MAT334 - Week 4 Problems

Additional Problems

1. Calculate each of the following:

   a) $\text{Arccos}(2) = \frac{1}{i}\text{Log}(2 + e^{1/2\text{Log}(3)}) = -i\text{Log}(2 + \sqrt{3}) = -i\ln(2 + \sqrt{3})$

   b) $\text{Arccos}(1+i) = \frac{1}{i}\text{Log}(1 + i + e^{1/2\text{Log}((1+i)^2-1)}) = -i\text{Log}(1 + i + e^{1/2\text{Log}(2i-1)}) = -i\text{Log}(1 + i + e^{1/2(\sqrt{5}+\arctan(-2))})$

   From here, it becomes much too difficult to keep working this.

   c) $\text{Arcsin}(i) = -i\text{Log}(i(i) + e^{1/2\text{Log}(1-(i^2))}) = -i\text{Log}(-1 + e^{1/2\text{Log}(2)}) = -i\text{Log}(\sqrt{2} - 1) = -i\ln(\sqrt{2} - 1)$

2. Find where the following functions are analytic by checking the Cauchy-Riemann equations. (Seriously, check the equations. On your test, if I say to use C-R and you don't do this you won't get the marks.)

   a) $f(z) = \frac{1}{z}$
   b) $f(z) = e^{z+1}$
   c) $f(z) = e^{(z^2)}$
   d) $f(z) = \frac{z^2}{z-1}$
   e) $f(z) = \sin(z)$
   f) $f(z) = \cos(z)$
   g) $f(z) = \sinh(z)$

   Solutions:

   a) First, write $f(z) = u + iv$. Well, $f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$.
   So $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = \frac{-y}{x^2+y^2}$. Then:

$$
\begin{aligned}
u_x &= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \\
v_y &= \frac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} = u_x \\
u_y &= -\frac{2yx}{(x^2+y^2)^2} \\
v_x &= -\frac{2x(-y)}{(x^2+y^2)^2} = -u_y
\end{aligned}
$$

   So C-R holds on the domain of $f(z)$, and also u, v, u_x, u_y, v_x, v_y are all continuous on $z \neq 0$. So $f(z)$ is analytic.

   b) $f(z) = e^{z+1} = e^{(x+1)+iy} = e^{x+1}(\cos y + i\sin y)$. So $u(x,y) = e^{x+1}\cos y = e(e^x \cos y)$ and $v(x,y) = e^{x+1}\sin y = e(e^x \sin y)$.

   We have already shown that C-R holds for $\frac{u(x,y)}{e}$ and $\frac{v(x,y)}{e}$, in lecture. The same calculation shows C-R holds here. The same facts about continuity also hold here.

   So e^{z+1} is entire.

c) $e^{z^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}(\cos(2xy) + i\sin 2xy)$.

$$\begin{aligned}
u_x &= 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) \\
v_y &= -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy) = u_x \\
u_y &= -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy) \\
v_x &= 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) = -u_y
\end{aligned}$$

So, C-R holds everywhere and again we have continuity of u, v and our partials everywhere. So f is entire.

d) $f(z) = \frac{z^2}{z-1} = \frac{x^2-y^2+2ixy}{(x-1)+iy}$. To get this in terms of $u+iv$, we need to deal with the denominator. You can't leave it as a complex number. So:

$$f(z) = \frac{x^2-y^2+2ixy}{(x-1)+iy}\frac{(x-1)-iy}{(x-1)-iy} = \frac{x^3+y^2x+y^2-x^2+i(x^2y+y^3-2xy)}{(x-1)^2+y^2}$$

This becomes a bit nasty. In this case, I would strongly suggest you use a computer algebra system, like Wolfram Alpha, to make sure you're on the right track.

$$\begin{aligned}
u_x &= \frac{x^4+2x^2y^2+y^4-4x^3-4xy^2+5x^2+3y^2-2x}{((x-1)^2+y^2)^2} \\
v_y &= \frac{x^4+2x^2y^2+y^4-4x^3-4xy^2+5x^2+3y^2-2x}{((x-1)^2+y^2)^2} \\
u_y &= \frac{-2y(x-1)}{((x-1)^2+y^2)^2} \\
v_x &= \frac{2y(x-1)}{((x-1)^2+y^2)^2}
\end{aligned}$$

So we see that C-R holds whenever $(x-1)^2+y^2 \neq 0$, i.e. when $z \neq 1$. And we still have continuity whenever the denominator is non-zero, i.e. when $z \neq 1$. So f is analytic on $\mathbb{C} \setminus \{1\}$.

e) $f(z) = \frac{e^{iz}-e^{-iz}}{2i} = -i\frac{(e^{-y}-e^{y})\cos(x)+(e^{-y}+e^{y})\sin x}{2}$. So:

$$\begin{aligned}
u(x,y) &= \frac{(e^{-y}+e^{y})\sin x}{2} \\
v(x,y) &= \frac{-(e^{-y}-e^{y})\cos x}{2}
\end{aligned}$$

We check the partial derivatives:

$$\begin{aligned}
u_x &= \frac{(e^{-y}+e^{y})\cos x}{2} \\
v_y &= \frac{-(-e^{-y}-e^{y})\cos x}{2} = \frac{(e^{-y}+e^{y})\cos x}{2} \\
u_y &= \frac{(-e^{-y}+e^{y})\sin x}{2} \\
v_x &= \frac{-(e^{-y}-e^{y})(-\sin x)}{2} = \frac{(e^{-y}-e^{y})\sin x}{2}
\end{aligned}$$

So f satisfies $C-R$ everywhere, and u, v and the partials are all continuous. So f is entire.

f) This works out almost identical to the previous part. Entire.

g) This also works out nearly the same. Entire.

3. For each of the functions in the previous question, find their derivatives. (We haven't talked about differentiation rules yet, so you need another approach. I gave you a formula.)

a) The formula in question is $f'(z) = u_x + iv_x$. So:

$$f'(z) = \frac{y^2 - x^2 + 2ixy}{(x^2+y^2)^2} = -\frac{x^2 - y^2 - 2ixy}{|z|^4} = -\frac{\overline{z}^2}{|z|^4} = -\frac{1}{z^2}$$

b)

$$f'(z) = e(e^x \cos y) + ie(e^x \sin y)) = ee^z = e^{z+1}$$

c)

$$\begin{aligned} f'(z) &= 2x(e^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) + i\left[2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy)\right] \\ &= 2x(e^{z^2}) + i2y(e^{z^2}) \\ &= 2ze^{z^2} \end{aligned}$$

d)

$$f'(z) = \frac{x^4 + 2x^2y^2 + y^4 - 4x^3 - 4xy^2 + 5x^2 + 3y^2 - 2x + i2y(x-1)}{((x-1)^2+y^2)^2}$$

You can verify, with a bunch of work, that this is equal to

$$\frac{2z(z-1) - z^2}{(z-1)^2} = \frac{z^2 - 2z}{(z-1)^2}$$

.

e) As we suspect, once we work it out, we get $f'(z) = \cos z$.

f) Similarly, $f'(z) = \sin z$.

g) And $f'(z) = \cosh(z)$.

4. Let $f(z) = 2^z$ be the principal branch of 2^z. Use the Cauchy-Riemann equations to show that $f(z)$ is entire and to find $f'(z)$.

Solution: The principal branch of 2^z is defined by $f(z) = e^{z\ln 2} = e^{x\ln 2}(\cos(y\ln 2) + i\sin(y\ln 2))$. Then we get the partial derivatives:

$$\begin{aligned} u_x &= \ln 2e^{x\ln 2}\cos(y\ln 2) \\ v_y &= \ln 2e^{x\ln 2}\cos(y\ln 2) \\ u_y &= -\ln 2e^{x\ln 2}\sin(y\ln 2) \\ v_x &= \ln 2e^{x\ln 2}\sin(y\ln 2) \end{aligned}$$

So C-R holds everywhere, and we further have continuity for u, v and our partials everywhere. So f is entire, and $f'(z) = \ln 2e^{x\ln 2}\cos(y\ln 2) + i(\ln 2e^{x\ln 2}\sin(y\ln 2)) = 2^z \ln 2$.

5. Let $f(z) = i^z$ be the principal branch of i^z. Use the Cauchy-Riemann equations to show that $f(z)$ is entire and to find $f'(z)$.

Solution: This works out almost identically to the previous problem, except that:

$$f(z) = i^z = e^{z\mathrm{Log}i} = e^{i\pi z/2} = e^{-y\pi/2}(\cos(x\pi/2) + i\sin(x\pi/2))$$

6. Let $a \in \mathbb{C}$. Show that for any branch of the function $f(z) = a^z$, $f(z)$ is always entire. Find $f'(z)$.

Solution: We covered this example in lecture using the chain rule.

To show this without the chain rule, we use a little trick. We don't know what $\log_1(a)$ is for our branch. So let $\log_1(a) = m + in$. Then:

$$f(z) = e^{(x+iy)(m+in)}$$

Once we do this, the same procedure from question 4 gives us that $f'(z) = \log_1(a)a^z$.

7. Determine if the following functions are harmonic or not:

a) $u(x,y) = xy$
b) $u(x,y) = x^2 + y^2$
c) $u(x,y) = x^2 - y^2$
d) $u(x,y) = ye^x$
e) $u(x,y) = e^{x-y}$
f) $u(x,y) = e^{-y}\cos(x) + e^y\cos(x)$

Solution:

a) $u_{xx} = 0$ and $u_{yy} = 0$. So $u_{xx} + u_{yy} = 0$. So harmonic. This is the imaginary part of $f(z) = z^2/2$.
b) $u_{xx} = u_{yy} = 2$. So not harmonic.
c) $u_{xx} = 2$ and $u_{yy} = -2$. So, harmonic. This is the real part of the analytic function $f(z) = z^2$.
d) $u_{xx} = ye^x$. $u_{yy} = 0$. So, not harmonic.
e) $u_{xx} = e^{x-y}$. $u_{yy} = - - e^{x-y} = e^{x-y}$. So, not harmonic.
f) $u_{xx} = -(e^{-y} + e^y)\cos x$. $u_y y = (e^{-y} + e^y)\cos x$. So, harmonic. This is the real part of $2\cos z$.

8. Which of the following sets are open or closed? (No proofs. We talked about an intuitive condition, so just use that.)

a) $\{z \in \mathbb{C} | |z| \geq 2\}$
b) $\{z \in \mathbb{C} | 1 < |z| \leq 3\}$
c) $\{z \in \mathbb{C} | \mathrm{Re}(z) + \mathrm{Im}(z) = 2\}$
d) $\{z \in \mathbb{C} | |e^z| < 2\}$

Solution:

a) Closed. Defined by a $\geq$ condition.
b) Neither. Defined by both a $<$ and a $\leq$ condition.
c) Closed. Defined by an $=$ condition.
d) Open. Defined by a $<$ condition.

9. For each of the sets in the previous question, find their boundary. (Again, we have an intuitive condition. No proofs.)

Solution: We find the boundary by changing each condition to an $=$.

a) The circle $\{z \in \mathbb{C} \mid |z| = 2\}$.

b) We have two conditions here: $|z| > 1$, which gives us the circle $|z| = 1$. And $|z| \leq 3$. Which gives us the circle $|z| = 3$. Which is the boundary? Well, the union of the two. This is easiest to see from a picture.

c) This set is its own boundary.

d) $\{z \in \mathbb{C} \mid |e^z| = 2\} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = \ln 2\}$

10. We know that $\mathrm{Log}(z^2 - 1)$ is defined when $z \notin [-1, 1] \cup \{iy | y \in \mathbb{C}\}$. Let's finish up finding the domain of $\mathrm{Arccos}(z)$.

a) What condition of $z + (z^2 - 1)^{1/2}$ ensures that $\mathrm{Arccos}(z)$ exists? (Don't peek.)

b) Suppose $z + (z^2 - 1)^{1/2} = r$ is in $(-\infty, 0]$. Rearrange this and show that we must have $z = \frac{r}{2} + \frac{1}{2r}$.

c) Now, z is a real number and is a function of the real number $r \leq 0$. Show that $z \in (-\infty, -1]$ by maximizing z as a function of r.

d) Show that if $z \in (-\infty, -1]$ that $z + (z^2 - 1)^{1/2} \in (-\infty, 0]$.

e) Conclude that $\mathrm{Log}(z + (z^2 - 1)^{1/2})$ is defined when $(z^2 - 1)^{1/2}$ is defined and $z \notin (-\infty, -1]$.

Solution:

a) We want $\mathrm{Log}(z + (z^2 - 1)^{1/2})$ to exist. This is true when $z + (z^2 - 1)^{1/2} \notin (-\infty, 0]$.

b) Suppose we have $z + (z^2 - 1)^{1/2} = r$. Rearranging:

$$(z^2 - 1)^{1/2} = r - z$$

Square both sides to get:

$$z^2 - 1 = (r - z)^2 = z^2 - 2rz + r^2$$

Simplifying:

$$2rz = r^2 + 1$$

And dividing by $2r$ gives:

$$z = \frac{r}{2} + \frac{1}{2r}$$

c) Now, we see that z is real and is a function of one real variable. So we can use single variable calc (hence, maximize) to see how big z can be.
Remember, to maximize we set $\frac{dz}{dr} = 0$. Well:

$$\frac{dz}{dr} = \frac{1}{2} - \frac{1}{2r^2} = 0$$

This occurs only when $r^2 = 1$. Since $r < 0$, this means that $r = -1$. So we have a critical point at $r = -1$, which corresponds to $z = -1$. Also, this is a local maximum, which we can see by noting that $\frac{d^2 z}{dr^2} = \frac{1}{r^3} < 0$.
So, since we have a differentiable function on $(-\infty, 0)$ that achieves exactly one critical point, which is a maximum, $z \leq -1$.

d) Now, if $z \in (-\infty, -1]$, then either $z = -1$ and $(z^2 - 1) = 0$ so that $(z^2-1)^{1/2}$ DNE, or $z < -1$. Now, if $z < -1$, we know that $0 < z^2 - 1 < z^2$. In particular, this means that $(z^2-1)^{1/2} < (z^2)^{1/2} = |z| = -z$ since $z < 0$. So $z + (z^2-1)^{1/2} < z + |z| = 0$. So if $z \in (-\infty, -1)$, then $z + (z^2-1)^{1/2} \in (-\infty, 0)$ as desired.

e) This follows immediately from our work: if $z + (z^2-1)^{1/2} < 0$, then $z \leq -1$ by part (c). And by part (d), if $z \in (-\infty, -1]$ then either $z + (z^2-1)^{1/2} < 0$ so that the logarithm doesn't exist, or $(z^2-1)^{1/2}$ does not exist.