University of Toronto
Department of Mathematics

MAT334 - Complex Variables - Midterm 1

June 25th, 2018

Examiner: T. Janisse

Start time: 9:10am

Duration 110 minutes. **No aids permitted.**

Last Name: ______________________________

First Name: ______________________________

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Student number: ______________________________

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Directions:

- This test is **1 hour and 50 minutes long.** You are not permitted the use of any aids, such as calculators, phones, textbooks, etc.
- This examination booklet contains a total of **10** pages. It is your responsibility to ensure that *no pages are missing from your examination.* Do not detach any pages from your examination.
- **Do not write on the QR code at the top of any of the pages.**
- You may write in either pen or pencil.
- Have your student card ready for inspection.
- There are **60** possible marks to be earned in this exam, and **2** possible bonus marks.
- For the long answer questions, answer the questions on the question pages themselves. You may use the two blank pages at the back of the test paper for rough work. The extra pages **WILL NOT BE MARKED** unless you *clearly* indicate otherwise on the question pages.
- Good luck!

PART A - TRUE OR FALSE QUESTIONS

Please answer the following questions as **TRUE** or **FALSE**. Write your answers in the boxes on the right side. No justification required.

1. **(2 points)** $-1 \leq |\sin(z)| \leq 1$ for any $z \in \mathbb{C}$. F

   Solutions: We saw this example in class. For $z = ir$ where $r \in \mathbb{R}$, we get that $\sin(z) = \frac{e^{-r}-e^{r}}{2i}$. As r gets very large, the e^r gets much larger than e^{-r}, and $|\sin(ir)|$ gets very large. $|\sin(ir)| \sim \frac{e^r}{2}$.

2. **(2 points)** For any non-zero $z \in \mathbb{C}$ and any logarithm $\log z$ of z, $e^{\log z} = z$. T

   Solution: This is just the definition of $\log(z)$.

3. **(2 points)** Every value of $(1+i)^i$ is real. F

   Solution: $(1+i)^i = e^{i\log(1+i)} = e^{i(\ln(\sqrt{2})+i(\frac{\pi}{4}+2k\pi)} = e^{-\frac{\pi}{4}-2k\pi}e^{i\ln(\sqrt{2})}$. Now, since $\ln(\sqrt{2}) \in (0,1)$, we know that $\ln(\sqrt{2}) \neq k\pi$ for any k, and so $e^{i\ln(\sqrt{2})}$ is not real.

   So, no value of $(1+i)^i$ is real.

4. **(2 points)** The function $f(z) = x^2 + iy^2$ is analytic on $\mathbb{C}$. F

   Solution: If $f = u + iv$ is analytic, then u, v are harmonic. But x^2 is not harmonic, since $u_{xx} = 2$ and $u_{yy} = 0$.

5. **(2 points)** The function $\frac{1}{z}$ has a primitive on the domain $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. T

   Solution: $\frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$. The domain $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is a subset of $\mathbb{C} \setminus \{0\}$, so $\frac{1}{z}$ is also analytic on this domain.

   Also, the domain $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is simply connected. So $\frac{1}{z}$ is analytic on a simply connected domain, and therefore has a primitive on that domain.

6. **(2 points)** If $f(z)$ is analytic on a domain D and γ is a closed curve in D, then $\int_\gamma f(z)dz = 0$. F

   Solution: Let $D = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$. Let γ be the unit circle, travelled once counterclockwise. We know from lecture that $\int_\gamma f(z)dz = 2\pi i \neq 0$.

PART B - LONG ANSWER QUESTIONS
Write your answer in the space provided. All answers must be justified.

1. a) (**5 points**) Find all solutions to the equation $z^6 = 4iz^2$.

Solution: Rearranging, we get that $z^6 - 4iz^2 = 0$. Factoring gives $z^2(z^4 - 4i) = 0$. So either $z = 0$ or $z^4 = 4i$.
We can write $4i$ as $4e^{i\pi/2}$. So if $z = re^{i\theta}$ we have:

$$r^4 e^{4i\theta} = 4e^{i\pi/2}$$

As such, $r = 4^{\frac{1}{4}} = \sqrt{2}$. And $\theta = \frac{\pi}{8} + \frac{k\pi}{2}$ for $k = 0, 1, 2, 3$.

So the solutions are $z = 0$ or $z = \sqrt{2}e^{i\left(\frac{\pi}{8}+\frac{k\pi}{2}\right)}$ for $k = 0, 1, 2, 3$.

b) (**5 points**) Find all solutions to the equation $e^{2z} - 5ie^z = 6$.

Solution: Let $w = e^z$. Then our equation becomes:

$$w^2 - 5iw - 6 = 0$$

The quadratic formula then gives:

$$\begin{aligned} w &= \frac{5i + ((5i)^2 - 4(1)(-6))^{\frac{1}{2}}}{2} \\ &= \frac{5i + (-25 + 24)^{\frac{1}{2}}}{2} \\ &= \frac{5i \pm i}{2} \end{aligned}$$

So $e^z = 2i$ or $3i$. Therefore $z = \log(2i)$ or $z = \log(3i)$.
Well, $\log(2i) = \ln(2) + i\left(\frac{\pi}{2} + 2k\pi\right)$ and $\log(3i) = \ln(3) + i\left(\frac{\pi}{2} + 2k\pi\right)$. So the solutions are:

$$z = \ln(2) + i\left(\frac{\pi}{2} + 2k\pi\right)$$
$$z = \ln(3) + i\left(\frac{\pi}{2} + 2k\pi\right)$$

2. a) (**4 points**) Find all values of $(2-i)^{2+i}$.

Solution:

$$(2-i)^{2+i} = e^{(2+i)\log(2-i)}$$

So what is this logarithm? Well, $2-i = \sqrt{5}e^{\arctan\left(\frac{-1}{2}\right)}$, so $\log(2-i) = \ln\sqrt{5}+i(\arctan\left(\frac{-1}{2}\right)+2k\pi)$. So:

$$\begin{aligned}(2-i)^{2+i} &= e^{(2+i)(\ln\sqrt{5}+i(\arctan\left(\frac{-1}{2}\right)+2k\pi))}\\ &= e^{2\ln\sqrt{5}-(\arctan\left(\frac{-1}{2}\right)+2k\pi)}e^{i(\ln\sqrt{5}+2(\arctan\left(\frac{-1}{2}\right)+2k\pi))}\\ &= e^{5-(\arctan\left(\frac{-1}{2}\right)+2k\pi)}e^{i(\ln\sqrt{5})}\left(\frac{(2-i)}{\sqrt{5}}\right)^2\\ &= \frac{(3-4i)e^{5-(\arctan\left(\frac{-1}{2}\right)+2k\pi)}e^{i(\ln\sqrt{5})}}{5}\end{aligned}$$

b) (**4 points**) Find the domain of $f(z) = \text{Log}(z^2+9)$ and show it is analytic on its domain. What is $f'(z)$?

Solution: We know that $\text{Log}(w)$ is defined for $w \notin (-\infty, 0]$.
So the domain is $\{z \in \mathbb{C} : z^2+9 \notin (-\infty, 0]\}$. This rearranges to $\{z \in \mathbb{C} : z^2 \notin (-\infty, -9]\}$.
Suppose $z^2 \in (-\infty, 9]$. Then $z^2 = re^{-i\pi}$ where $r \geq 9$. So then $z = \sqrt{r}e^{\frac{-i\pi}{2}+k\pi} = \pm\sqrt{r}i$. Since $r \geq 9$, then $\sqrt{r} \geq 3$.
So the domain is $\{z \in \mathbb{C} : z \neq ri, r \in (-\infty, -3] \cup [3, \infty)\}$.
As for checking that it is analytic, we could check Cauchy-Riemann. However, it is much easier to note that $\text{Log}(z)$ is analytic on its domain, and z^2+9 is entire. So the chain rule tells us that $\text{Log}(z^2+9)$ is analytic on its domain.
Its derivative is $\frac{2z}{z^2+9}$.

3. a) (**4 points**) Let γ_1 be the circle of radius 3 centered at $-i$, travelled once counterclockwise starting at $2i$. Find $\int_{\gamma_1} \frac{1}{z+i} dz$.

Solution: Since the discontinuity of $\frac{1}{z+i}$ occurs at $z = -i$, which is the center of the circle, the Cauchy Integral Theorem does not apply.

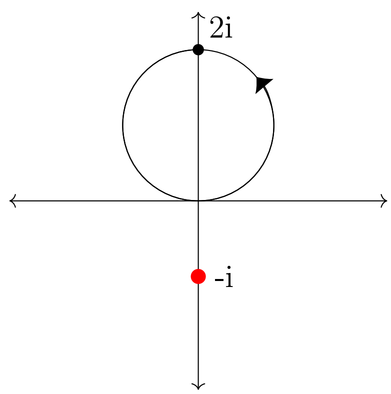
So we need to calculate from definition.

Let $\gamma_1(t) = 3e^{it} - i$ from $t = \frac{\pi}{2}$ to $\frac{5\pi}{2}$. Then:

$$\begin{aligned}\int_{\gamma_1} \frac{1}{z+i} dz &= \int_{\pi/2}^{5\pi/2} \frac{3e^{it} - i + i}{i} 3e^{it} dt \\ &= \int_{\pi/2}^{5\pi/2} i dt \\ &= 2\pi i\end{aligned}$$

b) (**4 points**) Let γ_2 be the circle of radius 1 centered at 1, travelled once clockwise starting at 0. Find $\int_{\gamma_2} \frac{1}{z+i} dz$.

Solution: Let us draw this situation:



We see that $\frac{1}{z+i}$ has a discontinuity, at $z = -i$, which lies outside our curve. Our curve is also smooth, simple, and closed.

So by the Cauchy Integral Theorem, $\int_{\gamma_2} \frac{1}{z+i} dz = 0$.

c) (**2 points**) Does there exist a domain D containing γ_1 (the curve from part a)) so that $f(z) = \frac{1}{z+i}$ has a primitive on D? Justify.

Solution: No such domain exists.

We know that if f has a primitive F on a domain D, and γ is any smooth closed curve in D, then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0$. So if $\frac{1}{z+i}$ has a primitive on a domain containing γ_1, it would follow that $\int_{\gamma_1} \frac{1}{z+i} dz = 0$. But we know this integral is $2\pi i \neq 0$. So $\frac{1}{z+i}$ cannot have a primitive.

4. a) (**3 points**) Let $u(x,y) = y + \arctan\left(\frac{y}{x}\right)$. Show that u is harmonic whenever $x \neq 0$.

Solution: We calculate:

$$\begin{aligned}
u_x &= \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} \\
u_{xx} &= \frac{-2xy}{(x^2+y^2)^2} \\
u_y &= 1 + \frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{1}{x} = 1 + \frac{x}{x^2+y^2} \\
u_{yy} &= \frac{2xy}{(x^2+y^2)^2}
\end{aligned}$$

So $u_{xx} + u_{yy} = 0$, and u is harmonic.

b) (**5 points**) Find a harmonic conjugate $v(x,y)$ for u such that $v(1,1) = -1 - \frac{1}{2}\ln(2)$.

Solution: We know that $v_y = u_x = \frac{-y}{x^2+y^2}$ and $v_x = -u_y = -1 - \frac{x}{x^2+y^2}$.
So $v(x,y) = \int v_y dy = \int \frac{-y}{x^2+y^2} dy$. We evaluate this integral by substituting for $t = x^2+y^2$. So:

$$v(x,y) = -\int \frac{1}{2t}dt = -\frac{1}{2}\ln t + C(x) = -\frac{1}{2}\ln(x^2+y^2) + C(x)$$

Now, this tells us that $v_x = \frac{-x}{x^2+y^2} + C'(x)$. We also know that $v_x = -1 - \frac{x}{x^2+y^2}$. So, equating the two expressions gives that $C'(x) = -1$, so $C(x) = -x + C$.
As such, we have that $v(x,y) = -\frac{1}{2}\ln(x^2+y^2) - x + C$.
We want $v(1,1) = -1 - \frac{1}{2}\ln(2)$. So, we evaluate our expression:

$$v(1,1) = -\frac{1}{2}\ln(1^2+1^2) - 1 + C = -1 - \frac{1}{2}\ln(2) + C$$

So $C = 0$, and $v(x,y) = -x - \frac{1}{2}\ln(x^2+y^2)$.

c) (**2 points**) Let $u(x,y)$ be as given in part (a) and $v(x,y)$ be the function you found in part (b). Let $f(z) = u(x,y) + iv(x,y)$. Show that $\frac{f(z)}{z+\mathrm{Log}(z)}$ is constant for any z in the first or fourth quadrants. (Hint: write $z + \mathrm{Log}(z)$ in terms of x, y first.)

Solution: In the first and fourth quadrants, we know that:

$$\begin{aligned}
\mathrm{Log}(z) &= \ln|z| + i\arctan\left(\frac{y}{x}\right) \\
&= \ln(\sqrt{x^2+y^2}) + i\arctan\left(\frac{y}{x}\right) \\
&= \frac{1}{2}\ln(x^2+y^2) + i\arctan\left(\frac{y}{x}\right)
\end{aligned}$$

So, plugging this into our expression gives:

$$\begin{aligned}
\frac{f(z)}{z+\mathrm{Log}(z)} &= \frac{y+\arctan\left(\frac{y}{x}\right)+i(-x-\frac{1}{2}\ln(x^2+y^2))}{x+\frac{1}{2}\ln(x^2+y^2)+i(y+\arctan\left(\frac{y}{x}\right))} \\
&= \frac{(-i)(i(y+\arctan\left(\frac{y}{x}\right)))+(-i)(x+\frac{1}{2}\ln(x^2+y^2))}{x+\frac{1}{2}\ln(x^2+y^2)+i(y+\arctan\left(\frac{y}{x}\right))} \\
&= \frac{-i\left[(x+\frac{1}{2}\ln(x^2+y^2)+i(y+\arctan\left(\frac{y}{x}\right))\right]}{x+\frac{1}{2}\ln(x^2+y^2)+i(y+\arctan\left(\frac{y}{x}\right))} \\
&= -i
\end{aligned}$$

So $\frac{f(z)}{z+Log(z)} = -i$, and hence is constant.

5. a) (**6 points**) Prove that if $f = u + iv$ is analytic and $u + v$ is constant on a domain D, then f is constant.

Solution: Suppose f is analytic and $u + v$ is constant. Let $u + v = C$. So $v = C - u$.
Now, this tells us that $v_x = -u_x$ and $v_y = -u_y$.
However, we also have the Cauchy-Riemann equations, $u_x = v_y$ and $v_x = -u_y$. Using them, with the above two equations, gives:

$$u_x = v_y = -u_y = v_x = -u_x$$

So $u_x = 0$. But then $v_y = u_y = v_x = 0$ as well, by following our chain of equalities.
So f is constant.

b) (**2 points**) Let $u(x, y) = x^2 + 7xy - y^2$ and $v(x, y) = 5 - x^2 - 7xy + y^2$. Show that $f = u + iv$ is not analytic on any domain D.

Solution: Notice that $u(x, y) + v(x, y) = 5$ is constant. Also, notice that u, v are not constant functions.
So by part (a), $f = u + iv$ cannot be analytic, since it is not constant.

c) (**2 points**) Suppose u and v are harmonic functions. Is $f = u + iv$ analytic? If so, justify. If not, provide a counter-example.

Solution: Notice that u, v from part (b) are harmonic.
Also, notice that $f = u + iv$ is not analytic. So the functions in part (b) provide a counter-example to the claim.

6. (Bonus: **2 points**) Where is the error in the following proof:

Theorem. *Every complex number is real.*

Proof. The claim is obviously true if $z \in \mathbb{R}$, so let $z \in \mathbb{C}$. Then $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in \mathbb{R}$ with $\theta \neq k\pi$ for any $k \in \mathbb{Z}$. So:

$$z = re^{i\theta} = re^{i\frac{2\pi\theta}{2\pi}} = r(e^{2\pi i})^{\frac{\theta}{2\pi}}$$

But $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$. So:

$$z = r(1^{\frac{\theta}{2\pi}}) = r \in \mathbb{R}$$

Since $z \in \mathbb{R}$ for any complex number z, every complex number is real.

$\square$

Solution: The error occurs across two steps. First, when we said that:

$$re^{i\frac{2\pi\theta}{2\pi}} = r(e^{2\pi i})^{\frac{\theta}{2\pi}}$$

We implicitly chose a branch of the function $z^{\frac{\theta}{2\pi}}$ to make this claim true. It is not true in general, as we have seen many times in other forms.

In particular, $(e^{2\pi i})^{\frac{\theta}{2\pi}} = e^{\frac{\theta}{2\pi}\log(e^{2\pi i})} = e^{\frac{\theta}{2\pi}2ki\pi}$.

Now, we want $e^{i\theta} = e^{\frac{\theta}{2\pi}2ki\pi}$, so it must be that we have chosen:

$$\theta = k\theta + 2j\pi$$

For some $k, j \in \mathbb{Z}$. So, we have fixed our branch to make this assumption.

The second half of the error is that we also want our branch of $z^{\frac{\theta}{2\pi}}$ to satisfy that $1^{\frac{\theta}{2\pi}} = 1$.

This amounts to $e^{\frac{\theta}{2\pi}\log(1)} = 1$. We already chose earlier that $\log(1) = 2k\pi$, so:

$$\frac{\theta}{2\pi}2k\pi = 2m\pi$$

This tells us that $k\theta = 2m\pi$. But we also know that $\theta = k\theta + 2j\pi$. So $\theta = 2(m+j)\pi$. This contradicts our assumption that $\theta \neq k\pi$ for any $k \in \mathbb{Z}$.

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