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MA4340 Probability Theory in Finance

Implied vs. Realized Volatility using GARCH and Heston Models

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Abstract

This project evaluates how well return-based volatility models—GARCH and the Heston model—replicate market-implied volatility, using the S&P 500 as the underlying asset and VIX as the benchmark. Both models are trained on historical returns, and their outputs are interpreted as implied volatility forecasts. We compare these forecasts to the VIX to assess alignment with market expectations. Through regression analysis and model comparison, we explore which approach better captures implied volatility behavior under varying market conditions.

Contents

1 Introduction

Volatility is one of the most crucial components in financial markets, influencing everything from risk management to the pricing of derivatives. Understanding and forecasting volatility accurately enables investors, portfolio managers, and traders to make informed decisions and hedge risks efficiently.
This project investigates two distinct but popular methodologies for measuring and predicting volatility.

- **Implied Volatility (IV):** A forward-looking measure derived from option prices, reflecting the market's consensus on future volatility.
- **Realized Volatility (RV):** A backward-looking measure based on historical returns that quantifies the actual price fluctuations of an asset over a given period.

We utilize and compare two volatility models:

- **GARCH:** A statistical model that captures volatility clustering using historical return data.
- **Heston Model:** A stochastic volatility model that models volatility as a random process itself.

In addition, the project evaluates market regime effects (stable vs. volatile periods) and develops a trading strategy based on the discrepancy between option-implied volatility (IV) and realized volatility (RV). These additions offer valuable insights into model adaptability and the potential for volatility arbitrage in financial markets.

2 Data Collection & Preprocessing

The dataset used in this analysis is taken from sp500_vix_data.csv

2.1 Preprocessing Steps

- **Compute Log Returns:** Log returns are calculated to standardize percentage changes and serve as inputs for volatility computation. The log return r_t is computed as:
$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$
where P_t is the closing price at time t, and P_{t-1} is the closing price at the previous time step.
- **Compute Realized Volatility (RV):** A 21-day rolling standard deviation of log returns.
The realized volatility $\sigma_{RV,t}$ at time t is given by:
$$\sigma_{RV,t} = \sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(r_{t-i} - \bar{r})^2}$$
where $N = 21$, r_{t-i} are the log returns over the past N days, and $\bar{r}$ is the mean return over that window.

- **Extract Implied Volatility (IV):** Derived from options chain data from VIX dataset.
- **Time Series Alignment:** Synchronized IV, RV, and risk-free rate, removing NaNs.

3 Model Implementation

3.1 Heston Stochastic Volatility Model

The Heston model describes the dynamics of an asset price S_t and its variance v_t using the following stochastic differential equations (SDEs):

$$dS_t = rS_t\,dt + \sqrt{v_t}S_t\,dW_t^S, \tag{1}$$
$$dv_t = \kappa(\theta - v_t)\,dt + \sigma\sqrt{v_t}\,dW_t^v, \tag{2}$$

where:

- r is the risk-free rate,
- κ is the speed of mean reversion,
- θ is the long-run variance,
- σ is the volatility of variance (volatility of volatility),
- dW_t^S and dW_t^v are two Wiener processes with correlation ρ:

$$\text{corr}(dW_t^S, dW_t^v) = \rho.$$

The model is calibrated by minimizing the mean squared error between the model-implied volatility and the realized volatility.

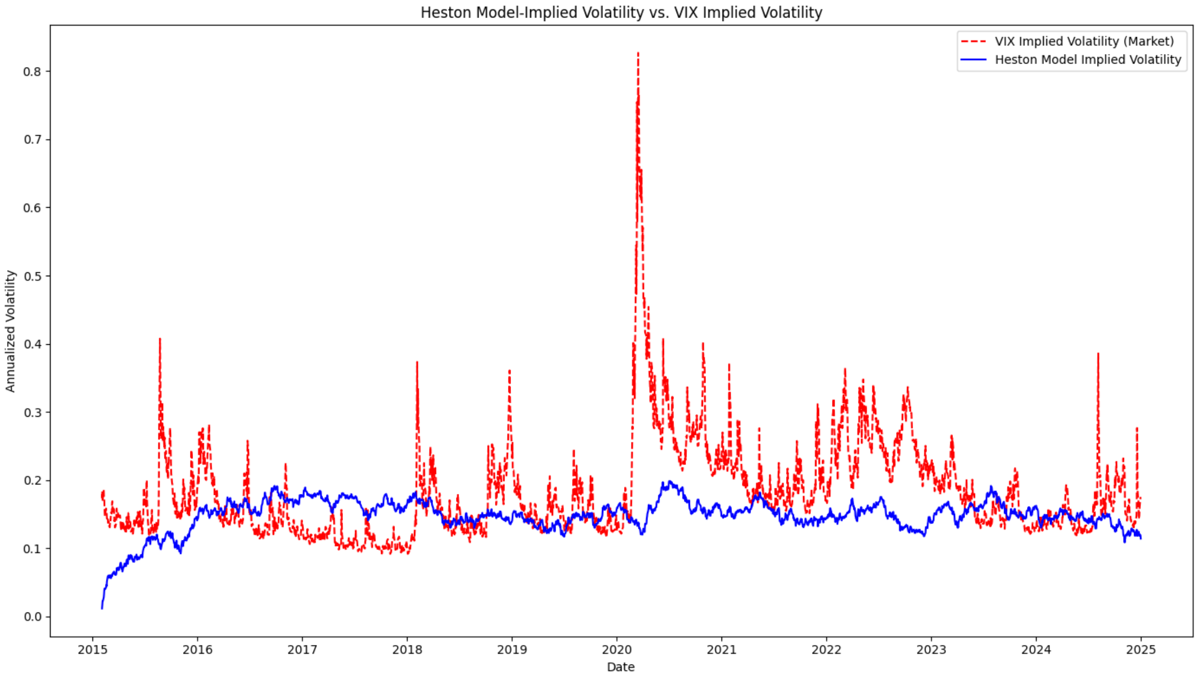
Implementation Details

The simulation iterates over N time steps. Asset prices are updated using a log-normal process that incorporates the risk-free rate and the instantaneous variance. The variance is updated using the mean-reverting term

$$dv_t = \underbrace{\kappa(\theta - v_{t-1})\,\Delta t}_{\text{mean reversion}} + \underbrace{\sigma\sqrt{v_{t-1}}\,dW_2}_{\text{stochastic component}},$$

Fitted Parameters

- kappa=2.0386
- theta=0.021211
- sigma=0.1029
- rho=0.0079



The Heston model underestimates sharp volatility spikes due to its relatively constrained stochastic structure. It captures the general level and trend of market volatility, but lacks responsiveness to sudden market changes as the model follows a mean-reverting smooth process which is unable to capture sudden movements.

Performance Metrics:Heston Model vs VIX Implied Volatility

- Root Mean Squared Error (RMSE): 0.0846
- Mean Absolute Error (MAE): 0.0575
- Correlation with VIX: -0.0068

3.2 GARCH Model

GARCH model captures the volatility clustering observed in financial returns. A common version is the GARCH model, GARCH(p, q) is given by:

$$\sigma_t^2 = \omega + \sum_i \alpha_i \epsilon_{t-i}^2 + \sum_i \beta_i \sigma_{t-i}^2,$$

where:

- σ_t^2 is the conditional variance at time t,
- ϵ_{t-i} is the shock (or innovation) from the previous period,
- ω is a constant,
- α_i represents the impact of recent spikes,
- β_i represents the persistence of past variances.

The model is typically estimated by maximum likelihood estimation.

Objective

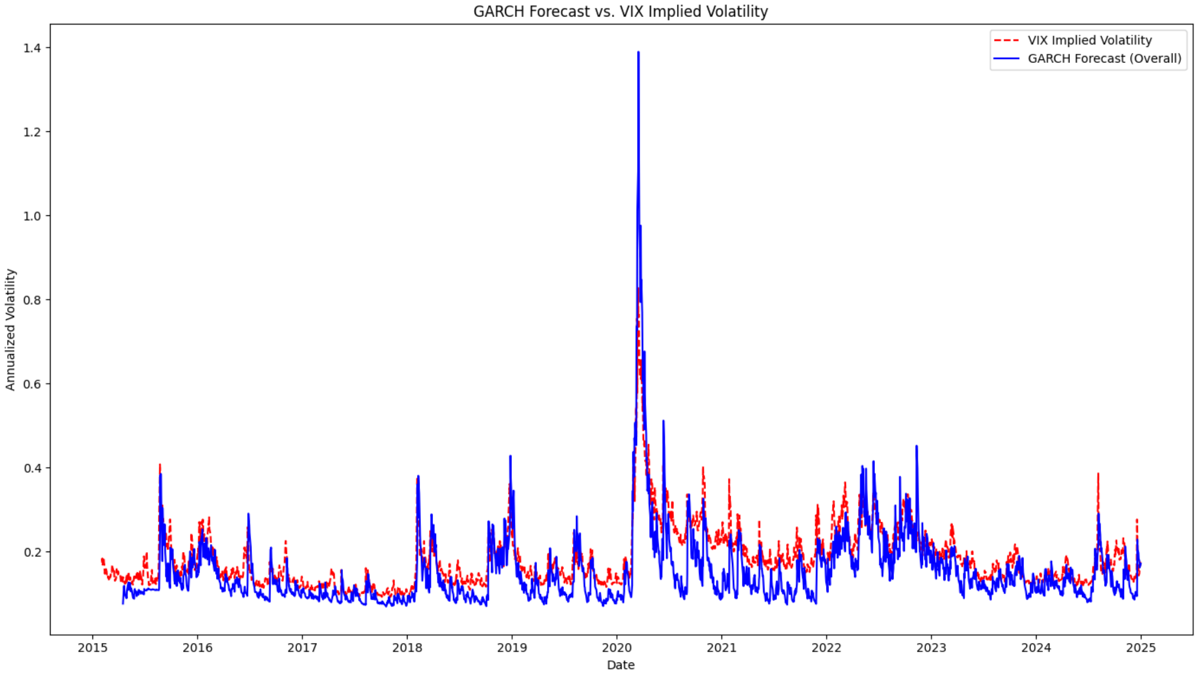
- Forecast one-step-ahead volatility using the GARCH(1,1) model.
- Compare GARCH forecasts with realized volatility over time.

Methodology

- **Model:** GARCH(1,1) with normal distribution (`dist='normal'`).
- **Data:** Log returns of asset prices, scaled by 100 for numerical stability.
- **Forecasting:** Rolling one-step-ahead forecasts, annualized using $\sqrt{252}$.

Implementation

- **Model Fitting:** Implemented using `arch_model()` from the `arch` package.
- **Rolling Forecasts:**
  - Begin forecasting after a burn-in period of `window + 30` observations.
  - For each time t, fit the model on data up to $t-1$, then forecast volatility at t.



GARCH, being a purely historical volatility model, lags in fast-moving markets but can overshoot in its reactivity. Compared to Heston, GARCH has a faster response to past realized volatility spikes.

Performance Metrics

- Root Mean Squared Error (RMSE): 0.06
- Mean Absolute Error (MAE): 0.0442
- Correlation with VIX: 0.8478

4 Market Regime Analysis

Market conditions vary and model performance can differ during stable and volatile periods. We segment the dataset using volatility thresholds (e.g., 30th and 70th percentiles of RV) and evaluate RMSE, MAE, and correlation separately for these regimes. Financial markets undergo different regimes: periods of calm and periods of high uncertainty. This analysis helps identify which model adapts better to market changes.

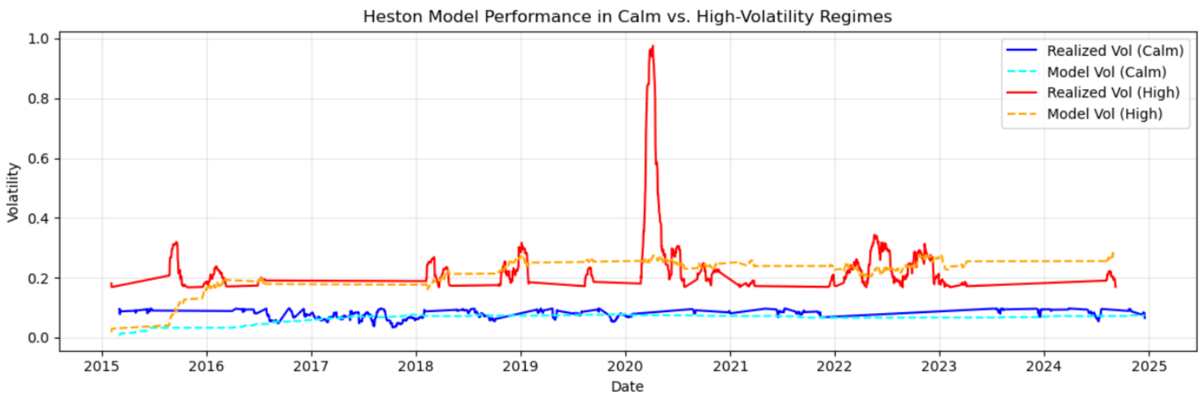
4.1 Regime Classification

We define market regimes based on the distribution of realized volatility as follows:

- **Calm Regime:** Observations in the bottom 30% of the realized volatility distribution.
- **High Volatility Regime:** Observations in the top 30% of the realized volatility distribution.

4.2 Comparison between both models

Heston Model Regime analysis



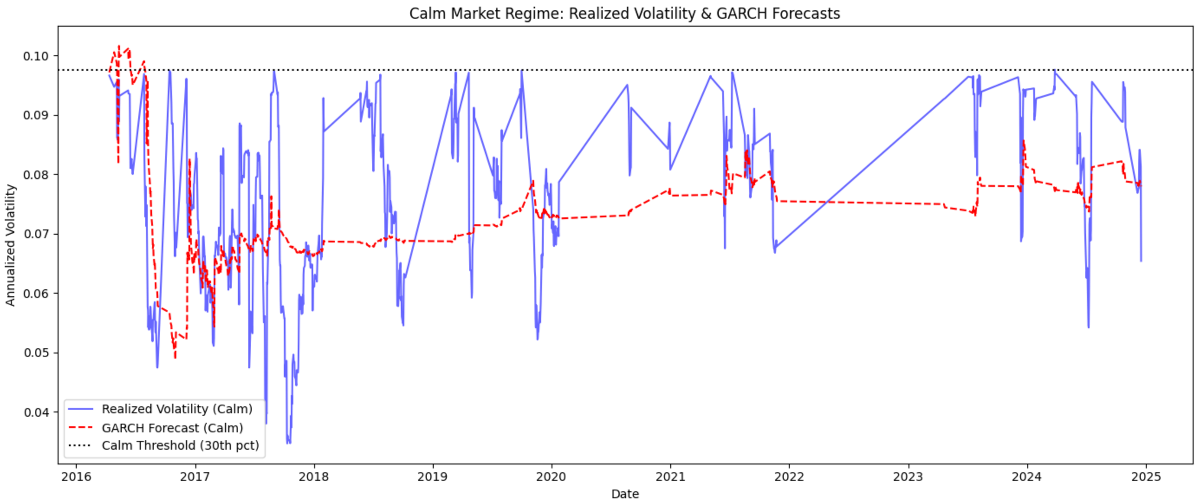
Metric	Calm Regime	High Volatility Regime
RMSE	0.021041	0.144777
MAE	0.015091	0.081875
Correlation	-0.0548	-0.0572

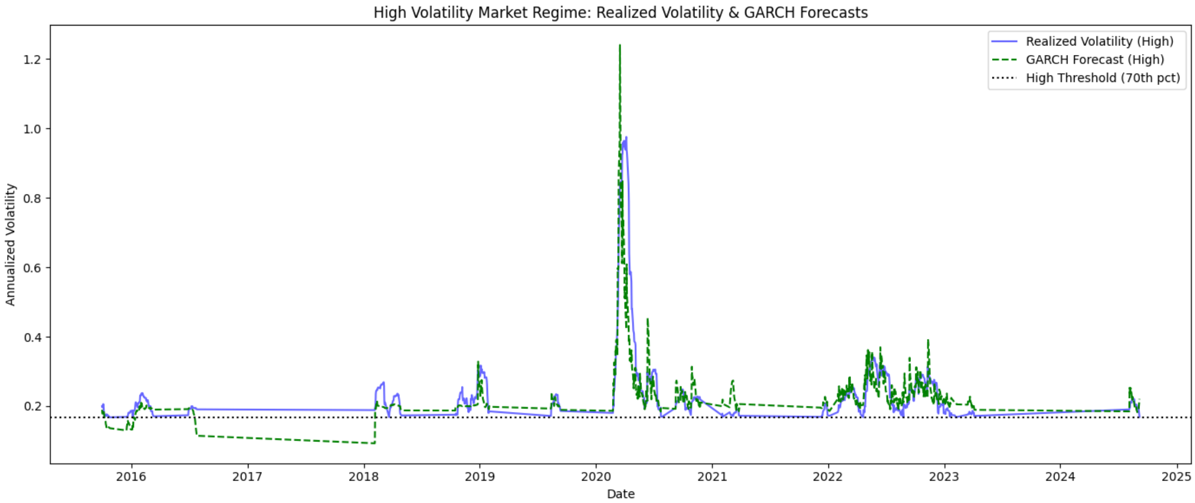
Table 1: Heston model performance under different market regimes.

Heston's Regime-wise Behavior:

- In calm periods, both realized and model-implied volatilities are low and stable. The model tracks realized volatility reasonably well.
- In high-volatility periods, realized volatility is highly erratic.The Heston model captures the general magnitude but lags during volatility spikes.

GARCH Model Regime analysis





Metric	Calm Regime	High Volatility Regime
RMSE	0.013713	0.072781
MAE	0.010850	0.040103
Correlation	0.3737	0.8327

Table 2: GARCH model performance under different market regimes.

GARCH Regime-wise Behavior:

- During calm regimes, the GARCH model maintains consistent forecasts, with slight underestimation of sharp dips in volatility.
- In high-vol regimes, GARCH is more responsive than Heston during spikes (e.g., 2020), but still slightly lags peak volatility.

4.3 Conclusions

:

- **GARCH consistently outperforms Heston** in both regimes.
- In the **calm regime**, GARCH shows lower RMSE and MAE with a positive correlation, whereas Heston exhibits a weak negative correlation.
- In the **high-volatility regime**, GARCH achieves significantly lower errors and a strong correlation (0.8327), while Heston remains weakly negatively correlated (-0.0572).
- Overall, the **GARCH model is more accurate and responsive** to both stable and turbulent market conditions.

Overall, for calm markets, both models are reliable, with Heston providing smooth long-term volatility estimates. For high-volatility regimes, GARCH may be preferable for its responsiveness, whereas Heston maintains structural consistency but with delayed adjustments.

5 Regression Analysis: RV vs. IV

5.1 Regression

To quantify the predictive relationship between implied volatility (IV) and realized volatility (RV), we perform an ordinary least squares (OLS) regression. This approach estimates how well IV can explain or predict RV.
We consider the following linear models:

Model 1: GARCH-Implied Volatility

$$RV_t = \beta_0 + \beta_1 \cdot \text{GARCH\_IV}_t + \varepsilon_t \tag{3}$$

Model 2: Heston-Implied Volatility

$$RV_t = \beta_0 + \beta_1 \cdot \text{Heston\_IV}_t + \varepsilon_t \tag{4}$$

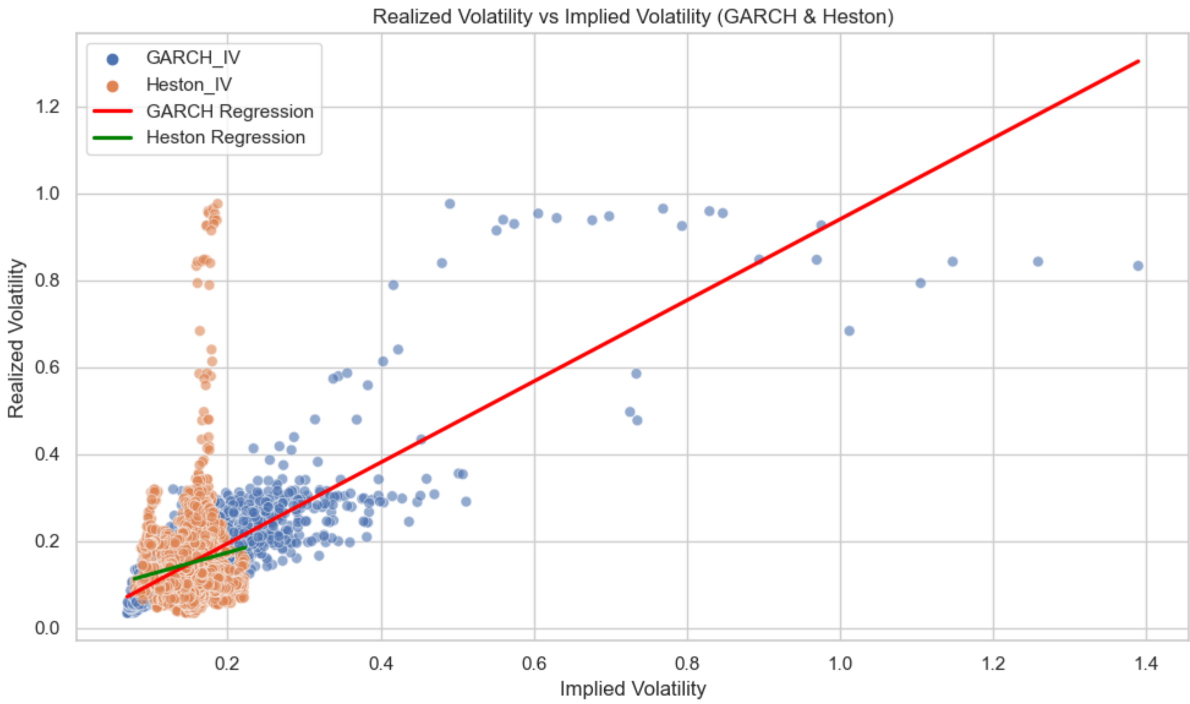
Model 3: GARCH + Heston-Implied Volatility

$$RV_t = \beta_0 + \beta_1 \cdot \text{GARCH\_IV}_t + \beta_2 \cdot \text{Heston\_IV}_t + \varepsilon_t \tag{5}$$

where:

- RV_t is the realized volatility at time t,
- β_0, β_1, β_2 are the regression coefficients,
- ε_t is the error term.

5.2 Results



- **GARCH-IV (blue points)** exhibits a strong linear relationship with realized volatility, as reflected by the steep red regression line.
- **Heston-IV (orange points)** shows a weaker correlation, with its green regression line being flatter and the data points more dispersed.
- The regression slope for GARCH-IV is notably higher than for Heston-IV, suggesting that GARCH-IV more accurately tracks variations in realized volatility.

Model	**Intercept (β_0)**	**Slopes**	R^2
GARCH-IV Only	0.0083	$\beta_1 = 0.9321$	0.754
Heston-IV Only	0.0748	$\beta_1 = 0.4957$	0.018
GARCH + Heston IV	-0.0065	$\beta_1 = 0.9284$, $\beta_2 = 0.1024$	0.755

Table 3: OLS Regression Results: Realized Volatility vs. Implied Volatility

Model	**R²**	**Key Takeaway**
RV - GARCH IV	0.754	GARCH IV strongly predicts RV with high slope
RV - Heston IV	0.018	Heston IV also predicts RV, but with a lower slope
RV - (GARCH IV + Heston IV)	0.755	Combined IVs improve predictive power slightly

5.3 Interpretation of Results

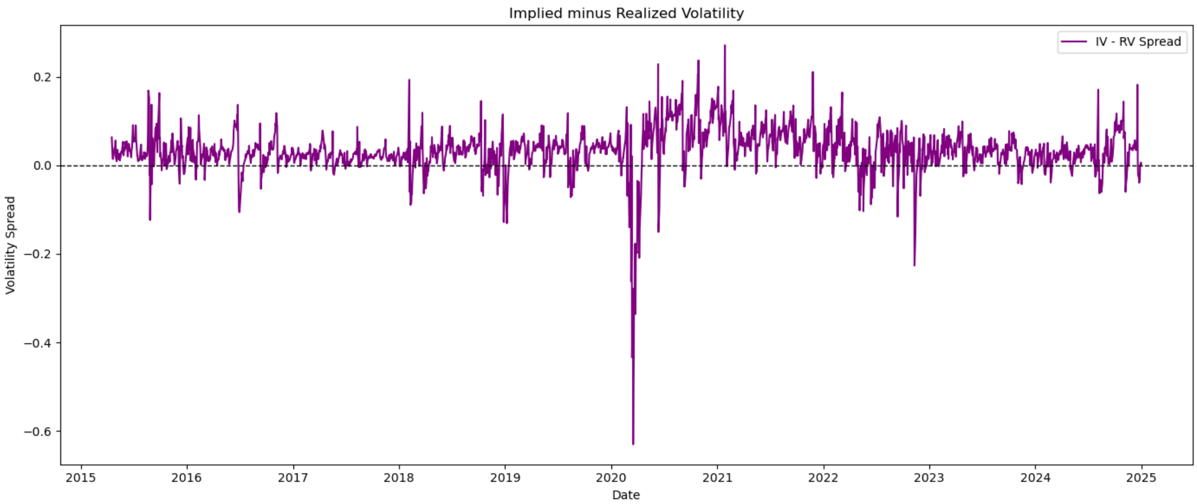
- **GARCH-IV Only:**
  - High $R^2 = 0.754$ indicates strong explanatory power.
  - Slope $\beta_1 = 0.9321$ suggests a near one-to-one relationship between GARCH-implied volatility and realized volatility.
  - Implies that GARCH-IV is a strong predictor of future realized volatility.
- **Heston-IV Only:**
  - Very low $R^2 = 0.018$ indicates poor fit.
  - Slope $\beta_1 = 0.4957$ shows a weaker relationship with realized volatility.
  - Suggests Heston-IV alone does not capture much of the variation in realized volatility.
- **Combined GARCH + Heston IV:**
  - Slightly improved $R^2 = 0.755$, marginally better than using GARCH-IV alone.
  - Slope for GARCH-IV remains high , while Heston-IV contributes modestly.
  - Indicates that Heston-IV adds some incremental value when combined with GARCH-IV.

Conclusion

GARCH-IV is the dominant predictor of realized volatility. While Heston-IV has limited standalone predictive power, it contributes marginally when used in conjunction with GARCH-IV.

6 Trading Strategy: IV-RV Spread Analysis

Discrepancies between Implied Volatility (IV) and Realized Volatility (RV) can provide trading signals. This section tests strategies that exploit these gaps.



Observations

Key Insights from IV-RV Spread Analysis

- **Positive Bias (Volatility Risk Premium):** IV generally exceeds RV, reflecting a consistent volatility risk premium in the market.
- **COVID-19 Shock (Mar 2020):** A sharp negative spike indicates realized volatility far surpassed implied volatility, highlighting market underestimation during crises.
- **Post-2020 Volatility:** Increased fluctuation in the spread reflects greater market uncertainty and potential model limitations.
- **Recent Stability (2023–2025):** The spread has stabilized at moderate positive levels, suggesting improved market calibration.

Conclusion

The IV-RV spread analysis confirms the presence of a volatility risk premium, with occasional sharp deviations during systemic events. Understanding this spread aids in volatility forecasting, risk management, and option pricing strategies.

Strategies

- If the spread is positive and the M.IV is greater than the RV, then we resort to a strategy known as **Long Straddle**
- If the spread is negative and the M.IV is less than the RV, then we resort to a strategy known as **Short Straddle**

7 Results & Conclusion

- The GARCH model yields smaller average forecasting errors, indicating higher predictive accuracy.
- A high positive correlation (approximately 0.85) between GARCH-implied volatility and the VIX suggests that the GARCH model effectively captures directional movements in market-implied volatility.
- These findings support the use of GARCH as a reliable proxy for VIX-based volatility expectations.
- The Heston model shows larger deviations from the VIX, making it less accurate in forecasting market-implied volatility.
- The lack of a consistent linear relationship with the VIX suggests that the Heston model fails to align with market expectations.

Final Takeaway

- **GARCH** is a strong performer in mimicking the VIX and can be considered for applications requiring implied volatility estimation.
- **Heston** model, as implemented here, does not align well with market-implied volatility, and may require better calibration techniques or integration with option price data.

8 Appendix

Model Implied Volatility (M.IV): The volatility that is predicted by our model (GARCH and HESTON)

Realized Volatility (RV): The volatility that is calculated from the dataset using the formula:

$$RV = \sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(r_{t-i} - \bar{r})^2} \tag{6}$$

Spread: The difference between the model implied volatility and the realized volatility.

Trading Strategy: The strategy that is used to trade based on the spread between the model implied volatility and the realized volatility.

Short Straddle: A trading strategy that involves sell a call and a put option with the same strike price and expiration date. This strategy profits from large price movements in either direction.

$$Profit = \text{Premium from Call} + \text{Premium from Put} - \text{Loss from the options} \tag{7}$$

Explantion:

- If the spread is negative then the volatility has been predicted to be lesser than the realized volatility. This means that the market is underestimating the volatility and hence we can profit from this by selling a call and a put option with the same strike price and expiration date.
- In this case the price movement of the market will not be that high which means the Loss term in the profit equation is very small.
- Hence allowing us to profit from the premiums of the call and put options.

Long Straddle: A trading strategy that involves buy a call and a put option with the same strike price and expiration date. This strategy profits from large price movements in either direction.

$$Profit = \text{Profit from the Option} - \text{Premium from Call} - \text{Premium from Put} \tag{8}$$

Explantion:

- If the spread is positive then the volatility has been predicted to be greater than the realized volatility. This means that the market is overestimating the volatility and hence we can profit from this by buying a call and a put option with the same strike price and expiration date.
- In this case the price movement of the market will be very high which means the Profit term in the profit equation is very high.
- Hence allowing us to profit from the options we bought.

References

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