Chapter 10 & 11 - In-Class Worksheet

MAT344 - Spring 2019

Definitions!

For a list of definitions from Chapters 9-12, please go to:

`Quercus` ⇒ `Week 10 Guide` ⇒ `Graph Theory Definitions`

A **tree** T (a connected graph with no cycles) can be made into a **rooted tree** by:

- Selecting a single vertex, **r**, called the **root vertex**.
- And *(optionally)* directing the edges away from the root towards the leaves (degree 1 vertices in T).

1

1.1 Justify why *any* vertex in a (plain) tree can be chosen as the *root vertex* to create a rooted tree.

Define the **height** of a vertex **v** in a (rooted) tree to be the distance from it to **r**.

And let the **height of** T be the height of the leaf of maximum height (i.e. the length of the longest path starting at r.)

1.2 Suppose T is a complete k-ary tree (all non-leaves have k children) with height h.

(a) Draw such a tree of height at least 3, with $k \geqslant 2$

(b) Use your drawing to conjecture the number of leaves in T.

(c) Prove your answer is correct via induction.

A vertex is a **centre vertex** in a simple, connected graph if the distance from it to any other vertex is minimal among all vertices in the graph.

The maximum length of a path from a given vertex $\mathbf{x}$ in a graph is called its **eccentricity**, written $\mathrm{ecc}(\mathbf{x})$. Thus, a vertex is a centre vertex if it has *(smallest/largest/?)* eccentricity of all vertices in the graph.

Claim: every tree T has exactly 1 or 2 centre vertices.

2

2.1 Find a tree with 1 centre vertex, and a tree with 2 centre vertices, each with at least 8 vertices. Label the eccentricities of each of the vertices.

2.2 Produce a tree whose centre vertex is a leaf (repeat for 2 centre vertices). Describe all such trees (justify your answer).

2.3 Given a tree T, define T' to be the tree obtained by deleting all of the **leaves** of T (unless T' would be empty).

Prove that if a vertex $\mathbf{x}$ is in T', then it has eccentricity one less in T' than when considered in T.

2.4 What does the previous result imply about the centre vertex/vertices of T'?

2.5 **Prove by induction on the number of vertices in** T that every tree has exactly 1 or 2 centre vertices.

We say that a graph G is k-colourable if there is an assignment of the elements of $[k]$ to the vertices of G. *(Note: G being k-colourable doesn't rule out G being j-colourable for some $j < k$.)*

We say that the **chromatic number** of G, written $\chi(G)$, is k if G is k-colourable, but not $k-1$-colourable.

3

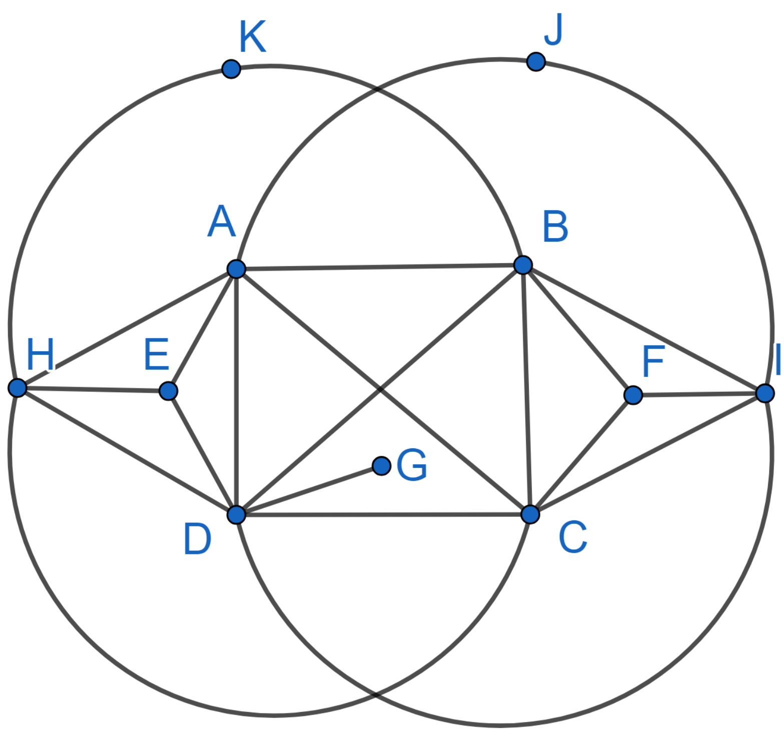
3.1 What is $\chi(K_n)$?

3.2 What is $\chi(C_n)$?

(Here, C_n is the graph consisting of a cycle with n vertices, and no other edges.)

3.3 What is the chromatic number of the following graph?

(You need to find (for some k) a k-colouring and ***prove*** *that there is no $k-1$-colouring.)*



For a (simple) graph G, we let $p_\chi(n, G)$ be the number of ways to n-colour G.

This quantity is a polynomial function of n (see Exercise 5 in Chapter 11), and so is called the **chromatic polynomial of** G.

4

4.1 Determine the chromatic polynomial of

(a) a single vertex,

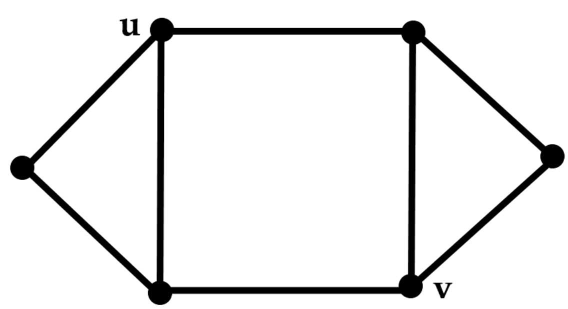
(b) a single edge,

(c) a 3-vertex path $\mathbf{a} - \mathbf{b} - \mathbf{c}$,

(d) K_n,

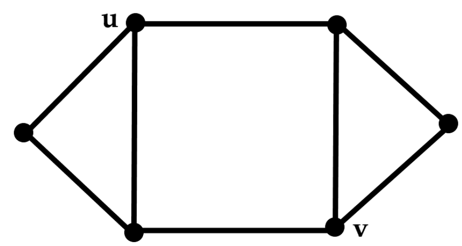
(e) C_n.

4.2 Try to determine $p_\chi(3, G)$ *by hand* for the following graph:



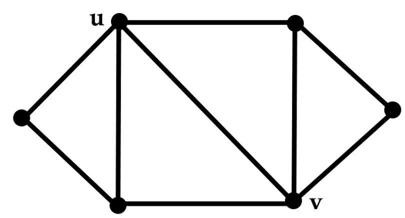
There are many 3-colourings... but if we knew $p_\chi(n, G)$, we could just plug in $n = 3$ to get the exact number. ... so we will find this now.

To find the chromatic polynomial we will count the number of n-colourings by considering **two disjoint cases** (and *add* the results.)



4.3 **Case 1:** colourings where **u** and **v** are assigned different colours ("Type 1").

Let G_1 be the result of adding edge **uv** to G:

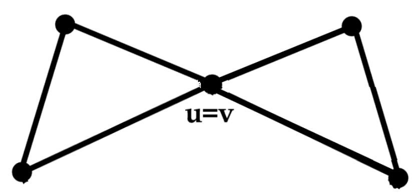


(a) What is the relationship between a colouring of G of Type 1 and a colouring of G_1?

(b) What is $p_\chi(n, G_1)$?

4.4 **Case 2:** colourings where **u** and **v** are assigned the same colour ("Type 2").

Let G_2 be the result of **contracting** the vertices **u** and **v** together into a single vertex (maintaining the adjacencies from both vertices):



(a) What is the relationship between a colouring of G of Type 2 and a colouring of G_2?

(b) What is $p_\chi(n, G_2)$?

4.5 So now, what is $p_\chi(n, G)$? *(And in particular, what is* $p_\chi(3, G)$*?)*

4.6 * Let G be a simple graph and k a positive integer. Prove that $(n - k)$ is a factor of $p_\chi(n, G)$ if and only if $k < \chi(G)$.