Path Integral Formulation Of A Free Particle

Term Paper
Classical Mechanics(PH31207)

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Abstract

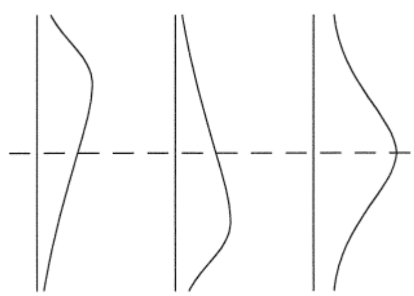
In this term paper, we try to understand the Path integral formulation based on the topics we learned in this course. We start with a double-slit experiment to understand the probabilistic behavior and the need for kernel description. We then follow the formulation of the path integral and try to explicitly solve the kernel of a free particle.

Double Slit experiment:

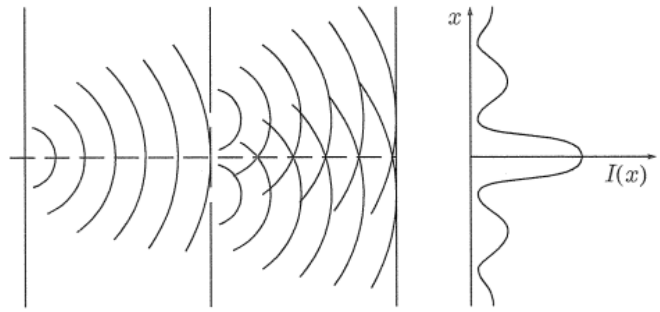
At atomic scales, the particles obey a different set of rules of probabilities, they are not just the simple addition of probabilities of the possible events. The double slit experiment is one such experiment that displays the quantum nature of the electrons. The electrons are ejected from source S with the same energy in all directions onto a screen with a window of two slits placed in between.

A single detector that is placed on the screen observes that the current that is produced at point x on the screen is not continuous but is proportional to the number of particles arriving at it through the path connecting the source to the point. Rather one can arrange all such sensitive detectors throughout the screen and observe that each detector clicks only when one single electron reaches it.

Based on classical ideas, the probability of an electron reaching a point x on the screen is the sum of the probabilities of P1 and P2 where P1 is the probability of the electron reaching the point x through slit 1 and P2 is the probability of the electron reaching the point through the slit 2.

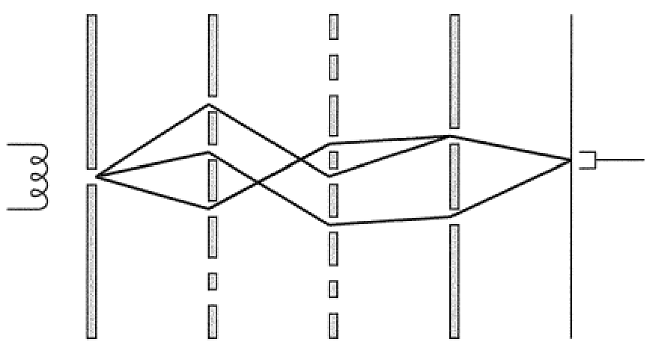


Rather to our surprise, what we expect the pattern to be and the pattern observed in the experiment was different. The pattern produced on the screen was the interference pattern. There were continuous maxima and minima observed on the screen. The probabilities weren't just the addition, it's the sum of the amplitudes square for the different paths. We needed to find the probability amplitudes that were associated with possible paths in order to have information about the whole kernel.



Kernel Description

The amplitude for an event is the sum of the amplitudes for the various alternative ways that the event can occur. The amplitude of moving from point a to b is given by the algebraic sum of amplitudes of individual paths. A simple space is analogous to infinite window slits. An electron moving through the free space can be thought of as moving through the screens with different slits placed at different distances. The position at any time can be specified by a coordinate 'x' as a function 't'. The path can be represented by a function x(t). If a particle starts at point x_a at time t_a and reaches point x_b at time t_b then the kernel of the path is defined by $K(b, a)$.



Contrary to the classical case of a particle following the trajectory with the action being maximized, the kernel consists of the contributions from all other paths. These individual trajectories contribute in equal magnitude but with different phases. These phases are proportional to the associated action S for their trajectories measured in the units of the quantum of action $\hbar$.

$$Phase = \frac{S[x(t)]}{\hbar}$$

The kernel is given by

$$K(b,a) = \Sigma\Phi[x(t)]$$

$$\Phi[x(t)] = ce^{\frac{S[x(t)]}{\hbar}}$$

$$\Phi[x(t)] = cons.\left[cos\frac{S[x(t)]}{\hbar} + isin\frac{S[x(t)]}{\hbar}\right]$$

By comparing the real values of amplitude for different values of S we get that they continuously oscillate between positive and negative values.

$$cos\left[\frac{a}{1.055 \times 10{-}34}\right]$$

Sno.	a	phase
1	0.1	0.671
2	0.2	-0.097
3	0.3	0.992
4	0.4	-0.982

From K(b,a)= $\Sigma\phi$, it is clear that only some particular paths becomes more important. As the real part of ϕ is cosine, it is equally likely to be positive and negative. When we vary the actual trajectory by a small quantity $\delta(dx)$ which is small on the classical scale, the change in S would also be small on the classical scale but not on the scale relative to $\hbar$. The small changes in the trajectory will make enormous changes in phase. The cosine/ sine parts would oscillate rapidly between positive and negative values. The total contribution would be added up to zero. When one path produces a positive real value, an infinitesimally close trajectory would make an equal negative real value resulting in no net contribution.

If one path say produces a positive cosine value, then a small change $\delta(dx)$ leads to a new path say which produces a negative change.

But this scenario is different for classical action. If the classical trajectory is slightly disturbed by $\delta(dx)$ in scales of$\hbar$ then action wouldn't vary and there will be no variation in the phase. So, they simply get added up.

"Only the paths closer to (t) (classical trajectory) will contribute the most"

When S is very large (classical) only the classical paths neighboring the classical trajectory contribute and the rest of the farther paths are canceled.

The approximation of classical physics that only the paths (t) need to be considered is valid when action is very large as compared to $\hbar$ i.e; classical mechanics.

All the contributions from the paths in the neighborhood region are nearly in the same phase $\frac{S_c l}{\hbar}$ and do not cancel out. "This is how classical laws of motion arise from quantum laws".

The phase of the Φ will also be changed when the endpoints (x_b, t_b) are varied slightly as $\delta_c l$ i.e.; K(b,a) will vary rapidly. To overcome this we can add a "smooth function".

$$K(b,a) = "smoothfunction".e^{i}\frac{\delta_c l}{\hbar}$$

Euler-Lagrange Equation:

$$S = \int_{t_a}^{t_b} L\left(\dot{x}, x, t\right)\, dt$$

$$\Rightarrow \delta S = S[\bar{x} + \delta x] - S[\bar{x}] = 0$$

$$\Rightarrow S[x + \delta x] = \int_{t_a}^{t_b} L(\dot{x} + \delta x, x + \delta x, t)\, dt$$

$$\Rightarrow S[x + \delta x] = \int_{t_a}^{t_b} \left[L(\dot{x}, x, t) + \delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x\frac{\partial L}{\partial x}\right]\, dt$$

$$\Rightarrow S[x + \delta x] = S[x] + \int_{t_a}^{t_b} \left[\delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x\frac{\partial L}{\partial x}\right]\, dt$$

$$\Rightarrow S[x+\delta x]-S[x]=\int_{t_a}^{t_b}\left[\delta\dot{x}\frac{\partial L}{\partial \dot{x}}+\delta x\frac{\partial L}{\partial x}\right]\,dt$$

$$\Rightarrow S[x+\delta x]-S[x]=\int_{t_a}^{t_b}\delta x\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}\right]\,dt$$

$$\Rightarrow \frac{S[x+\delta x]-S[x]}{\delta x}=\int_{t_a}^{t_b}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}\right]\,dt=0$$

$$\left[\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0\right]$$

<u>Deriving Path Integral For a Free Particle</u>

$$L=\frac{1}{2}m\dot{x}^2$$

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0\Rightarrow \ddot{x}=0\Rightarrow \bar{x}=kt+c$$

$$\Rightarrow S=\int_{t_a}^{t_b}Ldt=\int_{t_a}^{t_b}\frac{1}{2}mk^2dt$$

$$\Rightarrow S=\frac{1}{2}mk^2(t_b-t_a)$$

$$\Rightarrow S=\frac{1}{2}m\frac{(x_b-x_a)^2}{t_b-t_a}$$

Let the normalization constant be A and it needs to depend on ϵ . so A is function of ϵ i.e., A(ϵ)

$$\Rightarrow K(b,a)=\lim_{\epsilon\to 0}\frac{1}{A(\epsilon)}\int\frac{dx_1}{A(\epsilon)}.\frac{dx_2}{A(\epsilon)}...\frac{dx_{n-1}}{A(\epsilon)}.e^{\frac{im}{2\hbar\epsilon}\left[(x_1-x_a)^2+(x_2-x_1)^2+...+(x_b-x_{N-1})^2\right]}$$

We determine A(ϵ) to be:

$$A=\sqrt{\frac{2\pi\epsilon\hbar i}{m}}$$

$$\Rightarrow K(b,a)=\lim_{\epsilon\to 0}\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}}\int dx_1.dx_2...dx_{N-1}.e^{\frac{im}{2\hbar\epsilon}\left[(x_1-x_a)^2+(x_2-x_1)^2+...+(x_b-x_{N-1})^2\right]}$$

Integrating on x_1 i.e., dx_1

$$\int dx_1 . e^{\frac{im}{2\epsilon\hbar}\left[(x_1-x_a)^2+(x_2-x_1)^2\right]} = \int dx_1 . e^{\frac{im}{2\epsilon\hbar}\left[2x_1^2-2x_1(x_a-x_2)+x_a^2+x_2^2\right]}$$

$$= \int dx_1 . e^{\frac{im}{2\epsilon\hbar}\left[2\left[x_1^2-2x_1\left(\frac{x_a+x_2}{2}\right)+\frac{(x_a+x_2)^2}{4}\right]+x_a^2+x_2^2-\frac{(x_a+x_2)^2}{2}\right]}$$

$$= \int dx_1 . e^{\frac{im}{2\epsilon\hbar}\left[2\left(x_1-\left(\frac{x_a+x_2}{2}\right)^2\right)\right]} . \int e^{\frac{im}{2\epsilon\hbar}\left[\frac{2x_a^2+2x_2^2-x_a^2-x_2^2-2x_ax_2}{2}\right]}$$

$$\Rightarrow K(b,a) = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} \frac{1}{A_n}\sqrt{\frac{i\pi\epsilon\hbar}{im}} \int e^{\frac{-im}{i4\epsilon\hbar}\left[(x_a-x_2)^2\right]} dx_2 . dx_3 ... dx_{N-1} . e^{\frac{im}{2\hbar\epsilon}\left[(x_3-x_2)^2+...+(x_b-x_{N-1})^2\right]}$$

Integrating about x_2 i.e.; dx_2

$$\int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a-x_2)^2}{2}+(x_3-x_2)^2\right]} . dx_2 = \int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a)^2+3(x_2)^2+2(x_3)^2+2.x_a.x_2-4x_2x_3}{2}\right]} . dx_2$$

$$= \int e^{\frac{-im}{4\hbar\epsilon}\left[3(x_2-\frac{x_a+2x_3}{3})^2\right]} dx_2 . e^{\frac{im}{4\hbar\epsilon}\left[\frac{2(x_a)^2+2(x_3)^2-4x_ax_3}{3}\right]}$$

$$= \sqrt{\frac{4\pi\epsilon\hbar i}{3m}} . e^{\frac{im}{4\hbar\epsilon}\left[\frac{2(x_a)^2+2(x_3)^2-4x_ax_3}{3}\right]}$$

$$= \sqrt{\frac{4\pi\epsilon\hbar i}{3m}} . e^{\frac{im}{6\hbar\epsilon}[x_3-x_a]^2}$$

$$\Rightarrow K(b,a) = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} \frac{1}{A^n}\sqrt{\frac{2\pi\epsilon\hbar i}{2m}} . \sqrt{\frac{4\pi\epsilon\hbar i}{3m}} . \int e^{\frac{im}{6\hbar\epsilon}[x_3-x_a]^2} . e^{\frac{im}{2\hbar\epsilon}\left[(x_1-x_4)^2+...+(x_b-x_{N-1})^2\right]} . dx_3 ... dx_{N-1}$$

integrating with respect to dx_3

$$= \int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a-x_3)^2}{3}+(x_3-x_4)^2\right]} . dx_3 = \int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a)^2+3(x_4)^2+4(x_3)^2-2.x_a.x_3-6x_4x_3}{3}\right]} . dx_3$$

$$= \int e^{\frac{-4im}{6\hbar\epsilon}\left[(x_3 - \frac{x_a+3x_4}{4})^2\right]} dx_3 . e^{\frac{-im}{24\hbar\epsilon}\left[3(x_a)^2+3(x_4)^2-6x_a x_4\right]}$$

$$= \sqrt{\frac{6\pi\epsilon\hbar i}{4m}} . e^{\frac{im}{8\hbar\epsilon}\left[(x_a)^2+(x_4)^2-2x_a x_4\right]}$$

$$= \sqrt{\frac{6\pi\epsilon\hbar i}{4m}} . e^{\frac{im}{8\hbar\epsilon}[x_4-x_a]^2}$$

$$\Rightarrow K(b,a) = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} \frac{1}{A^n}\sqrt{\frac{2\pi\epsilon\hbar i}{2m}}.\sqrt{\frac{4\pi\epsilon\hbar i}{3m}}.\sqrt{\frac{6\pi\epsilon\hbar i}{4m}}.$$
$$\int e^{\frac{im}{8\hbar\epsilon}[x_4-x_a]^2} . e^{\frac{im}{2\hbar\epsilon}\left[(x_4-x_5)^2+...+(x_b-x_{N-1})^2\right]} .dx_4...dx_{N-1}$$

integrating with respect to dx_4

$$\int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a-x_4)^2}{4}+(x_4-x_5)^2\right]} .dx_4 = \int e^{\frac{im}{2\hbar\epsilon}\left[\frac{(x_a)^2+5(x_4)^2+4(x_5)^2-2.x_a.x_4-8x_4x_5}{4}\right]} .dx_4$$

$$= \int e^{\frac{-5im}{8\hbar\epsilon}\left[(x_4 - \frac{x_a+4x_5}{5})^2\right]} dx_4 . e^{\frac{im}{40\hbar\epsilon}\left[4(x_a)^2+4(x_5)^2-8x_a x_5\right]}$$

$$= \sqrt{\frac{8\pi\epsilon\hbar i}{5m}} . e^{\frac{im}{10\hbar\epsilon}\left[(x_a)^2+(x_5)^2-2x_a x_5\right]}$$

$$= \sqrt{\frac{8\pi\epsilon\hbar i}{5m}} . e^{\frac{im}{10\hbar\epsilon}[x_5-x_a]^2}$$

$$\Rightarrow K(b,a) = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} \frac{1}{A^n}\sqrt{\frac{2\pi\epsilon\hbar i}{2m}}.\sqrt{\frac{4\pi\epsilon\hbar i}{3m}}.\sqrt{\frac{6\pi\epsilon\hbar i}{4m}}.\sqrt{\frac{8\pi\epsilon\hbar i}{5m}}.$$
$$\int e^{\frac{im}{10\hbar\epsilon}[x_5-x_a]^2} . e^{\frac{im}{2\hbar\epsilon}\left[(x_6-x_5)^2+...+(x_b-x_{N-1})^2\right]} .dx_5...dx_{N-1}$$

From above one can clearly see a sequence i.e.,

$$\left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} . \left(\frac{\pi\epsilon\hbar i}{m}\right)^{\frac{N-1}{2}} . \left(\frac{2}{2}.\frac{2*2}{3}.\frac{2*4}{5}.\frac{2*5}{6}.......\frac{2*N-1}{N}\right)^{\frac{1}{2}}$$

$$= \left(\frac{m}{2\pi\epsilon\hbar i}\right)^{\frac{N}{2}} . \left(\frac{\pi\epsilon\hbar i}{m}\right)^{\frac{N-1}{2}} . \left(\frac{2^{-\frac{1}{2}}}{N^{\frac{1}{2}}}\right)$$

$$= \left[\frac{m}{2\pi i\hbar . N\epsilon}\right]^{\frac{1}{2}} . e^{\frac{im}{2\hbar N\epsilon}[x_n - x_0]^2}$$

$$K(b,a) = \left[\frac{m}{2\pi i\hbar . N\epsilon}\right]^{\frac{1}{2}} . e^{\frac{im}{2\hbar N\epsilon}[x_b - x_a]^2}$$

This is the kernel of the free particle.

Conclusion:

We obtained the kernel of a free particle through which the quantum properties can be derived from the formulation of the wave function from the kernel description. Feynman defined it as below,

$$\Psi(x_b, t_b) = \int_{-\infty}^{\infty} K(x_b, t_b; x_a, t_a)\psi(x_a, t_a)dx_a$$

References:

1)https://www.damtp.cam.ac.uk/user/tong/concepts/action.pdf
2)http://www-f1.ijs.si/ ramsak/km1/FeynmanHibbs.pdf