FINAL EXAM - EXTRA PRACTICE

MAT344 - FALL 2018

***This is not a practice exam, it's just a set of extra problems to practice with.* I don't claim that it is comprehensive.**

1

A **directed rooted tree** is a tree together with a particular vertex $\mathbf{r}$ selected and where all edges point "away" from the root: if $\mathbf{r}$ is adjacent to $\mathbf{v}$, then $\mathbf{r} \to \mathbf{v}$ and $\mathbf{v}$ is called a **child** of $\mathbf{r}$, if $\mathbf{w}$ is adjacent to a child $\mathbf{v}$ of $\mathbf{r}$, then $\mathbf{r} \to \mathbf{v} \to \mathbf{w}$ and $\mathbf{w}$ is called a **child** of $\mathbf{v}$, etc.

For $n \geqslant 1$, we let t_n be the number of directed rooted trees **with n vertices**,[1] and let $T(x)$ be the generating function for the sequence (t_n).

Draw all of the directed rooted trees (up to isomorphism) for $n \leqslant 3$. Then, by finding the closed form of $T(x)$ and then extracting the coefficient of x^n in the power series form of $T(x)$, **prove that** $t_n = \frac{1}{n}\binom{2n-2}{n-1}$ for all $n \geqslant 1$.

Hint: Any such tree can be made by gluing some unknown number of such trees to a single root vertex.

SOLUTION

Following the hint, we observe that a tree of our type is a root vertex $\mathbf{r}$ connected by some (unknown number) n edges to n separate trees T_i, where we connect our root $\mathbf{r}$ to each of the roots $\mathbf{r}_i$ of the T_i. This shows that the generating function $B(x)$ for just the "branches" of a tree (everything but the root) satisfies $B(x) = \frac{1}{1-T(x)}$ (it's just a sequence of trees of some unknown length.)

But also, a tree is just one vertex then the branches, so $T(x) = xB(x) = \frac{x}{1-T(x)}$. Thus $T(x)^2 - T(x) + x = 0$, and solving we get $T(x) = \frac{1-\sqrt{1-4x}}{2}$ (taking the negative root from the quadratic formula).

By the same reasoning we/you/the book did for Catalan numbers, you can derive the desired formula. *(Notice that $T(x)$ is just $xC(x)$, where $C(x)$ is the generating function for the Catalan numbers, so you are just showing that $t_n = C_{n-1}$ where C_k is the k^{th} Catalan number.)*

[1] We are not considering the vertices to be *labelled*; therefore t_1 is equal to 1, since a pair of vertices with one edge between them will be isomorphic without labels even when considering "different" choice of root vertex.

2

For each of the families of graphs listed below, determine the following (your answers may depend on n):

(a) Do the graphs have any Eulerian Cycles?

(b) Do the graphs have any Hamiltonian Cycles?

(c) What are the **chromatic number(s)** of the graphs?

(d) Are the graphs bipartite?

(e) Are the graphs planar? If they are planar, how many *faces* do they have?

(f) How many vertices and edges are there in the graphs?

(g) What is/are the centre vertice(s) in the graphs?

2.1 C_n, the **cycle graph** of length n, $n \geqslant 3$, consisting of just a cycle with n edges and vertices.

2.2 W_n, the **wheel graph** of length n, $n \geqslant 3$, consisting of a copy of C_n together with one new vertex which is connected to each of the vertices in the cycle.

2.3 K_n, the **complete graph** on n vertices, $n \geqslant 1$

2.4 $K_{n,n}$, the **complete bipartite graph** with $n+n$ vertices, $n \geqslant 1$

2.5 $G_{n,n}$, the $n \times n$ **grid graph** (i.e. like a $n \times n$ chessboard, with vertices at the corners of squares), $n \geqslant 1$.

SOLUTION

In general the answer to (d) is implied by your answer to (c): being bipartite is the same as being 2-colourable.

2.1 $C_n, n \geqslant 3$

(a) Yes, for all n.

(b) Yes, for all n.

(c) 2 for n even, 3 for n odd.

(d) Yes for n even, no for n odd.

(e) Yes, for all n. Two faces (include the "face" outside the graph).

(f) n of each.

(g) All vertices are centre vertices.

2.2 $W_n, n \geqslant 3$ (this should have been $n \geqslant 3$ not $n \geqslant 1$).

(a) No, the degrees of the vertices on the copy of C_n are always equal to 3, which is odd.

(b) Yes, for all n; just take the copy of C_n to start, then remove one edge, say edge AB, then add the edges AX and XB where X is new vertex added in the centre of the wheel. This is a HC.

(c) 3 for n even, 4 for n odd. (Colour the copy of C_n as per above, then colour the new vertex a new colour.)

(d) No.

(e) Yes, for all n. $n+1$ faces (include the "face" outside the graph).

(f) $n+1$ vertices, $2n$ edges.

(g) Just the new vertex added is a centre.

2.3 $K_n, n \geqslant 1$

(a) Yes, for $n \geqslant 3$ odd, since in this case, every vertex will have even degree (degrees will be $n-1$.)

(b) Yes, each K_n contains a copy of C_n as a subgraph.

(c) n

(d) Only for $n \leqslant 2$.

(e) Yes, for all $n \leqslant 4$. Otherwise, it contains a copy of K_5, which we know is not planar.

(f) n vertices, $\binom{n}{2}$ edges.

(g) All vertices are centre vertices.

2.4 $K_{n,n}, n \geqslant 1$

(a) Yes, for all n even (since the degree of each vertex will be n, which is even).

(b) Yes, for all n: just start at some vertex on one "side", go to an arbitrary vertex on the other, and continue, never going to a vertex that one has visited. This will always work since the graph from a vertex on one side has an edge to each vertex on the other side (you could do a proof by induction on n if you wanted.)

(c) 2 for all n

(d) Yes.

(e) No for all $n \geqslant 3$ ($K_{3,3}$ being non-planar was an example in class, and certainly all $K_{n,n}$ for $n > 3$ have $K_{3,3}$ as a subgraph so are also non-planar.) For $n = 1$, there is one face; for $n = 2$ there are two faces.

(f) $n+n$ vertices, n^2 edges.

(g) All vertices are centre vertices.

2.5 $G_{n,n}, n \geqslant 1$. *There was a bit of ambiguity in my definition; my answer here applies to the situation where each row and column has n vertices (not $n+1$). The solutions still apply, with some adjustments, if you took each row and column to have $n+1$ vertices.*

(a) For $n \geqslant 3$, there will be non-corner vertices that are on the edge; these will have degree 3; so for $n \geqslant 3$, no. But for $n = 2$ yes, for $n = 1$, technically no, since cycles need at least one edge.

(b) Yes for $n = 2$, no for $n = 1$ (again, the same technicality). For the remaining cases, call the vertex in row i and column j, $\mathbf{v}_{i,j}$. Now, for $n = 3$, no there are no HCs, since each of the eight edges at the corners must be in any HC (since the corner vertices are degree are degree 2), but then the four non-corner edge vertices, i.e. $\mathbf{v}_{1,2}, \mathbf{v}2, 1, \mathbf{v}2, 3$ and $\mathbf{v}3, 2$ have two edges "used" by the HC already, so their edges to the centre vertex, $\mathbf{v}_{2,2}$ cannot be used. But then it is impossible to reach this centre vertex.

For $n = 4$, yes, we can make one as follows: start at $\mathbf{v}_{1,1}$ and go to the end of the row; the go down and head back, but stop at $\mathbf{v}_{2,2}$ (i.e. don't go all the way back to $\mathbf{v}_{2,1}$.) Now go down to row three and go right to the end; go down, then head back all the way to $\mathbf{v}_{4,1}$; finally go up to $\mathbf{v}_{1,1}$, completing the HC.

In general, a by-hand argument can be used to verify that for $n \geqslant 3$ odd, there is no HC, while for $n \geqslant 4$ even, a similar construction will get you that there is an HC.

(c) 2 for all n

(d) Yes.

(e) Yes; there are $(n-1)^2 + 1$ faces.

(f) n^2 vertices, $2n(n-1)$ edges. You can prove the latter by induction; but I found this by noticing that if e_n is the number of edges in $G_{n,n}$, then $e_{n+1} = e_n + 4n - 4$, with $e_0 = e_1 = 0$. You can solve this using generating functions if you want :)

(g) For n odd there is a single centre vertex. For n even there are four.

3 Find the number of ways to distribute 90 candies to three children if the oldest child gets 30, the middle child gets 40, and the youngest child gets 20.

SOLUTION

$\binom{90}{30,40,20}$

The fact that they are ordered by age doesn't add anything here; it's just a way of distinguishing the children.

4 Give **combinatorial proofs** of the following statements:

4.1 $\binom{n}{m}\binom{n-m}{k} = \binom{n}{k}\binom{n-k}{m} = \binom{n}{m,k,n-m-k}$

4.2 $\sum_{k \leqslant m \leqslant n} \binom{n}{m} S(m,k) S(n-m,l) = \binom{k+l}{l} S(n, k+l)$, where k, l, n are fixed positive integers with $k + l \leqslant n$.

SOLUTION

4.1 I'll leave this more straightforward one for you.

4.2 The right-hand side is the number of ways of distributing n distinguishable things into $k+l$ boxes (with each box getting at least one), and then from the boxes, selecting l of them and distinguishing them from the rest of the boxes (say by putting an "A" on each of them.)

The left-hand side does the same, but in cases depending on how many things get put into the "A" boxes, and how many do not: m is the number of things that will end up in non-A-boxes (at least k, since we will have k such boxes); $\binom{n}{m}$ chooses the m things to put in these boxes; $S(m,k)$ counts the ways of then doing this distribution, with $S(n-m,l)$ accounting for the ways of distributing the remaining objects among the remaining boxes and then putting an "A" on these remaining l boxes.

5

5.1 Show that the number of partitions of m^2+n with Durfee square of size $m \times m$ is equal to

$$\sum_{k=0}^{n} p_{\leqslant m}(k) \cdot p_{\leqslant m}(n-k)$$

where $p_{\leqslant m}(k) = \sum_{i=0}^{m} p(k,i)$ is the number of partitions of k with at most m parts.

5.2 Use the previous part to show that $p(n^2+2n) > p(n)^2$, where $p(k)$ is the number of partitions of k. *(This second part of the question is Exercise 32 in the textbook in Chapter 5.)*

SOLUTION

5.1 Recall the terminology from earlier in the course: the portion of a partition to the right of the Durfee Square in the Ferrer's shape is called the *arm* and rest (below the square) is called the *leg*. We know that there are n blocks in these two pieces together; and one of the first observations we made about them is that their generating function is the same. So, we can count them both as partitions of some unknown number into *at most* m *parts.*

If we let k be the size of the arm, then there are $p_{\leqslant m}(k)$ choices for the arm and $p_{\leqslant m}(n-k)$ choices for the leg. Therefore, if we let k range over all possible values (0 to m), we get all the possible partitions.

5.2 From the previous problem, with the spacial case where $n = 2m$, we've shown that

$$p(m^2+2m) = \sum_{k=0}^{2m} p_{\leqslant m}(k) p_{\leqslant m}(2m-k)$$

But the right-hand side is strictly greater than any one of its terms, in particular the $k = m$ term, so we have

$$\ldots > p_{\leqslant m}(m) p_{\leqslant m}(m) = p(m)^2$$

6

6.1 Prove that the number of partitions with k distinct parts is equal to the number of partitions of n has exactly the numbers from 1 through k as parts, each occurring at least once. *(k is some fixed positive integer.)*

6.2 Prove that the amount (for fixed k) of partitions counted by both descriptions in the previous part is also equal to $p_{\leqslant k}\left(n - \binom{k+1}{2}\right)$.

Solution

6.1 You could prove this by writing down the generating function $G(x) = \prod_{i=1}^{k} \frac{x^i}{1-x^i}$ for the second type of partition, and then argue that this is equal to the generating function for the first type of partition as well using the fact that the conjugate of a partition of the first type is a partition of the second type. (This is because if you congugate a partition with exactly k rows (first type), you end up with a partition with exactly k columns, which is a partition with largest part k. But also, since the rows of the original partition were distinct, the columns of the conjugate will be; but then each row is a different number. Since there are k of them (columns), each of the numbers 1 to k must occur in the partition, so we have that the conjugate is a partition of the second type.)

6.2 You can prove this by creating a bijection: fix a k; then take a partition of n the first type (k distinct parts) and create a partition of $n - \binom{k+1}{2}$ into $\leqslant k$ parts by subtracting k from the first row (in the Ferrers shape), then $k-1$ from the second, ... until you subtract 1 from the last row. Then we will have subtracted a total of $k + (k-1) + ... + 1 = \binom{k+1}{2}$ from n, and certainly, there are still no more than k parts. This will be bijective: notice that the inverse is unique, because we are just adding k to the first row of the partition, $k-1$ to the second row, ... etc.

7

Let c_n be the number of weak compositions $r_1 + ... + r_n = n$ with the property that r_i is a multiple of i for $i = 1, ..., n$. Find the closed form generating function $C(x)$ for the sequence (c_n). What is the relationship between c_n and $p(n)$, the number of partitions of n?

Solution

For each i, we get a factor of $\frac{1}{1-x^i}$, so $C(x) = \prod_{i\geqslant 1} \frac{1}{1-x^i}$. But then this is equal to the generating function for partitions of n, i.e. the sequence $p(n)$; therefore, $c_n = p(n)$ for all n.

8

Prove the following identity by considering the Durfee square of a self-conjugate partition:

$$\sum_{m \geqslant 0} \frac{x^{m^2}}{(1-x^2)(1-x^4)\cdot \ldots \cdot (1-x^{2m})} = \prod_{i \text{ odd}} (1+x^i) = \prod_{i \geqslant 1} \frac{1}{1+(-x)^i}$$

Solution

For the second equality, there is a quick algebraic argument:

$$\prod_{i \text{ odd}} (1+x^i) = \prod_{i \text{ odd}} (1-(-x)^i) = \frac{\prod_{i \geqslant 1}(1-(-x)^i)}{\prod_{i \geqslant 1}(1-(-x)^{2i})}$$

$$= \frac{\prod_{i \geqslant 1}(1-(-x)^i)(1+(-x)^i)}{\prod_{i \geqslant 1}(1-(-x)^{2i})(1+(-x)^i)} = \prod_{i \geqslant 1} \frac{1}{1+(-x)^i}$$

For the first equality, notice that $\prod_{i \text{ odd}}(1+x^i)$ is the generating function for the number of partitions of n with distinct odd parts. But this, as we know, is the same as the number of partitions with self-conjugate Ferrers shapes (from Chapter 5), so they have the same GF. But a self-conjugate partition has some $m \times m$ Durfee square (for some (any!) $m \geqslant 1$) and then an arm and a leg with $\leqslant m$ parts, but with the rows of the arm matching the columns of the leg. Therefore, we can just take the GF for the arm, and substitute x^2 in to get the GF for the combined arm and leg: each block we put in the arm is mirrored in the leg. i.e. the denominator of the sum on the left-hand side of the identity we are proving. The sum then goes over all possible m, and includes a x^{m^2} term in the numerator to account for the blocks in the Durfee square.