Test 1 *SOLUTIONS*

MAT344 - Spring 2019

Prof. Alex Rennet

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

Instructions

There are **6 questions** on this test, some with multiple parts.

There are **22 points** available.

This test has **8 pages**, including this one.

No aids allowed. (i.e. no calculators, cheat sheets, devices etc.)

TUTORIAL SECTION (Leave blank if you can't remember)

WEDNESDAY	
3pm	☐ TUT101 - Arash
5pm	☐ TUT102 - Osaid
6pm	☐ TUT103 - Osaid

FOR MARKING (Leave This Blank)

/4	/3	/3	/3	/3	/6	/22

1 **(2 points each ⇒ 4 points total)**

Put your answers in this question in terms of terminology or notation that we've used in this course, like $\binom{n}{k}$ or $S(n, k)$, etc.
You do not need to justify your answers in this question.

In a science experiment gone wrong, a portal is created which connects our world to an unlimited number of *alternate realities*.

1.1 The scientists (in our world) that created the portal have already identified 5 alternate realities of special interest to investigate. You and nineteen others have been chosen to go on a mission to explore these realities.
How many ways can these 20 people be divided up amongst the 5 realities so that each world has at least one person going?

1.2 Suppose you end up on a team with 5 other people on it. In the alternate reality you visit, you each have a **duplicate**: someone who for all purposes is *identical* to you. After you and your teammates find your duplicates, all of you end up at a ceremony together to meet the Queen; all of you, along with 50 other people from their reality, will wait in line to shake hands with the Queen.
How many ways are there to form this line?

Solution

1.1 $5!S(20, 5)$ *(distinguishable into distinguishable)*

1.2 $\binom{62}{2,2,2,2,2,2,1,\ldots,1}$ *(ordering distinguishable objects, some with repeats.)*

For the remainder of the test, you must justify your answers.

2 **(3 points)**

For integers $1 \leqslant n \leqslant m$, how many functions $f : [n] \to [m]$ are there for which $f(1) < f(2) < \ldots < f(n)$?

SOLUTION

$\binom{m}{n}$

This is because once we pick n places from the m available to send the elements of $[n]$, the ordering in which they are mapped is completely determined by the restriction.

There are other, *much more complicated* ways to get the same answer (possibly written in a more complicated form). These, if correct, would get full credit.

3 **(3 points)** What is the coefficient on x^{2019} in the power series form of $f(x) = \sqrt[5]{2-x}$? *(Simplify your answer here as much as you can.)*

SOLUTION

First, we factor out a 2 from the inside of the function, and apply the Generalized Binomial Theorem:

$$\sqrt[5]{2-x} = 2^{1/5}(1+(-x/2))^{1/5} = 2^{1/5}\sum_{k\geqslant 0}\binom{1/5}{k}(-1/2)^k x^k$$

Then, we simplify the binomial coefficient:

$$\binom{1/5}{k} = \frac{\frac{1}{5}\cdot\frac{-4}{5}\cdot\frac{-9}{5}\cdot\cdots\cdot\frac{1-5(k-1)}{5}}{k!}$$

$$= \frac{(-1)^{k-1}1\cdot 4\cdot 9\cdot\cdots\cdot(5k-6)}{5^k k!}$$

(You could go straight from here to an expression for the coefficient (requiring a bit of simplification), but just for the record, I'll complete the expression for the power series, then *extract the coefficient.)* Substituting this in, and simplifying (pulling out the $k=0$ and $k=1$ terms, since $5k-6<0$ for these) we have:

$$\sqrt[5]{2}\left[1-(1/5)(1/2)x+\sum_{k\geqslant 2}\frac{(-1)^{k-1}1\cdot 4\cdot 9\cdot\cdots\cdot(5k-6)}{5^k k!}(-1/2)^k x^k\right]$$

$$= \sqrt[5]{2}\left[1-(1/5)(1/2)x+\sum_{k\geqslant 2}\frac{(-1)1\cdot 4\cdot 9\cdot\cdots\cdot(5k-6)}{2^k 5^k k!}x^k\right]$$

So the coefficient on x^{2019} is

$$-\sqrt[5]{2} \cdot \frac{1 \cdot 4 \cdot 9 \cdot \cdots \cdot (5(2019)-6)}{2^{2019} 5^{2019} 2019!}.$$

4 **(3 points)**

How many arrangements are there of the letters in the word

A R I T H M O P H E L I A

so that the arrangements have the M *immediately beside* at least one A (i.e. either as ...MA... or as ...AM... or as ...AMA...)?

SOLUTION

We can count the "AM" cases, add them to the "MA" cases, then subtract the overlapping "AMA" cases (each of the first two cases count the latter, so they will get double-counted). The AM cases can be counted by simplying gluing an A and the M together into a new symbol, giving 11 symbols to permute:

- AM cases: $\binom{12}{2,2,1,\ldots,1}$
- MA cases: same.
- AMA cases: $\binom{11}{2,2,1\ldots,1}$.
- Final answer: $2 \cdot \frac{12!}{2!2!} - \frac{11!}{2!2!}$

5 **(3 points)**

Prove the following identity using the **Binomial Theorem**:

$$\sum_{0 \leqslant k \text{ even}}^{n} \binom{n}{k} 2^k = \frac{3^n + (-1)^n}{2}$$

SOLUTION

This was taken word-for-word from the second Tutorial Worksheet (a solution was done in tutorial). Here is a write-up of it:

Use the Binomial Theorem with $x = 2, y = 1$, to get $3^n = \sum_k^n \binom{n}{k} 2^k \quad (*)$.

Use it again with $x = 2, y = -1$ to get $(-1)^n = \sum_k^n \binom{n}{k} 2^k (-1)^{n-k} \quad (**)$.

Computing $\frac{(*)+(**)}{2}$, we get:

$$\begin{aligned} \frac{3^n + (-1)^n}{2} &= \frac{1}{2}\left[\sum_k^n \binom{n}{k} 2^k + \sum_k^n \binom{n}{k} 2^k (-1)^{n-k}\right] \\ &= \frac{1}{2}\left[2 \sum_{k \text{ even}}^n \binom{n}{k} 2^k + 0 \cdot \sum_{k \text{ odd}}^n \binom{n}{k} 2^k (-1)^{n-k}\right] \\ &= \sum_{k \text{ even}}^n \binom{n}{k} 2^k, \quad \text{as desired.} \end{aligned}$$

6 **(3 points each ⇒ 6 points total)**

Prove each of the equations below using a **combinatorial proof**.

*(Recall that in a **combinatorial proof** you must explain in words why the two sides of the equation count the same thing in two different ways.)*

You will receive 0 points for an algebraic argument (for instance, don't rewrite $\binom{n}{k}$ in terms of $n!$ and $k!$ etc), or for attempting to use induction or a proof by contradiction, applying a theorem, etc.

6.1 **(3 points)**

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Hint: rewrite n *as* $\binom{n}{1}$.

6.2 **(3 points)**

$$S(n+1, k+1) = \sum_{i=k}^{n} \binom{n}{i} S(i, k)$$

SOLUTION

6.1 The LHS is the number of ways of picking a two-person committee from $2n$ people.

Rerwiting the RHS, we have $\binom{n}{2} + \binom{n}{2} + \binom{n}{1}\binom{n}{1}$. This exposes the structure of the cases on this side: we are still picking 2 from $2n$, but in three cases. The first case is where both choices come from the first n people, the second case is where the pair come from people $n+1$ to $2n$; lastly there is the case where 1 person comes from the first half and 1 person from the second half.

6.2 The LHS is the distribution of $n+1$ distinguishable books into $k+1$ identical boxes.

For the RHS, we choose i books from the first n books to put in k of the boxes (for i ranging from k to n). This ensures even with i minimal - when $i = k$ - that k of the $k+1$ boxes are non-empty. We then have to deal with (i) the remaining $n-i$ books from the first n, and (ii) the $n+1$-st book. We put these into the remaining (and currently empty) box.

(You might worry that the $n+1$-st book is always getting put in a particular box - the "$k+1$-st" box. But remember, the boxes are indistinguishable, so there is no "$k+1$-st" box; all that matters is what ends up with what. Also note that the $n+1$-st book can end up, as i varies, with any combination of the other books, since the remaining $n-i$ are grouped with it, and the choice of i is arbitrary.)