Tasks for 31/10/2022 and 07/11/2022:

(Problems marked with '*' needs to be attended during the class and later submitted as assignments. Other problems are practice problems.)

* **FMD 1:** Solve the 1st order ODE $y' + 2y + 3 = 0$ with both implicit and explicit Euler method with I.C. *y(0)=1* and compare with the analytical solution:

$y(t) = \frac{5}{2}e^{-2t} - \frac{3}{2}$.

By plotting analytical solutions for different values of Δt, show that the explicit method becomes unstable for $\Delta t > 0.5$ whereas implicit method is unconditionally stable.

FDM 2: Solve the following 2nd order ODE

$$\frac{d^2y}{dx^2} + 0.5\frac{dy}{dx} + 4y = 5$$

Initial condition: $y(0) = y'(0) = 0$;

by applying finite difference method for the x-range of $0 \leq x \leq 10$.

a) Determine a suitable h value from the stability criteria for oscillatory solution, *i.e.* $h < \frac{2\sqrt{\omega_0^2 - \alpha^2}}{\omega_0^2}$

b) Show that the solution becomes unstable, i.e. leads to growing instead of decaying oscillation for $h > \frac{2\sqrt{\omega_0^2 - \alpha^2}}{\omega_0^2}$.

c) Plot y(x) alongside the analytical solution and the one obtained from R-K 2nd order method for different h values for $h < 0.5$.

1D heat equation: Explicit form

$$\frac{\partial u(x,t)}{\partial t} = \alpha\frac{\partial^2 u(x,t)}{\partial x^2}$$

α is the constant for thermal conductivity. The equation may be rearranged in the discrete form by assuming forward difference in time and central difference in space as:

$$u(x, t + k) = \sigma u(x - h, t) + (1 - 2\sigma)u(x, t) + \sigma u(x - h, t)$$

$\sigma = \frac{\alpha k}{h^2}$, h *and* k are increments in *x* and *t*, respectively.

If we replace the variables with indices *i* for *x* and *n* for *t*, we have:

$$u(i, n + 1) = \sigma u(i - 1, n) + (1 - 2\sigma)u(i, n) + \sigma u(i - 1, n)$$

This is the **explicit form expression for heat equation which provides stable solution for $\sigma \leq 0.5$.** This can be put in form of a matrix equation as

$$u^{k+1}(x, t) = A(\sigma)u^k(x, t)$$

Where the index *k* is for k^{th} iteration and the matrix $A(\sigma)$ is a sparse matrix known as the propagator.

Heat equation has to be solved subject to the boundary condition: $u_{|x=x_1,t} = u_1$, $u_{|x=x_{2,t}} = u_2$ and initial condition of the form $u(x_n, t = 0) = u_n^o$. The propagation of heat profile with time may be monitored.

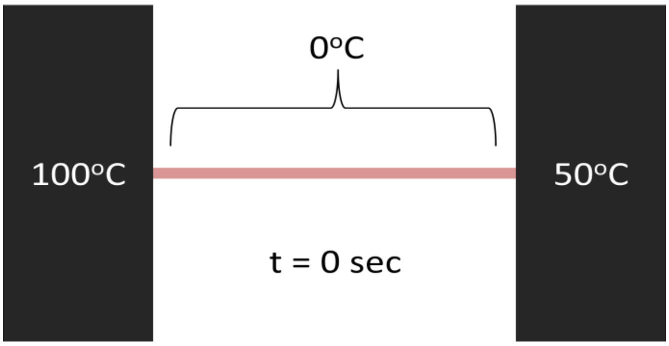
***Heat Equation 1:** Consider the system in which a thin rod of length L=10cm is placed between two heat reservoirs kept at 100 and 50°C, respectively. The initial temperature of the rod is 0°C. Write a code to compute heat evolution for 1^{st} 100sec in the rod using **explicit method** of solving heat equation. Plot the result

1) In a 2D contour or surface plot
2) And in animated version

for two values of σ (=κΔt/Δx^2) greater and less than 0.5.

Initial condition: u(0<x<L, t=0) = 0°C

Boundary condition: u(0,t)= 100°C and u(50,t) = 50°C at all t



1D heat Equation: Implicit form

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

If we use forward difference in time and central difference in space at time t+k, we get:

$$u(x,t) = -\sigma u(x-h, t+k) + (1+2\sigma)u(x, t+k) - \sigma u(x+h, t+k)$$

$\sigma = \frac{\alpha k}{h^2}, h\ and\ k$ are increments in *x* and *t*, respectively.

If we replace the variables with indices *i* for *x* and *n* for *t*, we have:

$$u(i,n) = -\sigma u(i-1, n+1) + (1+2\sigma)u(i, n+1) - \sigma u(i+1, n+1)$$

This is the **implicit form for heat equation and this form is unconditionally stable.**

A matrix representation for implicit form is

$$u(x,t) = A(\sigma)u(x, t+k)$$

Heat equation has to be solved subject to the boundary condition: $u_{|x=x_1,t} = u_1$, $u_{|x=x_{2,t}} = u_2$ and initial condition of the form $u(x_n, t=0) = u_n^o$. The propagation of heat profile with time may be monitored.

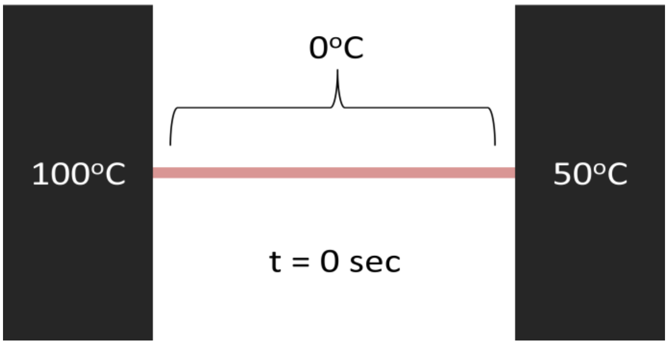
***Heat Equation 2:** Consider the system in which a thin rod of length L=10cm is placed between two heat reservoirs kept at 100 and 50^{o}C, respectively. The initial temperature of the rod is 0^{o}C. Write a code to compute heat evolution for 1st 100sec in the rod using **implicit method** of solving heat equation. Plot the result

1) In a 2D contour or surface plot
2) And in animated version

for two values of σ (=$\kappa\Delta t/\Delta x$^2) greater and less than 0.5.

Initial condition: $u(0<x<L, t=0) = 0^{o}C$

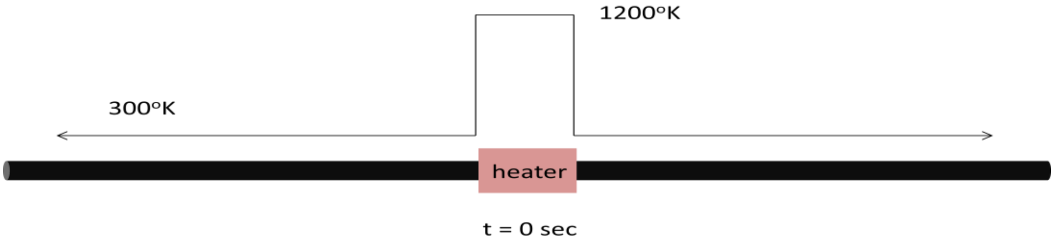
Boundary condition: $u(0,t)= 100^{o}C$ and $u(50,t) = 50^{o}C$ at all t



Heat equation 3: Consider the system in which an *l*=10 cm heating coil is placed at the centre of a long thin rod of length L=100cm. The initial temperature of the rod is 300K at that of the hater is 1200K. The heater is always on so that the temperature at the central part stays constant. The temperature of the end point does not change. Write a code to compute heat evolution for 1st 100sec in the rod using **implicit method** of solving heat equation. Plot the result in a real time animation for two values of $\sigma>0.5$ and $\sigma<0.5$.

I.C.: $u(-l/2<x<l/2, t=0) = 1200K$, $u(x, t=0)=300K$ for other x

B.C.: $u(-L/2,t)= u(L/2,t) = 300K$, $u(-l/2<x<l/2) = 1200K$ at all t



***Laplace equation:** Write a program to solve the two dimensional Laplace equation $T_{xx} + T_{yy} = 0$ describing the steady state temperature distribution on a **square plate of sides L=100 cm**. Use same length for x- and y- increment. Show the temperature distribution T(x, y) using a surface plot for an x-y grid of minimum 20×20 segments. Vary number of greed points to **fill the table below**.

The boundary conditions are:
T(x = 0) = T(x = L) = 0° C, T(y = 0) = -100°, T(y = L) = 100° at all time, corner points are assumed to have T equals to the average of adjoining sites.

Δx= ; Δy= ; error (ε) = ;

Sr. no.	No. of grid segments	No. of iterations to converge	Total no. of calculations
1.			
2.			
3.			
4.			
5.			
6.			
7.