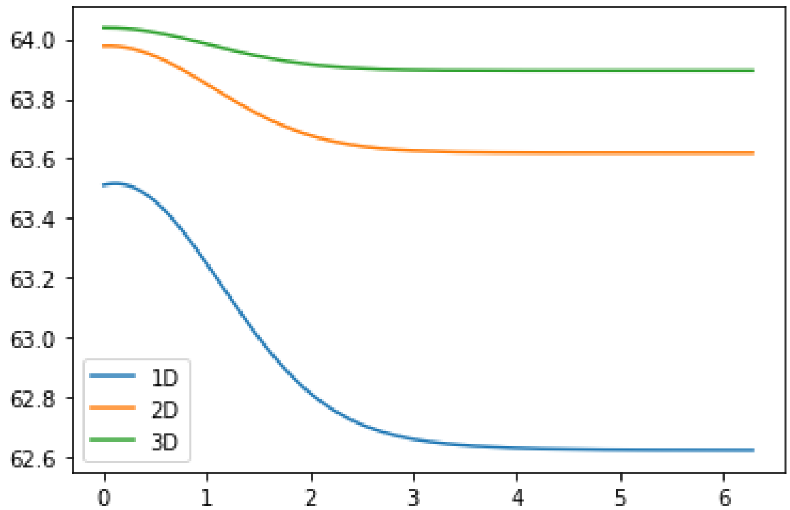

Figure 3: Experiment 3

4.1.2 Neiman‘s Method

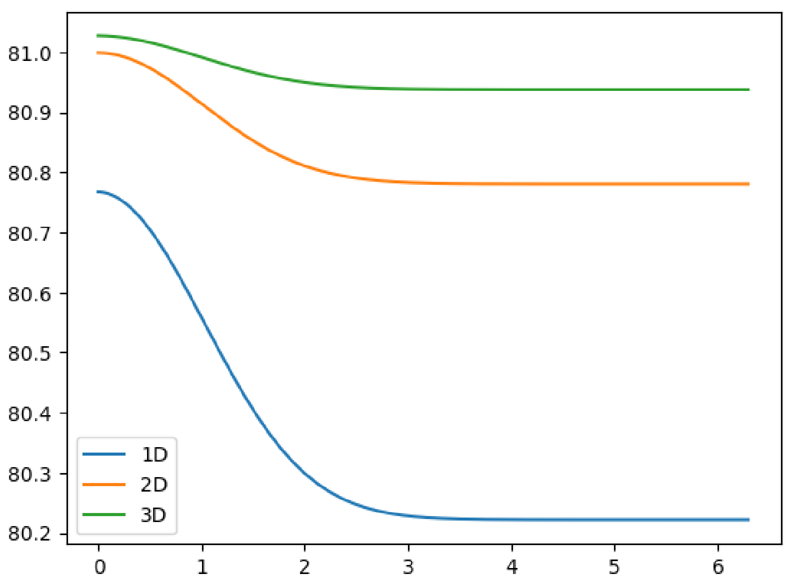

Figure 4: Experiment 1

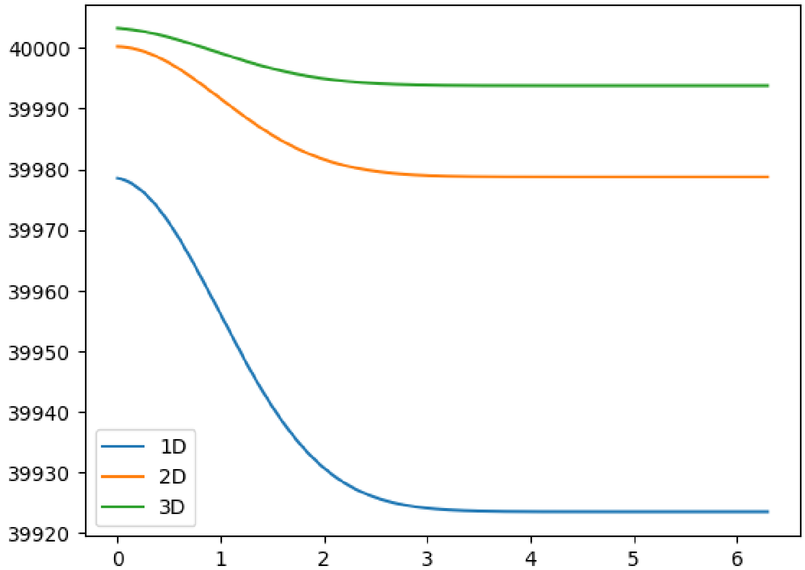

Figure 5: Experiment 2

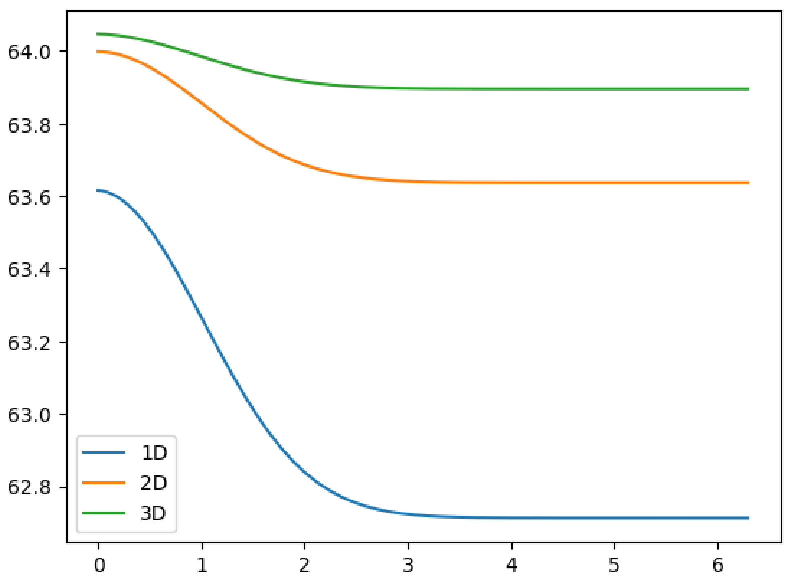

Figure 6: Experiment 3

5 Conclusion

This paper focuses on examining the equilibrium equation 8, which describes the first two spatial moments in a scenario where a single-species community is in a stable state. I derived a solution for the numerical method using Euler's Method to ensure that it is not superior to the Neiman's Method. The graphs demonstrate that the results are similar, but Neiman's Method remains faster.

References

[1] Michael J. Plank and Richard Law. Spatial point processes and moment dynamics in the life sciences: a parsimonious derivation and some extensions.

[2] Law R and Dieckmann U. Moment approximations of individual-based models. *Cambridge University Press*, pages 252–270, 2000.