Project Report

Intermediate Axis Theorem

Dhatri Raghunathan
20PH20009

Rohan Sidankar
20PH20038

Mohammad Arshad
20PH20022

DEPARTMENT OF PHYSICS
CLASSICAL DYNAMICS AND SPECIAL RELATIVITY COURSE

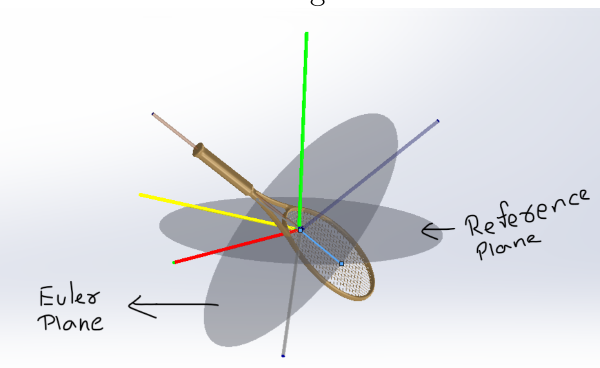
Prof. Krishna Kumar

Contents

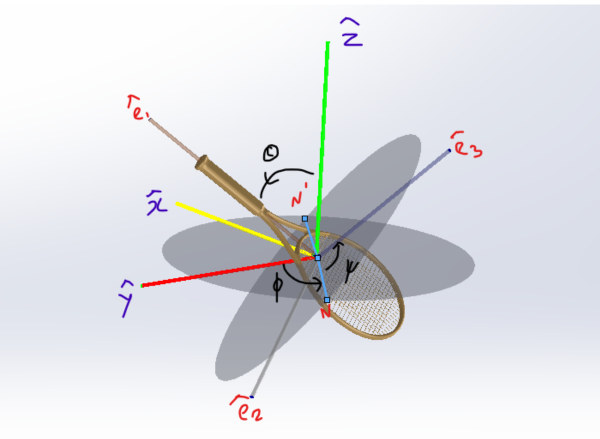
1 Introduction

The Tennis Racket Effect or Intermediate Axis Theorem is a result of classical mechanics describing the movement of a rigid body with three distinct principal moments of inertia. It is also dubbed the Dzhanibekov Effect, after Soviet cosmonaut Vladimir Dzhanibekov. Its name originates from the fact that, it can be easily observed in a standard tennis racket. This work focuses on the unified analytical and geometrical description of TRE along with a few real-life applications.

Representation of Euler angles:
The reference plane is the usual XY plane with Z-axis being the normal to it. For ease in describing the movement of the bat we choose both the plane of bat and Euler plane different as shown in figure.



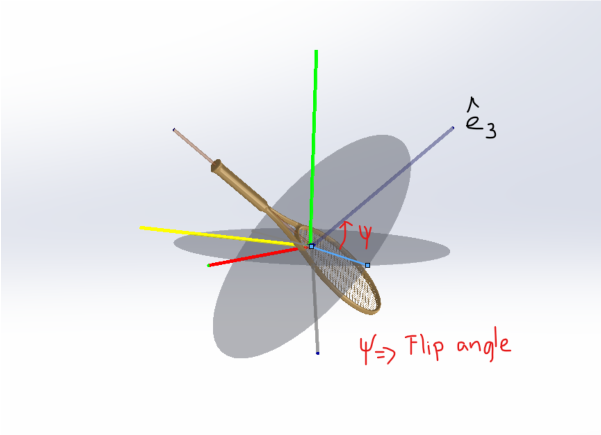
We define three Euler angles θ, Φ, Ψ.The angle Φ denotes the rotation about Z-axis, angle Ψ denotes the flip angle, θ denotes the inclination about Z-axis.The $\hat{e}_3$ and $\hat{e}_2$ are in the Euler plane and $\hat{e}_3$ is perpendicular to it.

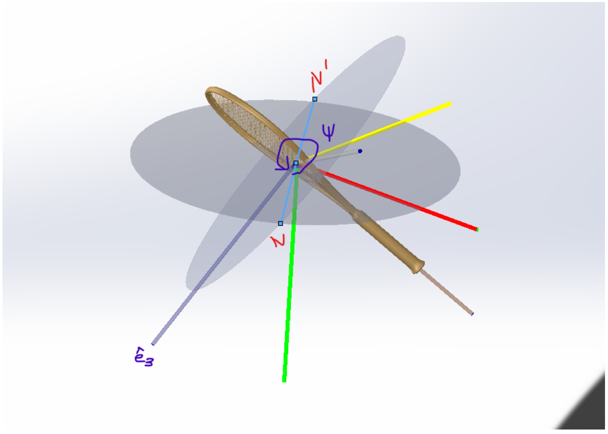


Most of the further discussion will be on the Flip angle as TRE states that for a 2π change in the Φ there will be a change in Ψ of π when there is a rotation about the intermediate axis.

$$\Delta\Psi = \pi, for \Delta\Phi = 2\pi$$

A visualization of the flipping angle Ψ showing the flipping of badminton is given below:





This flipping occurs in any body which consist 3 different moment of inertia $I_1 I_2 I_3$ and with one being the maximum moment of inertia and the other being the lowest moment of inertia and the intermediate moment of inertia.

$$I_1 < I_2 < I_3$$

The further discussions are based on the Asymmetric top assumptions $I_1 \neq I_2 \neq I_3$.

2 Rotational stability about e_1, e_2 and e_3 axes

Rotation about $\hat{e_1}$ axis:
Let us assume initial angular velocity is given only about $\hat{e_1}$ axis.i.e.;

$$\Omega = \Omega_1 \hat{e_1}$$

Due to the small pertubation , the angular velocities about the other two axes are also generated.So, Ω becomes

$$\Omega = \Omega_1 \hat{e_1} + \lambda \hat{e_2} + \mu \hat{e_3}$$

Euler equations:

$$\frac{d\Omega_1}{dt} = \dot{\Omega}_1 = \frac{I_2 - I_3}{I_1} \lambda \mu$$
$$\frac{d\lambda}{dt} = \dot{\lambda} = \frac{I_3 - I_1}{I_2} \Omega_1 \mu$$
$$\frac{d\mu}{dt} = \dot{\mu} = \frac{I_1 - I_2}{I_3} \Omega_1 \lambda$$

This gives the equations,

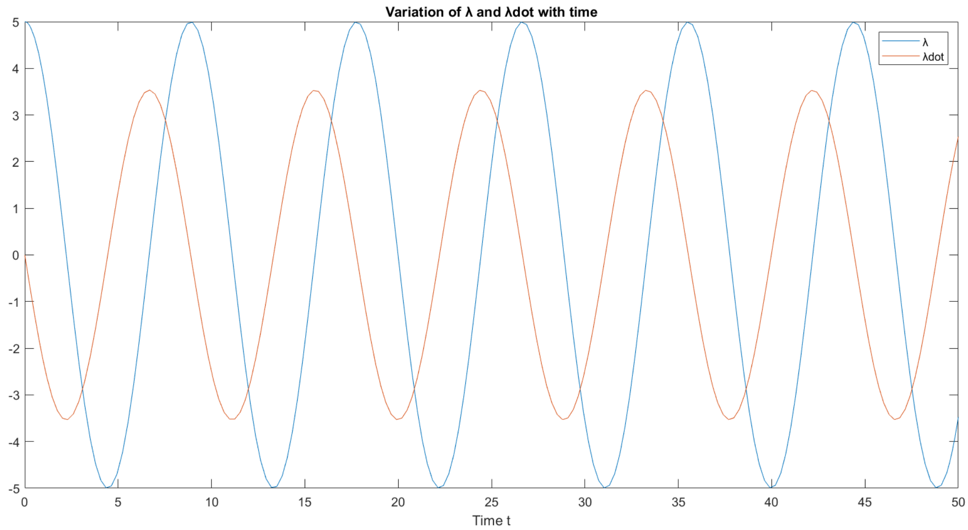
$$\dot{\lambda} = [\frac{I_3 - I_1}{I_2} \Omega_1] \mu$$

$$\dot{\mu} = [\frac{I_1 - I_2}{I_3} \Omega_1] \lambda$$

$$\ddot{\lambda} = (\frac{I_3 - I_1}{I_2})\dot{\Omega}_1\mu + (\frac{I_3 - I_1}{I_2})(\Omega_1\lambda\Omega_1)$$
$$\ddot{\lambda} = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\lambda\Omega_1^2$$
$$\ddot{\lambda} = -[\frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}\Omega_1^2]\lambda$$

$$\boxed{\lambda = Acos(kt + \alpha)}$$

This is a sinusoidal solution.



The body oscillates sinusoidally about its initial state with the angular frequency k. So the body is stable with respect to small perturbations when rotating about $\hat{e}_1$ axis. The perturbations do not grow with time.

$$\mu = \frac{\dot{\lambda}}{(\frac{I_3 - I_1}{I_2})\Omega_1}$$
$$= \frac{-\lambda_0 ksin(kt + \alpha)}{(\frac{I_3 - I_1}{I_2})\Omega_1}$$
$$= \mu_0 sin(kt + \alpha)$$

$$\mu = \mu_0 sin(kt + \alpha)$$
$$\lambda = \lambda_0 cos(kt + \alpha)$$

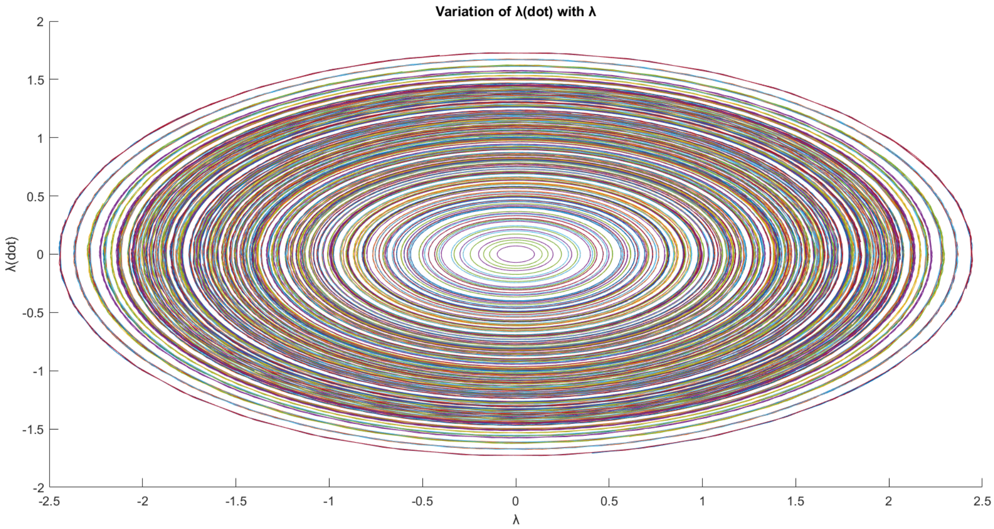
$$\begin{pmatrix} \lambda(t) \\ \mu(t) \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \mu_0 \end{pmatrix} \begin{pmatrix} e^{+ikt} & e^{-ikt} \end{pmatrix}$$

s=$\pm ik$

s is imaginary. The perturbations neither grow nor decay.

$$\boxed{\dot{\lambda} = -Aksin(kt + \alpha)}$$

Below is the phase potrait for the variation of $\dot{\lambda}$ with λ. As expected, (0,0) is the center and the graph is elliptical.



Rotation about $\hat{e_2}$ axis

Let us assume initial angular velocity is given only about $\hat{e_2}$ axis.

$\Omega = \Omega_2 \hat{e_2}$

After pertubation, Ω becomes

$\Omega = \lambda \hat{e_1} + \Omega_2 \hat{e_2} + \mu \hat{e_3}$

Euler equations:

$$\frac{d\lambda}{dt} = \dot{\lambda} = \frac{I_2 - I_3}{I_1}\Omega_2 \mu = -\frac{I_3 - I_2}{I_1}\Omega_2 \mu$$

$$\frac{d\Omega_2}{dt} = \dot{\Omega_2} = \frac{I_3 - I_1}{I_2}\lambda \mu$$

$$\frac{d\mu}{dt} = \dot{\mu} = \frac{I_1 - I_2}{I_3}\Omega_2 \lambda = -\frac{I_2 - I_1}{I_3}\Omega_2 \lambda$$

This gives the equations

$$\dot{\lambda} = -\frac{I_3 - I_2}{I_1}\Omega_2 \mu$$

$$\dot{\mu} = -\frac{I_2 - I_1}{I_3}\Omega_2 \lambda$$

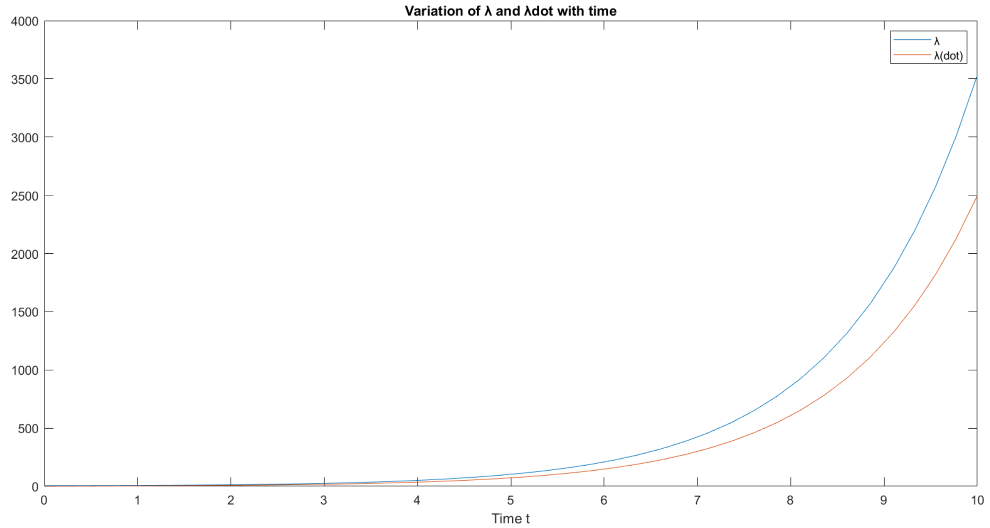
$$\ddot{\lambda} = -(\frac{I_3 - I_2}{I_1})\dot{\Omega_2}\mu + (\frac{I_3 - I_2}{I_1})(\Omega_2 \lambda \Omega_2)(\frac{I_2 - I_1}{I_3})$$

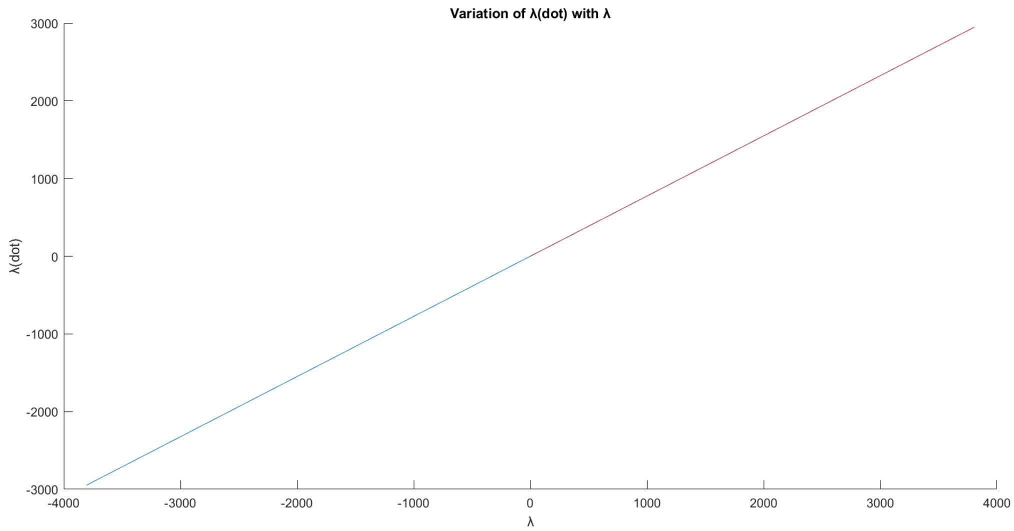
$$\ddot{\lambda} = [(\frac{I_3 - I_2}{I_1})(\frac{I_2 - I_1}{I_3})\Omega_2^2]\lambda$$

$$\boxed{\lambda = Ae^{kt} + Be^{-kt})}$$

$$\boxed{\mu = Ce^{kt} + De^{-kt})}$$

The perturbations consist exponential terms. They grow exponentially in-time, as shown in the graph below. Hence rotation about $\hat{e_2}$ axis is unstable.





Just like $\hat{e_1}$, the rotation about $\hat{e_3}$ is stable.
Hence, for a body, the rotations are stable about the minimum inertia axis and maximum inertia axis.

"The rotations are unstable about intermediate inertia axis."

3 Tennis Racket Effect

Let us assume a total angular momentum of M is given to the system. This could be in any arbitrary direction, but for ease in mathematical calculations let us take it along the Z-axis. The $\vec{M}$ can be represented along the $\vec{e_1}, \vec{e_2}, \vec{e_3}$ with a dependence on θ and Ψ.

$$M_1 = Mcos\theta,\ M_2 = -Msin\theta cos\Psi,\ M_3 = Msin\theta sin\Psi$$

The corresponding angular velocities $\vec{\Omega_1}, \vec{\Omega_2}, \vec{\Omega_3}$ are along $\vec{e_1}, \vec{e_2}, \vec{e_3}$ and given by

$$\Omega_1 = \frac{d\Psi}{dt} + \frac{d\Phi}{dt}cos\theta$$
$$\Omega_2 = \frac{d\theta}{dt}sin\Psi - \frac{d\Phi}{dt}sin\theta cos\Psi$$
$$\Omega_3 = \frac{d\theta}{dt}cos\Psi + \frac{d\Phi}{dt}sin\theta sin\Psi$$

From $\vec{M} = I\vec{\Omega}$ we can further write

$$M_1 = I_1(\frac{d\Psi}{dt} + \frac{d\Phi}{dt}cos\theta)$$
$$M_2 = I_2(\frac{d\theta}{dt}sin\Psi - \frac{d\Phi}{dt}sin\theta cos\Psi)$$
$$M_3 = I_3(\frac{d\theta}{dt}cos\Psi + \frac{d\Phi}{dt}sin\theta sin\Psi)$$

By solving for $\frac{d\theta}{dt}$ by eliminating the $\frac{d\Phi}{dt}$ term we get

$$\frac{d\theta}{dt} = M[\frac{1}{I_3} - \frac{1}{I_2}]sin\theta cos\Psi sin\Psi$$

And by re-substituting for $\frac{d\Phi}{dt}$ we get

$$\frac{d\Phi}{dt} = M[\frac{sin^2\Psi}{I_3} + \frac{cos^2\Psi}{I_2}]$$

Further solving for $\frac{d\Psi}{dt}$ we get

$$\frac{d\Psi}{dt} = M[\frac{1}{I_1} - \frac{sin^2\Psi}{I_3} - \frac{cos^2\Psi}{I_2}]cos\theta$$

Let us assume $a = \frac{I_2}{I_1} - 1, b = 1 - \frac{I_2}{I_3}$. From the above derived results ,

$$\frac{d\Psi}{dt} = \frac{[\frac{1}{I_1} - \frac{sin^2\Psi}{I_3} - \frac{cos^2\Psi}{I_2}]cos\theta}{\frac{sin^2\Psi}{I_3} + \frac{cos^2\Psi}{I_2}}$$
$$\frac{d\Psi}{dt} = \frac{[\frac{I_2}{I_1} - \frac{I_2 sin^2\Psi}{I_3} - cos^2\Psi]cos\theta}{\frac{I_2 sin^2\Psi}{I_3} + cos^2\Psi}$$
$$\frac{d\Psi}{dt} = \frac{[(\frac{I_2}{I_1} - 1) + sin^2\Psi(1 - \frac{I_2}{I_3})]cos\theta}{1 + sin^2\Psi(\frac{I_2}{I_3} - 1)}$$
$$\frac{d\Psi}{dt} = \frac{(a + bsin^2\Psi)cos\theta}{1 - bsin^2\Psi}$$

From energy conservation,

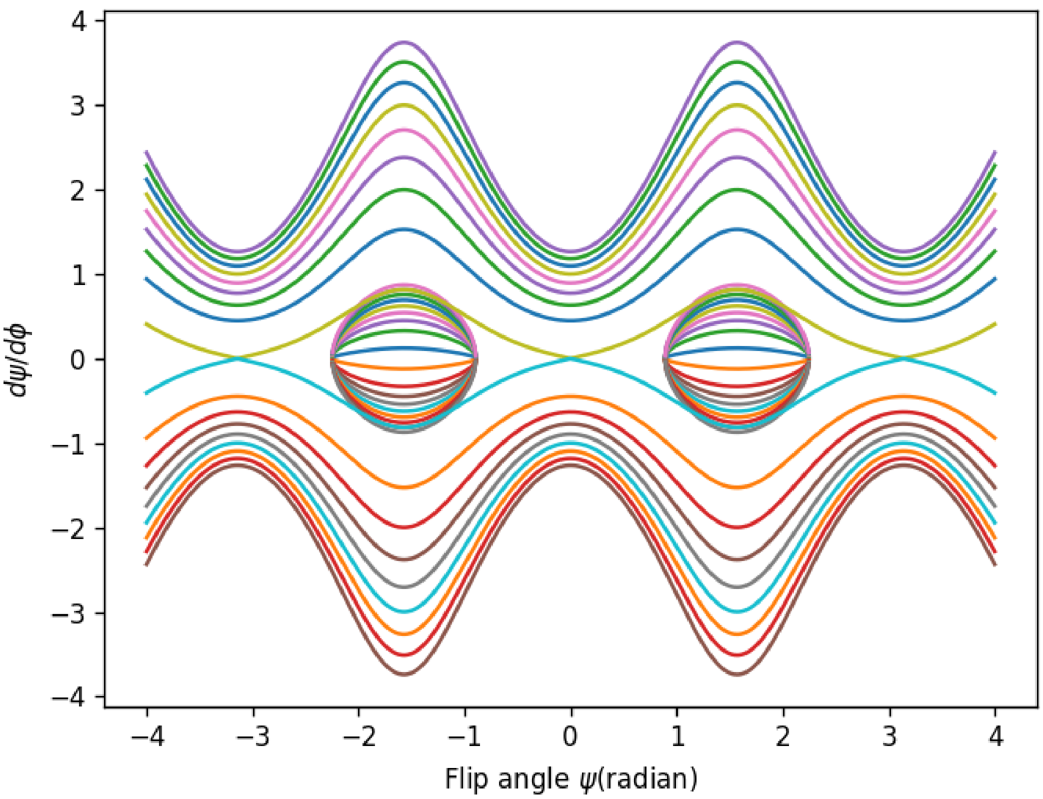
$$
\begin{aligned}
2E &= \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} \\
2E &= \frac{M^2 cos^2\theta}{I_1} + \frac{M^2 sin^2\theta cos^2\Psi}{I_2} + \frac{M^2 sin^2\theta sin^2\Psi}{I_3} \\
\frac{2EI_2}{M^2} &= cos^2\theta(\frac{I_2}{I_1}) + sin^2\theta cos^2\Psi + sin^2\theta sin^2\Psi(\frac{I_2}{I_3}) \\
\frac{2EI_2}{M^2} - 1 &= a - asin^2\theta - bsin^2\theta sin^2\Psi \\
Let, \gamma &= a - sin^2\theta(a + bsin^2\Psi) \\
\gamma &= \frac{2EI_2}{M^2} - 1
\end{aligned}
$$

In order to find how Ψ is varying with Φ we need to eliminate $cos\theta$ term.
From $\gamma = a - sin^2\theta(a + bsin^2\Psi)$,

$$cos\theta = \sqrt{\frac{bsin^2\Psi + \gamma}{a + bsin^2\Psi}}$$

By substituting in the equation of $\frac{d\Psi}{d\Phi}$
we get,

$$\frac{d\Psi}{d\Phi} = \frac{\sqrt{a + bsin^2\Psi}\sqrt{\gamma + bsin^2\Psi}}{1 - bsin^2\Psi}$$



From the plot the saddle points are (0,0),(π,0) which are unstable and ($\pi/2$,0) is centre which is stable. For a full change of Φ i.e.; '2π' there is a change of 'π' in Ψ.

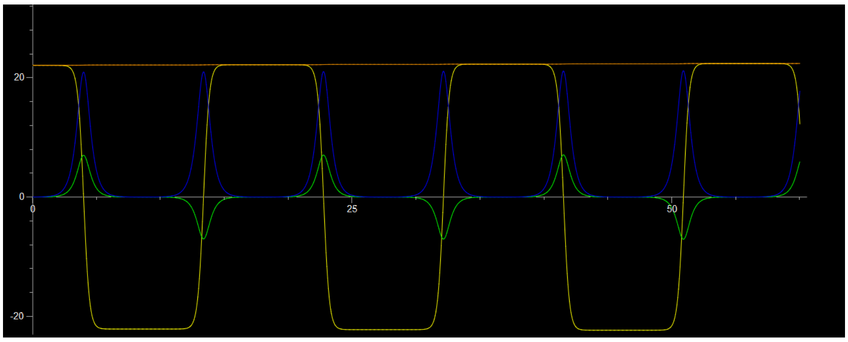
$$\Delta\Psi = \pi for \Delta\Phi = 2\pi$$

4 Angular momentum vs time

From Euler equations,

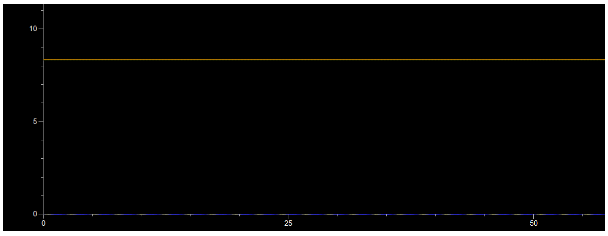
$$I_1\frac{d\omega_1}{dt} = \frac{I_2 - I_3}{I_1}\omega_2\omega_3$$
$$I_2\frac{d\omega_2}{dt} = \frac{I_3 - I_1}{I_2}\omega_1\omega_3$$
$$I_3\frac{d\omega_3}{dt} = \frac{I_1 - I_2}{I_3}\omega_1\omega_2$$

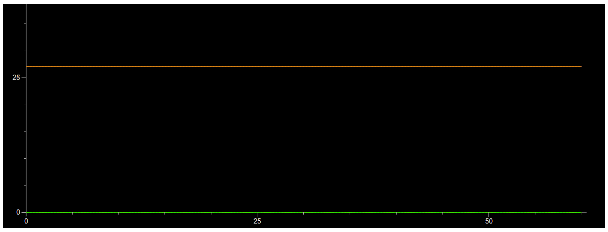
When angular momentum is given about Intermediate axis :



The Yellow represent the angular momentum about the intermediate axis vs time when initial angular velocity is given to the system about the intermediate axis. The energy gradually distributes about other two axis and the increase and decrease of the momentum about the intermediate axis is maintained.
The Blue and green represent the angular momentum vs time about the maximum moment of inertia axis and minimum moment of inertia axis.

Angular momentum is given about maximum,minimum moment of inertia axis :





When the initial angular momentum is given to the system about the minimum and maximum moment of inertia axis, the angular momentum is bound to that specific axis itself.

There is no distribution of angular momentum about the other two axis which suggests that the rotation about those axes are stable.

5 Applications

1. Using the Dzhanibekov effect in spacecraft for future space missions:
   We can employ periodic changes in the attitude orientation of the spacecraft. Controlling the dynamics of the spinning rotating spacecraft via active change of its inertial properties is possible.
   This, for example, enables for the spacecraft with initial stable axial spin at the desired time, to be transferred into the Dzhanibekov unstable flipping mode (maybe, for changing its head attitude by 180○ or observations) and then, if needed, to return back to the initial stable spin.
   Essentially, it is the method of controlled switching ON and OFF of the unstable periodic flipping motion of the spacecraft (known as "Dzhanibekov's effect") via controlled morphing of the spacecraft.

2. Rockfall Simulations and predictions:
   The stability properties of a freely rotating rigid body are governed by the intermediate axis theorem, i.e., rotation around the major and minor principal axes is stable whereas rotation around the intermediate axis is unstable.
   The stability of the principal axes is of importance for the prediction of rockfall.
   Based on the stability proof, a novel scheme which respects the stability properties of a freely rotating body and which can be incorporated in numerical schemes for the simulation of rigid bodies with frictional unilateral constraints.

6 References

1. https://arxiv.org/pdf/1606.08237.pdf
2. Course of Theoretical Physics by Landau-Lifshitz,volume 1.