SOLUTIONS

Tutorial Quiz 3

MAT344 - Spring 2019

> **Instructions:**
>
> Please record **each Group member's Name and Student Number**.
>
> Make sure to show your work, justifying where possible and annotating any interesting steps or features of your work. **Do not just give the final answer, and do not simplify your calculations (use notation from the course, like $\binom{n}{k}$ or $S(n,k)$ etc.)**

Recall that $\overline{G}$ is the **complement** of G: it has the same vertices as G, but two vertices are adjacent in $\overline{G}$ if and only if they are *not* adjacent in G.

We say that a (simple) graph G is **self-complimentary** if $\overline{G}$ is isomorphic to G.

1 **(2 points + 2 bonus points)**

1.1 **(2 points)** Find a **self-complimentary** graph G with 5 vertices and a self-complimentary graph with 4 vertices. *You do not need to prove that $G \cong \overline{G}$ in either case, but make sure to draw both G and $\overline{G}$.*

1.2 **(2 bonus points)** Prove that if G is **self-complimentary** and has n vertices, then $n = 4k$ or $n = 4k+1$ for some non-negative integer k (i.e. n is 0 or 1 mod 4).

Hint: G and $\overline{G}$ have the same number of edges.

Solution

1.1 **5 vertices:** a cycle of length 5.

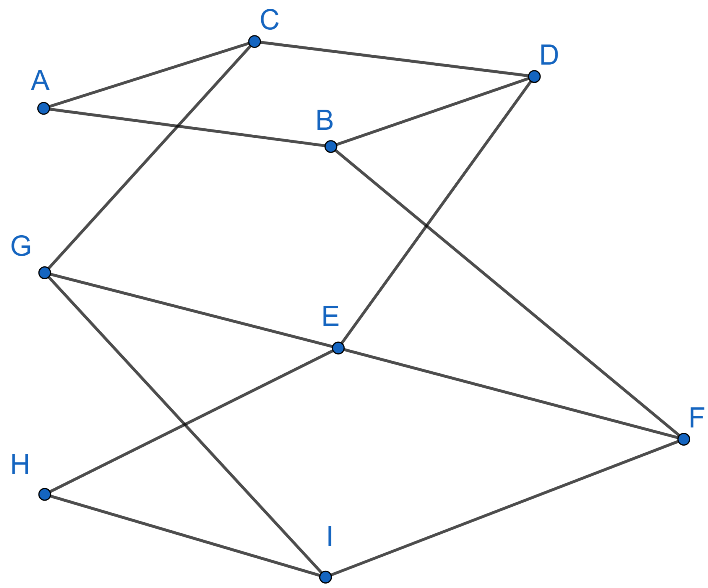
4 vertices: a path of length 3 (i.e. on 4 vertices), $\mathbf{a} - \mathbf{b} - \mathbf{c} - \mathbf{d}$.

1.2 Recall from our discussion of Q_n and $\overline{Q_n}$ in class, if we add the number of edges in a graph to the number of edges in its complement, we get the number of edges in K_n. This holds for any graph G, so letting E_G and $E_{\overline{G}}$ be the number of edges in G and its complement, we have $E_G + E_{\overline{G}} = \frac{n(n-1)}{2}$. Since G is self-complimentary, these numbers are the same, so we have $4E_G = n(n-1)$. Notice that the left-hand side is a multiple of 4, so the right hand must be as well - but if $n = 4k+2$ for instance, the right hand side is not divisible by 4: $(4k+2)(4k+1) = (16k^2 + 12k + 2) = 4(4k^2 + 3k) + 2$. (Same goes for $n = 4k+3$, but not the other two.)

(1+3 points & 2 + 1 bonus points) *(The first two parts of this question are identical to the individual version.)*

2.1 **(1 point)** Write down the definition of a **Hamiltonian cycle**.

2.2 **(3 points)** *Use a proof by contradiction* to show that the following graph does **not** have a **Hamiltonian cycle**:



2.3 **(2 bonus points)** In class we discussed a theorem, called *Dirac's Theorem*, which has the form:

"If a graph G is/has ... then G has a Hamiltonian cycle."

(a) **(1 point)** Fill in the blank in the statement of *Dirac's Theorem*.

(b) **(1 point)** Find the fewest number of additional edges we could add to the graph in this question (previous page) so that the quoted theorem would imply that the graph had a Hamiltonian cycle. *Explain your work.*

2.4 **(1 bonus point)** Find the *actual* fewest number of additional edges we could add to the graph to create a Hamiltonian cycle. *Justify your answer (for instance, if you think the answer is 10, you should prove that adding 9 more edges is not sufficient).*

SOLUTION

2.1 A cycle that visits every vertex. *(Note that a cycle is a kind of path, so this implies already that it doesn't use any vertex more than once.)*

2.2 Assume that $\mathcal{C}$ is a HC in the graph. Recall from class that in a HC $\mathcal{C}$, for any vertex $\mathbf{v}$ in the graph, *exactly* two edges at $\mathbf{v}$ are used in $\mathcal{C}$.

- There are two degree two vertices, A and H, and both edges at each must be in $\mathcal{C}$. *(Note that from this point there are a number of equally good arguments. For instance, you could consider two cases at vertex* I *instead.)*
- **Case 1:** $\mathsf{CG} \in \mathcal{C}$.
  - Then $\mathsf{CD} \notin \mathcal{C}$, so $\mathsf{BD}, \mathsf{DE} \in \mathcal{C}$.
  - Notice that since we have $\mathsf{HE}, \mathsf{DE} \in \mathcal{C}$, $\mathsf{GE}, \mathsf{EF} \notin \mathcal{C}$.
  - And since $\mathsf{AB}, \mathsf{BD} \in \mathcal{C}$, $\mathsf{BF} \notin \mathcal{C}$.

- But this is a contradiction: F has degree 3, but $\mathsf{EF}, \mathsf{BF} \notin \mathcal{C}$, so we can't take two edges at F in $\mathcal{C}$.

- **Case 2:** $\mathsf{CG} \notin \mathcal{C}$, which forces $\mathsf{CD} \in \mathcal{C}$.
  - Then there are only two edges available at G, so they are in $\mathcal{C}$: $\mathsf{GE}, \mathsf{GI} \in \mathcal{C}$.
  - Also $\mathsf{BD} \notin \mathcal{C}$ or else we get a cycle ABCDA. This forces $\mathsf{DE} \in \mathcal{C}$.
  - But this is a contradiction: three edges at E are being used, $\mathsf{GE}, \mathsf{HE}, \mathsf{DE}$.

2.3 (a) G with $n \geqslant 3$ vertices needs have that all vertices have degree at least $n/2$.

(b) Since there are 9 vertices, Dirac's Theorem would require that $\mathsf{deg}(\mathbf{v}) \geqslant 5$ for all vertices. By the Handshake Lemma, $\mathsf{deg}(\mathbf{v}) \geqslant 5$ for all $\mathbf{v}$, we would have $2|\mathsf{E}| = \sum_{\mathbf{v}} \mathsf{deg}(\mathbf{v}) \geqslant 45$, so that $|\mathsf{E}| \geqslant 23$. There are currently 13 edges, so we should be able to get away with adding only 10 more edges.

(You didn't have to confirm that this worked on the quiz, but really one should, and indeed if you add the following ten edges, the graph then has the property that $\mathsf{deg}(\mathbf{v}) \geqslant 5$ *for all vertices (there are many choices that work):* $\mathsf{AH}, \mathsf{AE}, \mathsf{AD}$, BG, BI, CF, CI, DH, FG, FH.

2.4 1 edge - from H to A, for instance.

Recall that a **tournament** is a copy of K_n which has had each of it's edges *directed* in one direction or the other (but not both). Also, the textbook defines a directed graph G to be **balanced** if for each vertex $\mathbf{v}$ in G, in-deg($\mathbf{v}$) = out-deg($\mathbf{v}$).

3 **(2 points)** *(This whole question is identical to the individual version.)*

For which values of $n \geqslant 1$ does there exist a **balanced tournament** with n vertices? *Justify your answer.*

SOLUTION

Notice that for any given vertex, there must be an even number of total edges at that vertex, or else the condition in-deg($\mathbf{v}$) = out-deg($\mathbf{v}$) is impossible to satisfy. This means that n cannot be even.

But for n odd, we can create a balanced tournament as follows: let $\mathbf{v}_0, \mathbf{v}_1, ...\mathbf{v}_{n-1}$ be the vertices in G. Now have vertex $\mathbf{v}_k$ beat the vertices with indices $k+1, k+2, ..., k+(n-1)/2$, with all those indices taken modulo n (i.e. cycling back to 0 after we go past $\mathbf{v}_{n-1}$.) *(You might not have phrased it this way; that's okay!)*

For example with $n = 5$, we'd have $\mathbf{v}_1 \to \mathbf{v}_2, \mathbf{v}_3$, $\mathbf{v}_2 \to \mathbf{v}_3, \mathbf{v}_4$, $\mathbf{v}_3 \to \mathbf{v}_4, \mathbf{v}_5$, $\mathbf{v}_4 \to \mathbf{v}_5, \mathbf{v}_1$ and $\mathbf{v}_5 \to \mathbf{v}_1, \mathbf{v}_2$.