Homework 5

Problem Set 9

Problem 1

Compute $E[X^2]$ for an exponential random variable X with parameter λ.
<u>Solution</u>:
As X is a exponential random variable

$$f_X(x) = \begin{cases} 0, & x \leq 0 \\ \lambda e^{-x\lambda}, & x > 0 \end{cases}$$

By definition of expectation

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx \implies E[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x}$$

By integration by parts

$$\int_0^{\infty} x^2 \lambda e^{-x\lambda x} = -x^2 e^{-\lambda x} - \frac{2}{\lambda} x e^{-\lambda x} - \frac{2e^{-\lambda x}}{\lambda^2}\Big|_0^{\infty} = \frac{2}{\lambda^2}$$

Therefore,

$$E[X^2] = \underline{\underline{\frac{2}{\lambda^2}}}$$

Problem 2

A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?
<u>Solution</u>:
Let X be a random variable that measures the distance between the location of the breakdown and city A. X is uniformly distributed from $(0, 100)$. The expected value of X tells the expected distance of a breakdown from city A.

$$f_X(x) = \frac{1}{100 - a} \implies E[X] = \int_0^1 00 \frac{x}{100} dx \implies E[X] = \frac{100}{2}$$

Therefore, it will be more beneficial to have a single service station at 50 miles from city A as it is more likely that a breakdown occurs at this location and building the other two stations will not be beneficial as the station at 50 miles will be servicing the majority of the busses.

Problem 3

A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

Solution:

Let X be a random variable that measures the distance between the target and the location of where the shot hits, X is uniformly distributed from $[0, 10]$. As the distance between the target and the shot determines the points earned, i.e. the points are a function of the random variable X, let this function be $g(X)$.

$$g(X) = \begin{cases} 10, & 0 \leq x \leq 1 \\ 5, & 1 < x \leq 3 \\ 3, & 3 < x \leq 5 \\ 0, & 5 < x \leq 10 \end{cases}$$

The expected number of points therefore are given by, $E[g(X)]$.As seen in lecture 22,

$$E[g(X)] = \int_{-\infty}^{\infty} g(X) f_X dx \implies E[g(X)] = \int_0^{10} g(X) \frac{1}{10} dx \implies$$

$$E[g(X)] = \int_0^1 (1) \frac{1}{10} dx + E[g(X)] + \int_3^5 (5) \frac{1}{10} dx + E[g(X)] = \int_3^5 (3) \frac{1}{10} dx + E[g(X)] + \int_5^1 0(0) \frac{1}{10} dx$$

$$\therefore E[g(X)] = \frac{10 \times 1}{10} + \frac{5 \times 2}{10} + \frac{3 \times 2}{10} + 0 = 2.6$$

Thus, the expected number of points scored is $\underline{\underline{2.6}}$.

Problem 4

In $10,000$ independent tosses of a coin, the coin landed on heads 5800 times. Is it reasonable to assume that the coin is not fair?

Solution:

Let X be a random variable that is equal to the number of heads that show up when a fair coin is tossed, $X \sim B(10000, 0.5)$. Therefore, the following quantities are known,

$$E[X] = np = 10000 \times 0.5 = 5000$$

$$\text{Var}(X) = np(1-p) = 10000 \times 0.5 \times 0.5 = 2500 \implies \sigma(\text{Standard Deviation}) = \sqrt{\text{Var}(X)} = 50$$

The standard deviation of a variable gives the expected deviation from the mean for random variable, therefore, after taking into account the standard deviation, the expected value of X must between 4950 and 5050 for a fair coin. But, the number of heads that are obtained(5800) is beyond this range and thus it is not reasonable to assume that the coin flipped was fair.

Problem 6

If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

Solution:

$Y = e^X \implies \ln Y = X$

$f_X(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & x \notin (0,1) \end{cases} \implies F_X(x) = x, x \in (0, 1)$

$F_Y(y) = P(Y \leq y) \implies P(e^X \leq y) \implies P(X \leq \ln y)$

$\implies F_Y(y) = F_X(\ln y) \implies f_Y(y) = \frac{1}{y} f_X(\ln y)$

$$\therefore f_X(x) = \begin{cases} \frac{1}{y}, & y \in (1, e) \\ 0, & y \notin (1, e) \end{cases}$$

Problem 8

Let X be a random variable that takes on values between 0 and c. That is, $P(0 \leq X \leq c) = 1$. Show that

$$\text{Var}(X) \leq \frac{c^2}{4}$$

Solution:
Proving Chebyshev inequality for continuous random variables,

$$\text{Var}\left(X\right) = \int_{-\infty}^{\infty} \left(x - \mu_x\right)^2 p_X dx$$

$$\implies \text{Var}\left(X\right) \geq \int_{-\infty}^{-(\varepsilon+\mu_X)} \left(x - \mu_x\right)^2 p_X dx + \int_{(\varepsilon+\mu_X)}^{\infty} \left(x - \mu_x\right)^2 p_X dx$$

$$\implies \text{Var}\left(X\right) \geq \varepsilon^2 \left(\int_{-\infty}^{-(\varepsilon+\mu_X)} p_X dx + \int_{(\varepsilon+\mu_X)}^{\infty} p_X dx\right) = \varepsilon^2 P\left(\{|X - \mu_X| \geq \varepsilon\}\right)$$

$$\therefore P\left(\{|X - \mu_X| \geq \varepsilon\}\right) \leq \frac{\text{Var}(X)}{\varepsilon^2}$$

If $\text{Var}(X) = 0$ is the trivial case as,

$$0 < \frac{c^2}{4}$$

Therefore, for $\text{Var}(X) \neq 0$, using Chebyshev inequality with, $\varepsilon = \frac{2\,\text{Var}(X)}{c}$,

$$P\left(\{|X - \mu_X| \geq \varepsilon\}\right) \leq \frac{\text{Var}(X)}{\varepsilon^2} \implies P\left(\{|X - \mu_X| \geq \varepsilon\}\right) \leq \frac{c^2}{4\,\text{Var}(X)}$$

As X could be any random variable this $P\left(\{|X - \mu_X| \geq \varepsilon\}\right)$ value could be anything between 0 and 1. Therefore, in the edge case where $P\left(\{|X - \mu_X| \geq \varepsilon\}\right) = 1$,

$$1 \leq \frac{c^2}{4\,\text{Var}(X)} \implies \text{Var}(X) \leq \frac{c^2}{4}$$

Problem 9

If X is an exponential random variable with mean $\frac{1}{\lambda}$, show that

$$E[X^k] = \frac{k!}{\lambda^k},\ k = 1, 2, \ldots$$

Solution:
$E[X^k] = \int x^k f_X(x)dx \implies E[X^k] = \int x^k \lambda e^{-\lambda x}$
Using integration by parts,

$$E[X^k] = -x^k e^{-\lambda x} - \int k x^{k-1} e^{-\lambda x} \implies E[X^k] = -x^k e^{-\lambda x}\Big|_0^{\infty} + \int \frac{k}{\lambda} x^{k-1} e^{-\lambda x}$$

$$-x^k e^{-\lambda x}\Big|_0^{\infty} = 0 \text{ by L'Hpital's rule as}$$

$$lim_{x\to\infty} \frac{x^{k-1}}{e^{-\lambda x}} = lim_{x\to\infty} \frac{k!}{\lambda^k e^{\lambda x}} = 0 \text{ after taking the limit multiple times}$$

$$\therefore E[x^k] = \frac{k}{\lambda} E[x^{k-1}]$$

Using this property recursively,

$$E[X^k] = \frac{k}{\lambda} E[x^{k-1}] = \frac{k(k-1)}{\lambda^2} E[X^{k-2}] \cdots = \frac{k!}{\lambda^k} \text{ as } E[X] = \frac{1}{\lambda}$$

$$\therefore E[x^k] = \frac{k!}{\lambda^k}$$

Problem 10

If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter $\frac{\lambda}{c}$.

<u>**Solution**</u>:
Let Y be a random variable such that $Y = cX$,

$$P(Y \leq y) = P(cX \leq y) = P(X \leq \frac{y}{c}) \implies F_X(\frac{y}{c})$$

$$\therefore F_Y(y) = F_X(\frac{y}{c}) \implies f_Y(y) = \frac{1}{c} f_X(\frac{y}{c})$$

As X is a exponential variable,

$$f_Y(y) = \begin{cases} \frac{\lambda}{c} e^{-\lambda \frac{y}{c}}, & y \in [0, \infty) \\ 0, & y \notin [0, \infty) \end{cases}$$

This can be rewritten as,

$$f_X(x) = \begin{cases} \frac{\lambda}{c} e^{-\lambda x}, & x \in [0, \infty) \\ 0, & x \notin [0, \infty) \end{cases}$$

Therefore, $cX(Y)$, is a exponential random variable with parameter $\frac{\lambda}{c}$.