HW2 — ARSHITHA BASAVARAJ

1. $\log_n \sqrt{n} < 70{,}000 < \log n^2 < (\log n)^2$

$< n^{1/5} < \sum_{k=1}^{n} \frac{n}{2^k} = \frac{7n}{10} < 3n^7 + 2$

$< 3^n < (5n+2)^n$

Justification

(i) $\log_n \sqrt{n} = \log_n (n)^{1/2}$

$= \frac{1}{2} \log_n n = \frac{1}{2}$

$\Rightarrow \frac{1}{2} < 70{,}000.$

(ii) By graphical comparison,

$\log n^2 < (\log n)^2$ only slightly though.

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If $f(n) = \log n^2$
$g(n) = (\log n)^2$ then

$f(n) \leq c\, g(n)$ where $\forall n > n_0$

$\log n^2 \leq c (\log n)^2$
Dividing by $\log n$ on both sides

$\Rightarrow$

$$\frac{2 \cancel{\log n}}{\cancel{\log n}} \leq c \frac{(\log n)^2}{\cancel{(\log n)}}$$

$\Rightarrow 2 \leq c . \log n$ $\forall n > 2$

$c = 2$

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$\Rightarrow 2 \leq 2 \log n \quad \forall n > 2$

$\Rightarrow f(n) \in O(g(n))$

$\Rightarrow \log n^2 \in O((\log n)^2)$

$\therefore \log n^2 \leq (\log n)^2$

(iii) $\sqrt[5]{n} = n^{1/5}$

(iv) Also,

$$\sum_{k=1}^{n} n/2^k$$

$$= n\left[\sum_{k=1}^{n} 1/2^k\right]$$

$$= n\left[\underbrace{1/2 + 1/2^2 + 1/2^3 + \cdots 1/2^n}_{\text{Geometric series, where } a = 1/2,\ r = 1/2}\right]$$

$$= n\left[\frac{a}{1-r}\right] \quad (\because 0 < r < 1)$$

$$= n\left[\frac{1/2}{1-1/2}\right] = \underline{\underline{n}}$$

$\frac{7n}{10}$ is of the same asymptotic order

(v) $n < 3n^7 + 2 < 3^n < (5n+2)^n$

~~order keeps increasing~~

order of n increases from 1 to 7.

$3^n \rightarrow$ 3 to the power of size of input.
rapid growth.

$(5n+2)^n \rightarrow$ both the base & the exponent
depend on the input size.

$f(n) = 3^n$

$g(n) = (5n+2)^n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3^n}{(5n+2)^n}$$

Taking ln on both numerator & denominator.

$$= \lim_{n\to\infty} \frac{\cancel{n} \ln 3}{\cancel{n} \ln(5n+2)} = \ln 3 \cdot \lim_{n\to\infty} \frac{1}{(5n+2)}$$

$$= \frac{\ln 3}{\ln(5n+2)} = \frac{\ln 3}{\infty} = \underline{\underline{0}}$$

$f(n) < c \cdot g(n) \quad \forall\, n > n_0$

$\Rightarrow f(n)$ is $o(g(n))$

$\Rightarrow f(n)$ is $O(g(n))$

$\therefore \; 3^n < (5n+2)^n$

A	B	O	o	Ω	ω	Θ
$n!$	e^{-n}	No	No	Yes	Yes	No
$\sum_{k=2}^{n} \frac{1}{\lg k}$	$\log(\log n)^n$	No	No	Yes	Yes	No
$10n^2+6n-2$	$\binom{n}{2}$	Yes	No	Yes	No	Yes
$\log(n^7)$	$e^{\ln(\ln n)}$	Yes	No	Yes	No	Yes
$\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k}\right)$	$\sqrt{n}$	Yes	Yes	No	No	No

2) (i) $f(n) = n!$

$g(n) = e^{-n}$

$g(n) \leq c \cdot f(n) \quad \forall \, n \geq n_0$

$e^{-n} \leq c \cdot n! \quad \forall \, n \geq n_0$

$$\frac{1}{e^n} \leq c \cdot \left(\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \right) \quad \forall \, n \geq n_0$$

Let $c = 1$ & $n_0 = 1$

$$\Rightarrow \frac{1}{e^n} \leq \sqrt{2\pi n} \left(\frac{n^n}{e^n} \right)$$

$\therefore$ Multiplying on both sides by e^n

$$\Rightarrow 1 \leq \sqrt{2\pi n} \, (n^n)$$

$$\Rightarrow 1 \leq n^n \sqrt{2\pi n} \quad \forall \, n \geq 1$$

$$\Rightarrow g(n) \in O(f(n))$$

$$\therefore f(n) \in \Omega(g(n))$$

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$\therefore f(n) \in \omega(g(n))$

If Ω & ω are true, then O, o & Θ aren't true.

(ii) $f(n) = 10n^2 + 6n - 2$

$g(n) = \binom{n}{2} = {}^nC_2$

$$g(n) = {}^nC_2 = \frac{n!}{(n-2)!\,2!}$$

$$= \frac{n(n-1)}{2!} = \frac{n^2-n}{2}$$

$$g(n) = \frac{n^2-n}{2}$$

a) $f(n) \leq c.g(n) \quad \forall \; n > n_0$

$$10n^2+6n-2 \leq c.\left(\frac{n^2-n}{2}\right) \quad \forall \; n > n_0$$

Let $c = 100$ & $n_0 = 5$

$$10n^2+6n-2 \leq 50n^2-50n \quad \forall \; n > 5$$

$\Rightarrow$ ~~$f(n) \in \Theta(g(n))$~~

b) $g(n) \le c \cdot f(n) \quad \forall\ n > n_0$

$$\frac{n^2 - n}{2} \le c \cdot (10n^2 + 6n - 2) \quad \forall\ n > n_0$$

Let $c = 10$ & $n_0 = 1$

$$\Rightarrow \frac{n^2 - n}{2} \le 100n^2 + 60n - 20 \quad \forall\ n > 1$$

$$\therefore \quad f(n) \in \Omega(g(n))$$

$$\Rightarrow f(n) \text{ is } \Theta(g(n))$$

If O, Θ & Ω are true, then o & ω are false.

(iv) $f(n) = \log(n^7) = 7 \log n$

$g(n) = e^{\ln(\ln n)}$

$= \ln n$

ay $f(n) \leq c \cdot g(n) \quad \forall \; n > n_0$

$7 \log n \leq c(\ln n)$

$\dfrac{7 \log_e n}{\log_e 10} \leq c \cdot \ln n$

$\dfrac{7}{\log_e 10} \leq c \quad \forall \; n > n_0$

If $c = 5$ & $n_0 = 1$

$\Rightarrow f(n)$ is $O(g(n))$

by $g(n) \le c \cdot f(n) \quad \forall n > n_0$

$\ln n \le c \cdot (7 \log n) \quad \forall n > n_0$

$\cancel{\ln n} \le 7c \left(\frac{\cancel{\ln n}}{\ln 10} \right)$

$1 \le \frac{7c}{\ln 10} \quad \forall n > 1$

$c = 8$

$1 \le \frac{7 \times 8}{\ln 10}$

$\Rightarrow g(n)$ is $O(f(n))$

$\therefore f(n)$ is $\Theta(g(n))$

$\Rightarrow O, \Theta, \Omega$ are true

$\Rightarrow \omega, o$ are false.

(v) $\sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k} \right) = f(n)$

$g(n) = \sqrt{n}$

$f(n) = \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \dots + \left(\frac{1}{n} - \frac{1}{(n-1)}\right)$

$= -1 + \frac{1}{n} = \frac{1}{n} - 1$

$g(n) = n^{1/2}$

a) $f(n) \le c.g(n) \quad \forall \; n > n_0$

$\frac{1}{n} - 1 \le c \cdot n^{1/2} \quad \forall \; n > n_0$

$\lim_{n \to \infty} \frac{\left(\frac{1}{n} - 1\right)}{\sqrt{n}} = 0$

$\Rightarrow$ $f(n)$ is $o(g(n))$

$\Rightarrow$ $f(n)$ is $O(g(n))$

$\therefore$ O, o are true

Ω, ω & Θ are false