MAT334 - Week 7 Problems

Textbook Problems 2.3: 2,3,4

Additional Problems

1. Which of the following functions have roots inside $|z| = 2$? (Remember that a root of f is a $z \in \mathbb{C}$ with $f(z) = 0$.) If so, what are they?

   a) $z^3 - \frac{1}{2}$
   b) $\sin(4z)$
   c) $z^3 - iz^2 - z + i$
   d) $e^{4z} - 1$
   e) $z^5 - 16$

2. Let γ_1 be the circle $|z| = 2$ travelled once counterclockwise, and γ_2 be the circle of radius 1 centered at 1. For which functions in question 1 is it possible to deform γ_1 into γ_2 without crossing any roots of the function?

3. Let γ be the circle $|z| = 1$ travelled once counterclockwise. Calculate the following integrals using the Cauchy Integral Formula.

   a) $\int_\gamma \frac{1}{z^2} dz$
   b) $\int_\gamma \frac{1}{z^2 - \frac{1}{4}} dz$
   c) $\int_\gamma \frac{\cos(z)e^z}{z - \frac{1}{2}} dz$
   d) $\int_\gamma \frac{\mathrm{Log}(z+2)}{z} dz$
   e) $\int_\gamma \frac{1}{z^2 - \frac{5}{2}z + 1} dz$
   f) $\int_\gamma \frac{1}{(z^2 - \frac{5}{2}z + 1)^n} dz$ for any $n \in \mathbb{N}$
   g) $\int_\gamma \frac{\cos(\sin(\cos(z)))}{z^2} dz$

4. So far we've seen how to handle simple closed curves travelled in positive orientation. What do we do if the curve isn't simple, or the orientation is negative?

   a) Suppose γ can be broken up into n simple closed curves $\gamma_1, ...\gamma_n$. How does $\int_\gamma f(z)dz$ relate to the integrals $\int_{\gamma_i} f(z)dz$? (No proof required, but it might help to draw a picture.)
   b) Suppose γ travels a simple closed curve n times. (Meaning that there is some simple closed curve Γ such that γ traces out Γ exactly -n times, all in the same direction.)

      Prove that $\int_\gamma f(z)dz = n \int_\Gamma f(z)dz$
   c) Suppose γ is negatively oriented. Justify why $-\gamma$ is positively oriented. Give a strategy for integrating over negatively oriented curves.

5. Suppose Γ is a simple, closed curve. Suppose γ travels Γ positively n_1 times, then negatively n_2 times, then positively n_3 times, and so on. For k even, γ travels Γ positively n_k times, and for each odd k it travels Γ negatively n_k times. Prove that:

$$\int_\gamma f(z)dz = \left(\sum_{k \text{ even}} n_k - \sum_{k \text{ odd}} n_k \right) \int_\Gamma f(z)dz$$

6. Let γ be the curve defined by travelling $|z| = 1$ twice clockwise, then once counterclockwise, then once clockwise again.

   Compute each integral from question 3 over this new curve.

7. Compute each of the following integrals, using any method you like.

   a) $\int_{|z|=1} \cot(z)dz$
   b) $\int_\gamma \frac{1}{z^2-4z}dz$ over the triangle with vertices $1, 1-i$, and $2+4i$
   c) $\int_\gamma \frac{1}{z^2-4z}dz$ over the triangle with vertices $-1, 1-i, 2+4i$
   d) $\int_{|z-1|=3} \frac{\cos(z)}{2z^5}dz$
   e) $\int_{|z-1|=\frac{1}{2}} \frac{1}{\sin(z)(z-1)^2}dz$
   f) $\int_{|z|=2} \frac{e^z-\sin(z)}{z^2-8z+15}dz$
   g) $\int_{|z|=2} \frac{1}{z^3-1}dz$ over the curve γ
   h) $\int_{|z-1|=1} \frac{\sin(z)e^z}{z^4-1}dz$
   i) $\int_\gamma \frac{\mathrm{Log}(z+i)}{(z^4+1)^2}dz$ over the square with vertices $-1, 1, 1+2i, -1+2i$

8. As a preview of things to come, let's see how we can use complex integration to calculate a real integral. We are going to calculate:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$$

   This is an improper integral, which is improper in two places: at $\pm\infty$. So let's recall what this integral is:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \lim_{R\to\infty}\int_{-R}^{0} \frac{1}{x^4+1}dx + \lim_{S\to\infty}\int_{0}^{S} \frac{1}{x^4+1}dx$$

   a) Let's start by getting this down to one limit. To begin, show that $\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$ exists, by comparing it to $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$. This shows that we can actually calculate the integral as:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^4+1}dx$$

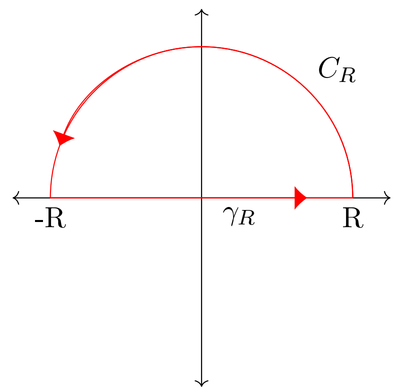
      Consider this step optional. It is a good idea to review your first year material on the convergence of improper integrals. We're going to need it in a bit.

   b) Let $R > 1$. Now, we can view this as the integral over $\gamma_R(t) = t$ for $t \in [-R, R]$ of:

$$\int_{\gamma_R} \frac{1}{z^4+1}dz$$

      Now, this doesn't help us much. We have a bunch of techniques for integrating closed curves, but this isn't closed. So let's define a related closed curve.

      Let C_R be the upper semicircle from R to $-R$, so that $\gamma_R + C_R$ is now a closed curve. So our curve is:



Find $\int_{\gamma_R + C_R} \frac{1}{z^4+1} dz$.

c) Now, this integral has two components, $\int_{\gamma_R} \frac{1}{z^4+1} dz$, which is the integral we care about, and $\int_{C_R} \frac{1}{z^4+1} dz$. We'd like to get rid of this second integral. Let's see what happens as we let $R \to \infty$.

   To do this, let's try to estimate this curve. We do this in stages.

   i) Show that on the curve $|z| = R$, that $|z^4 + 1| \geq R^4 - 1$. Use this to show that $\left|\frac{1}{z^4+1}\right| \leq \frac{1}{R^4-1}$.

   ii) Find the length of C_R.

   iii) Use our estimation of curves to show that $|\int_{C_R} \frac{1}{z^4+1} dz| \leq \frac{\pi R}{R^4-1}$.

d) Prove that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$.

e) Now, we know $\int_{\gamma_R + C_r} \frac{1}{z^4+1} dz$ is constant for $R > 1$, so $\lim_{R\to\infty} \int_{\gamma_R + C_r} \frac{1}{z^4+1} dz$ exists.

   We also know that $\lim_{R\to 0} \int_{C_R} \frac{1}{z^4+1} dz = 0$.

   Use these two facts to find $\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^4+1} dz$.

f) Show that $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$.

Note: you may be tempted to try to find an antiderivative for this by hand. That's really not easy. It is possible, but it involves partial fractions and factoring $x^4 + 1$ (it factors into a product of two quadratics). It's more work that what we just did.