Test 2 *SOLUTIONS*

MAT344 - Spring 2019

Prof. Alex Rennet

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

Instructions

There are **7 questions** on this test, one of them a **bonus question**.

There are **20 points** available, plus **2 bonus points**.

This test has **9 pages**, including this one.

No aids allowed. (i.e. no calculators, cheat sheets, devices etc.)

TUTORIAL SECTION (Leave blank if you can't remember)	
WEDNESDAY	
3pm	☐ TUT101 - Arash
5pm	☐ TUT102 - Osaid
6pm	☐ TUT103 - Osaid

FOR MARKING (Leave This Blank)

/2	/4	/4	/3	/4	/3	/+2	/20

1 **(1 point each $\Rightarrow$ 2 points total)**

You do not need to justify your answers in <u>this</u> question.

1.1 Suppose that $D(x) = \sum_{n \geqslant 0} d_n x^n$. Write a formula for $E(x) = \sum_{n \geqslant 0} d_{n+2}\, x^n$ in terms of $D(x)$ and the numbers d_n.

1.2 Determine a closed-form for the generating function $A(x) = \sum a_n x^n$, where a_n is the number of partitions of n with only odd-sized parts, and with three or less parts of any given size.

SOLUTION

1.1 $D(x) = d_0 + d_1 x + \sum_{n \geqslant 2} d_n x^n = d_0 + d_1 x + x^2 \sum_{n \geqslant 2} d_n x^{n-2} = d_0 + d_1 x + x^2 \sum_{n \geqslant 0} d_{n+2} x^n.$

Thus, $E(x) = (D(x) - d_0 - d_1 x)/x^2$.

1.2 $(1 + x + x^2 + x^3)(1 + x^3 + x^6 + x^9)(1 + x^5 + x^{10} + x^{15}) \cdot \ldots = \prod_{k \geqslant 1} (1 + x^k + x^{2k} + x^{3k})$

For the remainder of the test, you must justify your answers.

2 **(4 points)**

Let $B(x) = \sum_{n \geqslant 0} b_n x^n$, where the sequence (b_n) satisfies the recurrence relation

$$b_0 = 1,\; b_1 = 2, \;\text{ and for }\; n \geqslant 2, \quad b_n = -b_{n-1} + 6b_{n-2}.$$

Find a **closed-form** expression for $B(x)$ and then use this to find an **exact** expression for b_n in terms of n.

(Your final answer should involve n, but shouldn't involve other elements of the sequence like b_{n-1} etc, and it shouldn't contain "…" or any variable-length sums ("$\sum$") etc.)
You won't receive any points for solving for b_n using other techniques.

Solution

$$\begin{aligned}
B(x) = \sum_{n\geqslant 0} b_n x^n &= b_0 + b_1 x + \sum_{n\geqslant 2}[-b_{n-1} + 6b_{n-2}]x^n \\
&= 1 + 2x - \sum_{n\geqslant 2} b_{n-1}x^n + 6\sum_{n\geqslant 2} b_{n-2}x^n \\
&= 1 + 2x - x\sum_{n\geqslant 1} b_n x^n + 6x^2 \sum_{n\geqslant 0} b_n x^n \\
&= 1 + 2x - x(B(x) - 1) + 6x^2 B(x) \\
\Rightarrow (1 + x - 6x^2)B(x) = 1 + 3x &\Rightarrow B(x) = \frac{1+3x}{1+x-6x^2} \\
\Rightarrow B(x) &= \frac{1+3x}{(1+3x)(1-2x)} = \frac{1}{1-2x}
\end{aligned}$$

Therefore, $B(x) = \sum 2^n x^n$, so $b_n = 2^n$ for all n.

3 **(4 points)**

Let $C(x) = \sum_{n \geqslant 0} c_n x^n$ where $c_n = \sum_{k=0}^{n} 3^k (n-k+1)$.

Find a **closed-form** expression for $C(x)$ and re-express it as a sum of separate series to get an **exact** expression for c_n.

Hint: this is a product.

SOLUTION

The hint is pointing out that $C(x) = A(x)B(x)$ where $A(x) = \sum_{n \geqslant 0} a_n x^n$ where $a_n = 3^n$ and $B(x) = \sum_{n \geqslant 0} b_n x^n$ where $b_n = n+1$. This is by Lemma 8.4, since c_n is defined to be $\sum_{k=0}^{n} a_k b_{n-k}$, with these a_n's and b_n's.

Once we have that, we can write $A(x) = \frac{1}{1-3x}$ and $B(x) = \frac{1}{(1-x)^2}$ (the latter uses either the derivative technique from class or the generalized binomial theorem technique from class, neither of which you needed to reproduce), and use these to write $C(x) = \frac{1}{(1-3x)(1-x)^2}$.

Applying partial fractions, you should end up with

$$C(x) = \frac{9/4}{1-3x} - \frac{3/4}{1-x} - \frac{1/2}{(1-x)^2}$$

Converting to series forms we get

$$\frac{9}{4} \sum 3^n x^n - \frac{3}{4} \sum x^n - \frac{1}{2} \sum (n+1) x^n$$

So the final answer is $\frac{1}{4}[3^{n+2} - 5 - 2n]$.

4 **(3 points)**

That devious villain Kl'rt is at it again... his previous master plan was foiled, but he has a new one. This time Kl'rt will divide his army of n *Skrulls* in the following way:

- First, Kl'rt will split the n *Skrulls (named with elements of* $[n]$*)* into some as-of-yet unknown number of *consecutive, non-empty* groups.
- From each group, Kl'rt will choose a leader for the group.
- And then finally, Kl'rt chooses a subset of the groups to send to the University of Toronto (with the remainder going to the city of Toronto). *(The first set of groups will attempt to infiltrate the mathematics department again while the second set of groups causes a diversion again. Kl'rt is not an especially creative villain.)*

Find a **closed form** of the generating function $G(x) = \sum g_n x^n$ for the number of ways, g_n, for Kl'rt to do this. *You do not need to find an exact expression for* g_n.

Solution

Let $A(x) = \sum a_n x^n$, where $a_n = n$ and $B(x) = \sum b_n x^n$ where $b_n = 2^n$. Then (noting as is necessary that $a_0 = 0$ and $b_0 = 1$) we apply the Composition Formula to get $G(x) = B(A(x))$.

Since $A(x) = \frac{x}{(1-x)^2}$ (again using techniques from class) and $B(x) = \frac{1}{1-2x}$, we have

$$G(x) = \frac{1}{1 - 2\frac{x}{(1-x)^2}} = \ldots = \frac{(1-x)^2}{1-4x+x^2}$$

5 **(4 points)**

Definition

First, recall that for $n \geqslant 1$, $\mathbf{p_{\leqslant k}(n)}$ is the number of partitions of n with size at most k (or, equivalently, the number of partitions of n into at most k parts).

For $n \geqslant k \geqslant 1$, we define:

- $\mathbf{q_k}(n)$ to be the number of partitions of n which have exactly k parts, *each of them distinct.*
- $\mathbf{r_k}(n)$ to be the number of partitions of n which have parts of size at most k, but *each part from 1 to k occurs at least once.*

5.1 **(2 points)** Draw the *Ferrers shape* of each of the partitions enumerated by $q_3(10)$ and $r_3(10)$.

5.2 **(2 points)** Fix an integer $k \geqslant 1$, and prove that $q_k(n) = r_k(n)$ for all $n \geqslant k$.

Solution

5.1 $q_3(10)$ enumerates 7+2+1, 6+3 + 1, 5+3+2, and 5+4+1.

$r_3(10)$ enumerates 3+3+2+1+1, 3+2+2+2+1, 3+2+2+1+1+1, and 3+2+1+...+1.

5.2 In this question, you can just explain why the *conjugation* map (taking a Ferrers shape to it's conjugate) is a bijection from the partitions enumerated by $q_k(n)$ to the partitions enumerated by $r_k(n)$.

6 **(3 points)**

Fix an integer $k \geqslant 1$. Using the terminology of the previous question (and assuming it to be true), prove the following for all $n > \binom{k+1}{2}$:

$$q_k(n) = p_{\leqslant k}\left(n - \binom{k+1}{2}\right).$$

SOLUTION

From the previous question, $q_k(n) = r_k(n)$. So let π be one of the partitions of n enumerated by $r_k(n)$, and define a function by sending it to the following partition: remove one copy of each part size from π. i.e. remove a part of size k, a part of size $k-1$, a part of size $k-2$, ... a part of size 1. Then the remainder is a partition of $n - (k + (k-1) + (k-2) + ... + 1) = n - \binom{k+1}{2} \geqslant 1$, and has *at most* k parts (some of the sizes 1,...,k might only occur once in π, so they would not occur in the image of this function). The image of π is the unique partition left over from this operation (so it is injective)

This map is a bijection as well: the inverse takes a partition of the second type and simply adds a part of each size 1, ..., k to it, creating a unique partition of the type enumerated by $r_k(n)$.

An alternate solution uses generating functions - check out the solution to Assignment 7, #2 to get started.

7 **(2 BONUS points)**

For each fixed integer $k \geqslant 1$, let $F_k(x) = \sum_{n \geqslant k} S(n,k)x^n$. *(i.e. we are defining one generating function for each* $k \geqslant 1$ *here.)*

Use the fact that $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ to prove that for each k,

$$F_k(x) = \frac{x^k}{(1-x)(1-2x)\cdot \ldots \cdot (1-kx)}$$

Solution

The unusual feature of this question is that our expression for F_k will involve F_{k-1}, which, if expanded in the same way, would involve F_{k-2}, ... etc. Technically we would derive our equation $(*)$ below and then do an induction, but an argument like the one below would suffice:

$$\begin{aligned}
F_k(x) &= \sum_{n \geqslant k} [S(n-1,k-1) + k \cdot S(n-1,k)]x^n \\
&= \sum_{n \geqslant k} S(n-1,k-1)x^n + k \cdot \sum_{n \geqslant k} S(n-1,k)x^n \\
&= x \sum_{n \geqslant k} S(n-1,k-1)x^{n-1} + kx \sum_{n \geqslant k+1} S(n,k)x^n \\
&= xF_{k-1}(x) + kxF_k(x) \\
\Rightarrow (1-kx)F_k(x) &= xF_{k-1}(x) \Rightarrow F_k(x) = \frac{x}{1-kx} \cdot F_{k-1}(x) \\
\Rightarrow F_k(x) &= \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdot F_{k-2}(x) \\
&\vdots \\
\Rightarrow F_k(x) &= \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdot \cdots \cdot \frac{x}{1-x} \\
&= \frac{x^k}{(1-x)(1-2x)\cdot \ldots \cdot (1-kx)}, \quad \text{as desired.}
\end{aligned}$$