MAT334 Midterm 1 Review Sheet

0 Disclaimer

Any of the computation tricks you have learned in first or second year calculus are fair game. You should know how to integrate and differentiate. I will not ask anything really obscure, but if I've mentioned it in lecture or asked homework questions about it, then it's not obscure. This is a calculus course, and you need to be able to do single variable calculus to succeed.

1 Complex Algebra

You will need to be able to perform complex algebra fluently, including knowing when working in $x + iy$ form is appropriate vs. knowing when working in polar form is appropriate. You will need to be able to find nth roots of a complex number, and solve simple polynomial equations, including using the quadratic formula. You should know how to work with the modulus $|z|$ and the real and imaginary parts of z, and how they relate. You should be able to prove simple facts about complex numbers.

You should know the two different ways of writing polar form, and when it is appropriate to use each.

Question 1.1: Compute the following complex numbers

i) $\frac{(6-2i)^2}{1-3i} + 7i(3-4i)$

ii) $[(2+3i)(3+2i)]^{12}$

iii) $(\sqrt{3}-i)^{-11}$

iv) $[(18+2i)(9-i)]^{400}$ (There is a very simple way to do this that doesn't involve polar form or actually multiplying this out.)

Solutions:

i) $(6-2i)^2 = 32 - 24i$, so $\frac{(6-2i)^2}{1-3i} = \frac{(32-24i)(1+3i)}{10} = \frac{1}{5}(16-12i)(1+3i) = \frac{1}{5}(52+36i)$
So $\frac{(6-2i)^2}{1-3i} + 7i(3-4i) = \frac{52+36i}{5} + (21i + 28) = \frac{192}{5} + i\frac{141}{5}$.

ii) $(2+3i)(3+2i) = 6 + 6i^2 + 13i = 13i$. So $[(2+3i)(3+2i)]^{12} = 13^{12}i^{12} = 13^{12}$.

iii) $\sqrt{3} - i = 2e^{-\frac{i\pi}{6}}$, so $(\sqrt{3}-i)^{-11} = 2^{-11}e^{\frac{11\pi}{6}} = 2^{-11}e^{-\frac{\pi}{6}} = 2^{-12}(2e^{-\frac{\pi}{6}}) = 2^{-12}(\sqrt{3}-i)$

iv) Notice that $\overline{(9-i)} = 9 + i$, so $[(18+2i)(9-i)]^{400} = [2|9-i|^2]^{400} = 2^{400}(82)^{400} = 164^{400}$.

Question 1.2: Solve each of the following equations.

i) $z^5 = 1$

ii) $z^2 - 2iz = -1 + i$

iii) $z^7 - 2z^5 + iz^3 = 0$

iv) $z^8 - 2z^4 + 1 = 0$

Solutions:

i) $1 = 1e^{i0}$, so $z^5 = 1$ has the roots:

$$z = 1e^{i(\frac{2k\pi}{5})}$$

Where $k = 0, 1, 2, 3, 4$.

ii) $z^2 - 2iz = -1 + i$ rearranges to give:

$$z^2 - 2iz + 1 - i = 0$$

The quadratic formula gives:

$$z = \frac{2i + ((-2i)^2 - 4(1-i))^{\frac{1}{2}}}{2} = \frac{2i + (-8+4i)^{\frac{1}{2}}}{2} = i + (i-2)^{\frac{1}{2}}$$

We won't be able to get exact expressions for those roots. But, we can write it in a more exact form: $i - 2 = \sqrt{5}e^{\arctan(\frac{-1}{2}+\pi)}$, so $(i-2)^{\frac{1}{2}} = \pm 5^{\frac{1}{4}} e^{\frac{1}{2}(\arctan(\frac{-1}{2})+\pi)}$.

iii) Notice that we can do some factoring:

$$z^7 - 2z^5 + iz^3 = z^3(z^4 - 2z^2 + i) = 0$$

So, either $z = 0$ or $z^4 - 2z^2 + i = 0$. Now, this second equation is a quadratic equation in the variable z^2: if $w = z^2$ then $z^4 - 2z^2 + i = w^2 - 2w + i = 0$.

So, the quadratic formula tells us that:

$$w = \frac{2 + (4-4i)^{\frac{1}{2}}}{2}$$

Now, $4 - 4i = 2^{\frac{5}{2}} e^{-\frac{\pi}{4}}$, so:

$$w = 1 \pm 2^{\frac{1}{4}} e^{-\frac{\pi}{8}}$$

Then $z = w^{\frac{1}{2}} = (1 \pm 2^{\frac{1}{4}} e^{-\frac{\pi}{8}})^{\frac{1}{2}}$.

iv) Much like the last question, notice that $z^8 - 2z^4 + 1 = 0$ is a quadratic equation in z^4. Actually, it factors as $(z^4 - 1)^2 = 0$. So, this reduces to solving $z^4 = 1$. We know that the solutions to this are $e^{i0}, e^{\frac{i\pi}{2}}, e^{i\pi}, e^{\frac{i3\pi}{2}}$, which are $1, i, -1, -i$ respectively.

Question 1.3:

i) Prove that if $z = x + iy$, then $|z| - |x| \leq |y|$.

ii) Prove that $|Re^{i\theta}| = R$.

iii) Prove that $z\overline{z} = |z|^2$.

Solution:

i) Remember that $|z| \leq |x| + |y|$ by the triangle inequality. Subtracting $|x|$ from each side gives the desired inequality.

ii) $|Re^{i\theta}| = |R\cos\theta + iR\sin\theta| = \sqrt{R^2\cos^2\theta + R^2\sin^2\theta} = \sqrt{R^2} = R$

iii) We covered this in lecture.

2 Functions and Limits

You should know to find the domain of a function, the range of a function, and to see how some easy functions transform some easy shapes in the plane. You should be familiar with e^z and all of the functions based on it: $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, etc.

You should be able to take limits, including all the tricks from 135. When you think a limit doesn't exist, you should have some idea of which directions to approach from to get different values for the limit.

Question 2.1: Find the domains of the following functions

i) $z^5 - 3z + 8$

ii) $\frac{z^5-3z+8}{z^8-2z^4+1}$

iii) $\frac{z^8-2z^4+1}{z^8-2z^4+1}$ (Not, it's not $\mathbb{C}$.)

iv) $\frac{1}{\sin z}$

v) $e^{\frac{1}{z}} \cos\left(\frac{2z}{z^2-3}\right)$

Solution:

i) Polynomials are defined on all of $\mathbb{C}$.

ii) A rational function is defined everywhere except where the denominator is 0. We saw in question 1.2d) that $z^8 - 2z^4 + 1 = 0$ if and only if $z = \pm 1$ or $\pm i$.

So the domain of this function is $\mathbb{C} \setminus \{1, i, -1, -i\}$.

iii) This expression simplifies to 1 whenever it is defined. But it is defined only when the denominator is non-zero. So, for the same reason as the last question, the domain of this function is $\mathbb{C} \setminus \{1, i, -1, -i\}$.

iv) $\frac{1}{\sin(z)}$ is defined whenever $\sin(z) \neq 0$. We've seen as an exercise that $\sin(z) = 0$ if and only if $z = k\pi$ for $k \in \mathbb{Z}$. So the domain is $\mathbb{C} \setminus \{k\pi | k \in \mathbb{Z}\}$.

v) Let's look at this piece by piece. First, notice that e^w exists for all w, so $e^{\frac{1}{z}}$ exists if and only if $\frac{1}{z}$ exists. So, we have that $z \neq 0$.

Also, $\cos(w)$ exists for all w, so $\cos(\frac{2z}{z^2-3})$ exists if and only if $z^2 - 3 \neq 0$, so if $z \neq \pm\sqrt{3}$.

So the domain of this function is $\mathbb{C} \setminus \{0, \sqrt{3}i, -\sqrt{3}\}$.

Question 2.2: Find the ranges of the following functions:

i) $2z^2 - z + \frac{1}{z}$

ii) e^{2z+1}

iii) $e^{\frac{1}{z}}$

Solution:

i) The range of $f(z) = 2z^2 - z + \frac{1}{z}$ is the set of all $w \in \mathbb{C}$ such that $f(z) = w$ for some $z \in \mathbb{C}$.

So, to see which w can occur, we set $f(z) = w$ and solve for z.

Well, $f(z) = w$ tells us that:

$$2z^2 - z + \frac{1}{z} = w$$

Multiplying both sides by z (and now remembering that $z \neq 0$), we get the cubic equation:

$$2z^3 - z^2 - wz + 1 = 0$$

This is a polynomial equation, which has a root. (Remember, there is a formula for solving any cubic equation. We haven't talked about what this formula is, but I have mentioned it exists.)

So, there exists some z so that $2z^3 - z^2 - wz + 1 = 0$. If this z is non-zero, then $f(z) = w$. But notice that $2(0)^3 - 0^2 - w(0) + 1 = 1 \neq 0$, so 0 is never a root of this polynomial. And so $f(z) = w$ has a solution for all w.

So the range is $\mathbb{C}$.

ii) $g(z) = e^z$ has range $\mathbb{C} \setminus \{0\}$. Well, if $f(z) = e^{2z+1}$, then $f(z) = w$ if and only if $g(2z + 1) = w$, which occurs if and only if $w \neq 0$. So the range of e^{2z+1} is the same as the range for e^z: $\mathbb{C} \setminus \{0\}$.

iii) Again, we know that $e^{\frac{1}{z}} \neq 0$. For $w \neq 0$, $e^{\frac{1}{z}} = w$ if and only if:

$$z = \frac{1}{\log(w)}$$

For some logarithm of w. Now, $\log(w) = 0$ only occurs for $w = 1$. But we also have another logarithm for $w = 1$, namely $2\pi i$.

So, for any $w \neq 0$, there exists a logarithm $\log(w)$ which is non-zero. So $z = \frac{1}{\log(w)}$ can be chosen to exist for any $w \neq 0$, and so the range of $e^{\frac{1}{z}}$ is $\mathbb{C} \setminus \{0\}$.

Question 2.3: Compute the following limits:

i) $\lim_{z\to 2} z^2 - \sin \pi z$

ii) $\lim_{z\to i} \frac{z^4-1}{z^2+1}$

iii) $\lim_{z\to 0} \text{Arg}(z+1)$

iv) $\lim_{z\to i} z^{1/2}$ for the principal branch

v) $\lim_{z\to 0} e^{i\text{Arg}(z)}$

vi) $\lim_{z\to 0} \frac{x^2-y^2}{x+y}$

vii) $\lim_{z\to 0} \text{Re}(z^2-2)$

viii) $\lim_{z\to 2} \frac{z+\bar{z}}{z-\bar{z}}$

Solution:

i) This function is continuous, so the limit is $4^2 - \sin(2\pi) = 16$.

ii) $\lim_{z\to i} \frac{z^4-1}{z^2+1} = \lim_{z\to i} \frac{(z^2+1)(z^2-1)}{z^2+1} = \lim_{z\to i} z^2 - 1 = -2$

iii) We know that $\text{Arg}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$, so $\lim_{z\to 0} \text{Arg}(z+1) = \lim_{w\to 1} \text{Arg} w = \text{Arg} 1 = 0$

iv) The principal branch of $\text{Log} z$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$, and so the principal branch of z^a is also continuous on that set for any a. Therefore $\lim_{z\to i} z^{\frac{1}{2}} = i^{\frac{1}{2}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

v) Approach 0 from along the positive real axis. Along that ray, $\text{Arg}(z) = 0$, and so $e^{i\text{Arg} z} = e^0 = 1$. So $e^{i\text{Arg} z}$ approaches 1 along this ray.

   Approaching along the positive imaginary axis, we have $\text{Arg}(z) = \frac{\pi}{2}$, and so $e^{i\text{Arg}(z)} = e^{\frac{i\pi}{2}} = i$. So $e^{i\text{Arg}(z)}$ approaches i along this ray.

   Since $e^{i\text{Arg}(z)}$ approaches two different values, the limit DNE.

vi) $\frac{x^2-y^2}{x+y} = x - y$ except for when $x + y = 0$. So, approach along the line $z = x - ix$. Along this line, $x + y = x + (-x) = 0$, and so the function is not defined along this line through 0.

   But for the limit to exist, the function needs to be defined in an open ball around the point (except maybe at that point). Since the function isn't defined on that line, the limit DNE.

vii) $\text{Re}(z)$ and $z^2 - 2$ are continuous on $\mathbb{C}$, so the limit is $\text{Re}(0^2 - 2) = -2$.

viii) The numerator goes to 4. The denominator goes to 0. So the limit DNE.

3 Logarithms and Branches

You should know what $\log z = w$ means. You should know why log is not a function on $\mathbb{C}$ and be able to talk about taking a branch of the logarithm. Be able to find $\log_1 z$ for any branch of the logarithm.

You should know what a^z means, and why it has multiple branches. Know how it is related to $\log a$. Be able to find all the values of the expression a^z.

You should know what arcsin and arccos are in terms of logarithms.

Question 3.1: Find all values of the following expressions. Identify any values which are real.

i) $(1+i)^{2i}$

ii) $\left(\frac{1}{i}\right)^{2i}$

iii) $[(4+i)^2]^{1/2}$

iv) $\mathrm{Log}(\mathrm{Log}(i))$

v) $\arccos(4)$

Solution:

i) $(1+i)^{2i} = e^{2i\log(1+i)} = e^{2i\left(\ln(\sqrt{2})+i\frac{\pi}{4}+i2k\pi\right)} = e^{-\frac{\pi}{2}-4k\pi}e^{2i\ln\sqrt{2}}$

ii) $\left(\frac{1}{i}\right)^{2i} = (-i)^{2i} = e^{2i\log(-i)} = e^{2i\left(i\frac{3\pi}{2}+2ki\pi\right)} = e^{-3\pi-4k\pi}$

iii) $[(4+i)^2]^{\frac{1}{2}} = \pm 4 + i$

iv) $\mathrm{Log}(i) = i\frac{\pi}{2}$, so $\mathrm{Log}(\mathrm{Log}(i)) = \ln\left(\frac{\pi}{2}\right) + i\frac{\pi}{2}$

v) Recall that $\arccos(z) = -i\log(z + (z^2-1)^{\frac{1}{2}})$, so:

$$\arccos(4) = -i\log(4 \pm \sqrt{15})$$

Now, we know that $\sqrt{15} < \sqrt{16} = 4$, so $4 \pm \sqrt{15} > 0$ and so its argument is $2k\pi$. Therefore:

$$\arccos(4) = -i(\ln(4 \pm \sqrt{15}) + i2k\pi) = 2k\pi - i(\ln(4 \pm \sqrt{15}))$$

Question 3.2: Find a branch of the logarithm $\log_1 z$ making the following equations true, if possible.

i) $\log_1(i(1+i)) = \log_1(i) + \log_1(1+i)$

ii) $\log_1(4i) = \ln(4) + i\frac{19\pi}{2}$

iii) $\log_1(4i) = \ln(4) + i\frac{17\pi}{2}$

iv) $\log_1((1-i)(2+2i)) = \log_1(1-i) + \log_1(2+2i)$

v) $\log_1(z) = \mathrm{Log} z + 3i\pi$

vi) $\log_1(z) = \mathrm{Log} z + 4i\pi$

Solution:

i) Well, $i(1+i) = i-1$. Since $\mathrm{Arg}(i(1+i)) = \frac{3\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4} = \mathrm{Arg}(i) + \mathrm{Arg}(1+i)$, we know that $\mathrm{Log}(i(i+1)) = \mathrm{Log}(i) + \mathrm{Log}(1+i)$. So the appropriate branch here is the principal branch.

ii) We know that $4i = 4e^{i\frac{\pi}{2}}$, and so $\log(4i) = \ln(4) + i\left(\frac{\pi}{2} + 2k\pi\right)$.

So, if such a branch exists, we would need $\frac{19\pi}{2} = \frac{\pi}{2} + 2k\pi$. But $\frac{19\pi}{2} = \frac{\pi}{2} + 9\pi \neq \frac{\pi}{2} + 2k\pi$. So no such branch exists.

iii) Here, we know that $\frac{17\pi}{2} = \frac{\pi}{2} + 8\pi$, which means that such a branch exists.

To specify what the branch is, we need to give a range for $\arg(z)$ such that $\arg_1(4i) = \frac{17\pi}{2}$. We can choose $\arg_1(z) \in (8\pi, 10\pi)$. Then $8\pi < \frac{17\pi}{2} < 10\pi$, and so $\arg_1(4i) = \frac{17\pi}{2}$ as desired.

iv) We know that $\mathrm{Arg}(1-i) = -\frac{\pi}{4}$, and $\mathrm{Arg}(2+2i) = \frac{\pi}{4}$. Also, $(1-i)(2+2i) = 4$. So:

$$\mathrm{Arg}((1-i)(2+2i)) = \mathrm{Arg}(4) = 0 = \mathrm{Arg}(1-i) + \mathrm{Arg}(2+2i)$$

So the principal branch works perfectly fine here.

v) We saw that for any two logarithms $\log_1(z)$ and $\log_2(z)$, that $\log_1(z) - \log_2(z) = 2ki\pi$. However, here we would have $\log_1(z) - \mathrm{Log}(z) = 3\pi i \neq 2ki\pi$. So this is not possible.

vi) Since $\log_1(z) - \mathrm{Log}(z) = 4i\pi$, this is possible.

We need to be a bit careful. First, the domain of $\log_1(z)$ should be the domain of $\mathrm{Log}(z)$, namely $\mathbb{C} \setminus (-\infty, 0]$, so our branch cut needs to occur at $n\pi$ for n an odd integer.

Also, we need that $\arg_1(z) = \mathrm{Arg}(z) + 4\pi$. This tells us pretty strongly that we need $\arg_1(z) \in (3\pi, 5\pi)$. So we choose that to be our branch.

Question 3.3: Prove the following statements:

i) If $n \in \mathbb{Z}$, then z^n is single-valued.

ii) $z^0 = 1$ for any $z \in \mathbb{C}$, $z \neq 0$.

iii) Let $\arg_1 z$ be the branch taking values $\arg_1 z \in (\theta, \theta + 2\pi)$. Show that if $a = x + iy$ and $y \in (\theta, \theta + 2\pi)$, then $(e^a)^b = e^{ab}$ for any $b \in \mathbb{C}$.

Solution:

i) Suppose $z = re^{i\theta}$. Then, $z^n = e^{n\log(z)} = e^{n(\ln r + i\theta + i2k\pi)} = e^{n\ln r + in\theta}e^{2nki\pi}$

Now, since $n \in \mathbb{Z}$, we also have that $nk \in \mathbb{Z}$. So $e^{2nki\pi} = 1$, and so $z^n = e^{n\ln r + in\theta}$ is a single value.

ii) $z^0 = e^{0\log(z)} = e^0 = 1$

iii) $(e^a)^b = e^{b\log_1(e^a)} = e^{b\log_1(e^x e^{iy})}$

Now, we know that $y \in (\theta, \theta + 2\pi$, which is the range of angles given by $\arg_1(z)$. In particular, this means that $\arg_1(e^{iy}) = y$. Therefore:

$$(e^a)^b = e^{b(\ln(e^x) + i\arg_1(e^x e^{iy}))} = e^{b(x+iy)} = e^{ba} = e^{ab}$$

4 Differentiation and Analyticity

You should know the limit definition of a derivative. You should know the Cauchy-Riemann equations and what information they give youb. You should know what it means to be analytic on a domain, and how we check for that using Cauchy-Riemann. You should know that we never use the limit definition when Cauchy-Riemann suffices. Know what entire functions are, and which of our common functions are entire.

You should know all of the common derivatives: polynomials, rational functions, power z^a and exponentials a^z, logarithms, trig and hyperbolic trig functions. You should know which differentiation rules are valid, and what they say.

You should know what it means for a real function of two variables to be harmonic, and be able to find harmonic conjugates. You should know when two harmonic functions u, v give an analytic $f = u + iv$.

Question 4.1: Use Cauchy-Riemann to figure out where the following functions might be differentiable:

i) $f(z) = z + \overline{z}$

ii) $f(z) = |z|$

iii) $f(z) = \sin(x) - i\sin(y)$

iv) $f(z) = x^2 + i(y - x)$

v) $f(z) = (x - y)^2 + i2x^2y$

vi) $f(z) = x + (-1)^x y$

vii) $f(z) = z^z$

The key idea here is that if f is differentiable at z_0, then f satisfies C-R at z_0. So if f doesn't satisfy C-R, then f cannot be differentiable at that point.

Solution:

i) $f(z) = 2x$. So $u_x = 2$, and $v_y = 0$. Since $2 \neq 0$, f is not differentiable anywhere.

ii) $f(z)$ is real, so $u(x, y) = |z|$ and $v(x, y) = 0$. Then $v_x = v_y = 0$. If f satisfied Cauchy-Riemann, we would have that $u_x = u_y = 0$, and so $|z|$ would be constant. Since that is not true, $f(z)$ is not differentiable anywhere.

iii) $u_x = \cos x$ and $v_y = -\cos y$. So we need $\cos x = -\cos y$. This occurs only when $y = (2k + 1)\pi - x$. (The idea is the trig identity $\cos(\pi - x) = -\cos x$, so $\cos((2k + 1)\pi - x) = \cos(\pi - x) = -\cos x$.)

   So these are the only points where this function might be differentiable. (I didn't want you to actually check these points. The point was to get practice with C-R.)

iv) $u_y = 0$ and $v_x = -1$. Since $0 \neq 1$, C-R doesn't hold anywhere and f isn't differentiable anywhere.

v) $u_x = 2x - 2y$ and $v_y = 2x^2$. Also, $u_y = 2y - 2x$ and $v_x = 4xy$. So, we need to have:

$$2x - 2y = 2x^2$$
$$2y - 2x = -4xy$$

This gives us that $2x^2 = 4xy$. Two situations arise: if $x = 0$, then this is true. But then C-R would give $y = 0$. So $z = 0$ is one possible place for f to be differentiable.

And if $x \neq 0$, then $x = 2y$. Plugging this back into C-R, we get:

$$x = 2x^2$$

This reduces to $x = \frac{1}{2}$ and $y = \frac{1}{4}$. Plugging this into C-R:

$$u_x = 1 - \frac{1}{2} = \frac{1}{2} = 2\frac{1}{4} = 2\frac{1}{2}^2 = v_y$$
$$u_y = -\frac{1}{2} = -4\frac{1}{4}\frac{1}{2} = -v_x$$

So C-R holds at $z = 0$ and $z = \frac{1}{2} + \frac{i}{4}$. These are the only places f might be differentiable.

vi) This is actually fairly tricky. Remember that $(-1)^x = e^{x \log(-1)} = e^{x(i\pi + 2ki\pi)}$. For ease, we'll take the principal branch so that $(-1)^x = e^{i\pi x} = \cos(\pi x) + i\sin(\pi x)$.

So $f(z) = x + y(\cos(\pi x) + i\sin(\pi x)) = (x + y\cos(\pi x)) + iy\sin(\pi x)$. Then:

$$u_x = 1 - \pi y \sin(\pi x)$$
$$v_y = \sin(\pi x)$$
$$u_y = \cos(\pi x)$$
$$v_x = y\pi\cos(\pi x)$$

Putting this all together, we need:

$$1 = (\pi y + 1)\sin(\pi x)$$
$$0 = (1 + \pi y)\cos(\pi x)$$

Now, we cannot have both of these occuring when $\pi y + 1 = 0$, since then $1 = 0$. So $\cos(\pi x) = 0$, and so $x = \frac{n}{2}$ where n is an odd integer, so $x = \frac{(2k+1)}{2}$ for $k \in \mathbb{Z}$

Going to our first equation, we know that:

$$\sin\left(\frac{(2k+1)\pi}{2}\right) = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$

When k is even, we have $1 = \pi y + 1$, and so $y = 0$. When k is odd, we have $1 = -(\pi y + 1) = -\pi y - 1$, and so $y = -\frac{2}{\pi}$.

So, the points of interest are going to be:

$$z_k = \begin{cases} \frac{(2k+1)}{2}, & k \text{ even} \\ \frac{(2k+1)}{2} - i\frac{2}{\pi}, & k \text{ odd} \end{cases}$$

vii) Well, $z^z = e^{z\text{Log}z}$. Again, we'll take the principal branch to make life easy. Also, we'll restrict ourselves to the first and fourth quadrants.
So, $f(z) = e^{(x+iy)(\ln(x^2+y^2)+i\arctan(\frac{y}{x}))}$. So:

$$f(z) = e^{x\ln(x^2+y^2)-y\arctan(\frac{y}{x})}\left(\cos\left(y\ln(x^2+y^2)+x\arctan\left(\frac{y}{x}\right)\right)+i\sin\left(y\ln(x^2+y^2)+x\arctan\left(\frac{y}{x}\right)\right)\right)$$

Taking partials:

$$\begin{aligned} u_x = &\frac{\partial(x\ln(x^2+y^2)-y\arctan(\frac{y}{x}))}{\partial x}e^{x\ln(x^2+y^2)-y\arctan(\frac{y}{x})}\cos\left(y\ln(x^2+y^2)+x\arctan\left(\frac{y}{x}\right)\right) \\ &- e^{x\ln(x^2+y^2)-y\arctan(\frac{y}{x})}\sin\left(y\ln(x^2+y^2)+x\arctan\left(\frac{y}{x}\right)\right)\frac{\partial\left(y\ln(x^2+y^2)+x\arctan\left(\frac{y}{x}\right)\right)}{\partial x} \end{aligned}$$

At this point, I hope you recognize that this is basically impossible, and that there must be a better way. If something seems way too hard, don't be afraid to ask about it. I never intentionally assign a problem that isn't doable.

Question 4.2: Without using Cauchy-Riemann, determine where the following functions are analytic. Find $f'(z)$.

i) $f(z) = e^{z^2+z}$

ii) $f(z) = \sin(\tan(z))$

iii) $f(z) = \frac{\cos(z)+z^3}{z^5-1}$

iv) $f(z) = z^2\sin(\frac{1}{z})$

v) $f(z) = z^z$

Solution:

i) z^2+z and e^z are entire. So by the chain rule, e^{z^2+z} is entire.

$$f'(z) = (2z+1)e^{z^2+z}$$

ii) $\sin(z)$ is entire, so $\sin(\tan(z))$ is analytic wherever $\tan(z)$ is analytic. Now, we know that $\tan(z) = \frac{\sin(z)}{\cos(z)}$ is a ratio of entire functions. So by the quotient rule, $\tan(z)$ is analytic wherever $\cos(z) \neq 0$.

We know that $\cos(z) = 0$ if and only if $z = \frac{(2k+1)\pi}{2}$, and so $\sin(\tan(z))$ is analytic on the domain $\{z \in \mathbb{C} | z \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}\}$

$$f'(z) = \cos(\tan(z))\sec^2(z)$$

iii) $\cos(z)$, z^3 and $z^5 - 1$ are entire, so by the quotient rule $f(z)$ is analytic whenever $z^5 \neq 1$. We saw in question 1.1 that $z^5 = 1$ if and only if $z = e^{\frac{2ki\pi}{5}}$ for $k = 0, 1, 2, 3, 4$.. So $f(z)$ is analytic on $\mathbb{C} \setminus \{e^{\frac{2k\pi i}{5}} | k = 0, 1, 2, 3, 4\}$.

$$f'(z) = \frac{(3z^2 - \sin(z))(z^5 - 1) - z^4(\cos(z) + z^3)}{(z^5 - 1)^2}$$

iv) z^2 and $\sin(z)$ are entire. Also, $\frac{1}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$. So $f(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ by the chain rule.

$$f'(z) = 2z\sin\left(\frac{1}{z}\right) + z^2\cos\left(\frac{1}{z}\right)\frac{-1}{z^2} = 2z\sin\left(\frac{1}{z}\right) - \cos\left(\frac{1}{z}\right)$$

v) $f(z) = e^{z\log_1(z)}$ for any branch of the logarithm. Now, we know that z and e^z are entire. And $\log_1(z)$ is analytic on its domain. Suppose our branch of $\log_1(z)$ is given by $\arg_1(z) \in (\theta, \theta + 2\pi)$. Then the domain of $\log_1(z)$ is $\mathbb{C} \setminus \{re^{i\theta} | r > 0\}$ (remember that θ is a fixed angle now.) So this is the domain of this branch of z^z.

$$f'(z) = e^{z\log_1(z)}(\log_1(z) + 1) = (\log_1(z) + 1)z^z$$

Question 4.3: Determine which of the following functions are harmonic, and for those that are, find a harmonic conjugate so that $v(1, 1) = 1$

i) $u(x, y) = xy + x^3y - xy^3$

ii) $u(x, y) = x^2 + xy^2$

iii) $u(x, y) = x - y$

iv) $u(x, y) = \ln(x^2 + y^2)$

v) $u(x, y) = \arctan(y/x) - \arctan(x/y)$

vi) $u(x, y) = ax^2 + bxy - ay^2 + cx + dy + e$ for $a, b, c, d, e \in \mathbb{R}$.

Solution:

i) $u_{xx} = 6xy$ and $u_{yy} = -6xy$, so u is harmonic.

The harmonic conjugate is $\frac{y^2-x^2}{2} + \frac{6x^2y^2-x^4-y^4}{4} + C$

Choosing $v(1,1) = 1$ gives $C = 0$.

ii) $u_{xx} = 2$ and $u_{yy} = 2x$, so u is not harmonic except at $x = -1$. This does not give a domain, so we cannot find a harmonic conjugate.

iii) $u_{xx} = 0 = u_{yy}$, so u is harmonic.

The harmonic conjugate is $y + x + C$. Setting $v(1,1) = 1$ gives $C = -1$.

iv)

v) This is quiz 3A and 3B mashed together. This is harmonic, and its conjugate is $-\ln(x^2+y^2) + C$. And setting $v(1,1) = 1$ gives $C = 1 - \ln 2$.

Alternatively, you could note that $\arctan(\frac{1}{t}) = \frac{\pi}{2} - \arctan(t)$, and so:

$$u(x,y) = 2\arctan\frac{y}{x} + \frac{\pi}{2}$$

We've found harmonic conjugates for $\arctan\frac{y}{x}$ before, we get $2\ln(\sqrt{x^2+y^2}) + C = \ln(x^2+y^2) + C$.

vi) $u_{xx} = a$ and $u_{yy} = -a$. So u is harmonic.

Now, $v(x,y) = \int u_x dy = \int 2ax + by + cdy = 2axy + \frac{by^2}{2} + cy + C(x)$. We differentiate to get:

$$v_x = 2ay + C'(x)$$

Also, $v_x = -u_y = 2ay - bx - d$, so $C'(x) = -bx - d$. As such, $C(x) = -\frac{bx^2}{2} - bx + C$. Therefore:

$$v(x,y) = 2axy + b\frac{y^2-x^2}{2} + cy - bx + C$$

Setting $v(1,1) = 1$ gives:

$$2a + c - b + C = 1$$

So $C = 1 - 2a + b - c$.

Question 4.4: Prove the following facts.

i) If $f = u + iv$ is analytic and $u = 2v$, then f is constant.

ii) If f is entire, then $\frac{f(z)}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$.

iii) If f, g are entire and $f(g(z)) = g(f(z)) = z$ for any $z \in \mathbb{C}$, then:

$$g'(w) = \frac{1}{f'(g(w))}$$

iv) If f is analytic on D, then every branch of $a^{f(z)}$ is analytic on D.

v) If f is analytic and $g(z) = |f(z)|$ is analytic, then f is constant.

Solution:

i) Suppose f is analytic and $u = 2v$. Since f is analytic, we know it satisfies C-R:

$$u_x = v_y$$
$$u_y = -v_x$$

But also, since $u = 2v$, we know:

$$u_x = 2v_x$$
$$u_y = 2v_y$$

Using these equations one at a time, we get: $u_x = 2v_x = -2u_y = -4v_y = -4u_x$.

This implies that $u_x = 0$. So then $v_x = u_y = v_y = 0$ as well, and f is constant.

ii) The quotient rule says that $\frac{f(z)}{g(z)}$ is analytic whenever $f(z)$ and $g(z)$ are analytic and $g(z) \neq 0$. Since $z = 0$ only occurs when, well, $z = 0$, the quotient rule tells us that $\frac{f(z)}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$.

iii) Well, we know that $f(g(z)) = z$. Differentiating, we get:

$$f'(g(z))g'(z) = 1$$

This tells us that $f'(g(z)) \neq 0$, and so $g'(z) = \frac{1}{f'(g(z))}$.

(Why did I use w in the question? Well, this is really a question about f^{-1}, and so I used the range side variable w instead of domain side variable z since $g(w)$ maps from the range of f to the domain of f.)

iv) $a^{f(z)} = e^{f(z)\log_1(a)}$ for a branch $\log_1(z)$. But the chain rule immediately tells us this is analytic on D.

v) Suppose $f(z)$ and $g(z) = |f(z)|$ are analytic. Then $g(z)^2 = |f(z)|^2 = f(z)\overline{f(z)}$ is analytic.

Suppose f is non-constant. Then $f \neq 0$ on a domain, and so by the quotient rule $\frac{g(z)^2}{f(z)} = \overline{f(z)}$ is analytic.

Now, we know that $\overline{f(z)} = u - iv$. So applying C-R to $\overline{f(z)}$ gives:

$$u_x = (-v)_y = -v_y$$

$$u_y = -(-v)_x = v_x$$

But applying C-R to $f(z)$ also gives that:

$$u_x = v_y$$
$$u_y = -v_x$$

Putting these together, we see that $u_x = -u_x$, so $u_x = 0$. Similarly, $u_y = v_x = v_y = 0$, and $f(z)$ is constant. This contradicts our assumption that f is non-constant, so f must have been constant all along.

5 Line Integration

You should know the definition of the line integral $\int_\gamma f(z)dz$ over a piecewise smooth curve γ. You should be able to compute with this definition.

You should recognize when one of our many theorems applies: you should know when a function f has a primitive F and be able to find that primitive for reasonable functions. You should know some simple examples of functions f defined on a domain D that do not have a primitive.

You should know what a simply connected domain is and be able to identify some straightforward examples of simply connected domains. You should know what D being simply connected means in terms of f having a primitive F.

You should know how to use a primitive F to calculate $\int_\gamma f(z)dz$.

You should know the Cauchy Integral Theorem and be able to recognize when it is applicable. Seriously, you can save yourself a ton of time by correctly identifying uses of this theorem.

Question 5.1: Calculate the following integrals from definition:

i) $\int_\gamma z dz$ for γ the straight line segment from 1 to $4-5i$

ii) $\int_\gamma e^z dz$ for γ the straight line segment from 1 to $4-5i$

iii) $\int_\gamma z^2 dz$ for γ the unit circle travelled once CCW starting at 1

iv) $\int_\gamma \frac{1}{z^2}$dz for γ the unit circle travelled once CCW starting at 1

Solution:

i) Our curve is $\gamma(t) = (1-t) + t(4-5i) = 1 + 3t - 5it$, for $t = 0$ to 1.

   Then $\int_\gamma z dz = \int_0^1 [1 + (3-5i)t](3-5i)dt = (3-5i)\left[t + \frac{(3-5i)t^2}{2}\right]_0^1 = (3-5i)(1 + \frac{3-5i}{2}) = ...$

ii) Same curve as the previous question. Our integral is:

$$\int_\gamma e^z dz = \int_0^1 (3-5i)e^{1+(3-5i)t} dt = e^{1+(3-5i)t}\Big|_0^1 = e^{4-5i} - e^1$$

iii) $\gamma(t) = e^{it}$ for $t = 0$ to 12π. Then:

$$\int_\gamma z^2 dz = \int_0^{2\pi} e^{2it} i e^{it} dt = i \int_0^{2\pi} e^{3it} dt = i \int_0^{2\pi} \cos(3t) + i \sin(3t) dt = \frac{i}{3} \left[\sin(3t) - \cos(3t)\right]_0^{2\pi} = \frac{i}{3}(-1-(-1)) = 0$$

iv) $\gamma(t) = e^{it}$ for $t = 0$ to 12π. Then:

$$\int_\gamma z^{-2} dz = \int_0^{2\pi} e^{-2it} i e^{it} dt = i \int_0^{2\pi} e^{-it} dt = i \int_0^{2\pi} \cos(t) - i \sin(t) dt = i \left[\sin(t) + \cos(t)\right]_0^{2\pi} = i(1-1) = 0$$

Question 5.2: Use the $\mathbb{C}$FTC to calculate each of the previous integrals.

Solution:

i) $\frac{z^2}{2}$ is a primitive for z on $\mathbb{C}$, so:

$$\int_\gamma z dz = \left.\frac{z^2}{2}\right|_1^{4-5i} = \frac{(4-5i)^2}{2} - \frac{1}{2}$$

ii) e^z is a primitive for e^z on $\mathbb{C}$, so:

$$\int_\gamma e^z dz = e^z|_1^{4-5i} = e^{4-5i} - e^1$$

iii) $\frac{z^3}{3}$ is a primitive for z^2 on $\mathbb{C}$. And γ starts and ends at 1. So:

$$\int_\gamma z^2 dz = \left.\frac{z^3}{3}\right|_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

iv) $\frac{-1}{z}$ is a primitive for z^{-2} on $\mathbb{C} \setminus \{0\}$, and γ is contained in this set. So:

$$\int_\gamma z^{-2} dz = \left.\frac{-1}{z}\right|_1^1 = 0$$

Question 5.3: Identify which of the following functions has a primitive in the domain $\mathbb{C} \setminus \{0\}$.

i) $z^5 + z - 2$

ii) $e^{\sin z} \cos z - z^2 + 2 \sin(\cos(\sinh z))$

iii) $\frac{1}{z}$

iv) $\frac{1}{z^2}$

v) $\mathrm{Log} z$

vi) $\log_1 z$, given by the branch of $\arg z$ defined for $\theta \in (2\pi, 4\pi)$

Solution:

i) This function is entire, so it has a primitive on $\mathbb{C}$ (since $\mathbb{C}$ is simply connected.) So it has a primitive on this smaller set as well.

ii) This function is entire, so yes.

iii) No. We saw that $\int_{|z|=1} \frac{1}{z} dz = 2\pi i$, but if $\frac{1}{z}$ had a primitive this integral would have to be 0 by $\mathbb{C}$FTC.

iv) Yes. $\frac{-1}{z}$.

v) No. This function isn't even defined on all of this domain.

vi) No. This function isn't even defined on all of this domain.

Question 5.4: Let $f(z) = \frac{1}{z}$ and $g(z) = \frac{z^2}{z^4 - 2z^2 + 1}$. For each curve γ below, determine whether the Cauchy Integral Theorem applies over that curve or not for each function.

i) The unit circle travelled once counterclockwise (CCW) starting at 1.

ii) The triangle with vertices $2, i, -1-i$ travelled in that order.

iii) The circle of radius 1 centered at $\frac{3}{2}$ travelled once CCW starting at $\frac{5}{2}$.

iv) The circle of radius $\frac{1}{2}$ centered at 0 travelled twice clockwise (CW) starting at $-\frac{1}{2}$.

v) The pentagon formed by the solutions to $z^5 = i$.

Solution:

i) There is a discontinuity for $f(z)$ inside γ, so the theorem does not apply.

For $g(z)$, the discontinuities are when $z^4 - 2z^2 + 1 = 0$, which occurs when $z = \pm 1$. Since these discontinuities lay on the curve, the theorem does not apply.

ii) If you draw the triangle out, you'll see it contains 0 and 1, so the theorem doesn't apply for either function.

iii) 0 does not lay inside this circle, so CIT does apply for f. But 1 is inside the circle, so it doesn't apply for g.

iv) The circle contains 0, so CIT doesn't apply for f.

By looking at $D = \{z \in \mathbb{C} | \text{Re}(z) \in (-1, 1)\}$, which is a simply connected domain, we see that CIT applies for g.

v) Drawing this out will show you that this curve contains 0, but does not contain 1 or -1. So CIT doesn't apply for f, but does for g.

Question 5.5: Compute the following integrals using any method.

i) $\int_\gamma p(z)dz$ for $p(z)$ a polynomial and γ is the polygon with vertices $1, 1+4i, 2-3i, 7$.

ii) $\int_\gamma 3 + z^4 - (1+i)z^3 dz$ where γ is the line segment from $2+i$ to $1-3i$.

iii) $\int_\gamma \frac{1}{z^6} dz$ where γ is the unit circle travelled once fully around.

iv) $\int_\gamma e^{\frac{-1}{z^2}} dz$ where γ is the triangle with vertices $2, -1+i, i$ travelled once in that order.

v) $\int_\gamma \frac{1}{z^2-1} dz$ where γ is the circle of radius 1 centered at 1. (Hint: partial fractions might help.)

Solution:

i) $p(z)$ is entire and γ is a closed curve, so $\int_\gamma p(z)dz = 0$ by CIT.

ii) $3 + z^4 - (1+i)z^3$ has a primitive: $3z + \frac{z^5}{5} - \frac{1+i}{4}z^4$

So $\int_\gamma 3 + z^3 - (1+i)z^3 dz = \left[3z + \frac{z^5}{5} - \frac{1+i}{4}z^4\right]_{2+i}^{3-i}$.

iii) $\frac{-1}{5z^5}$ is a primitive for $\frac{1}{z^6}$ on $\mathbb{C} \setminus \{0\}$ which is a domain containing γ, so $\int_\gamma \frac{1}{z^6} dz = 0$ by CFTC.

iv) If you draw out this triangle, you'll see it doesn't contain 0. So, this is a simple, piecewise smooth curve contained in the domain of $e^{\frac{-1}{z^2}}$, and this function is analytic on D by the chain rule.

So $\int_\gamma e^{-\frac{1}{z^2}} dz = 0$ by CIT.

v) Recall: partial fractions lets us write $\frac{1}{z^2-1}$ as $\frac{A}{z-1} + \frac{B}{z+1}$. If we set $\frac{1}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1}$ and multiply both sides by $z^2 - 1$, we get:

$$1 = A(z+1) + B(z-1)$$

Setting $z = 1$ gives $A = \frac{1}{2}$. Setting $z = -1$ gives $B = -\frac{1}{2}$. So:

$$\frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z-1)}$$

So $\int_\gamma \frac{1}{z^2-1} dz = \int_\gamma \frac{1}{2(z-1)} - \frac{1}{2(z-1)} dz$. This first integral is a problem, since $z = 1$ is the center of the circle, and is the discontinuity for $\frac{1}{2(z-1)}$. But, by CIT:

$$\int_\gamma \frac{1}{2(z+1)} dz = 0$$

For the other integral, make the substitution $w = z - 1$. This translates our curve to being the unit circle, and so:

$$\int_\gamma \frac{1}{z^2-1} = \int_\gamma \frac{1}{2(z-1)} dz = \int_{|w|=1} \frac{1}{2w} dw = \pi i$$