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Интегральное преобразование Лапласа

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-\lambda t} dt = F(\lambda) \qquad \lambda = a + ib$$

$$\mathcal{L}^{-1}[F(\lambda)] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\lambda) e^{\lambda t} d\lambda = f(t)$$

Преобразование Лапласа можно применить если:

1) $f(t)$ определена на $t \in [0, \infty)$
2) $\forall A$ $f(t)$ - кусочно-непрерывная $t \in [0, A]$
3) $\exists M, c, T > 0$, что $|f(t)| \le M e^{ct}$ $\forall t > T$

$\Rightarrow \mathcal{L}[f(t)]$ определено $\forall \operatorname{Re}\lambda = a > c$

Св-ва.

1. Взаимно-однозначное
2. Линейное

$f(t);\ g(t);\ a, b$ - const

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

3. Теорема запаздывания

$$\mathcal{L}[f(t-c) \cdot H(t-c)] = e^{-\lambda c} F(\lambda) \qquad c - \text{const}$$

$$H(t-c) = \begin{cases} 0, & 0 < t < c \\ 1, & t \ge c \end{cases}$$

► $\mathcal{L}[f(t-c) H(t-c)] = \int_0^{\infty} f(t-c) H(t-c) e^{-\lambda t} dt =$

$= \int_c^{\infty} f(t-c) e^{-\lambda t} dt = [\xi = t - c] = \int_0^{\infty} f(\xi) e^{-\lambda \xi - \lambda c} d\xi =$

$$= \underbrace{\int_0^\infty f(\xi) e^{-\lambda\xi} \cdot e^{-\lambda c} d\xi}_{F(\lambda)} = e^{-\lambda c} F(\lambda)$$ ◄

4. Теорема смещения

$\mathcal{L}[e^{ct} f(t)] = F(\lambda - c)$ $\quad$ c - const

► $\mathcal{L}[e^{ct} f(t)] = \int_0^\infty e^{ct} f(t) e^{-\lambda t} dt = \int_0^\infty f(t) e^{-t(\lambda - c)} dt =$

$= F(\lambda - c)$ ◄

5 Свертка

$f(t); \; g(t)$

$\mathcal{L}[f(t)] = F(\lambda); \quad \mathcal{L}[g(t)] = G(\lambda)$

$\mathcal{L}^{-1}[F(\lambda) \cdot G(\lambda)] = f(t) * g(t)$ - свертка

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t g(\tau) f(t-\tau) d\tau$$

Д-ем, что

$\mathcal{L}[f(t) * g(t)] = F(\lambda) \cdot G(\lambda)$

► $\mathcal{L}[f * g] = \int_0^\infty (f * g) e^{-\lambda t} dt = \int_0^\infty e^{-\lambda t} \left[\int_0^t f(\tau) g(t-\tau) d\tau \right] dt =$

$=$ /* $H(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$ - зеркально отраженная ф-я Хевисайда */

$$= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty f(\tau) g(t-\tau) H(t-\tau) d\tau \right] dt =$$

$$\int_0^\infty f(\tau)\left[\int_0^\infty g(t-\tau)H(t-\tau)e^{-\lambda t}\,dt\right]d\tau =$$

$$\mathcal{L}[g(t-\tau)H(t-\tau)] = e^{-\lambda\tau}G(\lambda) \quad \text{т. запаздыв.}$$

$$= \int_0^\infty f(\tau)e^{-\lambda\tau}G(\lambda)\,d\tau = F(\lambda)\cdot G(\lambda) \quad \blacktriangleleft$$

6. преобразование производных

$u(t,x)$

$$\mathcal{L}[u(t,x)] = \int_0^\infty u(t,x)e^{-\lambda t}\,dt = U(\lambda,x)$$

/* $u_{tt} = a^2 u_{xx}$
$u_t = a^2 u_{xx}$ */

u_x:

$$\mathcal{L}[u_x(t,x)] = \int_0^\infty u_x(t,x)e^{-\lambda t}\,dt = \frac{\partial}{\partial x}\left(\int_0^\infty u(t,x)e^{-\lambda t}\,dt\right) =$$

$$= \frac{\partial}{\partial x}U(\lambda,x) = U_x(\lambda,x)$$

$$\mathcal{L}[u_{xx}(t,x)] = \int_0^\infty u_{xx}(t,x)e^{-\lambda t}\,dt = \frac{\partial^2}{\partial x^2}\left(\int_0^\infty u(t,x)e^{-\lambda t}\,dt\right) =$$

$$= U_{xx}(\lambda,x)$$

u_t:

$$\mathcal{L}[u_t(t,x)] = \int_0^\infty u_t(t,x)e^{-\lambda t}\,dt = \int_0^\infty e^{-\lambda t}\,du(t,x) =$$

$$= e^{-\lambda t} u(t,x) \Big|_{t=0}^{t\to\infty} - \int_0^\infty u(t,x)\, d e^{-\lambda t} =$$

0 при $t \to \infty$

в силу дост. усл. $\exists$ инт. преобр Лапласа

$$= -u(0,x) + \lambda \int_0^\infty u(t,x) e^{-\lambda t} dt = -u(0,x) + \lambda U(\lambda, x)$$

$$\mathcal{L}[u_{tt}(t,x)] = \int_0^\infty u_{tt}(t,x) e^{-\lambda t} dt = \int_0^\infty e^{-\lambda t} d u_t(t,x) =$$

$$= e^{-\lambda t} u_t(t,x) \Big|_{t=0}^{t\to\infty} - \int_0^\infty u_t(t,x) d e^{-\lambda t} =$$

$$= -u_t(0,x) + \lambda \int_0^\infty u_t(t,x) e^{-\lambda t} dt =$$

$$= -u_t(0,x) + \lambda \int_0^\infty e^{-\lambda t} d u(t,x) =$$

$$= -u_t(0,x) + \lambda e^{-\lambda t} u(t,x) \Big|_{t=0}^{t\to\infty} - \lambda \int_0^\infty u(t,x) d e^{-\lambda t} =$$

$$= -u_t(0,x) - \lambda u(0,x) + \lambda^2 \int_0^\infty u(t,x) e^{-\lambda t} dt =$$

$$= -u_t(0,x) - \lambda u(0,x) + \lambda^2 U(\lambda, x)$$

таблица преобразования производных

$$\mathcal{L}[u_x(t,x)] = U_x(\lambda, x) \qquad U(\lambda,x) = \int_0^\infty u(t,x)e^{-\lambda t}\,dt$$

$$\mathcal{L}[u_{xx}(t,x)] = U_{xx}(\lambda, x)$$

$$\mathcal{L}[u_t(t,x)] = -u(0,x) + \lambda U(\lambda, x)$$

$$\mathcal{L}[u_{tt}(t,x)] = -u_t(0,x) - \lambda u(0,x) + \lambda^2 U(\lambda, x)$$