MAT334 - Week 8 Problems

Textbook Problems 1.4: 1 - 8, 30 - 41

2.2: 1 - 6, 19a

2.4: 18, 20

Additional Problems

1. Which of the following sequences converge? Find their limits.

   (a) $a_n = \frac{n-1}{n+1}$

   (b) $a_n = (n^2+n-1)^{\frac{1}{2}} - (n^2)^{\frac{1}{2}}$ for the principal branch of the square root.

   (c) $a_n = \left(\frac{1+3i}{16}\right)^n$

   (d) $a_n = \left(1+\frac{1}{n}\right)^i$ for the principal branch of z^i. (Hint: this is not 0!)

   (e) $a_n = \left(1+\frac{1}{n}\right)^i$ for the branch of z^i given by $\arg_1(z) \in (\pi, 3\pi)$.

   (f) $a_n = i\sin\left(\frac{i}{n}\right)$

   (g) $a_n = \frac{e^{in}}{e^n}$

   (h) $a_n = e^{in\theta}$

   Solution:

   a) Our trick from lecture (from 135) gives a limit of 1.
   There is another way of doing this one though. Notice $n-1 = (n+1)-2$, so:

   $$a_n = 1 - \frac{2}{n+1}$$

   As $n \to \infty$, the second term goes to 0 and so $a_n \to 1$.

   b) This is another trick from 135:

   $$\begin{aligned} a_n &= (n^2+n-1)^{\frac{1}{2}} - (n^2)^{\frac{1}{2}}\left(\frac{(n^2+n-1)^{\frac{1}{2}} + (n^2)^{\frac{1}{2}}}{(n^2+n-1)^{\frac{1}{2}} + (n^2)^{\frac{1}{2}}}\right) \\ &= \frac{n^2+n-1-(n^2)}{(n^2+n-1)^{\frac{1}{2}} + (n^2)^{\frac{1}{2}}} \\ &= \frac{n-1}{(n^2+n-1)^{\frac{1}{2}} + (n^2)^{\frac{1}{2}}} \end{aligned}$$

   Now, the principal branch of the square root on positive reals is simply $r^{\frac{1}{2}} = \sqrt{r}$ (the normal square root function on $[0,\infty)$). So:

   $$\begin{aligned} a_n &= \frac{n-1}{\sqrt{n^2+n-1} + \sqrt{n^2}} \\ &= \frac{1-\frac{1}{n}}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}} + \sqrt{1}} \end{aligned}$$

   In the limit, we get $\frac{1}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}$.

c) For powers w^n, you always want to look at $|a_n|$. Here:

$$|a_n| = \left(\frac{\sqrt{10}}{16}\right)^n$$

As $n \to \infty$, we see that $|a_n| \to 0$ and so $a_n \to 0$.

d) Remember that $z^i = e^{i\text{Log}(z)}$ for the principal branch. So:

$$a_n = e^{i\text{Log}\left(1+\frac{1}{n}\right)} = e^{i\ln\left(1+\frac{1}{n}\right)}$$

In the limit, $1 + \frac{1}{n} \to 1$, and so $e^{i\ln\left(1+\frac{1}{n}\right)} \to e^{i0} = 1$.

e) Now, we have $z^i = e^{i\log_1(z)}$, so we need to be careful. Now, $\log_1\left(1+\frac{1}{n}\right) = i\arg_1\left(1+\frac{1}{n}\right) = 2\pi$. So:

$$a_n = e^{i(\ln\left(1+\frac{1}{n}\right)+2\pi i)} = e^{-2\pi + i\ln\left(1+\frac{1}{n}\right)}$$

As in the last problem, we see the limit is $e^{-2\pi}$.

f) Recall that $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. So:

$$a_n = \frac{e^{\frac{-1}{n}} - e^{\frac{1}{n}}}{2}$$

As $n \to \infty$, $e^{\frac{1}{n}} \to 1$ so the numerator goes to $1 - 1$, and so the limit is 0.

g) Careful here. We see that:

$$|a_n| = \frac{|e^{in}|}{|e^n|} = \frac{1}{e^n}$$

So, since $|a_n| \to 0$, we have that $a_n \to 0$.

h) This one is a visual puzzle. What you're doing is looping around the unit circle a whole bunch. Unless $e^{i\theta} = 1$, the limit does not exist. It never settles down. A picture suffices for this one.

2. Which of the following series converge? Justify.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{i}{2n^2}$

b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{i}{1+3i}\right)^n$

c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{1+3i}\right)^n$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{n^2+3n-1}$

e) $\displaystyle\sum_{n=1}^{\infty} \text{Log}\left(\frac{n}{n+2}\right)$

f) $\displaystyle\sum_{n=2}^{\infty} \frac{i}{n^2-1}$

g) $\displaystyle\sum_{n=0}^{\infty} e^{in\frac{\pi}{7}}$

h) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2^n}$

i) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2^n}$

Solution:

a) This is $\frac{i}{2}\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{i\pi^2}{12}$ (we know this sum from lecture).

b) This is a geometric series with $r = \frac{i}{1+3i}$, so $|r| = \frac{1}{\sqrt{10}} < 1$, and so the series converges.
Alternatively, apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{i}{1+3i}\right)^{n+1}}{\left(\frac{i}{1+3i}\right)^{n}} \right| = \left| \frac{i}{1+3i} \right| = \frac{1}{\sqrt{10}}$$

And so by the ratio test, our series converges absolutely.

c) Similar to the previous question, except now $|r| = \frac{5}{\sqrt{10}} > 1$, and so the series diverges.

d) Looking at the terms, we see that $\lim_{n\to\infty} \frac{n^2+1}{n^2+3n-1} = 1 \neq 0$. So the series diverges.

e) There's a trick to this. Recall that $\mathrm{Log}(r) = \ln r$ for all $r \in (0, \infty)$. So:

$$\mathrm{Log}\left(\frac{n}{n+2}\right) = \mathrm{Log}(n) - \mathrm{Log}(n+2)$$

Looking at our first few terms, we get: $\mathrm{Log}(1) - \mathrm{Log}(3) + \mathrm{Log}(2) - \mathrm{Log}(4) + \mathrm{Log}(3) - \mathrm{Log}(5)....$
If S_n is the nth partial sum, then $S_n = \mathrm{Log}(1) + \mathrm{Log}(2) - \mathrm{Log}(n+1) - \mathrm{Log}(n+2) = \ln(2) - \ln(n+1) - \ln(n+2)$. As $n \to \infty$, these negative ln terms each go to ∞, and so:

$$S_n \to -\infty$$

At least, when viewed as a real limit. We haven't talked about this yet, but for $\mathbb{C}$ we only have one ∞, which encompasses the infinity in every direction. So, in this case, the series actually diverges to ∞.

f) A similar trick works here. Recall that $\frac{1}{n^2-1} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$. So, using the same idea, we see that:

$$S_n = i\left(\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right)$$

Now, the difference here is that as $n \to \infty$, the two negative terms both go to 0. So:

$$\lim_n S_n = \frac{3i}{4}$$

So $\displaystyle\sum_{n=2}^{\infty} \frac{i}{n^2-1} = \frac{3i}{4}$. (Naturally, this means the series converges.)

g) $|a_n| = 1$, so $a_n \not\to 0$, so the series diverges.

h) $n!$ and 2^n are both the type of terms that the ratio test behaves nicely with. With both appearing, my first instinct is to run for the ratio test. So:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}} \right| = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

So, the ratio test tells us that the series diverges.

i) Similarly, we apply the ratio test. But instead, we get:

$$\lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} < 1$$

So by the ratio test, the series converges.

3. Find the radius of convergence for the following power series.

a) $\sum_{n=0}^{\infty} n(z-1)^n$

b) $\sum_{n=0}^{\infty} \frac{1}{n}(z-i)^n$

c) $\sum_{n=0}^{\infty} \frac{1}{2^n} z^n$

d) $\sum_{n=0}^{\infty} \frac{1}{w^n}(z+1)^n$ for any fixed $w \in \mathbb{C}$, $w \neq 0$

e) $\sum_{n=0}^{\infty} \frac{2^n}{n!}\left(\frac{1}{2}z+1\right)^n$

f) $\sum_{n=0}^{\infty} 4^n z^{2n}$ (Hint: the ratio test doesn't apply straight away. Try a substitution first.)

Solution:

a) The ratio test gives $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = 1$, so $R = 1$.

b) Similar to the last question, we get $R = 1$.

c) Here, we get $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{2^{n+1}}\right)}{\left(\frac{1}{2^n}\right)}\right| = \frac{1}{2}$, so $R = 2$.

d) Here, we get $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{w^{n+1}}\right)}{\left(\frac{1}{w^n}\right)}\right| = \frac{1}{|w|}$, so $R = |w|$.

e) Here, we get $\lim_{n\to\infty}\left|\frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)}\right| = \frac{2}{n+1} = 0$. When this occurs, remember that we say $R = \infty$.

f) If we try to apply the ratio test, we're going to have an issue. For n odd, $a_n = 0$! So, the ratio test is going to give a limit that DNE, which does not mean that $R = 0$ in this case.
To solve this, let $w = z^2$. Then:

$$\sum_{n=0}^{\infty} 4^n z^{2n} = \sum_{n=0}^{\infty} 4^n w^n$$

Is a power series in w, which we can apply the ratio test to. The ratio test gives:

$$\frac{1}{R} = \lim_{n\to\infty}\frac{4^{n+1}}{4^n} = 4$$

So, $\sum_{n=0}^{\infty} 4^n w^n$ coverges absolutely when $|w| < \frac{1}{4}$. But $|w| = |z|^2$, so the series converges when $|z| < \frac{1}{2}$, and $R = \frac{1}{2}$.

4. Suppose f is entire with $|f'(z)| < |f(z)|$ for all $z \in \mathbb{C}$. Prove that $f(z) = Ke^{Cz}$ for some $C \in \mathbb{C}$. (Hint: show that $f(z) \neq 0$ for any $z \in \mathbb{C}$. Then, try to use this to come up with a function to use Liouville's theorem on. You may assume that Ke^{Cz} is the only solution to $f'(z) = Cf(z)$, although we will prove this very soon.)

   Solution: This is a bounding condition, so you should think of Liouville right away. Let's see if we can get an entire function bounded by a constant.

   Consider $g(z) = \frac{f'(z)}{f(z)}$. Notice that as long as $f(z) \neq 0$, that $|g(z)| \leq 1$. But since $|f(z)| > |f'(z)| \geq 0$, we know $f(z) \neq 0$ for all z.

   So $g(z)$ is an entire function (by the chain rule) and is bounded by 1. So $g(z)$ is constant.

   Therefore, $g(z) = C$ for some $C \in \mathbb{C}$ with $|C| \leq 1$. This tells us:

$$f'(z) = Cf(z)$$

   We have shown in lecture that the only solutions to this are $f(z) = Ke^{Cz}$.

5. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$, whenever both series converge. This is called a Laurent series expression for $f(z)$.

   Unless $b_n = 0$ for all n, we see that this is not defined at $z = z_0$. So if R is the radius of convergence for $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, we cannot claim that $f(z)$ is defined for all $|z - z_0| < R$.

   Apply the ratio test to both series to show that there exist R_1, R_2 such that $f(z)$ is defined for all $R_1 < |z - z_0| < R_2$.

   Solution: Applying the ratio test to the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ gives that it converges for $|z-z_0| < R_2$. Assume $R_2 > 0$.

   Now, apply the ratio test to the other series. It converges when:

$$\lim_{n\to\infty} \left| \frac{\frac{b_{n+1}}{(z-z_0)^{n+1}}}{\frac{b_n}{(z-z_0)^n}} \right| = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| \frac{1}{|z-z_0|} < 1$$

   So, this series converges when $|z - z_0| > \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = R_1$.

   So the function is defined when $|z - z_0| > R_1$ and when $|z - z_0| < R_2$, as desired.