Assignment2

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1 Problem 1.

Consider, $x_0 \in$ R
$X^{-1}(x_0) = \{E \in \Omega \ : \ X(E) = x_0\}$

Proof (a):
$X^{-1}(\phi) = \{E \in \Omega \ : \ X(E) = \phi\} = \phi$

Proof (b):
for $X^{-1}(R)$, Consider $x_0 \in$ R, corresponding to each x_0 We either have an event $E \in \Omega$ such that $X(E) = x_0$ or no such mapping exists. Varying $x_0 \, from - \infty \, to \, \infty$ we get all possible events in the sample space Ω.
Hence, $X^{-1}(R) = \Omega$

2 Problem 2.

Support of Random variable X is defined as

$$Support(X) = \{x \in R \ : \ P(X = x) = 0\}$$

We know that P(X=x) = f(x) (Probability density function) Also,

$$f(x) = \frac{dF_X(x)}{dx}$$

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lambda e^{\lambda x} & \text{if } x > 0 \end{cases}$$

Clearly, $\forall\, x \in (-\infty, 0)$, P(X=x)=f(x)=0
Therefore, Support(X) = { x : x < 0 }

3 Problem 3.

$$Consider,\ Z = \begin{bmatrix} m \\ c \end{bmatrix}$$

$$\mathbf{J(costfunction)} = \frac{\sum_{i=1}^{N}(mx_i + c - y_i)^2}{N}$$

Goal : To minimize J w.r.t to Z or equivalently We want, $\frac{\partial J}{\partial m} = 0$ and $\frac{\partial J}{\partial c} = 0$

$$\frac{\partial \mathbf{J}}{\partial \mathbf{m}} = \frac{2}{N}\sum_{i=1}^{N}(mx_i + c - y_i)x_i = m\sum_{i=1}^{N}x_i^2 + c\sum_{i=1}^{N}x_i - \sum_{i=1}^{N}x_i y_i = 0$$

$$\frac{\partial \mathbf{J}}{\partial \mathbf{c}} = \frac{2}{N}\sum_{i=1}^{N}(mx_i + c - y_i) = m\sum_{i=1}^{N}x_i + Nc - \sum_{i=1}^{N}y_i = 0$$

Solving for m and c, We get

$$\mathbf{m} = \frac{N(\sum_{i=1}^{N}x_i y_i) - (\sum_{i=1}^{N}x_i)(\sum_{i=1}^{N}y_i)}{N(\sum_{i=1}^{N}x_i^2) - (\sum_{i=1}^{N}x_i)^2}$$

$$\mathbf{c} = \frac{(\sum_{i=1}^{N}y_i)(\sum_{i=1}^{N}x_i^2) - (\sum_{i=1}^{N}x_i)(\sum_{i=1}^{N}x_i y_i)}{N(\sum_{i=1}^{N}x_i^2) - (\sum_{i=1}^{N}x_i)^2}$$

To check that this is indeed a minima

$$We\ want\ M = \frac{\partial}{\partial Z^T}[\frac{\partial J}{\partial Z}]\ to\ be\ positive\ definite(b^T M b\ >\ 0\,\forall\, b\ \in\ R^n)$$

$$So, Let\ b = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

To show $b^T M b > 0\ \forall\ \beta_1\ and\ \beta_2 \in R\ such\ that$
Substituting b in the inequality We get,

$$\frac{2\beta_1^2}{N}\sum_{i=1}^{N}x_i^2 + \frac{4\beta_1\beta_2}{N}\sum_{i=1}^{N}x_i + 2\beta_2^2 > 0$$

$$\frac{\beta_1^2}{\beta_2^2}\sum_{i=1}^{N} x_i^2 + \frac{2\beta_1}{\beta_2}\sum_{i=1}^{N} x_i + N > 0$$

Substitute $\frac{\beta_1}{\beta_2} = t$

$$\sum_{i=1}^{N} x_i^2 t^2 + \sum_{i=1}^{N} x_i t + N > 0$$

Clearly, this is a quadratic in t with coefficient of $t^2 > 0$ and also discriminant is less the zero $\forall\ t \in R$ So,

$$b = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} Always\ satisfies\ the\ condition$$

Hence Proved