① $A(0) = 5^0 + 7 = 8 \equiv 0 \mod 4$ ✓

$A(n+1) = 5^{n+1} + 7 = 5 \cdot 5^n + 7 = \underbrace{4 \cdot 5^n}_{\text{durch 4 teilbar}} + \underbrace{5^n + 7}_{\text{durch 4 teilbar wegen Annahme}} \equiv 0$ ✓

② ⓐ $\frac{1}{17} = 17^{69}$

$17^1 = 17 \ (71) \quad 17^8 \equiv -14 \ (71) \quad 17^{64} \equiv 25 \ (71)$

$17^2 \equiv 5 \ (71) \quad 17^{16} \equiv -17 \ (71) \quad 17^{69} = 17^{64} \cdot 17^4 \cdot 17$

$17^4 \equiv 25 \ (71) \quad 17^{32} \equiv 5 \ (71) \quad \equiv 25 \cdot 25 \cdot 17 \equiv -14 \cdot 17 \equiv \underline{\underline{46 \ (71)}}$

ⓑ $21^1 = 21 \ (43) \quad 21^8 \equiv 21 \ (43) \quad 21^{35} = 21^{32} \cdot 21^2 \cdot 21$

$21^2 \equiv 11 \ (43) \quad 21^{16} \equiv 11 \ (43) \quad \equiv -8 \cdot 11 \cdot 21 \equiv -2 \cdot 21 \equiv \underline{\underline{1 \ (43)}}$

$21^4 \equiv -8 \ (43) \quad 21^{32} \equiv -8 \ (43)$

③ $\tilde{b}_1 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \quad \underline{\underline{\vec{b}_1 = \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}}}$

$\tilde{b}_2 = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} - \left\langle \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \quad \underline{\underline{\vec{b}_2 = \frac{1}{\sqrt{1}}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}}}$

$\tilde{b}_3 = \begin{pmatrix}1\\1\\1\\0\\2\end{pmatrix} - \left\langle \begin{pmatrix}1\\1\\1\\0\\2\end{pmatrix}, \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} - \left\langle \begin{pmatrix}1\\1\\1\\0\\2\end{pmatrix}, \frac{1}{\sqrt{1}}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{1}}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\\2\end{pmatrix} - \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix} \quad \underline{\underline{\vec{b}_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix}}}$

$\tilde{b}_4 = \begin{pmatrix}2\\1\\0\\2\\3\end{pmatrix} - \left\langle \begin{pmatrix}2\\1\\0\\2\\3\end{pmatrix}, \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{1}}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} - \left\langle \begin{pmatrix}2\\1\\0\\2\\3\end{pmatrix}, \frac{1}{\sqrt{1}}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{1}}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} - \left\langle \begin{pmatrix}2\\1\\0\\2\\3\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix}$

$= \begin{pmatrix}2\\1\\0\\2\\3\end{pmatrix} - \begin{pmatrix}2\\0\\0\\0\\0\end{pmatrix} - 0 - \frac{7}{5}\begin{pmatrix}0\\1\\0\\0\\2\end{pmatrix} = \begin{pmatrix}0\\-2/5\\0\\2\\1/5\end{pmatrix} \quad \underline{\underline{\vec{b}_4 = \frac{\sqrt{105}}{21}\begin{pmatrix}0\\-2/5\\0\\2\\1/5\end{pmatrix}}}$

④ $A = \begin{pmatrix}3 & 2 & -4\\-2 & -1 & 3\\2 & 0 & \alpha\end{pmatrix}, \ \vec{b} = \begin{pmatrix}-1\\1\\\beta\end{pmatrix}$

ⓐ $\left| \begin{pmatrix}3 & 2 & -4\\-2 & -1 & 3\\2 & 0 & \alpha\end{pmatrix} \right| = 2 \cdot \begin{vmatrix}2 & -4\\-1 & 3\end{vmatrix} + \alpha \cdot \begin{vmatrix}3 & 2\\-2 & -1\end{vmatrix} = 4 + \alpha \Rightarrow$ regulär für $\{\alpha \in \mathbb{R} \mid \alpha \neq -4\}$

ⓑ $\left(\begin{array}{ccc|c}3 & 2 & -4 & 0\\-2 & -1 & 3 & 0\\2 & 0 & \alpha & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & 1 & 0\\0 & 0 & \alpha+4 & 0\end{array}\right) \Rightarrow$ Lösungsmenge für reguläres $A = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

ⓒ $\left(\begin{array}{ccc|c}3 & 2 & -4 & -1\\-2 & -1 & 3 & 1\\2 & 0 & \alpha & \beta\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 1 & -1 & 0\\0 & 1 & 1 & 1\\0 & 0 & \alpha+4 & \beta+2\end{array}\right) \Rightarrow (\alpha+4) \cdot x_1 = \beta + 2$

$\Rightarrow$ keine Lösung für $\alpha = -4, \beta \neq -2$

ⓓ $\left(\begin{array}{ccc|c}3 & 2 & -4 & -1\\-2 & -1 & 3 & 1\\2 & 0 & 2 & 10\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 0 & 1 & 5\\0 & 1 & -7/2 & -8\\0 & 0 & 1 & 2\end{array}\right)$ $\quad \underline{\underline{x_3 = 2}}$ $\quad \underline{\underline{x_2 = -1}}$ $\quad \underline{\underline{x_1 = 3}}$

⑤ ⓐ $\begin{pmatrix}-5 & 8 & 6\\ -3 & 5 & 3\\ 0 & 0 & 1\end{pmatrix}^2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ ✓

ⓑ $\left(\begin{array}{ccc|c}-6 & 8 & 6 & 0\\ -3 & 4 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & -\frac{3}{4} & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $x_3 = r$, $x_2 = s$, $x_1 = \frac{3}{4}s + r$ $\Rightarrow r\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} + s\cdot\begin{pmatrix}4\\3\\0\end{pmatrix}$

$\left(\begin{array}{ccc|c}-4 & 8 & 6 & 0\\ -3 & 6 & 3 & 0\\ 0 & 0 & 2 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right)$ $x_3 = 0,\ x_2 = r,\ x_1 = 2r$

$\Rightarrow r\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}$

ⓒ $\left\langle \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\3\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle$ $\left(\begin{array}{ccc|c}2 & 4 & 1 & 0\\ 1 & 3 & 0 & 0\\ 1 & 0 & 1 & 0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1 & 2 & \frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & 0\end{array}\right)$ $\Rightarrow x_3 = 0,\ x_2 = 0,\ x_1 = 0$

$\Rightarrow$ linear unabhängig ✓

ⓓ Einfach alle Spaltenvektoren durch Basis darstellen

⑥ $A = \begin{pmatrix}-2 & -1 & 2\\ 4 & 5 & -4\\ 3 & 3 & -3\end{pmatrix}$

$$\left|\begin{pmatrix}-2-\lambda & -1 & 2\\ 4 & 5-\lambda & -4\\ 3 & 3 & -3-\lambda\end{pmatrix}\right| = (-2-\lambda)\cdot\begin{vmatrix}5-\lambda & -4\\ 3 & -3-\lambda\end{vmatrix} + \begin{vmatrix}4 & -4\\ 3 & -3-\lambda\end{vmatrix} + 2\cdot\begin{vmatrix}4 & 5-\lambda\\ 3 & 3\end{vmatrix}$$

$$= (-2-\lambda)\cdot\big((5-\lambda)\cdot(-3-\lambda) + 12\big) - 6 + 2\lambda$$
$$= (-2-\lambda)\cdot(-15-5\lambda+3\lambda+\lambda^2+12) - 6 + 2\lambda$$
$$= (-2-\lambda)\cdot(-3-2\lambda+\lambda^2) - 6 + 2\lambda$$
$$= \cancel{6} + 4\lambda \cancel{-2\lambda^2} + 3\lambda \cancel{+2\lambda^2} - \lambda^3 \cancel{-6} + 2\lambda$$
$$= 7\lambda + 2\lambda - \lambda^3 = 9\lambda - \lambda^3 = \lambda(\lambda^2 - 9) = \lambda\cdot(\lambda-3)\cdot(\lambda+3)$$

$\rightarrow\ 0,\ 3,\ -3$

$\lambda = 0$: $\left(\begin{array}{ccc|c}1 & \frac{1}{2} & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $x_3 = 1,\ x_2 = 0,\ x_1 = 1$ $\Rightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix} - \begin{pmatrix}-2 & -1 & 2\\ 4 & 5 & -4\\ 3 & 3 & -3\end{pmatrix} = \begin{pmatrix}5 & 1 & -2\\ -4 & -2 & 4\\ -3 & -3 & 6\end{pmatrix}$ $\left(\begin{array}{ccc|c}5 & 1 & -2 & 0\\ -4 & -2 & 4 & 0\\ -3 & -3 & 6 & 0\end{array}\right)$ $\left(\begin{array}{ccc|c}1 & 1 & -2 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $x_3 = 1,\ x_2 = 2,\ x_1 = 0$ $\Rightarrow \begin{pmatrix}0\\2\\1\end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix} - \begin{pmatrix}-2 & -1 & 2\\ 4 & 5 & -4\\ 3 & 3 & -3\end{pmatrix} = \begin{pmatrix}-1 & 1 & -2\\ -4 & -8 & 4\\ -3 & -3 & 0\end{pmatrix}$ $\left(\begin{array}{ccc|c}1 & -1 & 2 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ $x_3 = 1,\ x_2 = 1,\ x_1 = -1 \Rightarrow \begin{pmatrix}-1\\1\\1\end{pmatrix}$