MAT334 - Week 2 Problems

Additional Problems

1. Solve the following equations:

   a) $z^2 = 1 - i$
   b) $z^3 = 8$
   c) $z^4 = 12 - 5i$
   d) $z^5 = z$
   e) $z^6 = 27iz^3$

   Solution:

   a) $z = \pm(2^{1/4})(\cos(-\pi/8) + i\sin(-\pi/8))$.
   b) $z = 2$, $2(\cos(2\pi/3) + i\sin(2\pi/3)$ and $2(\cos(4\pi/3) + i\sin(4\pi/3))$.
   c) $z = (13^(1/4))(\cos(\arctan(-5/12)/4 + k\pi/2) + i\sin(\arctan(-5/12)/4 + k\pi/2))$ for $k = 0, 1, 2, 3$.
   d) $z = 0$ or $z = \pm 1, \pm i$
   e) $z = 0$ or $z = 3(\cos((2k+1)\pi/3) + i\sin((2k+1)\pi/3))$ for $k = 0, 1, 2$.

2. Solve $z^n = w$ for each w below, and each $n \in \mathbb{N}$.

   a) $w = 1$ (These are called the nth roots of unity.)
   b) $w = -1$
   c) $w = 3 - i\pi$

   Solution:

   a) $z = \cos(2k\pi/n) + i\sin(2k\pi/n)$ for $k = 0, 1, ..., n-1$.
   b) $z = \cos((2k+1)\pi/n) + i\sin((2k+1)\pi/n)$ for $k = 0, 1, ..., n-1$.
   c) $z = (9+\pi^2)^{1/n}(\cos(\arctan(-\pi/3)/n + 2k\pi/n) + i\sin(\arctan(-\pi/3)/n + 2k\pi/n))$ for $k = 0, 1, 2, ..., n-1$.

4. Find the ranges of the following functions:

   a) $f(z) = z$
   b) $f(z) = z^3$
   c) $f(x + iy) = xy$
   d) $f(x + iy) = x^2 + iy^2$
   e) $f(z) = 3z^2 + iz - 2$

   Solution:

   a) $\mathbb{C}$
   b) $\mathbb{C}$
   c) $\mathbb{R} = \{x + 0i | x \in \mathbb{R}\}$
   d) $\{a + ib \mid a \geq 0, b \geq 0\} = \{r(\cos(\theta) + i\sin(\theta) \mid \theta \in [0, \pi/2]\}$
   e) $\mathbb{C}$

6. Find the following limits:

   a) $\lim_{z\to 1} \frac{z^2-3z+2}{z-1}$
   b) $\lim_{z\to 0} Arg(z)$
   c) $\lim_{z\to 0} \frac{|z|}{z}$
   d) $\lim_{z\to i} Arg(z^2)$
   e) $\lim_{z\to 3} \frac{z}{\mathrm{Re}(z)}$
   f) $\lim_{z\to 3} \frac{z}{\mathrm{Im}(z)}$

   Solution:

   a) -1
   b) DNE
   c) DNE
   d) DNE
   e) 1
   f) DNE