SOLUTIONS

Tutorial Quiz 2

MAT344 - Spring 2019

Definition

First, recall that for $n \geqslant 1$, $\mathbf{p(n)}$ is the number of all partitions of n.

For $n \geqslant k \geqslant 1$, **we define:**

- $\mathbf{f_k(n)}$ to be the number of partitions of n whose *first* k *parts are equal*.
- $\mathbf{g_k(n)}$ to be the number of partitions of n whose parts *all* have size k *or more.*
- $\mathbf{s_k(n)}$ to be the number of partitions of n whose *last* part has size k.
  ($s_k(n)$ is needed only for Question 2.)

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1 **(2+3 points ⇒ 5 points)** *(This whole question is identical to the individual version.)*

1.1 **(1 point each)** Draw the *Ferrers shapes* of all of the partitions enumerated by

(a) $f_3(9)$

(b) $g_3(9)$

1.2 **(3 points)** Fix arbitrary $n \geqslant k \geqslant 1$. Prove using a bijection that $f_k(n) = g_k(n)$.

Solution

1.1 There are four of each; I'll save some time and just write them out in numbers.

For $f_3(9)$ we have 3+3+3, 2+2+2+2+1, 2+2+2+1+1+1, 1+...+1.

For $g_3(9)$ we have 3+3+3, 5+4, 6+3, and 9.

1.2 The bijection is conjugation: the Ferrers shape of a partition enumerated by f_k has a $k \times w$ block β as it's first k rows, for some $w \geqslant 1$. The rest of the partition consists of a partition, call it π, with parts of size no larger than w. When conjugated, the resulting partition will have w rows, each with size at least k, because the image of β is a $w \times k$ block. Now since π has no rows of size $\geqslant w$, the conjugate of π will not have more than w rows, so we get a proper partition to the right of the image of β. Notice that there is a unique partition of the first type that is sent to the second by conjugation; it's purely determined by the two pieces we mentioned above (β and π).

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2 **(3 points)** *(This whole question is identical to the individual version.)*

Fix an arbitrary $k \geqslant 2$. Prove that $f_k(n) = p(n) - \sum_{i=1}^{k-1} s_i(n)$.

Solution

We can count the partitions of n with all parts of size $\geqslant k$, by subtracting from *all* possible partitions of n (counted by $p(n)$), the partitions of n where the last part is size 1 or 2 or ... or $k-1$. Notice that each of these classes of partitions of n is distinct from the others (they don't overlap) and each is counted by $s_i(n)$ for the appropriate value of i. This proves that $g_k(n) = p(n) - \sum_{i=1}^{k-1} s_i(n)$. From the previous question, $f_k(n) = g_k(n)$, so we're done.

3 **(2+2 = 4 bonus points)**

3.1 **(1 point each)** Prove the following (for an arbitrary $n \geqslant 4$):

(a) $s_1(n) = p(n-1)$

(b) $s_2(n) = p(n-2) - p(n-3)$.

3.2 **(2 points)**
Use our observations so far to prove (for an arbitrary $n \geqslant 4$) that

$$f_3(n) = p(n) - p(n-1) - p(n-2) + p(n-3).$$

SOLUTION

3.1 (a) Define a bijection as follows: from a partition of n with last part 1, remove that part. The remainder is an arbitrary partition of $n-1$. (This is clearly bijective.)

(b) Define a bijection as follows: from a partition of n with last part 2, remove that part. The remainder is a partition of $n-2$ with the property that the last block cannot have size 1. (This is clearly bijective.)

This shows that $s_2(n) = p(n-2) - s_1(n-2)$, but from the previous part of the question, $s_1(n-2) = p(n-3)$, and the formula for $s_2(n)$ follows.

3.2 From Question 2, $f_3(n) = p(n) - s_1(n) - s_2(n)$. Plugging in the formulas from 3.1, we are done.

4 **(1 bonus point)**
Conjecture and prove a formula for $f_4(n)$ which only involves various $p(n)$'s, as we did for $f_3(n)$.

SOLUTION

From Question 2 we have $f_4(n) = p(n) - s_1(n) - s_2(n) - s_3(n)$.

So to repeat something like we did in 3.2, we need a formula for $s_3(n)$.

A similar argument to what we did in 3.1(b) shows that $s_3(n) = p(n-3) - p(n-4) - p(n-5) + p(n-6)$. If you prove that and then substitute, we get

$$f_4(n) = p(n) - p(n-1) - p(n-2) + p(n-4) + p(n-5) - p(n-6)!$$