Лекция 15 15.12.2020

Уравнение Лапласа в сферической системе координат.

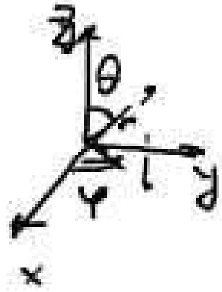
$\Delta u = 0$ $\quad u(r, \theta, \varphi)$ $\quad 0 \le \theta \le \pi$

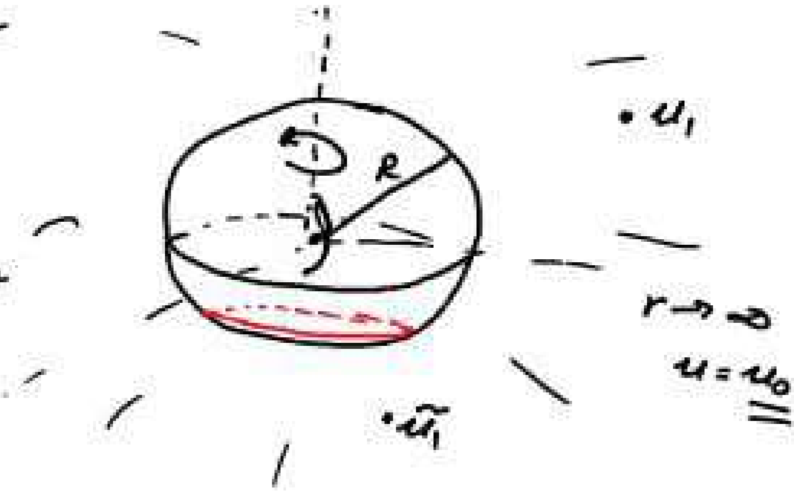
$0 \le \varphi < 2\pi$

$u(R, \theta, \varphi) = g(\theta, \varphi)$

а). внутренняя задача
$0 \le r < R$

б) внешняя задача
$r > R$





$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

I а) u - не зависит от θ, φ

$u(R, \theta, \varphi) = g$ $\quad 0 \le r < R$, $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0$$

$$r^2 \frac{\partial u}{\partial r} = C \Rightarrow \frac{du}{dr} = \frac{C}{r^2} \Rightarrow du = \frac{C\,dr}{r^2}$$

$u = \frac{A}{r} + B$ $\quad r = 0$ – особая т.

$A \equiv 0 \Rightarrow$ для ограничен. реш

$u(r, \theta, \varphi) = B$

$u(R, \theta, \varphi) = B = g \Rightarrow$ Ответ $u(r, \theta, \varphi) = g$

б) $r > R$

$0 \le \theta \le 0$

$0 \le \varphi < 2\pi$

$u = \frac{A}{r} + B$

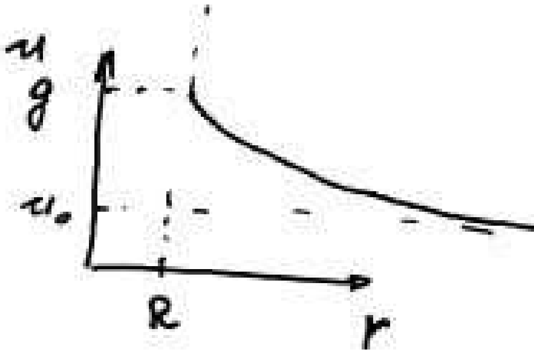
$u(r \to \infty, \theta, \varphi) = u_0$

$r \to \infty \quad B = u_0$

$u(R, \theta, \varphi) = \frac{A}{R} + u_0 = g$

$A = (g - u_0) R$

$u(r, \theta, \varphi) = \frac{(g - u_0) R}{r} + u_0$



II а) $u(r, \theta)$ не зависит от φ

$u(R, \theta, \varphi) = g(\theta)$

$0 \le r < R$

$0 \le \theta \le \pi$

$0 \le \varphi < 2\pi$

$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0$

$u(r, \theta) = X(r) \cdot Y(\theta)$

$Y \frac{\partial}{\partial r}(r^2 X') + \frac{X}{\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, Y') = 0 \quad \Big| : XY$

$\frac{1}{X} \frac{\partial}{\partial r}(r^2 X') = -\frac{1}{Y \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, Y') = C$

$r = r_0 \qquad \forall \theta \in [0, \pi]$

$\forall r \in [0, R) \qquad \theta = \theta_0$

$$\frac{\partial}{\partial r}\left(r^2 X'\right) = CX$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, Y'\right) = -CY$$

Решаем задачу для Y

$x = \cos\theta$

$Y'_\theta = Y'_x x'_\theta = Y'_x(-\sin\theta)$

$(\ )'_\theta = (\ )'_x x'_\theta = (\ )'_x(-\sin\theta)$

$$\frac{1}{\sin\theta}(-\sin\theta)\frac{\partial}{\partial x}\left(\sin\theta(-\sin\theta)Y'_x\right) = -CY$$

$$+\frac{\partial}{\partial x}\left((1-x^2)Y'_x\right) = -CY$$

$(1-x^2)Y'' - 2xY' + cY = 0$ – Классический вид ур-я Лежандра

$C = n(n+1) \quad n = 0, 1, \ldots$

Y_n – решения ур-я Лежандра

$Y_n = P_n(x)$ – полиномы Лежандра

$n=0 \quad P_0(x) = 1$

$n=1 \quad P_1(x) = x$

$n=2 \quad P_2(x) = \dfrac{3x^2-1}{2}$

$n=3 \quad P_3(x) = \dfrac{5x^3-3x}{2}$

$n=k \quad P_k(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

Полиномы Лежандра ортогональны на $[-1, 1]$

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = \begin{cases} 0 & n \neq k \\ \dfrac{2}{2k+1} & n = k \end{cases}$$

$$\int_0^{\pi} P_n(\cos\theta)\, P_k(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0 & n \neq k \\ \frac{2}{2k+1} & n = k \end{cases}$$

$$Y_n(\theta) = P_n(\cos\theta) \qquad n = 0, 1, \ldots$$

Решаем ур-е для X

$$\frac{\partial}{\partial r}\left(r^2 X' \right) = cX$$

$$c = n(n+1)$$

$$r^2 X'' + 2rX' - n(n+1)X = 0 \quad \text{- ур-е Эйлера}$$

$n \neq 0$

$$X \sim r^{\alpha}$$

$$r^2 \alpha(\alpha-1) r^{\alpha-2} + 2r\, r^{\alpha-1} \alpha - n(n+1) r^{\alpha} = 0$$

$$\alpha^2 - \alpha + 2\alpha - n(n+1) = 0$$

$$\alpha^2 + \alpha - n(n+1) = 0$$

$$\alpha = \frac{-1 \pm \sqrt{1 + 4n^2 + 4n}}{2} = \frac{-1 \pm (2n+1)}{2}$$

$$\alpha_1 = n; \quad \alpha_2 = -n-1$$

$$X_n = A_n r^n + B_n r^{-n-1} \qquad n = 1, 2, \ldots$$

$n = 0$

$$r^2 X'' + 2rX' = 0$$

$$V = X'$$

$$r^2 V' + 2rV = 0$$

$$\frac{dV}{dr} = -\frac{2r}{r^2} V$$

$$\frac{dV}{V} = -\frac{2dr}{r}$$

$$\ln V = \ln r^{-2} + \tilde{B}$$

$$V = \frac{D}{r^2}$$

$$X' = \frac{D}{r^2} \Rightarrow \quad \frac{dX}{dr} = \frac{D}{r^2} \Rightarrow \int dX = \int \frac{D\,dr}{r^2}$$

$$X_0 = \frac{C}{r} + F$$

т. о.

$$X_n = A_n r^n + B_n r^{-n-1} \qquad n = 0, 1, 2, \ldots\ldots$$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(A_n r^n + B_n r^{-n-1}\right)$$

<u>внутренняя задача</u>

$r = 0$ - особая т. $\quad B_n = 0$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \quad \checkmark$$

$$u(R, \theta, \varphi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) = g(\theta)$$

$$\Big|\int_0^{\pi} P_k(\cos\theta)\sin\theta d\theta \qquad k = 0, 1, \ldots$$

$$A_k R^k \frac{2}{2k+1} = \int_0^{\pi} g(\theta) P_k(\cos\theta)\sin\theta\, d\theta$$

$$A_k = \frac{2k+1}{2R^k} \int_0^{\pi} g(\theta) P_k(\cos\theta)\sin\theta\, d\theta \quad \checkmark$$

Внешняя задача

$A_n = 0$ для ограниченности решения при $r \to \infty$

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta)$$ ✓

$$u(R,\theta,\varphi) = \sum_{n=0}^{\infty} B_n R^{-n-1} P_n(\cos\theta) = g(\theta) \quad \Big| \int_0^{\pi} P_k(\cos\theta)\sin\theta\, d\theta$$

$$B_k R^{-k-1} \frac{2}{2k+1} = \int_0^{\pi} g(\theta) P_k(\cos\theta)\sin\theta\, d\theta$$

$$B_k = \frac{2k+1}{2} R^{k+1} \int_0^{\pi} g(\theta) P_k(\cos\theta)\sin\theta\, d\theta$$ ✓