Assignment 2

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Problem 1

Proof:

(a) $X^{-1}(\Phi) = \Phi$
Since $A = \Phi$ (the empty set), therefore:

$$X^{-1}(\Phi) = \{\omega \in \Omega \mid X(\omega) \in \Phi\}$$

Since Φ is the empty set, there are no elements in Φ, so no $\omega \in \Omega$ can satisfy $X(\omega) \in \Phi$. Thus,

$$X^{-1}(\Phi) = \Phi$$

(b) $X^{-1}(\mathbb{R}) = \Omega$
If $A = \mathbb{R}$ then:

$$X^{-1}(\mathbb{R}) = \{\omega \in \Omega \mid X(\omega) \in \mathbb{R}\}$$

Since X is a random variable, by definition, it maps every element $\omega \in \Omega$ to some real number in $\mathbb{R}$. Therefore, every $\omega \in \Omega$ satisfies $X(\omega) \in \mathbb{R}$, and:

$$X^{-1}(\mathbb{R}) = \Omega$$

Problem 2

The support of a random variable X means probability density function $f_X(x)$ is positive.

First, we find the probability density function $f_X(x)$ by differentiating the cumulative distribution function $F_X(x)$.

For $x < 0$:

$$F_X(x) = 0 \implies f_X(x) = \frac{d}{dx} 0 = 0$$

For $x \geq 0$:

$$F_X(x) = 1 - e^{-\lambda x} \implies f_X(x) = \frac{d}{dx}\left(1 - e^{-\lambda x}\right) = \lambda e^{-\lambda x}$$

The probability density function $f_X(x)$ is positive for all $x \geq 0$. Therefore, the support of the random variable X is:

$$\text{Support of } X = [0, \infty)$$

Problem 3

The cost function is equal to:

$$J(m, c) = \frac{1}{N} \sum_{i=1}^{N} (y_i - mx_i - c)^2$$

The goal is to minimize J with respect to m and c, i.e. $\frac{\partial J}{\partial Z} = 0$, where Z can be m or c.

$$\frac{\partial J}{\partial c} = \frac{2}{N} \sum_{i=1}^{N} (mx_i + c - y_i) x_i = 0$$

$$\frac{\partial J}{\partial m} = \frac{2}{N} \sum_{i=1}^{N} (mx_i + c - y_i) = 0$$

After solving for m(slope) and c(intercept) :

$$m = \frac{N(\sum_{i=1}^{N} x_i y_i) - (\sum_{i=1}^{N} x_i)(\sum_{i=1}^{N} y_i)}{N(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)^2}$$

$$c = \frac{(\sum_{i=1}^{N} y_i)(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)(\sum_{i=1}^{N} x_i y_i)}{N(\sum_{i=1}^{N} x_i^2) - (\sum_{i=1}^{N} x_i)^2}$$

Hence, proved.