1 Numerical Fokker-Planck

We look to solve the following partial differential equation with a robust yet simple numerical scheme,

$$\frac{\partial}{\partial t}p(x,v,t) = -v\frac{\partial}{\partial x}p(x,v,t) - \left[\frac{\mathrm{d}}{\mathrm{d}x}U(x)\right]\frac{\partial}{\partial v}p(x,v,t) + \frac{\partial}{\partial v}\left(vp(x,v,t) + \frac{\partial}{\partial v}p(x,v,t)\right) \tag{17}$$

$$= -v\frac{\partial}{\partial x}p(x,v,t) + p(x,v,t) + \left\{v - \frac{\mathrm{d}}{\mathrm{d}x}U(x)\right\}\frac{\partial}{\partial v}p(x,v,t) + \frac{\partial^2}{\partial v^2}p(x,v,t), \tag{18}$$

we denote $p^n_{i,j}$ an approximation of $p(x_i, v_j, t_n)$ where $t_n = n\Delta t$, $x_i = -x_{\max} + i\Delta x$ and $v_j = -v_{\max} + j\Delta v$ with in our case $x_{\max} = 8$ and $v_{\max} = 6$ and where Δt, Δx and Δv denote the time, position and velocity discretization grid respectively. We can rewrite Eq. (18) as

$$\frac{\partial}{\partial t}p(x,v,t) + v\frac{\partial}{\partial x}p(x,v,t) + \left[\frac{\mathrm{d}}{\mathrm{d}x}U(x)\right]\frac{\partial}{\partial v}p(x,v,t) = \frac{\partial}{\partial v}\left[vp(x,v,t) + \frac{\partial}{\partial v}p(x,v,t)\right] \tag{19}$$

The left-hand side of this equation can be seen as the transport whereas the right-hand side can be seen as the Fokker-Planck operator (Liouvillian).

Transport part

To approximate the transport, we use a decentered scheme dependent of the velocity sign.

Example : To approximate

$$\frac{\partial}{\partial t}p(x,v,t) + v\frac{\partial}{\partial x}p(x,v,t) = 0 \tag{20}$$

we get,

$$\frac{p^{n+1}_{i,j} - p^n_{i,j}}{\Delta t} + \frac{v_j + |v_j|}{2\Delta x}(p^n_{i,j} - p^n_{i-1,j}) + \frac{v_j - |v_j|}{2\Delta x}(p^n_{i+1,j} - p^n_{i,j}) = 0. \tag{21}$$

This scheme can be understood as follow : in function of v_j's sign, we choose an approximation of $\partial_x p(x_j)$ that is decentered in the good way, that's to say that we look for the information where it comes from :

- If $v_j > 0$ we search the information in x_{i-1} to approximate its derivative in x_i.
- If $v_j < 0$ we search the information in x_{i+1} to approximate its derivative in x_i.

Thus, the whole scheme for the transport part can be written as

$$\begin{aligned}\frac{p^{n+1}_{i,j} - p^n_{i,j}}{\Delta t} &+ \frac{v_j + |v_j|}{2\Delta x}(p^n_{i,j} - p^n_{i-1,j}) + \frac{v_j - |v_j|}{2\Delta x}(p^n_{i+1,j} - p^n_{i,j}) \\ &+ \frac{U'_i + |U'_i|}{2\Delta v}(p^n_{i,j} - p^n_{i,j-1}) + \frac{U'_i - |U'_i|}{2\Delta v}(p^n_{i,j+1} - p^n_{i,j}) = 0.\end{aligned} \tag{22}$$

Fokker-Planck operator part

$$\mathscr{L}p(x,v,t) \equiv \frac{\partial}{\partial v}\left[vp(x,v,t) + \frac{\partial}{\partial v}p(x,v,t)\right] = \frac{\partial}{\partial v}\left[M\frac{\partial}{\partial v}\left(\frac{p(x,v,t)}{M}\right)\right] \tag{23}$$

with

$$M(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right). \tag{24}$$

Indeed,

$$\frac{\partial}{\partial v} M = -vM \tag{25}$$

Thus,

$$M \frac{\partial}{\partial v}\left(\frac{p(x,v,t)}{M}\right) = \frac{\partial}{\partial v} p(x,v,t) - \frac{p(x,v,t)}{M} \frac{\partial}{\partial v} M = vp(x,v,t) + \frac{\partial}{\partial v} p(x,v,t) \tag{26}$$

and leading us to the result of Eq. (23).

We write,

$$D^+(p_j) = \frac{p_{j+1} - p_j}{\Delta v} \qquad \text{and} \qquad D^-(p_j) = \frac{p_j - p_{j-1}}{\Delta v} \tag{27}$$

We will also write $M_j = M(v_j)$. Thus we can write,

$$\mathscr{L}p(x_i, v_j, t_n) \equiv \mathscr{L}_j \approx \frac{1}{2}\left[D^+\left(M\left\{D^- \frac{p}{M}\right\}_j\right)_j\right] + \frac{1}{2}\left[D^-\left(M\left\{D^+ \frac{p}{M}\right\}_j\right)_j\right] \tag{28}$$

Which we can write differently,

$$\mathscr{L}_j \approx \frac{1}{2\Delta v^2}\left[p_{j-1}\left(1 + \frac{M_j}{M_{j-1}}\right) + p_j\left(-2 - \frac{M_{j+1} + M_{j-1}}{M_j}\right) + p_{j+1}\left(1 + \frac{M_j}{M_{j+1}}\right)\right] \tag{29}$$

Combination

Now that we have explicited both the transport and the Fokker-Planck operator part, we can combine them,

$$\begin{aligned} \frac{p_{i,j}^{n+1} - p_{i,j}^n}{\Delta t} &+ \frac{v_j + |v_j|}{2\Delta x}(p_{i,j}^n - p_{i-1,j}^n) + \frac{v_j - |v_j|}{2\Delta x}(p_{i+1,j}^n - p_{i,j}^n) \\ &+ \frac{U_i' + |U_i'|}{2\Delta v}(p_{i,j}^n - p_{i,j-1}^n) + \frac{U_i' - |U_i'|}{2\Delta v}(p_{i,j+1}^n - p_{i,j}^n) = \mathscr{L}_j, \end{aligned} \tag{30}$$

where $p_{i,j}^0$ is given by evaluation of the initial condition onto our phase space grid. This scheme require a stability condition. The transport part require

$$\max\left(\frac{v_{\max}\Delta t}{\Delta x}, \max\left(U'\right)\frac{\Delta t}{\Delta v}\right) < 1 \tag{31}$$

where the Fokker-Planck operator part require

$$\Delta t < \frac{\Delta v^2}{2} \tag{32}$$

We look for a time step that will satisfy both inequality.

For the situation at the boundaries, we require that $[-x_{\max}, +x_{\max}] \times [-v_{\max}, +v_{\max}]$ is *big enough* such that $p_{i,j}^n$ is very close to zero near the border.