Week 2

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Problem 1:

(a) $X^{-1}(\emptyset) = \emptyset$

Proof:

$$X^{-1}(\emptyset) = \{\omega \in \Omega : X(\omega) \in \emptyset\} = \emptyset$$

(b) $X^{-1}(R) = \Omega$

Proof:

$$X^{-1}(R) = \{\omega \in \Omega : X(\omega) \in R\} = \Omega$$

Problem 2:

Finding the Support of a Random Variable

Given the exponential distribution function $F_X(x)$:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

To find the support of X, we need to determine the range of values for which $F_X(x)$ is nonzero.

For $x < 0$, $F_X(x) = 0$. Therefore, the support starts from $x = 0$.

For $x \geq 0$, $F_X(x)$ is nonzero. Thus, the support extends indefinitely towards positive infinity.

Therefore, the support of X is the interval of nonnegative real numbers, which is $[0, \infty)$.

So, the support of X is $[0, \infty)$.

Problem 3:

Loss Function

The loss function (mean squared error, MSE) for linear regression is given by:

$$J(m, c) = \frac{1}{N} \sum_{i=1}^{N} (y_i - (mx_i + c))^2$$

Compute First-Order Partial Derivatives

Taking partial derivatives of $J(m, c)$ with respect to m and c:

$$\frac{\partial J}{\partial m} = \frac{1}{N} \sum_{i=1}^{N} -2x_i(y_i - (mx_i + c)) = 0$$

$$\frac{\partial J}{\partial c} = \frac{1}{N} \sum_{i=1}^{N} -2(y_i - (mx_i + c)) = 0$$

Solve the System of Equations

From the partial derivatives, we get two equations:

$$\sum_{i=1}^{N} x_i y_i - m \sum_{i=1}^{N} x_i^2 - c \sum_{i=1}^{N} x_i = 0$$

$$\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i - Nc = 0$$

Solving this system of equations for m and c:

$$m = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

$$c = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{N \sum x_i^2 - (\sum x_i)^2}$$

Verify Positive Definiteness of the Matrix

Second-Order Partial Derivatives

$$\frac{\partial^2 J}{\partial m^2} = \frac{2}{N} \sum_{i=1}^{N} x_i^2$$

$$\frac{\partial^2 J}{\partial c^2} = 2$$

$$\frac{\partial^2 J}{\partial m \partial c} = 2 \cdot \bar{x}$$

Matrix

The matrix A is composed of these second-order partial derivatives:

$$A = \begin{pmatrix} \frac{2}{N}\sum_{i=1}^{N} x_i^2 & 2 \cdot \bar{x} \\ 2 \cdot \bar{x} & 2 \end{pmatrix}$$

The eigenvalues of the matrix determine its positive definiteness. The eigenvalues are given by:

$$\lambda = \frac{\frac{2}{N}\sum_{i=1}^{N} x_i^2 + 2 \pm \sqrt{\left(\frac{2}{N}\sum_{i=1}^{N} x_i^2 - 2\right)^2 + 16\bar{x}^2}}{2}$$