Two theorems about the P versus NP problem

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Abstract

Two theorems about the P versus NP problem be proved in this article (1) There exists a language L, that the statement $L \in \mathbf{P}$ is independent of ZFC. (2) There exists a language $L \in \mathbf{NP}$, for any polynomial time deterministic Turing machine M, we cannot prove L is decidable on M.

1   Introduction

First of all, the results of this article neither imply $\mathbf{P} = \mathbf{NP}$ nor $\mathbf{P} \neq \mathbf{NP}$. Let me to explain.

The meaning of the first theorem is that we can define a language L formally in ZFC, and there are two models of ZFC, model A and model B, the sentence $L \in \mathbf{P}$ is true in model A and false in model B. Therefore we cannot prove or disprove $L \in \mathbf{P}$ in ZFC. The method for proving this result is to reduce the independence of $L \in \mathbf{P}$ to a sentence φ which is already known independent of ZFC.

More specifically, first,we can construct a function f with the property " $\forall n f(n) = 1$ is independent of ZFC", that is, there are two model of ZFC, model A and model B, the sentence $\forall n f(n) = 1$ is true in model A and false in model B. Then construct a Turing machine M depending on f, and define the language $L = \{n|\ M(n) = 1\}$. Finally, we prove the function f like a switch, such that $L \in \mathbf{P}$ is true in model A where $\forall n f(n) = 1$ is true, and $L \in \mathbf{P}$ is false in model B where $\forall n f(n) = 1$ is false, therefore the independence of $L \in \mathbf{P}$ be proved.

The meaning of the second theorem is that we can define a language L formally in ZFC, and we can prove $L \in \mathbf{P}$ in ZFC, **but** for any concrete polynomial time Turing machine M, we cannot prove M decide L in ZFC. **The statement is highly confusing, and what does this mean?!**

Let me to explain.

First, it can be proved in ZFC that there exist a language L and a non-ploynomial time Turing machine M which decide the language L, that is

$$L = \{n|\ M(n) = 1\} \text{ and } M \notin \mathbf{P}$$

.

Second, in order to prove $L \in \mathbf{P}$ in ZFC, there are actually two methods, constructive proof and non-constructive proof:

1. Constructive method: Proving a concrete ploynomial time Turing machine M_1 satisfies $\forall n(M(n) = M_1(n))$ in ZFC, i.e., we must construct a concrete Turing machine M_1 and give a proof of the sentence

$$(M_1 \in \mathbf{P}) \wedge \forall n(M(n) = M_1(n))$$

in ZFC, and more, from the soundness of first order logic, we know M_1 statisfies the conditions in every model of ZFC. **But** we will see that such constructive method does not exist, because following reason:

2. The second method is non-constructive. It prove in each model of ZFC, W, there exist a corresponding concrete ploynomial time Turing machine M_W, the sentence $(M_W \in \mathbf{P}) \wedge \forall n(M(n) = M_W(n))$ is true in model W, that is, in each model of ZFC, $L \in \mathbf{P}$ is true, therefore from the Gödel's completeness theorem, we can prove $L \in \mathbf{P}$ in ZFC. **But** we will prove the M_A in model A must be different from the M_B in model B, where the model A and model B are two different models of ZFC as in the first theorem of this article. Hence we cannot use the first constructive method to prove $L \in \mathbf{P}$ in ZFC. That means for any concrete polynomial time Turing machine M, we cannot prove M decide L in ZFC.

Since $\mathbf{P} \subset \mathbf{NP}$ and above explanation, we get the second theorem: there exists a language $L \in \mathbf{NP}$, for any polynomial time deterministic Turing machine M, we cannot prove L is decidable on M.

2   Preliminaries and Notation

This section is devoted to the exposition of basic preliminary material, notations and conventions which be used throughout of this article.

The notion of algorithm can be defined in terms of Turing machines by means of the ChurchTuring thesis, so the sentence "There is an algorithm ..." means "There is a Turing machine to compute ...", and sometimes Turing machine algorithms be described in very high level. If a function or a map is recursive, it means that the function or the map can be computed by a Turing machine.

Let the formal Zermelo-Fraenkel axiomatic set theory is denoted by ZF, and ZFC denotes the theory ZF with the Axiom of Choice, and ω represents the natural number set N in the formal ZFC system.

Definition 2.1. A Turing machine $\mathbf{M}$ is a 5-tuple, $(Q, \Gamma, \delta, q_0, q_{halt})$

Q is a finite set of states, i.e., $\exists i(i \in \omega \wedge \|Q\| = i)$,

Γ is the tape alphabet containing the blank symbol $\sqcup$, and the left end symbol $\triangleright$,

$\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, S, R\}$ is the transition function,
if $\delta(q, \triangleright) = (p, s, b)$, then $(s = \triangleright) \wedge (b = R)$,
if $\delta(q, a) = (p, s, b)$ and $a \neq \triangleright$, then $s \neq \triangleright$,
$q_0 \in Q$ is the start state,
$q_{halt} \in Q$ is the halt state, that is $\forall a \in \Gamma$:
$\delta(q_{halt}, a) = (q_{halt}, a, S)$, and
$\forall q \in Q(\forall a \in \Gamma(\delta(q, a) = \delta(q_{halt}, a)) \to (q = q_{halt}))$.

Unless otherwise indicated, it will always be assumed that the tape alphabet $\Gamma = \{0, 1, \sqcup, \triangleright\}$ throughout this article, and we assume the basic notions and results of mathematical logic, such as formula, sentence, the set of all formulas is recursive ..., etc.

Definition 2.2. (**time complexity**) Let M be a Turing machine that halts on all inputs. The running time or time complexity of M, denoted by t_M, is the function

$$t_M : \omega \to \omega$$

where $t_M(n)$ is the maximum number of steps that M uses on any input of length n.

3   A theorem about P class problems

Let the formal Zermelo-Fraenkel axiomatic set theory is denoted as ZF, and ZFC denotes the theory ZF with the Axiom of Choice, and ω represents the natural number set N in the formal ZFC system. In this section, a theorem about **P** class problems be proved: There exists a language L, that the statement $L \in \mathbf{P}$ is independent of ZFC.

Let D denotes the set of all deducible(provable) expressions of ZFC. Suppose ZFC is a consistent effective formal system, then D is enumerable[10]. It is obvious that the concepts and statements such as "Turing machine", "Turing machine M do not halts on input w", etc..., can be expressed in ZFC.

Lemma 3.1. There exists a Turing machine M_h and an input w_h, and let φ denotes the statement "M_h halts on input w_h", then:

1. φ cannot be proved in ZFC.

2. $\neg\varphi$ cannot be proved in ZFC.

3. There exist a model of ZFC, $\mathcal{A}$, φ is ture in it.

4. There exist a model of ZFC, $\mathcal{B}$, $\neg\varphi$ is ture in it.

hence φ is independent of ZFC.

Proof. Let

$$HALT_{\mathbf{TM}} = \{\langle M, w\rangle | M \text{ is a Turing machine and } M \text{ halts on input } w\}$$

Let Turing machine U as:
$U = $ "On input $\langle M, w\rangle$, where M is a Turing machine and w is a string:

1. Simulate M on input w.

2. At the same time, enumerate the deducible expressions from ZFC, and find whether the expression is "M do not halts on input w", one by one.

3. If M ever enters its halt states on input w, U accept $\langle M, w\rangle$ and halt.

4. If found a deducible expression is "M do not halts on input w", U reject $\langle M, w\rangle$ and halt."

It is well known that $HALT_{\mathbf{TM}}$ is undecidable[5, 6]. So U does not decide $HALT_{\mathbf{TM}}$, therefore there exist $\langle M_h, w_h\rangle$, U never halts on it. That means:

1. From the third item of the above definition of U and the fact that U never halts on $\langle M_h, w_h\rangle$, we can conclude Turing machine M_h never halts on the input w_h, so the statement: "M_h halts on input w_h", is not a deducible expression of ZFC, i.e., φ cannot be proved in ZFC.

2. From the fourth item of the above dedinition of U and the fact that U never halts on $\langle M_h, w_h\rangle$, the statement: "M_h does not halt on input w_h", is not a deducible expression of ZFC, i.e., $\neg\varphi$ cannot be proved in ZFC.

From the Gödel's completeness theorem[9] that means there is a model of ZFC, $\mathcal{A}$, φ is ture in it, and there is another model of ZFC, $\mathcal{B}$, $\neg\varphi$ is ture in $\mathcal{B}$. $\square$

Let strings in $\{0, 1\}^*$ be used to represent the nonnegative integers in the familiar binary notation.

Definition 3.1. $s = a_1 a_2 \ldots a_n \in \{0, 1\}^*$, let $\|s\|$ denotes the length of the string s, (i.e., $\|s\| = n$) and
   $\mathbf{num}(s) = a_1 \cdot 2^{n-1} + a_2 \cdot 2^{n-2} + \ldots + a_n$.

Definition 3.2. For any Turing machine M and an input w, we can define a Turing machine $O_{\langle M, w\rangle}$ on $\{0, 1\}^*$ as:
$O_{\langle M, w\rangle} = $ "On input $s = a_1 a_2 \ldots a_n \in \{0, 1\}^*$:

1. Compute the length of the string s: n $= \|s\|$.

2. Simulate M on input w.

3. if M halts within n steps, return 0, halt; else return 1, halt ."

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Lemma 3.2. For all $O_{\langle M,w \rangle}$ defined as definition 3.2, the following statements can be proved in ZFC:

1. $O_{\langle M,w \rangle} \in \mathbf{P}$

2. $(\forall s(s \in \{0,1\}^*) \rightarrow (O_{\langle M,w \rangle}(s) = 1))$, if and only if "$M$ does not halt on input w".

3. $(\exists m(\|s\| < m \rightarrow O_{\langle M,w \rangle}(s) = 1) \wedge (\|s\| \geq m \rightarrow O_{\langle M,w \rangle}(s) = 0))$, if and only if "$M$ halts at the mth step on the input w".

4. $(\exists m(\|s\| < m \rightarrow O_{\langle M,w \rangle}(s) = 1) \wedge (\|s\| \geq m \rightarrow O_{\langle M,w \rangle}(s) = 0)) \vee (\forall s(O_{\langle M,w \rangle}(s) = 1))$

5. $\|s\| > \|r\| \rightarrow (O_{\langle M,w \rangle}(s) = 1 \rightarrow O_{\langle M,w \rangle}(r) = 1)$

6. $\|s\| = \|r\| \rightarrow (O_{\langle M,w \rangle}(s) = O_{\langle M,w \rangle}(r))$

Proof. It is obviously from the definition of $O_{\langle M,w \rangle}$. $\qquad\square$

From the lemma 3.2, it is easy to get:

Corollary 3.1. For all $\langle M, w \rangle$, where M is a Turing machine and w is a input string:

1. The statement: "M does not halt on input w", can be proved in ZFC, if and only if $(\forall s(s \in \{0,1\}^*) \rightarrow (O_{\langle M,w \rangle}(s) = 1))$ can be proved in ZFC.

2. The statement: "M halts on input w", can be proved in ZFC, if and only if $(\exists m(\|s\| < m \rightarrow O_{\langle M,w \rangle}(s) = 1) \wedge (\|s\| \geq m \rightarrow O_{\langle M,w \rangle}(s) = 0))$ can be proved in ZFC.

3. The statement: "M halts on input w, or, M does not halt on input w" can be proved in ZFC.

Proof. It is easy to see:

The statement **1** corresponds to the statement **2** of lemma 3.2.
The statement **2** corresponds to the statement **3** of lemma 3.2.
The statement **3** is a $\phi \vee \neg\phi$ form, so it is ture, and corresponds to the statement **4** of lemma 3.2.

$\qquad\square$

Lemma 3.3. There exists a Turing machine M_h and an input w_h, let ϕ_1 denotes the statement $(\forall s(s \in \{0,1\}^*) \rightarrow (O_{\langle M_h,w_h \rangle}(s) = 1))$, and ϕ_2 denotes the statement $(\exists m(\|s\| < m \rightarrow O_{\langle M_h,w_h \rangle}(s) = 1) \wedge (\|s\| \geq m \rightarrow O_{\langle M_h,w_h \rangle}(s) = 0))$ then

1. $\phi_1 \vee \phi_2$ can be proved in ZFC,

2. ϕ_1 cannot be proved in ZFC,

3. ϕ_2 cannot be proved in ZFC,

4. ϕ_1 cannot be disproved in ZFC,

5. ϕ_2 cannot be disproved in ZFC,

6. There exist a model of ZFC, $\mathcal{A}$, in which ϕ_1 is true,

7. There exist a model of ZFC, $\mathcal{B}$, in which ϕ_2 is true.

Proof. From the **definition** 3.2, it is easy to see $\phi_1 \vee \phi_2$ can be proved in ZFC. From the lemma 3.1, there exists a Turing machine M_h and an input w_h:

1. "M_h does not halt on input w_h" cannot be proved in ZFC.

2. "M_h halts on input w_h" cannot be proved in ZFC.

3. There exist a model of ZFC, $\mathcal{A}$, "M_h does not halt on input w_h" is ture in it.

4. There exist a model of ZFC, $\mathcal{B}$, "M_h halts on input w_h" is ture in it.

Then from the corollary3.1, the corresponding formulas cannot be proved in ZFC, that is ϕ_1 cannot be proved in ZFC, and ϕ_2 cannot be proved in ZFC. From the definition of $O_{\langle M_h, w_h \rangle}(s)$, it is easy to that $\neg\phi_1 \leftrightarrow \phi_2$. Hence ϕ_1, ϕ_2 cannot be disproved in ZFC.

From the lemma 3.2, "M_h does not halt on input w_h" is ture in $\mathcal{A}$ deduce ϕ_1 is true in $\mathcal{A}$; "M_h halts on input w_h" is ture in $\mathcal{B}$ deduce ϕ_2 is true in $\mathcal{B}$ □

Let abbreviate the formula $(\forall s(s \in \{0,1\}^*) \rightarrow (M(s) = 1))$ to $\forall s(M(s) = 1)$.

Corollary 3.2. There exists a Turing machine M that it halts on every input and the following five formulas can be proved in ZFC:

$$\forall r, s \in \{0,1\}^* \ (\ \|s\| = \|r\| \rightarrow (M(s) = M(r)) \ ) \tag{1}$$

$$\forall r, s \in \{0,1\}^* \ (\ \|s\| > \|r\|) \rightarrow (M(s) = 1 \rightarrow M(r) = 1) \tag{2}$$

$$\forall r, s \in \{0,1\}^* \ (\ \|s\| > \|r\|) \rightarrow (M(r) = 0 \rightarrow M(s) = 0) \tag{3}$$

$$\forall s \in \{0,1\}^* \ (\ M(s) = 0 \vee M(s) = 1) \tag{4}$$

$$(\forall s(s \in \{0,1\}^*) \rightarrow (M(s) = 1)) \vee (\exists m(\|s\| < m \rightarrow M(s) = 1) \wedge (\|s\| \geq m \rightarrow M(s) = 0)) \tag{5}$$

but the formula $\forall s(M(s) = 1)$ is independent of ZFC, i.e., it cannot be proved in ZFC and its negation is also unprovable in ZFC.

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Proof. Let the machine M is the $O_{\langle M_h, w_h \rangle}$ as mentioned in **lemma** 3.3. Hence it is obvious. $\qquad\square$

Lemma 3.4. Let $L_1 \subseteq \{0,1\}^*, L_2 \subseteq \{0,1\}^*$.
If $\exists m \forall s (s \in \{0,1\}^*) \to (\|s\| \geq m \to (s \in L_1 \leftrightarrow s \in L_2))$,
then $L_1 \in \mathbf{P} \to L_2 \in \mathbf{P}$.

Proof. This lemma says that if there exist a number m, any string $s \in \{0,1\}^*$ with $\|s\| \geq m$ imply: "$s \in L_1$ if and only if $s \in L_2$", then from the $L_1 \in \mathbf{P}$, we can deduce $L_2 \in \mathbf{P}$.

Let M is the polynomial time deterministic Turing machine which decide L_1. Obviously, from the lemma's condition, if $\|s\| \geq m$, M can be used to decide whether $s \in L_2$, else if $\|s\| < m$, seaching table method can be used to decide whether $s \in L_2$. Hence $L_2 \in \mathbf{P}$.
$\qquad\square$

Definition 3.3. Let $L_1 \subseteq \{0,1\}^*, L_2 \subseteq \{0,1\}^*$, we define $L_1 \simeq L_2$ as
$\exists m \forall s (s \in \{0,1\}^*) \to (\|s\| \geq m \to (s \in L_1 \leftrightarrow s \in L_2))$.

It is easy to see that the "$\simeq$" is an equivalence relation, and the following result is easily derived from the lemma 3.4.

Corollary 3.3. Let $L_1 \subseteq \{0,1\}^*, L_2 \subseteq \{0,1\}^*$.
If $L_1 \simeq L_2$ then $L_1 \in \mathbf{P} \leftrightarrow L_2 \in \mathbf{P}$, and $L_1 \in \mathbf{NP} \leftrightarrow L_2 \in \mathbf{NP}$.

Proof. Obviously. $\qquad\square$

Definition 3.4. Let M_1 is a Turing machine, and w is an input string of M_1, M_2 is a Turing machine on $\{0,1\}^*$ and return 0 or 1, then $Q_{\langle M_1, M_2, w \rangle}$ is a Turing machine defined on $\{0,1\}^*$ as:
$Q_{\langle M_1, M_2, w \rangle} = $ "On input $s \in \{0,1\}^*$:

1. Use the machine $O_{\langle M_1, w \rangle}$ to compute on s.

2. If $O_{\langle M_1, w \rangle}(s) = 1$, return 1, and halt.

3. Else if $O_{\langle M_1, w \rangle}(s) = 0$, use the machine M_2 to compute on s and return as M_2 return, and halt."

$$L_{\langle M_1, M_2, w \rangle} = \{s \in \{0,1\}^* | Q_{\langle M_1, M_2, w \rangle}(s) = 1\}$$

Theorem 3.1. There exists a language $L \subseteq \{0,1\}^*$, that the statement $L \in \mathbf{P}$ is independent of ZFC.

Proof. It is easy to see that there is a language $L_1 \subseteq \{0,1\}^*, L_1 \notin \mathbf{P}$.

Therefore we suppose M_1 is a Turing machine which decide the language L_1. Because $L_1 \notin \mathbf{P}$, M_1 is not in the polynomial time class.

Let $\langle M_h, w_h \rangle$ is the machine and string pair as in the lemma 3.3 that is:

1. $(\forall s (s \in \{0,1\}^*) \to (O_{\langle M_h, w_h \rangle}(s) = 1))$ cannot be proved in ZFC.

2. $(\exists m(\|s\| < m \to O_{\langle M_h, w_h \rangle}(s) = 1) \land (\|s\| \geq m \to O_{\langle M_h, w_h \rangle}(s) = 0))$ cannot be proved in ZFC.

3. There exist a model of ZFC, $\mathcal{A}$, in which $(\forall s(s \in \{0,1\}^*) \to (O_{\langle M_h, w_h \rangle}(s) = 1))$ is true.

4. There exist a model of ZFC, $\mathcal{B}$, in which $(\exists m(\|s\| < m \to O_{\langle M_h, w_h \rangle}(s) = 1) \land (\|s\| \geq m \to O_{\langle M_h, w_h \rangle}(s) = 0))$ is true.

Then we can define a Turing machine $M_2 = Q_{\langle M_h, M_1, w_h \rangle}$, that is:
$M_2 = $ "On input $s \in \{0,1\}^*$:

1. Use the machine $O_{\langle M_h, w_h \rangle}$ to compute on s.

2. If $O_{\langle M_h, w_h \rangle}(s) = 1$, return 1, and halt.

3. Else if $O_{\langle M_h, w_h \rangle}(s) = 0$, use the machine M_1 to compute on s and return as M_1 return, and halt."

Let $L = \{s \in \{0,1\}^* | M_2(s) = 1\}$

In model $\mathcal{A}$, because $(\forall s(s \in \{0,1\}^*) \to (O_{\langle M_h, w_h \rangle}(s) = 1))$ is true, the machine M_2 never execute the 3rd instruction as above definition, and from the lemma 3.2, $O_{\langle M_h, w_h \rangle} \in \mathbf{P}$, therefore $L \in \mathbf{P}$ is true in model $\mathcal{A}$.

In model $\mathcal{B}$, because $(\exists m(\|s\| < m \to O_{\langle M_h, w_h \rangle}(s) = 1) \land (\|s\| \geq m \to O_{\langle M_h, w_h \rangle}(s) = 0))$ is true, on the $s \in \{0,1\}^* \land \|s\| \geq m$ the machine M_2 must execute the 3rd instruction as above definition: use the machine M_1 to compute on s and return as M_1 return.

That is $\|s\| \geq m \to M_2(s) = M_1(s)$. As suppose M_1 decide the language L_1, so $L \simeq L_1$.

Since the $L_1 \notin \mathbf{P}$ as above supposed and the corollary 3.3, $L \notin \mathbf{P}$ is true in model $\mathcal{B}$.

Hence $L \in \mathbf{P}$ is true in model $\mathcal{A}$, and $L \notin \mathbf{P}$ is true in model $\mathcal{B}$. That means $L \in \mathbf{P}$ is independent of ZFC.

□

Remark 3.1. The theorem 3.1 be proved by model method, indeed, it can be proved directly: if $L \in \mathbf{P}$ can be proved in ZFC, then from the $L_1 \notin \mathbf{P}$ and the definition of M_2, we can deduce $(\forall s(s \in \{0,1\}^*) \to (O_{\langle M_h, w_h \rangle}(s) = 1))$ in ZFC, contradicting the property of the selected $\langle M_h, w_h \rangle$, and if $L \notin \mathbf{P}$ can be proved in ZFC, also result in a contradiction. Therefore $L \in \mathbf{P}$ cannot be proved in ZFC and $L \notin \mathbf{P}$ cannot be proved in ZFC either. That is $L \in \mathbf{P}$ is independent of ZFC.

Remark 3.2. We can enumerate triplets $\langle M_1, M_2, w \rangle$ as the definition 3.4. For some corresponding languages $L_{\langle M_1, M_2, w \rangle}$, the question: "$L_{\langle M_1, M_2, w \rangle} \in \mathbf{P}$?" are easy, others are difficult. For example, let
M_2 is arbitrary non-$\mathbf{P}$ class Turing machine, and
$M_1 = $ "On input $s \in \{0,1\}^*$:

1. If $\|s\| > 1$, halt.

2. If $\|s\| = 1$, searching a even number n satisfy $n > 4$ and n is not the sum of two odd primes, do not halt until found."

Then the question: $L_{\langle M_1, M_2, 1 \rangle} \in \mathbf{P}$? is very difficult, indeed it is equivalent to Goldbach conjecture. But the theorem 3.1 says there exists language L, the question "$L \in \mathbf{P}$?" is even more difficult that we cannot prove or disprove it, it is independent of ZFC.

Remark 3.3. Let $L \in \mathbf{P}$ is independent of ZFC. It is obvious that $(L \in \mathbf{P}) \vee (L \notin \mathbf{P})$ is true, but no one can decide whether $(L \in \mathbf{P})$ or $(L \notin \mathbf{P})$, the language L is in a "correlated" state. More interesting quantum phenomena of computation will be study later, but now, another interesting theorem about $\mathbf{P}$ versus $\mathbf{NP}$ will be given in the following section.

4   A theorem about P versus NP problem

In this section, a theorem about $\mathbf{P}$ versus $\mathbf{NP}$ problem be proved: there exists a language $L \in \mathbf{NP}$, but for any polynomial time deterministic Turing machine M, we cannot prove M decide L in ZFC system.

There is an encoding system that can encode any $\langle M_1, M_2, P_1, P_2 \rangle$ as a natural number, where M_1, M_2 are Turing machines, P_1, P_2 are formula sequences, and there is a corresponding decode algorithm which can decode a number n to a corresponding quadruplet $\langle M_1, M_2, P_1, P_2 \rangle$ if n is an encoding number of it, else if n is not an encoding number of any quadruplet $\langle M_1, M_2, P_1, P_2 \rangle$, return 0. Think of encryption and decryption algorithms in modern cryptography, this assumption is obviously true. We fixed one encoding system throughout this section, denote the encode algorithm by T_{encode} and denote the corresponding decode algorithm by T_{decode}.

Definition 4.1. Let M_0 be a Turing machine on $\{0,1\}^*$ as in the **corollary** 3.2. Now we define a language L_{M_0}, such that $L_{M_0} \subset \omega$. Let
$L_{M_0} = \{n \in \omega |\ n = T_{encode}(\langle M_1, M_2, P_1, P_2 \rangle),\ M_1, M_2$ are Turing machines, and P_1 is a proof of $M_1 \in \mathbf{P}$ in ZFC, P_2 is a proof of the following statement in ZFC: M_2 compute a function: $\omega \to \omega$ and $M_0(M_2(n)) = M_1(n)\}$
It is obvious that L_{M_0} is a decidable language, let H_{M_0} is a Turing machine which decide the language L_{M_0}. Now we define a Turing machine T_{M_0} on ω as following:
$T_{M_0} = $ "On input n:

1. if $n = 0$, return 0 and halts, else:

2. use H_{M_0} to decide whether $n \in L_{M_0}$,

3. if $n \notin L_{M_0}$, return $1 + T_{M_0}(n-1)$ and halts, else:

4. if $n \in L_{M_0}$, use algorithm T_{decode} to decode the number n into $\langle M_1, M_2, P_1, P_2 \rangle$, and let T_{M_0} return $\mathbf{max}\{1 + T_{M_0}(n-1),\ 1 + M_2(n)\}$ and halts.

Then the property of T_{M_0} be described in following lemma:

Lemma 4.1. Let M_0, T_{M_0} as the above **definition** 4.1 and let the function $f(n) = M_0(T_{M_0}(n))$ then:

1. T_{M_0} compute a function from ω to ω,

2. $T_{M_0}(n-1) < T_{M_0}(n)$ can be proved in ZFC,

3. $\forall n\ (\ f(n) = 0 \vee f(n) = 1)$ can be proved in ZFC,

4. $\forall m, n\ (m > n) \rightarrow (f(m) = 1 \rightarrow f(n) = 1)$ can be proved in ZFC,

5. $\forall m, n\ (m > n) \rightarrow (f(n) = 0 \rightarrow f(m) = 0)$ can be proved in ZFC,

6. $(\forall n(f(n) = 1)) \vee (\exists m(\ (n < m \rightarrow f(n) = 1) \wedge (n \geq m \rightarrow f(n) = 0)))$ can be proved in ZFC,

7. the formula $\forall n(f(n) = 1)$ is independent of ZFC,

8. for any Turing machine M, and any formula sequences P_1 and P_2, if $n = T_{encode}(\langle M, T_{M_0}, P_1, P_2 \rangle)$, then $n \notin L_{M_0}$, that means if $M \in \mathbf{P}$ can be proved in ZFC (i.e., the statement "M is a polynomial-time Turing machine" can be proved in ZFC) then the statement:

$$f(n) = M(n)$$

cannot be proved in ZFC.

Proof. From the **definition** 4.1 and the **corollary** 3.2, it is easy to verify the statements from (1) to (7). Now we prove the statement (8) by contradiction.

Assume that there exist a Turing machine M and two formula sequences P_1 and P_2 which satisfy

$$n = T_{encode}(\langle M, T_{M_0}, P_1, P_2 \rangle),\ n \in L_{M_0}$$

Then from the **definition** 4.1,

$$T_{M_0}(n) = \mathbf{max}\{1 + T_{M_0}(n-1),\ 1 + T_{M_0}(n)\} \geq 1 + T_{M_0}(n) > T_{M_0}(n)$$

Thus we obtain a contradiction. $\qquad\square$

Theorem 4.1. There exist a language L which satisfies the following statements:

1. $L \in \mathbf{NP}$,

2. For any Turing machine T_M, if we can prove the statement "T_M decide the language L" in ZFC, then the statement "T_M is a polynomial-time Turing machine " cannot be proved in ZFC, and vice versa.

Proof. If $\mathbf{P} \neq \mathbf{NP}$, the theorem is obvious.

So we just need to show this theorem still holds under the assumption $\mathbf{P} = \mathbf{NP}$. Though $L \in \mathbf{P}$ under this assumption, we cannot prove any concrete polynomial-time Turing machine decide it in ZFC. Indeed, despite whether or not $\mathbf{P} = \mathbf{NP}$ there exist a language L satisfies this theorem.

Let $L_0 \subset \omega$ be a language, $L_0 \in \mathbf{NP}$. Since the assumption $\mathbf{P} = \mathbf{NP}$, we get $\mathbf{NP} = \mathbf{coNP}$, therefore $\overline{L_0} \in \mathbf{NP}$ where $\overline{L_0} = \omega \setminus L_0$. If L_0 satisfies the two statements of this theorem, the theorem be proved.

Else let there is a Turing machine T_{L_0}, we can prove the statement "T_{L_0} decide the language L_0" in ZFC, and $T_{L_0} \in \mathbf{P}$ can be proved in ZFC. It is easy to see that $1 - T_{L_0}$ decide $\overline{L_0}$.

Let $f(n)$ be defined as in the **lemma** 4.1. Now we define a Turing machine U on ω as following:

$U = $ "On input n:

1. First, compute $f(n)$,

2. If $f(n) = 1$, compute $T_{L_0}(n)$, return $T_{L_0}(n)$, and halts,

3. Else if $f(n) = 0$, compute $1 - T_{L_0}(n)$, return $1 - T_{L_0}(n)$, and halts.

Let L_U denote the language $\{n|\ U(n) = 1\}$. It is not hard to see that

$$U(n) = T_{L_0}(n) \text{ if and only if } f(n) = 1$$

thus we can prove the following formula in ZFC:

$$U(n) = T_{L_0}(n) \leftrightarrow f(n) = 1 \tag{6}$$

From the **lemma** 4.1, $f(n)$ satisfies:

$$(\forall n(f(n) = 1)) \vee (\exists m(\ (n < m \to f(n) = 1) \wedge (n \geq m \to f(n) = 0)))$$

Therefore from the definition of U we can prove the following formula in ZFC:

$$(\forall n(U(n) = T_{L_0}(n))) \vee (\exists m(\ (n < m \to U(n) = T_{L_0}(n)) \wedge (n \geq m \to U(n) = 1 - T_{L_0}(n))))$$

From the **definition** 3.3 and above formula, we get $(L_U \simeq L_0) \vee (L_U \simeq \overline{L_0})$. Under the assumption $\mathbf{P} = \mathbf{NP}$, we know that $L_0 \in \mathbf{NP}$ and $\overline{L_0} \in \mathbf{NP}$. Since the **corollary** 3.3, the formula $L_U \in \mathbf{NP}$ can be proved in ZFC. Though $L_U \in \mathbf{P}$ under the assumption $\mathbf{P} = \mathbf{NP}$, we now prove L_U satisfies the statement 2 of this theorem by contradiction.

Assume that there exist a concrete Turing machine T_M that we can prove the statement "T_M decide the language L_U" in ZFC and "T_M is a polynomial-time

Turing machine " can be proved in ZFC, too, i.e., we assumed the following two formulas be proved in ZFC:

$$\forall n(U(n) = T_M(n)) \tag{7}$$

$$T_M \in \mathbf{P} \tag{8}$$

Now, we define a polynomial-time Turing machine M compute $f(n)$, which will contradict the statement 8 of the **lemma** 4.1:

$M =$ "On input n:

1. Compute $T_M(n)$, (using polynomial-time as assumed)

2. Compute $T_{L_0}(n)$, (using polynomial-time as assumed)

3. Compare $T_M(n)$ to $T_{L_0}(n)$, if they are equal, return 1, and halts,

4. Else, return 0, and halts.

From the definition of M, and $T_M \in \mathbf{P}$, $T_{L_0} \in \mathbf{P}$ are all proved in ZFC as assumptions, it is easy to see the following formulas can be proved in ZFC:

$$M \in \mathbf{P} \tag{9}$$

$$\forall n\ (\ M(n) = 0 \vee M(n) = 1) \tag{10}$$

$$M(n) = 1 \leftrightarrow T_M(n) = T_{L_0}(n) \tag{11}$$

Therefore from the formula (7) and formula (11), the following formula can be proved in ZFC:

$$M(n) = 1 \leftrightarrow U(n) = T_{L_0}(n) \tag{12}$$

Then from the formula (6) and formula (12), the following formula can be proved in ZFC:

$$M(n) = 1 \leftrightarrow f(n) = 1 \tag{13}$$

Hence from the property of $f(n)$ in the **lemma** 4.1, formula (10) and (13) the following formula can be proved in ZFC:

$$f(n) = M(n) \tag{14}$$

Therefore, we can prove the formula (9): $M \in \mathbf{P}$ in ZFC, and the formula (14): $f(n) = M(n)$ can be proved in ZFC, too. That violates the 8th statement of the **lemma** 4.1 and is thus a contradiction.

So the assumption: the two statements "T_M decide the language L_U" and "T_M is a polynomial-time Turing machine " can be proved all in ZFC, is wrong. Thus we proved L_U satisfies the statement 2 of this theorem.

$\square$

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