Final Exam - EXTRA PRACTICE

MAT344 - Fall 2018

***This is not a practice exam, it's just a set of extra problems to practice with.* I don't claim that it is comprehensive.**

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1 A **directed rooted tree** is a tree together with a particular vertex $\mathbf{r}$ selected and where all edges point "away" from the root: if $\mathbf{r}$ is adjacent to $\mathbf{v}$, then $\mathbf{r} \to \mathbf{v}$ and $\mathbf{v}$ is called a **child** of $\mathbf{r}$, if $\mathbf{w}$ is adjacent to a child $\mathbf{v}$ of $\mathbf{r}$, then $\mathbf{r} \to \mathbf{v} \to \mathbf{w}$ and $\mathbf{w}$ is called a **child** of $\mathbf{v}$, etc.

For $n \geqslant 1$, we let t_n be the number of directed rooted trees **with n vertices**,[1] and let $T(x)$ be the generating function for the sequence (t_n).

Draw all of the directed rooted trees (up to isomorphism) for $n \leqslant 3$. Then, by finding the closed form of $T(x)$ and then extracting the coefficient of x^n in the power series form of $T(x)$, **prove that** $t_n = \dfrac{1}{n}\dbinom{2n-2}{n-1}$ for all $n \geqslant 1$.

Hint: Any such tree can be made by gluing some unknown number of such trees to a single root vertex.

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2 For each of the families of graphs listed below, determine the following (your answers may depend on n):

(a) Do the graphs have any Eulerian Cycles?

(b) Do the graphs have any Hamiltonian Cycles?

(c) What are the **chromatic number(s)** of the graphs?

(d) Are the graphs bipartite?

(e) Are the graphs planar? If they are planar, how many *faces* do they have?

(f) How many vertices and edges are there in the graphs?

(g) What is/are the centre vertice(s) in the graphs?

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[1] We are not considering the vertices to be *labelled*; therefore t_1 is equal to 1, since a pair of vertices with one edge between them will be isomorphic without labels even when considering "different" choice of root vertex.

2.1 C_n, the **cycle graph** of length n, $n \geqslant 3$, consisting of just a cycle with n edges and vertices.

2.2 W_n, the **wheel graph** of length n, $n \geqslant 1$, consisting of a copy of C_n together with one new vertex which is connected to each of the vertices in the cycle.

2.3 K_n, the **complete graph** on n vertices, $n \geqslant 1$

2.4 $K_{n,n}$, the **complete bipartite graph** with $n+n$ vertices, $n \geqslant 1$

2.5 $G_{n,n}$, the $n \times n$ **grid graph** (i.e. like a $n \times n$ chessboard, with vertices at the corners of squares), $n \geqslant 1$.

3

Find the number of ways to distribute 90 candies to three children if the oldest child gets 30, the middle child gets 40, and the youngest child gets 20.

4

Give **combinatorial proofs** of the following statements:

4.1 $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m} = \binom{n}{m,k,n-m-k}$

4.2 $\sum_{k \leqslant m \leqslant n} \binom{n}{m} S(m,k) S(n-m,l) = \binom{k+l}{l} S(n, k+l)$, where k, l, n are fixed positive integers with $k + l \leqslant n$.

5

5.1 Show that the number of partitions of $m^2 + n$ with Durfee square of size $m \times m$ is equal to

$$\sum_{k=0}^{n} p_{\leqslant m}(k) \cdot p_{\leqslant m}(n-k)$$

where $p_{\leqslant m}(k) = \sum_{i=0}^{m} p(k,i)$ is the number of partitions of k with at most m parts.

5.2 Use the previous part to show that $p(n^2 + 2n) > p(n)^2$, where $p(k)$ is the number of partitions of k. *(This second part of the question is Exercise 32 in the textbook in Chapter 5.)*

6

6.1 Prove that the number of partitions with k distinct parts is equal to the number of partitions of n has exactly the numbers from 1 through k as parts, each occurring at least once. *(k is some fixed positive integer.)*

6.2 Prove that the amount (for fixed k) of partitions counted by both descriptions in the previous part is also equal to $p_{\leqslant k}\left(n - \binom{k+1}{2}\right)$.

7

Let c_n be the number of weak compositions $r_1 + ... + r_n = n$ with the property that r_i is a multiple of i for $i = 1, ..., n$. Find the closed form generating function $C(x)$ for the sequence (c_n). What is the relationship between c_n and $p(n)$, the number of partitions of n?

8 Prove the following identity by considering the Durfee square of a self-conjugate partition:

$$\sum_{m \geqslant 0} \frac{x^{m^2}}{(1-x^2)(1-x^4)\cdot \ldots \cdot (1-x^{2m})} = \prod_{i \text{ odd}} (1+x^i) = \prod_{i \geqslant 1} \frac{1}{1+(-x)^i}$$