*Fill in your **Name** (as it appears on Portal) and **Student ID**, **sign** below, and select your tutorial.*

NAME (Last, First): ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

TUTORIALS - Indicate Your Registered Tutorial (✓)
WEDNESDAY
☐ TUT101 - 3pm
☐ TUT102 - 4pm
☐ TUT103 - 5pm

- There are **seven questions** on this test, some with multiple parts.
- There is a total of **30 available points.**
- **No aids are allowed.** (i.e. no calculators, cheat sheets, devices etc.)
- This test has **8 pages** including this page, and a **page of scrap**.
- **Nothing on the scrap page will be marked. You may remove them.**

MARKING - Leave This Blank							
/5	/4	/5	/3	/3	/4	/6	/30

QUESTION 1 **(5 points)**

Seventy-three points are given inside a hexagon with side lengths all 1. Prove that there are three of these points that span a triangle of area at most 1/8.

Note: the area of a triangle is $\frac{1}{2}\mathtt{bh}$.

SOLUTION Divide the hexagon into 6 regular triangles, with the centre of the hexagon as a shared vertex, then subdivide each of these into 4 regular triangles of side length 1/2. Such a triangle has area $\sqrt{3}/16 < \sqrt{4}/16 = 1/8$. By PHP, there are at least three points (four even) contained in one of these latter triangles. They together span a triangle of area less than its containing triangle, which we've seen is less than 1/8.

This was similar to Exercises 16 and 40 from Chapter 1.

QUESTION 2 **(4 points)**

Using a **double counting** argument, prove the following identity, for any *fixed* nonnegative integers n, and $r \leqslant n$:

$$\sum_{k=0}^{n-r+1} \binom{r+k}{r} = \binom{n+2}{r+1}$$

Your argument could be a "committee formation" argument, as in lecture, or an argument involving subsets, as in the textbook. But it shouldn't involve induction, or breaking down binomial coefficients into factorials, apply the Binomial Theorem, or give any sort of "algebraic" argument, etc.

SOLUTION This is almost exactly the same as Example 3c from the Chapter 4 (Part 1) Lecture notes: move a "check" or "yes" marker from left to right through the people, starting at the $r+1$st person, *including* the marked person, and *excluding* anyone to the right. This gives $\binom{r}{r} + \binom{r+1}{r} + ...\binom{n+1}{r}$, which is the LHS. These are all of the ways of choosing $r+1$ people, from $n+2$ people, which is the RHS.

QUESTION 3 **(5 points)**

Let $q_m(n)$ be the number of partitions of n with exactly m parts and with first part of size exactly m, and let $r_m(n)$ be the number of partitions of n with no part of size greater than $m-1$ and no more than $m-1$ parts.

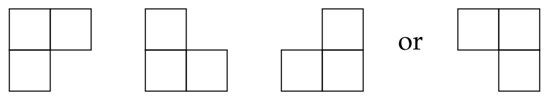
Prove that for any positive integers n, m with $2m+1 \leqslant n$, we have

$$q_m(n) = r_m(n-2m+1)$$

SOLUTION This uses a similar observation to the proof that there are the same number of self-conjugate partitions of n as partitions of n into distinct odd parts (from class and the book): if you look at the Ferrer's diagram of a partition of n with m parts and first part of size m, then it has an outer "hook" consisting of $m+m-1$ blocks. If we remove this hook, we leave behind a partition of $n-(2m-1) = n-2m+1$. The partition left behind cannot have any parts of size $> m-1$ since the first part of the original partition was size m. It also cannot have more parts than $m-1$, since the original partition didn't have more than m parts (so we've created a partition of $n-2m+1$ of the type counted by r_m). It is not too hard to show that this is a bijection (the inverse operation takes a partition of $n-2m+1$, and simply adds an outer hook of the above type, adding $2m-1$ blocks, and resulting in a partition of n with the condition we want.)

QUESTION 4 **(3 points)**

Prove that for any positive integer n, it is possible to tile any $2^n \times 2^n$ grid with exactly one square removed, using only "L"-shaped tiles with three squares, as in:



Hint: induction on n*.*

SOLUTION This is a proof by induction. Here's the inductive step: note that the grid is made of **four** distinct $2^n \times 2^n$ grids and that the removed square is in one of these four sub-grids. So apply the IH to tile this subgrid with L's. Now, place an "L" covering the three "inner" tiles from the remaining three subgrids. Finally, apply the IH to tile these three subgrids (each with a block removed!). In total we have tiled the larger grid (which has only one block removed).

QUESTION 5 **(3 points)**

Prove the following identity for any integer $n \geqslant 2$:

$$n(n-1)4^n = \sum ac2^{b+4}\binom{n}{a,b,c}$$

(Where the sum is taken over all a, b, c so that $a+b+c=n$.)

SOLUTION This statement is almost exactly what you get by taking partial derivatives of both sides of the equation in the multinomial theorem, with respect to x and z, and then plugging in $x=1, y=2, z=1$. You just need to multiply both sides by 2^4 afterwards to get the final answer. This is like a question from Quiz 2, and the Quick Check from section 4.2 in the book.

QUESTION 6 **(4 points)**

Recall that $S(n, k)$, *the Sterling numbers of the second kind*, stand for the number of partitions of $[n]$ into k non-empty subsets.

Prove, using a **combinatorial argument**, that for all positive integers $k \leqslant n$,

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

SOLUTION This is Theorem 5.8 from the book (we also covered this in class).

QUESTION 7 **(2 points per part - 6 points)**

You are on the sidewalk handing out 500 identical flyers (lucky you!). You hand them out to people who pass you in the street, *trying* to get one to each person. But **sometimes you miss people**, and **sometimes you accidentally hand out multiple copies at once**.

(a) Suppose that by the end, 200 people walked by you on the street, and that you managed to hand out all of the flyers. Furthermore, suppose that the first ten people all got exactly one flyer each, and that the last ten people each got at least two. Then how many ways are there for you to have handed out the flyers?

(b) Suppose that instead, you don't know how many people walked by you, but you do know that everyone who walked by got at least one flyer, and you know that the first ten people got exactly one flyer each again. How many ways are there for you to have handed out the flyers in this case?

(c) Finally, suppose that 250 people walk by, and that you've handed out all of the flyers. But unbeknownst to you, this time *each flyer had a (single) QR-code number* on it, and that there were exactly 25 different QR-codes, each appearing the same number of times amongst the flyers. This means, of course, that the flyers weren't actually identical after all. Now how many ways are there for you to have handed out the flyers?

SOLUTION (a) This is the weak compositions of $500 - 10 - 20$ into 190 parts (ignoring the first 10 flyers and people, and removing 20 flyers for the last ten people). As such, this is $\binom{470+190-1}{190-1}$.

(b) This time we're talking about all compositions (not weak compositions) of 500, with 1 each in the first ten boxes. Then this is compositions of 490 into an unknown number of boxes, i.e. 2^{489}.

(c) This is still weak compositions, as in (a), but first, we put the flyers into some ordering. (Note: there are 25 of each of the 20 types of flyers.) How many orderings? $\binom{500}{20,20,\ldots.,20}$ (where there are twenty-five 20's written along the bottom). Thus, the answer is $\binom{500}{20,20,\ldots.,20}\binom{500+100-1}{100-1}$.