NAME (PRINT):			
	Last/Surname	First /Given Name	
STUDENT #:		SIGNATURE:	

UNIVERSITY OF TORONTO MISSISSAUGA
DECEMBER 2018 FINAL EXAMINATION
MAT344H5F
Introduction to Combinatorics
Alex Rennet
Duration - 3 hours
Aids: None

The University of Toronto Mississauga and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of an exam. Clear, sealable, plastic bags have been provided for all electronic devices with storage, including but not limited to: cell phones, SMART devices, tablets, laptops, calculators, and MP3 players. Please turn off all devices, seal them in the bag provided, and place the bag under your desk for the duration of the examination. You will not be able to touch the bag or its contents until the exam is over.

If, during an exam, any of these items are found on your person or in the area of your desk other than in the clear, sealable, plastic bag, you may be charged with an academic offence. A typical penalty for an academic offence may cause you to fail the course.

Please note, once this exam has begun, you ***CANNOT*** *re-write it.*

There are **two parts** to this exam.

- In **Part A**, only your **final answers** will be marked.
- In **Part B**, you should **justify your solutions** and **show all of your work**.

This exam has **15 pages** including this page.

FOR MARKING (Leave This Blank)

#1-3 (/10)	#4 (/20)	#5 (/15)	#6 (/15)	#7 (/15)	(/75)

Part A

In this part of the exam, **only your final answers will be marked.**
Make sure to clearly indicate your final answer.
Each question (or part of a question) in this part is worth **2 points**, for a total of **10 points**.

1

1.1 **(2 points)** How many seven-digit *numbers* are there that contain no repeated digits?

1.2 **(2 points)** How many ways are there to pick a group of n people from 2018 people, and then pick a second group of m people from the remaining 2018 - n people, so that all people in the first group are *taller* than all of the people in the second group? *(Assume that the people all have different heights, and that n and m are fixed numbers with $1 \leqslant n \leqslant 2018$ and $0 \leqslant m \leqslant 2018 - n$.)*

2 **(2 points)** What is the coefficient of $x^{999}y^{1001}z^{18}$ in $(x-y+20z)^{2018}$?
(You do not need to simplify or expand your answer; just write it in terms of functions we've considered in this class, like $(n)_k$, $\binom{n}{k}$, $\binom{n}{a_1,\ldots,a_k}$, $B(n)$, $S(n,k)$, $p(n)$ *etc.)*

3 Find a **closed-form generating function** for each sequence below.
(Your answers should not involve n*, infinite sums ("*$\sum$*") or ellipsis ("...").).*

3.1 **(2 points)** a_n is the number of ways to make n dollars out of *5-*, *10-* and *20-dollar* bills.
(We set $a_0 = 1$*.)*

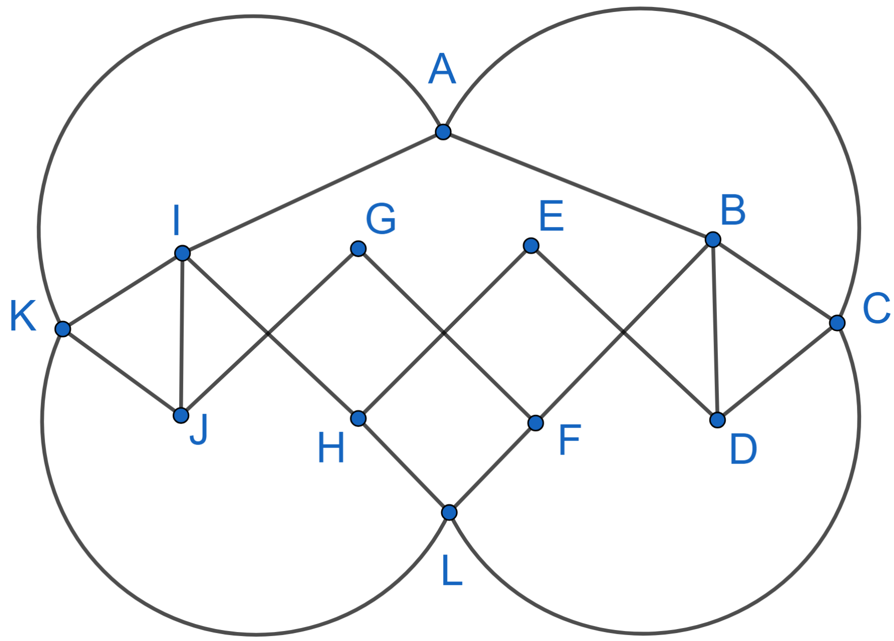
3.2 **(2 points)** b_n is the number of partitions of n with only even-sized parts, and where the number of parts of any given size is also even. *(We set* $b_0 = 1$*.)*

Part B

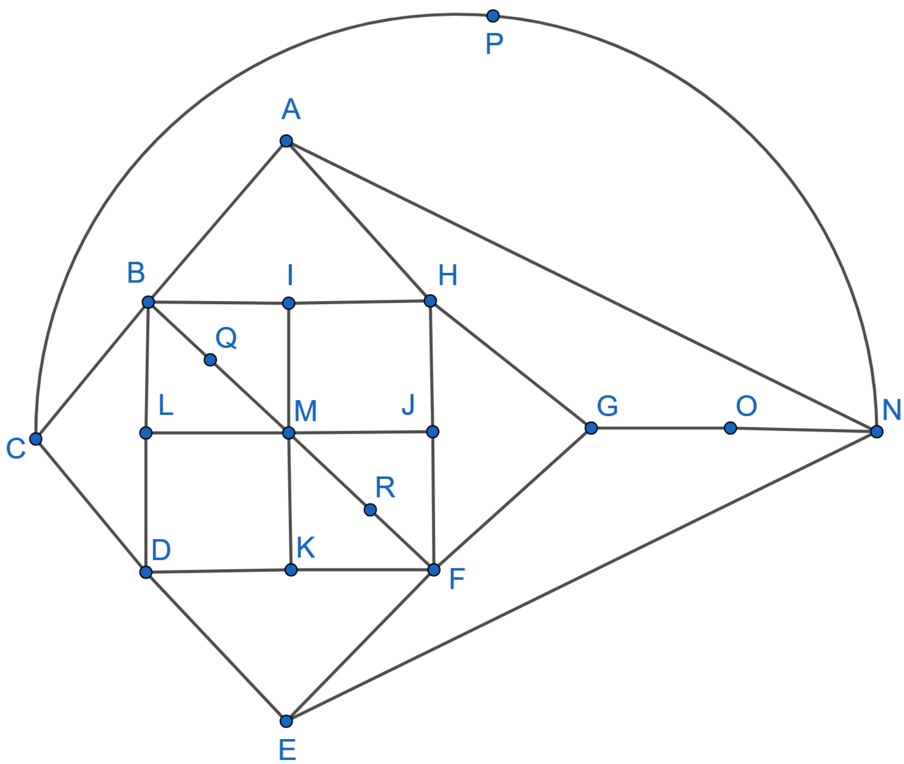
In this part of the exam, you should **justify your solutions** and **show all of your work.**

4 **(20 points total)**

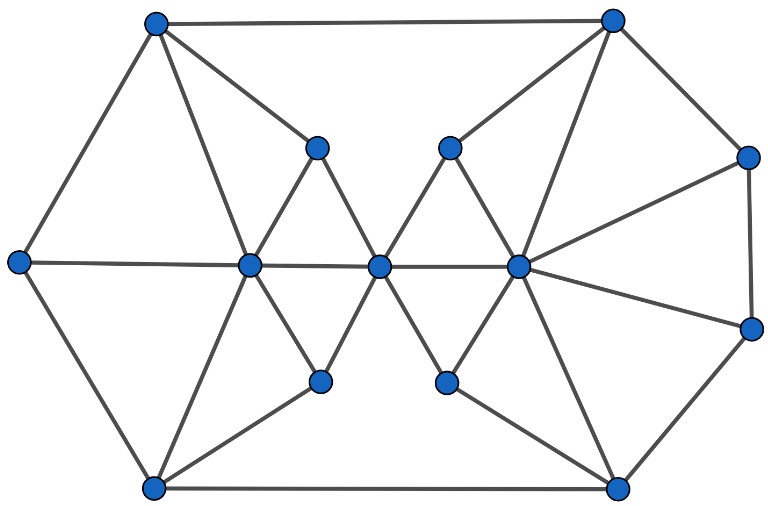
4.1 **(5 points)** Prove that the following graph is **not** planar.



4.2 **(5 points)** Explain why the following graph does **not** have a Hamiltonian cycle.



4.3 **(5 points)** Find the chromatic number of the following graph. *Do not apply the 4-Colour or 5-Colour Theorems as part of your answer.*



4.4 **(5 points)**

We say that a graph G is **colour critical** if $\chi(G) = k$ but $\chi(G - \mathbf{v}) < k$ for any vertex $\mathbf{v}$ in G. *(Recall that $G - \mathbf{v}$ is the graph G with the vertex $\mathbf{v}$ removed.)*

Prove that if G is **colour critical**, then G is connected.

5 **(15 points total)**

5.1 **(5 points)**

How many arrangements are there of the letters in the word “ARITHMOPHELIA” so that the arrangements have the “M” immediately beside a “A” (either as “MA” or as “AM” or as “AMA”)?

5.2 **(5 points)**

Prove the following using a **combinatorial proof**:

$$S(n,k) = \sum_{j=k-1}^{n-1} k^{n-j-1} \cdot S(j, k-1)$$

Note: *You must argue that the two sides are equal by explaining how they can be seen to count the same thing in two different ways. An algebraic argument, induction argument, or other alternate proof strategies will not receive any points.*

5.3 **(5 points)**

Let $A(n)$ be the number of partitions of $[n+1]$ in which no pair of consecutive numbers can be in the same part of the partition.

*For example, for $n = 3$, the partitions $\{\{1,2\},\{3\},\{4\}\}$ and $\{\{1,3,4\},\{2\}\}$ would **not** be counted, while $\{\{1,3\},\{2,4\}\}$ would be counted.*

Recalling that $B(n)$ is the n-th *Bell number*, i.e. the number of partitions of $[n]$, **prove** that

$$A(n) = B(n), \text{ for all } n \geqslant 1.$$

6 **(15 points total)**

6.1 **(2 points)** Find a **closed-form generating function** $A(x)$ for the sequence (a_n) given by

$$(2^0 + 2^1, 2^0 + 2^1 + 2^2, 2^0 + 2^1 + 2^2 + 2^3, \ldots)$$

Your answer should not involve n, *an infinite sum ("*$\sum$*") or ellipsis ("...")..*

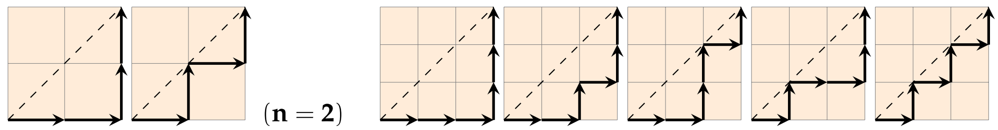
Hint: $A(x)$, *or something close to it, can be written as a product of two functions.*

6.2 **(3 points)** Find an exact expression for a_n using your closed-form expression for $A(x)$. *(No other approach to solving this problem will receive any points.)*

6.3 **(10 points)**

For this question, we define a **monotonic northeastern lattice path** to be a path through a square $n \times n$ grid that consists entirely of edges pointing up ($\uparrow$) or pointing to the right ($\rightarrow$), starts in the bottom left corner, ends in the top right corner, and that *never passes above the diagonal line from the lower-left corner to the upper-right corner* (touching the diagonal line is okay).

Here is a picture of all possible monotonic northeastern lattice paths for the cases $n = 2$ and 3:



$(\mathbf{n} = \mathbf{2})$ $(\mathbf{n} = \mathbf{3})$

For $n \geqslant 1$, we let m_n be the number of monotonic northeastern lattice paths in an $n \times n$ grid (setting $m_0 = 1$), and we let $M(x)$ be the generating function for the sequence (m_n).

By finding the closed form of $M(x)$ and then extracting the coefficient of x^n in the power series form of $M(x)$, **prove that** $m_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ for all $n \geqslant 0$.

Other approaches to solving this will not receive any points. You may use the following fact without justification:

$$\sqrt{1-4x} = 1 - 2x - \frac{2}{n}\sum_{n \geqslant 2}\binom{2n-2}{n-1}x^n$$

Extra space for the previous question...

7 **(15 points total)**

7.1 **(5 points)** Prove that for all connected simple planar graphs G,

$$E \leqslant 3V - 6$$

(Where E is the number of edges in G and V is the number of vertices in G).

7.2 **(5 points)** Use the previous part to prove that all simple planar graphs have a vertex of degree *at most* five.

7.3 **(5 points)** Use the previous part to prove the Six-Colour Theorem: that simple planar graphs have chromatic number at most six. *You are not allowed to appeal to the Five- or Four-Colour Theorems in your answer. Your argument must use the previous part non-trivially to receive any points.*