Chapter 12 - In-Class Worksheet

MAT344 - Spring 2019

Definitions!

For a list of definitions from Chapters 9-12, please go (on Quercus) to

`Graph Theory Definitions`.

1 Recall that a graph is **planar** if it *can* be drawn in the plane with no edges crossing (it has a "plane depiction"). For example, while a typical drawing of K_4 doesn't satisfy this condition, it nonetheless has a plane depiction, and so is planar.

The Circle-Chord Method

This method can be used to determine planarity (or non-planarity) of graphs, each of whose connected components has a Hamiltonian cycle.[a]

For each component do the following:

- Draw a Hamiltonian cycle as a circle with nodes labelled *in the order they appear in the HC.*
- Beginning from a particular starting node, add in additional edges from G to the circle diagram, with each edge an arc going inside or outside the circle.
- Continue, adding in the edges of G, at successive vertices, always attempting to keep edges from crossing. Stop if you are unable to add an edge without such a crossing, or if you have added all of the edges of G.
- In the first case, G is non-planar; in the second case, G is planar.

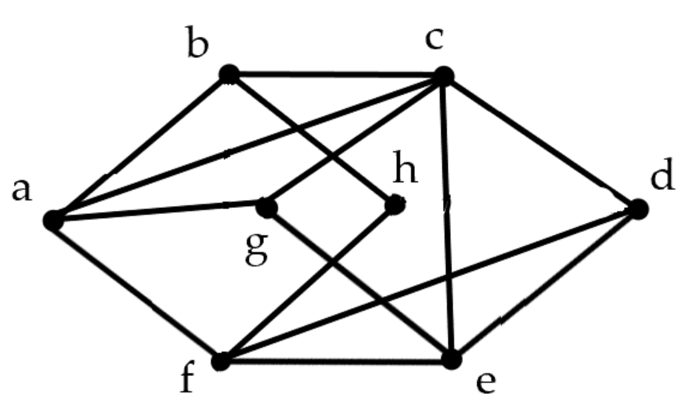
[a]Since determining whether a graph has a HC is a hard problem in general, this is not the greatest algorithm in practice, but you can use it to test some graphs for planarity "by hand", so it is somewhat satisfying to have at hand.

Use the C-C Method to determine, for each of the following, whether it is planar or not:

1.1 $K_{3,3}$,

1.2 K_5

1.3 G, the following graph:



Euler's Formula (EF):
If a simple, connected graph G is planar, with V vertices, E edges, and F faces (in a planar depiction of G), then $V + F = E + 2$.

2.1 Fix a simple connected graph G, and suppose I don't know if it is planar or not.

Is **Euler's Formula** something I can directly apply to test G for planarity?
Discuss with your neighbours.

2.2 While the answer to the above is "no", we can nonetheless derive a test for planarity from **EF** by eliminating consideration of the number of *faces* from the formula.

We say that the **degree of a face** is the number of edges we travel on to form the region it encloses.

(a) Draw a graph with only two faces: one of degree three and the other of degree five.

(b) Draw a graph with a face with degree 2.

(c) If a graph has at least two edges, can it have a face of degree 2?

2.3 Recall the *Handshake Lemma:* for any simple graph G, $\sum_{\mathbf{v}} \mathrm{deg}(\mathbf{v}) = 2E$.

There is a similar statement for faces: find it, and convince yourself it is true.

2.4 What is the smallest that a degree of a face can be in a graph with at least two edges? *Call your answer* d.

2.5 Combine the previous two results to get an inequality involving E and $d \cdot F$.

2.6 Rearrange **Euler's Formula** to solve for F and substitute the result into the inequality you wrote down in the previous part.

Simplify so that it is of the form "$E \leqslant \ldots$", with the right-hand side having *only* V as a variable.

Notice that for any (simple connected) graph G, ***if the inequality you derived is not satisfied, then it is not planar!***

2.7 (a) Use the formula to show that K_5 is non-planar again.

(b) Find a non-planar graph which satisfies the inequality. *This shows that the inequality is not a complete test of non-planarity.*
Hint: try making a graph with $V = 5$, $E = 8$.

3 The **Five-Colour Theorem** states that every planar graph is *5-colourable*. The proof goes by induction on n: we prove, for each n, that all planar graphs with n vertices have a 5-colouring.

3.1 To begin, state and mentally check-off the base case.

3.2 Next, explain why we don't insist on *connectedness* in the statement of the theorem.

3.3 Now, make our *Induction Hypothesis*, and consider some planar graph G with $n+1$ vertices. Prove that any such graph has at least one vertex $\mathbf{v}$ with degree $\leqslant 5$.

3.4 Let $\mathbf{v}$ be such a vertex, and let $G' = G - \{\mathbf{v}\}$ (i.e. remove a vertex of degree 5 or less). Complete the proof in the very special case where $\mathtt{deg}(\mathbf{v}) < 5$.

3.5 Now, assume $\mathtt{deg}(\mathbf{v}) = 5$ and complete the proof in the special case where we have a 5-colouring of G' with four or fewer colours used on vertices adjacent to $\mathbf{v}$ in G.

3.6 Finally, suppose that $\mathtt{deg}(\mathbf{v}) = 5$ and that in every 5-colouring of G', all five colours are used on the five vertices adjacent to $\mathbf{v}$ in G (label them $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ and colour them $1, 2, 3, 4, 5$ respectively). Here we consider a couple of cases:

(a) Suppose that there is no path $\mathbf{a} - ... - \mathbf{c}$ where the vertices are coloured in exactly the alternating pattern $1-3-1-3-...-3$. In this case, propagate a $1 \leftrightarrow 3$ colour swap through all of the paths starting at $\mathbf{a}$ which *are* coloured $1-3-1-3-...$ (there may be none - this is a very specific situation). *Make sure you understand how this would work by drawing a few possible situations and re-colouring as indicated. Then complete the proof of this sub-case!*

(b) Now, suppose that there *is* such a path $\mathbf{a} - ... - \mathbf{c}$; try the exact same thing with $\mathbf{b}$ and $\mathbf{d}$ and colours 2 and 4. This time, use the planarity of G to force ourselves to be in the first of the two subcases, and thereby complete the proof.