SOLUTIONS

Tutorial Quiz 1

MAT344 - Spring 2019

INSTRUCTIONS:

Please record **each Group member's Name and Student Number**.

Make sure to show your work, justifying where possible and annotating any interesting steps or features of your work. **Do not just give the final answer, and do not simplify your calculations (use notation from the course, like $\binom{n}{k}$ or $S(n,k)$ etc.)**

1 **(2 points each ⇒ 4 points)** *(This whole question is identical to the individual version.)*

3333 aliens head towards planet Earth from a distant star. They are from a race of identical clones, each completely indistinguishable from the others.

1.1 When the spaceships reach Earth, they split up into groups to visit some of Earth's 100 largest cities. Some cities could have no spaceships visit.
How many ways are there for the spaceships to split up in this way?

1.2 Now suppose a spaceship with 123 of the aliens arrives above Toronto. This time, for the sake of the humans, the aliens decide to give themselves each unique names , like "Ziggy" and "Marvin" (and they put on name tags) to distinguish themselves from each other.
Then they split into groups and get into landing pods to take them from the spacecraft down to the ground. These landing pods are *identical* to each other and are unlimited in number, but each pod that gets launched from the ship will have at least one alien in it.
How many ways can the 123 aliens split up into landing pods?

SOLUTION

1.1 $\binom{3333+99}{99} = \binom{3432}{99} = \binom{3432}{3333}$

1.2 $B(123)$

2 **(2 points + 1 bonus point available)**

2.1 **(2 points)** *(This part of the question is identical to #2 from the individual version.)*

How many ways are there to hand out 12 identical pieces of red licorice candy and 16 identical pieces of black licorice to four children, if each child must receive at least one piece of *black* licorice and must receive an even number of pieces of *red* licorice?

2.2 **(1 bonus point)** What if instead, there are the same starting amounts of candy, but only two children, and the restriction is that each child must receive an odd amount of licorice pieces *in total* (irrespective of colour)?

SOLUTION

2.1 $\binom{16-4+4-1}{4-1} = \binom{15}{3}$ for black licorice, and $\binom{(12/2)+4-1}{4-1} = \binom{9}{3}$ for the red. The choices don't affect each other, so we just multiply these for the final answer.

2.2 Let's call the children Alice and Bob. In this problem, we distribute an odd number to one of them, then the rest of the licorice to the other (so a distribution to one child fixes the entire distribution - so we focus only on what Alice receives. For example, if Alice receives an odd number of red pieces, then so does Bob since the total adds up to an even number, 12.)

In order to end up distributing an odd number of licorice to Alice, an odd number of red or black is given, and an even number of the other colour is given. Therefore, there are two cases for Alice's licorice distribution (and hence for the whole problem):

(i) (#red,#black)=(odd,even)

(ii) (#red,#black)=(even,odd)

In (i), we can count the ways by first giving Alice a single red licorice, then giving her even amounts of both colours. But then Alice gets either 0 or 2 or... or 10 red pieces, giving us 6 ways to give her red licorice. For the black pieces she can get 0 or 2 or ... or 16, giving us 9 ways, so $\mathbf{6} \cdot \mathbf{9} = \mathbf{54}$ ways.

(Notice that there were 12 red pieces, we gave 1 at the start, and then can either give 0, 2, 4, 6, 8 or 10 more pieces. But no matter what, there's at least 1 piece left for Bob, who must also get an odd number. For the black pieces, we give any even number to Alice.)

Case (ii) is counted in a similar fashion, giving us $\mathbf{7} \cdot \mathbf{8} = \mathbf{56}$.

The total number of ways is $\mathbf{54} + \mathbf{56} = \mathbf{110}$.

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3 **(2 + 1 bonus point available)**

Suppose there are $n \geqslant 2$ students to be split into $k \geqslant 1$ groups. We will count ways to form the groups, where we will only be concerned with which students end up together (i.e. the groups are not numbered or otherwise distinguishable from each other).

3.1 **(2 points)** *(This part of the question is identical to #3 from the individual version.)*

Determine a formula for $T(n,k)$, the number of ways to split up the students with the following restriction: there are two particular students, *Legolas* and *Gimli*, who cannot be in a group together.

3.2 **(1 bonus point)** Repeat the previous part, but this time *Legolas* and *Gimili* can *either* be in a group together with no one else *or* they have to be put in different groups from each other.

Solution

3.1 This can most immediately be counted by *counting the complement*: consider all (unrestricted) possibilities and then subtract the cases where the restriction is violated (i.e. where L & G *are* together). This gives $S(n,k) - S(n-1,k)$ (the second term because we can consider "LG" to be one object to be distributed instead of two.)

3.2 This is similar: we add the answer from the previous part to the (disjoint) cases where they go together with no one else: these are counted by $S(n-2,k-1)$, since we use up a box and two objects.

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4 **(2 bonus points)**

Let $N(n)$ be the number of all partitions of $[n]$ with *no* singleton blocks.

Let $A(n)$ be the number of all partitions of $[n]$ with *at least one* singleton block.

Prove that for all $n \geqslant 1$, $N(n+1) = A(n)$.

SOLUTION

We can do this as follows: describe how to turn a partition of the first type (1) into a partition of type (2) "bijectively" (yes we could make this more formal, but an informal argument like this was sufficient): **take all the singletons from the partition of $[n]$ and put them into a block together and add the number "$n + 1$" to it.**

First notice that the leftover partition has (a) no singletons, and is (b) a partition of $[n + 1]$ (so it is a partition of type (2)). i.e. we have defined a function from partitions of the first type to the second.

We can see that for any partition of type (2), we get it's "pre-image" under this construction by taking the block with $n + 1$ (which has at least one other element) and splitting any of the other elements of that block into singletons, thereby giving us a partition of type (1) (this proves "surjectivity").

Now notice that if we take two distinct partitions of type (1), then the result of this operation clearly creates distinct partitions of type (2): if they created the same thing, the two inputs would have to share the same singletons and share the same non-singleton blocks (hence all their blocks are the same!), making them the same partitions of $[n]$ (so this operation is "injective").