Fall 2017 | # MAT344 Midterm | October 19th

*Fill in your **Name** (as it appears on Portal) and **Student ID**, **sign** below, and select your tutorial.*

NAME (Last, First): ____________________

STUDENT ID: ____________________

SIGNATURE: ____________________

TUTORIALS - Indicate Your Registered Tutorial (✓)
WEDNESDAY
☐ TUT101 - 3pm
☐ TUT102 - 4pm
☐ TUT103 - 5pm

- There are **seven questions** on this test, some with multiple parts.
- There is a total of **30 available points.**
- **No aids are allowed.** (i.e. no calculators, cheat sheets, devices etc.)
- This test has **9 pages** including this page, and a **page of scrap**.
- **Nothing on the scrap page will be marked. You may remove them.**

MARKING - Leave This Blank							
/5	/4	/5	/3	/3	/4	/6	/30

QUESTION 1 **(5 points)**

Seventy-three points are given inside a hexagon with side lengths all 1. Prove that there are three of these points that span a triangle of area at most $1/8$.

Note: the area of a triangle is $\frac{1}{2}\mathtt{bh}$.

QUESTION 2 **(4 points)**

Using a **double counting** argument, prove the following identity, for any *fixed* nonnegative integers n, and $r \leqslant n$:

$$\sum_{k=0}^{n-r+1} \binom{r+k}{r} = \binom{n+2}{r+1}$$

Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.

QUESTION 3 **(5 points)**

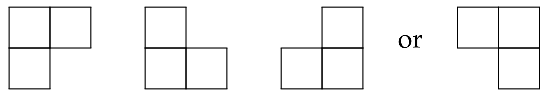
Let $q_m(n)$ be the number of partitions of n with exactly m parts and with first part of size exactly m, and let $r_m(n)$ be the number of partitions of n with no part of size greater than m and no more than m parts.

Prove that for any positive integers n, m with $2m+1 \leqslant n$, we have

$$q_m(n) = r_m(n-2m+1)$$

Question 4 **(3 points)**

Prove that for any positive integer n, it is possible to tile any $2^n \times 2^n$ grid with exactly one square removed, using only "L"-shaped tiles with three squares, as in:



Hint: induction on n*.*

QUESTION 5 **(3 points)**

Prove the following identity for any integer $n \geqslant 2$:

$$n(n-1)4^n = \sum ac2^{b+4}\binom{n}{a,b,c}$$

(Where the sum is taken over all a, b, c so that $a+b+c=n$.)

QUESTION 6 **(4 points)**

Recall that $S(n, k)$, *the Sterling numbers of the second kind*, stand for the number of partitions of $[n]$ into k non-empty subsets.

Prove, using a **combinatorial argument**, that for all positive integers $k \leqslant n$,

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

QUESTION 7 **(2 points per part - 6 points)**

You are on the sidewalk handing out 500 identical flyers (lucky you!). You hand them out to people who pass you in the street, *trying* to get one to each person. But **sometimes you miss people**, and **sometimes you accidentally hand out multiple copies at once**.

(a) Suppose that by the end, 200 people walked by you on the street, and that you managed to hand out all of the flyers. Furthermore, suppose that the first ten people all got exactly one flyer each, and that the last ten people each got at least two. Then how many ways are there for you to have handed out the flyers?

(b) Suppose that instead, you don't know how many people walked by you, but you do know that everyone who walked by got at least one flyer, and you know that the first ten people got exactly one flyer each again. How many ways are there for you to have handed out the flyers in this case?

(c) Finally, suppose that 250 people walk by, and that you've handed out all of the flyers. But unbeknownst to you, this time *each flyer had a (single) QR-code number* on it, and that there were exactly 25 different QR-codes, each appearing the same number of times amongst the flyers. This means, of course, that the flyers weren't actually identical after all. Now how many ways are there for you to have handed out the flyers?

The End

SCRAP PAPER

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