Assignment 2

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Solution 1

(a) $X^{-1}(\varnothing) = \varnothing$

The inverse image $X^{-1}(A)$ of a set A under a random variable X is defined as:

$$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\}$$

Consider $A = \varnothing$. Then, we have:

$$X^{-1}(\varnothing) = \{\omega \in \Omega \mid X(\omega) \in \varnothing\}$$

By definition, $X(\omega) \in \varnothing$ means there are no elements ω in the sample space Ω such that $X(\omega)$ belongs to the empty set $\varnothing$. Thus:

$$X^{-1}(\varnothing) = \varnothing$$

Therefore, we have proven:

$$X^{-1}(\varnothing) = \varnothing$$

(b) $X^{-1}(\mathbb{R}) = \Omega$

Using the definition of the inverse image again:

$$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\}$$

Consider $A = \mathbb{R}$. Then, we have:

$$X^{-1}(\mathbb{R}) = \{\omega \in \Omega \mid X(\omega) \in \mathbb{R}\}$$

Since X is a random variable, by definition, $X(\omega)$ must take values in $\mathbb{R}$ for every $\omega \in \Omega$. This implies that for all $\omega \in \Omega$:

$$X(\omega) \in \mathbb{R}$$

Therefore:

$$\{\omega \in \Omega \mid X(\omega) \in \mathbb{R}\} = \Omega$$

Thus, we have:

$$X^{-1}(\mathbb{R}) = \Omega$$

Therefore, we have proven:

$$X^{-1}(\mathbb{R}) = \Omega$$

Solution 2

The support of a random variable X, denoted as $\text{supp}(X)$, is defined as the set of values x belonging to the real numbers ($\mathbb{R}$) such that the probability density function (PDF) $f(x)$ is greater than zero.

$$\text{supp}(X) = \{x \in \mathbb{R} \mid f(x) > 0\}$$

The PDF $f(x)$ is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

From this, we can see that the PDF $f(x)$ is positive ($\lambda e^{-\lambda x} > 0$) for all $x \geq 0$ and zero otherwise. Therefore, the support of X is the set of all non-negative real numbers.

So, the support of X is:

$$\text{Support of } X = [0, \infty)$$

Solution 3

The formula of a linear model is

$$y = mx + c$$

Given a set of data points (x_i, y_i) where $i = 1, 2, \ldots, N$, we calculate the error term for each:

$$e_i = y_i - (mx_i + c)$$

The objective is to minimize the sum of the squared errors:

$$S = \sum_{i=1}^{N} (y_i - (mx_i + c))^2$$

Partial Derivative with Respect to m:

$$\sum_{i=1}^{N} x_i y_i - m \sum_{i=1}^{N} x_i^2 - c \sum_{i=1}^{N} x_i = 0$$

Partial Derivative with Respect to c:

$$\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i - Nc = 0$$

Solving these equations simultaneously for m and c:

Solving for c:

$$c = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{N(\sum x^2) - (\sum x)^2}$$

Solving for m:

$$m = \frac{N(\sum xy) - (\sum x)(\sum y)}{N(\sum x^2) - (\sum x)^2}$$

Consider $Z = \begin{pmatrix} m \\ c \end{pmatrix}$. The cost function is given as:

$$J(\text{costfunction}) = \frac{1}{N}\sum_{i=1}^{N}(mx_i + c - y_i)^2$$

To check if this is indeed a minimum, we want $M = \frac{\partial}{\partial Z^T}\left[\frac{\partial J}{\partial Z}\right]$ to be positive definite ($b^T M b > 0$ for all $b \in \mathbb{R}^n$).

Let $b = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$. To show that $b^T M b > 0$ for all β_1 and β_2 in $\mathbb{R}$, we substitute b in the inequality:

$$2\beta_1^2 \frac{1}{N}\sum_{i=1}^{N} x_i^2 + 4\beta_1\beta_2 \frac{1}{N}\sum_{i=1}^{N} x_i + 2\beta_2^2 > 0$$

$$\beta_1^2 \frac{1}{N}\sum_{i=1}^{N} x_i^2 + 2\beta_1\beta_2 \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{N}{2} > 0$$

Substitute $\beta_1\beta_2 = t$:

$$\frac{1}{N}\sum_{i=1}^{N} x_i^2 t^2 + \frac{1}{N}\sum_{i=1}^{N} x_i t + \frac{N}{2} > 0$$

Clearly, this is a quadratic in t with a coefficient of t^2 greater than zero and a discriminant less than zero for all t in $\mathbb{R}$. Therefore, $b = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ always satisfies the condition. Hence proved.