Homework 2

Problem Set 3

Problem 1

If 8 identical computers are to be divided among 4 lab rooms, how many divisions are possible? How many if each room must receive at least 1 computer?

Solution:
Distributing 8 computers among 4 labs is the same as grouping 8 computers into 4 distinct groups, we know that if we want to distribute n objects into m groups there are $\binom{n+m-1}{m-1}$ possible groupings. In this case $n = 8, m = 4$, therefore the total number of ways to group 8 computers into 4 distinct groups is $\binom{8+4-1}{4-1} = \binom{11}{3} = \underline{\underline{165}}$.
If there needs to be at least one computer in each lab then, we first distribute one computer to each lab, and as all the computers are indistinguishable there is only 1 way to do this. Once each lab has one computer we can just distribute the remaining computers in a similar manner as before. As 4 of the 8 total computers have already been distributed we just need to distribute the remaining 4 computers into the 4 labs. Using the same formula as before, the total number of ways in which 4 computers can be grouped into 4 distinct groups is $\binom{4+4-1}{4-1} = \binom{7}{3} = \underline{\underline{35}}$.

Problem 3

Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

by connecting the formula to a concrete situation (i.e. interpreting its meaning). Show that the formula implies

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Solution:
Let us consider a set of $n+m$ objects, to select r objects from this set the total number of ways to to do this are known to be $\binom{n+m}{r}$. But instead of this being a group of $n+m$ objects, we can also consider two sets of n and m objects each. From these two sets if we wish to choose r objects, we must first choose i objects from the first set and then choose the remaining $r-i$ objects from the other set, the value of i can range from 0 to r as we can choose all the objects from only one set or some from one and some from the other. As all the different values of i are separate cases we can say that the total number of ways to select r objects from two sets of n and m objects is $\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}$.
As these two cases are the same selection done in two different ways their values must be equal.

$$\therefore \binom{n+m}{r} = \sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i}$$

In the special case where $n = m = r$, we can rewrite the formula as the following:

$$\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$$

And as we know, $\binom{n}{m} = \binom{n}{n-m}$, this formula can be rewritten as,

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

Problem 7

A retail establishment accepts either the American Express or the VISA credit card. A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

Solution:
Let A be the event that the customer carries an American Express card and E be the event that the customer carries a Visa card. We know that $P(A \cup V) = P(A) + P(V) - P(A \cap V)$, and we are given the values of $P(A) = 0.24$, $P(V) = 0.61$ and $P(A \cap V) = 0.11$. We are told that the establishment accepts both cards so our goal is to find the probability that a customer is carrying at least one of these cards, i.e. $P(A \cup V)$. Using the formula we stated we can see that $P(A \cup V) = 0.24 + 0.61 - 0.11 = 0.74$, therefore $\underline{\underline{74\%}}$ of the customers carry a card that the establishment accepts.

Problem 9

Prove that

$$P(E \cap F^c) = P(E) - P(E \cap F)$$

Solution:
By Bayes' formula we know that,

$$P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c)$$

And by the definition of conditional probability we know,

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

Substituting the definition of conditional probability into Bayes' formula we get,

$$P(E) = P(E \cap F) + P(E \cap F^c)$$

Rearranging this we get,

$$P(E \cap F^c) = P(E) - P(E \cap F)$$

Problem Set 4

Problem 1

If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i for $i = 1, 2, ..., 12$?

Solution:
For $i \leq 6$ the probability of the first die landing on 6 is $\underline{\underline{0}}$ as if the first die landed at six the sum must be at least 7 as there are no non-positive numbers on the face of a fair die.
For $i = 7$, there are six possible outcomes out of which only 1 has six appearing on the first die. Therefore the probability is $\underline{\underline{\frac{1}{6}}}$.
For $i = 8$, there are five possible outcomes out of which only 1 has six appearing on the first die.

Therefore the probability is $\underline{\underline{\frac{1}{5}}}$.
For $i = 9$, there are four possible outcomes out of which only 1 has six appearing on the first die. Therefore the probability is $\underline{\underline{\frac{1}{4}}}$.
For $i = 10$, there are three possible outcomes out of which only 1 has six appearing on the first die. Therefore the probability is $\underline{\underline{\frac{1}{3}}}$.
For $i = 11$, there are two possible outcomes out of which only 1 has six appearing on the first die. Therefore the probability is $\underline{\underline{\frac{1}{2}}}$.
For $i = 7$, there is only one possible outcome, both dice have six appearing on them. Therefore the probability is $\underline{\underline{1}}$.

Problem 4

In a certain community, 36% of the families own a dog and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. What is

a. the probability that a randomly selected family owns both a dog and a cat?

b. the conditional probability that a randomly selected family owns a dog given that it owns a cat?

Solution:
Let D be the event that a family owns dog and C the event a family owns a cat, we know that $P(D) = 0.36$, $P(C \mid D) = 0.22$ and $P(C) = 0.30$.

a. By the definition of conditional probability we have,
   $P(C \mid D) = \frac{P(C \cap D)}{P(D)} \implies P(C \cap D) = P(C \mid D)P(D) = 0.0792$
   Therefore, $\underline{\underline{7.92\%}}$ of the families own both a dog and a cat.

b. Again using the definition of conditional probability we have,
   $P(D \mid C) = \frac{P(C \cap D)}{P(C)} \implies P(D \mid C) = \frac{0.0792}{0.30} = 0.264$
   Therefore, $\underline{\underline{26.4\%}}$ of the families that own a cat also own a dog.

Problem 7

Let $E \subset F$. Express the following probabilities as simply as possible

$$P(E \mid F), P(E \mid F^c), P(F \mid E^c)$$

Solution:
$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \implies P(E \mid F) = \underline{\underline{\frac{P(E)}{P(F)}}}$, as $E \subset F \implies E \cap F = E$
$P(E \mid F^c) = \frac{P(E \cap F^c)}{P(F^c)} \implies P(E \mid F^c) = \underline{\underline{0}}$, as $E \subset F \implies E \cap F^c = \emptyset$
$P(F \mid E^c) = \frac{P(F \cap E^c)}{P(E^c)} \implies P(F \mid E^c) = \underline{\underline{\frac{P(F) - P(E)}{1 - P(E)}}}$, as $F \cap E^c = F \setminus E$

Problem 10

The probability of getting a head on a single toss of a coin is p. Suppose that A starts and continues to flip the coin until a tail shows up, at which point B starts flipping. Then B continues to flip until a tail comes up, at which point A takes over, and so on. Let $P_{n,m}$ denote the probability that A accumulates a total of n heads before B accumulates m. Show that

$$P_{n,m} = pP_{n-1,m} + (1 - p)(1 - P_{m,n})$$

Solution:
Using Bayes' formula we can represent $P_{n,m}$ using the outcome of the first coin toss.
Case 1: the first coin toss by A is heads
The probability of this happening is p as this event is independent, now as A is in the lead with one head, he needs $n-1$ more heads before B accumulates m heads, and using the notation defined the probability of this occurring is $P_{n-1,m}$.
Case 2: the first coin toss by A is tails
The probability of this happening is $1-p$ as this event is independent, now as A is behind B as its not his turn anymore. For A to accumulate n heads before B accumulates m heads, B must not accumulate m heads before A accumulates n heads, again we can use the notation as effectively B is going first. Therefore the probability of B not reaching his goal is $1-P_{m,n}$.
As these are the only possible cases, we can now use Bayes' formula to state the following,

$$P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{m,n})$$