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Решение неоднородных уравнений для бесконечной струны

$$u_{tt} = a^2 u_{xx} + f(t,x) \qquad t>0 \qquad x\in\mathbb{R} \qquad (1)$$

$$u(0,x)=0$$
$$u_t(0,x)=0$$

Сформулируем вспомогательную задачу

$w(t,x,\tau)$

$$w_{tt} = a^2 w_{xx} \qquad t>\tau \qquad x\in\mathbb{R} \qquad (2)$$

$$w(t=\tau, x,\tau)=0$$
$$w_t(t=\tau,x,\tau) = f(\tau,x)$$

Докажем, что решение (1) определяется по формуле

$$u(t,x) = \int_0^t w(t,x,\tau)\,d\tau, \quad \text{где } w - \text{решение (2)}$$

/* $I(\alpha) = \int\limits_{a(\alpha)}^{b(\alpha)} f(\alpha,\xi)\,d\xi$

$$I'_\alpha = \int\limits_{a(\alpha)}^{b(\alpha)} f'_\alpha(\alpha,\xi)\,d\xi + f(\alpha,b(\alpha))\,b'_\alpha(\alpha) - f(\alpha,a(\alpha))\,a'_\alpha(\alpha)$$
*/

$$\underline{u_{xx}} = \int_0^t w_{xx}(t,x,\tau)\,d\tau$$

$$u_t = \int_0^t w_t(t,x,\tau)\,d\tau + \underbrace{w(t,x,\tau=t)\cdot 1}_{=0 \text{ из нач.усл (2)}} - 0 =$$

$$= \int_0^t w_t(t,x,\tau)\,d\tau$$

$$\underline{u_{tt}} = \int_0^t w_{tt}(t,x,\tau)\,d\tau + \underbrace{w_t(t,x,\tau=t)}_{=f(t,x)}\cdot 1 - 0$$

$$= \int_0^t w_{tt}(t,x,\tau)\,d\tau + f(t,x)$$

$$\underbrace{\int_0^t w_{tt}(t,x,\tau)\,d\tau + \cancel{f(t,x)}}_{u_{tt}} = a^2 \underbrace{\int_0^t w_{xx}(t,x,\tau)\,d\tau}_{u_{xx}} + \cancel{f(t,x)}$$

$$\left[\begin{array}{ll} w_{tt}(t,x,\tau) = a^2 w_{xx}(t,x,\tau) & t > \tau,\quad x \in \mathbb{R} \\ w(t=\tau, x, \tau) = 0 & \\ w_t(t=\tau, x, \tau) = f(\tau, x) & \end{array}\right.$$

$$t^* = t - \tau$$

$$w_{t^*t^*}(t^*, x, \tau) = a^2 w_{xx}(t^*, x, \tau) \qquad t^* > 0 \qquad x \in \mathbb{R}$$

$$w(t^*=0, x, \tau) = 0 \qquad = \varphi(x)$$

$$w_{t^*}(t^*=0, x, \tau) = f(\tau, x) \qquad = \psi(x)$$

$$w(t^*, x, \tau) = \frac{1}{2a} \int_{x-at^*}^{x+at^*} f(\tau, \xi)\,d\xi \;;$$

$$w(t, x, \tau) = \frac{1}{2a} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau, \xi)\,d\xi$$

$$\boxed{u(t,x) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau, \xi)\,d\xi\,d\tau}$$

Усложняем

$$u_{tt} = a^2 u_{xx} + f(t,x) \qquad t>0, \quad x \in R$$

$$u(0,x) = \varphi(x)$$

$$u_t(0,x) = \psi(x)$$

$$u = v + w$$

$$v_{tt} = a^2 v_{xx} \qquad t>0, \quad x \in R$$

$$v(0,x) = \varphi(x)$$

$$v_t(0,x) = \psi(x)$$

$$\Downarrow$$

$$v(t,x) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi$$

$$w_{tt} = a^2 w_{xx} + f(t,x) \qquad t>0, \quad x \in R$$

$$w(0,x) = 0$$

$$w_t(0,x) = 0$$

$$\Downarrow$$

$$w(t,x) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\, d\xi\, d\tau$$