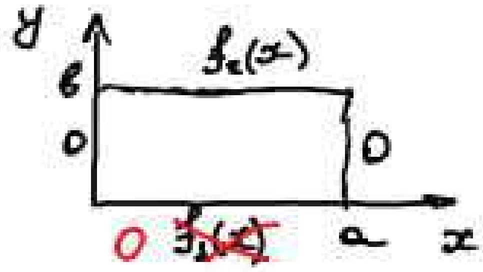
Решение уравнения Лапласа в прямоугольной области



$\Delta u = 0$

$u(x,y)$

$0 < x < a$

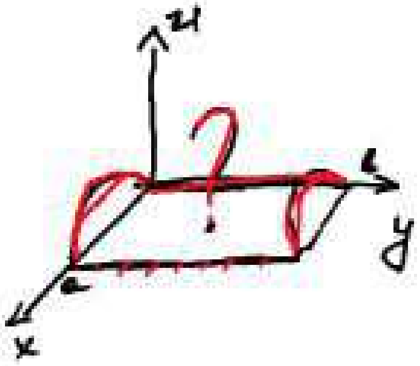
$0 < y < в$

$u_x(x,0) = \cancel{f_1(x)}\ 0$

$u_x(a,y) = 0$

$u(x,в) = f_2(x)$

$u(0,y) = 0$



$u(x,y) = X(x)\,Y(y)$

$\Delta u = 0 \;\Rightarrow\; u_{xx} + u_{yy} = 0$

$X''Y + Y''X = 0 \quad | : XY$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = C$$

x^* $\quad \forall y \in (0, b)$

$\forall x \in (0, a)$ $\quad y^*$

$X'' = CX$

$Y'' = -CY$

$\xi = C$!!!

Г.У

$X(x) \cdot Y(0) = f_1(x)$ $\quad 0$

$X(a) \cdot Y(y) = 0 \Rightarrow X(a) = 0$

$X(x)\, Y(b) = f_2(x)$ $\quad 0$

$X(0)\, Y(y) = 0 \Rightarrow X(0) = 0$

$Y(0) = 0$

$Y(b) = 0$

$$\begin{matrix} X'' = CX \\ X(a) = 0 \\ X(0) = 0 \end{matrix} \Rightarrow C = -\lambda^2 \Rightarrow \begin{matrix} X_n = \sin \lambda_n x \\ \lambda_n = \frac{\pi n}{a}\ ; \quad n = 1, 2, \ldots \end{matrix}$$

Задача для Y

$Y_n'' = \lambda_n^2 Y_n$

$\mu_n^2 = \lambda_n^2 \Rightarrow \mu_n = \pm \lambda_n$

$Y_n = A_n e^{-\lambda_n y} + B_n e^{\lambda_n y}$

$$\underline{u(x,y)} = \sum_{n=1}^{\infty} X_n Y_n = \underline{\sum_{n=1}^{\infty} \sin \lambda_n x \left(A_n e^{-\lambda_n y} + B_n e^{\lambda_n y} \right)}$$

$u(x, 0) = f_1(x)$

$u(x, b) = f_2(x)$

$$u(x, 0) = \sum_{n=1}^{\infty} \sin \lambda_n x\, (A_n + B_n) = f_1(x) \qquad \Big| \int_0^a \sin \lambda_k x\, dx$$

$$u(x,b)=\sum_{n=1}^{\infty}\sin\lambda_n x\left(A_n e^{-\lambda_n b}+B_n e^{\lambda_n b}\right)=f_2(x)$$

$$\begin{cases} \frac{a}{2}(A_k+B_k)=\int_0^a f_1(x)\sin\lambda_k x\,dx \\ \frac{a}{2}\left(A_k e^{-\lambda_k b}+B_k e^{\lambda_k b}\right)=\int_0^a f_2(x)\sin\lambda_k x\,dx \end{cases} \quad \forall k=1,2,\dots$$

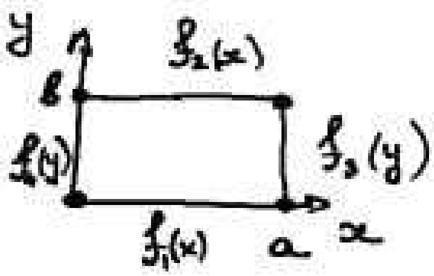
$\Downarrow$

Решение алгебраич. системы определяет A_k, B_k $\forall k=1,2,\dots$

Ответ $u(x,y)=\sum_{n=1}^{\infty}\sin\lambda_n x\left(A_n e^{-\lambda_n y}+B_n e^{\lambda_n y}\right)$

$\lambda_n=\frac{\pi n}{a}$; $n=1,2,\dots$

Общий случай



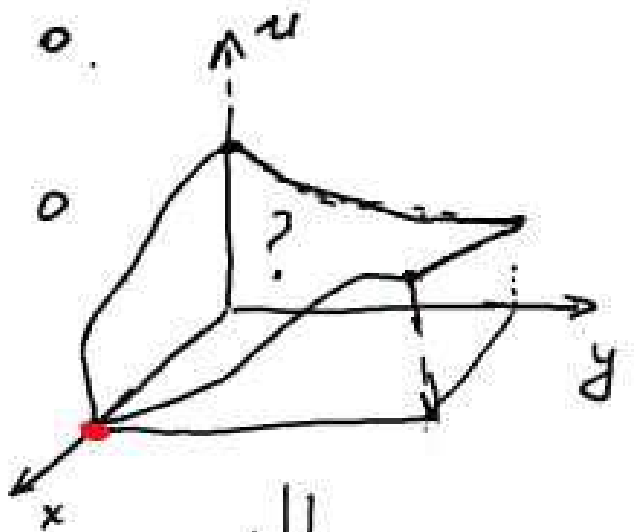
$\Delta u=0$ $\quad 0<x<a$, $\quad 0<y<b$

$u(x,0)=f_1(x)$ $\quad 0$

$u(a,y)=f_3(y)$ $\quad 0$

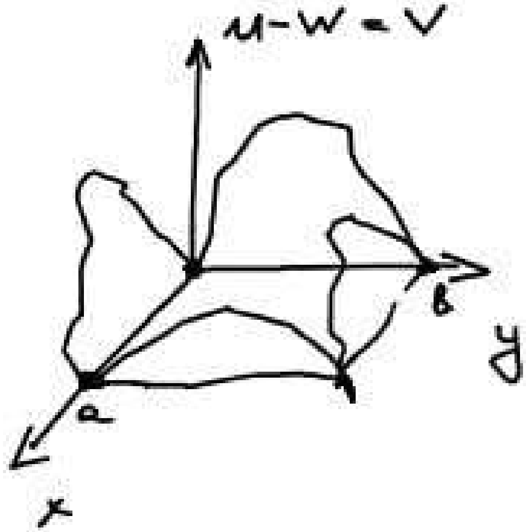
$u(x,b)=f_2(x)$ $\quad 0$

$u(0,y)=f_4(y)$ $\quad 0$



$\Downarrow$

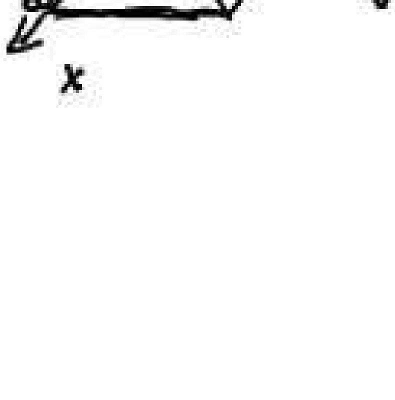
$u=V+W$

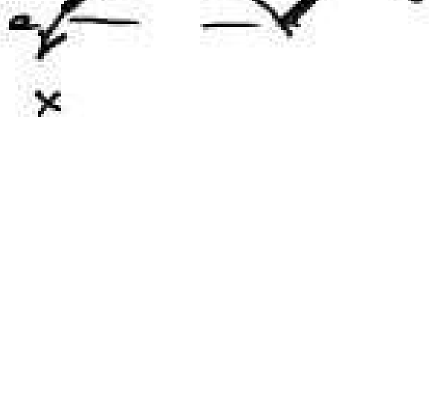


$V = V_1 + V_2$

$\Delta V_1 = 0$

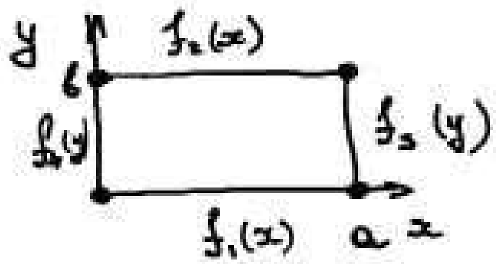
$\Delta V_2 = 0$





Лекция 13 01.12.2020

$\Delta u = 0$ $\quad 0 < x < a$, $\quad 0 < y < b$



$$u(x,0) = f_1(x)$$
$$u(a,y) = f_3(y)$$
$$u(x,b) = f_2(x)$$
$$u(0,y) = f_4(y)$$

$$u(x,y) = u_1(x,y) + Ax + By + Cxy + D$$

A, B, C, D – определяем из условий

$(0,0)$: $A \cdot 0 + B \cdot 0 + C \cdot 0 + D = f_1(0) \Rightarrow D = f_1(0)$

$(0,b)$: $A \cdot 0 + B \cdot b + C \cdot 0 + D = f_4(b) \Rightarrow Bb + D = f_4(b)$

(a,b): $A \cdot a + B \cdot b + Cab + D = f_2(a)$

$(a,0)$: $A \cdot a + B \cdot 0 + C \cdot 0 + D = f_3(0) \Rightarrow Aa + D = f_3(0)$.

$\Downarrow$

A, B, C, D – известные

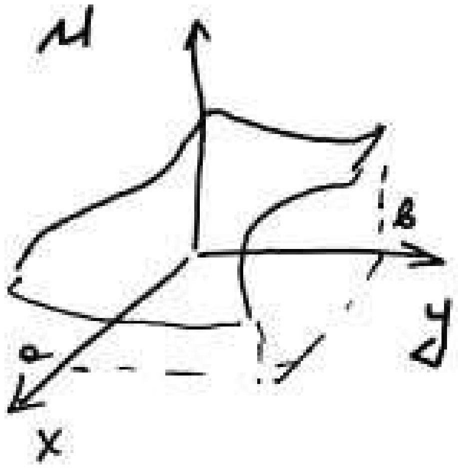
$\Delta u = 0 \Rightarrow \Delta u_1 = 0$ $\quad 0 < x < a$, $\quad 0 < y < b$

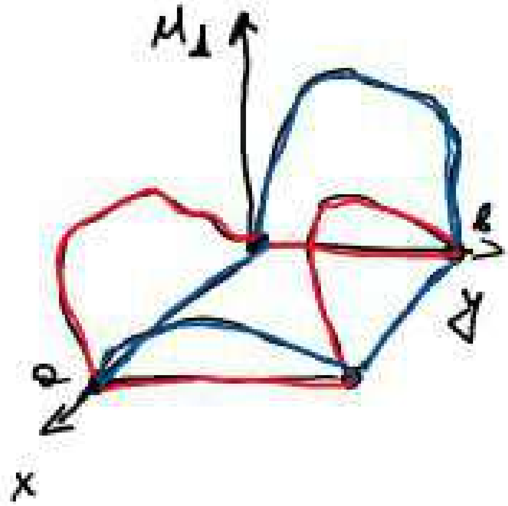
$u_1(x,0) + Ax + D = f_1(x)$

$u_1(a,y) + Aa + By + Cay + D = f_3(y)$

$u_1(x,b) + Ax + Bb + Cbx + D = f_2(x)$

$u_1(0,y) + By + D = f_4(y)$





$$u_1(x,y) = u_2(x,y) + u_3(x,y)$$

$\Delta u_2 = 0$ $\quad 0<x<a$, $0<y<b$

$u_2(x,0) + Ax + D = f_1(x)$

$u_2(a,y) = 0$

$u_2(x,b) + Ax + Bb + Cbx + D = f_2(x)$

$u_2(0,y) = 0$

$\Delta u_3 = 0$ $\quad 0<x<a$, $0<y<b$

$u_3(x,0) = 0$

$u_3(a,y) + Aa + By + Cay + D = f_3(y)$

$u_3(x,b) = 0$

$u_3(0,y) + By + D = f_4(y)$

Задачи для u_2 и u_3 решать умеем

Ответ: $u = u_2 + u_3 + Ax + By + Cxy + D$