Recurrence Relations

1)

$$e_n := \text{edges in a complete graph, where n is the number of nodes}$$

$$e_0 := 0$$

$$e_n := e_{n-1} + (n-1)$$

Backward Substitution:

$$\begin{aligned}
&e_n = e_{n-1} + (n-1) \\
&e_n = (e_{n-2} + (n-2)) + (n-1) \\
&e_n = ((e_{n-3} + (n-3)) + (n-2)) + (n-1) \\
&\quad \dots \\
&e_n = e_{n-n} + (n-n) + (n-(n-1)) + \dots + (n-2) + (n-1) \\
&e_n = e_0 + 0 + 1 + 2 + 3 + \dots + (n-2) + (n-1) \\
&e_n = \sum_{k=0}^{n-1} k \\
&e_n = \frac{n(n-1)}{2}
\end{aligned}$$

Forward Substitution:

$$\begin{aligned}
&e_1 = e_0 + (1-1) = 0 + 0 \\
&e_2 = e_1 + (2-1) = 0 + 1 \\
&e_3 = e_2 + (3-1) = 1 + 2 \\
&e_4 = e_3 + (4-1) = 3 + 3 \\
&e_5 = e_4 + (5-1) = 6 + 4 \\
&e_6 = e_5 + (6-1) = 10 + 5 \\
&\quad \dots \\
&e_n = e_{n-1} + (n-1) \\
&\quad = \sum_{k=0}^{n-1} k \\
&\quad = \frac{n(n-1)}{2}
\end{aligned}$$

Statement: $e_n = \frac{n(n-1)}{2}$

Proof: (not necessary)

Base Case: $n = 0$

$$e_0 = 0 = \frac{0 \cdot (-1)}{2}$$

Hypothesis: $e_n = \frac{n(n-1)}{2}$

Induction Step:

$$\begin{aligned}
e_{n+1} &= e_n + ((n+1) - 1) \\
e_{n+1} &= \frac{n(n-1)}{2} + n \\
e_{n+1} &= \frac{n(n-1) + 2n}{2} \\
e_{n+1} &= \frac{n^2 - n + 2n}{2} \\
e_{n+1} &= \frac{n^2 + n}{2} \\
e_{n+1} &= \frac{(n+1)n}{2}
\end{aligned}$$

2)

$$d_n := \text{diameter in a square-grid } (n := \#\text{nodes}, k := \#\text{nodes in a single line})$$

$$d_1 := 0$$

$$d_{k^2} := d_{(k-1)^2} + 2$$

Why $+2$?

Imagine you know the diameter for $k-1$, that is the shortest path from the bottom left to the top right corner. Now you increase k by adding a line to the top and right of he grid. To reach the new top right corner now, you go to the old top right corner and then take one step to the right and one to the top. Thus, there are two more nodes in this shortest path.

Backward Substitution:

$$
\begin{aligned}
d_{k^2} &= d_{(k-1)^2} + 2 \\
d_{k^2} &= d_{(k-2)^2} + 2 + 2 \\
d_{k^2} &= d_{(k-3)^2} + 2 + 2 + 2 \\
&\dots \\
d_{k^2} &= d_{(k-(k-1))^2} + 2(k-2) \\
d_{k^2} &= 2(k-2) \\
d_n &= 2(\sqrt{n}-1)
\end{aligned}
$$

Forward Substitution:

$$
\begin{aligned}
d_{2^2} &= d_{1^2} + 2 = 2 \\
d_{3^2} &= d_{2^2} + 2 = 4 \\
d_{4^2} &= d_{3^2} + 2 = 6 \\
d_{5^2} &= d_{4^2} + 2 = 8 \\
d_{6^2} &= d_{5^2} + 2 = 10 \\
&\dots \\
d_{k^2} &= 2(k-1)
\end{aligned}
$$

Statement: $d_{k^2} = 2(k-1)$

Unecessary: Recursion for n_k, showing that $n = k^2$

$$n_0 := 0$$

$$n_k := n_{k-1} + (k-1) + k$$

Backward Substitution:

$$
\begin{aligned}
n_k &= n_{k-1} + (k-1) + k \\
d_k &= (n_{k-2} + (k-2) + (k-1)) + (k-1) + k \\
d_k &= ((n_{k-3} + (k-3) + (k-2)) + (k-2) + (k-1)) + (k-1) + k \\
&\dots \\
n_k &= n_{k-k} + (k-k) + 2(k-k+1) + 2(k-k+2) + \dots + 2(k-1) + k \\
d_k &= (2 \cdot 0) + (2 \cdot 1) + (2 \cdot 2) + \dots + (2 \cdot (k-1)) + k \\
n_k &= \sum_{i=0}^{k-1} (2 \cdot i) + k \\
n_k &= k(k-1) + k \\
n_k &= k^2 - k + k \\
n_k &= k^2
\end{aligned}
$$

3)

$$c_h := \#\text{connections in an } h\text{-dimensional hyper-cube}$$

$$c_0 := 0$$

$$c_h := 2 \cdot c_{h-1} + v_{h-1}$$

where

$$v_h := \#\text{vertices in an } h\text{-dimensional hyper-cube}$$

It is obvious, that

$$v_h = 2^h$$

Thus

$$c_h = 2 \cdot c_{h-1} + 2^{h-1}$$

Backward Substitution:

$$
\begin{aligned}
c_h &= 2 \cdot c_{h-1} + 2^{h-1} \\
c_h &= 2 \cdot (2 \cdot c_{h-2} + 2^{h-2}) + 2^{h-1} \\
c_h &= 2 \cdot (2 \cdot (2 \cdot c_{h-3} + 2^{h-3}) + 2^{h-2}) + 2^{h-1} \\
c_h &= 2^3 \cdot c_{h-3} + 3 \cdot 2^{h-1} \\
&\dots \\
c_h &= 2^h \cdot c_{h-h} + h \cdot 2^{h-1} \\
c_h &= 2^h \cdot 0 + h \cdot 2^{h-1} \\
c_h &= h \cdot 2^{h-1}
\end{aligned}
$$

Statement: $c_h = h \cdot 2^{h-1}$