Theoretical Machine Learning - Assignment 2

Gaurav Kumar Rampuria

May 26, 2024

1. Problem 1

1.1.

$X^{-1}(\Phi) = X^{-1}(\text{Null set in } \mathbb{R}) = \phi = \text{Null Set in the sample space}$

1.2. Part 1

$X^{-1}(\mathbb{R}) = X^{-1}(\text{Set of } \mathbb{R}) = \Omega = \text{Entire Sample space}$

2. Problem 2

Support of X, $S_x =$ x: $f(x) > 0$, where $f(x) = \lim_{h\to 0} \frac{F(x)-F(x-h)}{h}$

Given, $F_X(x) = 1 - e^{-\lambda x}$, for $x \geq 0$.
$f(x) = \lim_{h\to 0} \frac{(1-e^{-\lambda x})-(1-e^{-\lambda(x-h)})}{h} = \lim_{h\to 0} \frac{e^{-\lambda(x-h)}-e^{-\lambda x}}{h} = \lim_{h\to 0} \frac{e^{-\lambda x}(e^{-\lambda h}-1)}{h} =$
$e^{-\lambda x} \lim_{h\to 0} \frac{e^{-\lambda h}-1}{h} = \lambda e^{-\lambda x}$.

So, $f(x) = \lambda e^{-\lambda x} \geq 0$, so $x \in [0, \infty)$.

Ans) $x \in [0, \infty)$

3. Problem 3

From, the L2 Norm we have L(m, c) $= \sum_{i=1}^{N} \frac{(y_i - mx_i - c)^2}{N}$. We have to minimize the L2 norm, hence we will differentiate it with m and c. Let $z = \begin{vmatrix} \text{m} \\ \text{c} \end{vmatrix}$

$\frac{\partial L}{\partial m} = \sum\limits_{i=1}^{N} \frac{-2x_i(y_i - mx_i - c)}{N}$ and $\frac{\partial L}{\partial c} = \sum\limits_{i=1}^{N} \frac{-2(y_i - mx_i - c)}{N}$

Now both $\frac{\partial L}{\partial m}$ and $\frac{\partial L}{\partial c}$ will be equal to 0 for minima.

Hence, we get $\sum\limits_{i=1}^{N} x_i \frac{(y_i - mx_i - c)}{N} = 0$ and $\sum\limits_{i=1}^{N} \frac{(y_i - mx_i - c)}{N} = 0$

$\sum\limits_{i=1}^{N} x_i y_i - m \sum\limits_{i=1}^{N} x_i^2 - c \sum\limits_{i=1}^{N} x_i = 0$

$\sum\limits_{i=1}^{N} y_i - m \sum\limits_{i=1}^{N} x_i - c \sum\limits_{i=1}^{N} 1 = 0$

$\sum\limits_{i=1}^{N} y - m \sum\limits_{i=1}^{N} x = cN \implies c = \frac{\sum\limits_{i=1}^{N} y - m \sum\limits_{i=1}^{N} x}{N}$

Substituting this value of c in equation (1) we get:-

$\sum\limits_{i=1}^{N} xy - m \sum\limits_{i=1}^{N} x^2 - \frac{\sum\limits_{i=1}^{N} y - m \sum\limits_{i=1}^{N} x}{N} \sum\limits_{i=1}^{N} x = 0$

$\sum\limits_{i=1}^{N} xy - m \sum\limits_{i=1}^{N} x^2 - \frac{(\sum\limits_{i=1}^{N} y)(\sum\limits_{i=1}^{N} x) - m \sum\limits_{i=1}^{N} x^2}{N} = 0$

$N \sum\limits_{i=1}^{N} xy - mN \sum\limits_{i=1}^{N} x^2 - (\sum\limits_{i=1}^{N} x)(\sum\limits_{i=1}^{N} y) + m \sum\limits_{i=1}^{N} x^2 = 0$

$N \sum\limits_{i=1}^{N} xy - (\sum\limits_{i=1}^{N} x)(\sum\limits_{i=1}^{N} y) = m(N \sum\limits_{i=1}^{N} x^2 - (\sum\limits_{i=1}^{N} x)^2)$

So, $m = \frac{N \sum xy - (\sum x)(\sum y)}{N \sum x^2 - \sum x^2}$

Substituting this value of m back in c, we get:-

So, $c = \frac{\sum y \sum x^2 - (\sum x)(\sum xy)}{N \sum x^2 - (\sum x)^2}$

For it to be a minima and not the maxima, we have to see the double derivative :-

$\frac{d}{dZ^T}(\frac{dL}{dZ}) = \begin{vmatrix} \frac{\partial^2 L}{\partial m^2} & \frac{\partial^2 L}{\partial m \partial c} \\ \frac{\partial^2 L}{\partial c \partial m} & \frac{\partial^2 L}{\partial c^2} \end{vmatrix} = \begin{vmatrix} \frac{2\sum x^2}{N} & \frac{2\sum x}{N} \\ \frac{2\sum x}{N} & \frac{2\sum 1}{N} \end{vmatrix}$

How do we define > 0 for a matrix? In that case, we have to see if a matrix is positive definite. For any n x n matrix M to be positive definite, we need that $\vec{b} \in \mathbb{R}^n, \vec{b}^T M \vec{b} > 0$

Let $\vec{b} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$\vec{b}^T M \vec{b} = \begin{vmatrix} 1 & 1 \end{vmatrix} \begin{vmatrix} \frac{2\sum x^2}{N} & \frac{2\sum x}{N} \\ \frac{2\sum x}{N} & \frac{2\sum 1}{N} \end{vmatrix} \vec{b} = \begin{vmatrix} \frac{2\sum x^2}{N} + \frac{2\sum x}{N} & \frac{2\sum x}{N} + \frac{2\sum 1}{N} \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$= \frac{2\sum x^2}{N} + \frac{2\sum x}{N} + \frac{2\sum x}{N} + \frac{2\sum 1}{N} = \frac{2}{N} \sum(x^2 + 2x + 1) = \frac{2}{N} \sum(x+1)^2 > 0$

Hence, the matrix is a positive definite, and the obtained result is a minima.