MAT334 - Week 2 Problems

Additional Problems

1. Solve the following equations:

   a) $z^2 = 1 - i$
   b) $z^3 = 8$
   c) $z^4 = 12 - 5i$
   d) $z^5 = z$
   e) $z^6 = 27iz^3$

   Solution: For parts a,b, and c, simply use the formula given in lecture:

   $$z = r^{1/n} e^{i(\theta + 2k\pi)/n}$$

   For part d, notice that $z^5 = z$ if and only if $z^5 - z = 0$. But $z^5 - z = z(z^4 - 1)$, so either $z = 0$ or $z^4 = 1$. Then apply the formula.

   For part e, a similar argument gives that $z^3(z^3 - 27i) = 0$ so $z = 0$ or $z^3 = 27i$.

2. Solve $z^n = w$ for each w below, and each $n \in \mathbb{N}$.

   a) $w = 1$ (These are called the nth roots of unity.)
   b) $w = -1$
   c) $w = 3 - i\pi$

   Solution: Use the formula.

3. Draw the following regions:

   a) $|z - 2| \geq 1$
   b) $0 < |z - 2i| < 1$
   c) The intersection of $1 < |z - 1|$ and $|z| < 2$
   d) The union of $1 < |z| < 3$ and $2 < |z| < 5$

   Solution:

   a) A filled in disk of radius 1 centered at 0.
   b) An open disk of radius 1 centered at 2i, but missing its center.
   c) The filled in disk of radius 2 centered at 0, but then remove the disk of radius 1 centered at 1. The two disks are tangent at 2.
   d) An open annulus centered at 0 with inner radius 1 and outer radius 5.

4. Find the ranges of the following functions:

   a) $f(z) = z$
   b) $f(z) = z^3$
   c) $f(x + iy) = xy$
   d) $f(x + iy) = x^2 + iy^2$
   e) $f(z) = 3z^2 + iz - 2$

Solution:

a) w is in the range of $f(z) = z$ means that there exists some $z \in \mathbb{C}$ with $f(z) = w$. Choose $z = w$. Then $f(z) = f(w) = w$, so w is in the range of f for any $w \in \mathbb{C}$. So the range of $f(z)$ is $\mathbb{C}$.

b) Again, for w to be in the range of $f(z) = z^3$, this means that we can find $z \in \mathbb{C}$ with $f(z) = z^3 = w$. Let $z = |w|^{1/3} e^{i \arg(w)/3}$. Then $z^3 = |w| e^{i \arg(w)} = w$. So the range of $f(z)$ is $\mathbb{C}$.

c) Notice that $xy \in \mathbb{R}$. Let $r \in \mathbb{R}$. Then choosing $x = r$ and $y = 1$ gives $f(z) = r$, so r is in the range of $f(z)$. So the range of $f(z)$ is $\mathbb{R}$.

d) Notice that $x^2 \geq 0$ and $y^2 \geq 0$. So $f(z)$ lies in the first quadrant. Now, let $a + ib$ lie in the first quadrant, so $a \geq 0$ and $b \geq 0$. Then $f(\sqrt{a} + i\sqrt{b}) = a + ib$, so the range of $f(z)$ is the entire first quadrant, $\{a + ib | a, b \geq 0\}$.

e) The quadratic formula tells us that $f(z) = w$ if and only if $z = \frac{-i \pm (-1-4(3)(-2-w))^{\frac{1}{2}}}{6}$. We know that such a z exists for any $w \in \mathbb{C}$, so the range of $f(z)$ is $\mathbb{C}$.

5. Find the range of z^n for all $n \in \mathbb{N}$. (Try to turn this into a question about solving an equation.)

Solution: We claim that the range of z^n is $\mathbb{C}$. To show this, let $w \in \mathbb{C}$. Then w is in the range of z^n if there exists some $a \in \mathbb{C}$ with $a^n = w$.

In lecture, we showed that $a^n = w$ always has solutions for any w, namely the nth roots of w. So w is in the range of z^n for any $w \in \mathbb{C}$, and so the range if $\mathbb{C}$.

6. Find the following limits:

a) $\lim_{z \to 1} \frac{z^2-3z+2}{z-1}$

b) $\lim_{z \to 0} Arg(z)$

c) $\lim_{z \to 0} \frac{|z|}{z}$

d) $\lim_{z \to i} Arg(z^2)$

e) $\lim_{z \to 3} \frac{z}{\mathrm{Re}(z)}$

f) $\lim_{z \to 3} \frac{z}{\mathrm{Im}(z)}$

Solution:

a) Notice that $z^2 - 3z + 2 = (z-1)(z-2)$. So $\lim_{z \to 1} \frac{z^2-3z+2}{z-1} = \lim_{z \to 1} z - 2 = -1$.

b) Approaching from $y = 0$ and $x > 0$ gives a limit of 0. Approaching from $y = 0$ and $x < 0$ gives a limit of π. Since we get two different limits, the limit DNE.

c) Approaching from $y = 0$ and $x > 0$ gives a limit of 1. Approaching from $y = 0$ and $x < 0$ gives a limit of -1. So the limit DNE.

d) Approaching from $y = 1$ and $x > 0$, notice that z has angle $\theta < \pi/2$ and is approaching $\pi/2$. The angle of z^2 is 2θ, which is approaching π from below. Now, $\mathrm{Arg}(w) \in (-\pi, \pi]$ for any $w \in \mathbb{C}$, so $\mathrm{Arg}(z^2) = 2\theta$. As θ approaches $\pi/2$, we see that $\mathrm{Arg}(z^2)$ approaches π.

   If we instead approach from $y = 1$ and $x < 0$, then z has angle $\pi/2 < \theta < \pi$. So z^2 has angle $2\theta \in (\pi, 2\pi)$. Since $\mathrm{Arg}(z^2) < \pi$, we see that $\mathrm{Arg}(z^2) = 2\theta - 2\pi$. Now, as θ heads towards $\pi/2$, we see that $\mathrm{Arg}(z^2)$ approaches $\pi - 2\pi = -\pi$.

   So the limit DNE.

e) The numerator and denominator are continuous functions, both approaching 3. So the limit is 1.

f) The numerator approaches 3, the denominator approaches 0, so the limit DNE.

7. Let $f(z) = |z|z$. For each of the following sets D, draw D and then draw where $f(z)$ sends D.

a) $D = \{z \in \mathbb{C} \mid |z| = 2\}$
b) $D = \{z \in \mathbb{C} \mid |z| < 1\}$
c) $D = \{r(\cos\theta + i\sin\theta) \mid r > 3, \theta \in [0, \pi]\}$
d) $D = \{z \in \mathbb{C} \mid \frac{1}{|z-1|} > 2\}$

Solution: The function $f(z)$ sends $z = re^{i\theta}$ to $r^2e^{i\theta}$. In particular, it stretches z if $|z| > 1$ and shrinks z if $|z| < 1$. For parts a,b, and c, this amounts to just squaring the radius of the circle.

For part d, draw a circle of radius 1 centered at 0. Everything inside the circle shrinks towards 0. Everything outside the circle is stretched away from 0. This should turn your circle into more of an egg shape.

8. Suppose $w \in \mathbb{C}$ and $w \neq 0$. Futhermore, suppose $a^n = w$ for some $a \in \mathbb{C}$. Let $\omega_1, \ldots, \omega_n$ be the n roots of 1. Prove that the roots of $z^n = w$ are exactly $\{a\omega_i | i = 1, 2, \ldots, n\}$.

Solution: First, note that $\omega_1, ..., \omega_n$ are n distinct complex numbers. So $a\omega_i \neq a\omega_j$ for $i \neq j$. So we have n different numbers.

All we need to do is show they actually are solutions to $z^n = w$. Well, $(a\omega_i)^n = a^n{\omega_i}^n$. But $a^n = w$ by definition, and ${\omega_i}^n = 1$ since ω_i is an nth root of 1. So $(a\omega_i)^n = w$.

So they are solutions, and we have n of them, which means they are the n solutions to $z^n = w$.

9. The Quadratic Formula: Let $a, b, c \in \mathbb{C}$ and $a \neq 0$. Suppose s_1 and s_2 are the square roots of $b^2 - 4ac$. Prove that the roots of $az^2 + bz + c = 0$ are exactly $z_1 = \frac{-b+s_1}{2a}$ and $z_2 = \frac{-b+s_2}{2a}$.

Solution: Suppose $az^2 + bz + c = 0$ and $a \neq 0$.

We begin by completing the square:

$$\begin{aligned}
az^2 + bz + c &= a(z^2 + \frac{b}{a}z) + c \\
&= a(z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c \\
&= a(z^2 + \frac{b}{a}z + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a} \\
&= a(z + \frac{b}{2a})^2 + c - \frac{b^2}{4a}
\end{aligned}$$

Then $az^2 + bz + c = 0$ if and only if $(z + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$.

Now, the square roots of $4a^2$ are $\pm 2a$, so we won't worry about the sign on that factor. By taking square roots, we get that $z + \frac{b}{2a} = \frac{s_1}{2a}$ or $\frac{s_2}{2a}$. So $az^2 + bz + c = 0$ if and only if $z = \frac{-b+s_1}{2a}$ or $\frac{-b+s_2}{2a}$.

10. We know that $f(z) = z$ is continuous. Even better, $f(x + iy) = x$ and $f(x + iy) = y$ are continuous. Use these facts, together with the limits laws you know, to prove that $f(z) = \overline{z}$, $g(z) = |z|$, and all polynomials $p(z) = a_0 + a_1z + \cdots + a_nz^n$ are continuous everywhere.

Solution: We are given that $f(z) = z$, $\Re(z)$ and $\Im(z)$ are continuous.

Since $\overline{z} = \Re(z) + i\Im(z)$, and $\Re(z)$ and $i\Im(z)$ are continuous, the limit sum law gives that $\lim_{z\to z_0} \overline{z} = [\lim_{z\to z_0} \Re(z)] - i[\lim_{z\to z_0} \Im(z)] = \Re(z_0) - i\Im(z_0) = \overline{z_0}$.

Since $\sqrt{x}$ is continuous on $[0, \infty)$, $\lim_{z\to z_0} |z| = \sqrt{\lim_{z\to z_0} x^2 + y^2} = \sqrt{x_0^2 + y_0^2} = |z_0|$.

To prove that all polynomials are continuous, using the limit sum law, the limit product law, and induct.

Base case: we know the constants are continuous, and that $f(z) = z$ is continous.

Suppose all polynomials p of degree $k \leq n$ are continous.

Let $q(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$. Then $q(z) = a_n z^n + f(z)$ where $f(z)$ is a polynomial of degree $n - 1$, which is continuous by our induction hypothesis.

Now, $a_n z^n = (a_n z^{n-1})z$. z is continuous and $a_n z^n - 1$ is continuous. The limit product law tells us that their product is continuous, and so $a_n z^n$ is continuous. Therefore, $q(z)$ is a sum of two continuous functions and is therefore continuous.