Chapter 8 (8.1) - In-Class Worksheet - Part 1

MAT344 - Spring 2019

A quick review... the **generating function** $A(x)$ for a sequence of integers (a_n) is the formal power series $\sum_{n \geqslant 0} a_n x^n$.

We don't care about convergence of the series; nonetheless we are usually interested in the **closed-form** for the generating function.

The core example of this is the "geometric" series you know and love from Calculus:

$$1 + x + x^2 + x^3 + \ldots = \sum x^n = \frac{1}{1-x}.$$

(What is the sequence a_n here?)

Work earnestly! Work in groups!
Don't be afraid to ask questions, or check your work!

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1 For each of the following sequences, find its generating function (in closed-form).

(Write both (a) the ***series form*** $\sum_{n \geqslant 0} a_n x^n$ *(i.e. identify what* a_n *is, but also make sure your index is "*$n \geqslant 0$*" and power on* x *is* n *inside the sum); and (b) in* ***closed form*** *(i.e. as a function of* x*, like a rational function, polynomial,* e^x*, etc.)*

1.1 $(2, 2, 2, 2, \ldots)$

1.2 $(1, 2, 4, 8, 16, \ldots)$

1.3 $(1, -1, 1, -1, \ldots)$

1.4 $(1, 2, 3, 4, \ldots)$

1.5 $(1^2, 2^2, 3^2, \ldots)$

1.6 (*) $(1, 1/2, 1/3, 1/4, 1/5, \ldots)$

(This is not a sequence of integers, but ignore that and use the opposite approach from the previous two cases to get this one.)

1.7 (*) $(2^3, 3^3, 4^3, \ldots)$

Fix a generating function $A(x) = \sum a_n x^n$.

Determine the **closed form** of the following generating functions in terms of $A(x)$:

2.1 $R(x) = \sum_{n \geqslant 1} a_{n-1} x^n$, i.e. the gf for the sequence $(r_n) = (0, a_0, a_1, a_2, \dots)$.

We could call this the result of *("shifting a_n to the right")*.

2.2 $L(x) = \sum_{n \geqslant 1} a_n x^{n-1}$, i.e. the gf for the sequence $(l_n) = (a_1, a_2, a_3, \dots)$.

We could call this the result of *("shifting a_n to the left")*.

2.3 $L_2(x) = \sum_{n \geqslant 2} a_n x^{n-2}$, i.e. the gf for the sequence $(l_{2,n}) = (a_2, a_3, a_4, \dots)$.

2.4 To what extent are $R(x)$ and $L(x)$ *inverses* of each other?

3 Recall (from Lemma 8.4): if $C(x) = \sum c_n x^n = A(x)B(x)$ (i.e. is the product of generating functions $A(x) = \sum a_n x^n$ and $B(x) = \sum b_n x^n$), then

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Find the closed form of each of the following generating functions:

3.1

$$C_A(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + ... = \sum_n c_n x^n$$

Here $A(x) = \sum a_n x^n$ is unknown; your answer should be in terms of $A(x)$.

3.2 $D(x) = C_A(x)$, but where $A(x) = 1/(1-x)$.

(Notice that this recovers for us one of the closed forms we found in Question 1.)

3.3

$$S(x) = \sum s_n x^n = 1 + (1+2)x + (1+2+3)x^2 + ...$$

Hint: use $D(x)$*.*

3.4

$$T(x) = \sum t_n x^n = 1 + (1+4)x + (1+4+9)x^2 + ...$$

3.5

$$(*)\ E(x) = \sum e_n x^n = \binom{0}{0} + \left(\binom{1}{0} + \binom{1}{1}\right)x + \left(\binom{2}{0} + \binom{2}{1} + \binom{2}{2}\right)x^2 + ...$$

This one is a bit of a trick question...

In the previous problem, we found that:

$$S(x) = (1-x)^{-3} = \sum s_n x^n = \sum_n \left(\sum_{k=0}^{n} (k+1) \right) x^n$$

$$\text{and} \;\; T(x) = (1+x)(1-x)^{-4} = \sum t_n x^n = \sum_n \left(\sum_{k=0}^{n} (k+1)^2 \right) x^n$$

Here $s_n = 1+2+...+n+(n+1)$ and $t_n = 1^2+2^2+...+n^2+(n+1)^2$ are the sums of the first $n+1$ numbers and the sum of the first $n+1$ squares, respectively.

We already know[1] nice **closed formulas** for s_n and t_n:

$$s_n = \frac{(n+1)(n+2)}{2} \; (\star) \; , \quad \text{and} \quad t_n = \frac{(n+1)(n+2)(2n+3)}{6} \; (\star\,\star)$$

We want to **derive** these formulas using our knowledge of the **Generalized Binomial Theorem**.

Recall that a special case of it tells us that for $k > 0$:

$$(1+x)^{-k} = \sum_{n \geqslant 0} \binom{-k}{n} x^n$$

4.1 Prove *algebraically* that $\binom{-k}{n}$ (here $k > 0$ still) is equal to $(-1)^n \binom{n+k-1}{k-1}$.

4.2 Use this to rewrite the **series forms** of $S(x)$ and $T(x)$ so that s_n and t_n are expressed as the formulas $(\star)$ and $(\star\,\star)$ above, i.e. we want to see

$$S(x) = \sum_n \frac{(n+1)(n+2)}{2} x^n, \quad \text{and } T(x) = \sum_n \frac{(n+1)(n+2)(2n+3)}{6} x^n$$

[1] You would have perhaps proved that these are true in MAT102 using induction, and you might have used them in Calculus when working with Riemann sums.

5 Fix an arbitrary gf $A(x) = \sum a_n x^n$.

5.1 What is b_n if $B(x) = \sum b_n x^n = \dfrac{A(\sqrt{x}) + A(-\sqrt{x})}{2}$?

5.2 What is c_n if $C(x) = \sum c_n x^n = \dfrac{A(\sqrt{x}) - A(-\sqrt{x})}{2\sqrt{x}}$?

5.3 Find the **closed form** of $D(x) = \sum d_n x^n$ where $(d_n) = (1, 0, 1, 0, 1, 0, \ldots)$.

5.4 Find the **closed form** of $E(x) = \sum e_n x^n$ where $(e_n) = (0, 1, 0, 1, 0, 1, \ldots)$.

Let $S(x) = \sum s_n x^n$ where s_n is the n-th Fibonnaci number,
i.e. s_n satisfies $s_0 = 1, s_1 = 1$ and $s_n = s_{n-1} + s_{n-2}$ for $n \geqslant 2$.

6.1 Derive a **functional equation** involving S, then solve it to show that $S(x) = \frac{1}{1-x-x^2}$.
Start by substituting the recurrence relation for s_n into "$S(x) = \sum s_n x^n$".

6.2 Use *partial fractions* to rewrite the closed-form of $S(x)$ as a sum of fractions of the form "$\frac{a}{1-{}^*x^*}$".
Here a is some number, and "${}^*x^*$" could be something more complicated, like x^2 or $2x$, etc.

Steps:

- Show that the roots of the denominator are $\phi = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and $\varphi = \frac{1}{2} - \frac{\sqrt{5}}{2}$.
- Write $1 - x - x^2 = (1-\phi x)(1-\varphi x)$ and do partial fractions with these factors. *Note that $\phi + \varphi = 1$.*
- You should get

$$\frac{1}{1-x-x^2} = \frac{A}{1-\phi x} + \frac{B}{1-\varphi x}, \quad \text{where } A = \frac{1}{\sqrt{5}} \cdot \phi, \ B = -\frac{1}{\sqrt{5}} \cdot \varphi$$

6.3 Rewrite the expression you got for S from the previous part as a sum of *series* by finding the **series form** of each partial fraction.

6.4 Extract a closed formula for s_n from the result in the previous part.

6.5 (*) Go back to 6.2 and instead of moving on to 6.3, instead rewrite the closed form in the form "$\frac{1}{1-(\ldots)}$", convert to series form, and then apply the **Binomial Theorem** to get a (messier) series form expression for s_n involving $\binom{n}{k}$'s. What is the relationship to *Pascal's Triangle?*