Chapter 7
Random-Number Generation

Banks, Carson, Nelson & Nicol

Discrete-Event System Simulation

Purpose & Overview

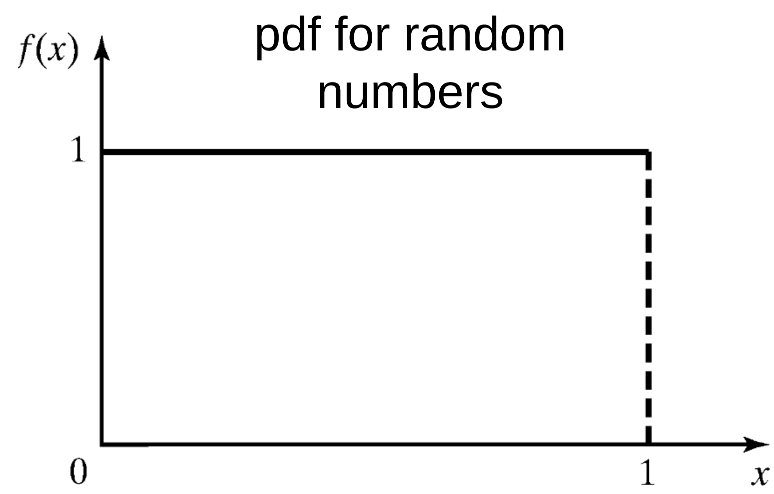
- Discuss the generation of random numbers.
  - Used to generate event times and other random variables

- Introduce the subsequent testing for randomness:
  - Frequency test
  - Autocorrelation test.

Properties of Random Numbers

- A sequence of random numbers R_1, R_2, …, must have two important statistical properties:
  - Uniformity
  - Independence.
- Random Number, R_i, must be independently drawn from a uniform distribution with pdf:

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



$$E(R) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$V(R) = \int_0^1 x^2 dx - [E(R)]^2 = \left.\frac{x^3}{3}\right|_0^1 - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Uniformity and Independence

- ***Uniformity:*** If the interval [0,1] is divided into n classes, or subintervals of equal length, the expected number of observations in each interval is N/n, where N is the total number of observations
- ***Independence:*** The probability of observing a value in a particular interval is independent of the previous value drawn

Generation of Pseudo-Random Numbers

- "Pseudo", because generating numbers using a known method removes the potential for true randomness.
  - If the method is known, the set of random numbers can be replicated!!
- ***Goal:*** To produce a sequence of numbers in [*0,1*] that simulates, or imitates, the ideal properties of random numbers (RN) - *uniform distribution and independence*.

Generation of PRNs (contd..)

- Problems that occur in generation of pseudo-random numbers (PRN)
  - Generated numbers might not be uniformly distributed
  - Generated numbers might be discrete-valued instead of continuous-valued
  - Mean of the generated numbers might be too low or too high
  - Variance of the generated numbers might be too low or too high
  - There might be dependence (i.e., correlation)

Generation of PRNs (contd..)

- Departure from uniformity and independence for a particular generation scheme can be tested.
- If such departures are detected, the generation scheme should be dropped in favor of an acceptable one.

Generation of PRNs (contd ..)

- Important considerations in RN routines:
  - *The routine should be fast*. Individual computations are inexpensive, but a simulation may require many millions of random numbers
  - *Portable to different computers* – ideally to different programming languages. This ensures the program produces same results
  - Have sufficiently *long cycle*. The *cycle length, or period* represents the length of random number sequence before previous numbers begin to repeat in an earlier order.
  - *Replicable*. Given the starting point, it should be possible to generate the same set of random numbers, completely independent of the system that is being simulated
  - *Closely approximate the ideal statistical properties* of uniformity and independence.

Random Number Generators

- Inventing techniques that seem to generate random numbers is easy
- Inventing techniques that really produce sequences that appear to be independent, uniformly distributed random numbers is very difficult
- Vast literature and rich theory is available on this topic
- Many hours of testing been devoted to establish properties of various generators

Techniques for Generating Random Numbers

- Linear Congruential Method (LCM).
  - Most widely used technique for generating random numbers
- Combined Linear Congruential Generators (CLCG).
  - Extension to yield longer period (or cycle)
- Random-Number Streams.

Linear Congruential Method [Techniques]

- To produce a sequence of integers, $X_1, X_2, \ldots$ between 0 and $m-1$ by following a recursive relationship:

$$X_{i+1} = (\underbrace{a}_{\text{The multiplier}} X_i + \underbrace{c}_{\text{The increment}}) \bmod \underbrace{m}_{\text{The modulus}}, \quad i = 0,1,2,\ldots$$

- X_0 is called the *seed (initial value)*
- The selection of the values for a, c, m, and X_0 drastically affects the statistical properties and the cycle length.
- If $c \neq 0$ then it is called *mixed congruential* method
- When $c=0$ it is called *multiplicative congruential* method

Linear Congruential Method

- The random integers are being generated in the range [*0,m-1*], and to convert the integers to random numbers:

$$R_i = \frac{X_i}{m}, \quad i = 1,2,...$$

Example [LCM]

- Use $X_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.
- The X_i and R_i values are:

  $X_1 = (17*27+43) \bmod 100 = 502 \bmod 100 = 2,$ $R_1 = 0.02;$
  $X_2 = (17*2+43) \bmod 100 = 77 \bmod 100 = 77,$ $R_2 = 0.77;$
  $X_3 = (17*77+43) \bmod 100 = 1352 \bmod 100 = 52$ $R_3 = 0.52;$
  …

- Notice that the numbers generated assume values only from the set $I = \{0,1/m,2/m,\ldots\ldots, (m-1)/m\}$ because each X_i is an integer in the set $\{0,1,2,\ldots\ldots,m-1\}$
- *Thus each Ri is discrete on I, instead of continuous on interval [0,1]*

Characteristics of a Good Generator

[LCM]

- Maximum Density
  - Such that the values assumed by R_i, $i = 1,2,\ldots$, leave no large gaps on $[0,1]$
  - Problem: Instead of continuous, each R_i is discrete
  - Solution: a very large integer for modulus m (e.g., $2^{31}-1$, 2^{48})
- Maximum Period
  - To achieve maximum density and avoid cycling.
  - Achieved by: proper choice of a, c, m, and X_0.
- Most digital computers use a binary representation of numbers
  - Speed and efficiency are aided by a modulus, m, to be (or close to) a power of 2.

Maximum Period or Cycle Length

- For m a power of 2, say $m=2^b$, and $c \neq 0$, the longest possible period is $P=m=2^b$, which is achieved when c is relatively prime to m (greatest common divisor of c and m is 1) and $a=1+4k$, where k is an integer
- For m a power of 2, say $m=2^b$, and $c=0$, the longest possible period is $P=m/4=2^{b-2}$, which is achieved if the seed X_0 is odd and if the multiplier a is given by $a=3+8k$ or $a=5+8k$ for some $k=0,1,\ldots$.
- For m a prime number and $c=0$, the longest possible period is $P=m-1$, which is achieved whenever the multiplier a has the property that the smallest integer k such that a^k-1 is divisible by m is $k=m-1$

Example

- Using the multiplicative congruential method, find the period of the generator for $a=13$, $m=2^6=64$ and $X_0=1,2,3$ *and* 4

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Xi	1	13	41	21	17	29	57	37	33	45	9	53	49	61	25	5	1
Xi	2	26	18	42	34	58	50	10	2								
Xi	3	39	59	63	51	23	43	47	35	7	27	31	19	55	11	15	3
Xi	4	52	36	20	4												

- $m=64$, $c=0$; Maximal period $P=m/4 = 16$ is achieved by using odd seeds $X_0=1$ and $X_0=3$ ($a=13$ is of the form $5+8k$ with $k=1$)
- With $X_0=1$, the generated sequence {1,5,9,13,…,53,57,61} has large gaps
- Not a viable generator !! Density insufficient, period too short

Example

- Speed and efficiency in using the generator on a digital computer is also a factor
- Speed and efficiency are aided by using a modulus *m* either a power of 2 ($=2^b$)or close to it
- After the ordinary arithmetic yields a value of aX_i+c, X_{i+1} can be obtained by dropping the leftmost binary digits and then using only the *b* rightmost digits

Example

- *$c=0$; $a=7^5=16807$; $m=2^{31}-1=2{,}147{,}483{,}647$ (prime #)*
- Period $P=m-1$ (well over 2 billion)
- Assume $X_0=123{,}457$

- $X_1=7^5(123457)\bmod(2^{31}-1)=2{,}074{,}941{,}799$
- $R_1=X_1/2^{31}=0.9662$
- $X_2=7^5(2{,}074{,}941{,}799) \bmod(2^{31}-1)=559{,}872{,}160$
- $R_2=X_2/2^{31}=0.2607$
- $X_3=7^5(559{,}872{,}160) \bmod(2^{31}-1)=1{,}645{,}535{,}613$
- $R_3=X_3/2^{31}=0.7662$
- ……….
- Note that the routine divides by *m+1* instead of *m.* Effect is negligible for such large values of *m.*

Combined Linear Congruential Generators

[Techniques]

- With increased computing power, the complexity of simulated systems is increasing, requiring longer period generator.
  - Examples: 1) highly reliable system simulation requiring hundreds of thousands of elementary events to observe a single failure event;
  - 2) A computer network with large number of nodes, producing many packets
- Approach: Combine two or more *multiplicative congruential generators* in such a way to produce a generator with good statistical properties

Combined Linear Congruential Generators

[Techniques]

- L'Ecuyer suggests how this can be done:
  - If $W_{i,1}$, $W_{i,2}, \ldots, W_{i,k}$ are any independent, discrete valued random variables (not necessarily identically distributed)
  - If one of them, say $W_{i,1}$ is uniformly distributed on the integers from 0 to m_1-2, then

$$W_i = \left(\sum_{j=1}^{k} W_{i,j} \right) \bmod m_1 - 1$$

is uniformly distributed on the integers from 0 to m_1-2

Combined Linear Congruential Generators

[Techniques]

- Let $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$, be the i^{th} output from k different multiplicative congruential generators.
  - The j^{th} generator:
    - Has prime modulus m_j and multiplier a_j and period is m_j-1
    - Produced integers $X_{i,j}$ is approx ~ Uniform on integers in $[1, m_j-1]$
    - $W_{i,j} = X_{i,j} -1$ is approx ~ Uniform on integers in $[0, m_j-2]$

Combined Linear Congruential Generators

[Techniques]

- Suggested form:

$$X_i = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \right) \bmod m_1 - 1 \qquad \text{Hence, } R_i = \begin{cases} \dfrac{X_i}{m_1}, & X_i > 0 \\ \dfrac{m_1 - 1}{m_1}, & X_i = 0 \end{cases}$$

  - The maximum possible period for such a generator is:

$$P = \frac{(m_1 - 1)(m_2 - 1)\dots(m_k - 1)}{2^{k-1}}$$

Combined Linear Congruential Generators

[Techniques]

- Example: For 32-bit computers, L'Ecuyer [1988] suggests combining $k = 2$ generators with $m_1 = 2{,}147{,}483{,}563$, $a_1 = 40{,}014$, $m_2 = 2{,}147{,}483{,}399$ and $a_2 = 40{,}692$. The algorithm becomes:

  Step 1: Select seeds

  - $X_{1,0}$ in the range $[1, 2{,}147{,}483{,}562]$ for the 1st generator
  - $X_{2,0}$ in the range $[1, 2{,}147{,}483{,}398]$ for the 2nd generator.

  Step 2: For each individual generator,

  $X_{1,j+1} = 40{,}014\ X_{1,j}$ mod $2{,}147{,}483{,}563$

  $X_{2,j+1} = 40{,}692\ X_{1,j}$ mod $2{,}147{,}483{,}399$.

  Step 3: $X_{j+1} = (X_{1,j+1} - X_{2,j+1})$ mod $2{,}147{,}483{,}562$.

  Step 4: Return

  $$R_{j+1} = \begin{cases} \dfrac{X_{j+1}}{2{,}147{,}483{,}563}, & X_{j+1} > 0 \\[2ex] \dfrac{2{,}147{,}483{,}562}{2{,}147{,}483{,}563}, & X_{j+1} = 0 \end{cases}$$

  Step 5: Set $j = j+1$, go back to step 2.

  - Combined generator has period: $(m_1 - 1)(m_2 - 1)/2 \sim 2 \times 10^{18}$

Random-Numbers Streams [Techniques]

- The *seed* for a linear congruential random-number generator:
  - Is the integer value X_0 that initializes the random-number sequence.
  - Any value in the sequence can be used to “seed” the generator.
- A *random-number stream*:
  - Refers to a starting seed taken from the sequence $X_0, X_1, \ldots, X_P$.
  - If the streams are *b* values apart, then stream *i* could defined by starting seed:

$$S_i = X_{b(i-1)} \text{ for } i = 1, 2, \cdots, \lfloor P/b \rfloor$$

  - Older generators: $b = 10^5$; Newer generators: $b = 10^{37}$.

Random-Numbers Streams (contd ..)

- A single random-number generator with *k* streams can act like *k* distinct virtual random-number generators
- To compare two or more alternative systems.
  - Advantageous to dedicate portions of the pseudo-random number sequence to the same purpose in each of the simulated systems.

Tests for Random Numbers

- Desirable properties of random numbers: *Uniformity* and *Independence*
- Number of tests can be performed to check whether these properties have been achieved or not
- Two type of tests:
  - *Frequency Test*: Uses the Kolmogorov-Smirnov or the Chi-square test to compare the distribution of the set of numbers generated to a uniform distribution
  - *Autocorrelation test:* Tests the correlation between numbers and compares the sample correlation to the expected correlation, *zero*

Tests for Random Numbers

- Two categories:
  - Testing for uniformity. The hypotheses are:

$$H_0: \quad R_i \sim U[0,1]$$

$$H_1: \quad R_i \nsim U[0,1]$$

    - Failure to reject the null hypothesis, H_0, means that evidence of non-uniformity has not been detected.
  - Testing for independence. The hypotheses are:

$$H_0: \quad R_i \sim \text{independently distributed}$$

$$H_1: \quad R_i \nsim \text{independently distributed}$$

    - Failure to reject the null hypothesis, H_0, means that evidence of dependence has not been detected.

Tests for Random Numbers

- For each test, a *Level of significance* α must be stated.
- The level α, is the probability of rejecting the null hypothesis H_0 when the null hypothesis is true:

$$\alpha = P(\text{reject } H_0 | H_0 \text{ is true})$$

- The decision maker sets the value of α for any test
- Frequently α is set to 0.01 or 0.05

Tests for Random Numbers

- When to use these tests:
  - If a well-known simulation languages or random-number generators is used, it is probably unnecessary to test
  - If the generator is not explicitly known or documented, e.g., spreadsheet programs, symbolic/numerical calculators, tests should be applied to many sample numbers.
- Types of tests:
  - *Theoretical tests*: evaluate the choices of *m, a*, and *c* without actually generating any numbers
  - *Empirical tests*: applied to actual sequences of numbers produced. *Our emphasis*.

Frequency Tests

- Test of uniformity
- Two different methods:
  - Kolmogorov-Smirnov test
  - Chi-square test
- Both these tests measure the degree of agreement between the distribution of a sample of generated random numbers and the theoretical uniform distribution
- Both tests are based on null hypothesis of no significant difference between the sample distribution and the theoretical distribution

Kolmogorov-Smirnov Test [Frequency Test]

- Non-parametric test
- Compares the continuous cdf, *F(x)*, of the uniform distribution with the empirical cdf, $S_N(x)$, of the *N* sample observations. $F(x) = x, \quad 0 \le x \le 1$
  - We know:
  - If the sample from the RN generator is $R_1, R_2, \ldots, R_N$, then the empirical cdf, $S_N(x)$ is:

$$S_N(x) = \frac{\text{number of } R_1, R_2, \ldots, R_n \text{ which are} \le x}{N}$$

- The cdf of an empirical distribution is a step function with jumps at each observed value (See example slide).

Kolmogorov-Smirnov Test [Frequency Test]

- Test is based on the largest absolute deviation statistic between $F(x)$ and $S_N(x)$ over the range of the random variable:

$$D = max|\, F(x) - S_N(x)|$$

- The distribution of D is known and tabulated (A.8) as function of N
- Steps:
  1. Rank the data from smallest to largest. Let $R_{(i)}$ denote i^{th} smallest observation, so that $R_{(1)} \leq R_{(2)} \leq \ldots \leq R_{(N)}$
  2. Compute $$D^+ = \max_{1 \leq i \leq N}\left\{\frac{i}{N} - R_{(i)}\right\}; \quad D^- = \max_{1 \leq i \leq N}\left\{R_{(i)} - \frac{i-1}{N}\right\}$$
  3. Compute $D = max(D^+, D^-)$
  4. Locate in Table A.8 the *critical value* $D\alpha$, for the specified significance level α and the sample size N *(degrees of freedom)*
  5. If the sample statistic D is greater than the critical value $D\alpha$, the null hypothesis is rejected. If $D \leq D\alpha$, conclude there is no difference

Kolmogorov-Smirnov Test [Frequency Test]

- Example: Suppose *5* generated numbers are *0.44, 0.81, 0.14, 0.05, 0.93*.

Step 1:

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
$i/N - R_{(i)}$	0.15	0.26	0.16	-	0.07
$R_{(i)} - (i-1)/N$	0.05	-	0.04	0.21	0.13

Step 2:

Arrange $R_{(i)}$ from smallest to largest

$D^{+} = \max \{i/N - R_{(i)}\}$

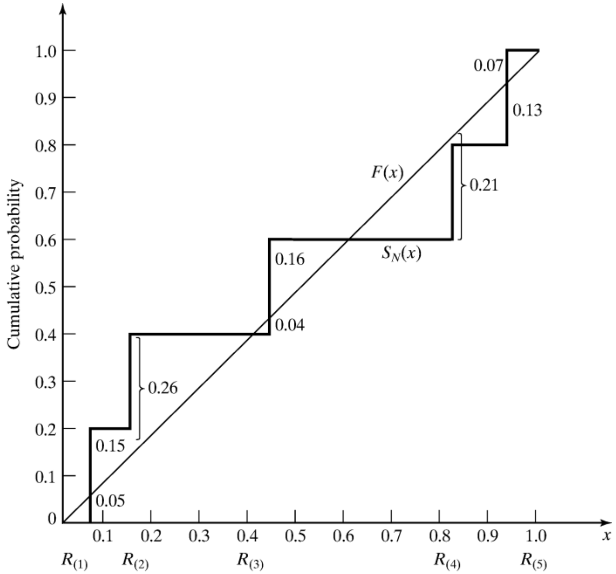
$D^{-} = \max \{R_{(i)} - (i-1)/N\}$

Step 3: $D = \max(D^{+}, D^{-}) = 0.26$

Step 4: For $\alpha = 0.05$,

$D_{\alpha} = 0.565 > D$

Hence, H_0 is not rejected.



Chi-square test

[Frequency Test]

- Chi-square test uses the sample statistic:

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

n is the # of classes

E_i is the expected # in the i^{th} class

O_i is the observed # in the i^{th} class

  - Approximately the chi-square distribution with *n-1* degrees of freedom (where the critical values are tabulated in Table A.6)
  - For the uniform distribution, E_i, the expected number in the each class is:

$$E_i = \frac{N}{n}, \quad \text{where N is the total \# of observation}$$

- Valid only for large samples, e.g. N >= 50
- Reject H_0 if $\chi_0^2 > \chi_{\alpha,N-1}^2$

Chi-square test

[Frequency Test]

- Example 7.7: Use Chi-square test for the data shown below with α=0.05. The test uses *n=10* intervals of equal length, namely [0,0.1),[0.1,0.2), …., [0.9,1.0)

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

Chi-square test

[Frequency Test]

- The value of χ_0^2=3.4; The critical value from table A.6 is $\chi_{0.05,9}^2$=16.9. Therefore the null hypothesis is not rejected

Table 7.3 Computations for Chi-Square Test

Interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1	8	10	−2	4	0.4
2	8	10	−2	4	0.4
3	10	10	0	0	0.0
4	9	10	−1	1	0.1
5	12	10	2	4	0.4
6	8	10	−2	4	0.4
7	10	10	0	0	0.0
8	14	10	4	16	1.6
9	10	10	0	0	0.0
10	11	10	1	1	0.1
	100	100	0		3.4

Tests for Autocorrelation

- The test for autocorrelation are concerned with the dependence between numbers in a sequence.
- Consider:

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

- Though numbers seem to be random, every fifth number is a large number in that position.
- This may be a small sample size, but the notion is that numbers in the sequence might be related

Tests for Autocorrelation

- Testing the autocorrelation between every *m* numbers (*m* is a.k.a. *the lag*), starting with the i^{th} number
  - The autocorrelation ρ_{im} between numbers: R_i, R_{i+m}, R_{i+2m}, $R_{i+(M+1)m}$
  - *M* is the largest integer such that $i+(M+1)m \leq N$
- Hypothesis:

$$H_0: \quad \rho_{im} = 0, \quad \text{if numbers are independent}$$

$$H_1: \quad \rho_{im} \neq 0, \quad \text{if numbers are dependent}$$

- If the values are uncorrelated:
  - For large values of M, the distribution of the estimator of ρ_{im}, denoted $\hat{\rho}_{im}$ is approximately normal.

Tests for Autocorrelation

- Test statistics is: $$Z_0 = \frac{\hat{\rho}_{im}}{\hat{\sigma}_{\hat{\rho}_{im}}}$$
  - Z_0 is distributed normally with mean = *0* and variance = *1*, and:

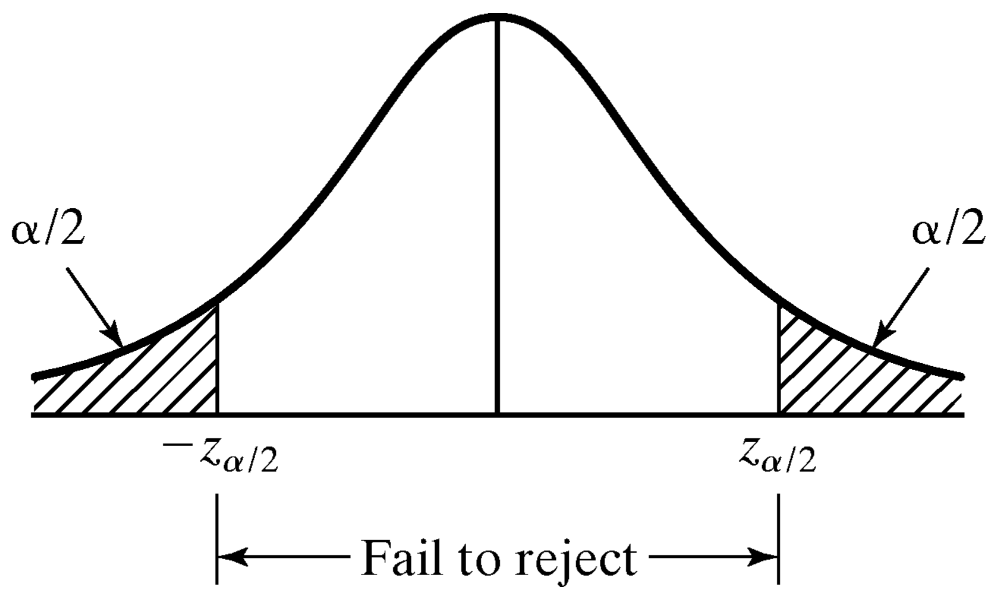
$$\hat{\rho}_{im} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

$$\hat{\sigma}_{\rho_{im}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

- If $\rho_{im} > 0$, the sub-sequence has positive autocorrelation
  - High random numbers tend to be followed by high ones, and vice versa.
- If $\rho_{im} < 0$, the sub-sequence has negative autocorrelation
  - Low random numbers tend to be followed by high ones, and vice versa.

Tests for Autocorrelation

- After computing Z_0, do not reject the hypothesis of independence if $-z_{\alpha/2} \leq Z_0 \leq z_{\alpha/2}$
- α is the level of significance and $z_{\alpha/2}$ is obtained from table A.3



Example

[Test for Autocorrelation]

- Test whether the *3rd, 8th, 13th*, and so on, for the output on Slide 37 are auto-correlated or not.
  - Hence, α = *0.05, i = 3, m = 5, N = 30,* and M = 4. M is the largest integer such that 3+(M+1)5≤30.

$$\hat{\rho}_{35} = \frac{1}{4+1}\begin{bmatrix}(0.23)(0.28)+(0.28)(0.33)+(0.33)(0.27) \\ +(0.27)(0.05)+(0.05)(0.36)\end{bmatrix} - 0.25$$

$$= -0.1945$$

$$\hat{\sigma}_{\rho_{35}} = \frac{\sqrt{13(4)+7}}{12(4+1)} = 0.128$$

$$Z_0 = -\frac{0.1945}{0.1280} = -1.516$$

  - From Table A.3, $z_{0.025} = 1.96$. Hence, the hypothesis is not rejected.

Shortcomings

- The test is not very sensitive for small values of *M*, particularly when the numbers being tested are on the low side.
- Problem when “fishing” for autocorrelation by performing numerous tests:
  - If $\alpha = 0.05$, there is a probability of 0.05 of rejecting a true hypothesis.
  - If 10 independent sequences are examined,
    - The probability of finding no significant autocorrelation, by chance alone, is $0.95^{10} = 0.60$.
    - Hence, the probability of detecting significant autocorrelation when it does not exist = *40%*

Summary

- In this chapter, we described:
  - Generation of random numbers
  - Testing for uniformity and independence

- Caution:
  - Even with generators that have been used for years, some of which still in use, are found to be inadequate.
  - This chapter provides only the basics
  - Also, even if generated numbers pass all the tests, some underlying pattern might have gone undetected.