Chapter 4 - In-Class Worksheet

MAT344 - Spring 2019

Work earnestly! Work in groups!
Don't be afraid to ask questions, or check your work!

A **combinatorial argument** proves an equality by saying that the two sides of the equation *count the same thing*, just in different ways.

- Combinatorial proofs rarely involve much manipulation of the quantities shown. So for example, you typically should not be converting *binomial coefficients* like $\binom{n}{k}$ into $\frac{n!}{(n-k)!k!}$ (even if there is a valid proof involving this).
- You are also *not* doing induction, or proof by contradiction, etc.
- In general, start by exploring the equation for low values of the variables; you are looking for a way to see the two sides as counting the same thing.
- Then, give separate characterizations of the "LHS" and "RHS" that make it clear they are counting the same thing in the general case (i.e. when the variables aren't specific numbers.)

For example, consider $2^n = \sum_{k=0}^{n} \binom{n}{k}$. This was proved using a *bijection* in Chapter 3, but there is an easy combinatorial proof...

1 **Warmup:** Why is $\binom{10}{4} = \binom{10}{6}$? *Prove it combinatorially, not algebraically.* (In general, why is $\binom{n}{k} = \binom{n}{n-k}$?)

Prove the following, for any $n \geqslant 1$, **using combinatorial arguments**:

1.1

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Suggestion: think of the left-hand-side as selecting a committee of n people from a larger group of $2n$ people.

1.2

$$n \cdot \binom{2n-1}{n-1} = \sum_{k=1}^{n} k \binom{n}{k}^2$$

Hint: n is the same as $\binom{n}{1}$.

1.3 (*) What other formulas can we derive and prove using similar reasoning?
i.e. how can we generalize the observation that we used to prove 1.2?

2.1 Prove the following, for any $n, r \geqslant 0$, **using a combinatorial argument**:

$$\binom{n+r+1}{r} = \sum_{k=0}^{r} \binom{n+k}{k}$$

Suggestion: start by exploring the equation for low values of the variables. Try to think of this in terms of committee selection, like the previous question.

2.2 (*) Why doesn't the same reasoning work if we replace the LHS with $\binom{n+r+2}{r}$ and leave the RHS the same?
(The RHS changing but the LHS increasing means it's certainly false... but why? And is there a way to correct it?)

3

3.1 What are the coefficients on z^3 in $(1+z)^4$ and $(1+y+z)^4$?

3.2 In the polynomial $(x+2y-z^2)^4$, what are the coefficients on the following?

$$(a)\ x^2y^2 \qquad (b)\ z^5 \qquad (c)\ x^2 \qquad (d)\ yz^6$$

3.3 (*) What is the coefficient of x^6 in $(2x^2-x+1)^7$?

4 Prove the following using the **Binomial Theorem** or the **Multinomial Theorem**:

4.1

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

4.2

$$5^n = \sum_{a_1+a_2+a_3=n} \binom{n}{a_1, a_2, a_3} 2^{a_1+a_3}$$

4.3

$$6^n = \sum_{0 \leqslant j+k \leqslant n} \binom{n}{j} \binom{n-j}{k} 2^j 3^k$$

5 Use the **Generalized Binomial Theorem** to find (i) the **power series expansion** and (ii) **the coefficient on** x^6 for the following:

$$(2-3x)^{-1/3}$$

You can write your answer with "!!" or variable-length products like $1 \cdot 7 \cdot 13 \cdot ...(1+6k)$*, etc.*