# Chapter 9
# Input Modeling

Banks, Carson, Nelson & Nicol

*Discrete-Event System Simulation*

# Purpose & Overview

- Input models provide the driving force for a simulation model.
- The quality of the output is no better than the quality of inputs.
- In this chapter, we will discuss the 4 steps of input model development:
  - Collect data from the real system
  - Identify a probability distribution to represent the input process
  - Choose parameters for the distribution
  - Evaluate the chosen distribution and parameters for goodness of fit.

# Data Collection

- One of the biggest tasks in solving a real problem. GIGO – garbage-in-garbage-out
- Suggestions that may enhance and facilitate data collection:
  - **Plan ahead:** begin by a practice or pre-observing session, watch for unusual circumstances
  - **Analyze the data as it is being collected:** check adequacy
  - **Combine homogeneous data sets**, e.g. successive time periods, during the same time period on successive days
  - **Be aware of data censoring:** the quantity is not observed in its entirety, danger of leaving out long process times
  - **Check for relationship** between variables, e.g. build scatter diagram
  - **Check for autocorrelation**
  - **Collect input data**, not performance data

# Identifying the Distribution when data is available

- Histograms
- Selecting families of distribution
- Parameter estimation
- Goodness-of-fit tests
- Fitting a non-stationary process

# Histograms

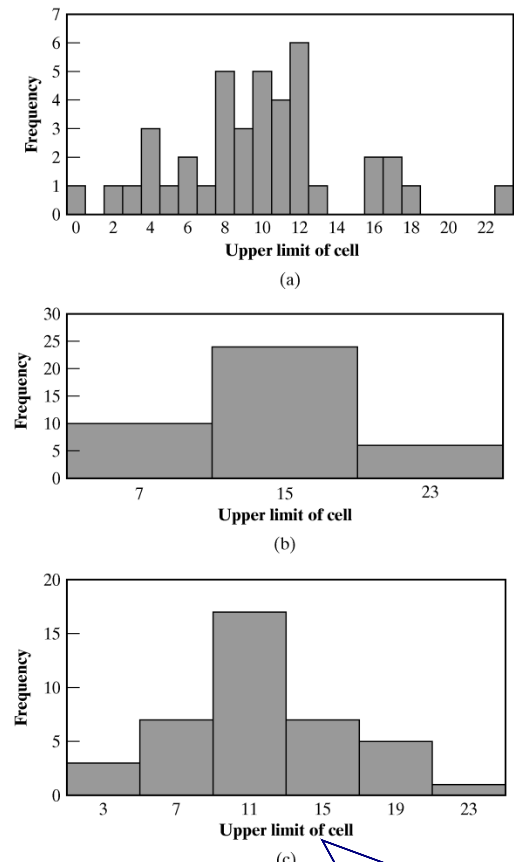
[Identifying the distribution]

- A frequency distribution or histogram is useful in determining the shape of a distribution
- The number of class intervals depends on:
  - The number of observations
  - The dispersion of the data
  - Suggested: the number intervals ≈ the square root of the sample size works well in practice
- If the interval is too wide, the histogram will be coarse or blocky and it’s shape and other details will not show well
- If the intervals are too narrow, the histograms will be ragged and will not smooth the data

# Histograms

[Identifying the distribution]

- For continuous data:
  - Corresponds to the probability density function of a theoretical distribution
  - A line drawn through the center of each class interval frequency should results in a shape like that of pdf
- For discrete data:
  - Corresponds to the probability mass function
- If few data points are available: combine adjacent cells to eliminate the ragged appearance of the histogram

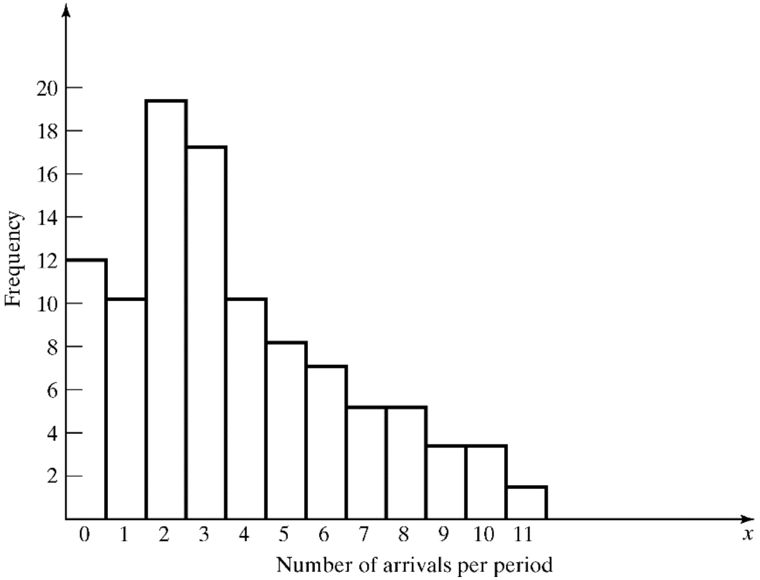



Same data with different interval sizes

# Histograms [Identifying the distribution]

- Vehicle Arrival Example: # of vehicles arriving at an intersection between 7 am and 7:05 am was monitored for *100* random workdays.

| Arrivals per Period | Frequency |
|---|---|
| 0 | 12 |
| 1 | 10 |
| 2 | 19 |
| 3 | 17 |
| 4 | 10 |
| 5 | 8 |
| 6 | 7 |
| 7 | 5 |
| 8 | 5 |
| 9 | 3 |
| 10 | 3 |
| 11 | 1 |




- There are ample data, so the histogram may have a cell for each possible value in the data range

# Selecting the Family of Distributions

[Identifying the distribution]

- A family of distributions is selected based on:
  - ***The context of the input variable***
  - ***Shape of the histogram***
    - The purpose of preparing a histogram is to infer a known pdf or pmf
- Frequently encountered distributions:
  - ***Easier to analyze:*** exponential, normal and Poisson
  - ***Harder to analyze:*** beta, gamma and Weibull

# Selecting the Family of Distributions

[Identifying the distribution]

- Use the physical basis of the distribution as a guide, for example:
  - *Binomial:* # of successes in *n* trials
  - *Poisson:* # of independent events that occur in a fixed amount of time or space
  - *Normal:* dist'n of a process that is the sum of a number of component processes
  - *Exponential:* time between independent events, or a process time that is memoryless
  - *Weibull:* time to failure for components
  - *Discrete or continuous uniform:* models complete uncertainty. All outcomes are equally likely.
  - *Triangular:* a process for which only the minimum, most likely, and maximum values are known. Improvement over uniform.
  - *Empirical:* resamples from the actual data collected

# Selecting the Family of Distributions

[Identifying the distribution]

- Do not ignore the physical characteristics of the process
  - Is the process naturally discrete or continuous valued?
  - Is it bounded or is there no natural bound?
- No “true” distribution for any stochastic input process
- **Goal:** to obtain a good approximation that yields useful results from the simulation experiment.

# Quantile-Quantile Plots [Identifying the distribution]

- ***Q-Q*** **plot is a useful tool for evaluating distribution fit**
- If $X$ is a random variable with cdf $F$, then the $q$-quantile of $X$ is the $\gamma$ such that

$$F(\gamma) = P(X \le \gamma) = q \text{ for } 0 < q < 1$$

  - When $F$ has an inverse, $\gamma = F^{-1}(q)$

**By a quantile, we mean the fraction (or percent) of points below the given value**

- Let $\{x_i, i = 1,2, \dots, n\}$ be a *sample of data* from $X$ and $\{y_j, j = 1,2, \dots, n\}$ be the observations in ascending order. The Q-Q plot is based on the fact that $y_j$ is an estimate of the $(j-0.5)/n$ quantile of X.

$$y_j \text{ is approximately } F^{-1}\left(\frac{j-0.5}{n}\right)$$

where $j$ is the ranking or order number

**percentile: 100-quantiles**
**deciles: 10-quantiles**
**quintiles: 5-quantiles**
**quartiles: 4-quantiles**

# Quantile-Quantile Plots [Identifying the distribution]

- The plot of $y_j$ versus $F^{-1}((j-0.5)/n)$ is
  - Approximately a straight line if $F$ is a member of an appropriate family of distributions
  - The line has slope 1 if $F$ is a member of an appropriate family of distributions with appropriate parameter values
  - If the assumed distribution is inappropriate, the points will deviate from a straight line
  - The decision about whether to reject some hypothesized model is subjective!!

# Quantile-Quantile Plots [Identifying the distribution]

- Example: Check whether the door installation times given below follows a normal distribution.
  - The observations are now ordered from smallest to largest:

| j | Value | j | Value | j | Value | j | Value |
|---|---|---|---|---|---|---|---|
| 1 | 99.55 | 6 | 99.82 | 11 | 99.98 | 16 | 100.26 |
| 2 | 99.56 | 7 | 99.83 | 12 | 100.02 | 17 | 100.27 |
| 3 | 99.62 | 8 | 99.85 | 13 | 100.06 | 18 | 100.33 |
| 4 | 99.65 | 9 | 99.9 | 14 | 100.17 | 19 | 100.41 |
| 5 | 99.79 | 10 | 99.96 | 15 | 100.23 | 20 | 100.47 |

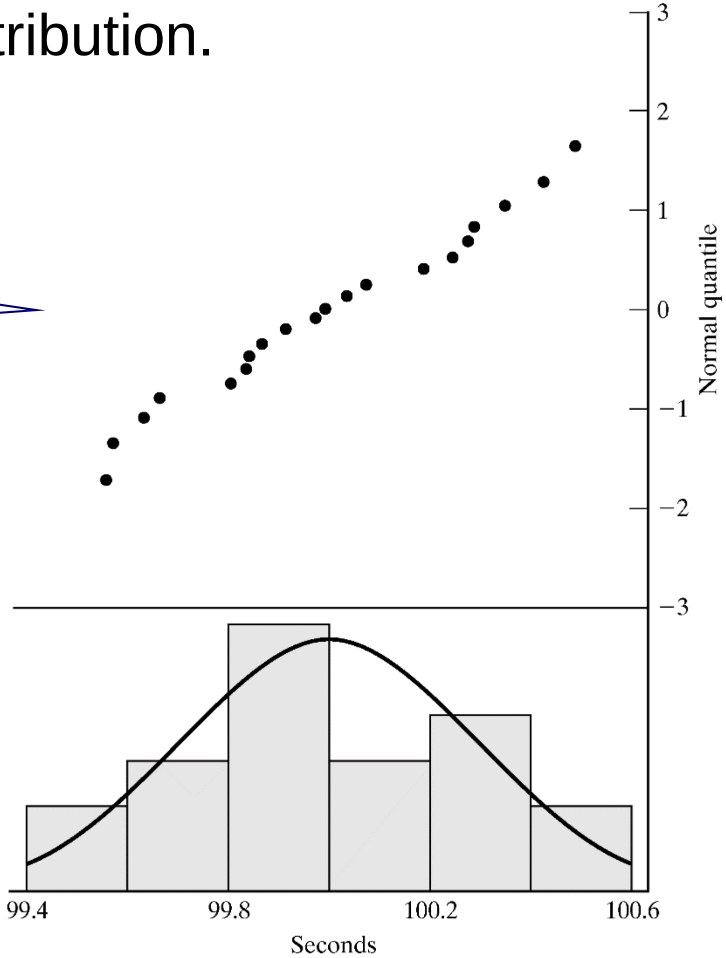
  - $y_j$ are plotted versus $F^{-1}((j-0.5)/n)$ where $F$ has a normal distribution with the sample mean *(99.99 sec)* and sample variance ($0.2832^2\ sec^2$)

# Quantile-Quantile Plots [Identifying the distribution]

- Example (continued): Check whether the door installation times follow a normal distribution.

Straight line, supporting the hypothesis of a normal distribution

Superimposed density function of the normal distribution



# Quantile-Quantile Plots [Identifying the distribution]

- Consider the following while evaluating the linearity of a *Q-Q* plot:
  - The observed values never fall exactly on a straight line
  - The ordered values are ranked and hence not independent, unlikely for the points to be scattered about the line
  - Variance of the extremes is higher than the middle. Linearity of the points in the middle of the plot is more important.
- *Q-Q* plot can also be used to check homogeneity
  - Check whether a single distribution can represent two sample sets
  - Plotting the order values of the two data samples against each other. A straight line shows both sample sets are represented by the same distribution

# Parameter Estimation [Identifying the distribution]

- Next step after selecting a family of distributions
- If observations in a sample of size $n$ are $X_1$, $X_2$, …, $X_n$ (discrete or continuous), the sample mean and variance are defined as:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

- If the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{n} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j X_j^2 - n\bar{X}^2}{n-1}$$

where $f_j$ is the observed frequency of value $X_j$

# Parameter Estimation [Identifying the distribution]

- When raw data are unavailable (data are grouped into class intervals), the approximate sample mean and variance are:

$$\bar{X} = \frac{\sum_{j=1}^{c} f_j m_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j m_j^2 - n\bar{X}^2}{n-1}$$

where $f_j$ is the observed frequency of in the $j^{th}$ class interval; $m_j$ is the midpoint of the $j^{th}$ interval, and $c$ is the number of class intervals

- **A parameter is an unknown constant, but an estimator is a statistic**.

# Suggested Estimators

| Distribution | Parameters | Suggested Estimator |
|---|---|---|
| Poisson | $\alpha$ | $\hat{\alpha} = \overline{X}$ |
| Exponential | $\lambda$ | $\hat{\lambda} = 1/\overline{X}$ |
| Normal | $\mu,\sigma^2$ | $\hat{\mu} = \overline{X}$<br>$\hat{\sigma}^2 = S^2$ (Unbiased) |

# Parameter Estimation [Identifying the distribution]

- Vehicle Arrival Example (continued): Table in the histogram example on slide 7 (Table 9.1 in book) can be analyzed to obtain:

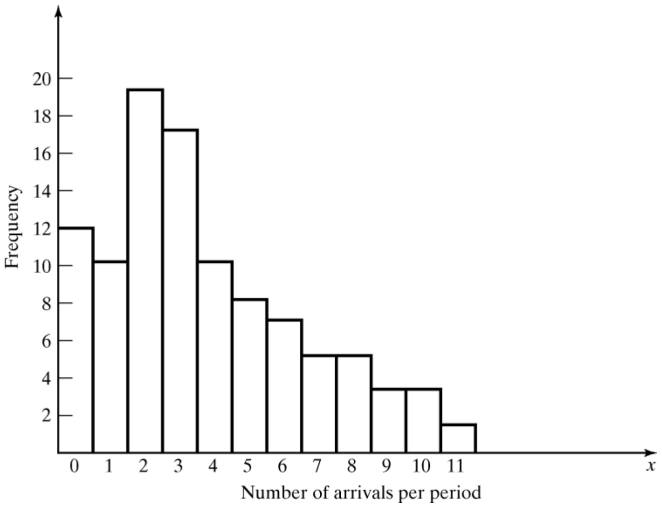
$$n = 100,\ f_1 = 12, X_1 = 0,\ f_2 = 10, X_2 = 1, \ldots,$$

$$\text{and } \sum_{j=1}^{k} f_j X_j = 364, \text{ and } \sum_{j=1}^{k} f_j X_j^2 = 2080$$

  - The sample mean and variance are

$$\bar{X} = \frac{364}{100} = 3.64$$

$$S^2 = \frac{2080 - 100 * (3.64)^2}{99} = 7.63$$




  - The histogram suggests *X* to have a Poisson distribution
    - However, note that sample mean is not equal to sample variance.
    - Reason: each estimator is a random variable, is not perfect.

# Goodness-of-Fit Tests [Identifying the distribution]

- **Conduct hypothesis testing** on input data distribution using:
  - Kolmogorov-Smirnov test
  - Chi-square test
- **Goodness-of-fit tests provide helpful guidance** for evaluating the suitability of a potential input model
- No single correct distribution in a real application exists.
  - If very little data are available, it is unlikely to reject any candidate distributions
  - If a lot of data are available, it is likely to reject all candidate distributions

# Chi-Square test

[Goodness-of-Fit Tests]

- Intuition: comparing the histogram of the data to the shape of the candidate density or mass function
- Valid for **large** sample sizes when parameters are estimated by maximum likelihood
- By arranging the *n* observations into a set of *k* class intervals or cells, the test statistics is:

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Observed Frequency

Expected Frequency

$E_i = n^*p_i$

where $p_i$ is the theoretical prob. of the $i$th interval.

*Suggested Minimum = 5*

which **approximately** follows the chi-square distribution with *k-s-1* degrees of freedom, where s = # of parameters of the hypothesized distribution estimated by the sample statistics.

# Chi-Square test



- The hypothesis of a chi-square test is:

  $H_0$: The random variable, *X*, conforms to the distributional assumption with the parameter(s) given by the estimate(s).

  $H_1$: The random variable *X* does not conform.

- If the distribution tested is discrete and combining adjacent cell is not required (so that $E_i$ > minimum requirement):
  - Each value of the random variable should be a class interval, unless combining is necessary, and

$$p_i = p(x_i) = P(X = x_i)$$

# Chi-Square test



- If the distribution tested is continuous:

$$p_i = \int_{a_{i-1}}^{a_i} f(x)\,dx = F(a_i) - F(a_{i-1})$$

where $a_{i-1}$ and $a_i$ are the endpoints of the $i^{th}$ class interval and $f(x)$ is the assumed pdf, $F(x)$ is the assumed cdf.

- Recommended number of class intervals (*k*):

| **Sample Size, n** | **Number of Class Intervals, k** |
|---|---|
| 20 | Do not use the chi-square test |
| 50 | 5 to 10 |
| 100 | 10 to 20 |
| > 100 | $n^{1/2}$ to n/5 |

- Caution: Different grouping of data (i.e., *k*) can affect the hypothesis testing result.

# Chi-Square test



- Vehicle Arrival Example (continued) (See Slides 7 and 19):
- The histogram on slide 7 appears to be Poisson
- From Slide 19, we find the estimated mean to be 3.64
- Using Poisson pmf:

$$p(x) = \begin{cases} \dfrac{e^{-\alpha}\alpha^x}{x!}, & x = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$$

- For $\alpha$=3.64, the probabilities are:

| | |
|---|---|
| p(0)=0.026 | p(6)=0.085 |
| p(1)=0.096 | p(7)=0.044 |
| p(2)=0.174 | p(8)=0.020 |
| p(3)=0.211 | p(9)=0.008 |
| p(4)=0.192 | p(10)=0.003 |
| p(5)=0.140 | p(≥11)=0.001 |

# Chi-Square test



- Vehicle Arrival Example (continued):

  $H_0$: the random variable is Poisson distributed.

  $H_1$: the random variable is not Poisson distributed.

| $x_i$ | Observed Frequency, $O_i$ | Expected Frequency, $E_i$ | $(O_i - E_i)^2/E_i$ |
|---|---|---|---|
| 0 | 12 | 2.6 | 7.87 |
| 1 | 10 | 9.6 | |
| 2 | 19 | 17.4 | 0.15 |
| 3 | 17 | 21.1 | 0.8 |
| 4 | 19 | 19.2 | 4.41 |
| 5 | 6 | 14.0 | 2.57 |
| 6 | 7 | 8.5 | 0.26 |
| 7 | 5 | 4.4 | 11.62 |
| 8 | 5 | 2.0 | |
| 9 | 3 | 0.8 | |
| 10 | 3 | 0.3 | |
| > 11 | 1 | 0.1 | |
| | 100 | 100.0 | 27.68 |

$$E_i = np(x) = n\frac{e^{-\alpha}\alpha^x}{x!}$$

Combined because of min $E_i$

- Degree of freedom is $k-s-1 = 7-1-1 = 5$, hence, the hypothesis is rejected at the *0.05* level of significance.

$$\chi_0^2 = 27.68 > \chi_{0.05,5}^2 = 11.1$$

# Chi-Square test



- Chi-square test can accommodate estimation of parameters
- Chi-square test requires data be placed in intervals
- Changing the number of classes and the interval width affects the value of the calculated and tabulated chi-sqaure
- A hypothesis could be accepted if the data grouped one way and rejected another way
- Distribution of the chi-square test static is known only approximately. So we need other tests

# Kolmogorov-Smirnov Test



- Intuition: formalize the idea behind examining a *q-q* plot
- Recall from Chapter 7.4.1:
  - The test compares the **continuous** cdf, $F(x)$, of the hypothesized distribution with the empirical cdf, $S_N(x)$, of the $N$ sample observations.
  - Based on the maximum difference statistics (Tabulated in A.8):

$$D = max|\, F(x) - S_N(x)|$$

- A more powerful test, particularly useful when:
  - *Sample sizes are small,*
  - *No parameters have been estimated from the data*.
- When parameter estimates have been made:
  - Critical values in Table A.8 are biased, too large.
  - More conservative.

# p-Values and "Best Fits"



- *p-value* for the test statistics
  - The significance level at which one would just reject $H_0$ for the given test statistic value.
  - A measure of fit, the larger the better
  - Large *p-value*: good fit
  - Small *p-value*: poor fit

- Vehicle Arrival Example (cont.):
  - $H_0$: data is Possion
  - Test statistics: $\chi_0^2 = 27.68$ , with *5* degrees of freedom
  - *p-value = 0.00004*, meaning we would reject $H_0$ with *0.00004* significance level, hence Poisson is a poor fit.

# p-Values and “Best Fits”

[Goodness-of-Fit Tests]

- Many software tools use *p-value* as the ranking measure to automatically determine the “best fit”.
  - Software could fit every distribution at our disposal, compute the test statistic for each fit and choose the distribution that yields largest *p*-value.
- Things to be cautious about:
  - Software may not know about the physical basis of the data, distribution families it suggests may be inappropriate.
  - Close conformance to the data does not always lead to the most appropriate input model.
  - *p-value* does not say much about where the lack of fit occurs
- Recommended: always inspect the automatic selection using graphical methods.

# Fitting a Non-stationary Poisson Process

- Fitting a NSPP to arrival data is difficult, possible approaches:
  - Fit a very flexible model with lots of parameters or
  - Approximate constant arrival rate over some basic interval of time, but vary it from time interval to time interval.

**Our focus**

- Suppose we need to model arrivals over time [0,T], our approach is the most appropriate when we can:
  - Observe the time period repeatedly and
  - Count arrivals / record arrival times.

# Fitting a Non-stationary Poisson Process

- The estimated arrival rate during the *i*th time period is:

$$\hat{\lambda}(t) = \frac{1}{n\Delta t}\sum_{j=1}^{n} C_{ij}$$

  - where n = # of observation periods, $\Delta t$ = *time interval length*

    $C_{ij}$ = # of arrivals during the $i^{th}$ time interval on the $j^{th}$ observation period

- Example: Divide a *10*-hour business day [*8am,6pm*] into equal intervals *k* = *20* whose length $\Delta t$ = ½, and observe over n =3 days

May be we can combine periods 8:30-9:00 and 9:00 to 9:30 as arrival rates are close

| Time Period | Number of Arrivals | | | Estimated Arrival Rate (arrivals/hr) |
|---|---|---|---|---|
| | Day 1 | Day 2 | Day 3 | |
| 8:00 - 8:30 | 12 | 14 | 10 | 24 |
| 8:30 - 9:00 | 23 | 26 | 32 | 54 |
| 9:00 - 9:30 | 27 | 18 | 32 | 52 |
| 9:30 - 10:00 | 20 | 13 | 12 | 30 |

For instance, 1/3(0.5)*(23+26+32) = 54 arrivals/hour

# Selecting Model without Data

- If data is not available, some possible sources to obtain information about the process are:
  - *Engineering data:* often product or process has performance ratings provided by the manufacturer or company rules specify time or production standards.
  - *Expert option:* people who are experienced with the process or similar processes, often, they can provide optimistic, pessimistic and most-likely times, and they may know the variability as well.
  - *Physical or conventional limitations:* physical limits on performance, limits or bounds that narrow the range of the input process.
  - *The nature of the process*.
- The uniform, triangular, and beta distributions are often used as input models.

# Selecting Model without Data

- Example: Production planning simulation.
  - Input of sales volume of various products is required, salesperson of product XYZ says that:
    - No fewer than *1,000* units and no more than *5,000* units will be sold.
    - Given her experience, she believes there is a *90%* chance of selling more than *2,000* units, a *25%* chance of selling more than *3,500* units, and only a *1%* chance of selling more than *4,500* units.
  - Translating these information into a cumulative probability of being less than or equal to those goals for simulation input:

| i | Interval (Sales) | Cumulative Frequency, $c_i$ |
|---|---|---|
| 1 | $1000 \leq x \leq 2000$ | 0.10 |
| 2 | $2000 < x \leq 3500$ | 0.75 |
| 3 | $3500 < x \leq 4500$ | 0.99 |
| 4 | $4500 < x \leq 5000$ | 1.00 |

# Multivariate and Time-Series Input Models

- **Multivariate:**
  - For example, lead time and annual demand for an inventory model, increase in demand results in lead time increase, hence variables are dependent.
- **Time-series:**
  - For example, time between arrivals of orders to buy and sell stocks, buy and sell orders tend to arrive in bursts, hence, times between arrivals are dependent.

Co-variance and Correlation are measures of the linear dependence of random variables

# Covariance and Correlation

[Multivariate/Time Series]

- Consider the model that describes relationship between $X_1$ and $X_2$:

$$(X_1 - \mu_1) = \beta(X_2 - \mu_2) + \varepsilon$$

$\varepsilon$ is a random variable with mean *0* and is independent of $X_2$

  - $\beta = 0$, $X_1$ and $X_2$ are statistically independent
  - $\beta > 0$, $X_1$ and $X_2$ tend to be above or below their means together
  - $\beta < 0$, $X_1$ and $X_2$ tend to be on opposite sides of their means

- Covariance between $X_1$ and $X_2$ :

$$\text{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E(X_1 X_2) - \mu_1 \mu_2$$

  - where $\quad cov(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases} \quad$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$

Co-variance can take any value between -∞ to ∞

# Covariance and Correlation

[Multivariate/Time Series]

- Correlation normalizes the co-variance to -1 and 1.
- Correlation between $X_1$ and $X_2$ (values between *-1* and *1*):

$$\rho = \mathrm{corr}(X_1, X_2) = \frac{\mathrm{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

  - where $corr(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$
  - The closer $\rho$ is to *-1* or *1*, the stronger the linear relationship is between $X_1$ and $X_2$.

# Auto Covariance and Correlation



- A "time series" is a sequence of random variables $X_1$, $X_2$, $X_3$, … , are identically distributed (same mean and variance) but dependent.
  - Consider the random variables $X_t$, $X_{t+h}$
  - cov($X_t$, $X_{t+h}$) is called the *lag-h autocovariance*
  - corr($X_t$, $X_{t+h}$) is called the *lag-h autocorrelation*
  - If the autocovariance value depends only on $h$ and not on $t$, the time series is covariance stationary

# Multivariate Input Models

[Multivariate/Time Series]

- If $X_1$ and $X_2$ are normally distributed, dependence between them can be modeled by the ***bi-variate normal distribution*** with $\mu_1$, $\mu_2$, $\sigma_1^2$, $\sigma_2^2$ and correlation $\rho$
  - To Estimate $\mu_1$, $\mu_2$, $\sigma_1^2$, $\sigma_2^2$, see "Parameter Estimation" (slide 16-18, Section 9.3.2 in book)
  - To Estimate $\rho$, suppose we have $n$ independent and identically distributed pairs $(X_{11}, X_{21})$, $(X_{12}, X_{22})$, … $(X_{1n}, X_{2n})$, then:

$$\hat{\text{cov}}(X_1, X_2) = \frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \hat{X}_1)(X_{2j} - \hat{X}_2)$$

$$= \frac{1}{n-1}\left(\sum_{j=1}^{n} X_{1j}X_{2j} - n\hat{X}_1\hat{X}_2\right)$$

$$\hat{\rho} = \frac{\hat{\text{cov}}(X_1, X_2)}{\hat{\sigma}_1\hat{\sigma}_2}$$

Sample deviation

# Multivariate Input Models

[Multivariate/Time Series]

- Algorithm to generate bi-variate normal random variables

1. Generate $Z_1$ and $Z_2$, two independent standard normal random variables (see Slides 38 and 39 of Chapter 8)
2. Set $X_1 = \mu_1 + \sigma_1 Z_1$
3. Set $X_2 = \mu_2 + \sigma_2(\rho Z_1 + Z_2\sqrt{1-\rho^2})$

- Bi-variate is not appropriate for all multivariate-input modeling problems
- It can be generalized to the *k*-variate normal distribution to model the dependence among more than two random variables

# Example (Lead time vs Demand)

- *X1* is the average lead time to deliver in months and *X2* is the annual demand for industrial robots.
- Data for this in the last 10 years is shown:

| *X1*: Lead time | *X2*: Demand |
|---|---|
| 6.5 | 103 |
| 4.3 | 83 |
| 6.9 | 116 |
| 6.0 | 97 |
| 6.9 | 112 |
| 6.9 | 104 |
| 5.8 | 106 |
| 7.3 | 109 |
| 4.5 | 92 |
| 6.3 | 96 |

# Example (Lead time vs Demand) contd ..

- From this data we can calculate:

$$\bar{X}_1 = 6.14, \qquad \hat{\sigma}_1 = 1.02; \qquad \bar{X}_2 = 101.8, \qquad \hat{\sigma}_2 = 9.93$$

- Correlation is estimated as:

$$\sum_{j=1}^{10} X_{1j} X_{2j} = 6328.5$$

$$\text{cov} = [6328.5 - (10)(6.14)(101.80)]/(10-1) = 8.66$$

$$\hat{\rho} = \frac{8.66}{(1.02)(9.93)} = 0.86$$

# Time-Series Input Models

[Multivariate/Time Series]

- If $X_1, X_2, X_3, \ldots$ is a sequence of identically distributed, but dependent and covariance-stationary random variables, then we can represent the process as follows:
  - ***Autoregressive order-1 model, AR(1)***
  - ***Exponential autoregressive order-1 model, EAR(1)***
    - Both have the characteristics that:

$$\rho_h = corr(X_t, X_{t+h}) = \rho^h, \quad \text{for } h = 1,2,\ldots$$

    - *Lag-h* autocorrelation decreases geometrically as the lag increases, *hence, observations far apart in time are nearly independent*

# AR(1) Time-Series Input Models

[Multivariate/Time Series]

- Consider the time-series model:

$$X_t = \mu + \phi(X_{t-1} - \mu) + \varepsilon_t, \quad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. normally distributed with $\mu_\varepsilon = 0$ and variance $\sigma_\varepsilon^2$

- If $X_1$ is chosen appropriately, then
  - $X_1, X_2, \ldots$ are normally distributed with *mean* = $\mu$, and *variance* = $\sigma_\varepsilon^2/(1-\phi^2)$
  - Autocorrelation $\rho_h = \phi^h$
- To estimate $\phi, \mu, \sigma_\varepsilon^2$ :

$$\hat{\mu} = \bar{X}, \qquad \hat{\sigma}_\varepsilon^2 = \hat{\sigma}^2(1-\hat{\phi}^2), \qquad \hat{\phi} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

# AR(1) Time-Series Input Models

[Multivariate/Time Series]

- Algorithm to generate AR(1) time series:

1. Generate $X_1$ from Normal distribution with *mean* = $\mu$, and *variance* = $\sigma_\varepsilon^2/(1-\phi^2)$. Set $t=2$
2. Generate $\varepsilon_t$ from Normal distribution with mean 0 and variance $\sigma_\varepsilon^2$
3. Set $X_t=\mu+\phi(X_{t-1}-\mu)+\varepsilon_t$
4. Set $t=t+1$ and go to step 2

# EAR(1) Time-Series Input Models

[Multivariate/Time Series]

- Consider the time-series model:

$$X_t = \begin{cases} \phi X_{t-1}, & \text{with probability } \phi \\ \phi X_{t-1} + \varepsilon_t, & \text{with probability } 1\text{-}\varphi \end{cases} \qquad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. exponentially distributed with $\mu_\varepsilon = 1/\lambda$, and $0 \le \phi < 1$

- If $X_1$ is chosen appropriately, then
  - $X_1, X_2, \ldots$ are exponentially distributed with *mean* = $1/\lambda$
  - Autocorrelation $\rho_h = \phi^h$, and only positive correlation is allowed.
- To estimate $\phi, \lambda$ :

$$\hat{\lambda} = 1/\bar{X} \quad , \qquad \hat{\phi} = \hat{\rho} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

# EAR(1) Time-Series Input Models



- Algorithm to generate EAR(1) time series:

1. Generate $X_1$ from exponential distribution with *mean* = $1/\lambda$. Set $t=2$
2. Generate $U$ from Uniform distribution [0,1].

    If $U \leq \phi$, then set $X_t = \phi X_{t-1}$.

    Otherwise generate $\varepsilon_t$ from the exponential distribtuion with mean $1/\lambda$ and set $X_t = \mu + \phi(X_{t-1} - \mu) + \varepsilon_t$
3. Set $t=t+1$ and go to step 2

# EAR(1) Time-Series Input Models

[Multivariate/Time Series]

- Example: The stock broker would typically have a large sample of data, but suppose that the following twenty time gaps between customer buy and sell orders had been recorded (in seconds): 1.95, 1.75, 1.58, 1.42, 1.28, 1.15, 1.04, 0.93, 0.84, 0.75, 0.68, 0.61, 11.98, 10.79, 9.71, 14.02, 12.62, 11.36, 10.22, 9.20. Standard calculations give

$$\bar{X} = 5.2 \text{ and } \hat{\sigma}^2 = 26.7$$

- To estimate the lag-1autocorrelation we need $\sum_{j=1}^{19} X_t X_{t+1} = 924.1$

- Thus, cov=[924.1-(20-1)(5,2)$^2$]/(20-1)=21.6 and

$$\hat{\rho} = 21.6/26.7 = 0.8$$

- Inter-arrivals are modeled as EAR(1) process with mean = 1/5.2=0.192 and $\phi$=0.8 provided that exponential distribution is a good model for the individual gaps

# Normal-to-Anything Transformation (NORTA)

- $Z$ is a Normal random variable with cdf $\phi(z)$
- We know $R=\phi(z)$ *is* uniform $U(0,1)$
- To generate any random variable X that has CDF F(x), we use the variate method:
- $X=F^{-1}(R)=F^{-1}(\phi(z))$
- To generate bi-variate non-normal:
- Generate bi-variate normal RVs *(Z1,Z2)*
- Use the above transformation
- Numerical approximations are needed to inverse

# Summary

- In this chapter, we described the 4 steps in developing input data models:
  - Collecting the raw data
  - Identifying the underlying statistical distribution
  - Estimating the parameters
  - Testing for goodness of fit