MAT334 - Week 5 Problems

Additional Problems

1. Which of the following functions are harmonic?

a) $u(x, y) = xy^2$
b) $u(x, y) = 4x^2 - 2xy - 4y^2$
c) $u(x, y) = x^3 - 2xy^2$
d) $u(x, y) = x^3 - 3xy^2$
e) $u(x, y) = x^4 - 6x^2y^2 + y^4$
f) $u(x, y) = \frac{2x}{x^2+y^2}$
g) $u(x, y) = \arctan(y/x)$
h) $u(x, y) = \ln(x^2 + y^2)$

Solutions:

a) $u_{xx} = 0$ and $u_{yy} = 2x$, so $u_{xx} + u_{yy} \neq 0$ for $x \neq 0$, so $u(x, y)$ is not harmonic on any domain.
b) $u_{xx} = 8$, $u_{yy} = -8$, so u is harmonic on $\mathbb{C}$.
c) $u_{xx} = 6x$, $u_{yy} = -4x$, so $u_{xx} + u_{yy} \neq 0$ for $x \neq 0$ as in (a), so not harmonic.
d) $u_{xx} = 6x$, $u_{yy} = -6x$, so $u_{xx} + u_{yy} = 0$ and u is harmonic on $\mathbb{C}$.
e) $u_{xx} = 12x^2 - 12y^2$, $u_{yy} = -12x^2 + 12y^2$, so $u_{xx} + u_{yy} = 0$ and u is harmonic on $\mathbb{C}$.
f) This is a bit harder to calculate, so we'll take it one derivative at a time.

$$u_x = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$u_{xx} = \frac{(-4x)(x^2 + y^2)^2 - (2y^2 - 2x^2)(2(x^2 + y^2)(2x))}{(x^2 + y^2)^4}$$

As for y:

$$u_y = \frac{-4xy}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{(-4x)(x^2 + y^2)^2 - (-4xy)(2(x^2 + y^2))(2y)}{(x^2 + y^2)^4}$$

$$
\begin{aligned}
u_{xx}+u_{yy} &= \frac{(-8x)(x^2+y^2)^2-4(2xy^2-4xy^2-2x^3)(x^2+y^2)}{(x^2+y^2)^4} \\
&= \frac{(-8x)(x^4+2x^2y^2+y^4)-4(-2x^3-2xy^2)(x^2+y^2)}{(x^2+y^2)^4} \\
&= \frac{-8x^5-16x^3y^2-8xy^4+8(x^3+xy^2)(x^2+y^2)}{(x^2+y^2)^4} \\
&= \frac{-8x^5-16x^3y^2-8xy^4+8x^5+8x^3y^2+8x^3y^2+8xy^4}{(x^2+y^2)^4} \\
&= 0
\end{aligned}
$$

So u is harmonic on $\mathbb{C}\setminus\{0\}$.

g)

$$
\begin{aligned}
u_x &= \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{-y}{x^2} \\
&= \frac{-y}{x^2+y^2} \\
u_{xx} &= \frac{-2xy}{(x^2+y^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
u_y &= \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{1}{x} \\
&= \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{x}{x^2} \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

$$
u_{yy} = \frac{2xy}{(x^2+y^2)^2}
$$

Then $u_{xx}+u_{yy}=0$ and u is harmonic on $\mathbb{C}\setminus\{x+iy|x=0\}$.

h)

$$
u_x = \frac{2x}{x^2+y^2}
$$

$$
\begin{aligned}
u_{xx} &= \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} \\
&= \frac{2y^2-2x^2}{(x^2+y^2)^2}
\end{aligned}
$$

$$
u_y = \frac{2y}{x^2+y^2}
$$

$$
\begin{aligned}
u_{yy} &= \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} \\
&= \frac{2x^2-2y^2}{(x^2+y^2)^2}
\end{aligned}
$$

So $u_{xx} + u_{yy} = 0$ and u is harmonic on $\mathbb{C} \setminus \{0\}$.

2. For each of the harmonic functions in the previous question, find a harmonic conjugate.

Solution:

a) This function is not harmonic.

b) We know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 8x - 2y$.
So $v(x,y) = \int \frac{\partial v}{\partial y} dy = 8xy - y^2 + C(x)$.
Then we have that $\frac{\partial v}{\partial x} = 8y + C'(x)$. But we also know that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 8y + 2x$. So$C'(x) = 2x$, and $C(x) = x^2 + C$.
Therefore, $v(x,y) = x^2 + 8xy - y^2 + C$.

c) This function is not harmonic.

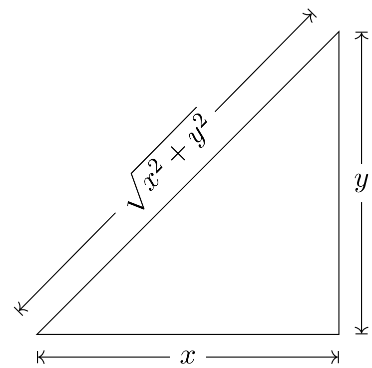
d) We know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$.
So $v(x,y) = \int \frac{\partial v}{\partial y} dy = 3x^2 y - y^3 + C(x)$.
Then we have that $\frac{\partial v}{\partial x} = 6xy + C'(x)$. But we also know that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy$. So$C'(x) = 0$, and $C(x) = C$.
Therefore, $v(x,y) = 3x^2 y - y^3 + C$.

e) We know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$.
So $v(x,y) = \int \frac{\partial v}{\partial y} dy = 4x^3 y - 4xy^3 + C(x)$.
Then we have that $\frac{\partial v}{\partial x} = 12x^2 y - 4y^3 + C'(x)$. But we also know that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 12xy^2 - 4y^3$. So$C'(x) = 0$, and $C(x) = C$.
Therefore, $v(x,y) = 4x^3 y - 4xy^3 + C$.

f) We know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$.
So $v(x,y) = \int \frac{\partial v}{\partial y} dy$. To compute this integral, we make the substitution $y = x\tan\theta$. So we have:

$$
\begin{aligned}
v(x,y) &= \int \frac{2x^2\tan^2\theta - 2x^2}{(x^2 + x^2\tan\theta)^2} x\sec^2\theta d\theta \\
&= 2\int \frac{x^3\tan^2\theta - x^3}{x^4\sec^4\theta} \sec^2\theta d\theta \\
&= \frac{2}{x}\int (\tan^2\theta - 1)\cos^2\theta d\theta \\
&= \frac{2}{x}\int \sin^2\theta - \cos^2\theta d\theta \\
&= \frac{2}{x}\int -\cos(2\theta) d\theta \\
&= \frac{-1}{x}\sin(2\theta) + C(x) \\
&= \frac{-2}{x}\sin\theta\cos\theta + C(x)
\end{aligned}
$$

Why $C(x)$? Well, we had a derivative in terms of y treating x as constant, so our constant still depends on x, even after changing to θ. Now, we know that $y = x\tan\theta$. So we have the triangle:



So it follows that:

$$\begin{aligned}
v(x,y) &= \frac{-2}{x}\sin\theta\cos\theta + C(x) \\
&= \frac{-2}{x}\frac{y}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}} + C(x) \\
&= \frac{-2y}{x^2+y^2} + C(x)
\end{aligned}$$

Then we have that $\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} + C'(x)$. But we also know that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$. So$C'(x) = 0$, and $C(x) = C$.
Therefore, $v(x,y) = \frac{-2y}{x^2+y^2}$.

g) We have already calculated the partial derivatives, in the previous question.
So, we have that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-x}{x^2+y^2}$, and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$. So:

$$v(x,y) = \int \frac{\partial v}{\partial x}dx = \int \frac{-x}{x^2+y^2}dx$$

By making the substitution $t = x^2+y^2$, we get:

$$v(x,y) = -\int \frac{1}{2t}dt = -\frac{1}{2}\ln(t) + C(y) = -\frac{1}{2}\ln(x^2+y^2) + C(y)$$

Differentiating in terms of y gives us that:

$$\frac{\partial v}{\partial y} = -\frac{y}{x^2+y^2} + C'(y)$$

But we already know that $\frac{\partial v}{\partial y} = \frac{-y}{x^2+y^2}$, so:

$$\frac{-y}{x^2+y^2} = \frac{-y}{x^2+y^2} + C'(y)$$

As such, $C'(y) = 0$, so we get $C(y) = C$ is constant. Therefore, $v(x,y) = -\frac{1}{2}\ln(x^2+y^2) + C$.

h) We know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$.
So $v(x,y) = \int \frac{\partial v}{\partial y}dy = \int \frac{2x}{x^2+y^2}dy$.
To integrate this, make the substitution $t = \frac{y}{x}$. Then:

$$
\begin{aligned}
v(x,y) &= \int \frac{2x}{x^2+y^2}dy \\
&= \int \frac{2x}{x^2+x^2t^2}xdt \\
&= 2\int \frac{1}{1+t^2}dt \\
&= 2\arctan(t) + C(x) \\
&= 2\arctan\left(\frac{y}{x}\right) + C(x)
\end{aligned}
$$

Then we have that $\frac{\partial v}{\partial x} = 2\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{-y}{x^2} = \frac{-2y}{x^2+y^2}$. But we also know that $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{-2y}{x^2+y^2}$. So$C'(x) = 0$, and $C(x) = C$.

Therefore, $v(x,y) = 2\arctan\left(\frac{y}{x}\right) + C$.

3. Suppose $f = u + iv$ is analytic. Then u and v are harmonic conjugates. Is it true that $g = v + iu$ is analytic?

Solution: Let's look at an example. Consider $f(z) = z$, which we know is entire. Then $u(x,y) = x$ and $v(x,y) = y$.

Now, $g(z) = v(x,y) + iu(x,y) = y + ix$. Is this analytic? Let's check the Cauchy-Riemann equations. Write $g(z) = a(x,y) + ib(x,y)$. Then

$$
\begin{aligned}
\frac{\partial a}{\partial x} &= 0 \\
\frac{\partial b}{\partial y} &= 0 \\
\frac{\partial a}{\partial y} &= 1 \\
\frac{\partial b}{\partial x} &= 1
\end{aligned}
$$

We do have that $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$, but we do not have that $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$. So the Cauchy-Riemann equation do not hold, and g is not analytic.

In general: $g(z) = i\overline{f(z)}$, so if f and g are analytic, then f and $\overline{f}$ are analytic, and so f is constant.

4. Give an explicit formula for each of the following curves:

a) The circle of radius 3 centered at $z_0 = -i + 1$.
b) The semicircle of radius 2 centered at 0 starting at 2 and ending at -2.
c) The quarter-circle of radius $\sqrt{2}$, centered at i, and going from $1 + 2i$ to 1.
d) The closed semicircle of radius R from R to $-R$. (Closed here means that once we reach -1, we head back to 1 along a straight line.)
e) The triangle with vertices $1 + i$, $2 - i$ and $\frac{1}{i}$ starting at $1 + i$ and moving counterclockwise.
f) The square with vertices $0, 1, 1 + i$, and i, travelled in that order.
g) The curve given first by following the circle of radius 2 centered at 0 from 2 to -2, counterclockwise. Then follow the line segment from -2 to -1. Then follow the circle of radius 1 centered at 0 around one full rotation, clockwise. Then follow the line segment from -1 back to -2. And then finish following the circle of radius 2 from -2 back to 2, still going counterclockwise.

Solution:

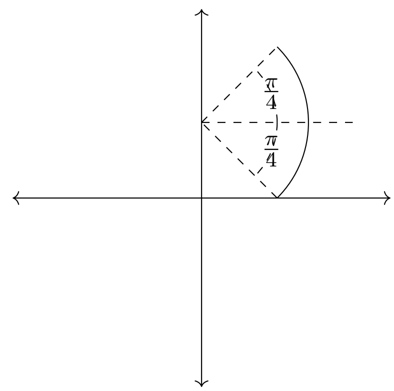
a) $\gamma(t) = 3e^{it} + (-i + 1)$ from $t = 0$ to $t = 2\pi$.

b) There are two options here, depending on if we take the upper semicircle or the lower one.

$$\gamma_{upper}(t) = 2e^{it}$$
$$\gamma_{lower}(t) = 2e^{-it}$$

Where t goes from 0 to 2π.

c) We need to draw this one out:



Then we can see that our angle goes from $-\pi/4$ to $\pi/4$. So $\gamma(t) = \sqrt{2}e^{it}$ from $t = -\pi/4$ to $\pi/4$.

d) This curve has two components: the semicircle from R to $-R$ and the line from $-R$ to R. We're going to assume that the semicircle is above the real axis, but the other curve is similar. The equations for our two curves are:

$$\gamma_{circle}(t) = Re^{it}$$
$$\gamma_{line}(t) = (1-t)(-R) + tR = -R + 2tR$$

Where γ_{circle} goes from $t = 0$ to $t = \pi$. And γ_{line} goes from $t = 0$ to $t = 1$.

Then $\gamma(t) = (\gamma_{circle} + \gamma_{line})(t) = \begin{cases} \gamma_{circle}(t), & t \in [0, \pi] \\ \gamma_{line}(t-\pi), & t \in [\pi, \pi+1] \end{cases} = \begin{cases} 2e^{it}, & t \in [0, \pi] \\ -R + 2R(t-\pi), & t \in [\pi, \pi+1] \end{cases}$.

e) This curve has three components:

$$\gamma_{l_1}(t) = (1-t)(1+i) + (2-i)t = 1 + i + (1-2i)t$$
$$\gamma_{l_2}(t) = (1-t)(2-i) - it = 2 - i - 2t$$
$$\gamma_{l_3}(t) = (1-t)(-i) + (1+i)t = -i + (1+2i)t$$

Where for each curve, t goes from 0 to 1. Then:

$$\gamma(t) = (\gamma_{l_1} + \gamma_{l_2} + \gamma_{l_3})(t) = \begin{cases} \gamma_{l_1}(t), & t \in [0,1] \\ \gamma_{l_2}(t-1), & t \in [1,2] \\ \gamma_{l_3}(t-2), & t \in [2,3] \end{cases} = \begin{cases} 1 + i + (1-2i)t, & t \in [0,1] \\ 2 - i - 2(t-1), & t \in [1,2] \\ -1 + (1+2i)(t-2) & , t \in [2,3] \end{cases}$$

f) This works out the same as the previous two parts:

$$\gamma(t) = \begin{cases} t, & t \in [0,1] \\ 1 + i(t-1), & t \in [1,2] \\ 1 + i - (t-2), & t \in [2,3] \\ -i(t-4), & t \in [3,4] \end{cases}$$

g)

$$\gamma(t) = \begin{cases} 2e^{it}, & t \in [0,\pi] \\ -2 + (t-\pi), & t \in [\pi, \pi+1] \\ e^{-i(t-1)}, & t \in [\pi+1, 3\pi+1] \\ -1 - (t - 3\pi + 1), & t \in [3\pi+1, 3\pi+2] \\ 2e^{i(t-2\pi-2)}, & t \in [3\pi+2, 4\pi+2] \end{cases}$$

5. For each of the **closed** curves in the previous question, determine whether the curve is positively or negatively oriented.

Solution:

a) I did not specify which direction to travel the circle, but assuming CCW, then γ is positively oriented.
b) This curve is not closed.
c) Not closed.
d) Positively oriented.
e) Positively oriented.
f) Positively oriented.
g) Not simple, but it does make sense to say this curve is postively oriented.

6. Let $\gamma_1(t)$ be the curve in 4d) and $\gamma_2(t)$ be the curve in 4f). For each $f(z)$ given below, evaluate $\int_{\gamma_1} f(z)dz$ and $\int_{\gamma_2} f(z)dz$.

a) $f(z) = z$
b) $f(z) = z^2$
c) $f(z) = e^z$
d) $f(z) = \frac{1}{z+i}$
e) $f(z) = 1$

You should get 0 for each of these integrals. Do not use any FTC like results, I want you to do these from the definition of the integral.

Solutions:

a) By CIT, the integral is 0. Or by $\mathbb{C}$FTC.
b) By CIT, 0.
c) By CIT, 0.
d) For the curve in 4d, CIT gives 0. $\frac{1}{z+i}$ is not defined at $z = -i$, which lies on the curve from 4g, so that integral isn't even defined.

e) CIT gives 0.

7. Recall that for a smooth curve $\gamma(t)$ on $[a, b]$, we defined $(-\gamma)(t)$ by:

$$(-\gamma)(t) = \gamma(a+b-t)$$

Prove that $\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$.

Solution: Remember how $-\gamma$ is defined:

$$(-\gamma)(t) = \gamma(a+b-t)$$

So: $\int_{-\gamma} f(z)dz = \int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))dt$. Let $s = a+b-t$. Then: $\int_b^a -f(\gamma(s))(-\gamma'(s))ds = -\int_a^b f(\gamma(s))\gamma'(s)ds = -\int_{\gamma} f(z)dz$.

8. Suppose $\gamma_1(t)$ is a smooth curve, and $\gamma_2(t)$ traverses $\gamma_1(t)$ backwards. Prove that:

$$\int_{\gamma_1+\gamma_2} f(z)dz = 0$$

Hint: this should be a 2 line argument.

Solution: First, note that $\gamma_2(t) = (-\gamma_1)(t)$. Then:

$$\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_1} f(z)dz = 0$$