Chapter 9 - In-Class Worksheet

MAT344 - Spring 2019

Definitions!

For a list of definitions from Chapters 9-12, please go to:

`Quercus` $\Rightarrow$ `Week 10 Guide` $\Rightarrow$ `Graph Theory Definitions`

Recall that a **Closed Eulerian Trail ("CET")** in a graph is a **sequence of edges** through a graph that has the **same starting vertex as ending vertex**, and **uses each edge exactly once**.

Recall **Theorem 9.2:** a graph has a CET if and only if the $\mathtt{deg}(\mathbf{v})$ is *even* for all $\mathbf{v}$.

So, apparently, the graph below has a CET.

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1.1 Confirm for yourself that indeed, all vertices in the graph below have *even degree.*

1.2 Theorem 9.2 not only proves a CET exists, but gives a "procedure" to create one:

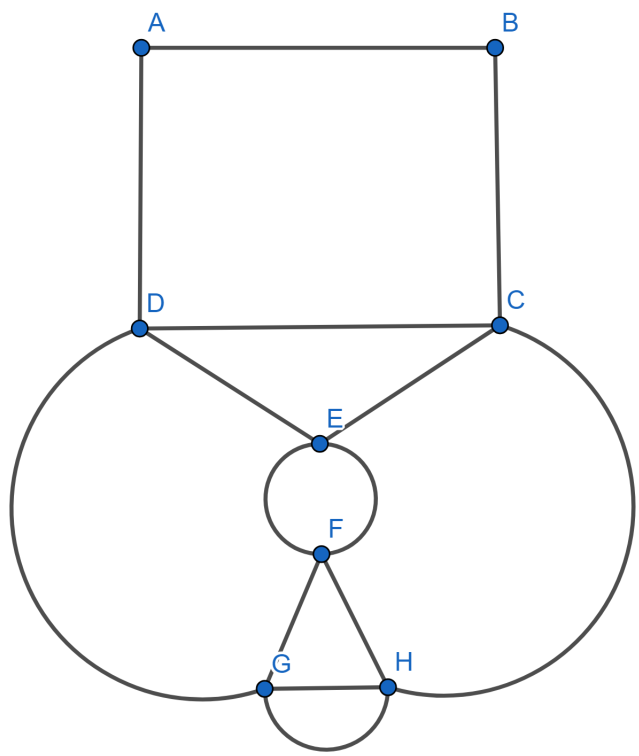
(a) Starting from vertex A, pick an edge to walk away from it on. Continue from there by picking some other edge. Continue until you've created *some* **closed trail**, and call it $\mathcal{C}_1$. *(Pick "randomly"; I specifically **don't want** you to get a CET yet.)*

(b) Why was there no possibility of getting "stuck"? That is, how do we know that we have either gotten back to vertex A, or can find a new edge to take?

(c) Choose any vertex in $\mathcal{C}_1$ that still has a "free edge" (an edge not already used in creating $\mathcal{C}_1$). *(Note here that this is possible since the graph is connected.)*
Walk along this free edge out of $\mathcal{C}_1$ and repeat the procedure from Step (a) to create a new closed trail $\mathcal{C}_2$ using edges not in $\mathcal{C}_1$.

(d) Continue to do this until **all** edges of the graph have been used in some finite number of closed trails, say $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$.

(e) Create a single **Eulerian** closed trail out of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$.



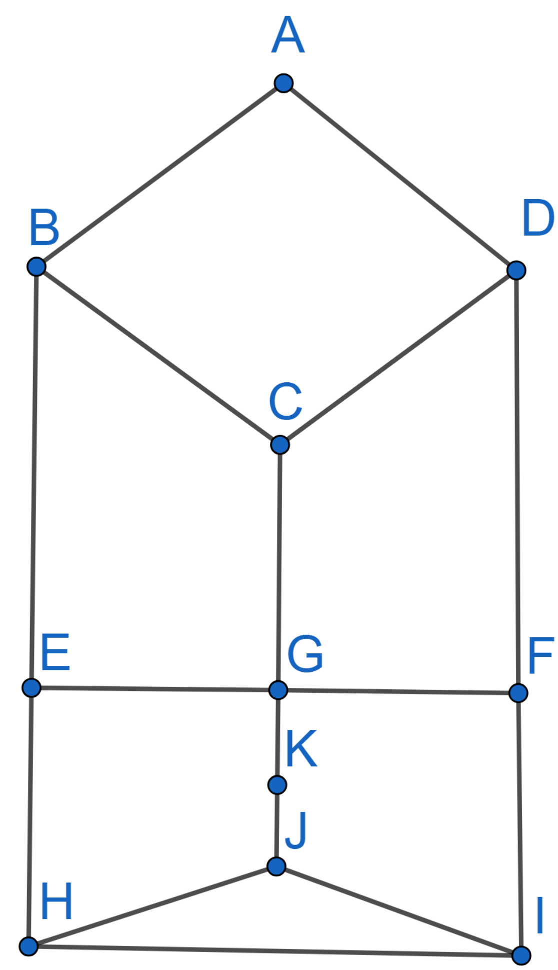
Recall that a **Hamiltonian cycle ("HC")** is a **closed** (start=end) *path* that visits every *vertex*.

...HC's are much *more complicated to find that CETs!*

2.1 Recall an observation from the reading quiz: suppose a graph G has a HC... what can we say about the vertices of degree 0, 1, and 2 in G? In general, what can we say about the edges "used" in an HC at a particular vertex **v**?

2.2 Using this observation, we can argue "by hand" that the graph below has **no Hamiltonian Cycle**.

To do this, start by *assuming for contradiction* that a HC, call it $\mathcal{C}$, exists in this graph; then argue which edges would have to be in $\mathcal{C}$ to arrive at a contradiction.



Theorem 9.5 is often called *Dirac's Theorem*. Recall that it gives us a **sufficient** condition to have a HC:

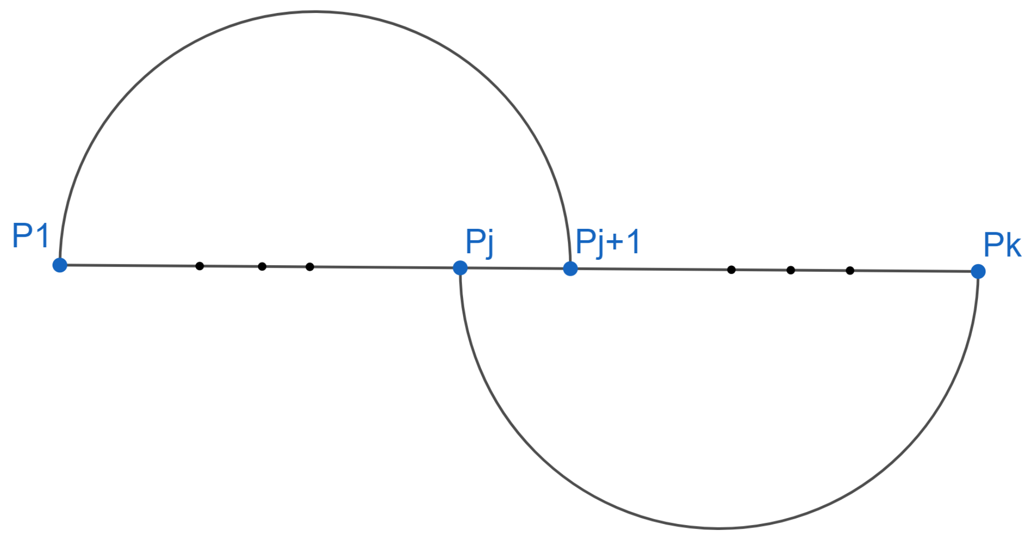
If $n \geqslant 3$, and G is a simple connected graph with n vertices, which has $\deg(\mathbf{v}) \geqslant n/2$ for all vertices $\mathbf{v}$, then G has a HC.

What follows is a similar, but different proof from the textbook's.

3.1 Let $\mathcal{P}$ be **some** path of maximum length in G, and write it $\mathbf{p}_1 ... \mathbf{p}_k$.

Note: we do not assume that $\mathcal{P}$ contains all *of the vertices of* G.

Now, consider the situation where there is $1 \leqslant j < k$ so that the following **"configuration"** exists in G (the rest of the edges/vertices of G are not shown):



Show that **if such a configuration occurs**, then there is a cycle $\mathcal{C}$ in the graph which uses all of the vertices in $\mathcal{P}$.

3.2 Then prove by contradiction that such a cycle $\mathcal{C}$ would have to be a HC. *(Again, this is in the case where a configuration exists.)*

Hint: for contradiction, let $\mathbf{v}$ *be a vertex outside of* $\mathcal{C}$*, and use the connectedness of* G *to get a path* $\mathcal{P}'$ *longer than* $\mathcal{P}$ *starting at* $\mathbf{v}$*.*

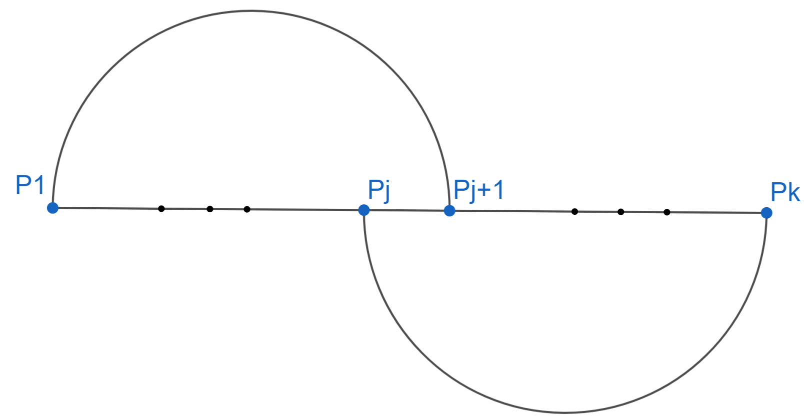
3.3 So far then, we've proven that if G has a configuration as above, then G has a HC.

So what do we need to do next to finish our proof?

What assumption about G have we not used yet?

We need to prove that a configuration *must* exist.

We'll prove this by contradiction, so **start by assuming there is no configuration**.



3.4 Focus on $\mathbf{p}_1$ and $\mathbf{p}_k$:

(a) What do we know about $\mathcal{P}$ that makes it impossible for there to be a vertex $\mathbf{v}$ in G outside of $\mathcal{P}$ which is adjacent to $\mathbf{p}_1$ or adjacent to $\mathbf{p}_k$?

(b) Why is it also impossible for $\mathbf{p}_1$ and $\mathbf{p}_k$ to be adjacent?

These restrict where edges incident at $\mathbf{v}_1$ or $\mathbf{v}_k$ can go: only to $\mathbf{p}_2, ..., \mathbf{p}_{k-1}$ in $\mathcal{P}$.

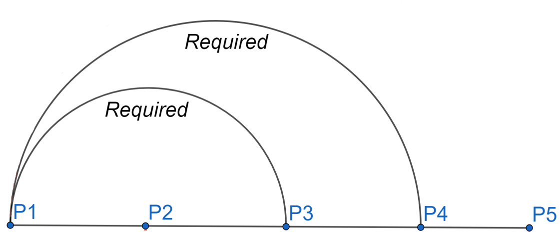
(c) What happens if $\mathbf{v}_1$ and $\mathbf{v}_6$ are adjacent?

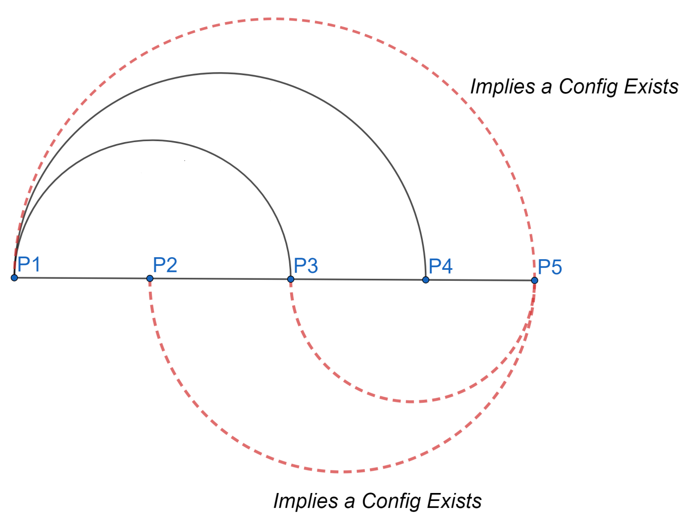
This further restricts where an edge incident at $\mathbf{v}_1$ can go...

(d) How can we use $\deg(\mathbf{v}_1)$ and $\deg(\mathbf{v}_k)$ get a contradiction?

Suggestion: Draw a version of $\mathcal{P}$ with $k = 5$ (note $n \geqslant k$ necessarily.)

Here's what the $k = 5$ case looks like (noting that $\deg(\mathbf{p}_1) \geqslant 5/2 > 2$):





Recall that a **complete** graph on n vertices, written K_n, is a graph with all possible edges included. *(How many edges is this?)*

A **tournament** is a copy of K_n for some n with each edge directed one way or the other. *(How many ways can we do this for a fixed n?)*

A **directed** cycle (in a *directed* graph) is a cycle $\mathbf{v}_1 \to \mathbf{v}_2 \to \cdots \to \mathbf{v}_k \to \mathbf{v}_1$.

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4.1 Prove that if a **tournament** T has no directed cycles of length 3 (i.e. a triple of vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and a path $\mathbf{x} \to \mathbf{y} \to \mathbf{z} \to \mathbf{x}$), then it has no directed cycles of any length.

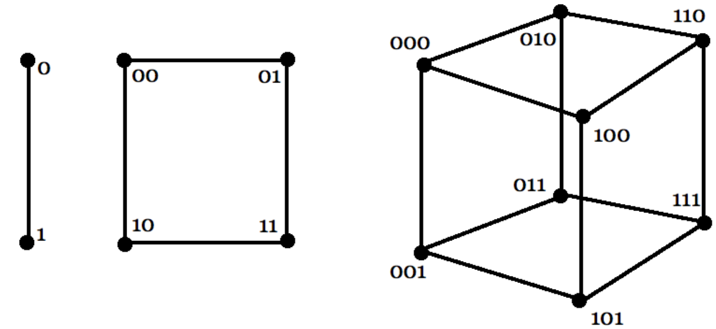
4.2 We say that a vertex $\mathbf{d}$ in a tournament T is a **dominator** if for all other vertices $\mathbf{v}$, either

(a) there is an edge $\mathbf{d} \to \mathbf{v}$; or

(b) there is another vertex $\mathbf{w}$ so that there is a path $\mathbf{d} \to \mathbf{w} \to \mathbf{v}$.

Prove that in any tournament T, every vertex of *maximum* ***out****-degree* is a dominator.

5 For $n \geqslant 1$, let Q_n, the n-**dimensional hypercube** graph, be the (simple) graph with vertex set $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid i \in [n],\ x_i = 0 \text{ or } 1\}$, and where two vertices are adjacent if they agree in $n-1$ coordinates.



Q_1 Q_2 Q_3

5.1 How many vertices does Q_n have, and what is/are their degree(s)?

5.2 (a) Explain how (for any fixed $n \geqslant 1$) the graph Q_{n+1} consists of "two copies" of the graph Q_n, together with some additional edges these two copies of Q_n to each other.

(b) How many edges are there in Q_n? What about in $\overline{Q_n}$?
*(The latter is the **complement** of* Q_n*).*
Hint: use the previous part to help you count!

5.3 For which values of $n \geqslant 1$ does Q_n have a Hamiltonian cycle? Justify your answer by constructing an actual HC in Q_n for the given value of n.

Hint: again, use 5.2(a).