Test 2

MAT344 - Spring 2019

Prof. Alex Rennet

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

Instructions

There are **7 questions** on this test, one of them a **bonus question**.

There are **20 points** available, plus **2 bonus points**.

This test has **9 pages**, including this one.

No aids allowed. (i.e. no calculators, cheat sheets, devices etc.)

TUTORIAL SECTION (Leave blank if you can't remember)

WEDNESDAY	
3pm	☐ TUT101 - Arash
5pm	☐ TUT102 - Osaid
6pm	☐ TUT103 - Osaid

FOR MARKING (Leave This Blank)

/2	/4	/4	/3	/4	/3	/+2	/20

1 **(1 point each ⇒ 2 points total)**

You do not need to justify your answers in this question.

1.1 Suppose that $D(x) = \sum_{n \geqslant 0} d_n x^n$. Write a formula for $E(x) = \sum_{n \geqslant 0} d_{n+2}\, x^n$ in terms of $D(x)$ and the numbers d_n.

1.2 Determine a closed-form for the generating function $A(x) = \sum a_n x^n$, where a_n is the number of partitions of n with only odd-sized parts, and with three or less parts of any given size.

For the remainder of the test, you must justify your answers.

2 **(4 points)**

Let $B(x) = \sum_{n \geqslant 0} b_n x^n$, where the sequence (b_n) satisfies the recurrence relation

$$b_0 = 1,\ b_1 = 2,\ \text{ and for } \ n \geqslant 2, \quad b_n = -b_{n-1} + 6b_{n-2}.$$

Find a **closed-form** expression for $B(x)$ and then use this to find an **exact** expression for b_n in terms of n.

(Your final answer should involve n*, but shouldn't involve other elements of the sequence like* b_{n-1} *etc, and it shouldn't contain "…" or any variable-length sums ("*$\sum$*") etc.)*
You won't receive any points for solving for b_n **using other techniques.**

3 **(4 points)**

Let $C(x) = \sum_{n \geqslant 0} c_n x^n$ where $c_n = \sum_{k=0}^{n} 3^k (n-k+1)$.

Find a **closed-form** expression for $C(x)$ and re-express it as a sum of separate series to get an **exact** expression for c_n.

Hint: this is a product.

4 **(3 points)**

That devious villain Kl'rt is at it again... his previous master plan was foiled, but he has a new one. This time Kl'rt will divide his army of n *Skrulls* in the following way:

- First, Kl'rt will split the n *Skrulls (named with elements of* $[n]$*)* into some as-of-yet unknown number of *consecutive, non-empty* groups.
- From each group, Kl'rt will choose a leader for the group.
- And then finally, Kl'rt chooses a subset of the groups to send to the University of Toronto (with the remainder going to the city of Toronto). *(The first set of groups will attempt to infiltrate the mathematics department again while the second set of groups causes a diversion again. Kl'rt is not an especially creative villain.)*

Find a **closed form** of the generating function $G(x) = \sum g_n x^n$ for the number of ways, g_n, for Kl'rt to do this. *You do not need to find an exact expression for* g_n.

5 **(4 points)**

Definition

First, recall that for $n \geqslant 1$, $\mathbf{p_{\leqslant k}(n)}$ is the number of partitions of n with size at most k (or, equivalently, the number of partitions of n into at most k parts).

For $n \geqslant k \geqslant 1$, we define:

- $\mathbf{q}_k(n)$ to be the number of partitions of n which have exactly k parts, *each of them distinct.*
- $\mathbf{r}_k(n)$ to be the number of partitions of n which have parts of size at most k, but *each part from 1 to k occurs at least once.*

5.1 **(2 points)** Draw the *Ferrers shape* of each of the partitions enumerated by $q_3(10)$ and $r_3(10)$.

Question continued on the next page...

5.2 **(2 points)** Fix an integer $k \geqslant 1$, and prove that $q_k(n) = r_k(n)$ for all $n \geqslant k$.

6 **(3 points)**

Fix an integer $k \geqslant 1$. Using the terminology of the previous question (and assuming it to be true), prove the following for all $n > \binom{k+1}{2}$:

$$q_k(n) = p_{\leqslant k}\left(n - \binom{k+1}{2}\right).$$

7 **(2 BONUS points)**

For each fixed integer $k \geqslant 1$, let $F_k(x) = \sum_{n \geqslant k} S(n,k)x^n$. *(i.e. we are defining one generating function for each* $k \geqslant 1$ *here.)*

Use the fact that $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$ to prove that for each k,

$$F_k(x) = \frac{x^k}{(1-x)(1-2x) \cdot ... \cdot (1-kx)}$$