# MAT334 - Week 9 Solutions

## Additional Problems

1. Find power series for the following functions:

   a) $p(z)$, where $p$ is any polynomial, centered at $z_0 = 0$.
   b) $e^{2z}$ centered at $z_0 = 0$.
   c) $e^{az+b}$, centered at $z_0 = 0$.
   d) $e^{az+b}$, centered at $z_0 = -\frac{b}{a}$.
   e) $z\sin(z^3)$, centered at $z_0 = 0$.
   f) $\frac{z^2+1}{1-(iz^3)}$, centered at $z_0 = 0$.
   g) $3z^3\cos(z^3) + \sin(z^3)$, centered at $z_0 = 0$.
   h) $\frac{nz^{n-1}}{1-z^n}$, centered at $z_0 = 0$.
   i) $\frac{1}{z}$, centered at $z_0 = 1$.
   j) $\frac{1}{z^n}$, centered at $z_0 = 1$.
   k) $\frac{1}{a-z}$, centered at $z_0 = 0$, for any $a \neq 0$.
   l) $\frac{1}{a^2-z^2}$, centered at $z_0 = 0$, for any $a \neq 0$.
   m) $\mathrm{Log}(z)$, centered at $z_0 = 1$.
   n) $\log_1(z)$, centered at $z_0 = 1$, where $\log_1$ is the branch with $\arg(z) \in (3\pi, 5\pi)$.
   o) $2z^2\log_1(z^3) - e^z$, centered at $z_0 = 1$, same branch as last part.

**Solution:**

a) Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0$ be an arbitrary polynomial. Then it is already a power series centered at 0, with $a_k = 0$ for $k > n$.

b) We know $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$. So $e^{2z} = \sum_{n=0}^\infty \frac{2^n z^n}{n!}$.

c) We have $e^{az+b} = e^b e^{az} = \sum_{n=0}^\infty \frac{e^b a^n z^n}{n!}$.

d) We could use the formula for coefficents in terms of derivatives. Or, we could notice:

$$e^{az+b} = e^{a(z-\frac{-b}{a})} = \sum_{n=0}^\infty \frac{a^n\left(z - \frac{-b}{a}\right)^n}{n!}$$

e) $\sin(z) = \sum_{n=0}^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. So:

$$z\sin(z^3) = z\sum_{n=0}^\infty \frac{(-1)^n (z^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^\infty \frac{(-1)^n z^{6n+4}}{(2n+1)!}$$

f) We know $\frac{1}{1-z} = \sum_{n=0}^\infty z^n$. So:

$$\frac{z^2+1}{1-(iz^3)} = (z^2+1)\sum_{n=0}^\infty i^n z^{3n} = \sum_{n=0}^\infty i^n(z^{3n+2} + z^{3n})$$

If we want to write this in typical form, we could define:

$$a_n = \begin{cases} i^k, & n = 3k \text{ or } n = 3k+2 \\ 0, & n = 3k+1 \end{cases}$$

And then $\frac{z^2+1}{1-(iz^3)} = \sum_{n=0}^\infty a_n z^n$.

g)

$$3z^3\cos(z^3)+\sin(z^3)=\sum_{n=0}^{\infty}\frac{3(-1)^n z^{6n+3}}{(2n)!}+\sum_{n=0}^{\infty}\frac{(-1)^n z^{6n+3}}{(2n+1)!}=\sum_{n=0}^{\infty}\left(\frac{3}{(2n!)}+\frac{1}{(2n+1)!}\right)(-1)^n z^{6n+3}$$

h) The same argument as for part (f) gives:

$$\frac{nz^{n-1}}{1-z^n}=\sum_{k=0}^{\infty} nz^{(k+1)n-1}$$

Now, if this problem had been what I meant, which was: $\frac{nz^{n-1}}{(1-z^n)^2}$. To solve this, let $w=z^n$. Then $nz^{n-1}=w'$. So:

$$\frac{nz^{n-1}}{(1-z^n)^2}=\frac{w'}{(1-w)^2}=\frac{d}{dz}\frac{1}{1-w}$$

This gives that:

$$\frac{nz^{n-1}}{(1-z^n)^2}=\frac{d}{dz}\sum_{k=0}^{\infty} z^{kn}=\sum_{k=1}^{\infty} knz^{kn-1}$$

i) Notice that $\frac{1}{z}=\frac{1}{1+(z-1)}=\frac{1}{1-(-(z-1))}=\sum_{n=0}^{\infty}(-1)^n(z-1)^n$.

j) This is a bit trickier. The same trick doesn't work. It would give us a series in terms of $z^n-1$, not $z-1$.
Notice that $\frac{d^{n-1}}{dz^{n-1}}\frac{1}{z}=(-1)^{n-1}(n-1)!\frac{1}{z^n}$. So:

$$\begin{aligned}\frac{1}{z^n}&=\frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\sum_{k=0}^{\infty}(-1)^k(z-1)^k\\&=\sum_{k=n-1}^{\infty}(-1)^{k+n-1}\frac{(k)(k-1)(k-2)...(k-(n-2))}{(n-1)!}(z-1)^{k-(n-1)}\\&=\sum_{k=n-1}^{\infty}(-1)^{k+n-1}\frac{k!}{(k-(n-1))!(n-1)!}(z-1)^{k-(n-1)}\\&=\sum_{k=n-1}^{\infty}(-1)^{k+n-1}\binom{k}{n-1}(z-1)^{k-(n-1)}\end{aligned}$$

k)

$$\frac{1}{a-z}=\frac{1}{a\left(1-\frac{z}{a}\right)}=\sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}$$

l) Using our answer from the last part:

$$\frac{1}{a^2-z^2}=\sum_{n=0}^{\infty}\frac{z^{2n}}{a^{2n+2}}$$

m) Recall that $\mathrm{Log}(z)$ is a primitive for $\frac{1}{z}$. So using our series from part (i), we get:

$$\mathrm{Log}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^{n+1}}{n+1}+C$$

Setting $z=1$ gives $C=0$.

n) Again, $\log_1(z)$ is a primitive for $\frac{1}{z}$. So again:

$$\log_1(z) = \sum_{n=0}^{\infty} \frac{(1-)^n (z-1)^{n+1}}{n+1} + C$$

The difference here is that $\log_1(1) = i4\pi$. So $C = i4\pi$.

o) This is very tricky. We can't just plug $z^3$ into our series from the last part and go on our merry way. First, let $w = z^3$. Then $\log_1(w)$ is a primitive for $\frac{w'}{w} = \frac{3z^2}{z^3} = \frac{3}{z} = \sum_{n=0} 3(-1)^n(z-1)^n$.
So, $\log_1(z^3) = \sum_{n=0}^{\infty} \frac{3(-1)^n(z-1)^{n+1}}{n+1} + C$, for $C = i4\pi$.

That takes care of the logarithm. What about the $z^2$? We can't just multiply it in, it's not centered at $z_0 = 1$. So we need to write:

$$z^2 = A(z-1)^2 + B(z-1) + C$$

Solving this, we see that $z^2 = (z-1)^2 + 2(z-1) + 1$. So:

$$\begin{aligned}
2z^2 \log_1(z^3) &= 2\left((z-1)^2 + 2(z-1) + 1\right) \sum_{n=0}^{\infty} \frac{3(-1)^n(z-1)^{n+1}}{n+1} + 4\pi i \\
&= 2\left[\left(\sum_{n=0}^{\infty} \frac{3(-1)^n(z-1)^{n+1}}{n+1} + 4\pi i\right)\right. \\
&\quad + 2\left(\sum_{n=0}^{\infty} \frac{3(-1)^n(z-1)^{n+2}}{n+1} + 4\pi i(z-1)\right) \\
&\quad \left. + \left(\sum_{n=0}^{\infty} \frac{3(-1)^n(z-1)^{n+3}}{n+1} + 4\pi i(z-1)^2\right)\right] \\
&= 8\pi i + (16\pi i + 6)(z-1) + (8\pi i + 12 - 6)(z-1)^2 \\
&\quad + \sum_{n=3}^{\infty} \left(\frac{3(-1)^{n-1}}{n} + \frac{3(-1)^{n-2}}{n-1} + \frac{3(-1)^{n-3}}{n-2}\right)(z-1)^n
\end{aligned}$$

Now, we know the series for $e^z = e^{z-1+1} = ee^{z-1}$. So:

$$\begin{aligned}
2z^2 \log_1(z^3) &= (8\pi i + e) + (16\pi i + 6 + e)(z-1) + (8\pi i + 6 + \frac{e}{2})(z-1)^2 \\
&\quad + \sum_{n=3}^{\infty} \left(\frac{3(-1)^{n-1}}{n} + \frac{3(-1)^{n-2}}{n-1} + \frac{3(-1)^{n-3}}{n-2} + \frac{e}{n!}\right)(z-1)^n
\end{aligned}$$

2. For each power series you found in the previous question, find the radius of convergence.

**Solution:** These series are all built out of series we have the radius of convergence for. Taking derivatives and finding primitives doesn't change the radius of convergence, so:

a) $R = \infty$

b) $R = \infty$

c) $R = \infty$

d) $R = \infty$

e) $R = \infty$

f) $R^3 = 1$, so $R = 1$.

g) $R = \infty$

h) $R^n = 1$, so $R = 1$.

i) $R = 1$

j) We're taking derivatives of a series with $R = 1$, this gives a series with $R = 1$.

k) Here, we have to have $\frac{1}{R} = \frac{1}{|a|}$, so $R = |a|$.

l) Now, $R^2 = |a|^2$, so $R = |a|$.

m) We're finding a primitive for a series with $R = 1$, so $R = 1$.

n) We're finding a primitive for a series with $R = 1$, so $R = 1$.

o) The smaller radius here is $R = 1$, so $R = 1$.

3. Each of the following power series is a function we're familiar with. What functions do these series represent, and where are they valid representations?

a) $\displaystyle\sum_{n=0}^{\infty} \frac{i^n z^n}{n!}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n z^n}{n}$

c) $\displaystyle\sum_{n=0}^{\infty} n i^n z^n$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{3^n (2n)!}$

e) $\displaystyle\sum_{n=0}^{\infty} (z+1)^n$

f) $\displaystyle\sum_{n=0}^{\infty} 2^n z^n$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)! - 1}{n!} z^n$

**Solution:**

a) Let $w = iz$. Then $\displaystyle\sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum_{n=0}^{\infty} \frac{w^n}{n!} = e^w = e^{iz}$.

b) Recall that $\text{Log}(1-z) = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1}$.

So: $\displaystyle\sum_{n=1}^{\infty} \frac{i^n z^n}{n} = \sum_{m=0}^{\infty} \frac{(iz)^{m+1}}{m+1} = \text{Log}(1-iz)$.

c) Recall that $\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$. So $\displaystyle\sum_{n=0}^{\infty} n i^n z^n = iz \sum_{n=1}^{\infty} n (iz)^{n-1} = \frac{iz}{(1-(iz))^2}$.

d) Since $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. Then $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{3^n (2n)!} = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z^2}{\sqrt{3}}\right)^n}{(2n)!} = z^2 \cos\left(\frac{z^2}{\sqrt{3}}\right)$.

e) $\displaystyle\sum_{n=0}^{\infty}(z+1)^n = \frac{1}{1-(z+1)} = \frac{1}{-z}$.

f) $\displaystyle\sum_{n=0}^{\infty}2^n z^n = \sum_{n=0}^{\infty}(2z)^n = \frac{1}{1-2z}$.

g) Remember that $\sum_{n=1}^{\infty}\frac{1}{n}z^n = \mathrm{Log}(1-z)$. So:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(n-1)!-1}{n!}z^n &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n!}\right)z^n \\
&= \mathrm{Log}(1-z) - \sum_{n=1}^{\infty}\frac{z^n}{n!} \\
&= \mathrm{Log}(1-z) - \sum_{n=1}^{\infty}\frac{z^n}{n!} + \frac{z^0}{0!} - \frac{z^0}{0!} \\
&= \mathrm{Log}(1-z) - \sum_{n=0}^{\infty}\frac{z^n}{n!} + \frac{z^0}{0!} \\
&= \mathrm{Log}(1-z) - e^z + 1
\end{aligned}$$

4. Which of the following sums converge? For those that do, find their value. (Hint: these should look very similar to power series you have seen before.)

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{6n+3}}{(2n+1)!}$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^{n/2}}{n!}$

c) $\displaystyle\sum_{n=0}^{\infty}\frac{n-1}{2^n}$

d) $\displaystyle\sum_{n=0}^{\infty}(n-1)2^n$

**Solution:**

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(\pi^3)^{2n+1}}{(2n+1)!} = \sin(\pi^3)$.

b) $\displaystyle\sum_{n=0}^{\infty}\frac{\pi^{n/2}}{n!} = e^{\sqrt{\pi}}$

c) $\displaystyle\sum_{n=0}^{\infty}\frac{n-1}{2^n} = \frac{1}{2}\sum_{n=0}^{\infty}n\left(\frac{1}{2}\right)^{n-1} - \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\frac{1}{\left(1-\frac{1}{2}\right)^2} - \frac{1}{\left(1-\frac{1}{2}\right)} = 0.$

d) Notice that as $n\to\infty$, that $(n-1)2^n\to\infty$. So since the terms do not go to 0, the series diverges.

5. Use power series to solve the following ODEs.

a) $f'(z) = f(z)+1$

b) $f'(z) = f(z)+z$

c) $f'(z) = f(z)+1+z$

d) $f'(z) = f(z) + iz^2$

e) $f'(z) = f(2z)$ (Hint: check the ratio test once you have the series.)

f) $f'(z) = 2zf(z)$

g) $zf'(z) = f(z)$

h) $f''(z) + f'(z) = 2f(z)$

**Solution:** Before we solve these, first we need to clarify what I'm actually asking for. Since I haven't specified a domain that I wish the function to be analytic on, assume we are looking for an entire solution.

(a) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by expanding at 0. This gives:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = (a_0 + 1) + \sum_{n=1}^{\infty} a_n z^n$$

If we look at the $z^{n-1}$ terms, we see that $a_1 = a_0 + 1$, and for $n > 1$, we see that $na_n = a_{n-1}$. So for $n > 0$:

$$a_n = \frac{a_0 + 1}{n!}$$

Therefore, $f(z) = a_0 + \sum_{n=1}^{\infty} \frac{a_0+1}{n!} z^n = (-1) + \sum_{n=0}^{\infty} \frac{a_0+1}{n!} z^n = (a_0 + 1)e^z - 1$.

(b) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by expanding at 0. This gives:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_0 + (a_1 + 1)z + \sum_{n=2}^{\infty} a_n z^n$$

If we look at the $z^{n-1}$ terms, we see that $a_1 = a_0$ and $2a_2 = a_1 + 1$, and for $n > 2$, we see that $na_n = a_{n-1}$. So for $n > 1$:

$$a_n = \frac{a_0 + 1}{n!}$$

Therefore, $f(z) = a_0 + a_1 + \sum_{n=2}^{\infty} \frac{a_0+1}{n!} z^n = (-1 - z) + \sum_{n=0}^{\infty} \frac{a_0+1}{n!} z^n = (a_0 + 1)e^z - 1 - z$.

(c) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by expanding at 0. This gives:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = (a_0 + 1) + (a_1 + 1)z + \sum_{n=2}^{\infty} a_n z^n$$

If we look at the $z^{n-1}$ terms, we see that $a_1 = a_0 + 1$ and $2a_2 = a_1 + 1 = a_0 + 2$, and for $n > 2$, we see that $na_n = a_{n-1}$. So for $n > 1$:

$$a_n = \frac{a_0 + 2}{n!}$$

Therefore, $f(z) = a_0 + a_1 + \sum_{n=2}^{\infty} \frac{a_0+2}{n!} z^n = (-2 - z) + \sum_{n=0}^{\infty} \frac{a_0+1}{n!} z^n = (a_0 + 2)e^z - 1 - z$.

(d) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by expanding at 0. This gives:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = a_0 + a_1 z + (a_2 + i)z^2 + \sum_{n=3}^{\infty} a_n z^n$$

If we look at the $z^{n-1}$ terms, we see that $a_1 = a_0$, $2a_2 = a_1 = a_0$, and $3a_3 = a_2 + i = \frac{a_0}{2!} + i$. So $a_3 = \frac{a_0}{3!} + \frac{2i}{3!}$. And for $n > 2$, we see that $na_n = a_{n-1}$. So for $n > 3$:

$$a_n = \frac{a_0 + 2i}{n!}$$

Therefore, $f(z) = a_0 + a_1 z + a_2 z^2 + \sum_{n=3}^{\infty} \frac{a_0+2i}{n!} z^n = i(-2-2z-z^2) + \sum_{n=0}^{\infty} \frac{a_0+2i}{n!} z^n = (a_0+2i)e^z - i(2+2z+z^2)$.

(e) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by expanding at 0. This gives:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} a_n 2^n z^n$$

So, we see that $n a_n = 2^{n-1} a_{n-1}$. This tells us that:

$$a_n = \frac{2^{n-1}}{n} a_{n-1}$$

We could solve for a general form for $a_n$, but right now we can already calculate the radius of convergence for the series. Notice that $\frac{a_n}{a_{n-1}} = \frac{2^{n-1}}{n}$. So:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^n}{n+1} = \infty$$

So $R = 0$, and therefore the function is not analytic on any domain containing 0 (otherwise we would have a series of radius $R > 0$.) This means that this equation has no solution.

(f) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ by expanding at 0. Then:

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} 2 a_n z^{n+1}$$

Looking at the $z^{n-1}$ terms, we see that $1a_1 = 0$. And for $n > 1$:

$$n a_n = 2 a_{n-2}$$

So $a_3 = \frac{2}{3} a_1 = 0$. And similarly $a_5 = 0$, and so on. $a_{2k+1} = 0$ for all $k \in \mathbb{N}$.
And for $n = 2k$, we have:

$$\begin{aligned} a_2 &= a_0 \\ a_4 &= \frac{2}{4} a_2 = \frac{1}{2!} a_0 \\ a_6 &= \frac{2}{6} a_4 = \frac{1}{3!} a_0 \end{aligned}$$

Continuing the pattern, we have that $a_{2k} = \frac{a_0}{k!}$. So $f(z) = \sum_{n=0}^{\infty} \frac{a_0}{k!} z^{2k} = a_0 \sum_{n=0}^{\infty} \frac{(z^2)^k}{k!} = a_0 e^{z^2}$.

(g) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ by expanding at 0. Then:

$$\sum_{n=1}^{\infty} n a_n z^n = \sum_{n=0}^{\infty} a_n z^n$$

So for $n \neq 0$, we have that $n a_n = a_n$, and $a_0 = 0$. As such, for $n > 1$, $a_n = 0$.
So $f(z) = a_1 z$. We easily verify that these are actually solutions.

(h) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ by expanding at 0. Then:

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} 2a_n z^n$$

By comparing the $z^{n-2}$ terms, we see that:

$$n(n-1)a_n + (n-1)a_{n-1} = 2a_{n-2}$$

Therefore, $a_n = \frac{-a_{n-1}}{n} + \frac{2a_{n-2}}{n(n-1)}$. Let's try to see if a pattern is obvious. Looking at the first few terms:

$$\begin{aligned}
a_2 &= \frac{2a_0 - a_1}{2} \\
a_3 &= \frac{-a_2}{3} + \frac{2a_1}{(3)(2)} = \frac{3a_1 - 2a_0}{3!} \\
a_4 &= \frac{-a_3}{4} + \frac{2a_2}{(4)(3)} = \frac{2a_0 - 3a_1}{4!} + \frac{-2a_1 + 4a_0}{4!} = \frac{6a_0 - 5a_1}{4!} \\
a_5 &= \frac{-a_4}{5} + \frac{2a_3}{(5)(4)} = \frac{5a_1 - 6a_0}{5!} + \frac{6a_1 - 4a_0}{5!} = \frac{11a_1 - 10a_0}{5!} \\
a_6 &= \frac{-a_5}{6} + \frac{2a_4}{(6)(5)} = \frac{10a_0 - 11a_1}{6!} + \frac{12a_0 - 10a_1}{6!} = \frac{22a_0 - 21a_1}{6!}
\end{aligned}$$

This seems difficult to analyse at first. Let's let $b_n$ be the coefficient of $a_0$ and $c_n$ the coefficient of $a_1$ in the expressions for $a_n$. Then

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= 0 = b_0 - 1 \\
b_2 &= 2 = b_1 + 2 \\
b_3 &= -2 = 2 - 4 = b_2 - 4 \\
b_4 &= 6 = -2 + 8 = b_3 + 8 \\
b_5 &= -10 = b_4 - 16
\end{aligned}$$

And so on. In general, $b_n = b_{n-1} + (-1)^n 2^{n-1}$.

And for $c_n$, a similar analysis gives $c_n = c_{n-1} + (-1)^{n-1} 2^{n-1}$ with $c_0 = 0$. We can prove by induction that these formulas are correct. We won't, but we could.

So, $b_n = 1 - \sum_{k=0}^{n-1} (-2)^k$. To figure out what this is explicitly, we need to remember a formula for finite geometric sums:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

So, $b_n = 1 - \frac{1-(-2)^n}{1+2} = \frac{2+(-2)^n}{3}$. And similarly, $c_n = \frac{1-(-2)^n}{3}$.

Plugging these formulas back into our expression for $a_n$ gives:

$$a_n = \frac{1}{3}\left(\frac{(2+(-2)^n)a_0}{n!} + \frac{(1-(-2)^n)a_1}{n!}\right)$$

And now putting those back into our power series:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{(2+(-2)^n)a_0}{n!} + \frac{(1-(-2)^n)a_1}{n!}\right) z^n \\ &= \frac{2a_0+a_1}{3}e^z + \frac{a_0-a_1}{3}e^{-2z} \end{aligned}$$

6. Let $f(z)$ be analytic on the domain $\{z \in \mathbb{C} | |z-z_0| < R\}$ with $R > 0$. Prove that $f^n(z_0) = 0$ for all $n > 0$ if and only if $f$ is constant on $D$.

   **Solution:** ($\Rightarrow$) Let $f(z)$ be as given. Then since $f$ is analytic on this open ball, $f(z)$ is equal to its power series centered at $z_0$ on all of $D$:

   $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0)$$

   So $f$ is constant.

   ($\Leftarrow$) Since $f$ is contant, $f^{(n)}(z_0) = 0$ on $D$ since $f'(z) = 0$ on $D$.

7. Prove that if $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ on the domain $\{z \in \mathbb{C} | |z-z_0| < R\}$ and $R > 0$, then $a_n = b_n$ for all $n \geq 0$.

   **Solution:** Let $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n - \sum_{n=0}^{\infty} b_n(z-z_0)^n$. Since the series are equal on the open ball, $g(z) = 0$ on the open ball. This means that $g^{(n)}(z_0) = 0$ on the open ball by the previous question. However, $g^{(n)}(z_0) = \frac{a_n - b_n}{n!}$. So $a_n = b_n$ for all $n$.