Решение ур-я колебаний струны на отрезке

Метод разделения переменных (метод Фурье)

$u_{tt} = a^2 u_{xx}$ $\quad t > 0, \quad 0 < x < l$

н.у. $u(0, x) = \varphi(x)$

$u_t(0, x) = \psi(x)$

г.у. $u(t, 0) = 0$

$u(t, l) = 0$

Решение ищем в виде

$u(t,x) = X(x)\cdot T(t)$ /* ищем нетривиальные решения */

$T''X = a^2 T X''$

н.у. $T(0)X(x) = \varphi(x)$

$T'(0)X(x) = \psi(x)$

г.у. $T(t)\cdot X(0) = 0$ $\Rightarrow$ $X(0) = 0$

$T(t)\cdot X(l) = 0$ $\quad X(l) = 0$

ур-е:

$T''X = a^2 T X''$ $\quad |:a^2 T X$

$\underbrace{\frac{T''}{a^2 T}}_{t=t_0} = \underbrace{\frac{X''}{X}}_{\forall x\in(0,l)} = C$

$\forall t>0 \quad x = x_0$

$\begin{cases} X'' = \overset{?}{C}X \\ X(0) = 0 \\ X(l) = 0 \end{cases}$ Задача - Штурма - Лиувилля

$T'' = a^2 C T$

1) $C>0 \quad C = \lambda^2$

$X'' = \lambda^2 X$ $\quad \mu^2 = \lambda^2$

$X(0) = 0$ $\quad \mu = \pm\lambda$

$X(l) = 0$ $\quad X = Ae^{\lambda x} + Be^{-\lambda x}$

$\left.\begin{array}{l} X(0) = A + B = 0 \\ X(l) = Ae^{\lambda l} + Be^{-\lambda l} = 0 \end{array}\right\}$ $\quad A = -B$

$B(e^{-\lambda l} - e^{\lambda l}) = 0 \Rightarrow B = 0$

$A = 0 \quad \varnothing$

2) $C = 0$

$X'' = 0 \Rightarrow X = Ax + B$

$X(0) = 0$

$X(\ell) = 0$

$X(0) = B = 0$

$X(\ell) = A\ell = 0 \Rightarrow A = 0$ $\qquad \varnothing$

3) $C < 0 \qquad C = -\lambda^2$

$X'' = -\lambda^2 X \qquad \mu^2 = -\lambda^2$

$X(0) = 0 \qquad \mu = \pm i\lambda$

$X(\ell) = 0$

$X = A\sin\lambda x + \cancel{B\cos\lambda x}$

$X(0) = B = 0$

$X(\ell) = A\sin\lambda\ell = 0$

$\sin\lambda\ell = 0 \Rightarrow \lambda\ell = \pi n, \quad n = 1, 2, \ldots$

$\lambda_n = \frac{\pi n}{\ell}$ - собств. зн-е задачи Ш-Л

$X_n = \underline{\underline{\sin\lambda_n x}}$ - собств. ф-ии з. Ш-Л.

$\{X_n\}$ - образуют ортогональную сист. ф-ий

$$\int_0^\ell X_n \cdot X_k \, dx = \begin{cases} 0, & n \neq k \\ \frac{\ell}{2}, & n = k \end{cases}$$

$n \neq k$

$$\int_0^\ell \sin\lambda_n x \cdot \sin\lambda_k x \, dx = \int_0^\ell \sin\frac{\pi n x}{\ell} \sin\frac{\pi k x}{\ell} \, dx =$$

$$= /* \ \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \ */ =$$

$$= \frac{1}{2}\int_0^\ell \cos\frac{\pi x}{\ell}(n-k) - \cos\frac{\pi x}{\ell}(n+k) \, dx =$$

$$= \frac{1}{2}\left[\frac{l}{\pi(n-k)} \sin\frac{\pi x}{l}(n-k)\Big|_0^l - \frac{l}{\pi(n+k)} \sin\frac{\pi x}{l}(n+k)\Big|_0^l\right] =$$

$$= \frac{1}{2}\left[\frac{l}{\pi(n-k)} \underbrace{\sin \pi(n-k)}_{=0} - \frac{l}{\pi(n+k)} \underbrace{\sin \pi(n+k)}_{=0}\right] = 0$$

$n=k$

$$\int_0^l \sin^2 \lambda_k x \, dx = \int_0^l \sin^2 \frac{\pi k x}{l} dx = /* \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha) */$$

$$= \frac{1}{2}\int_0^l 1 - \cos\frac{2\pi k x}{l} dx = \frac{1}{2}\left[x\Big|_0^l - \frac{l}{2\pi k} \sin\frac{2\pi k x}{l}\Big|_0^l\right] =$$

$$= \frac{1}{2}\left[l - \frac{l}{2\pi k} \overbrace{\sin\frac{2\pi k \cancel{l}}{\cancel{l}}}^{=0}\right] = \frac{l}{2}$$

$$u_{tt} = a^2 u_{xx} \qquad t>0; \qquad x \in (0, l)$$

н.у. $u(0,x) = \varphi(x)$

$u_t(0,x) = \psi(x)$

г.у $u(t,0) = 0$

$u(t,l) = 0$

$$u(t,x) = T(t)X(x)$$

$X'' = CX$, $C = -\lambda^2$ $\qquad X_n(x) = \sin\lambda_n x$ ✓

$X(0) = 0$ $\Rightarrow$ $\lambda_n = \frac{\pi n}{l}$ $\qquad n = 1, 2 \dots$

$X(l) = 0$

$$\int_0^l X_k(x) X_n(x)\, dx = \begin{cases} 0 & n \neq k \\ \frac{l}{2} & n = k \end{cases}$$

$$T'' = a^2 C T \Rightarrow T_n'' = -a^2 \lambda_n^2 T_n$$

$$\mu_n^2 = -a^2 \lambda_n^2 \Rightarrow \mu_n = \pm i a \lambda_n$$

$T_n = D_n \sin a\lambda_n t + E_n \cos a\lambda_n t$ ✓

$$\underline{u(t,x)} = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \underline{\sum_{n=1}^{\infty} (D_n \sin a\lambda_n t + E_n \cos a\lambda_n t) \cdot \sin\lambda_n x}$$

— общее решение исходной задачи

н.у.

$$u(0,x) = \sum_{n=1}^{\infty} E_n \sin\lambda_n x = \varphi(x) \qquad \Big| \int_0^l \sin\lambda_k x\, dx$$

$$\int_0^l \sum_{n=1}^{\infty} E_n \sin\lambda_n x \cdot \sin\lambda_k x\, dx = \int_0^l \varphi(x) \sin\lambda_k x\, dx$$

$$\underbrace{\mathcal{E}_k = \frac{2}{\ell}\int_0^{\ell} \varphi(x)\sin\lambda_k x\, dx} \qquad k = 1, 2, \dots$$

$$u_t(0,x) = \psi(x)$$

$$u_t(t,x) = \sum_{n=1}^{\infty}\left(\mathcal{D}_n a\lambda_n \cos a\lambda_n t - \mathcal{E}_n a\lambda_n \sin a\lambda_n t\right)\sin\lambda_n x$$

$$u_t(0,x) = \sum_{n=1}^{\infty}\mathcal{D}_n a\lambda_n \sin\lambda_n x = \psi(x) \qquad \Big| \int_0^{\ell}\sin\lambda_k x\, dx$$

$$\sum_{n=1}^{\infty}\int_0^{\ell}\mathcal{D}_n a\lambda_n \underbrace{\sin\lambda_n x}\cdot\underbrace{\sin\lambda_k x}\, dx = \int_0^{\ell}\psi(x)\sin\lambda_k x\, dx$$

$$\mathcal{D}_k a\lambda_k \frac{\ell}{2} = \int_0^{\ell}\psi(x)\sin\lambda_k x\, dx \qquad k = 1, 2, \dots$$

$$\underbrace{\mathcal{D}_k = \frac{2}{a\lambda_k \ell}\int_0^{\ell}\psi(x)\sin\lambda_k x\, dx}$$

Решения з. Штурма - Лиувилля

$X_n'' = -\lambda_n^2 X_n \qquad x \in (0, l)$

ЛГ \ ПГ	I рода	II рода
I рода	$X_n = \sin \lambda_n x$ $\lambda_n = \frac{\pi n}{l}$; $n = 1, 2, \ldots$	?
II рода	$X_n = \cos \lambda_n x$ $\lambda_n = \frac{\pi}{2l} + \frac{\pi n}{l}$, $n = 0, 1, 2 \ldots$	$X_n = \cos \lambda_n x$ $\lambda_n = \frac{\pi n}{l}$ $\quad n = 0, 1, \ldots$