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Homework 3

Problem Set 5

Problem 1

A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that there is an 80% chance that she will get the job if she receives a strong recommendation, a 40% chance if she receives a moderately good recommendation, and a 10% chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are 0.7, 0.2, and 0.1, respectively.

a. How certain is she that she will receive the new job offer?

b. Given that she does receive the offer, how likely should she feel that she received a strong recommendation? A moderate recommendation? A weak recommendation?

c. Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? A moderate recommendation? A weak recommendation?

Solution:
Let S be the event that a strong recommendation is given, M be the event that a moderate recommendation is given and W be the event a weak recommendation. E be the event she gets the job.

a. The sample space is partitioned into 3 subsets, using Bayes' formula,

$$P(E) = P(E \mid S)P(S) + P(E \mid M)P(M) + P(E \mid W)P(W)$$

$$\implies P(E) = (0.8 \times 0.7) + (0.4 \times 0.2) + (0.1 \times 0.1)$$

$$\therefore P(E) = \underline{\underline{0.65}}$$

b. Using the definition of conditional probability,

$$P(S \mid E) = \frac{(0.8 \times 0.7)}{P(E)} = \underline{\underline{\frac{56}{65}}}$$

$$P(M \mid E) = \frac{(0.4 \times 0.2)}{P(E)} = \underline{\underline{\frac{8}{65}}}$$

$$P(W \mid E) = \frac{(0.1 \times 0.1)}{P(E)} = \underline{\underline{\frac{1}{65}}}$$

c. Using the definition of conditional probability,

$$P(S \mid E^c) = \frac{((1 - 0.8) \times 0.7)}{1 - P(E)} = \underline{\underline{\frac{14}{35}}}$$

$$P(M \mid E^c) = \frac{((1 - 0.4) \times 0.2)}{1 - P(E)} = \underline{\underline{\frac{12}{35}}}$$

$$P(W \mid E^c) = \frac{((1 - 0.1) \times 0.1)}{1 - P(E)} = \underline{\underline{\frac{9}{35}}}$$

Problem 2

Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits a wooden duck target with probability p_1, while each shot of Dianne's hits it with probability p_2. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that

a. both shots hit the duck?

b. Barbara's shot hit the duck?

Solution:
Let B be the event that Barbara's shot hit and D be the event that Dianne's shot hits. As both these events are independent $P(B \cap D) = p_1 p_2$.

a. The probability that at least one shot hits given that the target is knocked over,

$$P(B \cup D) = P(B) + P(D) + P(B \cap D) = p_1 + p_2 + p_1 p_2$$
$$\implies P(B \cap D \mid B \cup D) = \underline{\underline{\frac{p_1 p_2}{p_1 + p_2 + p_1 p_2}}}$$

b. The probability that Barbara's shot knocked over the target given that the target is knocked over,

$$P(B \mid B \cup D) = \underline{\underline{\frac{p_1}{p_1 + p_2 + p_1 p_2}}}$$

Problem 5

Independent trials that result in a success with probability p are successively performed until a total of r successes is obtained. Show that the probability that exactly n trials are required is

$$\binom{n-1}{r-1} p^r (1-p)^{n-r}$$

Solution:
If the experiment has to be have r successes in exactly n trails then the last trail has to yield a success as if we get r successes before the n^{th} trail then the condition is not met. Therefore, we are only free to distribute $r-1$ successes over $n-1$ trials as the last trial has a fixed outcome. The number of ways to select $r-1$ positions form $n-1$ positions is given by $\binom{n-1}{r-1}$ and as all trails are independent events the probability of getting r successes in n trials means that there were $n-r$ failures, therefore, the probability of each such instance is $p^r \times (1-p)^{n-r}$.
Therefore, the total probability is $\underline{\underline{\binom{n-1}{r-1} p^r (1-p)^{n-r}.}}$

Problem 9

Extend the definition of conditional independence to more than 2 events.

Solution:
Let $Q(E)$ be the probability of an event E given that previously an event F already occurred. As $Q(E)$ is a well defined probability function, using the definition of independence for more than two events,

$$Q(E_1 \cap E_2 \cap ...E_n) = Q(E_1) \times Q(E_2) \times ...Q(E_n)$$

As $Q(E) = E(E \mid F)$, this can be rewritten as,

$$P(E_1 \cap E_2 \cap ...E_n \mid F) = P(E_1 \mid F) \times P(E_2 \mid F) \times ...P(E_n \mid F)$$

Problem Set 6

Problem 2

Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

Solution:

Let h be the number of heads and t be the number of heads obtained when a coin is tosses n times. As there are n tosses and the each toss can either be heads or tails, $n = h + t$. Therefore,

$$X = h - t \implies X = n - 2t$$

As t can take all values from 0 to n the values of X range from $-n$ to n increments of 2.

$$X \in \{-n, -n+2, -n+4 \ldots n-2, n\}$$

Problem 4

Suppose that the distribution function X of a random variable is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{4}, & 0 \le x < 1, \\ \frac{1}{2} + \frac{x-1}{4}, & 1 \le x < 2, \\ \frac{11}{12}, & 2 \le x < 3, \\ 1, & 3 \le x. \end{cases}$$

Find $P(X = i)$ for $i = 1, 2, 3$ and $P(\frac{1}{2} < X < \frac{3}{2})$.

Solution:

$P(X) = F_X'(x)$ when the function is continuous, therefore, for

$P(X = 1) = \underline{\underline{\frac{1}{4}}}$

As $F_X(x)$ is not continuous at 2and 3, $P(X) = F_X(b) - lim_{x \to b^-} F_X(x)$, therefore, for

$P(X = 2) = \frac{11}{12} - \frac{1}{4} = \underline{\underline{\dfrac{1}{4}}}$

$P(X = 3) = 1 - \frac{11}{12} = \underline{\underline{\dfrac{1}{12}}}$

$P(a < X < b) = F_X(b) - F_X(a)$, therefore,

$P(\frac{1}{2} < X < \frac{3}{2}) = \frac{5}{8} - \frac{1}{8} = \underline{\underline{\dfrac{1}{2}}}$

Problem 7

A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are of the same color, then you win \$1.10. If they are of different colors, then you loose \$1.00. Calculate

a. the expected value of the amount you win.

b. the variance of the amount you win.

Solution:

Let X be the random variable that represents the value of the amount of money earned.

The number of ways getting balls of the same color $= 2\binom{5}{2}$, the number of ways of getting one ball of each color $= \binom{5}{1} \times \binom{5}{1}$, the total number of ways of drawing 2 balls $= \binom{10}{2}$. Therefore, the probability of getting two balls of the same color $= \frac{4}{9}$ and the probability of one ball of each color $= \frac{5}{9}$.

a. $E(X) = (1.1 \times \frac{4}{9}) + ((-1) \times \frac{5}{9}) = \underline{\underline{\frac{-1}{15}}}$

b. $Var(X) = (1.1^2 \times \frac{4}{9}) + ((-1)^2 \times \frac{5}{9}) - E(X)^2 = \underline{\underline{\frac{49}{45}}}$

Problem 9

Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of

$$Y = \frac{X - \mu}{\sigma}$$

Solution:
As $E(aX + b) = aE(X) + b$,(lecture 16)

$$E(Y) = \frac{E(X)}{\sigma} - \frac{\mu}{\sigma} = \underline{\underline{\frac{E(X) - \mu}{\sigma}}}$$

As $Var(aX + b) = a^2Var(X)$,(lecture 16)

$$Var(Y) = \underline{\underline{\frac{Var(X)}{\sigma^2}}}$$