Calculus 1 Workbook Solutions

Derivative theorems

krista king MATH

MEAN VALUE THEOREM

1. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,5]$.

$$f(x) = x^3 - 9x^2 + 24x - 18$$

Solution:

First, $f(x)$ is continuous and differentiable on the interval $[1,5]$. The problem says to find c in the interval such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Find the values you need for the numerator.

$$f(5) = 5^3 - 9(5)^2 + 24(5) - 18 = 2$$

$$f(1) = 1^3 - 9(1)^2 + 24(1) - 18 = -2$$

Then

$$\frac{f(5) - f(1)}{5 - 1} = \frac{2 - (-2)}{4} = 1$$

Take the derivative $f'(x) = 3x^2 - 18x + 24$, so $f'(c) = 3c^2 - 18c + 24$, then set $f'(c) = 1$ and solve for c.

$$3c^2 - 18c + 24 = 1$$

$$3c^2 - 18c + 23 = 0$$

$$c = \frac{18 \pm \sqrt{18^2 - 4(3)(23)}}{2(3)} = \frac{18 \pm \sqrt{48}}{6} = \frac{18 \pm 4\sqrt{3}}{6} = \frac{9 \pm 2\sqrt{3}}{3}$$

Verify that the slope of the tangent line at these two x-values is 1.

$$f'\left(\frac{9 - 2\sqrt{3}}{3}\right) = 3\left(\frac{9 - 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 - 2\sqrt{3}}{3}\right) + 24 = 1$$

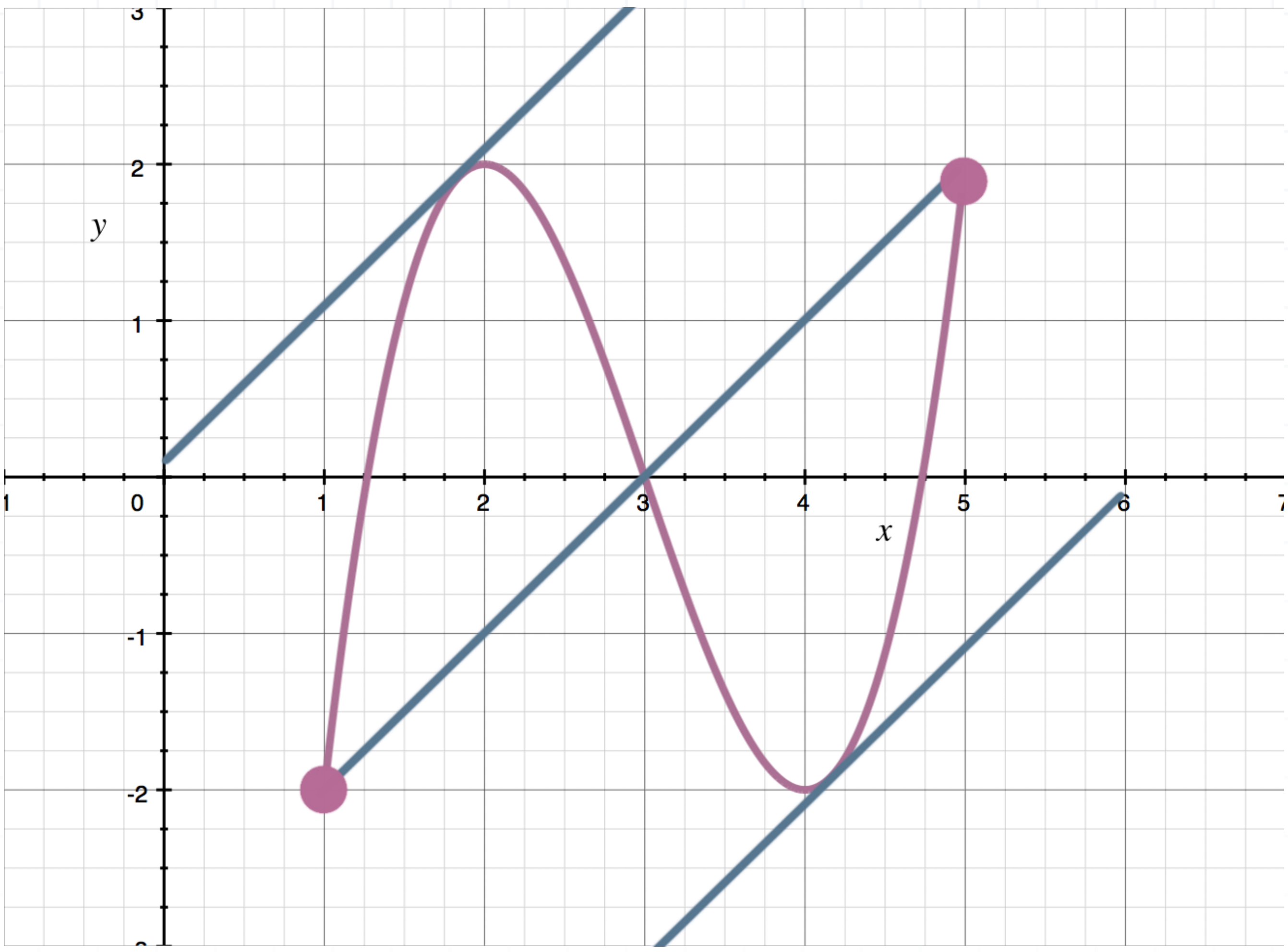
$$f'\left(\frac{9 + 2\sqrt{3}}{3}\right) = 3\left(\frac{9 + 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 + 2\sqrt{3}}{3}\right) + 24 = 1$$

And both values are in the interval $[1,5]$.

$$c = \frac{9 + 2\sqrt{3}}{3} \approx 4.2$$

$$c = \frac{9 - 2\sqrt{3}}{3} \approx 1.8$$

Therefore, the values of c are $(9 \pm 2\sqrt{3})/3$. The figure illustrates how these two points satisfy the Mean Value Theorem.



■ 2. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,4]$.

$$g(x) = \frac{x^2 - 9}{3x}$$

Solution:

First, $g(x)$ is continuous and differentiable on the interval $[1,4]$. The problem says to find c in the interval such that

$$g'(c) = \frac{g(4) - g(1)}{4 - 1}$$

Find the values you need for the numerator.

$$g(4) = \frac{4^2 - 9}{3(4)} = \frac{16 - 9}{12} = \frac{7}{12}$$

$$g(1) = \frac{1^2 - 9}{3(1)} = \frac{1 - 9}{3} = -\frac{8}{3}$$

Then

$$\frac{g(4) - g(1)}{4 - 1} = \frac{\frac{7}{12} - \left(-\frac{8}{3}\right)}{3} = \frac{\frac{13}{4}}{3} = \frac{13}{4} \cdot \frac{1}{3} = \frac{13}{12}$$

Take the derivative,

$$g'(x) = \frac{(3x)(2x) - (x^2 - 9)(3)}{(3x)^2} = \frac{6x^2 - 3x^2 + 27}{9x^2} = \frac{3x^2 + 27}{9x^2} = \frac{x^2 + 9}{3x^2}$$

$$g'(c) = \frac{c^2 + 9}{3c^2}$$

then set $g'(c) = 13/12$ and solve for c.

$$\frac{c^2 + 9}{3c^2} = \frac{13}{12}$$

$$12c^2 + 108 = 39c^2$$

$$27c^2 = 108$$

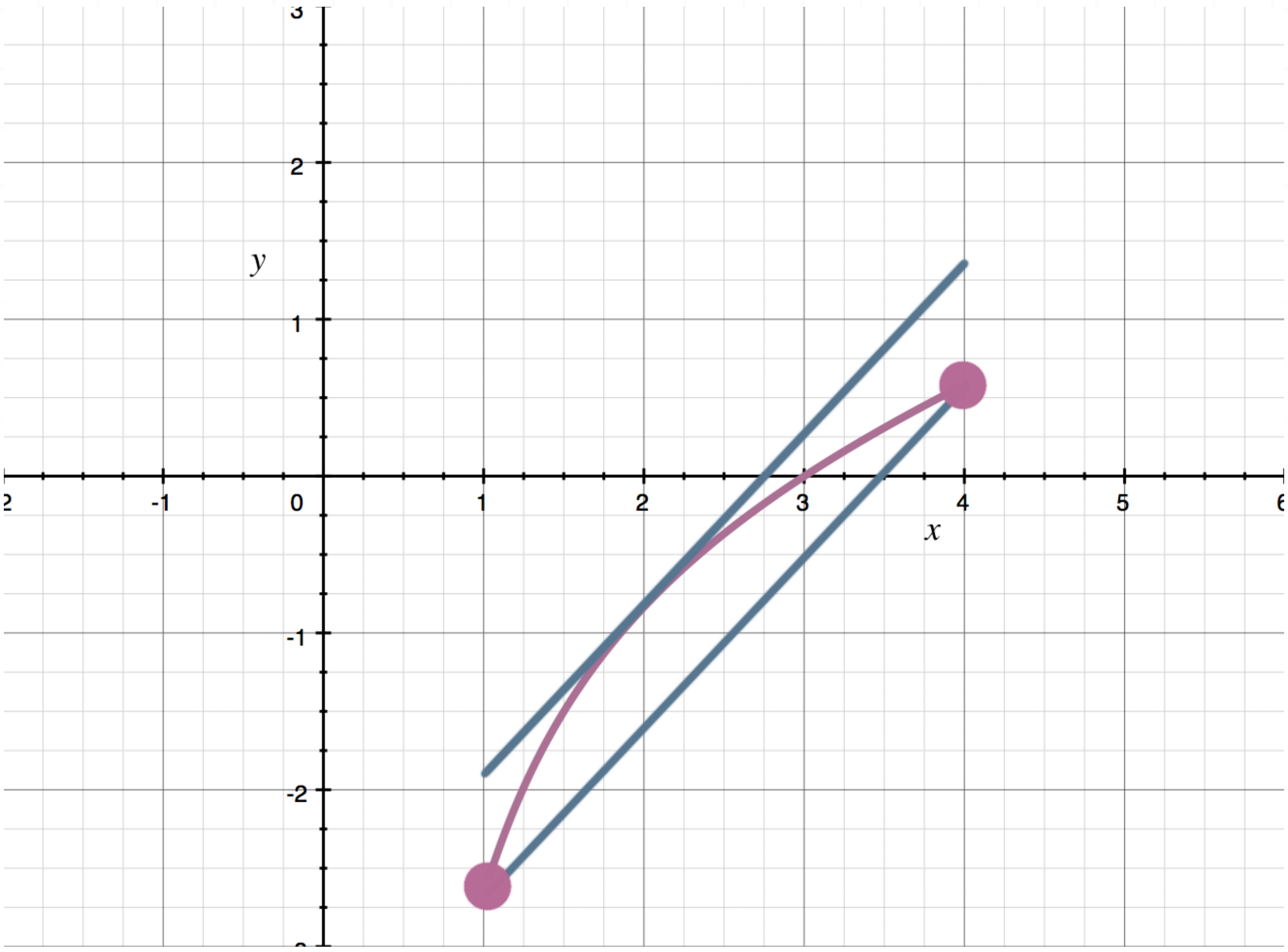
$$c^2 = 4$$

$$c = \pm 2$$

Only $c=2$ is in the given interval. Verify that the slope of the tangent line at this value is $13/12$.

$$g'(2)=\frac{2^2+9}{3(2)^2}=\frac{4+9}{3(4)}=\frac{13}{12}$$

Therefore, the value of c is 2. The figure illustrates how this point satisfies the Mean Value Theorem.



■ 3. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[0,5]$.

$$h(x)=-\sqrt{25-5x}$$

Solution:

First, $h(x)$ is continuous and differentiable on the interval $[0,5]$. The problem says to find c in the interval such that

$$h'(c) = \frac{h(5) - h(0)}{5 - 0}$$

Find the values you need for the numerator.

$$h(5) = -\sqrt{25 - 5(5)} = -\sqrt{0} = 0$$

$$h(0) = -\sqrt{25 - 5(0)} = -\sqrt{25} = -5$$

Then

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - (-5)}{5} = 1$$

Take the derivative,

$$h'(x) = -\frac{-5}{2\sqrt{25 - 5x}} = \frac{5}{2\sqrt{25 - 5x}}$$

$$h'(c) = \frac{5}{2\sqrt{25 - 5c}}$$

then set $h'(c) = 1$ and solve for c.

$$\frac{5}{2\sqrt{25 - 5c}} = 1$$

$$5 = 2\sqrt{25 - 5c}$$

$$\frac{5}{2} = \sqrt{25 - 5c}$$

$$\frac{25}{4} = 25 - 5c$$

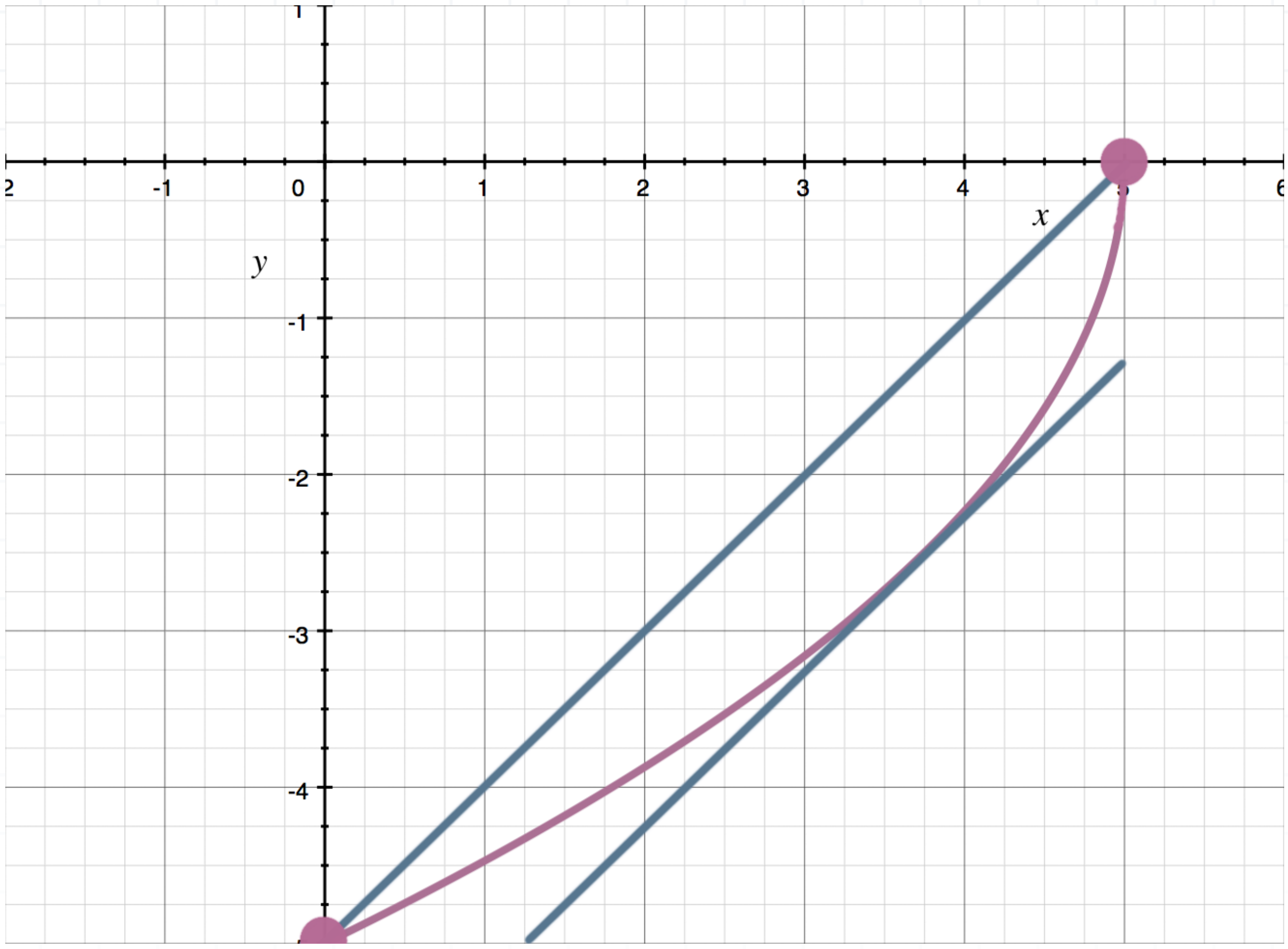
$$c = \left(\frac{25}{4} - 25\right) \div -5$$

$$c = \frac{15}{4}$$

Verify that the slope of the tangent line at this value is 1.

$$h'\left(\frac{15}{4}\right) = \frac{5}{2\sqrt{25 - 5\left(\frac{15}{4}\right)}} = \frac{5}{2\sqrt{\frac{25}{4}}} = \frac{5}{2\left(\frac{5}{2}\right)} = \frac{5}{5} = 1$$

Therefore, the value of c is $15/4$. The figure illustrates how this point satisfies the Mean Value Theorem.



4. If we know that $g(x)$ is continuous and differentiable on $[2,7]$, $g(2) = -5$ and $g'(x) \le 15$, find the largest possible value for $g(7)$.

Solution:

Use the Mean Value Theorem

$$g'(c) = \frac{g(7) - g(2)}{7 - 2}$$

or

$$g(7) - g(2) = g'(c)(7 - 2)$$

$$g(7) = g'(c)(7 - 2) + g(2)$$

$$g(7) = 5g'(c) - 5$$

Now we know that $g'(x) \leq 15$, so we know that $g'(c) \leq 15$.

$$g(7) = 5g'(c) - 5 \leq 5(15) - 5 \leq 70$$

This means that the largest possible value for $g(7)$ is 70.

5. If we know that $f(x)$ is continuous and differentiable on $[-4,3]$, $f(3) = 12$ and $f'(x) \leq 4$, find the smallest possible value for $f(-4)$.

Solution:

Use the Mean Value Theorem

$$f'(c) = \frac{f(3) - f(-4)}{3 - (-4)}$$

or

$$f(3) - f(-4) = f'(c)(3 - (-4))$$

$$12 - f(-4) = 7f'(c)$$

Now we know that $f'(x) \leq 4$, so we know that $f'(c) \leq 4$.

$$12 - f(-4) = 7f'(c)$$

$$12 - f(-4) \le 7(4)$$

$$12 - f(-4) \le 28$$

$$-f(-4) \le 16$$

$$f(-4) \ge -16$$

This means that the smallest possible value for $f(-4)$ is -16.

■ 6. When a cake is removed from an oven and placed in an environment with an ambient temperature of 20° C, its core temperature is 180° C. Two hours later, the core temperature has fallen to 30° C. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 75° C per hour.

Solution:

Because the cake cools down for 2 hours, we can set the interval at $t = [0,2]$. If we plug these values into the Mean Value Theorem, we get

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(c) = \frac{30 - 180}{2 - 0}$$

$$f'(c) = \frac{-150}{2}$$

$$f'(c) = -75$$

This result tells us that, at least at some point, the temperature is decreasing at a rate of 75° C per hour.

ROLLE'S THEOREM

1. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = x^3 - 2x^2 - x - 3$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-1,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-1,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(2) = f(-1)$.

$$f(2) = 2^3 - 2(2)^2 - 2 - 3 = -5$$

$$f(-1) = (-1)^3 - 2(-1)^2 - (-1) - 3 = -5$$

Thus, Rolle's Theorem applies. Next, find $f'(x) = 3x^2 - 4x - 1$ and set $f'(c) = 0$ and solve for c using the quadratic formula.

$$3c^2 - 4c - 1 = 0$$

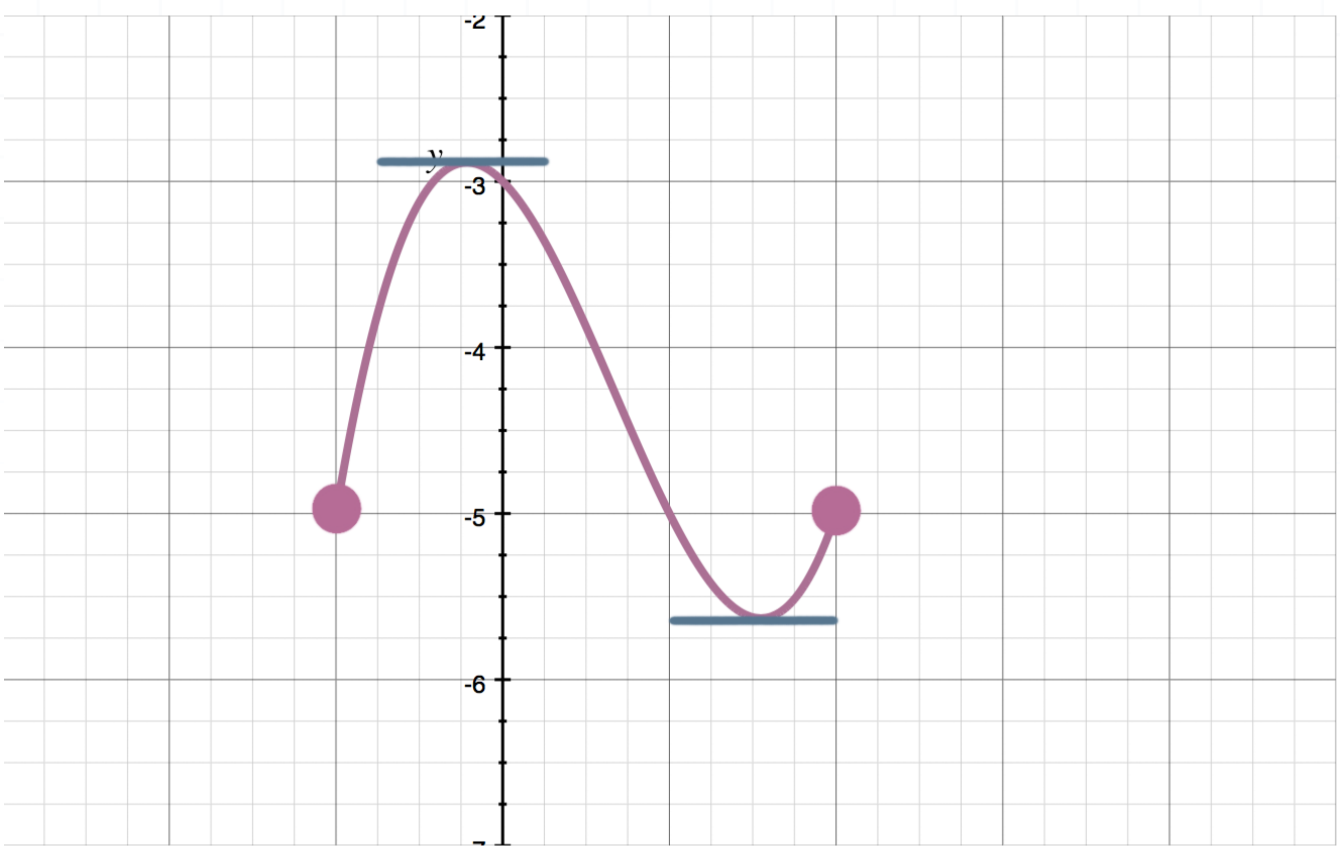
$$c = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} = \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

Verify that the slope of the tangent line at these two x-values is 0.

$$f'\left(\frac{2-\sqrt{7}}{3}\right)=3\left(\frac{2-\sqrt{7}}{3}\right)^2-4\left(\frac{2-\sqrt{7}}{3}\right)-1=0$$

$$f'\left(\frac{2+\sqrt{7}}{3}\right)=3\left(\frac{2+\sqrt{7}}{3}\right)^2-4\left(\frac{2+\sqrt{7}}{3}\right)-1=0$$

Both values are in the interval. Therefore, the values of c such that $f'(c)=0$ are $(2\pm\sqrt{7})/3$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 2. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$g(x)=\frac{x^2-2x-15}{6-x}$$

Solution:

The function $g(x)$ is continuous and differentiable on the interval $[-3,5]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-3,5]$, such that $g'(c) = 0$.

To use Rolle's Theorem, show that $g(5) = g(-3)$.

$$g(5) = \frac{5^2 - 2(5) - 15}{6 - 5} = \frac{0}{1} = 0$$

$$g(-3) = \frac{(-3)^2 - 2(-3) - 15}{6 - (-3)} = \frac{0}{9} = 0$$

Thus, Rolle's Theorem applies. Next, find

$$g'(x) = \frac{(6 - x)(2x - 2) - (x^2 - 2x - 15)(-1)}{(6 - x)^2} = \frac{-x^2 + 12x - 27}{(6 - x)^2}$$

and set $g'(c) = 0$ and solve for c using the quadratic formula.

$$-c^2 + 12c - 27 = 0$$

$$-(c^2 - 12c + 27) = 0$$

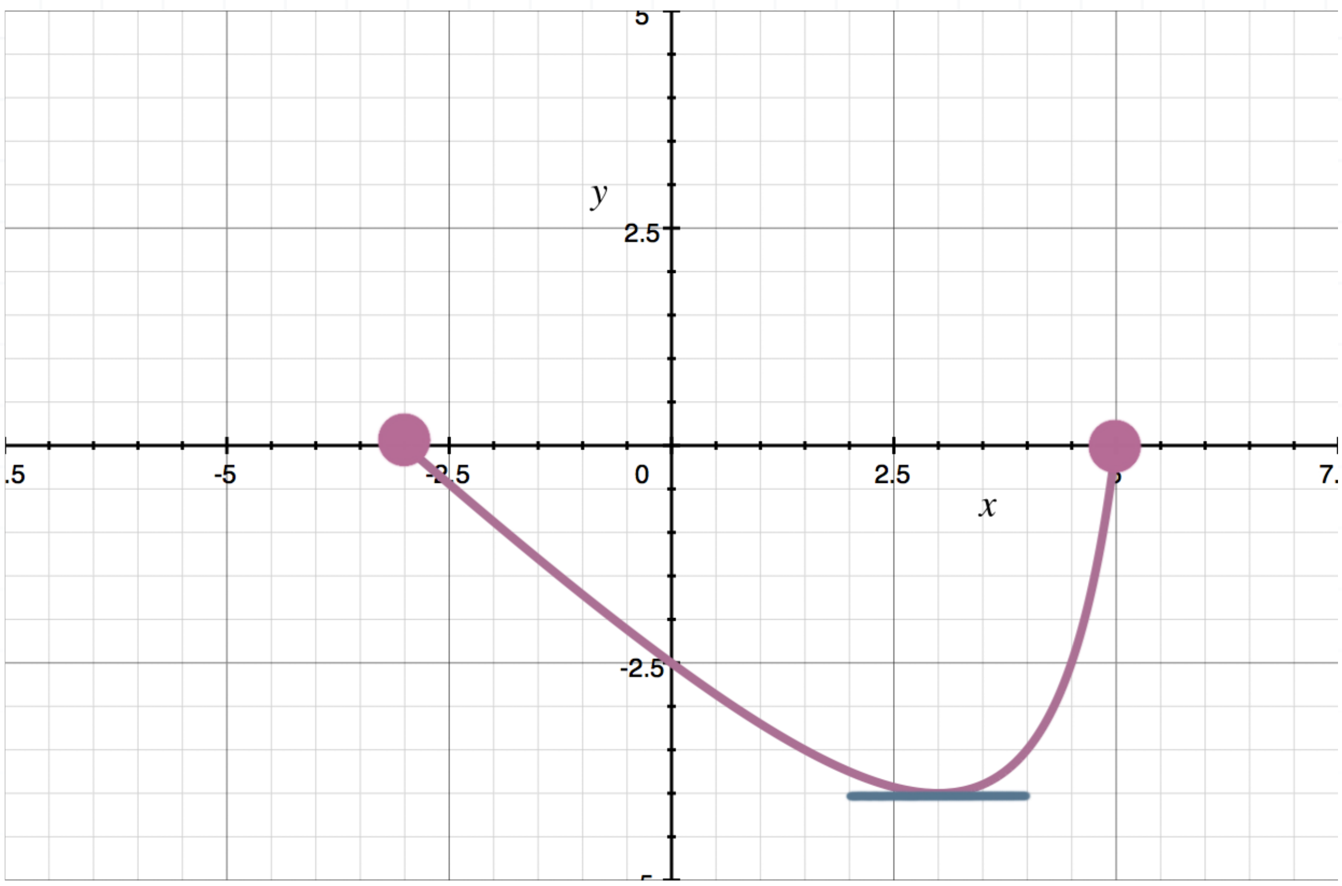
$$-(c - 3)(c - 9) = 0$$

$$c = 3, 9$$

The value $c = 9$ is outside of the given interval. Verify that the slope of the tangent line at $c = 3$ is 0.

$$g'(3) = \frac{-3^2 + 12(3) - 27}{(6-3)^2} = \frac{0}{9} = 0$$

Therefore, the value of c such that $f'(c) = 0$ is 3. The figure illustrates how this point satisfies Rolle's Theorem.



■ 3. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-\pi/2,\pi/2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$h(x) = \sin(2x)$$

Solution:

The function $h(x)$ is continuous and differentiable on the interval $[-\pi/2,\pi/2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-\pi/2,\pi/2]$, such that $h'(c)=0$.

To use Rolle's Theorem, show that $h(\pi/2)=h(-\pi/2)$.

$$h\left(\frac{\pi}{2}\right)=\sin\left(2\cdot\frac{\pi}{2}\right)=\sin(\pi)=0$$

$$h\left(-\frac{\pi}{2}\right)=\sin\left(2\cdot-\frac{\pi}{2}\right)=\sin(-\pi)=0$$

Thus, Rolle's Theorem applies. Next, find $h'(x)=2\cos(2x)$ and set $h'(c)=0$ and solve for c.

$$2\cos(2c)=0$$

$$\cos(2c)=0$$

$$\arccos(0)=2c$$

$$2c=\pm\frac{\pi}{2}$$

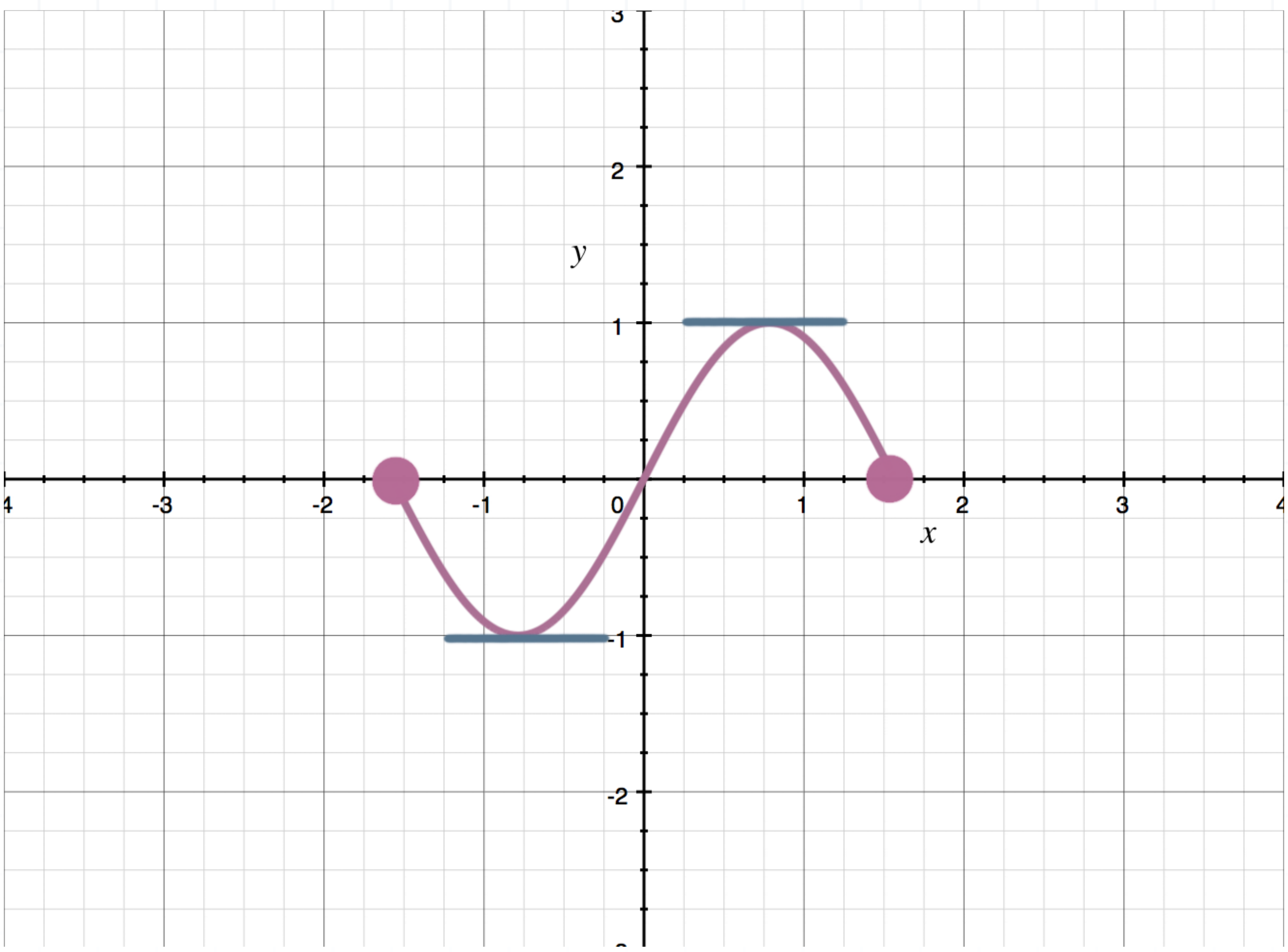
$$c=\pm\frac{\pi}{4}$$

Verify that the slope of the tangent line at these two c-values is 0.

$$h'\left(-\frac{\pi}{4}\right)=2\cos\left(-\frac{\pi}{2}\right)=2\cdot 0=0$$

$$h'\left(\frac{\pi}{4}\right)=2\cos\left(\frac{\pi}{2}\right)=2\cdot 0=0$$

Therefore, the values of c such that $f'(c) = 0$ are $\pm\pi/4$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 4. Determine whether Rolle's Theorem can be applied to $f(x) = \sqrt{4 - x^2}$ on the interval $[-2,2]$. If Rolle's Theorem applies, find the value(s) of c in the interval such that $f'(c) = 0$.

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-2,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-2,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(-2)=f(2)$.

$$f(-2)=\sqrt{4-(-2)^2}=\sqrt{4-4}=\sqrt{0}=0$$

$$f(2)=\sqrt{4-(2)^2}=\sqrt{4-4}=\sqrt{0}=0$$

Because theses values are equivalent, Rolle's Theorem applies. Next, find the derivative and set $f'(c)=0$ to solve for c.

$$f'(x)=-\frac{x}{\sqrt{4-x^2}}$$

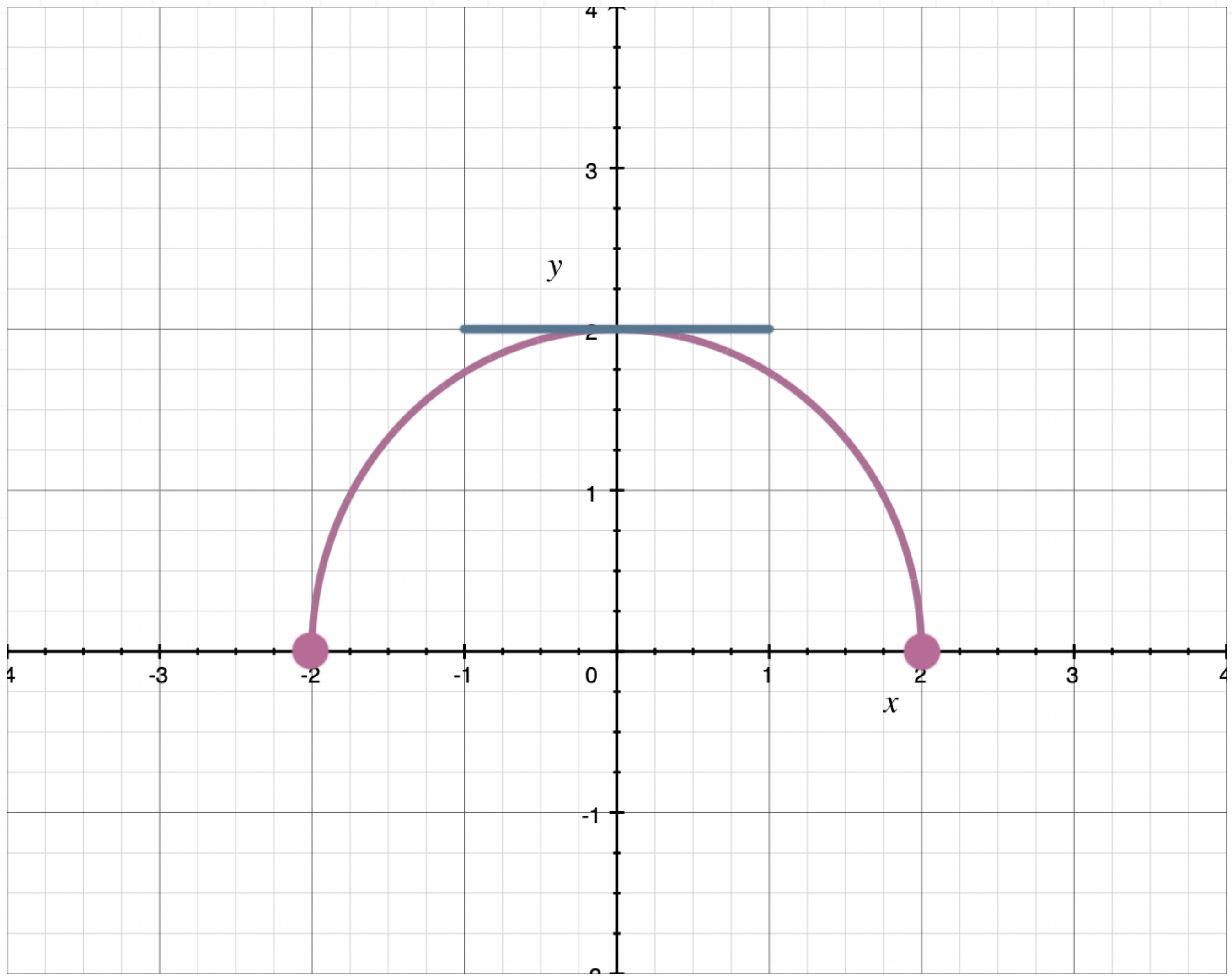
$$-\frac{c}{\sqrt{4-c^2}}=0$$

$$c=0$$

Verify that the slope of the tangent line at $c=0$ is 0.

$$f'(0)=-\frac{0}{\sqrt{4-0^2}}=0$$

Therefore, $c=0$ makes $f'(c)=0$. The figure illustrates how this point satisfies Rolle's Theorem.



■ 5. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = |x - 2|$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[3,5]$. The problem says to use Rolle's Theorem to find c in the interval $[3,5]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(3) = f(5)$.

$$f(3) = |3 - 2| = |1| = 1$$

$$f(3) = |5 - 2| = |3| = 3$$

Because the function doesn't have the same value at the endpoints of the interval, Rolle's Theorem can't be applied, at least not on this particular interval.

■ 6. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,1]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = \ln(9 - x^2)$$

Solution:

The domain of the function is $(-3,3)$. Therefore the function $f(x)$ is continuous and differentiable on the interval $[-1,1]$. The problem says to use Rolle's Theorem to find c in the interval $[-1,1]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(1) = f(-1)$.

$$f(-1) = \ln(9 - (-1)^2) = \ln(9 - 1) = \ln 8$$

$$f(1) = \ln(9 - (1)^2) = \ln(9 - 1) = \ln 8$$

Because theses values are equivalent, Rolle's Theorem applies. Next, find the derivative and set $f'(c) = 0$ to solve for c.

$$f'(x) = -\frac{2x}{9 - x^2}$$

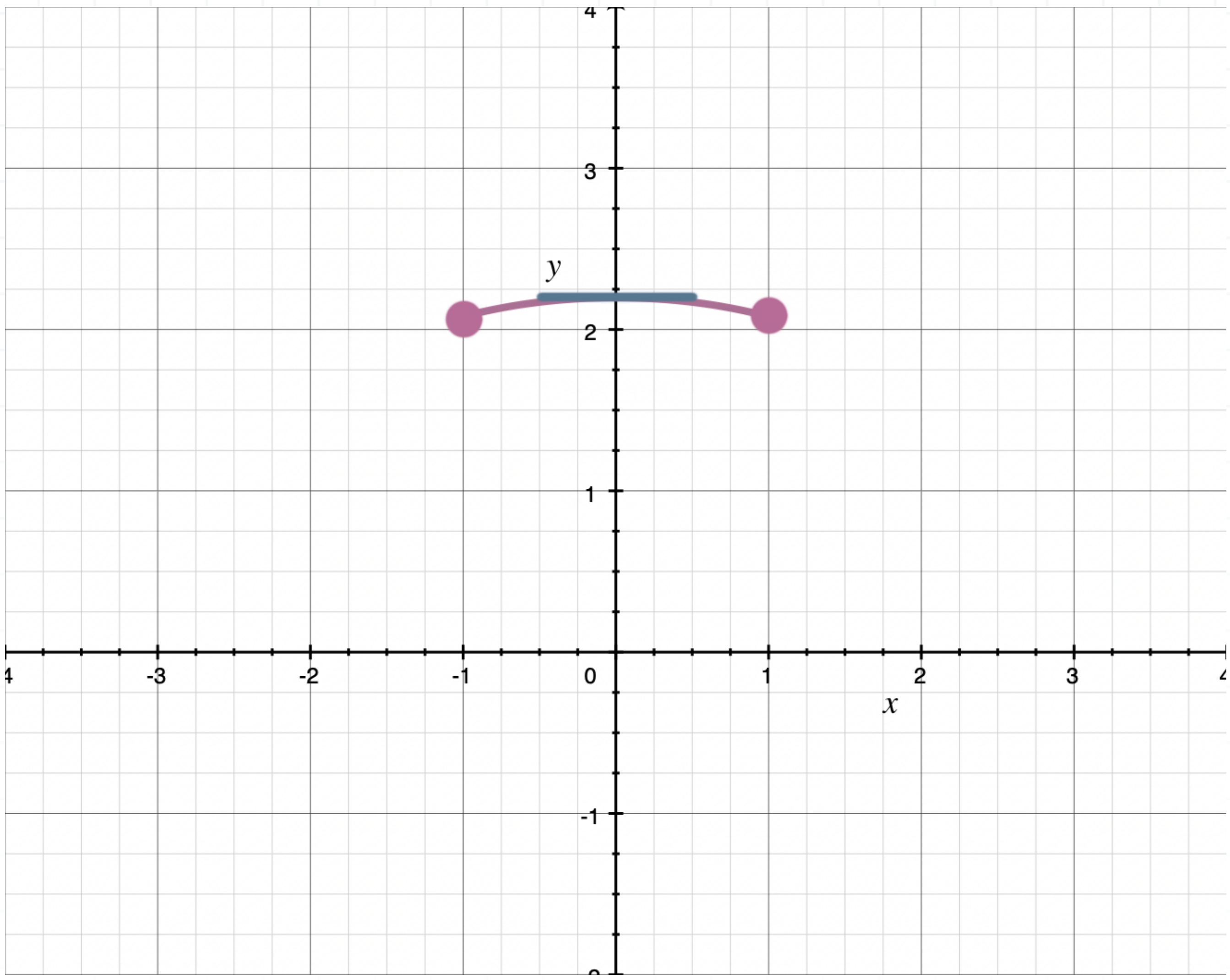
$$-\frac{2c}{9 - c^2} = 0$$

$$c = 0$$

Verify that the slope of the tangent line at $c = 0$ is 0.

$$f'(0) = -\frac{2(0)}{9 - 0^2} = 0$$

Therefore, $c = 0$ makes $f'(c) = 0$. The figure illustrates how this point satisfies Rolle's Theorem.


y
x
4
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-1
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2
3

NEWTON'S METHOD

1. Use four iterations of Newton's Method to approximate the root of $g(x) = x^3 - 12$ in the interval $[1,3]$ to the nearest three decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^3 - 12 = 0$$

If $g(x) = x^3 - 12$ and $g'(x) = 3x^2$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (3 + 1)/2 = 2$. Then $g(2) = -4$ and $g'(2) = 12$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$x_1 = 2 - \frac{-4}{12} \approx 2.333$$

Next, $g(2.333) = 0.698$ and $g'(2.333) = 16.329$. So

$$x_2 = 2.333 - \frac{0.698}{16.329} = 2.290$$

Next, $g(2.290) = 0.009$ and $g'(2.290) = 15.732$. So

$$x_3 = 2.290 - \frac{0.009}{15.732} = 2.289$$

Next, $g(2.289) = -0.007$ and $g'(2.289) = 15.719$. So

$$x_4 = 2.289 - \frac{-0.007}{15.719} = 2.289$$

■ 2. Use four iterations of Newton's Method to approximate the root of $f(x) = x^4 - 14$ in the interval $[-2, -1]$ to the nearest four decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^4 - 14 = 0$$

If $f(x) = x^4 - 14$ and $f'(x) = 4x^3$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (-1 - 2)/2 = -3/2$. Then $f(-1.5) = -8.9375$ and $f'(-1.5) = -13.5$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = -1.5 - \frac{-8.9375}{-13.5} = -2.1620$$

Next, $f(-2.1620) = 7.8501$ and $f'(-2.1620) = -40.4229$. So

$$x_2 = -2.1620 - \frac{7.8501}{-40.4229} = -1.9678$$

Next, $f(-1.9678) = 0.9957$ and $f'(-1.9678) = -30.4792$. So

$$x_3 = -1.9678 - \frac{0.9957}{-30.4792} = -1.9352$$

Next, $f(-1.9352) = 0.0245$ and $f'(-1.9352) = -28.9893$. So

$$x_4 = -1.9352 - \frac{0.0245}{-28.9893} = -1.9343$$

■ 3. Use four iterations of Newton's Method to approximate the root of $h(x) = 3e^{x-3} - 4 + \sin x$ in the interval $[2,4]$ to the nearest four decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$3e^{x-3} - 4 + \sin x = 0$$

If $h(x) = 3e^{x-3} - 4 + \sin x$ and $h'(x) = 3e^{x-3} + \cos x$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (4+2)/2 = 3$. Then $h(3) = -0.8589$ and $h'(3) = 2.0100$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$$

$$x_1 = 3 - \frac{-0.8589}{2.0100} = 3.4273$$

Next, $h(3.4273) = 0.3175$ and $h'(3.4273) = 3.6399$. So

$$x_2 = 3.4273 - \frac{0.3175}{3.6399} = 3.3401$$

Next, $h(3.3401) = 0.0181$ and $h'(3.3401) = 3.2349$. So

$$x_3 = 3.3401 - \frac{0.0181}{3.2349} = 3.3345$$

Next, $h(3.3345) = 0.00001$ and $h'(3.3345) = 3.2103$. So

$$x_4 = 3.3345 - \frac{0.00001}{3.2103} = 3.3345$$

■ 4. Use four iterations of Newton's Method to approximate $\sqrt[65]{100}$ to four decimal places.

Solution:

$$\sqrt[65]{100} = x$$

$$100 = x^{65}$$

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^{65} - 100 = 0$$

Take the derivative of the function.

$$f(x_n) = x_n^{65} - 100$$

$$f'(x_n) = 65x_n^{64}$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^{65} - 100}{65x_n^{64}}$$

Let's start with $x_n = 1$, and work our problem with the number of decimal places we were asked for.

$$x_0 = 1$$

$$x_1 = 1 - \frac{1^{65} - 100}{65(1)^{64}} = 2.5231$$

$$x_2 = 2.5231 - \frac{2.5231^{65} - 100}{65(2.5231)^{64}} = 2.4843$$

$$x_3 = 2.4843 - \frac{2.4843^{65} - 100}{65(2.4843)^{64}} = 2.4460$$

$$x_4 = 2.4460 - \frac{2.4460^{65} - 100}{65(2.4460)^{64}} = 2.4084$$

■ 5. Use Newton's Method to approximate to three decimal places the root of the function in the interval $[3,7]$.

$$5x^2 + 3 = e^x$$

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$5x^2 + 3 - e^x = 0$$

Take the derivative of the function.

$$f(x_n) = 5x_n^2 + 3 - e^{x_n}$$

$$f'(x_n) = 10x_n - e^{x_n}$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{5x_n^2 + 3 - e^{x_n}}{10x_n - e^{x_n}}$$

Since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (7+3)/2 = 5$, and work our problem with the number of decimal places we were asked for.

$$x_0 = 5$$

$$x_1 = 5 - \frac{5(5)^2 + 3 - e^5}{10(5) - e^5} = 4.793$$

$$x_2 = 4.793 - \frac{5(4.793)^2 + 3 - e^{4.793}}{10(4.793) - e^{4.793}} = 4.754$$

$$x_3 = 4.754 - \frac{5(4.754)^2 + 3 - e^{4.754}}{10(4.754) - e^{4.754}} = 4.753$$

$$x_3 = 4.753 - \frac{5(4.753)^2 + 3 - e^{4.753}}{10(4.753) - e^{4.753}} = 4.753$$

Since these last two approximations are identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 4.753$.

■ 6. Use Newton's Method to find an approximation of the root of the function to four decimal places.

$$2 \ln x = \cos x$$

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$2 \ln x - \cos x = 0$$

Take the derivative of the function.

$$f(x_n) = 2\ln(x_n) - \cos(x_n)$$

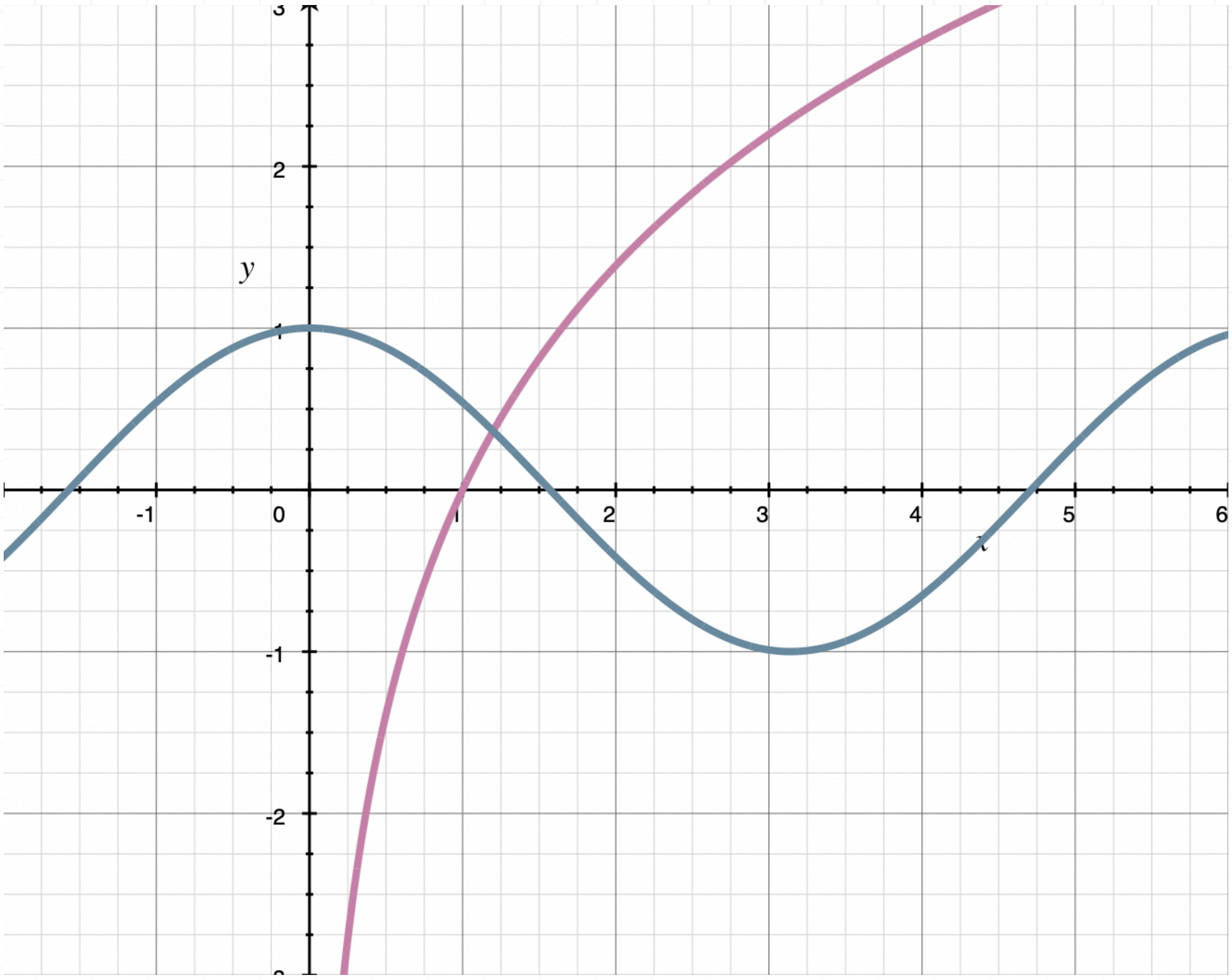
$$f'(x_n) = \frac{2}{x_n} + \sin(x_n)$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2\ln(x_n) - \cos(x_n)}{\frac{2}{x_n} + \sin(x_n)}$$

If we don't know an initial approximation to the solution x_0, we can sketch the graphs of $2\ln x$ and $\cos x$ and use their intersection point to get an estimate of the solution, which we can then use as x_0.



From the graphs, we can see that the two functions intersect one another near $x = 1$, so we can take $x_0 = 1$.

$$x_0 = 1$$

$$x_1 = 1 - \frac{2\ln(1) - \cos(1)}{\frac{2}{1} + \sin(1)} = 1.19$$

$$x_2 = 1.19 - \frac{2\ln(1.19) - \cos(1.19)}{\frac{2}{1.19} + \sin(1.19)} = 1.199$$

$$x_3 = 1.199 - \frac{2\ln(1.199) - \cos(1.199)}{\frac{2}{1.199} + \sin(1.199)} = 1.199$$

Since these last two approximations are identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.199$.

L'HOSPITAL'S RULE

1. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to 0} \frac{2\sqrt{x+4} - 4 - \frac{1}{2}x}{x^2}$$

Solution:

Evaluating the limit as $x \to 0$ gives the indeterminate form $0/0$, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to 0} \frac{\frac{1}{\sqrt{x+4}} - \frac{1}{2}}{2x}$$

But evaluating this $x \to 0$ still gives $0/0$, so we'll apply L'Hospital's rule again.

$$\lim_{x \to 0} \frac{-\frac{1}{2\sqrt{(x+4)^3}}}{2} = \lim_{x \to 0} -\frac{1}{4\sqrt{(x+4)^3}}$$

Then we can evaluate as $x \to 0$.

$$-\frac{1}{4\sqrt{(0+4)^3}} = -\frac{1}{4\sqrt{64}} = -\frac{1}{4(8)} = -\frac{1}{32}$$

■ 2. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{3 + \tan x}$$

Solution:

Evaluating the limit as $x \to \pi/2$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \lim_{x \to \frac{\pi}{2}} \sin x$$

Then we can evaluate as $x \to \pi/2$.

$$\sin \frac{\pi}{2} = 1$$

■ 3. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{\ln x}{4\sqrt{x}}$$

Solution:

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{2}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{\sqrt{x}}{2} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}}$$

Then we can evaluate as $x \to \infty$.

$$\lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0$$

4. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

Solution:

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \infty} \frac{e^x}{2x}$$

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll apply L'Hospital's Rule again.

$$\lim_{x \to \infty} \frac{e^x}{2}$$

Now evaluating the limit gives ∞, which means that the limit does not exist.

■ 5. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to 0^+} \cos x^{\cot x}$$

Solution:

If we try substitution to evaluate at $x = 0^+$, we get an indeterminate form.

$$1^\infty$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to 0^+} \cos x^{\cot x}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to 0^+} \ln(\cos x^{\cot x})$$

$$\ln y = \lim_{x \to 0^+} \cot x \ln(\cos x)$$

$$\ln y = \lim_{x \to 0^+} \frac{\ln(\cos x)}{\tan x}$$

With the limit rewritten, we'll apply L'Hospital's Rule to the fraction.

$$\ln y = \lim_{x \to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{\sec^2 x}$$

$$\ln y = \lim_{x \to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{\frac{1}{\cos^2 x}}$$

$$\ln y = \lim_{x \to 0^+} -\sin x \cos x$$

Evaluate the limit,

$$\ln y = -(0)(1)$$

$$\ln y = 0$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^0$$

$$y = 1$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to 0^+} \cos x^{\cot x}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to 0^+} \cos x^{\cot x} = 1$$

■ 6. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

Solution:

If we try substitution to evaluate at $x = \infty$, we get an indeterminate form.

$$\infty^0$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to \infty} \ln(e^x + 4x)^{\frac{4}{x}}$$

$$\ln y = \lim_{x \to \infty} \frac{4}{x} \ln(e^x + 4x)$$

$$\ln y = \lim_{x \to \infty} \frac{4 \ln(e^x + 4x)}{x}$$

We get the indeterminate form ∞/∞ when we evaluate the limit. With the limit rewritten, we'll apply L'Hospital's Rule to the fraction.

$$\ln y = \lim_{x\to\infty} \frac{\frac{4}{e^x + 4x}(e^x + 4)}{1}$$

$$\ln y = \lim_{x\to\infty} \frac{4(e^x + 4)}{e^x + 4x}$$

$$\ln y = \lim_{x\to\infty} \frac{4e^x + 16}{e^x + 4x}$$

We get an indeterminate form ∞/∞ when we evaluate the limit, so we'll apply L'Hospital's Rule again.

$$\ln y = \lim_{x\to\infty} \frac{4e^x}{e^x + 4}$$

We get an indeterminate form ∞/∞ when we evaluate the limit, so we'll apply L'Hospital's Rule one more time and evaluate the limit,

$$\ln y = \lim_{x\to\infty} \frac{4e^x}{e^x}$$

$$\ln y = \lim_{x\to\infty} 4$$

$$\ln y = 4$$

and then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^4$$

$$y = e^4$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}} = e^4$$

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