Solving Differential Equations using Power Series

A common technique for solving differential equations translates fairly well to complex differential equations. Recall the following two facts:

- If $f(z)$ is analytic on D and $z_0 \in D$, then $f(z)$ has a power series expansion at z_0 with positive radius of convergence.
- If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $R > 0$ is its radius of convergence, then $f'(z) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(z - z_0)^n$

This lets us assume that the solution to a differential equation has a power series expansion, and then manipulate it without concerns about convergence.

Example. Solve the differential equation $f'(z) = Cf(z)$ on $\mathbb{C}$. (Hopefully you have already seen that the solutions to this are Ke^{Cz}, but we'll work it out.)

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We want a solution on all $\mathbb{C}$, so we can center our power series anywhere. Centering it at $z = 0$ is the easy choice.

So, if $f'(z) = Cf(z)$, this tells us that:

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n = \sum_{n=0}^{\infty} Ca_n z^n$$

Now, remember that two power series agree on D only if their coefficients are all equal so looking at the z^n term gives:

$$a_{n+1} = \frac{C}{n+1}a_n$$

So: $a_1 = Ca_0$. Then $a_2 = \frac{C}{2}a_1 = \frac{C^2}{2}a_0$. Continuing in this way, $a_3 = \frac{C^3}{3!}a_0$, etc.

We conjecture that $a_n = \frac{C^n}{n!}a_0$. We can prove this by induction. We have already shown enough base cases. So suppose $a_n = \frac{C^n}{n!}a_0$.

Then $a_{n+1} = \frac{C}{n+1}a_n = \frac{C}{n+1}\frac{C^n}{n!}a_0 = \frac{C^n}{(n+1)!}a_0$. By induction, the claim holds.

So, $f(z) = \sum_{n=0}^{\infty} \frac{C^n a_0 z^n}{n!} = \sum_{n=0}^{\infty} a_0 \frac{(Cz)^n}{n!}$. The ratio test tells us this series has radius of convergence $R = \infty$, so it is a valid operation to say:

$$f(z) = a_0 \sum_{n=0}^{\infty} \frac{(Cz)^n}{n!} = a_0 e^{Cz}$$

Example. That's a fairly straightforward differential equation. Let's handle something a bit more difficult. Solve $f'(z) = f(z) - i\sin(z)$.

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$. So, if $f'(z) = f(z) - i\sin(z)$, this tells us that:

$$\sum_{n=1}^{\infty} na_n z^{n-1} = \sum_{n=0}^{\infty}\left(a_n z^n + \frac{(-1)^n i}{(2n+1)!}z^{2n+1}\right)$$

Now we need to take a bit of care. If n is even, the $i\sin z$ doesn't contribute. But if n is odd, we need to worry about its terms.

By looking at the z^{n-1} terms, we get:

$$na_n = \begin{cases} a_{n-1}, & n \text{ odd} \\ a_{n-1} + \frac{(-1)^{\frac{n-1}{2}} i}{(n-1)!}, & n \text{ even} \end{cases}$$

Now, this looks complicated, but let's see a few examples.

$$\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 + i \\ 3a_3 &= a_2 \\ 4a_4 &= a_3 - \frac{i}{3!} \\ 5a_5 &= a_4 \\ 6a_6 &= a_5 + \frac{i}{5!} \end{aligned}$$

So, $a_2 = \frac{a_0+i}{2!}$. Then $a_3 = \frac{a_0+i}{3!}$. Then $a_4 = \frac{a_3}{4} - \frac{i}{4!} = \frac{a_0+i}{4!} - \frac{i}{4!} = \frac{a_0}{4!}$.

Continuing, we find that:

$$a_n = \begin{cases} \frac{a_0}{n!}, & n = 4k \text{ or } 4k+1 \\ \frac{a_0+i}{n!}, & n = 4k+2 \text{ or } 4k+3 \end{cases}$$

So $f(z) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} z^n + i(\frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{6!}z^6 + \frac{1}{7!}z^7 + ...) = a_0 e^z + i(\frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{6!}z^6 + \frac{1}{7!}z^7 + ...)$.

This last sum is a bit of a pain. Let's break it into two pieces:

$$g_1(z) = \sum_{k=0}^{\infty} \frac{z^{4k+2}}{(4k+2)!} = \frac{z^2}{2!} + \frac{z^6}{6!} + ...$$

$$g_2(z) \sum_{k=0}^{\infty} \frac{z^{4k+3}}{(4k+3)!} = \frac{z^3}{3!} + \frac{z^7}{7!} + ...$$

Now, notice that $g_1(z)$ is just the negative terms from $\cos(z)$. So maybe we can make $g_1(z)$ out of $\cos(z)$ pieces. Indeed:

$$\cos(z) - \cos(iz) = \sum_{n=0}^{\infty} \frac{z^{2n} - i^{2n} z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(1-(-1)^n) z^{2n}}{(2n)!}$$

Looking at a few terms, we see that $(1-(-1)^n) = 0$ when n is even and 2 when n is odd. I.e., when $n = 2k+1$ So:

$$\cos(z) - \cos(iz) = \sum_{n \text{ odd}} \frac{2z^{2n}}{(2n)!} = \sum_{k=0}^{\infty} \frac{2z^{2(2k+1)}}{(2(2k+1))!} = \sum_{k=0}^{\infty} \frac{z^{4k+2}}{2(4k+2)!} = g_1(z)$$

In a similar way:

$$\sin(z) + i\sin(iz) = \sum_{n=0}^{\infty} \frac{z^{2n+1} + i^{2n+2} z^{2n+1}}{(2n+1)!} = \sum_{k=0}^{\infty} \frac{z^{4k+3}}{2(4k+3)!} = g_2(z)$$

Therefore: $f(z) = a_0 e^z + ig_1(z) + ig_2(z) = a_0 e^z + \frac{i}{2}(\cos(z) - \cos(iz) + \sin(z) + i\sin(iz))$.

We can simplify this a little bit:

$$\cos(iz) = \frac{e^{-z} + e^z}{2}$$

$$i\sin(iz) = \frac{e^{-z} - e^z}{2}$$

So $i\sin(iz) - \cos(iz) = -\frac{e^z}{2}$.

So, we actually have that $f(z) = (a_0 - \frac{i}{4})e^z + \frac{i}{2}(\cos(z) + \sin(z))$.

So, a take away: sometimes, finding the coefficients isn't too hard. Sometimes, the real work is in recognizing what the function is from its power series.