TML Week 2

Shrasti Dwivedi

May 2024

1 Problem 1

1.1

Random variable assigns numbers to the elements in the sample space hence for an empty set no element in sample space is assigned.So
$X^{-1}(\phi) = \phi$

1.2

Similarly,
$X^{-1}(R) = X^{-1}(\text{Set of } \mathbb{R}) = \Omega = \text{Entire Sample Space})$

2 Problem 2

The support will be the points where probability density function is ¿ 0 i.e. derivative is > 0. $x \geq 0$ The support of X, S_X, is defined as:

$$S_X = \{x : f(x) > 0\}$$

where $f(x)$ is the probability density function (pdf).

Given $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.

The pdf $f(x)$ is obtained as the limit:

$$f(x) = \lim_{h \to 0} \frac{F(x) - F(x-h)}{h}$$

Substituting the cumulative distribution function (CDF):

$$f(x) = \lim_{h \to 0} \frac{(1 - e^{-\lambda x}) - (1 - e^{-\lambda(x-h)})}{h}$$

Simplify the expression:

$$f(x) = \lim_{h \to 0} \frac{e^{-\lambda(x-h)} - e^{-\lambda x}}{h}$$

Factor out $e^{-\lambda x}$:

$$f(x) = \lim_{h\to 0} \frac{e^{-\lambda x}(e^{-\lambda h} - 1)}{h}$$

Recognize the limit as a known derivative:

$$f(x) = e^{-\lambda x} \lim_{h\to 0} \frac{e^{-\lambda h} - 1}{h}$$

The limit $\lim_{h\to 0} \frac{e^{-\lambda h}-1}{h}$ is $-\lambda$:

$$f(x) = e^{-\lambda x}(-\lambda)$$

Thus, the pdf is:

$$f(x) = -\lambda e^{-\lambda x}$$

Also, $\forall x \in (-\infty, 0), f(x) = 0$
So, Support of X is $(-\infty,\infty)$.

3 Problem 3

We have $L(m, c) = \frac{\sum_{i=1}^{N}(y_i - mx_i - c)^2}{N}$

We are trying to minimise this wrt m & c. Let's define the vector $Z = \begin{bmatrix} m \\ c \end{bmatrix}$

We need $\frac{dL}{dZ} = \vec{0}$

$\frac{\partial L}{\partial m} = \frac{\sum_{i=1}^{N} -x_i(y_i - mx_i - c)}{N}$ ....(i)

$\frac{\partial L}{\partial c} = \frac{\sum_{i=1}^{N} -(y_i - mx_i - c)}{N}$ ....(ii)

To minimize loss, we need both of these to be equal to 0, putting (ii) equal to 0 gives us,
$c = \frac{(\sum y_i) - m(\sum x_i)}{N}$

Putting this value in (i),we get,
$\frac{\partial L}{\partial m} = \frac{N(\sum x_i y_i) - mN(\sum x_i^2) - (\sum x_i)(\sum y_i) + m(\sum x_i)^2}{N^2} = 0$

Thus finally getting,
$m = \frac{N(\sum xy) - (\sum x)(\sum y)}{N(\sum x^2) - (\sum x)^2}$

And putting this back , we get ,
$c = \frac{(\sum y)(\sum x^2) - (\sum x)(\sum xy)}{N(\sum x^2) - (\sum x)^2}$
To check if a 2 dimensional function is concave, we have to see the double

derivative -
$\frac{d}{dZ^T}[\frac{dL}{dZ}] = \begin{bmatrix} \frac{\partial^2 L}{\partial m^2} & \frac{\partial^2 L}{\partial m \partial c} \\ \frac{\partial^2 L}{\partial c \partial m} & \frac{\partial^2 L}{\partial c^2} \end{bmatrix}$

To define > 0 for a matrix, we have to see if a matrix is positive definite. For any $n \times n$ matrix M to be positive definite, we need that $\vec{b} \in \mathbb{R}^n$, $\vec{b}^T M \vec{b} > 0$.

For simplicity, we take $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hence, the expression simplifies to:

$$\frac{2}{N}\left(\sum(x^2 + 2x + 1)\right) = \frac{2}{N}\left(\sum(x+1)^2\right) > 0$$

Thus, the matrix M is positive definite, and the obtained result is a minimum.