Chapter 5 (5.3) - In-Class Worksheet

MAT344 - Spring 2019

Work earnestly! Work in groups!
Don't be afraid to ask questions, or check your work!

1

1.1 In terms of *partitions*, describe the number of many ways to take n people and divide them into groups, where only groups of size 3, 5, or 7 can be formed. **Answer in two ways:** (a) if people (as usual) are treated distinguishably, and (b) if all we care about is how many groups of each size there are.

1.2 How many ways are there to do the division in the second way (where we only care about the number of groups of each size), if $n = 17$?

1.3 Suppose that 745 people are divided into groups, and specifically that there are 100 groups of 7, 3 groups of 5 and 10 groups of 3. How many ways are there to do this in the first way (where people are distinguishable)?

2.1 Draw the **Ferrers shapes** of the following partitions of 12:

$$(6, 4, 2), \quad (4, 4, 2, 2), \quad (6, 2, 2, 2), \quad (3, 3, 2, 2, 1, 1), \quad (4, 4, 1, 1, 1, 1)$$

2.2 Prove, using an argument involving **Ferrers shapes**, that the number of partitions of n *with even-size parts* is equal to the number of partitions of n in which *each part occurs an even number of times*.

Hint: what is an operation we can perform on Ferrers shapes?

2.3 (*) What can we say about partitions of n with *odd-size parts*? (i.e. is there something similar we can do?) Try toying with the case $n = 7$.

3

Recall that $p_k(n)$ is the number of partitions of n into *exactly* k parts.

For the sake of this question, we define $p_{k,<m}(n)$ to be the number of partitions of n into k parts, where each of the parts is of size $< m$.

3.1 Show that $p_3(7) = p_{3,<7}(14)$ by finding all of the partitions of each kind "by hand".

3.2 Now prove that $p_3(n) = p_{3,<n}(2n)$ for all integers $n \geqslant 3$.

Hint: start by working with $n = 7$, and finding a way to "combine" two partitions of 7 into 3 parts into a single partition of 14 (again into 3 parts) in such a way that none of the parts is larger than 7. If you find a nice way to do this, you'll be able to create a bijection between the partitions of the first type and the partitions of the second type. You'll still need to prove this is indeed a bijection.

3.3 (*) Try to generalize the situation to more than 3 parts. (i.e. is there a formula involving $p_k(n)$ and $p_{k,<m}(l)$ for some m and l related to k and n? And can you prove it in a similar way?)

Throughout this question, n and m are positive integers.

4.1 Let $q_m(n)$ be the number of partitions of n with m parts, and with first *(and therefore largest!)* part of size exactly m.

(a) Compute $q_1(n)$, $q_2(n)$, and $q_n(n)$.

(b) Draw the Ferrers shapes of some of the partitions enumerated by $q_3(10)$ and $q_4(14)$.

4.2 For $m > 1$, define $r_m(n)$ be the number of partitions of n with strictly less than m parts and all parts of size strictly less than m.

(a) Compute $r_2(n)$, and $r_3(n)$.

(b) Draw the Ferrers shapes of some of the partitions enumerated by $r_3(5)$ and $r_4(7)$.

We require that n and m are integers satisfying $1 < m \leqslant n$.)

Prove that if $1 < m \leqslant n$ and additionally $2 \leqslant 2m - 1 \leqslant n$, then

$$q_m(n) = r_m(n - (2m - 1)).$$

It may help to compare your answers to 4.1(b) and 4.2(b).