MAT334 - Week 7 Problems

Textbook Problems 2.3: 2,3,4

Additional Problems

1. Which of the following functions have roots inside $|z| = 2$? (Remember that a root of f is a $z \in \mathbb{C}$ with $f(z) = 0$.) If so, what are they?

   a) $z^3 - \frac{1}{2}$
   b) $\sin(4z)$
   c) $z^3 - iz^2 - z + i$
   d) $e^{4z} - 1$
   e) $z^5 - 16$

   Solution:

   a) The roots of $z^3 - \frac{1}{2}$ are $\frac{1}{\sqrt[3]{2}} e^{i\frac{2k\pi}{3}}$ for $k = 0, 1, 2$. These are all inside the circle.
   b) The roots of $\sin z$ are $k\pi$ for $k \in \mathbb{Z}$. So the roots of $\sin(4z)$ are $\frac{k\pi}{4}$ for $k \in \mathbb{Z}$. Of these, $\pm\frac{\pi}{4}, \pm\frac{\pi}{2}$, and 0 are inside the circle.
   c) We can factor: $z^3 - iz^2 - z + i = (z-i)z^2 + (z-i)(-1) = (z-i)(z^2-1) = (z-i)(z-1)(z+1)$. The roots are i and ± 1, all of which are inside the circle.
   d) $e^z = 1$ for $z = 2k\pi i$, so the roots of $e^{4z} - 1$ are $\frac{k\pi i}{2}$ for $k \in \mathbb{Z}$. As such, $\pm\frac{\pi}{2}$ and 0 are the roots inside the circle.
   e) The roots of $z^5 - 16$ are $2^{\frac{4}{5}} e^{i\frac{2k\pi}{5}}$ for $k = 0, 1, 2, 3, 4$. All of these are inside the circle.

2. Let γ_1 be the circle $|z| = 2$ travelled once counterclockwise, and γ_2 be the circle of radius 1 centered at 1. For which functions in question 1 is it possible to deform γ_1 into γ_2 without crossing any roots of the function?

   a) Not possible. Two of the roots ($k = 1, 2$) are outside γ_2 but inside γ_1.
   b) Not possible. 0 is actually on γ_2.
   c) Not possible. -1 is inside γ_1 but outside γ_2.
   d) Not possible. 0 is on γ_2.
   e) Not possible. For $k = 2, 3$, the roots are outside of γ_2 but inside γ_1. (It might also be true for $k = 1, 4$, but this is difficult to check.)

3. Let γ be the circle $|z| = 1$ travelled once counterclockwise. Calculate the following integrals using the Cauchy Integral Formula.

   a) $\int_\gamma \frac{1}{z^2} dz$
   b) $\int_\gamma \frac{1}{z^2 - \frac{1}{4}} dz$
   c) $\int_\gamma \frac{\cos(z)e^z}{z - \frac{1}{2}} dz$
   d) $\int_\gamma \frac{\operatorname{Log}(z+2)}{z} dz$
   e) $\int_\gamma \frac{1}{z^2 - \frac{5}{2}z + 1} dz$
   f) $\int_\gamma \frac{1}{(z^2 - \frac{5}{2}z + 1)^n} dz$ for any $n \in \mathbb{N}$
   g) $\int_\gamma \frac{\cos(\sin(\cos(z)))}{z^2} dz$

Solution:

a) Let $f(z) = 1$. $f(z)$ is entire. The roots of z^2 are $z = 0$, which is inside γ_1. γ_1 is positively oriented, and smooth, closed, etc..
All together, CIF applies:

$$\int_{\gamma_1} \frac{1}{z^2} dz = 2\pi i f'(0) = 0$$

b) The roots of $z^2 - \frac{1}{4}$ are $\pm\frac{1}{2}$, which are both inside γ_1. As such, we need a deformation of curves to separate these roots.
Let γ_2 be the circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$, and γ_3 be the circle of radius $\frac{1}{2}$ centered at $\frac{-1}{2}$.
Let $f_2(z) = \frac{1}{z+\frac{1}{2}}$. Then $f_2(z)$ is analytic on $\mathbb{C} \setminus \{-\frac{1}{2}\}$, which contains γ_2 and its inside. $\frac{1}{2}$ is inside γ_2, and γ_2 is positively oriented. Applying CIF gives:

$$\int_{\gamma_2} \frac{1}{z^2 - \frac{1}{4}} dz = 2\pi i f_2(\frac{1}{2}) = \frac{2\pi i}{\frac{1}{2} + \frac{1}{2}} = 2\pi i$$

As for γ_3, let $f_3(z) = \frac{1}{z-\frac{1}{2}}$. We do the same process as above (do as I say, not as I do - actually write it out) and we apply CIF to get:

$$\int_{\gamma_2} \frac{1}{z^2 - \frac{1}{4}} dz = 2\pi i f_3(\frac{-1}{2}) = \frac{2\pi i}{-\frac{1}{2} - \frac{1}{2}} = -2\pi i$$

So, we get that $\int_{\gamma_1} \frac{1}{z^2-\frac{1}{4}} dz = 2\pi i - 2\pi i = 0$.

c) Let $f(z) = \cos(z)e^z$. Then $\frac{1}{2}$ is inside $|z| = 1$, and $f(z)$ is entire. So CIF applies to give:

$$\int_{\gamma} \frac{\cos(z)e^z}{z - \frac{1}{2}} = 2\pi i f\left(\frac{1}{2}\right) = 2\pi i \cos\left(\frac{1}{2}\right) e^{\frac{1}{2}}$$

d) We need to find the discontinuities of $\mathrm{Log}(z + 2)$. We know $\mathrm{Log}(z)$ is analytic on $\mathbb{C} \setminus \{(-\infty, 0]\}$, so $\mathrm{Log}(z + 2)$ is analytic on $\mathbb{C} \setminus \{(-\infty, -2]\}$, which is a domain containing γ and inside γ. As always here, γ is positively oriented. So CIF gives:

$$\int_{\gamma} \frac{\mathrm{Log}(z + 2)}{z} dz = 2\pi i \mathrm{Log}(0 + 2) = 2\pi \ln(2) i$$

e) The quadratic formula gives $z^2 - \frac{5}{2}z + 1$ has roots $\frac{1}{2}$ and 2. Of these, only $z = \frac{1}{2}$ is inside γ. So let $f(z) = \frac{1}{z-2}$. This is analytic on $\mathbb{C} \setminus \{2\}$, which is a domain containing γ and inside γ. And, as before, γ is positively oriented. So:

$$\int_{\gamma} \frac{1}{z^2 - \frac{5}{2}z + 1} dz = 2\pi i f\left(\frac{1}{2}\right) = 2\pi i \frac{1}{\frac{1}{2} - 2} = -\frac{4}{3}\pi i$$

f) We perform the same analysis as in the previous part, except now $f(z) = \frac{1}{(z-2)^n}$. So we get, from CIF, that:

$$\int_{\gamma} \frac{1}{(z^2 - \frac{5}{2}z + 1)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}\left(\frac{1}{2}\right)$$

What is the $n - 1$st derivative of $f(z)$? Try a few out. Hopefully you see a pattern. You get:

$$f^{(n-1)}(z) = \frac{(-1)^{n-1}(n)(n+1)(n+2)...(n+(n-1)-1)}{(z-2)^{2n-1}} = \frac{(-1)^{n-1}(2n-2)!}{(n-1)!(z-2)^{2n-1}}$$

So, all together, we get:

$$\int_\gamma \frac{1}{(z^2 - \frac{5}{2}z + 1)^n} dz = \frac{(-1)^{n-1} 2\pi i (2n-2)!}{((n-1)!)^2 \left(\frac{-3}{2}\right)^{2n-1}} = \frac{2^{2n}\pi i (2n-2)!}{((n-1)!)^2 3^{2n-1}}$$

g) And lastly, we have only one discontinuity at $z = 0$, inside the curve. Let $f(z) = \cos(\sin(\cos(z)))$. Then $f(z)$ is entire. CIF then applies to give:

$$\int_\gamma \frac{\cos(\sin(\cos(z)))}{z^2} dz = 2\pi i f'(0) = 2\pi i(-\sin(\sin(\cos(0)))\cos(\cos(0))(-\sin(0))) = 0$$

4. So far we've seen how to handle simple closed curves travelled in positive orientation. What do we do if the curve isn't simple, or the orientation is negative?

a) Suppose γ can be broken up into n simple closed curves $\gamma_1, ...\gamma_n$. How does $\int_\gamma f(z)dz$ relate to the integrals $\int_{\gamma_i} f(z)dz$? (No proof required, but it might help to draw a picture.)

b) Suppose γ travels a simple closed curve n times. (Meaning that there is some simple closed curve Γ such that γ traces out Γ exactly -n times, all in the same direction.)

Prove that $\int_\gamma f(z)dz = n \int_\Gamma f(z)dz$

c) Suppose γ is negatively oriented. Justify why $-\gamma$ is positively oriented. Give a strategy for integrating over negatively oriented curves.

Solution:

a)

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_n} f(z)dz$$

b) The curve γ is simply $\Gamma + \Gamma + ... + \Gamma$ (n times). As such:

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_\Gamma f(z)dz = n\int_\Gamma f(z)dz$$

c) Suppose γ is negatively oriented. Let $\vec{v} = \gamma'(t_0)$ be the tangent vector to the curve at some t_0. The definition of negatively oriented is that "the inside of γ is to the right of γ". What this means specifically is that $e^{-\frac{i\pi}{2}}\vec{v}$ passes through the inside of γ (the multiplication rotates $\vec{v}$ by 90 degrees clockwise - i.e. to its right.

To see if $-\gamma$ is positively oriented, let's look to its right. It should be pointing at the *outside* of γ. So, let's now look at $e^{-\frac{i\pi}{2}}(-\gamma)'(t_1)$, where $a + b - t_1 = t_0$ (so that $(-\gamma)(t_1) = \gamma(t_0)$ - i.e. so that we're at the same point on the curve). Remember that we're travelling the curve backwards, so the times can't match up, that's why we need to adjust like this.

Anyway, $(-\gamma)'(t_1) = \frac{d}{dt}\gamma(a+b-t)|_{t_1} = -\gamma'(a+b-t_1) = -\gamma'(t_0) = -\vec{v} = e^{i\pi}\vec{v}$.

As such, $e^{-\frac{i\pi}{2}}(-\gamma)'(t_1) = e^{\frac{i\pi}{2}}\vec{v}$, which is now the vector emerging from $\gamma(t_0)$ to the *left* of γ. Since γ is negatively oriented, the outside of γ is to its left.

So, that means that the outside of $-\gamma$ is to its right, and therefore $-\gamma$ is positively oriented.

5. Suppose Γ is a simple, closed curve. Suppose γ travels Γ positively n_1 times, then negatively n_2 times, then positively n_3 times, and so on. For k even, γ travels Γ positively n_k times, and for each odd k it travels Γ negatively n_k times. Prove that:

$$\int_\gamma f(z)dz = \left(\sum_{k \text{ odd}} n_k - \sum_{k \text{ even}} n_k\right)\int_\Gamma f(z)dz$$

Solution: Let $n_i\Gamma = \Gamma + ... + \Gamma$, n_i times. We have already shown that $\int_{n_i\Gamma} f(z)dz = n_i \int_\Gamma f(z)dz$. Now, our curve γ is $n_1\Gamma + n_2(-\Gamma) + n_3\Gamma + n_4(-\Gamma)$.... Each odd integer has corresponds to a Γ and each negative integer corresponds to a $-\Gamma$. As such:

$$\int_\gamma f(z)dz = \sum_k n_k \int_{(-1)^{k+1}\Gamma} f(z)dz = \sum_k n_k(-1)^{k+1} \int_\Gamma f(z)dz = \left(\sum_{k \text{ odd}} n_k - \sum_{k \text{ even}} n_k\right) \int_\Gamma f(z)dz$$

(It is worth pointing out: the original formula in the question had the signs backwards. Take note of that, you have to be careful to match the signs to where they belong.)

6. Let γ be the curve defined by travelling $|z| = 1$ twice clockwise, then once counterclockwise, then once clockwise again.

   Compute each integral from question 3 over this new curve.

   Solution: Let $n_1 = 2$, $n_2 = 1$ and $n_3 = 1$. Then by the previous question $\int_\gamma f(z)dz = (2 - 1 + 1)\int_{\gamma_1} f(z)dz = 2\int_{\gamma_1} f(z)dz$. So take your answers from question 3 and double them.

7. Compute each of the following integrals, using any method you like.

   a) $\int_{|z|=1} \tan(z)dz$
   b) $\int_\gamma \frac{1}{z^2-4z}dz$ over the triangle with vertices $1, 1-i$, and $2+4i$
   c) $\int_\gamma \frac{1}{z^2-4z}dz$ over the triangle with vertices $-1, 1-i, 2+4i$
   d) $\int_{|z-1|=3} \frac{\cos(z)}{2z^5}dz$
   e) $\int_{|z-1|=\frac{1}{2}} \frac{1}{\sin(z)(z-1)^2}dz$
   f) $\int_{|z|=2} \frac{e^z-\sin(z)}{z^2-8z+15}dz$
   g) $\int_{|z|=2} \frac{1}{z^3-1}dz$ over the curve γ
   h) $\int_{|z-1|=1} \frac{\sin(z)e^z}{z^4-1}dz$
   i) $\int_\gamma \frac{\text{Log}(z+i)}{(z^4+1)^2}dz$ over the square with vertices $-1, 1, 1+2i, -1+2i$

   Solution: All of the curves in this problem are positively oriented, so I will not repeatedly state that assumption to use CIF.

   a) $\tan(z)$ has discontinuities at $z = \frac{\pi}{2} + k\pi$, none of which are inside γ. So CIT applies and the integral is 0.
   b) The roots of $z^2 - 4z$ are 0 and 4, neither of which are inside the triangle. So the integral is 0 by CIT.
   c) This triangle contains 0 but not 4. So let $f(z) = \frac{1}{z-4}$, which is analytic on $\mathbb{C}\setminus\{4\}$, a domain containing γ and inside γ.
      So, by the CIF:
      $$\int_\gamma \frac{1}{z^2-4}dz = 2\pi i f(0) = -\frac{\pi i}{2}$$
   d) Let $f(z) = \frac{\cos(z)}{2}$. This function is entire. Also, γ contains 0. So by CIF:
      $$\int_{|z-1|=3} \frac{\cos(z)}{2z^5}dz = \frac{2\pi i f^{(4)}(0)}{4!} = \frac{2\pi i \cos(0)}{(2)(4!)} = \frac{\pi i}{24}$$

e) The discontinuities of $\frac{1}{\sin(z)(z-1)^2}$ occur at $z = 1$ and $z = k\pi$ for $k \in \mathbb{Z}$. Of these, only $z = 1$ is inside the curve.

So, let $f(z) = \frac{1}{\sin(z)}$. Then f is analytic on $\mathbb{C} \setminus \{k\pi | k \in \mathbb{Z}\}$, which is a domain containing the curve and its inside.

By CIF: $\int_{|z-1|=\frac{1}{2}} \frac{f(z)}{(z-1)^2} dz = 2\pi i f'(1) = -\frac{2\pi i \cos(1)}{\sin^2(1)}$.

f) The roots of $z^2 - 8z + 15$ are $z = 3$ and $z = 5$. Both are outside the curve, and so CIT applies and the integral is 0.

g) This is a bit nasty. First, solve $z^3 = 1$ to get $z = 1, e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$. Or, more simply:

$$1, \ \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

All three of these roots are inside the curve. So, we deform to three small circles around each, say C_1 around 1, C_2 around $\frac{1}{2} + \frac{\sqrt{3}i}{2}$, and C_3 around the last root. A radius of $\frac{1}{2}$ should be small enough to ensure each circle only contains one root.
For C_1, let $f_1(z) = \frac{1}{\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\right)\left(z - \left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\right)} = \frac{1}{z^2+z+1}$.
Then $f_1(z)$ is analytic on a domain containing C_1 and its inside (by design, we're circling the root that doesn't come from f_1!) As such, CIF applies to give:

$$\int_{C_1} \frac{1}{z^3 - 1} dz = 2\pi i f_1(1) = \frac{2\pi i}{3}$$

For C_2, let $f_2(z) = \frac{1}{(z-1)\left(z - \left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\right)}$. The same arguments apply here, so let's just jump into CIF:

$$\int_{C_2} \frac{1}{z^3 - 1} dz = 2\pi i f_2\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = \frac{2\pi i}{\sqrt{3}i\left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right)} = -\frac{2\pi}{\sqrt{3}}\left(\frac{\sqrt{3}i}{2} + \frac{1}{2}\right)$$

And similarly, for C_3 we let $f_3(z) = \frac{1}{(z-1)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)\right)}$. Applying CIF gives:

$$\int_{C_3} \frac{1}{z^3 - 1} dz = 2\pi i f_3\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) = \frac{2\pi i}{-\sqrt{3}i\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)} = \frac{2\pi}{\sqrt{3}}\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$$

All together, we get:

$$\begin{aligned}\int_{|z|=2} \frac{1}{z^3 - 1} dz &= \frac{2\pi i}{3} - \frac{2\pi}{\sqrt{3}}\left(\frac{\sqrt{3}i}{2} + \frac{1}{2}\right) + \frac{2\pi}{\sqrt{3}}\left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) \\ &= \frac{2\pi i}{3} - 2\pi i = -\frac{4\pi i}{3}\end{aligned}$$

(This looks familiar, but this is a rather involved calculation, so I may have made a mistake somewhere. If you see something, please let me know.)

h) To begin with, we get the roots of $z^4 + 1$, which are $z = \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. Of these, $\pm\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ are the roots inside the curve. As in (3d), the logarithm isn't a problem here.
So, let C_1 be a small circle (say radius $\frac{1}{\sqrt{2}}$ centered at $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ and similarly C_2 a small circle centered at $-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.
We consider the two functions:

$$f_1(z) = \frac{\mathrm{Log}(z+i)}{\left(z - \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2}$$

$$f_2(z) = \frac{\mathrm{Log}(z+i)}{\left(z - \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2}$$

Then, after checking the conditions of CIF as usual, we get:

$$\int_\gamma \frac{\mathrm{Log}(z+i)}{(z^4+1)^2} dz = 2\pi i f_1'\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + 2\pi i f_2'\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

Well... let's calculate these derivatives. We can make life much simpler if we instead look at a general situation: let $g(z) = \frac{1}{(z-a)^2(z-b)^2(z-c)^2}$. Then $g'(z) = \frac{2}{(z-a)(z-b)^2(z-c)^2} + \frac{2}{(z-a)^2(z-b)(z-c)^2} + \frac{2}{(z-a)^2(z-b)^2(z-c)}$. And, instead of using the quotient rule, we'll use the product rule:

$$\begin{aligned}
f_1'(z) = {} & \frac{1}{(z+i)\left(z - \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2} \\
& + \mathrm{Log}(z+i)\left(\frac{2}{\left(z - \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right) \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2}\right. \\
& + \frac{2}{\left(z - \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right) \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2} \\
& \left. + \frac{2}{\left(z - \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{-1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)^2 \left(z - \left(\frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}\right)\right)}\right)
\end{aligned}$$

Okay, so this is a bit excessive, but it does simplify down a bit when we plug in our values:

$$\begin{aligned}
f_1'\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = {} & \frac{1}{\left(\frac{1+\sqrt{2}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)(2)(4i)(-2)} \\
& + \mathrm{Log}\left(\frac{1+\sqrt{2}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{2}{\frac{2}{\sqrt{2}}(4i)(-2)} + \frac{2}{(2)\left(\frac{2}{\sqrt{2}} + \frac{2i}{\sqrt{2}}\right)(-2)} + \frac{2}{(2)(4i)\frac{2i}{\sqrt{2}}}\right)
\end{aligned}$$

A similarly grueling computation gives:

$$\begin{aligned}
f_2'\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = {} & \frac{1}{\left(\frac{-1+\sqrt{2}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)(2)(-4i)(-2)} \\
& + \mathrm{Log}\left(\frac{-1+\sqrt{2}}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{2}{\frac{-2}{\sqrt{2}}(-4i)(-2)} + \frac{2}{(2)\left(\frac{2}{\sqrt{2}} + \frac{2i}{\sqrt{2}}\right)(-2)} + \frac{2}{(2)(-4i)\frac{2i}{\sqrt{2}}}\right)
\end{aligned}$$

8. As a preview of things to come, let's see how we can use complex integration to calculate a real integral. We are going to calculate:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

This is an improper integral, which is improper in two places: at $\pm\infty$. So let's recall what this integral is:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \lim_{R\to\infty} \int_{-R}^{0} \frac{1}{x^4+1} dx + \lim_{S\to\infty} \int_{0}^{S} \frac{1}{x^4+1} dx$$

a) Let's start by getting this down to one limit. To begin, show that $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$ exists, by comparing it to $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$. This shows that we can actually calculate the integral as:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1} dx$$

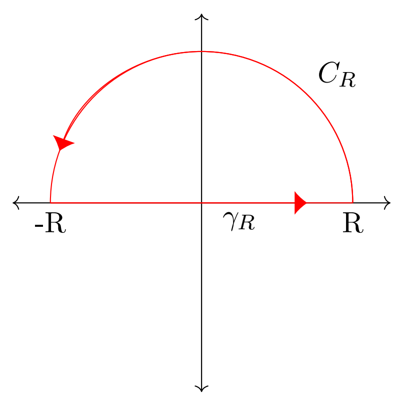
Consider this step optional. It is a good idea to review your first year material on the convergence of improper integrals. We're going to need it in a bit.

b) Let $R > 1$. Now, we can view this as the integral over $\gamma_R(t) = t$ for $t \in [-R, R]$ of:

$$\int_{\gamma_R} \frac{1}{z^4+1} dz$$

Now, this doesn't help us much. We have a bunch of techniques for integrating closed curves, but this isn't closed. So let's define a related closed curve.

Let C_R be the upper semicircle from R to $-R$, so that $\gamma_R + C_R$ is now a closed curve. So our curve is:



Find $\int_{\gamma_R + C_R} \frac{1}{z^4+1} dz$.

c) Now, this integral has two components, $\int_{\gamma_R} \frac{1}{z^4+1} dz$, which is the integral we care about, and $\int_{C_R} \frac{1}{z^4+1} dz$. We'd like to get rid of this second integral. Let's see what happens as we let $R \to \infty$.
To do this, let's try to estimate this curve. We do this in stages.

i) Show that on the curve $|z| = R$, that $|z^4 + 1| \geq R^4 - 1$. Use this to show that $\left|\frac{1}{z^4+1}\right| \leq \frac{1}{R^4-1}$.

ii) Find the length of C_R.

iii) Use our estimation of integrals to show that $|\int_{C_R} \frac{1}{z^4+1} dz| \leq \frac{\pi R}{R^4-1}$.

d) Prove that $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$.

e) Now, we know $\int_{\gamma_R+C_R} \frac{1}{z^4+1} dz$ is constant for $R > 1$, so $\lim_{R\to\infty} \int_{\gamma_R+C_R} \frac{1}{z^4+1} dz$ exists.

We also know that $\lim_{R\to 0} \int_{C_R} \frac{1}{z^4+1} dz = 0$.

Use these two facts to find $\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^4+1} dz$.

f) Show that $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$.

Solution:

a) We know that $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \lim_{r\to\infty} \arctan r - \lim_{s\to-\infty} \arctan s = \pi$.
By comparison, since $\frac{1}{x^4+1} \leq \frac{1}{x^2+1}$ for $|x| > 1$, the comparison test for improper integrals tells us that $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$ converges.
So, since the lmit exists, we can take any relationship between R and S. So let $R = S$, giving the desired equality.

b) First, factor $z^4 + 1$. Let $z = re^{it}$. Then $r^4 e^{4it} = e^{i\pi}$, and so our four roots are: $e^{i\frac{\pi}{4}}$, $e^{i\frac{3\pi}{4}}$, etc. Of these, only the first two are inside our curve. The first is a root of $z^2 - i$ and the second is a root of $z^2 + i$. It's actually fairly useful to write these roots out in rectangular form. The roots are:

$$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$
$$\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$
$$\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$
$$\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

You want to be very careful here to pick the right roots. Here, we want the roots with positive imaginary part.

$$\begin{aligned}
\int_{\gamma_R+C_R} \frac{1}{z^4+1} dz &= 2\pi i \left(\left[\frac{1}{(z^2+i)\left(z - \frac{-1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right)} \right]_{z=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} + \left[\frac{1}{(z^2-i)\left(z - \frac{1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right)} \right]_{z=\frac{-1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} \right) \\
&= 2\pi i \left(\frac{1}{2i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{-1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right)} + \frac{1}{-2i\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\right)} \right) \\
&= \pi \left(\frac{1}{2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} - \frac{1}{2\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} \right) \\
&= \pi \left(\frac{\sqrt{2}(1-i)}{4} - \frac{\sqrt{2}(-1-i)}{4} \right) \\
&= \pi \frac{2\sqrt{2}}{4} \\
&= \frac{\pi}{\sqrt{2}}
\end{aligned}$$

c) We know that $|z^4| = R^4$. Also, $|z^4| = |z^4 + 1 - 1| \leq |z^4 + 1| + 1$ by the triangle inequality. So $R^4 \leq |z^4 + 1| + 1$. Rearranging gives $|z^4 + 1| \geq R^4 - 1$. Then we simply divide to get the desired inequality.

Now, C_R is a semicircle of radius R, so it has length πR.

Estimation of integrals gives $\left|\int_{C_R} \frac{1}{z^4+1} dz\right| \leq ML = \frac{\pi R}{R^4-1}$ as desired.

d) Well, $0 \leq \lim_{R\to\infty} \left|\int_{C_R} \frac{1}{z^4+1} dz\right| \leq \lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0$.

So by the squeeze theorem, $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz = 0$.

e) We know that for $R > 0$, $\int_{\gamma_R + C_R} \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}$. So, $\lim_{R\to\infty} \int_{\gamma_R+C_R} \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}$.

On the other hand, $\int_{\gamma_R+C_R} \frac{1}{z^4+1} dz = \int_{\gamma_R} \frac{1}{z^4+1} dz + \int_{C_R} \frac{1}{z^4+1} dz$. Taking the limit as $R \to \infty$, and using the limit from part (d), we get:

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty} \int_{\gamma_R+C_R} \frac{1}{z^4+1} dz = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^4+1} dz$$

f) Since $\int_{\gamma_R} \frac{1}{z^4+1} dz = \int_{-R}^{R} \frac{1}{x^4+1} dx$, we have just shown that:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1} dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}$$

Solution using partial fractions: So, this is difficult. You need to recognize that $x^4 + 1$ factors into two quadratics (which I have told you), but then you need to actually factor it. It clearly doesn't have any roots in $\mathbb{R}$, so we can't write it as $(x - a)p(x)$ for any polynomial $p(x)$ with real coefficients. So it has to factor as a product of quadratics if it does factor.

So suppose $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$. Then we have the equations:

$$\begin{aligned}
&\text{From the } x^0 \text{ terms:} & bd &= 1 \\
&\text{From the } x^1 \text{ terms:} & ad + bc &= 0 \\
&\text{From the } x^2 \text{ terms:} & b + d + ca &= 0 \\
&\text{From the } x^3 \text{ terms:} & a + c &= 0
\end{aligned}$$

So $b = \frac{1}{d}$ and $c = -a$. This gives us the equations:

$$\begin{aligned}
ad - \frac{a}{d} &= 0 \\
\frac{1}{d} + d - a^2 &= 0
\end{aligned}$$

Rearranging:

$$\begin{aligned}
ad^2 &= a \\
1 + d^2 - a^2 d &= 0
\end{aligned}$$

Since $1 + d^2 > 0$, we know that $a \neq 0$, so $d^2 = 1$. This gives us that $2 = a^2 d$, from our second equation, which also tells us that $d > 0$. So $d = 1$, which also says that $b = 1$.

Lastly, since $d = 1$, we get $a^2 = 2$, so $a = \pm\sqrt{2}$. We can choose either, it doesn't matter. We get:

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2} + 1)$$

Which you can verify is actually correct.

Now we do partial fractions. We write:

$$\frac{1}{x^4+1} = \frac{1}{(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2} + 1)} = \frac{Ax + B}{(x^2 - \sqrt{2}x + 1)} + \frac{Cx + D}{(x^2 + \sqrt{2} + 1)}$$

If we multiply both sides by $x^4 + 1$, we get:

$$1 = (Ax + B)(x^2 + \sqrt{2}x + 1) + (Cx + D)(x^2 - \sqrt{2}x + 1)$$

And now we need to solve for A, B, C, D, which is not pleasant. Our trick for solving these quickly doesn't work, since we don't have access to the roots. So we need to actually multiply it out:

$$\begin{array}{lr} \text{From the } x^0 \text{ terms:} & B + D = 1 \\ \text{From the } x^1 \text{ terms:} & A + C + \sqrt{2}B - \sqrt{2}D = 0 \\ \text{From the } x^2 \text{ terms:} & B + D + \sqrt{2}A - \sqrt{2}C = 0 \\ \text{From the } x^3 \text{ terms:} & A + C = 0 \end{array}$$

So $A = -C$. Plugging that into the second equation gives $B = D$. So $B = D = \frac{1}{2}$ from the first equation. And from the third, we get $1 + 2\sqrt{2}A = 0$, so $A = \frac{-1}{2\sqrt{2}} = -C$.

This lets us write:

$$\frac{1}{x^4+1} = \frac{\frac{-1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 - \sqrt{2}x + 1} + \frac{\frac{1}{2\sqrt{2}}x + \frac{1}{2}}{x^2 + \sqrt{2}x + 1} = \frac{1}{2\sqrt{2}}\left(\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} - \frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1}\right)$$

So, this is a bit of a mess. And it's about to get worse. To integrate a function like $\frac{Ax+B}{ax^2+bx+c}$, the normal procedure is to complete the square on the denominator. So:

$$\frac{x + \sqrt{2}}{x^2 + \sqrt{2}x + 1} = \frac{x + \sqrt{2}}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}$$

$$\frac{x - \sqrt{2}}{x^2 - \sqrt{2}x + 1} = \frac{x - \sqrt{2}}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}$$

Now, $\frac{d}{dx}\left(x \pm \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} = 2\left(x \pm \frac{\sqrt{2}}{2}\right)$.

This lets us write:

$$\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\frac{2x+\sqrt{2}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{\frac{\sqrt{2}}{2}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right.$$
$$\left. -\frac{1}{2}\frac{2x-\sqrt{2}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{\frac{\sqrt{2}}{2}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\right)$$

Now, recall that the integral of $\frac{f'(x)}{f(x)}$ is $\ln(f(x))$, and that the integral of $(x-a)^2+b^2$ is $\frac{1}{b}\arctan\left(\frac{x-a}{b}\right)$. Using this, we get:

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \lim_{r\to\infty}\lim_{s\to-\infty}\frac{1}{2\sqrt{2}}\left(\frac{1}{2}\ln(x^2+\sqrt{2}x+1) - \frac{1}{2}\ln(x^2-\sqrt{2}x+1)\right.$$
$$\left.\left. + \frac{\sqrt{2}}{2}(\sqrt{2}\arctan(\sqrt{2}x+1)) + \frac{\sqrt{2}}{2}(\sqrt{2}\arctan(\sqrt{2}x-1))\right)\right|_s^r$$

Now, let's look at this piece by piece. Notice that $\ln(x^2+\sqrt{2}x+1) - \ln(x^2-\sqrt{2}x+1) = \ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)$. As $x \to \pm\infty$, this logarithm goes to $\ln 1 = 0$. So:

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \lim_{r\to\infty}\lim_{s\to-\infty}\frac{1}{2\sqrt{2}}\left.\left(\frac{\sqrt{2}}{2}(\sqrt{2}\arctan(\sqrt{2}x+1)) + \frac{\sqrt{2}}{2}(\sqrt{2}\arctan(\sqrt{2}x-1))\right)\right|_s^r$$
$$= \lim_{r\to\infty}\lim_{s\to-\infty}\frac{1}{2\sqrt{2}}\left.\left(\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right)\right|_s^r$$

Now, recall that as $x \to \infty$, $\arctan x \to \frac{\pi}{2}$, and as $x \to -\infty$, $\arctan x \to -\frac{\pi}{2}$. So, we see that this limit is really:

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2} - \frac{-\pi}{2} + \frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{4\pi}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

The moral: complex integration is nicer. Notice, we used a lot more theory to solve the complex integral, but that let us avoid so much nasty work. Both the factoring and the integrating for the partial fraction method were horrific.