Chapter 8 (8.1) - In-Class Worksheet - Part 2

MAT344 - Spring 2019

Recall **Theorem 8.5**:

- Let (a_n) and (b_n) each be sequences of numbers which for a given n, count the number of ways to do something to an n-element set *(e.g. pick 2, or put them in an order or colour each object one of three colours, or do nothing, or ... etc.)*.

  (The book calls this *building a structure* on a set of size n.
  Call our structures *type* A and *type* B.)

- Let (c_n) be the number of ways to take n distinguishable things (elements of $[n]$, say), split it into two consecutive (possibly empty) pieces, and build a structure of *type A* on the first piece and *type B* on the second piece.

- ... then $C(x) = A(x)B(x)$.

  (Where $C(x) = \sum c_n x^n,\ A(x) = \sum a_n x^n,\ B(x) = \sum b_n x^n$*.)*

Warmup 1:

What combinatorial observations can we make using the **special case** where

$$A(x) = B(x) = \frac{1}{1-x} = \sum x^n \ ?$$

Warmup 2:

What is the natural generalization to k-many functions $A_1(x), \ldots, A_k(x)$?

1.1 Pick functions $W_1(x), ..., W_k(x)$ so that $W(x) = \sum w_n x^n = \prod_{i=1}^{k} W_i(x)$, where w_n is the number of *weak compositions* of n into k boxes.

Hint: relate the creation of compositions to dividing up $[n]$ *into consecutive segments.*

1.2 Write $W(x)$ in closed form, then convert it to a series expression, manipulating it so that you get the usual expression for w_n.

1.3 Repeat the previous two parts, but for $C(x) = \sum c_n x^n$, where c_n is the number of *compositions* of n into k parts.

Hint: you can do this by modifying the functions C_i.

1.4 (*) Is there a way that *all* compositions of n can be treated in a similar fashion?

Recall **Theorem 8.13**:

- Let (a_n) be the number of ways to do build a structure ("of *type A*") on an n-element set, and assume $a_0 = 0$.
- Let (h_n) be the number of ways to take $[n]$ and split it into *an unknown number* of consecutive (*non*-empty) pieces, and build a structure of *type A* on each piece.
- ... then $H(x) = \dfrac{1}{1 - A(x)}$.

  (Where $H(x) = \sum h_n x^n, \; A(x) = \sum a_n x^n$*.)*

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2 How can we do something similar to the previous question, *but using Theorem 8.13 instead*, so that (h_n) is the number of *all compositions* of n?

Recall that w_n, the number of *weak compositions* of n into k boxes is equal to the number of (non-negative) integer solutions to the equation $a_1 + ... + a_k = n$.

Now let f_n be the number of solutions in the case where $k = 3$, *and* a_3 *must be odd.* Let $F(x) = \sum f_n x^n$.

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3.1 Find the closed form for $F(x)$ by writing it as a *product*.

Hint: You should the coefficient of x^n be for $F_3(x)$? To then get a closed-form for F_3, try writing it out *term-by-term* as a series (i.e. not in $\sum$-notation).

3.2 Use partial fractions to rewrite the closed form you got in the previous part. *(Yes, it will be a bit messy. Try it, then I will put up the answer.)*

3.3 Convert the partial fractions into their series forms, and use this to extract the total coefficient on x^n from the sum of these series. *This is a formula for* f_n*!*

3.4 Verify by hand that our formula is correct for $n = 1, 2, 3, 4$.

3.5 Think about how you would have tried to determine f_n using only Chapter 5 techniques. Would you get an equivalent answer? Aren't generating functions *great*?

4 Similarly to the previous question, let g_n be the number of positive integer solutions to $a_1 + 2a_2 + 3a_3 + ... + ka_k = n$, and let $G(x) = \sum g_n x^n$.

4.1 What is g_n equal in terms of Chapter 5 concepts?

4.2 Use a similar technique to the previous problem to get a closed form for $G(x)$. *Do not attempt to extract g_n this time.*

4.3 What happens if we try to do the same thing for the equation $\sum_{i \geqslant 1} ia_i = n$?

(i.e. What does this have to do with Ch5 concepts, and what is the closed form for the generating function $P(x) = \sum p_n x^n$, where p_n is the number of solutions to this new equation?)

5 Find closed-form generating functions for the following sequences, where the n-term, a_n, is the number of partitions of n with ...

5.1 ... only *even-sized* parts.

5.2 ... *distinct* even-sized parts.

5.3 ... with *at most two* parts of any given size (odd or even).

To simplify your closed form, fill in the blank and use the resulting identity:

$$(1 + x + x^2 + \ldots + x^n) = \frac{(\ldots ? \ldots)}{1 - x}.$$

5.4 Simplify your previous answer sufficiently so that you can see that the number of partitions of n with at most two parts of any size is equal to the number of partitions of n with no part of size *(fill in the blank)*.

5.5 * Try generalizing the previous two parts to "at most three parts", "at most four parts", etc.

Now, we'll use generating functions to prove an interesting result that has nothing to do with either (a) counting or (b) generating functions!

Theorem: Every positive integer can be written *uniquely* as a sum of *distinct* powers of 2.

6.1 Write the numbers 4, 35, and 678 as sums of distinct powers of 2, as the theorem above says we can do. *(Yes, this is just a form of the binary representation of the number.)*

6.2 For $n \geqslant 1$, let t_n be the number of ways we can write n as a sum of distinct powers of 2, and let $t_0 = 1$. And let $T(x)$ be the generating function for (t_n).

Write $T(x)$ in closed-form using a technique similar to what we've been doing in the previous two questions.

6.3 Use a similar technique to the last part of the previous problem to prove that $T(x) = \frac{1}{1-x}$.

6.4 What does this mean t_n is equal to, and why does this prove the theorem?

7 Consider the following kind of "drawing" in the plane ($\mathbb{R}^2$): it consists of finitely many "arrows", $\nearrow$ and $\searrow$, the first arrow is drawn with the tail at the origin, and successive arrows are drawn tip-to-tail (like vector addition); the last arrow must be a $\searrow$ with the tip on the x-axis; and finally, no arrow is allowed to have the tip below the x-axis. We call such a drawing a **mountain range**.

7.1 Draw the **mountain range** corresponding to the sequence $\nearrow\searrow\nearrow\nearrow\nearrow\searrow\nearrow\searrow\searrow\searrow$.

Now let r_n be the number of **mountain ranges** with $2n$ arrows, and set $r_0 = 1$.

7.2 Draw all **mountain ranges** for $n = 1, 2$ and 3 (i.e. with 2, 4, or 6 total arrows).

7.3 Explain why every **mountain range** has an equal number of $\nearrow$'s and $\searrow$'s occurring in it.

7.4 Call a **mountain range** just a **mountain** if at no point between it's beginning and it's end does an arrow touch the x-axis. Draw a **mountain range** consisting of exactly two consecutive **mountains**.

7.5 Explain why every **mountain** has the form

$$\text{“}\nearrow \textit{ mountain range } \searrow\text{”}.$$

Let $R(x) = \sum r_n x^n$ be the generating function for **mountain ranges**. And let $M(x) = \sum m_n x^n$, where m_n is the number of **mountains** of length $2n$ (setting $m_0 = 0$).

7.6 Find a functional equation which expresses $R(x)$ in terms of $M(x)$ which specifically captures the observation of the previous part of this question.

Hint: this will require treating "x" as accounting for a pair of arrows $\nearrow$ ... $\searrow$ in a ***mountain range.***

7.7 Now, convince yourself that the following *additional* functional equation is true using features of **mountain ranges** and **mountains**:

$$R(x) = \frac{1}{1 - M(x)}$$

Hint: fill in the sentence "Every ***mountain range*** *is just ...", and then relate this to the expression "$1 + M(x) + M^2(x) + M^3(x) + ...$"*

7.8 Next, substitute the first formula you obtained into the one given in the previous part to get a functional equation involving only $R(x)$.

7.9 Solve this equation using the quadratic formula (treating $R(x)$ as the variable), to get a closed-form expression for $R(x)$.

(Technically we get two roots for $R(x)$ here - why take the negative root only? Justify by plugging $x = 0$ into the equation you have for $R(x)$.)

7.10 Refer to Example 4.16 to express $R(x)$ as a series with a nice closed-form expression.

(Example 4.16 tells us that $\sqrt{1-4x} = 1 - 2x - 2\sum_{n \geqslant 2} \frac{1}{n}\binom{2n-1}{n-1}x^n$. *)*