**NAME (PRINT):** ______________________________
Last/Surname First /Given Name

**STUDENT #:** **SIGNATURE:**

**UNIVERSITY OF TORONTO MISSISSAUGA**
**DECEMBER 2017 FINAL EXAMINATION**
**MAT344H5F**
**Introduction to Combinatorics**
**Alex Rennet**
**Duration - 3 hours**
**Aids: None**

*The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of an exam. Clear, sealable, plastic bags have been provided for all electronic devices with storage, including but not limited to: cell phones, SMART devices, tablets, laptops, calculators, and MP3 players. Please turn off all devices, seal them in the bag provided, and place the bag under your desk for the duration of the examination. You will not be able to touch the bag or its contents until the exam is over.*

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*Please note, once this exam has begun, you* ***CANNOT*** *re-write it.*

**Justify all of your answers and show all of your work.**

This exam has **13 pages** including this page.

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1. **(2 points each = 8 points total)**

   (a) Find the number of arrangements of the letters in the word ENTANGLEMENT.

   (b) How many of the arrangements in part (a) either begin or end with a pair of Es? (i.e. are of the form "EE..." or "...EE".)

(c) How many of the arrangements in part (a) have no consecutive Es? (i.e. "EE" and "EEE" are not allowed anywhere in the arrangements.)

(d) Find the number of ways to distribute 300 identical toys to 70 (distinct) children.

2. **(4 points)** Let $f$ be a function from $[18]$ to $[4]$. Prove that there must be five distinct elements $v, w, x, y, z \in [18]$ so that $f(v) + f(w) + f(x) + f(y) + f(z)$ is divisible by 5.
*Hint: Pigeonhole Principle.*

3. **(4 points)** Recall that $S(n, k)$, *the Sterling numbers of the second kind*, stand for the number of partitions of $[n]$ into $k$ non-empty subsets.

   Using a **double counting** argument, prove the following, for any positive integers $n \geqslant k$:

   $$k \cdot S(n, k) = \sum_{m=0}^{n-1} \binom{n}{m} S(m, k-1)$$

   *Your argument shouldn't involve induction, applications of the Binomial Theorem, or give any sort of "algebraic" argument, etc.*

4. **(4 points)** Recall that $B(n)$, the $n$-th Bell number, is the number of all partitions of $[n]$ into any number of non-empty blocks. Let $F(n)$ be the number of all set partitions of $[n]$ into non-empty blocks, *with no singleton blocks* (i.e. no blocks of size 1).
Prove that $B(n) = F(n) + F(n+1)$ for all positive integers $n$.

5. **(7 points total)**

   (a) **(3 points)** Let $q_n$ denote the number of partitions of $n$ in which each part is repeated *at most* twice. Find a formula for the generating function $\sum_n q_n x^n$.

   (b) **(4 points)** Prove that $q_n$ (from part (a)) is equal to $r_n$, the number of partitions of $n$ with no parts divisible by 3.
   *Hint:* $(1 + x + ... + x^k)(1 - x) = (1 - x^{k+1})$.

6. **(4 points)** All $n$ students in a class stand in a line. The professor splits the line at several places, forming smaller (non-empty) groups. Then the professor chooses a (possibly empty) subset of the newly formed groups to work on Problem #1, while the rest of the groups work on Problem #2.

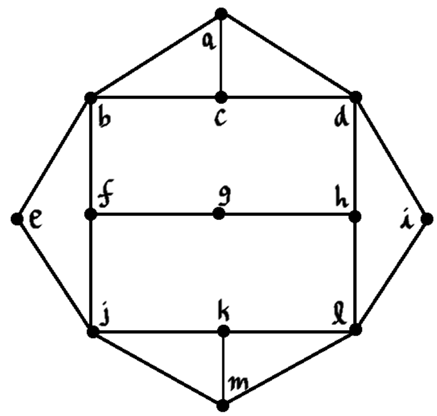
   *Using an argument involving the composition of generating functions,* determine the number of ways the professor can do this.

7. **(4 points)** Recall the following definitions from lecture:

   - Given a partition of $n$, the **Durfee square** of that partition is the largest $d \times d$ square of blocks that can be fit into the Ferrer's diagram of the partition; we say that the **Durfee number** of the partition is $d$ in this case.
   - The set of blocks in the Ferrer's diagram to the right of the Durfee square is called the **arm** of the partition, and the set of blocks below the Durfee square are called the **leg** of the partition.

   Derive a formula for the generating function $S_d(x) = \sum_n s_{n,d} x^n$ for $s_{n,d}$, the number of partitions of $n$ with (fixed) Durfee number $d$.

8. **(4 points)** Show that the following graph *does not* have a Hamiltonian cycle:



9. **(4 points each = 8 points total)**

   (a) Prove that for every connected planar graph $G$, if $G$ has twelve or fewer vertices, then $G$ has a vertex of degree *at most* four. *Hint: use a proof by contradiction.*

   (b) Prove that every connected planar graph $G$, if $G$ has twelve or fewer vertices, then $G$ can be 4-coloured. *Hint: use part (a), and mimic the proof of the 5-Colour Theorem.*

10. **(4 points)** Let $G = (X \cup Y, E)$ be a bipartite graph with its vertices split between the (disjoint) sets $X$ and $Y$. Suppose that there is a number $k \geqslant 1$ so that for all $x \in X$, $\deg(x) \geqslant k$ and for all $y \in Y$, $\deg(y) \leqslant k$. Prove that $G$ that has an $X$-matching.

11. **(4 points)** Complete the following proof that every tree $G$ has at most one perfect matching (a matching $M$ is perfect if it covers every vertex):

    *Suppose there are two perfect matchings,* $M$ *and* $M'$ *in* $G$.

    *Consider the graph* $G'$ *with the same vertices of* $G$ *but with only those edges from* $G$ *which are in* ***exactly one of*** $M$ ***or*** $M'$.

The End