MAT334 - Week 3 Problems

Additional Problems

1. Let $z_1 = 2+i$ and $z_2 = -3+i$. Assume that z^a is the principal branch. Show that:

   a) $\text{Log}(z_1 z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$

   b) $(z_1 z_2)^{1/2} \neq z_1^{1/2} z_2^{1/2}$

   c) Show that $((z_1 z_2)^{1/2})^2 = z_1 z_2$ and that $(z_1^{1/2} z_2^{1/2})^2 = z_1 z_2$.

   d) Find $z_1^{z_2}$ and $z_2{}^{z_1}$.

   Solution:

   a) Well, let's calculate these logarithms.
   We know that $2+i = \sqrt{5}e^{i\arctan(1/2)}$ and $-3+i = \sqrt{10}e^{i(\pi - \arctan(-1/3))}$ (remember that we need to adjust when our angle is in the second or third quadrant.
   So, $\text{Log}(z_1) + \text{Log}(z_2) = \ln(\sqrt{5}) + \ln(\sqrt{10}) + i(\pi + \arctan(1/2) - \arctan(-1/3))$. In particular, it should be clear by drawing a picture that the sum of these angles is ≥ 0.
   On the other hand, $z_1 z_2 = -7-i = \sqrt{50}e^{i(\arctan(1/7)-\pi)}$ and so $\text{Log}(z_1 z_2) = \ln(\sqrt{50}) + i(\arctan(1/7) - \pi)$. However, $\arctan(1/7) - \pi < 0$. So the angles for our two expressions do not match up.

   b) We can see this easily from the last part: $(z_1 z_2)^{1/2}$ has angle $\frac{\arctan(1/7)-\pi}{2}$. Notice that this lies in $(-\pi/2, 0)$.
   On the other handle, $z_1^{1/2}$ has angle $\arctan(1/2)/2$ and $z_2^{1/2}$ has angle $(\pi - \arctan(-1/3))/2$. If we add these together, we get that $z_1^{1/2} z_2^{1/2}$ has angle $(\pi - \arctan(-1/3) + \arctan(1/2))/2$ which lies in the interval $(\pi/2, \pi)$.
   So, since we can't add 2π to move our angle from $(-\pi/2, 0)$ to $(\pi/2, \pi)$, these must be different complex numbers.

   c) We can use a nice fact here: $(z^{1/2})^2 = z$. In particular:

   $$(z^{1/2})^2 = (e^{1/2\text{Log}(z)})^2 = e^{\text{Log}(z)} = z$$

   So, $((z_1 z_2)^{1/2})^2 = z_1 z_2$. And $(z_1^{1/2})^2 = z_1$, $(z_2^{1/2})^2 = z_2$, so

   $$(z_1^{1/2} z_2^{1/2})^2 = (z_1^{1/2})^2 (z_2^{1/2})^2 = z_1 z_2$$

   d) We're looking for $(2+i)^{-3+i}$ and $(-3+i)^{2+i}$ using the principal branch.

   $$\begin{aligned}(2+i)^{-3+i} &= e^{(-3+i)\text{Log}(2+i)} \\ &= e^{(-3+i)(\ln(\sqrt{5}) + i\arctan(1/2))} \\ &= e^{-3\ln(\sqrt{5}) - \arctan(1/2)} e^{i(\ln(\sqrt{5}) - 3\arctan(1/2))}\end{aligned}$$

   Similary, $(-3+i)^(2+i) = e^{2\ln(\sqrt{10}) - (\pi - \arctan(-1/3))} e^{i(\ln(\sqrt{10}) + 2(\pi - \arctan(1/3)))}$.

2. Let $w \in \mathbb{C}$ such that $w^4 = z$. Show that there exists a branch of the 4th root function $f(z) = z^{1/4}$ so that $f(z) = w$.

Solution: Let $w = re^{i\theta}$. Then $z = r^4 e^{4i\theta}$.

Let's look at all possible values of $z^{1/4}$. We have:

$$\begin{aligned} z^{1/4} &= e^{\frac{1}{4}\log(z)} \\ &= e^{\frac{1}{4}(\ln|z|+i\arg(z))} \\ &= e^{\frac{1}{4}(\ln(r^4)+i(4\theta+2k\pi))} \\ &= e^{\frac{1}{4}4\ln(r)+i(\theta+\frac{k\pi}{2})} \\ &= e^{\ln r}e^{i\theta+i\frac{k\pi}{2}} \\ &= re^{i\theta}e^{i\frac{k\pi}{2}} \qquad = we^{i\frac{k\pi}{2}} \end{aligned}$$

So, to get a branch where $z^{1/4} = w$, we need to choose $\arg(z)$ so that $k = 0$ (or $\pm 4, \pm 8, ...$; the choice isn't unique).

What the means is that we choise choose a branch of the argument so that $\arg(z) = 4\theta$. So we need to set $\arg(a) \in (\Psi, \Psi + 2\pi)$ so that $\Psi < 4\theta < \Psi + 2\pi$.

So, choose the branch of the argument so that $\arg(a) \in (4\theta - \pi, 4\theta + \pi)$. This ensures that 4θ is in the range of angles we want, which gives us that $\arg(z) = 4\theta$ as desired.

3. Let $a \in (0, \infty)$. Find all $z \in \mathbb{C}$ so that $\mathrm{Arg}(z^a) = a\mathrm{Arg}(z)$, where z^a is the principal branch.

Solution: First, notice that $a\mathrm{Arg}(z)$ is an argument for z^a. Why?

$$z^a = e^{a(\ln|z|+i\mathrm{Arg}(z))} = |z|^a e^{ia\mathrm{Arg}(z)}$$

Which tells us that z^a has angle $a\mathrm{Arg}(z)$.

So what's the issue? Well, sometimes $a\mathrm{Arg}(z)$ can land outside the range $(-\pi, \pi)$. If it is in this range, then $a\mathrm{Arg}(z) = \mathrm{Arg}(z^a)$ since it is an argument inside the correct range.

This means that $a\mathrm{Arg}(z) = \mathrm{Arg}(z^a)$ if and only if $-\pi < a\mathrm{Arg}(z) < \pi$, or: $\frac{-\pi}{a} < \mathrm{Arg}(z) < \frac{\pi}{a}$.

4. Let $a, b \in \mathbb{C}$. Show that $z^a z^b = z^{a+b}$ and $\frac{z^a}{z^b} = z^{a-b}$.

Solution: $z^a z^b = e^{a\log(z)}e^{b\log(z)} = e^{(a+b)\log(z)} = z^{a+b}$

And $\frac{z^a}{z^b} = e^{a\log(z)}e^{-b\log(z)} = e^{(a-b)\log(z)} = z^{a-b}$.

5. Show that $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$ if and only if $\mathrm{Arg}(z) + \mathrm{Arg}(w) \in (-\pi, \pi)$. Use this to show that $(zw)^a = z^a w^a$ if and only if $\mathrm{Arg}(z) + \mathrm{Arg}(w) \in (-\pi, \pi)$.

Solution: To start, $\mathrm{Log}(zw) = \ln(|zw|) + i\mathrm{Arg}(zw)$

But $\mathrm{Log}(z) + \mathrm{Log}(w) = \ln(|z|) + i\mathrm{Arg}(z) + \ln(|w|) + i\mathrm{Arg}(w) = \ln(|zw|) + i(\mathrm{Arg}(z) + \mathrm{Arg}(w)$.

As in question, $\mathrm{Arg}(z) + \mathrm{Arg}(w)$ is an argument for zw. But it is only the principal argument if $\mathrm{Arg}(z) + \mathrm{Arg}(w) \in (-\pi, \pi)$. So $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$ only if $\mathrm{Arg}(z) + \mathrm{Arg}(w) = \mathrm{Arg}(zw)$, which occurs only when $\mathrm{Arg}(z) + \mathrm{Arg}(w) \in (-\pi, \pi)$.

For the second part, note that:

$$(zw)^a = e^{a\mathrm{Log}(zw)} = e^{a(\ln(|zw|)+i\mathrm{Arg}(zw))}$$
$$z^a w^a = e^{a\mathrm{Log}(z)} e^{a\mathrm{Log}(w)} = e^{a(\ln|z|+i\mathrm{Arg}(z))} e^{a(\ln|w|+i\mathrm{Arg}(w))} = e^{a(\ln(|zw|)+i\mathrm{Arg}(z)+i\mathrm{Arg}(w)}$$

These two quantities are equal if and only if:

$$a\mathrm{Arg}(zw) = a(\mathrm{Arg}(z) + \mathrm{Arg}(w)) + 2k\pi$$

But if $\mathrm{Arg}(z) + \mathrm{Arg}(w) \notin (-\pi, \pi)$ (which would be fine, since then $k = 0$ works), we have $\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) \pm 2\pi$. For the above to be true, it would need to be the case that $a \in \mathbb{Z}$.

So, a more correct condition is that if $a \notin \mathbb{Z}$, then we need $\mathrm{Arg}(z) + \mathrm{Arg}(w) \in (-\pi, \pi)$.

6. Show that $((zw)^{1/2})^2 = zw$ and $(z^{1/2}w^{1/2})^2 = zw$. This shows that while they may not be the same square root, these two expressions both give square roots for zw.

   Solution: I proved this in general in my solutions to 1b.

7. Let $n \in \mathbb{N}$. Prove that $z^n = e^{n\log(z)}$ for any choice of logarithm $\log(z)$.

   Solution: Remember that if $z = re^{i\theta}$, then $z^n = r^n e^{i(n\theta)}$. In particular, notice that $\theta = \arg(z)$ for some branch of the argument.

   On the other hand, $e^{n\log(z)} = e^{n(\ln(r)+i\theta)} = e^{\ln(r^n)} e^{in\theta} = r^n e^{in\theta} = z^n$.

8. Find the range of $\sin(z)$ and of $\tan(z)$.

   Solution: Let's start with $\sin(z)$. Let $w \in \mathbb{C}$. We're trying to determine if there exists some $z \in \mathbb{C}$ with $\sin(z) = w$.

   Well, remember that $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, so if $\sin(z) = w$ we have:

$$\frac{e^{iz} - e^{-iz}}{2i} = w$$

   Multiply both sides by $2ie^{iz}$ to get:

$$(e^{iz})^2 - 1 = 2iwe^{iz}$$

   Rearranging gives:

$$(e^{iz})^2 - 2iwe^{iz} - 1 = 0$$

   This occurs if and only if $e^{iz} = \frac{2iw+(-4w^2+4)^{1/2}}{2} = iw + (1 - w^2)^{1/2}$ for some square root of $1 - w^2$, which we know always exists.

   When does this occur? Well, the range of the function e^{iz} is $\mathbb{C} \setminus \{0\}$, so we can find such a z if and only if:

$$iw + (1 - w^2)^{1/2} \neq 0$$

   So which w make this 0. Assume that $iw + (1 - w^2)^{1/2} = 0$. Rearranging gives:

$$(1-w^2)^{1/2} = -iw$$

Squaring both sides yields:

$$1 - w^2 = -w^2$$

Which never occurs, since $1 \neq 0$.

So $iw + (1-w^2)^{1/2} \neq 0$ for all w, and so $e^{iz} = iw + (1-w^2)^{1/2}$ always has a solution.

Therefore, the range of $\sin(z)$ is $\mathbb{C}$.

For $\tan(z)$, remember that $\tan(z) = \frac{\sin(z)}{\cos(z)} = i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$.

So, $w = \tan(z)$ if and only if $\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = iw$.

If we multiply both sides by $e^{iz}(e^{iz}+e^{-iz})$, we get:

$$((e^{iz})^2 - 1) = iw((e^{iz})^2 + 1)$$

Rearranging gives:

$$(1 - iw)e^{iz} = 1 + iw$$

So $\tan(z) = w$ if and only if $e^{iz} = \frac{1+iw}{1-iw}$.

This can only occur when the quotient is defined and non-zero (since $e^{iz} \neq 0$). So:

$$1 - iw \neq 0$$

$$1 + iw \neq 0$$

Solving for w yields that $w \neq \pm i$. So the range of $\tan(z)$ is $\mathbb{C} \setminus \{i, -i\}$.